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The Walras Core of an Economy and Its Limit Theorem*

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Abstract

The *Walras core* of an economy is the set of allocations that are attainable for the consumers when their trades are constrained to be based on some agreed set of prices, and such that no alternative price system exists for any sub-coalition that allows all members to trade to something better. As compared with the *Edgeworth core*, both coalitional improvements and being a candidate allocation for the Walras core become harder. The Walras core may even contain allocations that violate the usual Pareto efficiency. Nevertheless, the competitive allocations are the same under the two theories, and the equal-treatment Walras core allocations converge under general conditions to the competitive allocations in the process of replication.

KEYWORDS: Competitive allocation, coalition, Edgeworth core, Walras core. (JEL C71, D5)

1 Introduction

Fundamental to the Walrasian model of exchange is the requirement that transactions be governed by a uniform price system. If commodities pass between consumers i and j in a certain ratio, then they cannot pass between consumers k and l in a different ratio. That is, the *law of one price* is imposed in the Walrasian model. However, prices in the Walrasian model are given *ex machina* and are not responsive to the consumers' buying and selling decisions. Consumers are passive with respect to prices at which they trade. A theory is

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therefore needed to give an account of the circumstances in which passive behavior with respect to prices (price-taking behavior) will be expected.

Edgeworth (1881) modeled market competition as consisting of people getting together in coalitions and working out deals cooperatively to their mutual advantage. The Edgeworth core of an economy – the core usually studied – is the set of allocations which cannot be improved upon by any coalition of consumers through *recontracting*. Market competition presented by the Edgeworth core provides a foundation of competitive equilibrium analysis, because the Edgeworth core is defined without reference to quoted prices, it is plausible for both small and large economies, and the Edgeworth core allocations converge to the competitive allocations in the process of replication. There is now a huge literature on Edgeworth core convergence initiated by Shubik (1959), followed by Debreu and Scarf (1963), among others.[1]

In this paper we contribute to the literature on core convergence, hence on foundations of competitive equilibrium analysis, by requiring that transactions within any given coalition resemble the Walrasian model of market exchange. More specifically, as with the Edgeworth recontracting, we allow the possibility for the consumers in an economy to trade cooperatively by organizing themselves into coalitions. However, we do not allow price discrimination nor do we assume the existence of a clearing house, so that the consumers transacting in a given coalition have to coordinate their decisions by agreeing on a suitable uniform price system to balance their trades. As such, we would get a different cooperative game – the *Walras market game* – from the unrestricted-barter or the *Edgeworth market game*. Correspondingly, we would also get a different core – the Walras core – from the Edgeworth core.

Consideration of an economy as a coalitional game that resembles the Walrasian model of exchange was suggested in Shapley (1976, p. 169):

> It might surprise some people to learn that the core of the cooperative game that most closely resembles the Walrasian model of exchange (the game in which the players negotiate a marketwide price system to govern all transactions) is different from the core usually studied. We do not know of any treatment of this core in the literature.

[1] Aumann (1964) formulated a model of a continuum of agents, in which he showed that the Edgeworth core coincides with the set of competitive allocations. This definition of perfect competition required the introduction of measure theory – notably Lyapunov's Theorem – into economics. The Edgeworth core convergence in more general forms has been pursued by economists since (Vind 1964 and Hildenbrand 1968, 1974).

In the mid 1970s, the Edgeworth core convergence was joined in the study of the rate of convergence of the Edgeworth core to the set of competitive allocations. Shapley (1975) showed that the convergence could be arbitrarily slow and concluded with the conjecture that for any fixed concave utility functions only a set of initial allocations of measure zero will yield cores that converge more slowly than the inverse of the number of agents. Debreu (1975) proved that such is indeed the case provided that the utility functions are C^2 and the indifference surfaces have positive Gaussian curvature. Grodal (1975) and Cheng (1981) subsequently extended this result to more general sequences of economies. Aumann (1979) showed that the arbitrarily slow convergence can occur even if the utility functions are infinitely differentiable. See also Cheng (1982).

Further remarks were also made in Shapley and Shubik (1977, p. 944):

> It is not generally realized that there is a distinctive "Walras" cooperative game, with a more restrictive characteristic function than the Edgeworth game. Despite the possible failure of superadditivity, and hence of balancedness, the existence of the core in the Walras market game is not threatened, since the competitive equilibrium must still have the core property. But the Walras core and Walras-Pareto set will in general be different from the Edgeworth core and Edgeworth-Pareto set.

In this paper we show that the restriction that trades within a given coalition be subject to the law of one price in the Walras market game is significant. When a pure exchange economy has three or fewer consumers, an allocation in the Walras core of the economy is either in the Edgeworth core or Pareto dominated by some allocation in the Edgeworth core. We give an example of a three-person pure exchange economy to illustrate that an allocation in the Walras core may sometimes be strictly Pareto dominated by an allocation in the Edgeworth core. In contrast with the first welfare theorem, the welfare loss at an allocation in the Walras core is resulted from the fact that the bundles in the allocation do not necessarily maximize consumers' preferences subject to budget constraint. We also provide an example of a four-person pure exchange economy to illustrate that an allocation in the Walras core may sometimes be neither in the Edgeworth core nor Pareto dominated by any allocation in the Edgeworth core. It follows that in general the Walras core of an economy is not included in or Pareto dominated by the Edgeworth core of the economy.

The intersection of the Edgeworth core and the Walras core contains all competitive allocations. However, we provide an example of a three-person pure exchange economy to illustrate that the intersection may sometimes contain allocations different from competitive allocations.

We consider a limit property under replication of the *equal-treatment allocations* in the Walras core for economies with household production, as in Hurwicz (1960), Rader (1964), Shapley (1973), Billera (1974), among others. The economies considered in Debreu and Scarf (1963) can be viewed as economies with household production. In addition, each Arrow-Debreu economy can be naturally converted into an economy with household production having same competitive allocations. We show that the equal-treatment Walras core allocations converge to competitive allocations in the process of replication for economies with general closed convex consumption and household production possibility sets and with preferences satisfying local nonsatiation, continuity, and a weaker notion of monotonicity than the usual weak monotonicity.[2]

The rest of the paper is organized as follows. Section 2 introduces the competitive, the Edgeworth core, and the Walras core allocations. Section 3 discusses differences between the Edgeworth core and the Walras core of an economy. Section 4 proves the convergence

[2]Florenzano (1990) establishes, among other results, convergence of the equal-treatment fuzzy core and hence the equal-treatment Edgeworth core allocations to competitive allocations for economies with general convex consumption and production possibility sets and with locally nonsatiated preferences.

of the equal-treatment Walras core allocations to competitive allocations in the process of replication, and section 5 concludes the paper.

2 The Competitive, Edgeworth Core, and Walras Core Allocations

Given any positive integer $0 < k < \infty$, we adopt the following notation: $\Re^k$ denotes the k-dimensional Euclidean space; $a, b \in \Re^k$, $a \geq b$ and $a \gg b$ mean $a_h \geq b_h$ and $a_h > b_h$ for $h = 1, 2, \cdots, k$, respectively; $\Re^k_+ = \{a \in \Re^k \mid a \geq 0\}$, $\Re^k_{++} = \{x \in \Re^k \mid a \gg 0\}$, and $\Delta^k = \{a \in \Re^k_+ \mid \sum_{h=1}^k a_h = 1\}$.

An *economy* with l goods, n consumers, and with household production is an array $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$, where $N = \{1, \cdots, n\}$ is the consumer set, $X^i \subseteq \Re^l$ is the consumption set of consumer i, $\succeq^i$ is i's preference relation, w^i is his endowment bundle, and $Y^i \subseteq \Re^l$ is his household production possibility set.[3] An element y^i in Y^i represents a production plan that i can carry out. As usual, inputs into production appear as negative components of y^i and outputs as positive components.

Let P^i denote the strict preference correspondence generated from preference relation $\succeq^i$. That is, for $x^i \in X^i$, $P^i(x^i)$ is the set of bundles $x'^i \in X^i$ such that $x'^i \succ^i x^i$. The preference relation $\succeq^i$ is *reflexive* if $x^i \succeq^i x^i$ for all $x^i \in X^i$; $\succeq^i$ is *transitive* if for any three bundles $x^i, x'^i, x''^i \in X^i$, $x^i \succeq^i x'^i$ and $x'^i \succeq^i x''^i$ imply $x^i \succeq^i x''^i$; and $\succeq^i$ is *locally non-satiated* if for each $x^i \in X^i$, x^i is in the closure $\mathrm{cl}P^i(x^i)$ of $P^i(x^i)$ relative to X^i.

The following assumptions will be made throughout the paper: For any $i \in N$,

A1: X^i is closed and convex;

A2: $\succeq^i$ is reflexive and locally non-satiated, P^i is open-valued (i.e., for each $x^i \in X^i$, $P^i(x^i)$ is an open set relative to X^i), and for any $x^i \in X^i$ and $x'^i \in P^i(x^i)$, there exists a bundle $e^i \in \Re^l_{++}$ such that $x'^i + \lambda e^i \in P^i(x^i)$ for $\lambda \geq 0$;

A3: Y^i is closed and convex;

A4: For any $x^i \in X^i$, $P^i(x^i) - Y^i + \Re^l_+ \subseteq P^i(x^i) - Y^i$.

Assumptions A1, A2, and A3 are implied by the standard assumptions. In particular, the third condition in A2 is implied by the usual weak monotonicity of $\succeq^i$.[4] Note also that A4 is satisfied if either $\succeq^i$ is weakly monotonic or $Y^i - \Re^l_+ \subseteq Y^i$. This assumption guarantees that competitive equilibrium prices are all nonnegative.

[3] As mentioned in the introduction, this model of an economy was considered in Hurwicz (1960), Rader (1964), Shapley (1973), Billera (1974), among others. See Qin (1993) for an application of this model to the study of competitive outcomes in the Edgeworth cores of NTU market games.

[4] We say that preference relation $\succeq^i$ is *weakly monotonic* if for each $x^i \in X^i$, $P^i(x^i) + \Re^l_+ \subseteq P^i(x^i)$.

We assume that for each coalition $S \subseteq N$, its production possibility set is simply $Y^S = \sum_{i \in S} Y^i$. If Y^i is a convex cone Y with vertex at the origin for all $i \in N$, then $Y^S = Y$. Consequently, economies in Debreu and Scarf (1963) can be viewed as economies with household production. When $Y^i = \{0\}$ for all $i \in N$, we call $\mathcal{E}$ a *pure exchange economy*, which we denote by $\mathcal{E} = \{(X^i, \succeq^i, w^i)\}_{i \in N}$.

2.1 *Competitive Allocations*

With household production, a production plan changes a consumer's initial endowment before trading with the others on the market. Hence, selection of a production plan by an individual is guided by preference maximization instead of profit maximization. However, under the Walrasian model of market exchange, preference maximization implies profit maximization.

Definition 1: A *competitive equilibrium* for economy $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$ is a point $((x^{*i}, y^{*i})_{i \in N}, p^*) \in (\mathrm{X}_{i \in N}(X^i \times Y^i)) \times \Delta^l$ such that

(i) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot w^i + p^* \cdot y^{*i}$ and $x^i \in P^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot w^i + p^* \cdot y^i$ for all $y^i \in Y^i$;

(ii) $\sum_{i \in N} x^{*i} = \sum_{i \in N} w^i + \sum_{i \in N} y^{*i}$.

We call $x^* = (x^{*i})_{i \in N}$ a *competitive allocation* (of consumption bundles).

In the Arrow-Debreu model of an economy with $l < \infty$ goods, there are a set of $n < \infty$ consumers with each consumer i characterized by the triplet $(X^i, \succeq^i, w^i)$ and a set of J producers with producer j characterized by a production possibility set Y^j. In addition, each consumer i is also endowed with a relative share θ_{ij} of firm j's profit for $j = 1, 2, \cdots, J$ (see Arrow and Debreu 1954, Debreu 1959). With a slight abuse of notation, we let J also denote the set of the J firms. Symbolically, an *Arrow-Debreu economy* is an array $\mathcal{E} = \{\{(X^i, \succeq^i, w^i)\}_{i \in N}, \{Y^j\}_{j \in J}, \{\theta_{ij}\}_{i \in N, j \in J}\}$.

The relative shares θ_{ij} may be interpreted as representing private proprietorships of the production possibilities and facilities. With this interpretation, we can think of consumer i as owning the technology set $\theta_{ij} Y_j$ at his disposal in firm j. Consequently, we may think of consumer i as owning the following production possibility set in the Arrow-Debreu economy:

$$\tilde{Y}^i = \sum_{j \in J} \theta_{ij} Y_j. \tag{1}$$

We denote elements in $\tilde{Y}^i$ by $\tilde{y}^i = \sum_{j \in J} \theta_{ij} y^{ij}$ for some $y^{ij} \in Y^j$, $j \in J$. The reader is referred to Rader (1964, pp. 160–163) and Nikaido (1968, p. 285) for a justification of this understanding of the consumers' ownership shares. With equation (1), the Arrow-Debreu economy $\mathcal{E}$ is converted into economy $\tilde{\mathcal{E}} = \{(X^i, \succeq^i, w^i, \tilde{Y}^i)\}_{i \in N}$.

Definition 2 (Arrow and Debreu 1954, Debreu 1959): A *competitive equilibrium* for an Arrow-Debreu economy $\mathcal{E} = \{\{(X^i, \succeq^i, w^i)\}_{i\in N}, \{Y^j\}_{j\in J}, \{\theta_{ij}\}_{i\in N, j\in J}\}$ is a point

$$((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*) \in (\mathrm{X_{i\in N} X^i}) \times (\mathrm{X_{j\in J} Y^j}) \times \Delta^1$$

such that

(i′a) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot w^i + \sum_{j\in J} \theta_{ij} p^* \cdot y^{*j}$ and $x^i \in P^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot w^i + \sum_{j\in J} \theta_{ij} p^* \cdot y^{*j}$;

(i′b) For $j \in J$, $p^* \cdot y^{*j} \geq p^* \cdot y^j$ for $y^j \in Y^j$;

(ii′) $\sum_{i\in N} x^{*i} = \sum_{i\in N} w^i + \sum_{j\in J} y^{*j}$.

Rader showed that an Arrow-Debreu economy $\mathcal{E}$ with convex production possibility sets is equivalent to economy $\tilde{\mathcal{E}}$, in the sense that the competitive allocations are the same across the two economies (see Rader 1964, pp. 160–163).

Theorem 1 *Let $\mathcal{E} = \{\{(X^i, \succeq^i, w^i)\}_{i\in N}, \{Y^j\}_{j\in J}, \{\theta_{ij}\}_{i\in N, j\in J}\}$ be an Arrow-Debreu economy and let $\tilde{\mathcal{E}} = \{(X^i, \succeq^i, w^i, \tilde{Y}^i)\}_{i\in N}$. If $((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*)$ is a competitive equilibrium for $\mathcal{E}$, then there are production plans $\tilde{y}^{*i} \in \tilde{Y}^i$, $i \in N$, such that $((x^{*i}, \tilde{y}^{*i})_{i\in N}, p^*)$ is a competitive equilibrium for $\tilde{\mathcal{E}}$. Conversely, if $((x^{*i}, \tilde{y}^{*i})_{i\in N}, p^*)$ is a competitive equilibrium for $\tilde{\mathcal{E}}$, then there are production plans y^{*j}, $j \in J$, such that $((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*)$ is a competitive equilibrium for $\mathcal{E}$.*

It is also worth mentioning at this point yet another equivalent model of an economy. This is the model considered by McKenzie (1959), in which production is aggregated with the aggregate production set generated by a set of linear activities instead of a set of firms as in Arrow-Debreu model. However, by considering consumers' ownership shares of a firm as their shares of an entrepreneurial factor which is private to the firm and not marketed, so that profits above the firm's payments to hired inputs can be imputed to the entrepreneurial factor, an Arrow-Debreu economy can be converted into a McKenzie economy with the same competitive allocations (see McKenzie 1959, pp. 66-67).

On the other hand, since technology in a McKenzie economy exhibits constant returns to scale, the maximum profit with respect to the aggregate production set is necessarily zero in competitive equilibrium. Thus, a McKenzie economy can also be converted into an Arrow-Debreu economy with the same competitive allocations, in which the aggregate production set is assigned to one firm whose ownership structure can be arbitrarily specified.

2.2 *The Edgeworth Core and the Walras Core*

Given a coalition $S \subseteq N$, an *S-allocation of consumption bundles* in economy $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$ is an S-tuple $x^S = (x^i)_{i \in S}$ of consumption bundles $x^i \in X^i$, for $i \in S$. We refer an N-allocation x^N as an allocation for the economy and we denote it simply as x.

Definition 3: An S-allocation of consumption bundles x^S is *Edgeworth-feasible* if there exist production plans $y^i \in Y^i$ for $i \in S$ such that $\sum_{i \in S} x^i = \sum_{i \in S} w^i + \sum_{i \in S} y^i$; x^S is *Walras-feasible* if there exist production plans $y^i \in Y^i$ for $i \in S$ and a price system $p \in \Delta^l$ such that $\sum_{i \in S} x^i = \sum_{i \in S} w^i + \sum_{i \in S} y^i$ and $p \cdot x^i = p \cdot w^i + p \cdot y^i$ for $i \in S$.

Let $E(S)$ and $W(S)$ denote respectively the set of Edgeworth- and Walras-feasible allocations of consumption bundles for coalition $S \subseteq N$.

Definition 4: Coalition S can improve upon an allocation x with Edgeworth-feasible allocations if there is an S-allocation $x'^S \in E(S)$ such that $x'^i \in P^i(x^i)$ for all $i \in S$; coalition S can improve upon allocation x with Walras-feasible allocations if there is an S-allocation $x'^S \in W(S)$ such that $x'^i \in P^i(x^i)$ for all $i \in S$.

Note that Definition 3 implies $W(S) \subseteq E(S)$ for any coalition $S \subseteq N$. Consequently, when a coalition cannot improve upon an allocation with Edgeworth-feasible allocations, neither can it improve upon the allocation with Walras-feasible allocations.

Definition 5: The *Edgeworth core* of an economy $\mathcal{E}$ is the set of allocations $x \in E(N)$ such that no coalition can improve upon with Edgeworth-feasible allocations; The *Walras core* of an economy $\mathcal{E}$ is the set of allocations $x \in W(N)$ such that no coalition can improve upon with Walras-feasible allocations.

Let $((x^{*i}, y^{*i})_{i \in N}, p^*)$ be a competitive equilibrium for $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$. By Definitions 3 and 4, if coalition S can improve upon $(x^{*i})_{i \in N}$ with Edgeworth-feasible allocations, then there are pairs $(x^i, y^i) \in X^i \times Y^i$, $i \in S$, such that $\sum_{i \in S} x^i = \sum_{i \in S} w^i + \sum_{i \in S} y^i$ and $x^i \in P^i(x^{*i})$ for all $i \in S$. By condition (i) in Definition 1, $p^* \cdot x^i > p^* \cdot w^i + p^* \cdot y^i$ for $i \in S$. This implies $p^* \cdot \sum_{i \in S} x^i > p^* \cdot \sum_{i \in S} w^i + p^* \cdot \sum_{i \in S} y^i$, which contradicts the condition of $\sum_{i \in S} x^i = \sum_{i \in S} w^i + \sum_{i \in S} y^i$. It follows that no coalition can improve upon a competitive allocation with Edgeworth-feasible allocations. Competitive allocations are thus Edgeworth core allocations. By Definitions 1 and 3, competitive allocations are Walras-feasible for coalition N. Hence, competitive allocations are also Walras core allocations. This shows that the intersection of the Edgeworth core and the Walras core contains all competitive allocations. It will be shown in the next section that the intersection may also contain allocations different from competitive allocations.

2.3 *Edgeworth and Walras Market Games with Utility Representations*

Suppose that for $i \in N$, the preference relation $\succeq^i$ is represented by utility function u^i. This means that for every two bundles $x^i, x'^i \in X^i$, $x^i \succeq^i x'^i$ if and only if $u^i(x^i) \geq u^i(x'^i)$. Let $\Re^N$ denote the n-dimensional Euclidean space with coordinates indexed by $i \in N$, and let $V_e(S)$ and $V_w(S)$ denote the utility possibility sets that consumers of coalition S can obtain from their Edgeworth- and Walras-feasible allocations, respectively. Then $V_e(S)$ is the set of utility vectors $u = (u_i)_{i \in N} \in \Re^N$ such that $u_i = 0$ for $i \notin S$ and for some S-allocation $x^S \in E(S)$, $u_i \leq u^i(x^i)$ for $i \in S$, and $V_w(S)$ is the set of utility vectors $u = (u_i)_{i \in N} \in \Re^N$ such that $u_i = 0$ for $i \notin S$ and for some S-allocation $x^S \in W(S)$, $u_i \leq u^i(x^i)$ for $i \in S$. The pairs (N, V_e) and (N, V_w) are *games in coalitional form.*

The pair (N, V_e) is customarily called the *Edgeworth market game.* In comparison, we call the coalitional game (N, V_w) the *Walras market game.* Definitions 3–5 imply that allocation x is in the Edgeworth core (resp. in the Walras core) if and only if the utility vector $(u^i(x^i))_{i \in N}$ is in the core of the Edgeworth market game (N, V_e) (resp. in the core of the Walras market game).

3 The Edgeworth Core versus the Walras Core

As noticed before, the intersection of the Edgeworth core and the Walras core contains all competitive allocations. The intersection of the Edgeworth core with the Walras core in the following example of a three–person pure exchange economy contains allocations different from competitive allocations. It follows that in general the intersection of the Edgeworth core with the Walras core does not coincide with the set of competitive allocations.

Example 1 (Scarf 1960): Consider a three–person exchange economy $\mathcal{E} = \{(X^i, \succeq^i, w^i)\}_{i \in N}$, where $N = \{1, 2, 3\}$, $X^i = \Re^3_+$ for $i \in N$, $w^1 = (1, 0, 0)$, $w^2 = (0, 1, 0)$, $w^3 = (0, 0, 1)$, and $\succeq^1$, $\succeq^2$, and $\succeq^3$ are respectively represented by utility functions:

$$u^1(x) = \min\{x_1, x_2\},\ u^2(x) = \min\{x_2, x_3\},\ u^3(x) = \min\{x_1, x_3\}.$$

The unique competitive allocation assigns bundles $\hat{x}^1 = (1/2, 1/2, 0)$ to consumer 1, $\hat{x}^2 = (0, 1/2, 1/2)$ to consumer 2, and $\hat{x}^3 = (1/2, 0, 1/2)$ to consumer 3 (see Scarf 1960). Now consider an alternative allocation from which consumer 1 receives $\bar{x}^1 = (1/3, 1/3, 0)$, 2 receives $\bar{x}^2 = (0, 2/3, 2/3)$, and 3 receives $\bar{x}^3 = (2/3, 0, 1/3)$. Then, $\bar{p} \cdot \bar{x}^1 = \bar{p} \cdot w^1$, $\bar{p} \cdot \bar{x}^2 = \bar{p} \cdot w^2$, and $\bar{p} \cdot \bar{x}^3 = \bar{p} \cdot w^3$, where $\bar{p} = (1/4, 1/2, 1/4)$. Thus, $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ is Walras-feasible. From the consumers' endowments and utility functions it follows easily that neither one–player nor two–player coalitions can improve upon the allocation $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ with Edgeworth-feasible allocations.

To show that $\bar{x}$ is in both the Edgeworth and the Walras cores, it suffices to show that the grand coalition cannot improve upon $\bar{x}$ with Edgeworth-feasible allocations. Suppose on the

contrary that there exists an allocation $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in E(N)$ such that $u^i(\tilde{x}^i) > u^i(\bar{x}^i)$, for $i = 1, 2, 3$. Then, $\min\{\tilde{x}_1^1, \tilde{x}_2^1\} > 1/3$ and $\min\{\tilde{x}_2^2, \tilde{x}_3^2\} > 2/3$. This shows $\tilde{x}_2^1 + \tilde{x}_2^2 > 1$. Hence $\tilde{x}$ is not Edgeworth-feasible for the grand coalition N, which is a contradiction. ■

When the number of consumers in a pure exchange economy is less than or equal to three, a welfare comparison between the Edgeworth core and the Walras core is possible. We now have:

Theorem 2 *Let $\mathcal{E} = \{(X^i, \succeq^i, w^i)\}_{i \in N}$ be a pure exchange economy. Assume that $\mathcal{E}$ has three or fewer consumers. Assume further that for each $i \in N$, the preference relation $\succeq^i$ is transitive and weakly monotonic. Then, for any allocation $\hat{x}$ in the Walras core, there is an allocation $\bar{x}$ in the Edgeworth core such that $\bar{x}^i \succeq^i \hat{x}^i$ for $i \in N$.*

Proof. If allocation $\hat{x}$ is in the Edgeworth core, then we choose $\bar{x}$ to be $\hat{x}$, and the proof would be completed. Now suppose that coalition S can improve upon $\hat{x}$ with Edgeworth-feasible allocations. Denote by s the number of consumers in S. By Definition 3, Edgeworth-feasible allocations are the same as Walras-feasible allocations for each one–consumer coalition. It follows that no one–consumer coalition can improve upon $\hat{x}$ with Edgeworth-feasible allocations. This shows $s > 1$. Suppose $s = 2$ and $S = \{i, j\}$. Then, by Definition 4, there exists an S-allocation $(\tilde{x}^i, \tilde{x}^j)$ such that $\tilde{x}^i \in X^i$, $\tilde{x}^j \in X^j$, and

$$\tilde{x}^i + \tilde{x}^j = w^i + w^j, \ \tilde{x}^i \in P^i(\hat{x}^i), \ \tilde{x}^j \in P^j(\hat{x}^j). \tag{2}$$

Since $\hat{x}$ is in the Walras core, $w^i \notin P^i(\hat{X}^i)$. Thus (2) together with the transitivity of $\succeq^i$ implies $\tilde{x}^i \in P^i(w^i)$ and hence, by the weak monotonicity of $\succeq^i$, $H_+^i = \{h \mid \tilde{x}_h^i > w_h^i\} \neq \emptyset$. On the other hand, (2) together with the transitivity of $\succeq^j$ also implies $\tilde{x}^j \in P^j(w^j)$ and hence, by the weak monotonicity of $\succeq^j$, $H_-^i = \{h \mid \tilde{x}_h^i < w_h^i\} \neq \emptyset$. This implies that there exists $p \in \Delta^l$ such that $p \cdot \tilde{x}^i = p \cdot w^i$ and $p \cdot \tilde{x}^j = p \cdot w^j$. Consequently, $(\tilde{x}^i, \tilde{x}^j)$ is also Walras-feasible for S. We conclude that coalition $S = \{i, j\}$ can also improve upon $\hat{x}$ in the Walras market game, which contradicts the assumption that $\hat{x}$ is in the Walras core. Consequently, it must be $s > 2$. Since $\mathcal{E}$ has no more than three consumers, it must be that the grand coalition of all three consumers can improve upon $\hat{x}$ with Edgeworth-feasible allocations. Choose a Pareto efficient allocation $\bar{x} \in E(N)$ such that $\bar{x}^i \in P^i(\hat{x}^i)$ for all $i \in N$. Then, the Pareto efficiency of $\bar{x}$ and the result that no coalition with two or fewer consumers can improve upon $\bar{x}$ with Edgeworth-feasible allocations imply that $\bar{x}$ is in the Edgeworth core. ■

The Pareto dominance of an Edgeworth core allocation over a Walras core allocation can sometimes be strict. This is illustrated in the following example.

Example 2: Let $\mathcal{E}$ be a three–person pure exchange economy in which $X^1 = X^2 = X^3 = \Re_+^2$, $w^1 = w^2 = (9, 1)$, $w^3 = (17, 3)$, and consumers' preference relations are respectively represented by utility functions $u^1(x) = u^2(x) = \sqrt{x_1 x_2}$, and $u^3(x) = x_1 + x_2$.

Consider an allocation $\hat{x}$ with $\hat{x}^1 = (6, 2)$, $\hat{x}^2 = (3, 3)$, and $\hat{x}^3 = (26, 0)$. This allocation is supported by price system $\hat{p} = (1/4, 3/4)$. Since $\sum_{i\in N} \hat{x}^i = \sum_{i\in N} w^i$, the allocation is Walras-feasible for the grand coalition N. Denote by $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ the utility vector associated with allocation $\hat{x}$. Then $\hat{u}_1 = \sqrt{12}$, $\hat{u}_2 = 3$, and $\hat{u}_3 = 26$, and thus $\hat{u}_i \geq u^i(w^i)$, $i \in N$. Notice that for $i = 1, 2$, since the utility possibility set $V_e(\{i, 3\})$ is bounded above by the plane $2u_i + u_3 = 30$, coalition $\{i, 3\}$ cannot improve upon $\hat{x}$ in the Edgeworth market game. Because consumers 1 and 2 are identical, it is easily checked that coalition $\{1, 2\}$ cannot improve upon $\hat{x}$ either. Note also that both $\hat{x}^1$ and $\hat{x}^2$ are interior bundles, but the marginal rate of substitution of consumer 1 at $\hat{x}^1$ is not the same as that of consumer 2 at $\hat{x}^2$. Thus, $\hat{x}$ is not Pareto efficient in the Edgeworth market game.[5] Since coalitions with two or fewer consumers cannot improve upon $\hat{x}$ with Edgeworth-feasible allocations and since $\hat{x}$ is not a Pareto efficient allocation in $E(N)$, we conclude that there is an Edgeworth core allocation $\bar{x}$ that strictly Pareto dominates $\hat{x}$.

We now show that $\hat{x}$ is in fact a Walras core allocation. To this end, notice first that the analysis in the previous paragraph implies that it suffices to show the the grand coalition cannot improve upon $\hat{x}$ in the Walras market game. Suppose on the contrary that the grand coalition can improve upon $\hat{x}$ in the Walras market game. Then, there exist an allocation $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ and a strictly positive price system,[6] $p = (1, \rho)$ with $\rho > 0$, such that

$$\sqrt{\tilde{x}_1^1 \tilde{x}_2^1} > \sqrt{12}, \quad \sqrt{\tilde{x}_1^2 \tilde{x}_2^2} > 3, \quad \tilde{x}_1^3 + \tilde{x}_2^3 > 26, \tag{3}$$

$$\tilde{x}_1^1 = 9 + \rho - \rho\tilde{x}_2^2, \quad \tilde{x}_1^2 = 9 + \rho - \rho\tilde{x}_2^2, \quad \tilde{x}_1^3 = 17 + 3\rho - \rho\tilde{x}_2^3, \tag{4}$$

and

$$\tilde{x}_1^1 + \tilde{x}_1^2 + \tilde{x}_1^3 = 35, \quad \tilde{x}_2^1 + \tilde{x}_2^2 + \tilde{x}_2^3 = 5. \tag{5}$$

Conditions (3) and (4) are respectively consumers' utility and budget constraints. By consumer 3's utility and budget constraints, $\rho \neq 1$, and

$$17 + 3\rho + (1 - \rho)\tilde{x}_2^3 > 26. \tag{6}$$

If $\rho < 1$, then (6) implies $(1 - \rho)(\tilde{x}_2^3 - 9) > 6\rho$; hence $\tilde{x}_2^3 > 9$. Since $\tilde{x}_2^1 \geq 0$ and $\tilde{x}_2^2 \geq 0$, $\tilde{x}_2^3 > 9$ contradicts (5). Now suppose $\rho > 1$. Then (6) implies $17 + 3\rho > 26$; hence $\rho > 3$. By consumer 1's utility and budget constraints in (3) and (4), $\rho(x_2)^2 - (9 + \rho)x_2 + 12 < 0$. It follows from the discriminant of this inequality that $(9 + \rho)^2 - 48\rho = (27 - \rho)(3 - \rho) > 0$. Since $\rho > 3$, we must have $\rho > 27$. With $\rho > 27$, consumers 1 and 2's budget constraints in (4) and the positivity of $\tilde{x}_1^1$ and $\tilde{x}_1^2$ imply $\tilde{x}_2^1 < 4/3$ and $\tilde{x}_2^2 < 4/3$. Thus by 1 and 2's utility constraints in (3), $\tilde{x}_1^1 > 9$ and $\tilde{x}_1^2 > 6.75$, which by (5) implies both $\tilde{x}_1^3 < 19.25$ and

[5] For example, as compared to allocation $\hat{x}$, allocation $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ with $\bar{x}^1 = (9 - \frac{9}{\sqrt{5}}, 5 - \sqrt{5})$, $\bar{x}^2 = (\frac{9}{\sqrt{5}}, \sqrt{5})$, $\bar{x}^3 = (26, 0)$ makes both consumers 1 and 2 better off and keeps consumer 3 indifferent.

[6] Since there are only two commodities, for $i \in N$ and for commodity $h = 1, 2$, $\tilde{x}_h^i \neq w_h^i$ in order for the grand coalition to improve upon allocation $\hat{x}$. Thus the supporting price system must be in $\Re_{++}^2$.

$\tilde{x}_2^3 \leq 5$. These last two inequalities contradict consumer 3's utility constraint in (3). Hence the allocation $\hat{x}$ is in the Walras core. ■

Although consumers can be better off cooperating than if they behave individually, consumers in the Walras market game cannot independently choose arbitrary bundles satisfying budget constraint determined by the price system they agreed to. Thus the bundle a consumer receives from a Walras core allocation does not necessarily maximize the consumer's utility subject to his budget constraint. This implies that the Pareto efficiency of Walras core allocations are not implied by the first welfare theorem.

Example 3 below is constructed to show that in general there can be Walras core allocations that are neither Edgeworth core allocations nor Pareto dominated by any Edgeworth core allocation. Thus, the welfare comparison of theorem 1 may not be extended to economies with a general number of consumers.

Example 3: Consider a four–person pure exchange economy, where for $i = 1, 2, 3$, the triplet $(X^i, \succeq^i, w^i)$ is the same as in Example 2, $X^4 = \Re^2_+$, $w^4 = (0, 1)$, and $\succeq^4$ is representable by $u^4(x) = x_1 + (1/a)x_2$ with

$$(i)\ 2 < a < \frac{27}{13}, \qquad (ii)\ \sqrt{\frac{(27-9a)(11a-9)}{a}} > (13-3a),$$

$$(iii)\ \frac{10a-18}{4a} + \frac{27(3-a)}{(9+a)} > 2, \quad (iv)\ \frac{9+a}{2\sqrt{a}} + \sqrt{\frac{(27-9a)(11a-9)}{4a}} > \sqrt{3(18-a)}.$$

Conditions (i)–(iv) are consistent because, for example, they are all satisfied when $a = 2.01$. Now consider the allocation $x(a) = (x^1(a), x^2(a), x^3(a), x^4(a))$ and the price system $p(a) \in \Delta^l$, where

$$x^1(a) = (\frac{9+a}{2}, \frac{9+a}{2a}), \quad x^2(a) = (\frac{27-9a}{2}, \frac{11a-9}{2a}),$$

$$x^3(a) = (17+3a, 0), \quad x^4(a) = (a, 0), \quad p(a) = (\frac{1}{1+a}, \frac{a}{1+a}).$$

The allocation $x(a)$ is supported by the price system $p(a)$. Furthermore, $\sum_{i\in N} x^i(a) = \sum_{i\in N} w^i$. The allocation is therefore Walras-feasible for the grand coalition.

Step 1: No coalition with two or fewer consumers can improve upon $x(a)$ in the Edgeworth market game and hence in the Walras market game.

Note that $u^i(x^i(a)) \geq u^i(w^i)$ for $i \in N$. This shows that no one–consumer coalition can improve upon $x(a)$. Note also that consumers' endowments imply that neither coalition $\{1, 2\}$ nor coalition $\{3, 4\}$ can improve upon $\hat{x}(a)$ in the Edgeworth market game. Furthermore, for $i = 1, 2$ and for $(u_i, u_4) \in V_e(\{i, 4\})$, $u_4 \geq a$ implies $u_i \leq \sqrt{2(9-a)}$. On the other

hand, $u^4(x^4(a)) = a$ and by condition (i) on a, $u^2(x^2(a)) > \sqrt{2(9-a)}$. This shows that coalition $\{2,4\}$ cannot improve upon $x(a)$ in the Edgeworth market game. Since consumers 1 and 2 are identical and since $u^1(x^1(a)) > u^2(x^2(a))$, coalition $\{1,4\}$ cannot improve upon $x(a)$ in the Edgeworth market game. Since the utility possibility set $V_e(\{i,3\})$, $i = 1,2$, is bounded above by the plane $2u_i + u_3 = 30$, $u_i, u_3 \geq 0$ as noticed in Example 2, to prove that neither of the coalitions $\{1,3\}$ and $\{2,3\}$ can improve upon $x(a)$ in the Edgeworth market game, it suffices to show $2u^2(x^2(a)) + u^3(x^3(a)) \geq 30$. This inequality holds if and only if $\sqrt{(27-9a)(11a-9)/a} \geq 13 - 3a$. This last inequality is satisfied under condition (ii) on a.

Step 2: No three–consumer coalitions can improve upon $x(a)$ in the Walras market game.

Observe that $u^1(x^1) \geq 3$ and $u^3(x^3) \geq 26$ imply $x_1^1 + x_2^1 \geq 6$ and $x_1^3 + x_2^3 \geq 26$. Since coalition $\{1,3,4\}$ has total endowment $(26,5)$, it cannot improve upon $x(a)$ in the Edgeworth market game. By analogy, neither can coalition $\{2,3,4\}$ improve upon $x(a)$. Next, with condition (iii) on a, the analysis in Example 2 can be applied to show that coalition $\{1,2,3\}$ cannot improve upon $x(a)$ in the Walras market game. Next, $(u_1,u_2,u_4) \in V_e(\{1,2,4\})$ and $u_4 \geq a$ imply $u_1 + u_2 \leq \sqrt{3(18-a)}$. However, by condition (iv), $u^1(x^1(a)) + u^2(x^2(a)) > \sqrt{3(18-a)}$. This shows that coalition $\{1,2,4\}$ cannot improve upon $x(a)$ in the Edgeworth market game.

Step 3: The grand coalition cannot improve upon $x(a)$ in the Walras market game.

For any bundle $x^4 \in X^4$, $u^4(x^4) > u^4(x^4(a))$ and $p \cdot x^4 = p \cdot w^4$ imply $(p_2/p_1) > a$. With this restriction on the supporting price ratio, a similar analysis as in Example 2 shows that the grand coalition cannot improve upon $x(a)$ in the Walras market game. This concludes that $x(a)$ is a Walras core allocation.

Step 4: Allocation $x(a)$ is neither an Edgeworth core allocation nor Pareto dominated by any Edgeworth core allocation.

It suffices to show that Edgeworth core allocations all fail to satisfy

$$u^i(x^i) \geq u^i(x^i(a)),\ i \in N. \tag{7}$$

Suppose on the contrary that there is an Edgeworth core allocation x satisfies (7). Then, by the utility functions of consumers 1 and 2, x^1 and x^2 in allocation x must be interior bundles. Hence, if consumer 4's bundle x^4 in the allocation is also an interior bundle, then the Pareto efficiency of allocation x would imply that the marginal rates of substitution of consumers 1, 2, and 4 must be the same. In that case, $x_2^1 = ax_1^1$, $x_2^2 = ax_1^2$, and so by (7),

$$x_2^1 \geq \sqrt{a}u^1(x^1(a)), \quad x_2^2 \geq \sqrt{a}u^2(x^2(a)). \tag{8}$$

Since $u^1(x^1(a)) + u^2(x^2(a)) > 6$, (8) together with condition (i) on a implies $x_2^1 + x_2^2 > 6$. This is impossible because the total endowment of commodity 2 is 6. It follows that x^4 cannot be an interior bundle, implying either $x_1^4 = 0$ or $x_2^4 = 0$. By (7), $x_1^4 + (1/a)x_2^4 \geq a$. Consequently, $x_1^4 = 0$ implies $x_2^4 \geq a^2 > 4$. However, with $x_2^4 > 4$, we have either $x_2^1 < 1$ or $x_2^2 < 1$. In either case, (7) implies $x_1^1 + x_2^1 + x_1^2 + x_2^2 > 15$,[7] which in turn implies that consumer 3's bundle x^3 in allocation x satisfies $x_1^3 + x_2^3 < 22$. This violates (7) for $i = 3$ because $u^3(x^3(a)) = 17 + 3a > 23$. Consequently, $x_1^4 \geq a$ and $x_2^4 = 0$. Similarly, consumer 3's bundle x^3 must satisfy $x_1^3 \geq 17 + a$ and $x_2^3 = 0$. Set $u_i = u^i(x^i)$, $i \in N$. By the Pareto efficient of x, $u_3 = 35 - (u_1 + u_2)^2/6 - u_4$. Since $u_4 \geq a$,

$$u_3 \leq 35 - a - \frac{(u_1 + u_2)^2}{6}. \tag{9}$$

Since $u_3 > 17 + 3a$ and $a > 2$, (9) implies $(u_1 + u_2)^2 < 60$. On the other hand, with consumer 1 having a utility level of $u_1 > 0$ and consumer 2 having a utility level of $u_2 > 0$ such that $(u_1 + u_2)^2 < 175$, there exists a utility level u_3' for player 3 such that $(u_1, u_2, u_3') \in V_e(\{1, 2, 3\})$ and [8]

$$u_3' \geq 35 - \frac{(u_1 + u_2)^2}{5},$$

which together with (9) implies

$$u_3' - u_3 \geq a - \frac{(u_1 + u_2)^2}{30}. \tag{10}$$

Since $(u_1+u_2)^2 < 30a$, it follows from (10) that $u_3' > u_3$, which means that the utility vector (u_1, u_2, u_3) is below the Pareto frontier of $V_e(\{1, 2, 3\})$. Consequently, coalition $\{1, 2, 3\}$ can improve upon the allocation x in the Edgeworth market game. This contradicts the assumption that allocation x is in the Edgeworth core. ■

In the Walras market game (N, V_w), a coalition may not achieve those allocations that are achievable when the coalition is divided into two or more disjoint sub-coalitions. That is, the Walras market game sometimes may not be *superadditive*.[9] Superadditivity can be restored to the Walras market game by the device of taking the superadditive cover.[10] The

[7] For example, $x_2^1 < 1$ implies $x_1^1 > 9$ for $\sqrt{x_1^1 x_2^1} \geq 3$ to hold. On the other hand, $x_1^2 + x_2^2 \geq 2\sqrt{x_1^2 x_2^2}$. Thus by (7), $x_1^2 + x_2^2 \geq 6$. This shows $x_1^1 + x_2^1 + x_1^2 + x_2^2 > 15$.

[8] To see why the following inequality holds, consider bundles $x^1 = (\frac{u_1(u_1+u_2)}{5}, \frac{5u_1}{u_1+u_2})$, $x^2 = (\frac{u_2(u_1+u_2)}{5}, \frac{5u_2}{u_1+u_2})$, and $x^3 = (35 - \frac{(u_1+u_2)^2}{5}, 0)$. Then, the tuple (x^1, x^2, x^3) is Edgeworth-feasible for coalition $S = \{1, 2, 3\}$ and $u_1(x^1) = u_1$, $u^2(x^2) = u_2$, and $u^3(x^3) = 35 - \frac{(u_1+u_2)^2}{5}$.

[9] We say that a coalitional game (N, V) is *superadditive* if $V(S) + V(T) \subseteq V(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

[10] Let (N, V) be a coalitional game. For $S \subseteq N$, set $\bar{V}(S) = \sum_{P \in \mathcal{P}(S)} (\sum_{T \in P} V(T))$, where $\mathcal{P}(S)$ denotes the set of partitions of S. The pair $(N, \bar{V})$ is the *superadditive cover* of the coalitional game (N, V) (see Shapley and Shubik 1969).

core of (N, V_w) is contained in the core of its superadditive cover $(N, \bar{V}_w)$.[11] Thus Example 3 also shows that the superadditive cover of the Walras market game is still different from the Edgeworth market game. This result confirms the point on superadditivity of the Walras market game raised in Shapley and Shubik (1977).

4 A Limit Theorem on the Walras Core

In this section, we establish convergence of the equal-treatment Walras core allocations of an economy $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$ to competitive allocations in the process of replication. To consider the process of replication, we now reinterpret N as the set of *types* with type i characterized by the quadruple $(X^i, \succ^i, w^i, Y^i)$. For each positive integer r, $\mathcal{E}^r$ denotes the economy obtained from replicating economy $\mathcal{E}$ for r times, so that there are r consumers of each type in $\mathcal{E}^r$. For $i \in N$ and for $1 \leq m \leq r$, the m-th consumer of type i in $\mathcal{E}^r$ is denoted by im. An allocation for $\mathcal{E}^r$ is denoted by $x(x^{im})_{m=1,i=1}^{r,n}$.

For $r = 1, 2, \cdots, \infty$, we say that an allocation $(x^{im})_{m=1,r=1}^{r,n}$ is an *equal-treatment allocation* for $\mathcal{E}^r$ if $x^{im} = x^{im'}$, for all $i = 1, 2, \cdots, n$ and for all $m, m' = 1, \cdots, r$. Equal treatment property is interesting not only for the equal treatment it asserts, but also because it simplifies the description of the allocations in replicated economies. Equal-treatment allocations for $\mathcal{E}^r$ can all be conveniently represented by allocations for $\mathcal{E}$ that are composed of representative bundles for the various types instead of bundles for the various consumers. Thus, equal-treatment allocations for $\mathcal{E}^r$ can be viewed as allocations in the allocation space for $\mathcal{E}$, the dimensions of which remains fixed as the economy is replicated. Conversely, allocations in the allocation space for $\mathcal{E}$ can also be viewed as equal-treatment allocations for $\mathcal{E}^r$ for all r by letting the consumers of each type all have the representative bundle for that type. In the literature the Edgeworth core convergence is established typically for equal treatment Edgeworth core allocations.

We show that if an allocation for $\mathcal{E}$ is in the Walras core of $\mathcal{E}^r$ for $r = 1, 2, \cdots, \infty$, then it must be a competitive allocation for $\mathcal{E}$ under assumptions A1-A4 and the following assumption A5.

A5: For $i \in N$, $w^i \in \text{int}(X^i - Y^i)$ (the interior of $X^i - Y^i$).

[11] Let $u^* = (u_i^*)_{i \in N}$ be any utility vector in the Walras core. Suppose u^* is not in the core of $(N, \bar{V}_w)$, the superadditive cover of (N, V_w). Then for some coalition S, there exists a utility vector $u \in \bar{V}_w(S)$ such that $u_i > u_i^*$ for all $i \in S$. By definition of superadditive cover, there exists a partition $P \in \mathcal{P}(S)$ such that $u \in \sum_{T \in P} V_w(T)$. Let $u^T \in V_w(T)$ for $T \in P$ such that $u = \sum_{T \in P} u^T$. Then for $T \in P$ and for $i \in T$, $u_i = u_i^T$. This shows that every coalition in P can improve upon u^* in the Walras market game (N, V_w), which contradicts the assumption that u^* is in the core of (N, V_w). This shows that u^* is in the core of $(N, \bar{V}_w)$.

Given an allocation x for $\mathcal{E}$, let $Z^i(x) = P^i(x^i) - \{w^i\} - Y^i$ for $i \in N$ and

$$Z(x) = \left\{ \left\{ \sum_{k=1}^{K} \alpha_k z^k \;\middle|\; \alpha \in \Delta^K, z^k \in Z^{i_k}(x), i_k \in N \right\} \;\middle|\; 1 \leq K < \infty \right\}. \tag{11}$$

Then, $Z(x)$ is a convex subset of $\Re^l$.

Lemma 1 *Let $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$ be an economy satisfying A1-A5 and let $\hat{x} = (\hat{x}^i)$ be an allocation for $\mathcal{E}$ such that for some $\hat{y}^i \in Y^i$, $i \in N$, $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} w^i + \sum_{i \in N} \hat{y}^i$. If $0 \notin Z(\hat{x})$, then $\hat{x}$ is a competitive allocation for $\mathcal{E}$.*

Proof. Since $0 \notin Z(\hat{x})$, the Separating Hyperplane Theorem implies that there is a vector $\hat{p} \in \Re^l$ such that $\hat{p} \neq 0$ and

$$\hat{p} \cdot z \geq 0, \text{ for } z \in Z(\hat{x}). \tag{12}$$

Since $Z^i(\hat{x}) \subseteq Z(\hat{x})$ for $i \in N$, it follows from (12) that $\hat{p} \cdot z^i \geq 0$ for $z^i \in Z^i(\hat{x})$ or equivalently

$$\hat{p} \cdot x^i \geq \hat{p} \cdot w^i + \hat{p} \cdot y^i \text{ for } (x^i, y^i) \in P^i(\hat{x}^i) \times Y^i. \tag{13}$$

By A4, $P^i(\hat{x}^i) - Y^i + \Re^l_+ \subseteq P^i(\hat{x}^i) - Y^i$, which together with (13) implies $\hat{p} \in \Re^l_+$. We may thus assume $\hat{p} \in \Delta^l$. Next, by A2, $\hat{x}^i \in \mathrm{cl}P^i(\hat{x}^i)$ and hence by (13), $\hat{p} \cdot \hat{x}^i \geq \hat{p} \cdot w^i + \hat{p} \cdot \hat{y}^i$. Since $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} w^i + \sum_{i \in N} \hat{y}^i$, we have $\hat{p} \cdot \hat{x}^i = \hat{p} \cdot w^i + \hat{p} \cdot \hat{y}^i$ for $i \in N$. Thus to show that $((\hat{x}^i, \hat{y}^i), \hat{p})$ is a competitive equilibrium for $\mathcal{E}$, by Definition 1 it only remains to check that for $i \in N$ and for $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$, $\hat{p} \cdot x^i > \hat{p} \cdot w^i + \hat{p} \cdot y^i$.

Fix $i \in N$ and $(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$. By A5, $w^i \in \mathrm{int}(X^i - Y^i)$ which implies that there exists a pair $(x'^i, y'^i) \in X^i \times Y^i$ such that $\hat{p} \cdot x'^i < \hat{p} \cdot w^i + \hat{p} \cdot y'^i$. By A1 and A3, $(1-t)(x'^i, y'^i) + t(x^i, y^i) \in X^i \times Y^i$ for $t \in [0,1]$. Since $x^i \in P^i(\hat{x}^i)$ and since $P^i(\hat{x}^i)$ is open relative to X^i by A2, we have $t(x'^i, y'^i) + (1-t)(x^i, y^i) \in P^i(\hat{x}^i) \times Y^i$ for small $t \in (0,1)$. Thus, it follows from (13) that $t\hat{p} \cdot (x'^i - w^i - y'^i) + (1-t)\hat{p} \cdot (x^i - w^i - y^i) \geq 0$ for small $t \in (0,1)$. Since $\hat{p} \cdot (x'^i - w^i - y'^i) < 0$ by construction, it must be $\hat{p} \cdot (x^i - w^i - y^i) > 0$. ■

We are now ready to state and prove a limit property for the Walras core in the equal-treatment allocation space. We establish the property by applying Lemma 1 and a claim which we prove in an appendix.

Theorem 3 *Let $\mathcal{E} = \{(X^i, \succeq^i, w^i, Y^i)\}_{i \in N}$ be an economy satisfying A1-A5 and let $\hat{x} = (\hat{x}^i)$ be an allocation for $\mathcal{E}$. If $\hat{x}$ is in the Walras core of $\mathcal{E}^r$, for all $r = 1, 2, \cdots, \infty$, then it must be a competitive allocation for $\mathcal{E}$.*

Proof. Since $\hat{x}$ is in the Walras core of $\mathcal{E}$, it follows from Definitions 3 and 5 that there exists a production plan $\hat{y}^i \in Y^i$ for for each $i \in N$ such that $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} w^i + \sum_{i \in N} \hat{y}^i$. By Lemma 1, it suffices to show that $Z(\hat{x})$ does not contain the origion $0 \in \Re^l$.

Suppose on the contrary $0 \in Z(\hat{x})$. Then, by (11), there exist an integer $\bar{K} \geq 1$, an element $\bar{\alpha} \in \Delta^{\bar{K}}$, a sequence $\{\bar{i}_k\}_{k=1}^{\bar{K}}$ of elements $\bar{i}_k \in N$, and a sequence $\{(\bar{x}^k, \bar{y}^k)\}_{k=1}^{\bar{K}}$ of pairs $(\bar{x}^k, \bar{y}^k) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) \times Y^{\bar{i}_k}$ such that [12]

$$\sum_{k=1}^{\bar{K}} \bar{\alpha}_k(\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k) = 0. \tag{14}$$

By A5, we can choose a pair $(\tilde{x}^{\bar{i}_k}, \tilde{y}^{\bar{i}_k}) \in X^{\bar{i}_k} \times Y^{\bar{i}_k}$ such that

$$w^{\bar{i}_k} = \tilde{x}^{\bar{i}_k} - \tilde{y}^{\bar{i}_k} \text{ for } 1 \leq k \leq \bar{K},\ i \in N. \tag{15}$$

For $1 \leq k \leq \bar{K}$, choose a compact cube $B^{\bar{i}_k}$ in $\Re^l$ containing production plans $\bar{y}^k$ and $\tilde{y}^{\bar{i}_k}$ as its interior points. Then the closedness and convexity of Y^i imply that $\bar{Y}^i = Y^i \cap B^i$ is compact convex.

Since $\bar{x}^k \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$, A2 implies that there is an element $\bar{e}^k \in \Re^l_{++}$ such that $\bar{x}^k + \lambda \bar{e}^k \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$ for $\lambda \geq 0$. By the compactness of $\bar{Y}^{\bar{i}_k}$, we can choose a scalar $0 < \bar{\lambda}^k < \infty$ such that

$$\bar{x}^k + \bar{\lambda}^k \bar{e}^k \gg w^{\bar{i}_k} + y^{\bar{i}_k},\ y^{\bar{i}_k} \in \bar{Y}^{\bar{i}_k}. \tag{16}$$

Set

$$x^{0k} = \bar{x}^k + \bar{\lambda}^k \bar{e}^k, \text{ for } 1 \leq k \leq \bar{K}. \tag{17}$$

Next, for $1 \leq k \leq \bar{K}$, choose a compact cube $C^{\bar{i}_k} \subset \Re^l$ containing bundles $\tilde{x}^{\bar{i}_k}$, $\bar{x}^k$, and x^{0k} as its interior points. Then the closedness and convexity of $X^{\bar{i}_k}$ implies that $\bar{X}^{\bar{i}_k} = X^{\bar{i}_k} \cap C^{\bar{i}_k}$ is compact convex. Since $w^{\bar{i}_k} = \tilde{x}^{\bar{i}_k} - \tilde{y}^{\bar{i}_k}$ by (15) and since $\tilde{x}^{\bar{i}_k}$ and $\tilde{y}^{\bar{i}_k}$ are respectively in the interior of $\bar{X}^{\bar{i}_k}$ and $\bar{Y}^{\bar{i}_k}$ by construction, $w^{\bar{i}_k} \in \text{int}(\bar{X}^{\bar{i}_k} - \bar{Y}^{\bar{i}_k})$ (the interior of $\bar{X}^{\bar{i}_k} - \bar{Y}^{\bar{i}_k}$). In addition, since $x^{0k} = \bar{x}^k + \bar{\lambda}^k \bar{e}^k$ by (17) and since $\bar{x}^k + \bar{\lambda}^k \bar{e}^k \gg w^{\bar{i}_k} + y^{\bar{i}_k}$, for $y^{\bar{i}_k} \in \bar{Y}^{\bar{i}_k}$ by (16), we have

$$p \cdot x^{0k} > p \cdot \bar{x}^k \text{ and } p \cdot x^{0k} > p \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p) \text{ for } p \in \Delta^l. \tag{18}$$

Claim: There exists $((x^{*k})_{k=1}^{\bar{K}}, (y^{*k})_{k=1}^{\bar{K}}, p^*, \alpha^*) \in (\mathrm{X}_{k=1}^{\bar{K}} \bar{X}^{\bar{i}_k}) \times (\mathrm{X}_{k=1}^{\bar{K}} \bar{Y}^{\bar{i}_k}) \times \Delta^l \times \Delta^{\bar{K}}$ *such that*

$$x^{*k} \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) \cap \bar{X}^{\bar{i}_k},\ p^* \cdot x^{*k} = p^* \cdot w^{\bar{i}_k} + p^* \cdot y^{*k} \text{ when } \alpha_k^* > 0 \tag{19}$$

and

$$\sum_{k:\alpha_k^* > 0} \alpha_k^*(x^{*k} - w^{\bar{i}_k} - y^{*k}) \leq 0. \tag{20}$$

Proof of Claim is in the Appendix to this paper.

[12] Notice that it is possible that $\bar{i}_k = i_{k'}$ for two integers $k \neq k'$ and $1 \leq k, k' \leq \bar{K}$.

Let $z^* = \sum_{k:\alpha_k^*>0} \alpha_k^*(x^{*k} - w^{\bar{i}_k} - y^{*k})$. Then,

$$\sum_{k:\alpha_k^*>0} \alpha_k(x^{*k} - w^{\bar{i}_k} - y^{*k} - z^*) = 0. \tag{21}$$

Furthermore, by (19) and (20),

$$z^* \leq 0 \text{ and } p^* \cdot z^* = 0. \tag{22}$$

Since $x^{*k} - y^{*k} - z^* \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) - Y^{\bar{i}_k} + \Re_+^l$ and since $P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) - Y^{\bar{i}_k} + \Re_+^l \subseteq P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) - Y^{\bar{i}_k}$ by A4, there is a pair $(\hat{x}^{*k}, \hat{y}^{*k}) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) \times Y^{\bar{i}_k}$ such that $x^{*k} - y^{*k} - z^* = \hat{x}^{*k} - \hat{y}^{*k}$. Thus, by (19), (21), and (22),

$$p^* \cdot \hat{x}^{*k} = p^* \cdot w^{\bar{i}_k} + p^* \cdot \hat{y}^{*k} \text{ when } \alpha_k^* > 0 \tag{23}$$

and

$$\sum_{k:\alpha_k^*>0} \alpha_k^*(\hat{x}^{*k} - w^{\bar{i}_k} - \hat{y}^{*k}) = 0. \tag{24}$$

Fix $1 \leq k \leq \bar{K}$ with $\alpha_k^* > 0$. For each positive integer t, let $\lambda_k(t)$ be the smallest integer greater than or equal to $t\alpha_k^*$. Then, $0 \leq t\alpha_k^*/\lambda_k(t) \leq 1$ and $t\alpha_k^*/\lambda_k(t) \to 1$ as $t \to \infty$. Let $(x^k(t), y^k(t)) = [t\alpha_k^*/a_k(t)](\hat{x}^{*k}, \hat{y}^{*k}) + [1 - (t\alpha_k^*/a_k(t))](\tilde{x}^{\bar{i}_k}, \tilde{y}^{\bar{i}_k})$ for $t = 1, \cdots, \infty$. Then, $(x^k(t), y^k(t)) \in X^{\bar{i}_k} \times Y^{\bar{i}_k}$ by A1 and A3. By (15) and (23),

$$p^* \cdot x^k(t) = p^* \cdot w^{\bar{i}_k} + p^* \cdot y^k(t). \tag{25}$$

On the other hand, $\sum_{k:\alpha_k^*>0} \lambda_k(t)[x^k(t) - w^{\bar{i}_k} - y^k(t)] = \sum_{k:\alpha_k^*>0} \lambda_k(t)\{[t\alpha_k^*/\lambda_k(t)][\hat{x}^{*k} - w^{\bar{i}_k} - \hat{y}^{*k}] + [1 - (t\alpha_k^*/\lambda_k(t))][\tilde{x}^{\bar{i}_k} - w^{\bar{i}_k} - \tilde{y}^{\bar{i}_k}]\}$. Consequently, by (15) and (24),

$$\sum_{k:\alpha_k^*>0} \lambda_k(t)[x^k(t) - w^{\bar{i}_k} - y^k(t)] = t \sum_{k:\alpha_k^*>0} \alpha_k^*[\hat{x}^{*k} - w^{\bar{i}_k} - \hat{y}^{*k}] = 0. \tag{26}$$

Note that $x^k(t) \to \hat{x}^{*k}$ as $t \to \infty$ because $t\alpha_k^*/\lambda_k(t) \to 1$ as $t \to \infty$. Thus, since $\hat{x}^{*k} \in P^i(\hat{x}^{\bar{i}_k})$ by construction, A2 implies $x^k(t) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$ for large enough t.

Take t so large that $x^k(t) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$ for $1 \leq k \leq \bar{K}$ with $\alpha_k^* > 0$. For $i \in N$, let $r_i = \sum_{k:\bar{i}_k=i} \lambda_k(t)$ and let $r = \max\{r_i \mid i \in N\}$. Now consider coalition S_r consisting of r_i consumers of type i for $i \in N$. Then, by (25) and (26), the allocation, which assigns bundle $x^k(t)$ to $\lambda_k(t)$ consumers of type i with $\bar{i}_k = i$ for $1 \leq k \leq \bar{K}$, is Walras-feasible for S_r in economy $\mathcal{E}^r$. This together with the inclusions of $x^k(t) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$ for $1 \leq k \leq \bar{K}$ with $\alpha_k > 0$ implies that coalition S_r can improve upon allocation $\hat{x}$ with Walras-feasible allocations in economy $\mathcal{E}^r$. We have therefore established the needed contradiction. ■

Assumptions A1–A5 are weaker than the standard assumptions on the elements of an economy in general equilibrium theory. These assumptions do not guarantee that the

Walras core allocations all satisfy the equal-treatment property. They would if in addition consumers' preferences are strictly convex, and the strong coalitional improvements in the definition of the Walras core are replaced with weak coalitional improvements, as follows: coalition S *weakly improves* upon an allocation x in the Walras market game if there is an S-allocation $x^S \in W(S)$ such that $x^i \notin P^i(x'^i)$ for all $i \in S$ and $x'^i \in P^i(x^i)$ for at least one consumer $i \in S$. With strict convexity imposed on preferences in addition to assumptions A1-A5, it is known that the Edgeworth core with weak coalitional improvements allocations also have equal-treatment property.

5 Conclusion

In this paper we considered a model of market exchange – The Walras market game – in which consumers can trade cooperatively by organizing themselves into coalitions. However, unlike the unrestricted barter, trades within each coalition are subject to the law of one price in our model. The law of one price was shown to be a significant restriction, in that it makes the resulting Walras-Pareto set – and hence the Walras core – substantially different from their counterparts of the Edgeworth unrestricted barter.

Allocations in the Walras core are supported by price systems. However, bundles in a Walras core allocation do not necessarily maximize consumers' utilities subject to budget constraint. Consequently, the first welfare theorem does not necessarily apply to Walras core allocations.

We have shown by example that the Walras core may contain allocations that violate the usual Pareto efficiency. Nevertheless, the intersection of the Edgeworth and the Walras cores is nonempty, and contains all competitive allocations. Moreover, the equal-treatment Walras core allocations converge in the process of replication to the competitive allocations under fairly general conditions on the elements of an economy. This convergence result reinforces the price-taking behavior and hence the law of one price for large economies.

One of the issues that is left undiscussed in this paper is the rate of convergence of the Walras core. As mentioned before, the generic rate of convergence of the Edgeworth core of a sufficiently smooth economy has the same order as the reciprocal of the number of agents (footnote 2). It remains to be explored at what rate the Walras core converges under replication to the set of competitive allocations.

Appendix: Proof of Claim

Let $\bar{K} \geq 1$, $\bar{\alpha} \in \Delta^{\bar{K}}$, $\{\bar{i}_k\}_{k=1}^{\bar{K}}$ with $\bar{i}_k \in N$, and $\{(\bar{x}^k, \bar{y}^k)\}_{k=1}^{\bar{K}}$ with $(\bar{x}^k, \bar{y}^k) \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k}) \times Y^{\bar{i}_k}$ be as in the proof of Theorem 2. We begin by constructing several mappings that enable us to apply the Kakutani Fixed-Point Theorem.

First, for $1 \leq k \leq \bar{K}$, define mappings $\pi^{\bar{i}_k} : \Delta^l \to \Re$ and $\Pi^{\bar{i}_k} : \Delta^l \to \bar{Y}^{\bar{i}_k}$ by $\pi^{\bar{i}_k}(p) = \max\{p \cdot y^{\bar{i}_k} \mid y^{\bar{i}_k} \in \bar{Y}^{\bar{i}_k}\}$ and

$$\Pi^{\bar{i}_k}(p) = \{y^{\bar{i}_k} \in \bar{Y}^{\bar{i}_k} \mid p \cdot y^{\bar{i}_k} = \pi^{\bar{i}_k}(p)\}. \tag{27}$$

By the Maximum Theorem in Berge (1963), $\pi^{\bar{i}_k}$ is continuous and $\Pi^{\bar{i}_k}$ is upper semicontinuous (henceforth shortened to u.s.c.). Furthermore, $\Pi^{\bar{i}_k}$ is compact- and convex-valued (i.e., $\Pi^{\bar{i}_k}(p)$ is a compact convex subset of $\bar{Y}^{\bar{i}_k}$ for $p \in \Delta^l$).

Next, define mappings $\mu^k : \Delta^l \to \Re$ and $d^k : \Delta^l \to \bar{X}^i$ by

$$\mu^k(p) = \frac{p \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p) - p \cdot \bar{x}^k}{\bar{\lambda}^k p \cdot \bar{e}^k}, \tag{28}$$

and

$$d^k(p) = \begin{cases} \bar{x}^k & \text{if} \quad p \cdot \bar{x}^k > p \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p), \\ (1 - \mu^k(p))\bar{x}^k + \mu^k(p) x^{0k} & \text{if} \quad p \cdot \bar{x}^k \leq p \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p). \end{cases} \tag{29}$$

By (18) and by the continuity of $\pi^i(p)$, μ^i and d^i are well-defined and continuous.

Define correspondences $Q : (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{X}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{Y}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times \Delta^{\bar{\mathrm{K}}} \to \Delta^{\mathrm{l}}$ and $A : \Delta^l \to \Delta^{\bar{K}}$ by [13]

$$Q(x, y, \alpha) = \{p \in \Delta^l \mid p \cdot \sum_{k=1}^{\bar{K}} \alpha_k (x^k - w^{\bar{i}_k} - y^k) \geq p' \cdot \sum_{k=1}^{\bar{K}} \alpha_k (x^k - w^{\bar{i}_k} - y^k),\ p' \in \Delta^l\}, \tag{30}$$

and

$$A(p) = \{\alpha \in \Delta^{\bar{K}} \mid p \cdot \sum_{k=1}^{\bar{K}} \alpha_k (\bar{x}^k - w^k - \bar{y}^k) \leq p \cdot \sum_{k=1}^{\bar{K}} \alpha'_k (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k),\ \alpha' \in \Delta^{\bar{K}}\}. \tag{31}$$

Clearly, both Q and A are compact- and convex-valued and by the Maximum Theorem, they are also u.s.c.

Finally, define correspondence Φ from $(\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{X}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{Y}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times \Delta^{\mathrm{l}} \times \Delta^{\bar{\mathrm{K}}}$ to itself by

$$\Phi(x, y, p, \alpha) = (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \{\mathrm{d}^{\mathrm{k}}(\mathrm{p})\}) \times (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \Pi^{\bar{\mathrm{i}}_{\mathrm{k}}}(\mathrm{p})) \times \mathrm{Q}(\mathrm{x}, \mathrm{y}, \alpha) \times \mathrm{A}(\mathrm{p})$$

for $(x, y, p, \alpha) \in (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{X}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{Y}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times \Delta^{\mathrm{l}} \times \Delta^{\bar{\mathrm{K}}}$. Then, Φ is u.s.c. nonempty-, compact-, and convex-valued. Since the domain of Φ is nonempty, compact, and convex, the Kakutani Fixed-Point Theorem implies that there is a point $(x^*, y^*, p^*, \alpha^*) \in (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{X}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times (\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{Y}}^{\bar{\mathrm{i}}_{\mathrm{k}}}) \times \Delta^{\mathrm{l}} \times \Delta^{\bar{\mathrm{K}}}$ such that $(x^*, y^*, p^*, \alpha^*) \in \Phi(x^*, y^*, p^*, \alpha^*)$.

By construction, $x^{*k} = d^k(p^*)$ and $y^{*k} \in \Pi^{\bar{i}_k}(p^*)$ for $1 \leq k \leq \bar{K}$, $p^* \in Q(x^*, y^*, \alpha^*)$, and $\alpha^* \in A(p^*)$. From (31), $\alpha^* \in A(p^*)$ implies $p^* \cdot \sum_{k=1}^{\bar{K}} \alpha_k^* (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k) \leq p^* \cdot \sum_{k=1}^{\bar{K}} \bar{\alpha}_k (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k)$. Since $\sum_{k=1}^{\bar{K}} \bar{\alpha}_k (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k) = 0$ by (14), we have $p^* \cdot \sum_{k=1}^{\bar{K}} \alpha_k^* (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k) \leq 0$. This shows $p^* \cdot (\bar{x}^k - w^k - \bar{y}^k) \leq 0$ for at least one $1 \leq k \leq \bar{K}$ with $\alpha_k^* > 0$. Consequently, by the construction of the correspondence, A, we have $p^* \cdot (\bar{x}^k - w^{\bar{i}_k} - \bar{y}^k) \leq 0$ for $1 \leq k \leq \bar{K}$ with $\alpha_k^* > 0$. Since $\bar{y}^k \in \bar{Y}^{\bar{i}_k}$, $p^* \cdot \bar{x}^k \leq p^* \cdot w^{\bar{i}_k} + p^* \cdot \bar{y}^k \leq p^* \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p^*)$. Hence, by (17) and (29), $x^{*k} = d^k(p^*) = (1 - \mu^k(p^*))\bar{x}^k + \mu^k(p^*) x^{0k} = \bar{x}^k + \bar{\lambda}^k \mu^k(p^*) \bar{e}^k$. This, together with (28), implies $p^* \cdot x^{*k} = p^* \cdot w^{\bar{i}_k} + \pi^{\bar{i}_k}(p^*)$. Furthermore, by the choice of $\bar{e}^k$, $x^{*k} \in P^{\bar{i}_k}(\hat{x}^{\bar{i}_k})$

[13] Elements in $\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{X}}^{\bar{\mathrm{i}}_{\mathrm{k}}}$ are denoted by $x = (x^k)_{k=1}^{\bar{K}}$ and those in $\mathrm{X}_{\mathrm{k}=1}^{\bar{\mathrm{K}}} \bar{\mathrm{Y}}^{\bar{\mathrm{i}}_{\mathrm{k}}}$ by $y = (y^k)_{k=1}^{\bar{K}}$.

and by (27), $y^{*k} \in \Pi^{\bar{i}_k}(p^*)$ implies $p^* \cdot y^{*k} = \pi^{\bar{i}_k}(p^*)$. This establishes (19) from which it follows that

$$p^* \cdot \sum_{k:\alpha_k^*>0} \alpha_k^*(x^{*k} - w^{\bar{i}_k} - y^{*k}) = 0.$$

Finally, by (30) and the above equality, $p^* \in Q(x^*, y^*, \alpha^*)$ implies $p' \cdot \sum_{k:\alpha_k^*>0} \alpha_k^*(x^{*k} - w^{\bar{i}_k} - y^{*k}) \leq 0$ for $p' \in \Delta^l$, which in turn implies (20). ■

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