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Essays on Heterogeneous Preferences, Persuasion, and Planner-doer Games

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ABSTRACT OF THE DISSERTATION

Essays on Heterogeneous Preferences, Persuasion, and Planner-doer Games

By

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This dissertation consists of three essays that study heterogeneous preferences in economic activities and their implications for welfare. Chapter one axiomatizes the choice behavior that implies a finite distribution of underlying quasi-linear utilities (types). The choice alternatives are pairs of goods and their prices. Given the choice data over such choice alternatives, this model can uniquely construct the underlying types and their distribution and establish the existence of a quasi-linear tie-breaking rule. This identification gives a unique social welfare aggregator consistent with the Pareto efficiency criterion.

Chapter two is on signaling games with an imperfectly informed victim and a perfectly informed defendant with respect to whether the defendant is actually liable to the victim. A two-agent game and a three-agent extension where the victim can hire a lawyer who is perfectly informed but pursues a selfish objective in his advice are compared. In particular, a lawyer affects a victim's information environment in a way that is similar to Bayesian persuasion (Kamenica & Gentzkow, 2011). Overall, this analysis captures some stylized empirical patterns of the legal system such as the litigious tendency due to different parameters, and identifies both the positive and negative welfare effects of lawyers' advice.

Delegation is common in decision-making settings. Delegation usually comes with some costs since the planner needs to motivate doers to make appropriate choices. Such costs

can result from hidden actions such as private commitments or potential future verifications and thus can be unobservable to outsiders. Chapter three axiomatizes the planner's ex-ante preferences over finite menus and derives the planner's hidden delegation costs from such preferences. A special case when the delegation cost is binary (0 or ∞) is studied, and an algorithm to check whether ex-post choices conform to the delegation model with binary costs is provided.

Chapter 1

Random Quasi-linear Utility

In this chapter, I refine the random utility model (RUM) of Block and Marschak (1959) to represent stochastic choice data with quasi-linear types. In my framework, choices are observed across pairs of goods and money. The random quasi-linear utility function is identified uniquely in my model. This identification implies a unique social welfare aggregator that is consistent with the Pareto efficiency criteria. In general, the uniqueness of the quasi-linear tie-breaking rule is not guaranteed, but it can be obtained in a special case where the tie-breaking is uniform. I also characterize a special case where the set of possible types is binary.

1.1 Introduction

Stochastic choice data are common in empirical settings and can naturally arise from heterogeneous preferences. Such heterogeneity can be revealed by any generic group of agents due to taste differences. Even a single agent who faces repeated choices can often vary her responses when affected by psychological factors like perception, anchoring, framing, and so on (see, e.g., McFadden, 2001).

Block and Marschak (1959) model stochastic choice data via the random utility model (RUM), where the probability distribution of the choices reflects some endogenous distributions of types. In general, Block and Marschak assume that each type has a utility representation, but they impose no other constraints. Using a constructive proof, Falmagne (1978) shows that the sufficient condition for the rationalizability of the choice probabilities is the non-negativity of Block-Marschak polynomials. In Falmagne's construction, the marginal distribution of the types are recovered uniquely, although the joint distribution is not. An equivalent condition that characterizes the rationalizability of the choice probabilities is the axiom of revealed stochastic preferences (ARSP) proposed by McFadden and Richter (1990) (see also Stoye, 2019).

The general RUM provides a framework to identify heterogeneous preferences. However, the general model can be problematic for several reasons. First, the identification of heterogeneous types in the general RUM is not unique, and therefore the general RUM does not necessarily distinguish the objective distribution of types (see more discussions in Turansick, 2021 and McClellon, 2015).[1] Furthermore, RUM is agnostic about ties, and each possible type is derived as a linear (total) order over the finite domain X. For example, when applied to the domain of lotteries where agents have expected utilities, RUM would only allow a finite number of lotteries, and the lotteries would not have cash equivalents. Since many applications would involve infinite choice domains and weak orders on the choice alternatives, the RUM assumptions are overly restrictive. Finally, a related problem is that the general model does not provide a way to impose useful structures on the types.

To address these problems, the recent literature has studied various refinements of RUM. When additional structures are imposed on the distribution of ordinal types, the identifica-

[1]Any general RUM with full support is nonunique (McClellon, 2015; Turansick, 2021). In particular, Block-Marschack polynomials, $q(x, A)$, characterize the probability weight put on orders for which the strict upper contour set of x is exactly $X \setminus A$. One can construct a probability flow that respects observed probabilities from Block-Marschak polynomials. However, such a construction is nonunique since the flow at a two-in and two-out branching can go two different ways, and thus can be generated by different random utility functions. See the detailed discussion in Turansick (2021).

tion of types can be unique. These structures include the single-crossing property discussed by both Apesteguia, Ballester, and Lu (2017) and Filiz-Ozbay and Masatlioglu (2020) or the random attention restrictions discussed by Manzini and Mariotti (2014).[2] Uniqueness can also be obtained when the space of observations is enriched, for example, when choices are observed across lotteries (Gul & Pesendorfer, 2006)[3] or across state-contingent acts (Lu, 2016, 2021; McClellon, 2015), when choices are observed both ex ante and ex post (Ahn & Sarver, 2013) or when dynamic choices are observed over time (Duraj, 2018; Frick, Iijima, & Strzalecki, 2019; Lu & Saito, 2018).

In this chapter, I extend the line of refinements that enriches the underlying space of observations and study the random utility model with quasi-linear types. To do so, I augment choice alternatives with prices (or equivalently, wealth). In particular, the choice domain in my model is defined on $Z \times \mathbb{R}$, where $Z = \{0, 1, ..., n\}$ is a finite set of goods, and $\mathbb{R}$ includes the possible prices for goods in Z. The interpretation of the set, Z, is flexible. When Z consists of different goods, the model estimates private values on different goods. When Z consists of different quantities of the same good, the model captures the nonlinear pricing. When Z consists of different combinations of a set of physical goods, the model captures the bundling effect.

My analysis assumes that data are observed for price vectors in the entire real vector space of dimension $|Z|$. By observing the choices at various prices, the space of available observations is greatly expanded. In practice, variations in prices can be observed across different locations or over time. For example, in the marketplace, commodity prices change constantly. In auctions, there can be a large amount of bidding data that reflects demand for different goods at different prices.

[2] Aguiar (2015) generalizes the random attention model to capture both the similarity effect, which extends Manzini and Mariotti (2014) model, and the attraction effect, which violates regularity and thus do not nest with RUM. Cattaneo, Ma, Masatlioglu, and Suleymanov (2020) discuss a large class of nonparametric random attention rules.

[3] However, RUM with some classes of risk preferences still has the nonuniqueness problem; see Lin (2020).

I restrict all types to be quasi-linear in terms of price. Quasi-linearity is natural in many settings in mechanism design, auction theory, bargaining theory, public welfare analysis, and so on. Many work in these fields assume quasi-linear utility (see, e.g., Border, 1991; Che, Kim, & Mierendorff, 2013; Demange, Gale, & Sotomayor, 1986). Additionally, in empirical demand analysis, it is common assume that prices and incomes enter the utilities linearly (see, e.g., S. Berry, Levinsohn, & Pakes, 1995; McFadden, 1997). In my model, each quasi-linear type has the representation $U : Z \times \mathbb{R} \to \mathbb{R}$ as follows:

$$U(i, c_i) = v_i - c_i, \tag{1.1.1}$$

where $c_i \in \mathbb{R}$ is the price of the good i, and v_i is the agent's reserve value for good i. My model captures the case when observed choices are made by a collection of quasi-linear types. It identifies the underlying types and their distribution. Alternatively, my model can characterizes the choice behavior of one quasi-linear agent whose type changes stochastically. I call this model the *random quasi-linear utility model* (RQUM).

The added richness of primitives and the RQUM specification make the identification of my model unique. To illustrate how RQUM gives uniqueness, we first consider a dataset of the ordinal RUM without monetary variations. Let $Z = \{a, b, c, d\}$, and $\rho(x, A)$ be a stochastic choice function that represents the probability of choosing x from A for $A \subseteq Z$. In RUM,

$$\rho(x, A) = \sum_{R \in \mathcal{T}} \pi(R) \mathbf{1}_{x = R(A)},$$

where $\mathcal{T}$ is the set of types (linear orders) on Z, $R(A)$ is the best element in A according to $R \in \mathcal{T}$, and the random utility function (RUF) $\pi \in \Delta(\mathcal{T})$ is the finite distribution of types. Using this setup, we consider the example given by Fishburn (1998) (see also Turansick, 2021)[4]:

$$\pi_1 = \begin{cases} a \succ b \succ c \succ d \quad w.p. \quad 0.5 \\ \\ b \succ a \succ d \succ c \quad w.p. \quad 0.5 \end{cases}, \quad \text{and} \quad \pi_2 = \begin{cases} a \succ b \succ d \succ c \quad w.p. \quad 0.5 \\ \\ b \succ a \succ c \succ d \quad w.p. \quad 0.5 \end{cases}.$$

It is easy to verify that π_1 and π_2 both generate the same stochastic choice function $\rho(x, A)$. For example, $\rho(a, \{a, b, c, d\}) = \rho(b, \{a, b, c, d\}) = 0.5$ for both π_1 and π_2.

The money dimension in RQUM produces many more observations. Suppose the utility function is cardinal, and each good is associated with a cost vector that can vary. Consider the cardinal *random quasi-linear utility function* (RQUF) π_3:

$$\pi_3 = \begin{cases} v(a, b, c, d) = (5, 3, 2, 1) \quad w.p. \quad 0.5 \\ \\ v(a, b, c, d) = (3, 4, 1, 2) \quad w.p. \quad 0.5 \end{cases}. \tag{1.1.2}$$

One can verify that π_3 generates the same observations as π_1 and π_2 when the associated wealth vector is $(0, 0, 0, 0)$. However, many more observations can be produced when the cost vector changes. Notice that when the cost vector is in the support of RQUF, i.e., $c = (5, 3, 2, 1)$ or $c = (3, 4, 1, 2)$, there is a full tie. Hence, the support of RQUF is uniquely identified as the cost vector that induces full ties, and RQUF is uniquely constructed as the distribution of such full ties.

Furthermore, with applications of inclusion-exclusion principle, the probability of ties on any subset can also be identified in my model. Hence, one can study tie-breaking rules more closely. I study the uniform tie-breaking in Theorem 1.2 as a special case. In general, the tie-breaking rule in RUM can be hard to characterize. The ordinal RUM in Block and Marschak (1959) and the random expected utility model (henceforth, REU) in Gul and Pesendorfer (2006) are both restricted to the case where there are no ties. Gul and

[4] w.p. denotes with probability.

Pesendorfer (2013) study the ordinal RUM with weak orders and characterize ties with the "double total monotonicity" condition. Lu (2016) treats ties as nonmeasurable sets in the Anscombe-Aumann framework. Piermont and Teper (2019) discuss the tie-breaking in random expected utility models using choice capacities, which are quantities that are not necessarily observable. Compared to those models, the identification of ties in RQUM is much more transparent.

The uniqueness of identification in RQUM and the quasi-linearity of types are particularly convenient for aggregating social welfare.[5] Note that the ordinal types that appear in the general RUM will always make Pareto comparisons incomplete – sometimes extremely so. For example, if opposite types like $a \succ b \succ c \succ d$ and $d \succ c \succ b \succ a$ are possible, then we cannot make Pareto comparisons on any two distinct alternatives. Pareto aggregation is still problematic in the REU, where Pareto aggregation has many free parameters (Harsanyi, 1955). By contrast, the Pareto criterion in RQUM delivers the *unique* aggregate social welfare for any choice alternative paired with costs:

$$W(i, c_i) = \sum_{v \in V} \pi(v)(v_i - c_i) = \sum_{v \in V} \pi(v) v_i - c_i, \tag{1.1.3}$$

where V is the set of types, and the probability, $\pi \in \Delta(V)$, represents the proportion of agents of each type. Hence, when a planner observes how a population votes on the choice alternatives in Z, and she constructs the types in the population with Theorem 1.1, she can use (1.1.3) to choose an option that maximizes the welfare of the population.[6]

My main result (Theorem 1.1) characterizes RQUM in terms of an observable stochastic choice function, and it establishes the existence of a tie-breaking rule that is wealth in-

[5]Furthermore, the welfare prediction in utility maximization models (i.e., Marshallian models) are only correct with quasi-linear assumptions (Willig, 1976).

[6]RQUM can also have applications in the measurement of marketing effectiveness. For example, an analyst can use various framing techniques to promote certain feasible options. By constructing RQUF with RQUM model under these different situations, the analyst can understand how a marketing technique changes the public's preferences.

variant using four behavioral axioms. The most intriguing axiom in my model is axiom 3, Alternation. Alternation guarantees the weights on types calculated from the observations to be nonnegative. Alternation is analogous to the alternating condition in the theory of capacities which characterizes a capacity function (Barthélemy, 2000; Grabisch, 2015; Zhou, 2013).[7] As illustrated in the example with (1.1.2), RQUF in my model is the same as the probability on full ties. To identify RQUF, we observe that by reducing the value of c on subsets $A \subseteq Z$ slightly, one obtains the probability that at least one element in A is a maximizer. By the inclusion-exclusion principle, one can construct the probability that all elements in Z are maximizers, i.e., the probability of a full tie. The construction is slightly complicated by the tie-breaking (see the detailed discussion in Section 1.2).

To summarize, this chapter refines the RUM by restricting the types to be quasi-linear. This specification allows the identification of both the types and the tie-breaking rule. Furthermore, the identification of the types is unique. The remainder of this chapter is organized as follows. Section 1.2 introduces the primitives and the axioms and develops Theorem 1.1. Additionally, Section 1.2 provides the construction of RQUF and illustrates the role of Axiom 1.3 with two examples. Finally, Section 1.2 provides a sketch of proof for Theorem 1.1. Theorem 1.1 shows that the tie-breaking rule exists and is wealth independent, but do not further characterize it. Theorem 1.2 refines this result and characterizes the unique uniform tie-breaking rule for RQUM. Choices with binary types are common in situations like household decisions or decisions made with dual cognitive systems (see more discussions in Manzini & Mariotti, 2018). Theorem 1.3 characterizes RQUM for binary types. These two special cases, RQUM with uniform tie-breaking and RQUM with binary types, are discussed in Section 1.3. Section 1.4 discusses the relation of RQUM to other models, including the multinomial logit model and models for path independent choice functions. The proofs for Theorem 1.1 are provided in Section 1.5, and the proofs for Theorems 1.2 and 1.3 are

[7] Capacity functions are not new in the stochastic choice literature. For example, capacity functions can capture ties in stochastic choices (Gul & Pesendorfer, 2013; Piermont & Teper, 2019) and can be used to measure attention on a set (Aguiar, 2015).

provided in Section 1.6.

1.2 Main Model

To begin, let $Z = \{0, 1, ..., n\}$ be a finite domain of goods, and let $\Delta(Z)$ be the simplex of the probability distributions on Z. As money is desirable, any good should be selected with its lowest available cost. Therefore, without loss of generality, we associate each good with one cost.

A function, $\rho : \mathbb{R}^{n+1} \to \Delta(Z)$, is called a *random choice rule* (RCR) on Z. One can interpret the probability $\rho_i(c)$ as the observed likelihood of good i when the cost vector is c. For any $A \subseteq Z$,

$$\rho_A(c) = \sum_{i \in A} \rho_i(c)$$

is the combined likelihood of all $i \in A$ at cost vector c.

A utility function, $U : Z \times \mathbb{R}$, is called *quasi-linear* if

$$U(i, c_i) = v_i - c_i \tag{1.2.1}$$

for some vector $v \in \mathbb{R}^{n+1}$. Let $\mathbb{R}_0^{n+1}$ be the set of all vectors $v \in \mathbb{R}^{n+1}$ with $v_0 = 0$. Obviously, the value of good $0 \in Z$ can be restricted to zero without loss of generality. Good $0 \in Z$ can be interpreted as a default option. The private values on the choice alternatives are thus relative and can be either positive or negative. As a result, the prices(costs) are also relative and can be either positive or negative.

Let us say that $t : \mathbb{R}_0^{n+1} \times \mathbb{R}^{n+1} \to \Delta(Z)$ is a tie-breaking rule if

$$t_i(v, c) > 0 \implies i \in M(v, c),$$

where $M(v, c)$ is the set of all goods that maximizes the quasi-linear utility at the cost vector $c \in \mathbb{R}^{n+1}$,

$$M(v, c) = \arg\max_{i \in Z}(v_i - c_i). \tag{1.2.2}$$

In other words, t assigns positive probabilities only to the optimal choices in Z according to the quasi-linear utilities. If the maximizer i is unique, then $t_i(v, c) = 1$. Let T be the set of all quasi-linear tie-breaking rules, such that

$$t(v, c) = t(v, c + \alpha\mathbf{1}), \forall \alpha \in \mathbb{R}, \tag{1.2.3}$$

where $\mathbf{1} = \{1, .., 1\} \in \mathbb{R}^{n+1}$, and $\alpha\mathbf{1}$ is the constant vector $(\alpha, ..., \alpha)$. This property is analogous to the quasi-linear structure of utility functions.

Now, let Π be the set of all probability distributions that have finite support in $\mathbb{R}_0^{n+1}$. Let us say that $(\pi, t) \in \Pi \times T$ is a *random quasi-linear representation (RQR)* for ρ if for any $c \in \mathbb{R}^{n+1}$,

$$\rho(c) = \sum_{v \in \mathbb{R}_0^{n+1}} \pi(v) t(v, c). \tag{1.2.4}$$

The RQUF, $\pi \in \Pi$, is the probability distribution on the finite types. Since $t_i(v, c) > 0$ only if i is a maximizer for v, the likelihood of i at cost vector c, $\rho_i(c)$, is the weighted sum of the probabilities of types for which i is a maximizer.

To characterize representation (1.2.4), we can consider several conditions for the RCR, ρ. First, we adapt a standard invariance property from the quasi-linear utility model.

Axiom 1.1 (Wealth Invariance (WI)). *For all* $c \in \mathbb{R}^{n+1}$, $\alpha \in \mathbb{R}$,

$$\rho(c) = \rho(c + \alpha\mathbf{1}).$$

It is assumed here that the optimal choice, as well as the tie-breaking rule, is invariant of the constant wealth variations $\alpha\mathbf{1}$.

For any $A \subseteq Z$, $c, c' \in \mathbb{R}^{n+1}$, we write $c \gg_A c'$ if $c_i > c'_i$ for $i \in A$ and $c_j \leq c'_j$ for $j \notin A$.

Axiom 1.2 (Monotonic Demand (MD)). $c \gg_A c' \implies \rho_A(c) \leq \rho_A(c')$.

This condition assumes that the aggregate demand for goods in set $A \subseteq Z$ should not decrease if all goods in A become cheaper without reducing the costs of any other good. In the standard theory of demand, this condition corresponds to positive price effects. This axiom does not say anything about substitution patterns in terms of product characteristics.

For any subset $A \subseteq Z$, let the characteristic vector $\mathbf{1}_A \in \mathbb{R}^{n+1}$ be equal to 1 if $i \in A$ or equal to 0 if $i \notin A$.

Axiom 1.3 (Alternation). *For all* $\beta > \alpha > 0$,

$$\sum_{A \subseteq Z, |A| \text{ is odd}} \rho_A(c - \beta\mathbf{1}_A) \geq \sum_{A \subseteq Z, |A| \text{ is even}} \rho_A(c - \alpha\mathbf{1}_A). \tag{1.2.5}$$

Axiom 1.3 is analogous to the *alternating* property of capacity (see, e.g., Grabisch, 2016). I distinguish between α and β to account for tie-breaking. This point will be made more clear in the discussion of the construction of π in Section 1.2.1. To reach some level of intuition for Axiom 1.3, we rewrite (1.2.5) as

$$\sum_{A \subseteq Z, |A| \text{ is odd}} \sum_{i \in A} \left[\rho_i(c - \beta\mathbf{1}_A) - \rho_i(c)\right] \geq \sum_{A \subseteq Z, |A| \text{ is even}} \sum_{i \in A} \left[\rho_i(c - \alpha\mathbf{1}_A) - \rho_i(c)\right].$$

The above inequality is equivalent to (1.2.5) because ρ is an additive set function, and for any $i \in Z$, $|\{A \subseteq Z : i \in A, A \text{ is odd}\}| = |\{A \subseteq Z : i \in A, A \text{ is even}\}|$. The quantity $\rho_i(c - \alpha \mathbf{1}_A) - \rho_i(c), i \in A$ is the demand change for good i when it is in the set of goods that have price discounts. Axiom 1.3 imposes restrictions on such changes. When $Z = \{0, 1, 2\}$, Axiom 1.3 requires

$$\sum_{i=0}^{2} (\rho_i(c_i - \beta, c_i) - \rho_i(c)) - (\rho_i(c) - \rho_i(c_i + \alpha, c_{-i})) \geq 0. \tag{1.2.6}$$

This is very similar to the convexity condition. It requires that the average demand increase when price fall is greater than the average demand decrease when price rise.

The last axiom is technical and restricts the number of types to be finite.

Axiom 1.4 (Finite Range (FR))**.** *For any $A \subseteq Z$, $c \in \mathbb{R}^{n+1}, \alpha \in \mathbb{R}$, the function $\rho_A(c+\alpha \mathbf{1}_A)$ has a finite range in $[0, 1]$ that includes both 0 and 1.*

Taken together, Axiom 1.2 and Axiom 1.4 imply that $\rho_A(c + \alpha \mathbf{1}_A)$ is a monotonic step function, such that $\rho_A(c + \alpha \mathbf{1}_A)$ is decreasing in α. Due to tie-breaking, $\rho_j(c + \alpha \mathbf{1}_A)$ is not necessarily increasing for $j \notin A$. To see this, consider a type that has i, j, k as maximizers at c, and the tie-breaking rule is choosing j. At $(c_i + \alpha, c_{-i}), \alpha > 0$, the type has only two maximizers, j, k. The tie-breaking rule is to choose k. Then, $\rho_j(c_i + \alpha, c_{-i}) < \rho_j(c), \varepsilon \to 0$.

Regularity, which requires that the observed probability of a choice alternative increases as the menu enlarges, follows from the maximization of RUM (Block & Marschak, 1959) and of REU (Gul & Pesendorfer, 2006). Here, a menu is a collection of pairs of goods and their finite cost vectors. The lack of monotonicity in ρ will cause violations of regularity in RQUM. This is illustrated in Example 1.1 below.

Example 1.1. *Suppose* $Z = \{0, 1, 2\}$, *and*

$$\pi = \begin{cases} v_1 = (0,3,1) & w.p.\ \frac{1}{2}, \\ v_2 = (0,2,1) & w.p\ \frac{1}{2}. \end{cases}$$

Under price $c^1 = (0, 2, 1)$, *type* v_1 *chooses physical good* 1, *but type* v_2 *is indifferent across the three options. In this case, suppose type* v_2 *chooses the status quo. Hence* $\rho(c^1) = (\frac{1}{2}, \frac{1}{2}, 0)$.

Now, suppose $c^2 = (0, 2, \infty)$. *In effect, good* 2 *becomes unavailable. In this case, type* v_1 *still chooses physical good* 1. *Type* v_2 *is indifferent between the status quo and physical good* 1, *and she chooses each with equal probability. In this case,* $\rho(c^2) = (\frac{1}{4}, \frac{3}{4}, 0)$. *These observations satisfy Axioms 1 through 4. However, regularity is violated:* $\rho_1(c^2) < \rho_1(c^1)$. *In fact, the tie-breaking rule is not regular:* $t_1(v_1, c^1) > t_1(v_1, c^2)$. *This corresponds to a type of decoy effect: the presence of good 2 increases the choice probability for good 1.*

Since Axiom 1.4 restricts changes in $\rho_A(\alpha, c + \alpha\mathbf{1}_A)$ to occur at finitely many points, $\rho_j(c + \alpha\mathbf{1}_A)$ is increasing at all but finitely many points. Moreover, RCR satisfies the following asymptotic properties:

$$\lim_{\alpha\to\infty} \rho_A(c + \mathbf{1}_A\alpha) = \lim_{\alpha\to-\infty} \rho_j(c + \mathbf{1}_A\alpha) = 0, \text{ and } \lim_{\alpha\to-\infty} \rho_A(c + \mathbf{1}_A\alpha) = 1 \quad \forall A \subseteq Z, j \in Z \setminus A.$$

My main result is as follows:

Theorem 1.1. *An RCR* ρ *satisfies Axioms 1.1–1.4 if and only if* ρ *is represented by the RQR* $(\pi, t) \in \Pi \times T$. *Moreover, this* π *is unique.*

Theorem 1.1 characterizes my main model in terms of the observable properties of RCR. The representation has two components: RQUF π, the distribution of quasi-linear types $v \in \mathbb{R}_0^{n+1}$, and the tie-breaking rule $t \in T$. π in RQUM has the desirable property of being unique and thus it overcomes the identification issues in the RUM model. The uniqueness of π allows us to define a Pareto efficient utility aggregator, as shown in (1.1.3). In contrast to both RUM and REU, the RQUM model allows ties, and one can impose more properties on ties. (I discuss the uniform tie-breaking rule further in Section 1.3.2.)

1.2.1 Construction of π and Necessity of Axiom 1.3

To understand why the distribution π is unique and to see the necessity of Axiom 1.3, I demonstrate the construction of π. For $c \in \mathbb{R}^{n+1}$, $\varepsilon > 0$, I denote

$$E_\varepsilon(c) := \{v \in \mathbb{R}_0^{n+1} : v_i - c_i \text{ is within } \varepsilon \text{ from } \max_{j \in Z} v_j - c_j \ \forall i \in Z\}, \tag{1.2.7}$$

$$F_{\varepsilon,A}(c) := \{v \in \mathbb{R}_0^{n+1} : \exists\, i \in A \text{ s.t. } v_i - c_i \text{ is within } \varepsilon \text{ from } \max_{j \in Z} v_j - c_j\},\ \ A \subseteq Z, \tag{1.2.8}$$

$$G_{\varepsilon,i}(c) := \{v \in \mathbb{R}_0^{n+1} : v_i - c_i \text{ is within } \varepsilon \text{ from } \max_{j \in Z} v_j - c_j\}. \tag{1.2.9}$$

Hence,

$$F_{\varepsilon,A}(c) = \cup_{i \in A} G_{\varepsilon,i}(c), \quad E_\varepsilon(c) = \cap_{i \in Z} G_{\varepsilon,i}(c).$$

By the inclusion-exclusion principle,

$$\pi(E_\varepsilon(c)) = \sum_{A \subseteq Z} (-1)^{|A|+1} \pi(F_{\varepsilon,A}(c)). \tag{1.2.10}$$

Let $\beta > \alpha > 0$. For all types $v \in F_{\alpha,A}(c)$, $M(v, c - \beta \mathbf{1}_A) \subseteq A$, where $M(\cdot,\cdot)$ is the set of maximizers defined in (1.2.2). By the definition of the representation in (1.2.4), all types in $F_{\alpha,A}(c)$ would choose $M(v, c - \beta \mathbf{1}_A) \subseteq A$ given the cost $c - \beta \mathbf{1}_A$.[8] Hence,

$$\rho_A(c - \beta \mathbf{1}_A) \geq \pi(F_{\alpha,A}(c)).$$

[8]For $v \in F_{\gamma,A}(c)$ where $\alpha < \gamma \leq \beta$, $M(v, c - \beta \mathbf{1}) \subseteq A$. Hence v choose $M(v, c - \beta \mathbf{1}) \subseteq A$ and contributes to $\rho_A(c - \beta \mathbf{1}_A)$ according to the representation (1.2.4).

For any type $v \in F_{\alpha,A}(c)$, $M(v, c - \alpha\mathbf{1}_A) \cap A \neq \emptyset$. By the definition of the representation in (1.2.4), the type v with $(v_i - c_i) + \alpha = \max_{j \in Z} v_j - c_j$ for $i \in A, j \in Z \setminus A$ would choose between i and j with a tie-breaking rule t with $t_i \in [0, 1]$ under cost $c - \alpha\mathbf{1}_A$. Hence,

$$\rho_A(c - \alpha\mathbf{1}_A) \leq \pi(F_{\alpha,A}(c)).$$

Therefore, (1.2.10) implies that for any $0 < \alpha < \beta$,

$$\pi(E_\alpha(c)) \leq \sum_{A \subseteq Z, |A| \text{ is odd}} \rho_A(c - \beta\mathbf{1}_A) - \sum_{A \subseteq Z, |A| \text{ is even}} \rho_A(c - \alpha\mathbf{1}_A). \tag{1.2.11}$$

By definition, $E_\alpha(c)$ is a shrinking set when α or β decrease. WLOG let $\alpha = \varepsilon, \beta = 2\varepsilon$. Note a full tie, $v = c$, can be achieved when taking $\varepsilon \to 0$ in $E_\varepsilon(c)$. Indeed, $\pi(c) = \pi(v \in \mathbb{R}_0^{n+1} : v = c) = \pi\{v \in \mathbb{R}_0^{n+1} : v_i - c_i = v_j - c_j \ \forall i, j \in Z\} = \lim_{\varepsilon \to 0} \pi\{v \in \mathbb{R}_0^{n+1} : v_i - c_i \text{ is within } \varepsilon \text{ from} \max_{j \in Z} v_j - c_j \ \forall i \in Z\} = \lim_{\varepsilon \to 0} \pi(E_\varepsilon(c))$. Hence,

$$\pi(c) = \lim_{\varepsilon \to 0} \pi(E_\varepsilon(c)). \tag{1.2.12}$$

For any $c \in \mathbb{R}^{n+1}$, $\pi(E_\varepsilon(c))$ monotonically decreases in ε. Therefore, $\pi(E_\varepsilon(c)) \geq \pi(c) = \lim_{\varepsilon \to 0} \pi(E_\varepsilon(c))$. For $\pi(c) \geq 0$ to hold, $\pi(E_\varepsilon(c)) \geq 0$ must hold, and hence, the right-hand side of (1.2.11) must be nonnegative. This requirement is imposed by Axiom 1.3.

Furthermore, (1.2.11) and (1.2.12) implies construction for π. By Axiom 1.4, $\rho_A(c - \varepsilon\mathbf{1}_A), \varepsilon \in \mathbb{R}$ has a finite range. Thus, $\lim_{\varepsilon \to 0} \rho_A(c - \varepsilon\mathbf{1}_A) = \lim_{\varepsilon \to 0} \rho_A(c - 2\varepsilon\mathbf{1}_A)$. Therefore, we have

$$\lim_{\beta \to 0} \rho_A(c - \varepsilon\mathbf{1}_A) = \pi(F_{\varepsilon,A}(c)) = \lim_{\beta \to 0} \rho_A(c - 2\varepsilon\mathbf{1}_A). \tag{1.2.13}$$

To understand (1.2.13) intuitively, notice that since there are only a finite number of ties, when ε is small enough, we can avoid having ties between the elements in A and $Z \setminus A$. If there are is a tie at c between $i \in A$ and $j \in Z \setminus A$, $c - \varepsilon \mathbf{1}_A$ or $c - 2\varepsilon \mathbf{1}_A$ breaks the tie in favor of i without creating new ties. Next, we denote

$$\rho_A^+(c) = \lim_{\varepsilon \to 0} \rho_A(c - \varepsilon \mathbf{1}_A) = \lim_{\varepsilon \to 0} \sum_{i \in A} \rho_i(c - \varepsilon \mathbf{1}_A), \tag{1.2.14}$$

and

$$\rho_A^-(c) = \lim_{\varepsilon \to 0} \rho_A(c + \varepsilon \mathbf{1}_A) = \lim_{\varepsilon \to 0} \sum_{i \in A} \rho_i(c + \varepsilon \mathbf{1}_A). \tag{1.2.15}$$

By (1.2.13), $\pi(F_{\varepsilon,A}(c)) = \rho_A^+(c)$, and $\rho_A^+(c)$ is the revealed probability that at least one of the goods in the A is optimal. Combining (1.2.10), (1.2.12) and (1.2.14), $\pi(v = c)$ can be written in terms of ρ^+:

$$\pi(c) = \sum_{A \subseteq Z} (-1)^{|A|+1} \rho_A^+(c). \tag{1.2.16}$$

The set of types is

$$\text{supp}(\pi) = \{c \in \mathbb{R}_0^{n+1} : \pi(c) > 0\}. \tag{1.2.17}$$

1.2.2 Examples for Axiom 1.3

I provide an example below to show the importance of Axiom 1.3 in RQUM. Example 1.2 demonstrates that Axiom 1.3 is independent from Axioms 1.1, 1.2, and 1.4.

Example 1.2. *Let $Z = \{0, 1, 2\}$, and let us assume a uniform tie-breaking rule. Consider the set of types $\{v^i, i = 1, 2, 3, 4\}$ associated with a charge measure as follows:*

	0	1	2	charge
v^1	0	0	0	-0.5
v^2	0	-1	0	0.5
v^3	0	0	-1	0.5
v^4	0	1	1	0.5

Figure 1.1 illustrates the types on the plane $\mathbb{R}^2$, where each point on the plane represents a vector (0,x,y) for $x, y \in \mathbb{R}$.

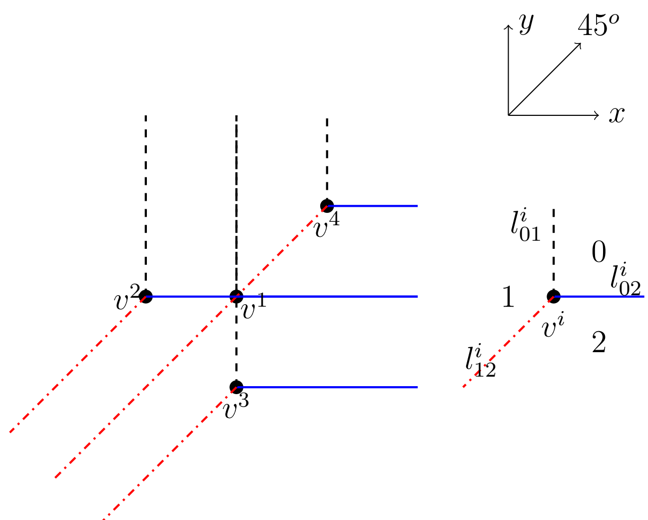

Figure 1.1: Negative Charge

Each type $v^i, i = 1, 2, 3, 4$ is associated with three half-lines starting from it: l^i_{01}, l^i_{02}, and l^i_{12}. These three half-lines divide $\mathbb{R}^2$ into three regions: $0, 1, 2$. Type v^i chooses k when c falls in the region k that is associated with it, $k = 0, 1, 2$, and v^i has a tie between j, k on the half-line l^i_{jk}, $j, k = 0, 1, 2$.

Observations ρ are always nonnegative. Indeed, notice that if c falls in region k for v^1, then it always falls in the same region for two other types. Hence, whenever type v^1 chooses k, two other types also choose k, and thus the RCR ρ is non-negative. It is easy to verify that Axioms 1.1, 1.2 and 1.4 hold given that ρ is generated by the four quasi-linear types.

However, since $\pi(v^1) < 0$, (1.2.16) *implies that Axiom 1.3 is violated.*

Example 1.3 shows that Axiom 1.3 separates RQUM from the logit models.

Example 1.3. *Consider* $Z = \{0, 1, 2\}$*. Let*

$$\rho_i(c) = \frac{e^{-c_i}}{\sum_{i \in Z} e^{-c_i}}.$$

It is easy to verify that ρ *satisfies Axioms 1.1 and 1.2. Let* $c = (0, 0, 0)$*. Then,* $\rho_i(x, c_{-i}) = \frac{e^{-x}}{2+e^{-x}}$ *for* $i = 0, 1, 2$*.* $\rho_i(x, c_{-i})$ *is concave for* $x < -\log 2$*, and is convex for* $x > -\log 2$*.*

However, Axiom 1.3 requires that

$$\sum_{i=0}^{2} \rho_i(c_i - \beta, c_i) - 2\rho_i(c) + \rho_i(c_i + \alpha, c_{-i}) \geq 0. \tag{1.2.18}$$

Condition (1.2.18) *does not hold for some parameters* α, β*, for example,* $\beta = 2, \alpha = 1$*.*

The above discussions illustrates that Axiom 1.3 is indispensible in RQUM.

1.2.3 Sketch of Proof for Theorem 1.1

By Axiom 1.1, we can fix the cost of good 0 to be 0. When $Z = \{0, 1\}$, $\rho_0(0, \alpha)$ can be considered an increasing, one-variable step function by Axiom 4. I define a distribution function F from $\rho_0(0, \alpha)$ by changing the values of the discontinuity points of ρ_0 to make it right-continuous. The realizations of the random variable are the types. I obtain the distribution of types that are differentiated by their values on good 1 with the idea of the Skorokhod construction of random variables.

This construction shows that the types are quasi-linear as in (1.2.1), and a discontinuity in $\rho_0(0, \alpha)$ means that a type is indifferent between 0 and 1. The behaviors of the types at the discontinuities are determined by a tie-breaking rule. Axiom 1.1 ensures that the tie-breaking rule is wealth invariant. Hence, Axioms 1.1, 1.2, and 1.4 impose quasi-linearity on the types. It follows that they ensure the existence of RQUM representation when $|Z| = 2$:

Lemma 1. *When $Z = \{0, 1\}$, an RCR ρ satisfies Axioms 1.1, 1.2, and 1.4 if and only if ρ maximizes RQR $(\pi, t) \in \Pi \times T$. Moreover, such (π, t) can be uniquely determined.*

The complete proof of Lemma 1 is found in Section A.1.

For $|Z| \geq 2$, when $\{i \in Z : \rho_i(c) > 0\} = \{0, i\}$, WLOG, it is useful to consider $Z = \{0, i\}$ – a case that relies on Axioms 1.1, 1.2, and 1.4 only. Lemma 1 guarantees that the types are quasi-linear, and recover the marginal distribution of the quasi-linear types in this case. For general $c \in \mathbb{R}_0^{n+1}$, (1.2.16) identifies the unique joint distribution of types. In Lemma 2, found in Section A.1, I show that π constructed in (1.2.16) satisfies the consistency conditions necessary for a probability measure.

Next, I show that π constructed in (1.2.16) represents ρ. To simplify the notation, define $c \in \mathbb{R}^{n+1}$ to be *generic* if

$$\pi(c \in \mathbb{R}^{n+1} : |M(v, c)| > 1) = 0. \tag{1.2.19}$$

The points in $\mathbb{R}^{n+1}$ satisfying condition (1.2.19) consists of all $c \in \mathbb{R}^{n+1}$ that induces a unique maximizer for each type. As π has finite support, the set of cost vectors c that violate (1.2.19) – the set of nongeneric points – is dense when restricted to $\mathbb{R}_0^{n+1}$. Given Axiom 1.1, the set of non-generic points is dense in the entire $\mathbb{R}^{n+1}$. Therefore, take any two points c and c' in $\mathbb{R}^{n+1}$, one can always find a path on $\mathbb{R}^{n+1}$ between them, such that $|M(v, \tilde{c})| \leq 2$ for all $v \in \text{supp}(\pi)$ and all $\tilde{c}$ on the path; and if $|M(v, \tilde{c})| = |M(v', \tilde{c})| = 2$, $M(v, \tilde{c}) = M(v', \tilde{c})$. In words, there are at most one two-way tie at any point $\tilde{c}$. We also require that there are no turns at points with a tie. I illustrate such a path when $|Z| = 3$ in Figure 1.2.

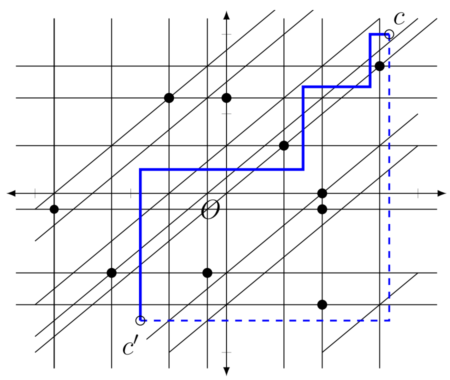

Figure 1.2: Path with At Most Two-way Ties

In Figure 1.2, the solid points are the cost vectors that are in the support of π. These points induce a full tie among the three goods in Z and represent the types. A horizontal ray starting at a solid point and extends to the right includes cost vectors with ties between 0 and 2 for the type represented by the solid point; an upward vertical ray starting at a solid point includes cost vectors with ties between 0 and 1 for the type represented by the solid point; and a ray starting at a solid point and pointing to the northeast includes cost vectors with ties between 1 and 2 for the type represented the solid point. By avoiding the solid points and crossing at most one ray at a time, the path from c to c' consists of at most one two-way tie at each point on the path. This picture can be generated to higher dimensions.

Take the point $c \in \mathbb{R}^{n+1}$ to be such that $c_i \to \infty$ for all $i \in Z \setminus \{0\}$. Then, $\rho_0(c) = 1$ by Axioms 1.2 and 1.4. Take any generic $c' \in \mathbb{R}^{n+1}$ and take a path between c and c' as described earlier. Take $\rho'(c) = \rho(c)$. We construct ρ' for generic points on the path according to RQUM, and show that ρ' agrees with the actual observation ρ. In particular, ρ' stays unchanged when the path passes generic points. At a point $\tilde{c}$ with two-way tie between i and j where the direction of change is on the i^{th} coordinate, the tie is broken in favor of good i. Therefore,

$$\begin{aligned} {\rho'}_i^{+}(\tilde{c}_i) &= {\rho'}_i^{-}(\tilde{c}_i) + \pi(v \in \text{supp}(\pi) : M(v,\tilde{c}) = \{i,j\}), \\ \lim_{\varepsilon \to 0} \rho'_j(\tilde{c}_i - \varepsilon, \tilde{c}_{-i}) &= \lim_{\varepsilon \to 0} \rho'_j(\tilde{c}_i + \varepsilon, \tilde{c}_{-i}) - \pi(v \in \text{supp}(\pi) : M(v,\tilde{c}) = \{i,j\}), \\ \lim_{\varepsilon \to 0} \rho'_k(\tilde{c}_i - \varepsilon, \tilde{c}_{-i}) &= \lim_{\varepsilon \to 0} \rho'_k(\tilde{c}_i + \varepsilon, \tilde{c}_{-i}) \end{aligned} \tag{1.2.20}$$

where ${\rho'}^{+}(\tilde{c})$ and ${\rho'}^{-}(\tilde{c})$ are defined as in (1.2.14) and (1.2.15) for $\rho'(\tilde{c})$.

To show that the construction $\rho'(\tilde{c})$ is the same as the observation $\rho(\tilde{c})$, we first define the *gap function* for any $c \in \mathbb{R}^{n+1}, i \in Z$, as follows:

$$gap_i(c) = \rho_i^{+}(c) - \rho_i^{-}(c). \tag{1.2.21}$$

In RQUM, the function $gap_i(c)$ is the probability measure on the set of types for which i is a weak maximizer at c. Hence, if i is the direction of change at $\tilde{c}$, $\rho_i^{+}(\tilde{c}) = \rho_i^{-}(\tilde{c}) + gap_i(\tilde{c})$. Lemma 3 in Section A.1 shows that $gap_i(c)$ can be calculated as local perturbations of $\pi(c)$ for any $c \in \mathbb{R}^{n+1}$. Lemma 4 in Section A.1 shows that at a non-generic point $\tilde{c}$ on the path, $\rho(\tilde{c}_i - \varepsilon)$ can be obtained from $\rho(\tilde{c}_i + \varepsilon)$ in exactly the same way as in (1.2.20) given the axioms and Lemma 3. If $\tilde{c}$ is a generic point on the path, $gap_i(\tilde{c}) = 0$ for all $i \in Z$ so $\rho(\tilde{c})$ is unchanged in the neighborhood of $\tilde{c}$. Therefore, starting from the point c where $\rho_0(c) = \rho'_0(c) = 1$, ρ' agrees with ρ at all generic points on the path.

At nongeneric points, types may have multiple maximizers, and the set of maximizers of different types may overlap. I show that for the π constructed with (1.2.16), the tie-breaking rule, $t \in T$, exists for the nongeneric points. To do so, I transform the problem into a matching problem that can be solved using Hall's marriage theorem (Hall, 1935) on rationals.[9] I then use compactness to approximate any real numbers. The nodes in the matching prob-

[9] Hall's marriage theorem is also discussed in Demange et al. (1986) to show the the existence of a equilibrium assignment in a multi-unit auction.

lems are “units” of RQUF π (a-units) and “units” of RCR ρ (p-unites), so that the number of a-units and the number of p-units are integers. A unit is defined as follows:

Definition 1.1. *A unit for $\pi, \rho \in \mathbb{Q}$ at $c \in \mathbb{R}^{n+1}$ is $\frac{1}{k}$, where $k \in \mathcal{N}$, such that*

$$k\pi(v) \in \mathcal{N} \ \forall \ v \in supp(\pi), \quad and \quad k\rho_A(c) \in \mathcal{N} \ \forall A \subseteq Z.$$

I show in Lemma 5 that there exists a perfect matching between a-units and p-units for any $c \in \mathbb{R}^{n+1}$. The tie-breaking rule $t_i(v, c)$ is the number of matchings between the a-units associate with v and the p-units of i divided by the total number of all a-units associated with v. Since the matching is not guaranteed to be unique, the tie-breaking rule is not necessarily unique.

1.3 Special Cases

In this section, I discuss two special cases of the general RQUM: uniform tie-breaking, and binary types.

1.3.1 Regular and Uniform Tie-Breaking

General RUM respects regularity. That is, the probability of choosing any good should not increase when more goods are added to the menu. In general, the tie-breaking and the RCR in RQUM does not need to be regular, as illustrated by Example 1.1 in Section 1.2. Furthermore, the tie-breaking rule in RQUM is not guaranteed to be unique.

In actual applications, it may be desirable have a more precise tie-breaking rule that satisfies regularity. A natural refinement is uniform tie-breaking, where an agent puts $\frac{1}{M(v,c)}$ choice probability on good $i \in M(v, c)$. With uniform tie-breaking, I define *uniform tie-breaking*

RQUM representation (U-RQUM representation) as follows:

$$\rho_i(c) = \sum_{A\subseteq Z:i\in A} \frac{1}{|A|}\pi(v \in \mathbb{R}_0^{n+1} : M(v,c)) = A). \tag{1.3.1}$$

In U-RQUM, the choice probability $\rho_i(c)$ is the weighted sum of the probabilities on agents $v \in \mathbb{R}_0^{n+1}$ for which $i \in M(v,c)$, where the weight is $\frac{1}{M(v,c)}$. The following axiom corresponds to the uniform tie-breaking:

Axiom 1.5 (Uniform)**.** *For all* $0 < \alpha$*,* $i \in Z$*, and* $c \in \mathbb{R}^{n+1}$*,*

$$\rho_i(c) \geq \sum_{A\subseteq Z, i\in A} \frac{1}{|A|} \sum_{A'\subseteq A} (-1)^{|A\setminus A'|}\rho_{A'}(c + \alpha\mathbf{1}_{A'}). \tag{1.3.2}$$

Theorem 1.2. *An RCR* ρ *satisfies Axioms 1-5 if and only if* ρ *maximizes the U-RQUM representation. Moreover, the representation is unique.*

To see the relation between (1.3.1) and (1.3.2), notice that $\rho^-_A(c)$ is the measure on $v \in \mathbb{R}_0^{n+1}$ such that $M(v,c) \subseteq A$. Equivalently,

$$\rho^-_A(c) = \sum_{A'\subseteq A} \pi(\{v \in \mathbb{R}_0^{n+1} : M(v,c) = A'\}).$$

By Möbius transform (see, e.g., p. 41, Grabisch, 2016),

$$\pi(\{v \in \mathbb{R}_0^{n+1} : M(v,c) = A\} = \sum_{A'\subseteq A} (-1)^{|A\setminus A'|}\rho^-_{A'}(c).$$

Furthermore, $\lim_{\alpha\to 0}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}) = \sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho^-_{A'}(c)$. The full proof of Theorem 1.3.1 is found in Section A.2.1.

1.3.2 Binary Types

There are many cases where choices are made by two types, for example, household decisions, dual cognitive systems, and so on. (For more discussion of such cases, see Manzini and Mariotti (2018)). Here, I consider the case where there are two quasi-linear types.

Axiom 3' (Two steps). *For any $i \in Z$, the function $\rho_i(\alpha, c_{-i})$ belongs to one of the following two cases:*

1. *$\rho_i(\alpha, c_{-i})$ is a nondecreasing step function with one point of discontinuity as α increases: $\rho_i(\alpha, c_{-i}) = 0$ before this point, and $\rho_i(\alpha, c_{-i}) = 1$ after this point.*

2. *$\rho_i(\alpha, c_{-i})$ is a nondecreasing step function with two points of discontinuity as α increases: $\rho_i(\alpha, c_{-i}) = 0$ before the first point, $\rho_i(\alpha, c_{-i}) = a$ or 1-a between the first and second points, and $\rho_i(\alpha, c_{-i}) = 1$ after the second point, for some fixed $a \in (0, 1)$.*

If the RCR ρ satisfies Axiom 3', then $\rho_i(c) \in \{0, a, 1-a, 1\}$ for all $c \in \mathbb{R}^{n+1}$, $i \in Z$. We can construct two types with probabilities $(a, 1-a)$.

Theorem 1.3. *An RCR ρ satisfies Axioms 1.1, 1.2, and 3', if and only if ρ is represented by the RQR $(\pi, t) \in \Pi \times T$, where π has binary support and takes value in $\{a, 1-a\}, a \in (0, 1)$. The binary RQR is unique.*

1.4 Comparison with Other Models

1.4.1 The Luce Model and Multinomial Logit Models

The primitive in RUM (Block & Marschak, 1959) is a choice system (X, ρ), where X is the choice domain, and $\rho : 2^X \times X \to \Delta(X)$ is such that $\sum_{x \in A} \rho_A(x) = 1$ for any $A \subseteq X$. RUM has two interpretations. In the first interpretation, the observations can be explained by a collection, U, consisting of injective functions $u : X \to \mathbb{R}$, and a probability measure,

$\pi \in \Delta(U)$, such that

$$\rho_A(x) = \pi(\{u \in U : x \text{ is best in } A \text{ according to } u\}). \tag{1.4.1}$$

In the second interpretation, there is an average utility, $v : X \to \mathbb{R}$, but it is perturbed by a noise term, $\varepsilon : X \to \mathbb{R}$, and

$$\rho_A(x) = Prob(v(x) + \varepsilon(x) \text{ is a maximizer in } A). \tag{1.4.2}$$

The Luce model (Luce, 1959) is an important refinement of RUM. Given the choice system (X, ρ), if there is a function $u : X \to \mathbb{R}_+$ that satisfies the Luce condition

$$\rho_A(y) = \frac{u(y)}{\sum_{x \in A} u(A)} \quad \forall A \subseteq X, y \in A, \tag{1.4.3}$$

then this choice system conforms to the Luce model. The Luce model is rationalizable by heterogeneous preferences, as in (1.4.1) (see, e.g., Theorem 7.6 in Chambers & Echenique, 2016).

McFadden (1974, 1980) shows that the Luce model can be written as an average utility plus a random error as in (1.4.2), where the error term follows a Gumbel distribution while the average utility has the logistic form.[10] Hence, the Luce model is also referred to as the multinomial logit model. The multinomial logit model is the foundation of the discrete choice literature, and it is the basis for many empirical demand estimation papers in empirical industrial organization (see, e.g., McFadden, 1974, 1980).[11]

[10] RQUM can also be written as in (1.4.2). To see this, we simply define the mean utility $\overline{u} = \sum_{u \in U} \pi(u)u$, and let $\varepsilon = \{u - \overline{u}, u \in U\}$ be distributed with the same $\pi \in \Delta(U)$. However, this construction is merely a normalization of the private values, and it lacks the statistical interpretation of the error term, as found in the Luce model (Luce, 1959).

[11] In this literature, each agent only chooses one choice alternative. Given market-level data, one can reconstruct the market share using the average utility, where the average utility is a function of the observed and unobserved individual and product characteristics. In the estimation, one uses statistical methods to find the coefficients for the characteristics so that the difference between the constructed market share and

The Luce condition (1.4.3) implies the independence of irrelevant alternatives (IIA) property. To investigate this further, let $\rho_A(x)$ denote the probability that the choice alternative x is chosen from menu A. Let $\{x, y\} \subseteq A$.

$$\frac{\rho_{\{x,y\}}(x)}{\rho_{\{x,y\}}(y)} = \frac{\rho_A(x)}{\rho_A(y)}. \tag{1.4.4}$$

Empirically, we see that the IIA condition is problematic.[12]

RQUM does not suffer from IIA. Intuitively, when a choice alternative associated with a relatively low price level (the price can be negative, interpreted as an increase in wealth) is added to the choice set, all agents may change their choices in favor of this new choice alternative. For example, suppose $Z = \{a, b, c\}$, and also suppose that there are two quasi-linear agents, with types $v_1 = (1, 3, 2)$, $v_2 = (1, 2, 3)$. When the price vector associated with the three goods is $c = (0, 0, 0)$, IIA is satisfied for positive choice probabilities. However, if the price level is $c = (0, 3, 3)$, IIA is violated in sets $\{b, c\}$ and $\{a, b, c\}$.

RQUM is distinct from the Luce model with respect to some other properties, as well. As discussed in Section 1.2 and in Example 1.1, the Luce model violates Axiom 1.3. In addition, RQUM restricts the number of types to be finite, while the Luce model does not. In fact, the types in the Luce model are not uniquely identified, even though their average utility is uniquely determined. Finally, the Luce model and its extensions (e.g., Fudenberg et al., 2015) respect stochastic transitivity due to the Luce condition (1.4.3). However, RQUM and general RUM do not respect stochastic transitivity.

the observed market share is minimized. The quasi-linearity is necessary for calibrating the parameters in many applications (see, e.g., S. Berry et al., 1995; S. Berry & Pakes, 2007; S. T. Berry, 1994; Nevo, 2000).

[12] Suppose the agent faces the choice set {bus, car}, and has $\frac{1}{2}$ probability of choosing either transportation. When the choice set is enlarged to be {blue bus, red bus, car}, IIA requires the agent to choose each transportation with probability $\frac{1}{3}$ (see, Debreu, 1960). This requirement is not reasonable since the color of the bus need not affect the choice of means of transportation. Specifications of the Luce modelLuce (1959), which does not suffer from the blue-bus, red-bus fallacy, has been studied by various researchers (e.g., Fudenberg, Iijima, & Strzalecki, 2015; Gul, Natenzon, & Pesendorfer, 2014).

1.4.2 Path Independent Choice Functions

In a multi-utility model, choices are made by a collection of types, and the choice function $\varphi : 2^X \to X$, $\varphi(A) \subseteq X$ consists of all maximizers of all types (see, e.g., Moulin, 1985a). In my setting, $X = Z \times \mathbb{R}^{n+1}$, and a menu here is of the form $A = \{(i, c_i), i \in Z, c_i < \infty\}$. For a given cost c, $\varphi(A) = \cup_{v \in \text{supp}(\pi)} \{(i, c_i) : i \in M(v, c)\}$. Hence, RCR ρ can be mapped onto a multi-utility model choice function with

$$\varphi(A) = \{(i, c_i) : \rho_i(c) > 0, i \in A\}. \tag{1.4.5}$$

Instead of tie-breaking, φ consists of all maximizers without referring to the probabilities.

Proposition 1.4.1. *If $\rho : Z \times \mathbb{R} \to \mathbb{R}$ is regular, then φ as defined in* (1.4.5) *satisfies the following properties:*

1. *Sen's α: $A \subseteq B$, then $\varphi(B) \cap A \subseteq \varphi(A)$.*
2. *Aizerman and Malishevski (AM): $\varphi(B) \subseteq A \subseteq B \implies \varphi(A) \subseteq \varphi(B)$.*

Proof. Let $c \in \mathbb{R}^{n+1}$ be associated with A, i.e., $A = \{(i, c_i), i \in Z, c_i < \infty)\}$, and $c' \in \mathbb{R}^{n+1}$ be associated with B, i.e., $B = \{(i, c'_i), i \in Z, c'_i < \infty)\}$. Take $A \subseteq B$. Hence, $c_i < \infty \implies c'_i < \infty$ and $c'_i = c_i$.

Show Sen's α. Since $(i, c'_i) \in \varphi(B)$, $\rho_i(c') > 0$ and therefore $c'_i < \infty$. If $(i, c'_i) \in A$, then $c_i = c'_i$. Hence $\rho_i(c) \geq \rho_i(c') > 0$ by regularity, and therefore $(i, c_i) \in \varphi(A)$.

Show AM. $\varphi(B) \subseteq A$ requires $\rho_i(c') > 0 \implies c'_i = c_i < \infty$. By regularity, $\rho_i(c) \geq \rho_i(c') > 0$. Since $\sum_{i \in Z : \rho_i(c') > 0} \rho_i(c') = 1$, then $\sum_{i \in Z : \rho_i(c') > 0} \rho_i(c) \geq 1$. Since ρ is a probability measure, it follows that $\rho_i(c) = \rho_i(c')$ if $\rho_i(c') > 0$. Therefore, $\varphi(A) = \varphi(B)$. □

A choice function φ that satisfies Sen's α and AM is path independent, and can be decomposed into the union of choices of a finite collection of types (Aizerman & Malishevski, 1981b). From Proposition 1.4.1, we see that in the settings of this chapter, RQUM and path independent choice models does not necessarily overlap. Any regular ρ can be mapped to a path independent choice function; while a non-regular ρ with RQUM representation cannot be mapped into a path independent choice function.

1.5 Conclusion

This chapter discusses the unique identification of a random quasi-linear utility model with complete data when there is a finite distribution of underlying types. This identification can be useful in practical settings. For example, if a marketing planner needs to know the private values of a population to better design a campaign but only observe its consumption choices, my model can be helpful. In practice, monetary variations are often discrete, so only a finite number of data are available. Hence, a useful extension of this chapter is to study the model in a finite domain. On the other hand, it is easy to characterize the existence of a general joint distribution with some modification of Axiom 1.3 (See Theorem 1.1.6 in Durrett, 2019). To avoid characterizing the tie-breaking rule, one can require the distribution of types to be continuous. In this case, ties are everywhere, but all types have a probability of 0, so any tie-breaking rule is acceptable. However, such a characterization may not be useful empirically.

Chapter 2

Persuasion in Signaling Games

This chapter models litigation in signaling games with an imperfectly informed victim and a perfectly informed defendant. I compare a two-agent game and a three-agent extension where the victim can hire a lawyer who is perfectly informed but pursues a selfish objective in his advice. In particular, a lawyer affects a victim's information environment in a way that is similar to Bayesian persuasion Kamenica and Gentzkow (2011). Overall, this analysis captures some stylized empirical patterns of the legal system, and identifies both the positive and negative welfare effects of lawyers' advice on the number of cases filed and litigated, victim's trial winning rates, and defendants' safety costs.

2.1 Introduction

In many areas of the legal world, lawyers are associated with the practice of aggressive solicitation of clients. The extreme of such a practice is found in personal injury cases, and the term "ambulance chasing" describes its extent Anderson (1957). In general, overly aggressive client solicitation is frowned upon, and the solicitation of clients by lawyers is subject to regulations under lawyers' professional ethical standards.[1] However, it is important to ask whether lawyers' aggressive solicitation of clients to encourage litigation really is that bad. Considering the lack of access to legal resources and the information problem faced with many claim-holders, the answer seems to be more complicated Schwab (2010).

Admittedly, it is clear that too much litigation takes place in the U.S. today. For instance, in 2016, there were 83 million incoming civil cases in state courts, and, surprisingly, this number is found to be on the lower side when looking at recent history.[2] Ligation also consumes many resources. One study [3] shows that in 2016, the U.S. tort system alone cost $429 billion, the equivalent of 2.3% of the United States' GDP that year, although tort cases account for only about 7% of the total civil cases in state and federal courts.

Still, claimants' litigious tendencies differ by category. There are two main groups of potential claimants: corporate and individual. Where corporate claimants tend to be quite litigious, individual claimants typically only litigate a small portion of their legal problems. For example, only 1% of job discrimination cases, 10% of tort problems, and 36% of real property problems induce individuals to seek lawyers, and only around 14 % of people injured by alleged medical malpractice end up filing suits Cramton (1993); Hylton (2007).[4]

Individual claim-holders' under-litigation often stems from a lack of information. Claim-

[1]See ABA Rule 7.3 ("Solicitation of Clients"), `https://www.americanbar.org/groups/professional_responsibility/publications/model_rules_of_professional_conduct/rule_7_3_direct_contact_with_prospective_clients/comment_on_rule_7_3/`, last accessed September 11, 2019

[2]For more caseload statistics, see the Court Statistics Project, http://www.courtstatistics.org/, last accessed Feb 28,2020.

[3]U.S. Chamber of Commerce Institute for Legal Reform, "Costs and Compensation of the U.S. Tort System", Oct. 2018. `https://www.instituteforlegalreform.com/research/2018-costs-and-compensation-of-the-us-tort-system`, last accessed Feb 28, 2020.

[4]The court also acknowledges individual claimants' under-litigation problem in tort cases and uses punitive damages and class action suits to alleviate this issue. Punitive damages augment the amount of damages paid by a defendant after considering the probability of escaping liability. In class action suits, one member can sue on behalf of the entire class to make up for the fact that many injured individuals are not suing. Punitive damages and class action suits are unusual under tort laws because the function of tort laws is compensation and restitution, rather than punishment and deterrence. However, the consideration to alleviate under-litigation is more important than the consideration to limit the scope of the function of the tort law Galligan Jr (2005); Lens (2014); Polinsky and Shavell (1998)

holders may not fully aware of the extent of liability of the defendant, may fail to establish the legal character of their problems, or may fail to understand the relevant legal resources available to them Cramton (1993). In such situations, lawyers can supply the needed information and thus reduce under-litigationCalvani, Langenfeld, and Shuford (1988); Cramton (1993); Greiner and Matthews (2015).[5]

At the same time, lawyers' fees constitute a large portion of the total sums involved in litigation. Out of the total cost of $429 billion in the U.S. tort system in 2016, 57% went to the victims while 32% went to the lawyers[6]; That year, lawyers made $137 billion in tort cases alone – about 0.74% of the U.S.'s GDP Doroshow and Gottlieb (2016).

The ambiguity of lawyers' effects on an individual claimant's inclination to take her case to court is the main motivation for this study. The two main objectives of this chapter are the following:

1. To model lawyers' effects on the amount of litigation and victim winning rate in suitable signaling games.
2. To evaluate lawyers' effects on social welfare.

To realize these objectives, this chapter explicitly considers how a victim's lawyer affects an uninformed individual victim's decisions in a three-agent signaling game. Such analysis takes a different perspective from mainstream game theory models in law and economics regarding litigation choices, which focus on games between two opposing parties–the plaintiff and the defendant Bebchuk (1984); Hubbard (2017); Reinganum and Wilde (1986).

In the current study, a victim ("she") is injured from an interaction with a defendant ("he").

[5]Lawyers are helpful to their clients because they can obtain better information (rather than because they can supply superior oral arguments at trial) Duvall (2007). Most cases settle and never go to trial. If a trial does occur, a case represented by legal counsel has a higher winning probability and typically better outcome terms compared to a case without legal representationGreiner, Pattanayak, and Hennessy (2012). However, a judge usually forms opinions on the case's merits before a trial begins based on the information supplied by the parties, rather than based on oral arguments at trial Duvall (2007).

[6]Supra footnote 2.

There are two possible states: the defendant is either liable or non-liable for the injury to the victim. The defendant is perfectly informed of the state. However, the victim might or might not recognize that she is injured, and also might or might not be able to tell whether the defendant is liable. This situation is modeled as a victim who receives a noisy signal on the realization of the state after being injured.

In a two-agent model, if the victim gets a signal indicating that the defendant is liable, she files a claim against him, and litigation begins; otherwise, there is no litigation. If litigation occurs, there is first a settlement stage; however, if there is no settlement, then a trial occurs. The settlement stage is summarized as a signaling game, where the perfectly informed defendant offers either a positive settlement amount or offers zero to the imperfectly informed victim. A positive amount signals the defendant is liable; where as a settlement offer of zero signals that the defendant is not liable. The victim can either reject or accept the defendant's offer. If she accepts his offer, a settlement is reached; if she rejects it, the settlement fails, and a trial begins. This process represents the general chronology of a typical lawsuit.[7].

The setup of such a game is discussed in detail in section 2.3, where the timeline and game tree are given in Figure 2.1 and Figure 2.2. In a two-agent game where an imperfectly informed victim is not represented by a lawyer, the victim might either (1) never recognize the harm done to them, and thus never file a claim; or (2) be stuck in a pooling equilibrium where they are not compensated. The two-agent baseline model in section 2.3 discusses such situations.

Section 2.4 adds a third player: a perfectly informed, profit-driven lawyer ("he"), who can affect the victim's information environment and payoffs. In particular, section 2.4 discusses a three-agent model where a victim is represented by a lawyer who is perfectly informed of

[7]See, e,g. United State Courts, ("About Federal Courts", "Civil Cases"). `https://www.uscourts.gov/about-federal-courts/types-cases/civil-cases`, last accessed Feb 28, 2020

the realization of the state (liable or non-liable) and of the victim's signal, and who collects fees for his services. This lawyer sends a signal to the victim. The lawyer's optimal signal structure is described in Proposition 2.4.1. A lawyer's degree of information disclosure is determined by the signaling equilibrium in the settlement game between the victim and the defendant.

Overall, in equilibrium, lawyers may (1) inform victims that defendants are liable when the victims receive a wrong signal; (2) eliminate the pooling equilibrium where victims are never compensated; and (3) help the victims select liable cases to go to litigation, and thus, increase the victims' numbers of positive settlements and the overall trial success rate. Under appropriate settings, most victims can be compensated either through a settlement or via a trial. However, lawyers' fees decrease the settlement amounts and trial payoffs. Furthermore, lawyers induce more case filings for non-liable cases. A numerical example of this situation is given at the end of section 2.4 to illustrate such effects.

When the extent to which the defendant exercises costly caution generally in his interaction with potential victims determines his prior probability of being liable, the presence of a profit-driven lawyer may increase the defendant's precaution level and thus decrease the defendant's prior probability of being liable by affecting the signaling equilibrium. In particular, when a defendant's prior probability of being liable is a function of the cost he spends in preventing injuring others, the presence of a victim's lawyer can induce a defendant to spend more on safety measures under a set of realistic parameters by increasing the amount of litigation. Detailed analysis is in Section 2.5.

This chapter also discusses several variations of the three-agent model. First, when lawyers are imperfectly informed, they can still increase the number of case filings and the amount of litigation, even though their solicitation efforts will be less effective. Second, if lawyers not only have financial interests in the cases but also internalize the utility of the victims to some degree above a threshold, they will report their true information to the victims. Here, the game becomes a complete information game. Finally, if trials do not reveal the true state, and the informed defendants can manipulate the judges' decision at trial, victims will lose many cases in which defendants are liable. However, if a victim is represented by a perfectly informed lawyer who can also persuade the judge, the trial becomes truth-revealing. These situations are discussed in detail in section 2.6.

2.2 Related Literature

The Bayesian persuasion framework originally developed in Kamenica and Gentzkow (2011) considers the setting of one sender and one receiver, where the receiver's action determines the payoffs for both parties. The sender can strategically conduct experiments to determine the states, and commit to truthfully reveal the results to the receiver. The receiver uses the signal from the sender to updates her posterior belief on the states according to Bayes' rule. By the choice of experiments, when the state space is small, the sender in effect can choose any posterior belief for the receiver, as longs the expectation of the posterior beliefs induced by a signal is same as the prior belief. The sender benefits as long as the receiver's action is discontinuous in beliefs.

Since Kamenica and Gentzkow (2011), there has been many applications of Bayesian persuasion setting and its variants. The applications include optimal experimentation by a politician for different electoral rules Alonso and Câmara (2016), optimal grading in schools Boleslavsky and Cotton (2015), optimal design of online ads Rayo and Segal (2010), and optimal competitive information disclosure in costly search markets Board and Lu (2018), to name just a few. This chapter also considers an application of a variant of Bayesian persuasion. The setting is related to the one-sender-one-receiver situation of the depositor-regulator case with private signals in Bergemann and Morris (2016). However, I combine persuasion with a signaling model, which makes the analysis more comprehensive.

Specifically, a victim (receiver) receives a private noisy signal indicating the state, and the lawyer (sender) knows the true state and the victim's prior belief and private signal. In the terminology of Bergemann and Morris (2016), the lawyer is *omniscient*. Bergemann and Morris (2016) also define *Obedience* as the condition that the receiver always follows the sender's advice.[8] To achieve this, the receiver's expected utility of following sender's advice needs to be higher than not following. This constrains the receiver's decision rule and the sender's signals. In the model presented in this chapter, in equilibrium, the victim's decision rule is obedient. When the lawyer tells the victim that the defendant is liable, the victim files a suit.[9] A lawyer makes a victim's decision rule obedient by choosing a signal to cause the victim's belief that the defendant is liable to be just above the threshold for her to file a case; where such threshold is determined in her signaling game with a defendant. A lawyer profits from persuasion by controlling the victim's belief when entering the signaling equilibrium and therefore inducing separating equilibrium where victims file suits and go to trial often.

The current chapter contributes to the discussion in the legal literature on socially optimal levels of litigation and settlement (e.g. Shavell (1999)) by studying such issues in a more sophisticated signaling game setup. Additionally, this chapter contributes to the discussion on the regulation of injurers' precaution levels. Some of the legal literature focus on regulation of tort injurers' activities and precaution levels via different liability standards Gilles (1992); Hylton (2002); Polinsky and Shavell (2000); Rosenberg (2007). The present chapter adds to this discussion from a different angle, namely, how lawyers can help enforce any predetermined liability standards by helping victims recognize liable defendants.

[8]See also Definition 1 in Bergemann and Morris (2019)

[9]In the language of Bergemann and Morris (2019), the situation that the victim always follows lawyer's signals, is called a "truthful mechanism"

2.3 Baseline Model: Two-Agent Settlement Game

The models of this chapter consider a victim's litigation decisions in a tort case after she is injured.[10] We first establish a two-agent model of a settlement game, where the players are a victim who is injured during her interaction with a defendant, and the defendant.

We first consider a complete information situation to determine the equilibrium settlement amounts from the defendant in the settlement game. Suppose the fair compensation to the victim, i.e., the adjudication amount from trial, is d. Suppose the court cost for the defendant is c_d, and for the victim is c_v. Therefore, a victim's payoff from trial is $d - c_d$, while her settlement payoff is σ. To avoid trial, a liable defendant will have to offer a settlement amount of $\sigma \geq d - c_v$. The equilibrium settlement amount would be $\sigma^* = d - c_v$ as the defendant maximizes his payoff, $-\sigma$. When the defendant is not liable, the fair compensation to the victims is zero, and the victim's payoff from litigation is $-c_v$. Thus, the victim will not go to trial even when offered zero settlement, and therefore the lowest non-negative payoff that V would accept would be $\sigma = 0$.

In the settlement game considered here, the victim is uncertain whether the defendant is liable to her or not, while the defendant is perfectly informed. In the game, the defendant offers either zero or σ^* to the victim. The former signals non-liable, and the latter signals liable. Therefore, this game is a typical signaling game.

2.3.1 Primitives

The defendant (D) is either liable (l) or non-liable (nl) to the victim (V). Thus, the state space includes two states: $\Omega = \{$ liable (ω_l), non-liable $(\omega_{nl})\}$ The realization of the state is exogenous.

[10] In a tort case, the victim seeks monetary compensation (damages) from the injurer. Personal injury cases that arise from motor vehicle accidents, slip-and-fall accidents, medical malpractice, and injuries caused by defective products all belong to this category.

V is imperfectly informed of the realization of the state. Her prior belief that D is liable is $p_0 = p(\omega_l)$, where $p(\omega_l)$ is the percentage of liable cases in the pool of all cases of the same type.[11] After the injury, V receives a noisy signal, z, whereby $z = 1$ indicates ω_l, and $z = 0$ indicates ω_{nl}. However, a false-negative occurs with probability β_0, and a false-positive occurs with probability β_1. Thus, each state generates z as follows:

$$\begin{aligned}
&P(z = 1|\omega_l) = 1 - \beta_0, \\
&P(z = 0|\omega_l) = \beta_0 \text{ (a false-negative, or type II error)}, \\
&P(z = 1|\omega_{nl}) = \beta_1 \text{ (a false-positive, or type I error)}, \\
&P(z = 0|\omega_{nl}) = 1 - \beta_1.
\end{aligned} \tag{2.3.1}$$

.

Therefore, when $z = 1$, D is liable with probability

$$p_s = P(\omega_l|z = 1) = \frac{p_0(1 - \beta_0)}{(1 - \beta_0)p_0 + \beta_1(1 - p_0)} \tag{2.3.2}$$

When $z = 0$, D is liable with probability

$$p'_s = P(\omega_l|z = 0) = \frac{\beta_0 p_0}{\beta_0 p_0 + (1 - p_0)(1 - \beta_1)} < p_s \tag{2.3.3}$$

In this model, the defendant is perfectly informed of the realization of the state and the victim's signal z. The victim files a claim against the defendant if and only if $z = 1$. Therefore, the probability of V filing a claim is the same as the probability that signal $z = 1$. Such a probability is determined by the prior probability, p_0, and the errors in the signal as

[11] Such prior probabilities can be obtained in survey data or from insurance contracts.

below:

$$P(z = 1) = (1 - \beta_0)p_0 + \beta_1(1 - p_0)$$

After the victim files a claim, a settlement negotiation occurs between the victim and the defendant. In the event of a settlement, the defendant offers a settlement, σ, to the victim. If the victim accepts the settlement offer σ, the defendant transfers the agreed-upon settlement amount σ to the victim, and the case concludes. However, if the victim disagrees with the defendant and rejects the settlement offer σ, then the settlement breaks down, and a trial ensues. If $z = 0$, the victim does not file a claim, and there is no litigation. To summarize, the timeline of the development of a case is illustrated in Figure 2.1.

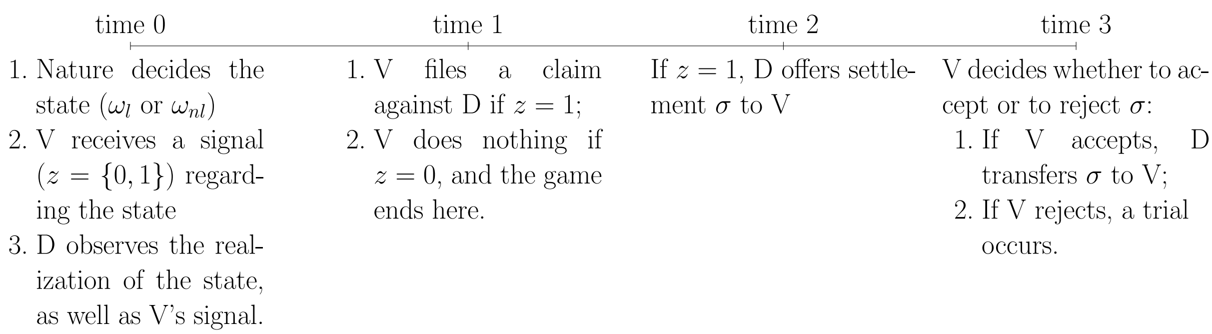

Figure 2.1: Timeline of the two-agent game

2.3.2 Notation

The notation used in the settlement model of this section are listed below.

1. ω – two possible states for defendant D's liability in a case: liable ω_l, or non-liable ω_{nl};
2. z – V's signal about the state: $z = 1$ signals ω_l, $z = 0$ signals ω_{nl};
3. β_0 – the probability of $z = 0$ when the true state is ω_l;
4. β_1 – the probability of $z = 1$ when the true state is ω_{nl};
5. p_0 – V's prior belief of ω_l;

6. p_s – V's belief that D is liable when $z = 1$;
7. p'_s – V's belief that D is liable when $z = 0$;
8. d – damages, which is the same as the amount of fair compensation from a liable D to V for the injury;
9. c_v – V's litigation costs;
10. c_d – D's litigation costs;
11. σ – settlement offered by D to V;
12. x – probability of D offering $\sigma = 0$ when in state ω_l;
13. r – probability of V rejecting an offer of $\sigma = 0$.

2.3.3 The Settlement Game

A settlement negotiation occurs if the victim, V, files a claim; and V files a claim only when her signal is $z = 1$. Thus, in the settlement game, V's belief that D is liable is p_s, as in equation (2.3.2). The settlement game adopts the framework of a standard signaling game Cho and Kreps (1987). In this game, there are two types of D – liable and non-liable. V is imperfectly informed of D's type, and D uses the value of σ to signal his type to V. In the separating equilibrium, with some probability, a liable D pretends to be non-liable, and send the signal of being a non-liable type; whereas in a pooling equilibrium, both types of D send the same signal.

In such a game, a settlement is successful if and only if V accepts σ offered by D. V's settlement payoff is the settlement amount σ, and D's settlement payoff is $-\sigma$. If V rejects D's offer of σ, the settlement fails, and a trial occurs. Here, a trial will reveal the true state. For the victim, the court costs are c_v; whereas for the defendant, the court costs are c_d. Thus, V's trial payoff is $d - c_v$ if D is liable, and $-c_v$ if D is not liable; and D's trial payoff is $d + c_d$ if he is liable, and is $-c_d$ if he is not liable. As described in the beginning of this section, the equilibrium value of σ is binary: either 0 or σ^*, where $\sigma^* = d - c_v$.

Note that when the victim is perfectly informed, that is, $\beta_0 = \beta_1 = 0$, V would accept zero settlement offers from a non-liable D with probability 1, and reject zero offers from a non-liable D with probability 1. Therefore, a liable D is better off offering σ^*. Hence, a liable D always offers σ^*, and a non-liable D always offers zero settlement; and V would always accept such settlement offers. Because of this situation, there would be no trial.

To summarize, when V is imperfectly informed ($\beta_0 > 0$ and/or $\beta_1 > 0$), the outcome of the settlement process can be modeled by a signaling game. D's strategy space includes offering two kinds of settlements – a settlement offer of zero or a positive amount, this can be denoted as $\sigma = \{\sigma = 0, \sigma = \sigma^*\}$. There are two actions available to V, accept or reject. If the victim accepts the settlement amount, σ, the settlement is successful, and this settlement amount is transferred from D to V. However, if V rejects settlement σ, then the settlement breaks down, and a trial occurs.

Assumption 1. *A trial will reveal D's true type.*

Admittedly, this assumption is very strong. However, the focus of this chapter is the effect of lawyer solicitation, rather than the actual litigation process. In addition, we relax this assumption in section 2.5.3.

The true state and their own court costs determine D's and V's trial payoffs. The game tree in Figure 2.2 summarizes the settlement game.

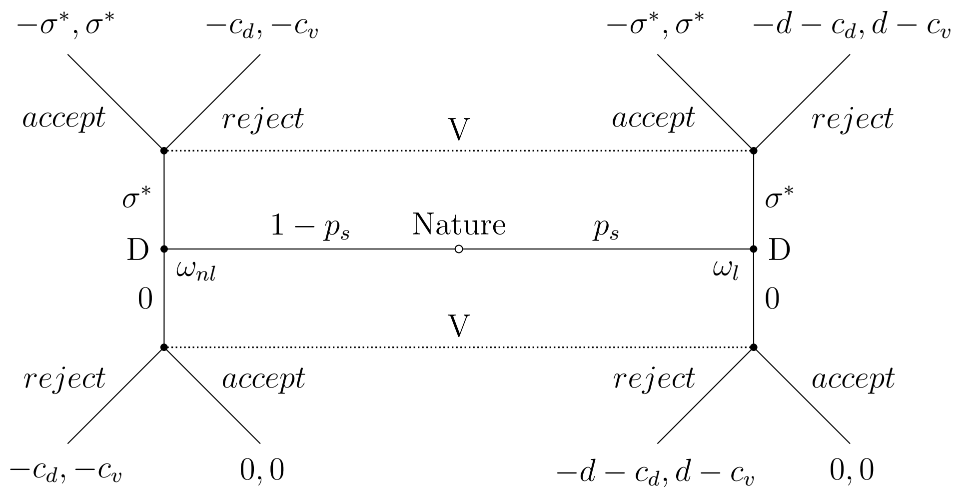

Figure 2.2: Two-agent signaling game

2.3.4 Equilibrium Characterization

The strategies of D ($\sigma = 0$ or $\sigma = \sigma^*$) and V (accept or reject) of the signaling game in a settlement negotiation are determined in equilibrium. The solution methods in this model are sequential equilibria.

Assumption 2. *Where there are multiple equilibria, the D1 criterion found in in Cho and Kreps (1987) is applied to refine the results.*[12]

Proposition 2.3.1 summarizes the equilibrium results. The detailed calculations are found in Appendix B.1.

Proposition 2.3.1. *(Equilibrium in the two-agent signaling model) When V is imperfectly informed ($\beta_0, \beta_1 > 0$), the equilibrium in the two-agent signaling model when a trial is truth-revealing has the following properties:*

(1) When $p_s \leq c_v/d$, there exists a pooling equilibrium where V is not compensated and there is no litigation.

[12] D1 criterion eliminate the pooling equilibria where one type defects whenever the other type defects.

(2) Litigation occurs if and only if $p_s > c_v/d$. When $p_s > c_v/d$, a non-liable D offers $\sigma = 0$. A liable D offers $\sigma = 0$ with probability x, and $\sigma = d - c_v$ with probability $1 - x$. A settlement is successful when $\sigma = d - c_v$, and a trial occurs when $\sigma = 0$ with probability r. x and r are determined as follows:

$$x = \frac{1 - p_s}{p_s}\frac{c_v}{d - c_v} = \frac{\beta_1}{1 - \beta_0}\frac{1 - p_0}{p_0}\frac{c_v}{d - c_v},$$
$$r = \frac{d - c_v}{d + c_d}.$$

(3) V's winning probability in a trial is c_v/d.

Proposition 2.3.1(1) suggests that when $p_s d < c_v$, V will never file a claim because V's expected payoff from litigation would not justify her court costs. Such a non-compensation pooling equilibrium can occur when p_s – the proportion of liable cases in all cases filed – is small. By equation (2.3.2), when either p_0 – the prior probability that D is liable – is very small, or when V's signal z is very noisy, this non-compensation pooling equilibrium is more likely to occur. This pooling equilibrium can also occur if the damages amount is small relative to the victim's court costs.

Proposition 2.3.1(2) suggests that when V's expected payoff from litigation is greater than her court costs, litigation occurs, and there is a separating equilibrium with randomization between a settlement and a trial. For both the two types of D, offering $\sigma = 0$ obtains a higher payoff, and V accepts such a settlement. Therefore, a liable defendant will randomize between the two settlements, $\sigma = 0$ and $\sigma = d - c_v$. In this game, x is the probability of D offering a settlement of zero when liable, and r is the probability of V rejecting an offer of zero, which is equivalent to the probability of the occurrence of a trial.[13]

Proposition 2.3.1(3) suggests that the ratio of V's court costs to the damages amount com-

[13]There is a pooling equilibrium whereby $\sigma = pd - c_v$ when $\frac{c_v}{d} < p < \frac{c_v + c_d}{d}$. However, this equilibrium is eliminated by the D1 criterion because whenever liable D wishes to defect, non-liable D wishes to do so as well.

pletely determines V's winning probability at trial. The intuition is that if the winning rate is higher than c_v/d, then the expected payoff from a trial would be $p(win)d - c_v > 0$, and V would want to try more cases in court. However, if the winning rate is low and $p(win)d - c_v < 0$, V would want to settle.

In fact, because a trial reveals the true state, such a winning rate is the true proportion of liable cases among the litigated cases. That is, the ratio between the liable cases in rejected zero settlement offers (xp_sr) , and the total number of rejected zero-settlement cases ($(1 - p_s + xp_s)r$):

$$\frac{xp_sr}{(1 - p_s + xp_s)r} = c_v/d. \tag{2.3.4}$$

2.4 Three-Agent Model

In reality, a lawyer provides information to affect a victim's legal decision-making in all stages of a case. Lawyers often solicit victim-clients after an accident, and most victims hire lawyers to help them decide whether to litigate their case, which may happen either before or after receiving a defendant's settlement offer. In some types of disputes, lawyers even aid victims in making a decision 100% of the time.[14]

In the model of this section, victims need to be represented by lawyers in the legal process, and thus need to pay lawyers' fees. To isolate the effects of solicitation by a profit-driven lawyer, we assume the following are the only things that the lawyer(L) can do in the game:

Assumption 3. *In this three-agent game, L sends a signal m to V that affects V's belief at the beginning of the signaling game, and charges fees determined by V's choices and the true*

[14] See, e.g. Strickland et. al.,"Virginia Self-Represented Litigant Study," National Center for State Courts, 2017. `https://ncsc.contentdm.oclc.org/digital/collection/accessfair/id/811/`, last accessed Feb.29, 2020.

state.

We further make the following assumptions about the lawyers.

Assumption 4 (Assumptions about L)**.** *(1) L is informed:*

(a) L observes the true state;

(b) L knows the prior p_0, and observes V's private signal, z. Therefore, L knows p_s;

(2) L is entirely profit-driven, and collects the following fees:

(a) a flat fee, f_0, when V files a claim against D;

(b) a flat fee, f, when the there is a court trial;

(c) $(1-\xi)d$, which is the contingency fee when V wins at trial.

These fees are exogenous to the game.

Admittedly, these assumptions are restrictive. However, the focus of this chapter is on studying the effects of solicitation by a profit-driven lawyer on an imperfectly informed victim's litigation choices and welfare; and these assumptions about L are isolate such effects.[15]

If the victim's own noisy signal is $z = 0$, indicating the state is non-liable, a lawyer can strategically send a solicitation $m = 1$ to the victim, which is interpreted as a signal stating that the defendant is liable. The lawyer's optimal signal when $z = 0$ is described in section 2.4.1. We introduce new notations in section 2.4.2, and describe the signaling between the victim and the defendant in section 2.4.3. The signaling equilibrium is described in section 2.4.4. If the victim's own signal is $z = 1$, she voluntarily hires the lawyer and enters the signaling game if expected payoff is greater than zero. This situation is described in section 2.4.5. Finally, we describe the lawyer's welfare effects in section 2.4.6, and provide a numerical illustration. The calculations and proofs this section are relegated to Appendix B.2

[15]We relax Assumption 4 in subsection 2.5.1, where L is imperfectly informed, and in subsection 2.5.3, where L is altruistic.

(for the case when $z = 0$) and Appendix B.3 (for the case when $z = 1$). Assumptions 1 to 4 are assumed in the model of this section.

2.4.1 The Structure of L's Optimal Solicitation Signal

L is entirely motivated by service fees. When V files a claim, she pays f_0 to L. When V rejects D's settlement offer, automatically leading to a trial, V pays f to L. When a tried case is of the liable type, V wins and pays $(1 - \xi)d$ to L.

As discussed in section 2.3, the victim will not voluntarily file a claim when she receives the private signal $z = 0$, leading to a belief of p'_s as in (3). L solicits V in this situation. Intuitively, a profit-driven L will want V to file and litigate not only all liable cases but also non-liable cases as well. However, L can only "lie" to a certain degree without losing credibility. In an extreme situation, in equilibrium, if V discovers during litigation that L solicits all cases and thus does not provide her with any information, V would act according her own signal and belief. In contrast, if his signal reflects the true state all the time, then knowing this strategy, the victim becomes perfectly informed. There is some room in-between these two extreme situations where lawyer solicit all liable cases as well as some non-liable cases. Kamenica and Gentzkow (2011) describes the optimal signal in send-receiver game that described the situation here. Such optimal signal achieves the following:

Definition 2.4.1 (Optimal Signal)**.** *After updating with the Bayesian optimal signal, V only have two posterior beliefs: whether $\mu_s(\omega_l) = 0$, or $\mu_s(\omega_l) = \mu_t$, where μ_t is the threshold probability for V to file a case.*

That is, after L solicitation, V either believes that D is liable with the threshold probability that is just enough for her to file a claim, or believes D is completely non-liable. V will file a claim in the former, and will not file a claim in the latter. By Kamenica and Gentzkow (2011), since V's actions are binary, under this posterior belief, V takes the sender-optimal action–filing a claim–with the highest probability.

Suppose L sends a signal, $m = \{0, 1\}$. $m = 1$ indicates the defendant is liable; $m = 0$ indicates the defendant is not liable. The following proposition characterizes L's optimal signal according to Definition 2.4.1. The proof is in Appendix B.2.1.

Proposition 2.4.1. *(L's optimal solicitation signal) Demote $P(\omega_l) = \mu_0$, $P(\omega_{nl}) = 1 - \mu_0$. V has the correct belief. The threshold belief for V to file a claim is $\mu_t > \mu_0$. Then L's optimal signal is as follows:*

$$\begin{aligned} &P(m = 1 \mid \omega_l) = 1, \\ &P(m = 0 \mid \omega_l) = 0, \\ &P(m = 1 \mid \omega_{nl}) = \frac{\mu_0}{1 - \mu_0} \frac{1 - \mu_t}{\mu_t}, \\ &P(m = 0 \mid \omega_{nl}) = 1 - \frac{\mu_0}{1 - \mu_0} \frac{1 - \mu_t}{\mu_t}. \end{aligned} \tag{2.4.1}$$

If V's original belief μ_0 is insufficient for her to file a claim ($\mu_0 < \mu_t$), for suitable costs, V follows the recommendation for L under L's optimal signal structure as described in Proposition 2.4.1. This decision rule of V is termed *Obedience* in Bergemann and Morris (2016). Analysis in section 2.3 suggests that Proposition 2.4.1 applies when $\mu_0 = p'_s < \mu_t$, that is, when $z = 0$. We will discuss this situation in subsection 2.4.4. When $z = 1$, $\mu_0 = p_s > \mu_t$. In this case, V hires L voluntarily without solicitation. This situation is described in subsection 2.4.5.

2.4.2 Additional Notations

To develop the three-agent model, we first introduce some new notations. The additional notations required for the three-agent signaling game are listed below.

1. m – the signal from L to V if $z = 0$. $m = 1$ denotes L's solicitation;

2. f_0 – V's flat payment to L if V files a claim;

3. f – the flat court trial fee paid to L when V rejects D's settlement offer and a trial occurs;

4. ξ – the portion of d that V keeps if she wins at trial, after paying $(1-\xi)d$ to L as contingency fee;

5. μ_s – when $z = 1$, V's posterior belief after L's signal; the specific meaning of μ_s depends on the context;

6. μ'_s – when $z = 0$, V's posterior belief after L's signal; the specific meaning of μ'_s depends on the context;

7. μ_t – the threshold $P(\omega_l)$ for V to file a suit;

8. $\bar{p}$ – the proportion of liable cases in settlement when $z = 0$ and $m = 1$;

9. p – the proportion of liable D in a signaling game used in the game tree.

2.4.3 The Three-Agent Signaling Game

In the model, V first gets her own signal $z = \{0, 1\}$. When $z = 1$, V hires L and files a claim against D without any solicitation signal from L. If $z = 0$, L sends a signal $m = \{0, 1\}$ to V. V hires L and files a claim against D if $m = 1$; this reflects the lawyer's solicitation. If V files a claim, V and D enter a settlement game. The timeline of this model is shown in Figure 2.3.

time 0	time 1	time 2	time 3
1. Nature decides the state (ω_l or ω_{nl}) 2. V receives a signal ($z = \{0, 1\}$ regarding the state 3. D and L observe the realization of the state, as well as V's signal, z.	1. L sends a signal $m = \{0, 1\}$ to V if $z = 0$; 2. V hires L if $m = 1$ or if $z = 1$, and files a claim against D. 3. V does nothing if $z = 0$ and $m = 0$, and the game ends here.	If $z = 1$ or $m = 1$ (when V files a case), D offers settlement σ to V.	V decides whether to accept or to reject σ: 1. If V accepts, D transfers σ to V; 2. If V rejects, a trial occurs which reveals the true state.

Figure 2.3: Timeline of the three-agent game

If $z = 0$, the three-agent signaling model in this section solves L's optimal information disclosure in the signaling equilibrium. There are two steps in the signaling game. First, L strategically sends a signal m to V whenever V's private signal $z = 0$. Thus L determines the proportion of liable D's in the game.[16] This is V's prior belief of ω_l in this Bayesian game. Second, the signaling between V and D are carried out. Here, Assumption 1 applies, and therefore litigation is assumed to reveal the true state.[17] If $z = 1$, we only need the second step in the game.

Given L's fee structure, L also affects V's payoffs in the game in the second step. The settlement offer σ from D changes accordingly. When interacting with a liable D, V's settlement payoff is $\sigma - f_0$, and the trial payoff is $\xi d - f_0 - f - c_v$. Therefore, the lowest settlement that V would accept is $\sigma = \xi d - f - c_v$. When interacting with a non-liable D, V's settlement payoff is $\sigma - f_0$, and the trial payoff is $-c_v - f - f_0$. Thus, the lowest non-negative settlement that V would accept is 0. We denote $\sigma^* = \xi d - f - c_v$. D's strategy is $\sigma = \{\sigma^*, 0\}$, and V's

[16]Each signal realization leads to a distribution $\Delta(\Omega)$ over posterior beliefs. Thus, given any signal, the distribution of posterior belief is $\tau \in \Delta(\Delta(\Omega))$. A sender can effectively choose any posterior belief μ_s for the receiver, as longs the expectation of the posterior beliefs induced by a signal is same as the prior belief μ_0, i.e. $\sum_{\text{supp}(\tau)} \mu_s \tau(\mu_s) = \mu_0$. See Kamenica and Gentzkow (2011).

[17]Proposition 2.6.5 shows that Bayesian persuasion of L and D toward a judge reveals the true state in litigation.

strategy is to either accept or reject.

The game tree found in Figure 2.4 summarizes the game with lawyer representation. The node L means that L affects V's belief when entering the game. When $z = 1$, L does not affect the V's belief, and thus $p = p_s$. When $z = 0$, p represents $\mu_s'(m = 1, z = 0)$ in section 2.4.4.

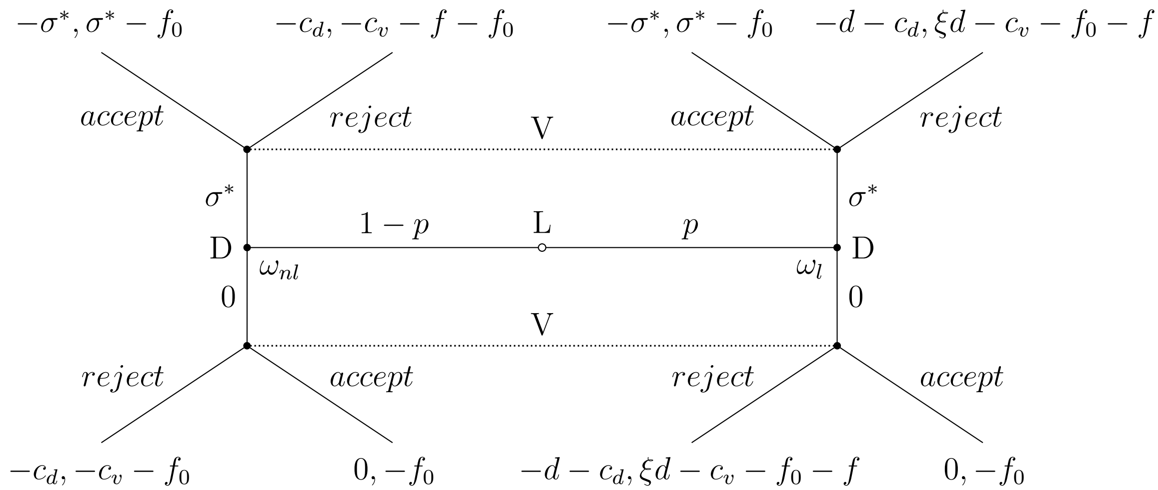

Figure 2.4: Three-agent signaling game

Notice that when $z = 1$, $p = p_s$. When $z = 0, m = 1$, $p = \bar{p}$.

2.4.3.1 Equilibrium Characterization

Given the cost of a lawyer's representation, a victim would only hire a lawyer when the expected payoff from litigation or the settlement would justify the lawyer's fees. In other words, we would have the following.

1. When $z = 0$, and when $m = 1$, V would accept L's solicitation and file a case. She believes that D is liable with probability $\mu_s(z = 0, m = 1)$, where $\xi d\mu_s'(z = 0, m = 1) - f - f_0 - c_v \geq 0$. For $0 < \mu_s'(z = 0, m = 1) < 1$, we must require $\xi d - c_v - f - f_0 > 0$.

2. When $z = 1$, V believes D is liable with probability p_s. V would hire L when the cost is justified, meaning $\xi dp_s - f - f_0 - c_v \geq 0$.

Next, subsection 2.4.4 discusses L's solicitation $z = 0, m = 1$, and subsection 2.4.5 studies the effect of the addition of L on the equilibrium when $z = 1$.

2.4.4 Lawyer's Solicitation ($m = 1$) When $z = 0$, $\xi d - c_v - f - f_0 > 0$

When the victim is unaware of the injury or the legal resources available to her ($z = 0$), a lawyer's solicitation ($m = 1$) provides such relevant information to her and motivates her to file a case. The lawyer's incentive is to encourage as many case filings and and as much litigation as possible without losing credibility. L uses the signal as described in proposition 2.4.1. V's posterior belief after receiving L's signal is as follows.

$$\mu_s'(\omega_l|m = 1, z = 0) = \frac{c_v + f + f_0}{\xi d} = \mu_t,$$
$$\mu_s'(\omega_l|m = 0, z = 0) = 0.$$

Since $0 < \mu_t < 1$, $0 < c_v + f + f_0 < \xi d$. The signaling equilibrium determines L's specific information disclosures and V's litigation decisions.

Proposition 2.4.2. *(V's belief after L's solicitation) When $m = 1$, V believes D is liable with the threshold probability for her to file a case, which is $\mu_s'(\omega_l|m = 1, z = 0) = \mu_t = \bar{p} = \frac{c_v+f+f_0}{\xi d}$; when $m = 0$, V believes that D is completely non-liable, $\mu_s'(\omega_l|m = 0, z = 0) = 0$. In this case, V files the maximum number of cases possible given the prior belief p_s'.*

V files a case when $m = 1$. Therefore among the cases filed, the probability that D is liable is

$$\bar{p} = \mu_s'(\omega_l|m = 1, z = 0) = \frac{c_v + f + f_0}{\xi d}. \tag{2.4.2}$$

V's decisions in the settlement game is summarized below.

Proposition 2.4.3. *(V's and D's strategies in a three-agent signaling equilibrium after L's solicitation)*

(1) V hires L only when $\xi d - c_v - f - f_0 > 0$.

(2) In such a case, there is only a separating equilibrium where,

(i) if non-liable, D offers zero settlement and there is no trial;

(ii) if liable, D randomizes between two offers: zero with probability

$$x = \frac{1-\bar{p}}{\bar{p}} \frac{c_v + f}{\xi d - c_v - f} = \frac{\xi d - c_v - f - f_0}{\xi d - c_v - f} \frac{c_v + f}{c_v + f + f_0}.$$

and a positive settlement amount,

$$\sigma^* = \xi d - c_v - f$$

with probability 1-x;

(iii) V accepts positive offer σ^**, and rejects zero offers with probability*

$$r = \frac{\xi d - c_v - f}{d + c_d}$$

Appendix B.2 provides a detailed calculation of the results for Propositions 2.4.2 and 2.4.3.

2.4.4.1 Effect of L's Solicitation

Because V does not file a claim without L when $z = 0$, the claims filed due to L's solicitation is a net increase of the total claims filed. L increases the total number of cases filed by

$$\begin{aligned} &P(z=0) * (p'_s + P(m=1 \mid \omega_{nl})(1-p'_s)) = P(z=0)p'_s \frac{\xi d}{c_v + f + f_0} \\ &= \beta_0 p_0 \frac{\xi d}{c_v + f + f_0}. \end{aligned} \tag{2.4.3}$$

The trial probability if the case is filed is

$$(1 - \bar{p} + x\bar{p})r = \frac{\xi d - c_v - f - f_0}{d + c_d}$$

Thus, the number of trials increased is the product of the above two equations:

$$\begin{aligned} &P(z=0) * (p'_s + P(m=1 \mid \omega_{nl})(1-p'_s)) * (1 - \bar{p} + x\bar{p})r \\ &= \beta_0 p_0 \frac{\xi d}{c_v + f + f_0} \frac{\xi d - c_v - f - f_0}{d + c_d} \end{aligned} \tag{2.4.4}$$

The probability of winning at trial here is also higher than the probability of winning in the two-agent equilibrium without a lawyer discussed in section 2.3:

$$\frac{x\bar{p}}{1 - \bar{p} + x\bar{p}} = \frac{c_v + f}{\xi d} > \frac{c_v}{d}. \tag{2.4.5}$$

Further, there is no pooling equilibrium in this three-agent signaling game.

2.4.5 Legal Representation When $z = 1$ and $\xi d p_s \geq c_v + f + f_0$

When $z = 1$, V would voluntarily file a case against D; thus, solicitation from L is unnecessary, and V would receive no signal from L. However, if L represents V, L eliminates the possible pooling equilibrium and also increases V's trial winning rate. Incentive compatibility requires that V hires L only when $\xi d p_s \geq c_v + f + f_0$.

The following proposition characterizes the equilibrium.

Proposition 2.4.4. *(Equilibrium in the three-agent model when $z = 1$)*

(1) V only hires L when $\xi d p_s \geq c_v + f + f_0$.

(2) There is only a separating equilibrium where,

(a) if non-liable, D offers zero settlement;

(b) if liable, D randomizes between two offers: zero with probability

$$x = \frac{1-p_s}{p_s}\frac{c_v+f}{\xi d - c_v - f} = \frac{1-p_0}{p_0}\frac{\beta_1}{1-\beta_0}\frac{c_v+f}{\xi d - c_v - f},$$

and a positive settlement amount,

$$\sigma^* = \xi d - c_v - f$$

with probability $1 - x$.

(c) V accepts positive offers σ^*, and rejects zero offers with probability

$$r = \frac{\xi d - c_v - f}{d + c_d}.$$

2.4.5.1 Effect of Being Represented by L

A trial occurs when V rejects D's zero offers. The probability of a trial is:

$$P(z=1) * (1 - p_s + x p_s) r = (1 - p_s)\frac{\xi d}{d + c_d} P(z=1). \tag{2.4.6}$$

The probability of winning at trial is

$$\frac{xp_s}{1 - p_s + xp_s} = \frac{c_v + f}{\xi d}. \tag{2.4.7}$$

2.4.6 The Welfare Effects of Lawyer's Solicitation

Lawyer's solicitation is more likely to be successful when the stakes of V in the case is high relative to lawyer's costs, and when V's private signal is noisy in a certain way. Specifically,

Proposition 2.4.5. *V is more likely to hire L under the following conditions:*

(1) When the cost of hiring L is relatively low, and the damages amount, d, is relatively high, or

(2) When V's private signal, z, is conducive to hiring L. Specifically,

(a) when $z = 0$, if V is more likely to fail to recognize a liable case;

(b) when $z = 1$, if V's signal is more precise.

Overall, L's solicitation when $z = 0$ helps V to recognize more liable cases, and at the same time, sends more non-liable cases into litigation. When L represents V in litigation, the cases at trial here are more likely to be liable compared to those discussed in Section 2.3, and thus, V's trial winning rate becomes higher. There is no pooling equilibrium when L represents V. However, a lawyer's representation is costly, and it reduces the net compensation award, from $d - c_v$ to $\xi d - c_v - f - f_0$. Furthermore, many victims in non-liable cases still file cases and some even go to trial. These victims pay lawyers fees and get no compensation.

Based on the results from the two-agent and three agent models, we propose the following results.

Proposition 2.4.6. *(L's welfare effect is ambiguous) L adds value in the following ways:*

(1) Under appropriate conditions, L helps V recognize liable cases and help her get compensated in the following ways:

(a) L claims filed in the amount of $\beta_0 p_0 \frac{\xi d}{c_v+f+f_0}$*;*

(b) if V hires L, the pooling equilibrium where V is not compensated is eliminated.

(2) L increases the overall trial winning rate from $\frac{c_v}{d}$ *to* $\frac{c_v+f}{\xi d}$*;*

(3) However, lawyers also increase case filings and trials in non-liable cases, and this reduces a victim's net payoff in litigated cases after considering lawyer's fees.

The numerical example in the next subsection illustrates such ambiguous welfare effects.

2.4.6.1 Numerical Example to Illustrate Proposition 2.4.6

We use a numerical example to illustrate Proposition 2.4.6. The calculation are based on Propositions 2.3.1, 2.4.2, 2.4.3, and 2.4.4. The detailed calculation for this example is in Appendix B.3.5.

Consider the situation where a collection of victims (V) is injured when interacting with a collection of defendants (D), and each injury leads to a medical bill of \$1000. Each V interacts with one D and wants D to compensate her for the medical expenses. However, each V is not sure whether the particular D she faces is liable. Litigation is costly – the court costs for V are \$50, and are \$100 for D.

Assume there are 1000 such injuries in total. It is common knowledge that D is liable for 100 of the injuries, and not liable for 900 of them. This suggests $p_0 = 0.1$. When D is actually liable, V knows with probability 0.7 ($\beta_0 = 0.3$). When D is not liable, V mistakenly believes D is liable with probability 0.1 ($\beta_1 = 0.1$).

Without lawyers, among the 100 liable cases, victims obtain a settlement of \$950 in 65 of them, and they win around 5 of them in court trials to obtain \$950. There are 160 total cases filed, and around 82 of them go trials, and V's trial winning rate is around 6%.

Assume that lawyers (L) approach V. L gets \$20 when V sues D, an additional \$100 if the case goes to trial, as well as 30% of V's damages award if V wins at trial (\$1000∗30% = \$300). When represented with lawyers, the victims obtain a settlement of \$550 in around 45+5 = 50 of the 100 liable cases, and win around $12 + 13 = 25$ cases in court trials to obtain \$550. In aggregate, around 75 victims receive compensation in the amount of 530 after factoring in the lawyers initial fee of f_0. In total, there are $160 + 124 = 284$ cases filed, $57 + 60 = 117$ court trials, and the winning rate at trial is 21%. V loses in $117 - 25 = 92$ cases, where they must pay \$70 to L. In $58 + 59 = 117$ cases, V pays consultation fees of \$20 to L.

This example show that lawyers help most victims in liable cases get compensation, and they increase litigation. However, lawyers are costly, which reduces the net award to the victims who obtain compensation. Lawyers increase total number of cases filed as will as the total number of ligation. Therefore, many more victims in non-liable cases also hire lawyers and litigate, resulting in a net payment to lawyers.

2.5 L's Effect on D's Endogenous Safety Costs and Prior Liability

This section analyzes the lawyer's effect of regulating the defendant's precaution level. In this setting, the defendant's prior probability of being liable, p_0, is an endogenous function of his safety costs, y, which are the costs associated with taking precautions to avoid injury to others. For simplicity, the following functional form is used:

$$p_0 = \frac{1}{y+1} \tag{2.5.1}$$

Thus, p_0 is a convex, decreasing function of y. When $y = 0$, $p_0 = 1$; and when $y \to \infty$, $p \to 0$. This function shows that if the total safety costs are 0, the defendant is definitely liable; increasing the safety costs decrease the defendant's prior probability of being liable, but the marginal effect of such a reduction decreases:

$$p_0' = -\frac{1}{(1+y)^2} < 0, \quad p_0'' = \frac{2}{(1+y)^3}.$$

The safety costs must approach infinity to reduce the prior probability in order to be close to 0. A defendant chooses the optimal level of safety costs y by balancing safety investments and the expected expenditure he would incur in litigation.

When both the victim and the defendant are completely informed, a liable defendant's payoff from a court trial is $-d - c_d$, so he would prefer to pay a settlement of $d - c_v$ to V. Because V would not obtain more from a trial, she would accept such a settlement. V would not file a claim against a non-liable D, as her payoff from a court trial would be $-c_v$, and D would only offer a settlement of 0 to V. Thus, there is no trial, and D offers a settlement of $d - c_v$ if and only if he is liable. D solves the following problem to minimize his expected cost:

$$\begin{aligned} &\min_y \; p(d - c_v) + y \\ &s.t. p = \frac{1}{y+1}, y > 0, 1 > p > 0 \end{aligned} \tag{2.5.2}$$

Let p_0^* and y^* be D's equilibrium safety cost and prior probability of being liable, respectively. From the first order condition, we get

$$y^* = \sqrt{d - c_v} - 1, \quad p_0^* = \frac{1}{\sqrt{d - c_v}}.$$

The following subsections discuss the defendant's problem in the signaling games when a victim only have incomplete information regarding whether a defendant is liable.

2.5.1 Two-Agent Signaling Game

We consider the setting of the baseline two-agent signaling game discussed in section 2.3. The defendant is perfectly informed, but the victim only obtains a noisy signal. There are two cases: the pooling equilibrium with no litigation, and separating equilibrium with some litigation. Denote the equilibrium safety cost $y^{'**}$ in the first case, and y^{**} in the second case. Let p_0^{**} the D's equilibrium prior probability of being liable.

2.5.1.1 Case 1: Pooling Equilibrium with No Litigation

In order to avoid litigation entirely, by proposition 2.3.1, the following condition must be satisfied:

$$p_s < \frac{c_v}{d}.$$

We substitute in p_s from equation (2.3.2), and obtain the following restriction on y:

$$\begin{aligned}
&\frac{p_0(1-\beta_0)}{(1-\beta_0)p_0 + \beta_1(1-p_0)} < \frac{c_v}{d} \\
\Longrightarrow\ & p_0 < p_0^{'**}, p_0^{'**} = \frac{\beta_1 c_v}{(1-\beta_0)(d-c_v) + \beta_1 c_v} \\
\Longrightarrow\ & y > y^{'**}, y^{'**} = \frac{1-\beta_0}{\beta_1}\frac{d-c_v}{c_v}.
\end{aligned}$$

Such safety costs can be quite high if $d \gg c_v$ or when $\beta_1 \to 0$ – that is, when V can almost perfectly recognize liable cases. In fact, $y^{'**} \to \infty$ and $p_0 \to 0$ if $\beta_1 \to 0$.

2.5.1.2 Case 2: Separating Equilibrium with Randomization in Litigation

Considering the safety costs, we can see from case 1 that litigation is practically inevitable when V can recognize nearly all liable cases, as this this requires very high safety costs, $\lim_{\beta_1 \to 0} y^{'**} = \infty$. Suppose there is litigation. The defendant's minimization problem is as follows:

$$\min_y \; p_0(1-\beta_0)\left[xr(d+c_d)+(1-x)(d-c_v)\right]+(1-p_0)\beta_1 r c_d + y$$
$$s.t. p_0 = \frac{1}{y+1} \tag{2.5.3}$$
$$0 < p_0 < 1, y > 0.$$

We substitute in r, x from Section 2.3, proposition 2.3.1. Thus, the defendant solves the following for $0 < p_0 < 1$:

$$\begin{aligned}
&\min_{p_0} \; p_0(1-\beta_0)\left[\frac{\beta_1}{1-\beta_0}\frac{1-p_0}{p_0}c_v + d - c_v - \frac{\beta_1}{1-\beta_1}\frac{1-p_0}{p_0}c_v\right] + (1-p_0)\beta_1\frac{d-c_v}{d+c_d}c_d + \frac{1}{p_0} - 1 \\
=&\min_{p_0} \; p_0(1-\beta_0)(d-c_v) + (1-p_0)\beta_1\frac{d-c_v}{d+c_d}c_d + \frac{1}{p_0} - 1 \\
=&\min_{p_0} \; p_0(d-c_v)\left[1-\beta_0-\frac{c_d}{d+c_d}\beta_1\right] + \frac{1}{p_0} - 1 + \frac{d-c_v}{d+c_d}\beta_1 c_d.
\end{aligned}$$

By first order condition, we get

$$\begin{aligned}
p_0^{**} &= \frac{1}{\sqrt{(d-c_v)(1-\beta_0-\frac{c_d}{d+c_d}\beta_1)}} > \frac{1}{\sqrt{d-c_v}} = p_0^* \\
y^{**} &= \sqrt{(d-c_v)(1-\beta_0-\frac{c_d}{d+c_d}\beta_1)} - 1 < \sqrt{d-c_v} - 1 = y^*.
\end{aligned}$$

There is no litigation when both V and D are completely informed. However, avoiding litigation entirely might not be possible when V is not perfectly informed. When there is litigation, $y^{**} < y^*$, and $p_0^{**} > p^*$. Thus, when the victim is imperfectly informed, the defendant's optimal safety costs will be lower, and the optimal probability of prior liability will be higher.

2.5.2 Three-Agent Signaling Game

We denote the defendant's equilibrium safety costs and prior probability of being liable as y^{***} and p_0^{***}, respectively. For litigation with lawyerly representation to occur, the following must hold.

$$\xi d - c_v - f - f_0 > 0.$$

If the above is satisfied, then there is always litigation. We then apply the results found in section 2.4 for the three-agent game for the separating equilibrium with randomization. The defendant's problem is as follows:

$$\begin{aligned}\min_{p_0}\ & p_0\left[1-\beta_0+\beta_0\right]\left[xr(d+c_d)+(1-x)(\xi d-c_v-f)\right]+\\ &+(1-p_0)\left[\beta_1+(1-\beta_1)\frac{\beta_0}{1-\beta_1}\frac{p_0}{1-p_0}\frac{\xi d-c_v-f-f_0}{c_v+f+f_0}\right]xrc_d+\frac{1}{p_0}-1.\end{aligned}$$

Substitute in x, r,

$$\min_{p_0}\ p_0(\xi d-c_v-f)+(1-p_0)\left[\beta_1+\beta_0\frac{p_0}{1-p_0}\frac{\xi d-c_v-f-f_0}{c_v+f+f_0}\right]\frac{\xi d-c_f-f}{d+c_d}c_d+\frac{1}{p_0}-1.$$

By solving the first order condition and restricting $0 < p_0 < 1$, we get the following.

$$\begin{aligned} p_0^{***} &= \frac{1}{\sqrt{\xi d-c_v-f}}\frac{1}{\sqrt{1+\left(\beta_0\frac{\xi d-c_v-f-f_0}{c_v+f+f_0}-\beta_1\right)\frac{c_d}{d+c_d}}} > \frac{1}{\sqrt{\xi d-c_v-f}},\\ y^{***} &= \sqrt{\xi d-c_v-f}\sqrt{1+\left(\beta_0\frac{\xi d-c_v-f-f_0}{c_v+f+f_0}-\beta_1\right)\frac{c_d}{d+c_d}}-1 > \sqrt{\xi d-c_v-f}-1.\end{aligned}$$

L's effects on D's safety costs and the prior probability of being liable is ambiguous. If V is unlikely to mistake a liable defendant for a non-liable one, the presence of a lawyer actually increases the prior probability of D being liable, as L's fees make a court trial less likely. When a victim is more likely to mistake a liable case for a non-liable one, and when the lawyers' flat case filing fee is sufficiently low, the presence of the lawyer increases the defendant's precaution cost, and thus, decreases the defendant's prior probability of being liable.

Proposition 2.5.1. *L's effects on D's safety costs, and thus, the prior probability of being liable, depend on specific parameters.*

Proof. Compare the two-agent signaling game and the three-agent signaling game.

$$\left(\frac{p_0^{**}}{p_0^{***}}\right)^2 = \frac{\xi d - c_v - f}{d - c_v} \frac{1 - \beta_1 \frac{c_d}{d+c_d} + \beta_0 \frac{\xi d - c_v - f - f_0}{c_v + f + f_0} \frac{c_d}{d+c_d}}{1 - \beta_1 \frac{c_d}{d+c_d} - \beta_0}$$

In reality, it is quite possible that $d \gg c_d$ and $d \gg c_v$, thus

$$\left(\frac{p_0^{**}}{p_0^{***}}\right)^2 \to \frac{\xi d - f}{d} \frac{1}{1 - \beta_0}$$

Therefore,

$$p_0^{**} > p_0^{***} \iff f < (\beta_0 - (1 - \xi))d$$

In reality, $0.33 < 1 - \xi < 0.4$. [18] Suppose $1 - \xi = 0.4$. If $0 < \beta_0 < 0.4$, then the right-hand side is less than 0, and thus $p_0^{***} > p_0^{**}$. That is, when the victim do not make much type II mistake, and recognize most liable cases, the presence of the lawyer actually increases the defendant's prior probability of being liable. In such situation, D's safety costs are decreased.

However, if $0.4 < \beta_0 < 1$, then $p_0^{***} < p_0^{**}$ if and only if $f < (\beta_0 - (1 - \xi))d$. In reality, when a lawyer charges a contingency fee, there is usually no flat case filing fee, that is $f = 0$. In such situations, when the victim makes a type II error and mistakes a liable case for a non-liable case sufficiently often (i.e.$1 \geq \beta_0 > 1 - \xi$), the presence of a lawyer increases the precaution cost of the defendant. □

Thus, the presence of a lawyer may increase D's safety costs and decrease D's prior probability

[18] Lawyer's contingency fees are usually between 33% and 40%. For example, see "Lawyers' Fees in Your Personal Injury Case", `https://www.alllaw.com/articles/nolo/personal-injury/lawyers-fees.html`, last access March 1, 2020.

of being liable, but may also do the opposite. The intuitions is that a lawyer has two effects: (1) helps a victim recognize liable cases; (2) increases a victim's litigation costs. The first effect encourages litigation and thus increases D's safety costs and reduces D's prior probability of being liable; and the second effect does the opposite. The lawyer's overall effect depends on the strength of the two opposing effects, which in turn depends on the specific parameters.

2.6 Three Extensions

Next, we consider three extensions to the three-agent model: (1) a lawyer's solicitation when he is imperfectly informed, (2) a lawyer solicitation when he is altruistic, and (3) persuasion during litigation where the defendant is able to persuade the judge.

2.6.1 Extension 1: Solicitation by an Imperfectly Informed L

This extension weakens the assumption that L is completely informed. The assumptions regarding L's information and signaling strategy are as follows.

(1) L is imperfectly informed before being hired:

(a) L's prior belief that a case is liable is p_0. L gets a noisy signal, s:

$$P(s=1|\omega_l) = 1-\theta_0$$

$$P(s=0|\omega_l) = \theta_0 \text{ (a false-negative, or a type II error)}$$

$$P(s=1|\omega_{nl}) = \theta_1 \text{ (a false-positive, or a type I error)}$$

$$P(s=0|\omega_{nl}) = 1-\theta_1;$$

(b) L observes V's signal z;

(c) L's signal is better than V's signal: $\theta_0 < \beta_0$, $\theta_1 < \beta_1$;

(2) L conditions his optimal signaling strategy $m = \{0, 1\}$ on s:

$$P(\omega, m \mid s) = P(\omega \mid s)P(m \mid s)$$

Therefore, the state ω and L's signal m are mutually independent, conditional on L's signal s. Intuitively, L can distinguish the states ω_l and ω_{nl} only as well as his signal, s. If L's signal is $s = 1$, he tells V that D is liable; and if L's signal is $s = 0$, with some probability, he tells V that L is liable.

(3) L becomes fully informed if hired. Thus, the equilibrium when $z = 1$ is not affected, as V voluntarily hires L in such a situation.

The next sub-subsection presents the equilibrium of this setup. The detailed calculations are provided in Appendix B.4.

2.6.1.1 Equilibrium Characterization

(1) L's signaling strategy is as follows:

$$\begin{aligned}
&P(m = 1|s = 1) = 1,\\
&P(m = 0|s = 1) = 0,\\
&P(m = 1|s = 0) = \frac{\xi d(1-\theta_0)p'_s - (c_v + f + f_0)[(1-\theta_0)p_0 + \theta_1(1-p_0)]}{(c_v + f + f_0)[\theta_0 p_0 + (1-p_0)(1-\theta_1)] - \xi d\theta_0 p'_s},\\
&P(m = 0|s = 0) = \frac{c_v + f + f_0 - \xi d p'_s}{(c_v + f + f_0)[\theta_0 p_0 + (1-p_0)(1-\theta_1)] - \xi d\theta_0 p'_s}.
\end{aligned}$$

(2) V's and D's signaling equilibrium is as follows:

The signaling equilibrium in terms of V's and D's strategies and V's beliefs are exactly the same as those in section 2.4. In fact, L chooses the signal that results in the same signaling equilibrium as that in the perfect information situation.

2.6.1.2 The Imperfectly Informed L's Effects

The winning rate from a trial is still $\frac{c_v+f}{\xi d}$. Given the signaling equilibrium characterized in subsection 2.4.3, the amount of claims filed increases by:

$$\begin{aligned} P(m=1)*P(z=0) &= \frac{\xi d p_s'(1-p_0)(1-\theta_0-\theta_1)}{[\theta_0 p_0+(1-p_0)(1-\theta_1)](c_v+f+f_0)-\xi d\theta_0 p_s'} * P(z=0), \\ =& \frac{\xi d\beta_0 p_0(1-p_0)(1-\theta_0-\theta_1)}{[\theta_0 p_0+(1-p_0)(1-\theta_1)](c_v+f+f_0)-\xi d\theta_0 p_s'}. \end{aligned} \tag{2.6.1}$$

When $\theta_1 = \theta_0 = 0$, the number of cases filed is $\frac{\xi d p_s'}{c_v+f+f_0}$. This is the situation when L is fully informed. When $\theta_0, \theta_1 \in (0, 0.5)$, $P(m=1) < \frac{\xi d p_s'}{c_v+f+f_0}$.[19] Thus, the number of cases filed after a lawyer's solicitation when L is imperfectly informed is fewer than that when L is perfectly informed.

The probability of a court trial increases by:

$$\begin{aligned} & P(z=0)P(m=1)(1-\bar{p_v}+x\bar{p_v})r \\ & = P(z=0)P(m=1)\frac{\xi d - c_v - f - f_0}{d+c_d}. \end{aligned} \tag{2.6.2}$$

Proposition 2.6.1. *An imperfectly informed L can also increase litigation. The effectiveness of L's solicitation increases as L becomes more informed. When L becomes increasingly more informed, the equilibrium converges to the three-agent case where L is completely informed as in section 2.4.*

Proof. Compare the probability of number of case filings increase by L here (in equation (2.6.2)) to that in Section 2.4 (in equation (2.4.5))), we find that there is an additional term

[19] $\theta_i = 0.5, i = 0, 1$ is the completely uninformed case.

$0 \leq P(m=1) \leq 1$ here, where

$$P(m=1) = \frac{\xi d p_s'(1-p_0)(1-\theta_0-\theta_1)}{[\theta_0 p_0 + (1-p_0)(1-\theta_1)](c_v+f+f_0) - \xi d\theta_0 p_s'} > 0.$$

The detailed calculations are shown in Appendix B.4. Therefore, we see that L can always increase the amount of litigation, although to a lesser extent compared to the situation where L is perfectly informed. If L is almost perfectly informed, then $\theta_0 = \theta_1 = 0$, and $P(m=1) = \frac{\xi d p_s'}{c_v+f+f_0}$, and the number of cases increased is the same as the perfectly informed L setting in equation (2.4.4). □

2.6.2 Extension 2: Solicitation by Altruistic Lawyers

In reality, L's personal and professional ethics may require him to be concerned about V's payoff. Moreover, L may value his reputation, which depends on how much he can help V. Therefore, L derives utility from both his profit and V's payoff:

$$U_L = (1-\delta)\pi_L + \delta\pi_V, \quad \delta \in [0,1].$$

Such internalization of the other party's utility is labeled "altruism". Altruism need not be driven by pure emotion. The parameter δ captures the degree of L's altruism.

Proposition 2.6.2. *There is a threshold degree of altruism that determines L's action. If L is more altruistic than the threshold level, he will always report the truth to V, thus eliminating a court trial. Otherwise L acts as if he is entirely profit-driven.*

Proof. This Proposition easily follows the equilibrium of a three-agent game with an altruistic lawyer, presented in the below. Detailed calculations are found in Appendix B.5. We can find a threshold degree of altruism,

$$\delta^* = \frac{1}{1+\kappa},$$
$$\kappa = \frac{c_v + f + f_0}{f_0 + \frac{1}{1+\frac{c_d}{d}}(f + (1-\xi)c_v)},$$

such that

(i) when $\delta \geq \delta^*$, L truthfully report the state to V. The signal from L to V, then, is the following:

$$P(m=1|\omega_l) = 1, \quad P(m=0|\omega_l) = 0,$$
$$P(m=1|\omega_{nl}) = 0, \quad P(m=0|\omega_{nl}) = 1.$$

Thus, this game is equivalent to the two-agent perfect information game, where V files a suit against D whenever D is liable, and D provides positive settlement $\sigma^* = \xi d - c_v - f - f_0$ to V whenever V sues him. There are no court trials;

(ii) when $\delta < \delta^*$, the equilibrium is the same as that found in section 2.4.

□

2.6.3 Extension 3: Bayesian Persuasion in a Court Trial

In this subsection, a trial does not necessarily reveal the truth. Instead, a judge (J) makes a binary decision, liable or non-liable, based on his belief and a certain threshold standard. In the two-agent signaling game, where informed defendants can manipulate the judges' decisions in trials by using the optimal signal characterized in proposition 2.4.1 to persuade J, victims will lose many trials in cases where defendants are liable in equilibrium. However, with the addition of a lawyer, the persuasion from L and D causes a trial to reveal the truth.

2.6.3.1 Two-Agent Model when Litigation Does Not Reveal the True State

In an actual trial, a judge considers information and the arguments supplied by both sides and determines the case outcome. This subsection assumes that in a trial, J shares V's prior, $\mu_0(\omega_l) = p_s$, and that D sends a signal to affect J's belief. J makes a decision based on his posterior belief and a decision standard. For example, if J adopts a "more likely than not" criterion, J rules D liable if and only if he believes that D is liable with a probability higher than 50%. However, if J is more pro-defendant, then the threshold probability becomes higher.

2.6.3.2 J's Decision Rule and D's Optimal Signal in A Trial

In a trial, J derives zero utility when he makes a correct decision, and derives a negative utility when he makes a wrong decision. Normalizing the utility from wrongfully ruling against a non-liable D to be 1, J's utility from ruling is:

$$\begin{aligned} &u(V\ wins|\omega_l) = 0, \quad u(V\ wins|\omega_{nl}) = -1, \\ &u(D\ wins|\omega_l) = -\gamma, \quad u(D\ wins|\omega_{nl}) = 0. \end{aligned} \tag{2.6.3}$$

The persuasion setting here between D and J is analogous to that between L and V described in proposition 2.4.1. We assume J takes a sender-optimal action when J is indifferent. J's action $\hat{v}(p_s)$ is binary: J will rule against V if $p_s = \mu_0(\omega_l) \leq \frac{1}{\gamma+1}$, and will rule against D if $p_s > \mu_0(\omega_l) < \frac{1}{\gamma+1}$. Thus, D only benefits from persuasion when $p_s > \frac{1}{\gamma+1}$.

Two new variables are introduced into the two-agent baseline model to capture the effect of J's decision rule on the signaling equilibrium:

1. α – V's trial winning probability for a liable case;
2. β – V's trial winning probability for a non-liable case (as we will see later, $\beta = 0$).

Let $\mu_0(\omega_l)$ be the probability of D being liable in a trial. In the two-agent model, $\mu_0(\omega_l) = p_s$ as in section 2.3.

The following proposition summarizes the equilibrium in the entire signaling game.

Proposition 2.6.3.

1. *When* $\mu_0(\omega_l) > \frac{1}{\gamma+1}$,

$$\beta = 0, \qquad \alpha = \frac{1+\gamma}{d/c_v + \gamma};$$

2. *When* $\mu_0(\omega_l) < \frac{1}{\gamma+1}$*, D always wins, and* $\alpha = \beta = 0$.

Proposition 2.6.4. *The equilibrium in the two-agent model when the defendant persuades the judge during a trial has the following properties:*

(1) There is a pooling equilibrium where there is no litigation and V is not compensated when $p_s < \frac{1+\gamma c_v/d}{1+\gamma}$.

(2) Litigation occurs if and only if $p_s \geq \frac{1+\gamma c_v/d}{1+\gamma}$*. When this condition is satisfied, a non-liable D offers* $\sigma = 0$*. A liable D offers* $\sigma = 0$ *with probability* x*, and* $\sigma = \alpha d - c_v = \frac{1+\gamma}{d/c_v+\gamma} d - c_v$ *with probability* $1 - x$*. A settlement is successes when* $\sigma = d - c_v$*; and a trial occurs when* $\sigma = 0$ *with probability* r*, where*

$$x = \frac{1-p_s}{p_s}\frac{c_v/d - \beta}{\alpha - c_v/d} = \frac{1-p_s}{p_s}\frac{1/\gamma + c_v/d}{1 - c_v/d},$$
$$r = \frac{1 - c_v/\alpha d}{1 + c_v/\alpha d} = \frac{\gamma(1-c_v/d)}{1+\gamma+c_d/c_v+\gamma c_d/d}.$$

(3) J's degree of sympathy towards V affects the signaling equilibrium: a higher γ *(when J is more sympathetic to V) works in V's favor;*

(4) V's winning rate at trial is c_v/d.

Such results are obtained in the signaling equilibrium, as in Appendix B.6.

Comparing the two-agent signaling equilibrium in section 2.3 and this extension, we see that ceteris paribus, if $c_v < d$, the pooling equilibrium where V is never compensated is more likely when D can persuade J during a trial.

We applying the result of this extension to the numerical example in subsection 2.3.6.1. Suppose that J adopts a "more likely than not" standard, i.e., V wins if D is 50% liable. Because $\frac{1+\gamma c_v/d}{1+\gamma} = 0.68 > p_s = 0.163$, there is a pooling equilibrium with no litigation and no compensation for V.

In the separating equilibrium, the positive settlement amount offered by D is not dependent on V's belief, p_s, but is dependent on J's degree of sympathy towards V and V's trial winning rate. In general, V is better off when J is more sympathetic towards her, i.e., when γ is large. Specifically, as γ increases:

(1) A separating equilibrium is more likely because the threshold $\frac{1+\gamma c_v/d}{1+\gamma}$ decreases.

(2) The trail rate $(1 - p_s + p_s x)r$ increases.

(3) The threshold $\mu_s^*(\omega_l) = \frac{1}{1+\gamma}$ for V to win decreases, and V's trial winning rate in liable cases α increases.

(4) D is more likely to offer σ^* because x decreases, and the settlement amount σ^* increases. V is thus more likely to accept the settlement;

However, no matter whether a trial reveals the true state, V's trial winning rate is the same, and is equal to the ratio of V's court costs and the damages amount; this because the effect of the litigation process on D's and V's randomization strategies x and r each other cancel out.

2.6.3.3 Three-Agent Model Bayesian Persuasion in Litigation Reveals the True State

When L is in the game, there are two senders, L and D, who affect J's decision during a trial. Suppose that during a trial, L and D both use optimal signals, as in proposition 2.4.1, to persuade J. In such a setting, a trial reveals the truth for any decision rule p^* and any prior belief of J. Therefore, at trial, $\alpha = 1$ and $\beta = 0$.

Proposition 2.6.5. *If perfectly informed L and D both use the signaling structure in proposition 2.4.1 to persuade J in during trial, the trial will reveal the true state.*

The proof of Proposition 2.6.5is in Appendix B.7. Gentzkow and Kamenica (2017) also suggests a similar result in their Bayesian persuasion framework. They then go on to provide a framework of Bayesian persuasion by multiple senders where the senders also strategically interact with one another. Such a framework provides richer information equilibria. In this extension here, the strategic interaction between L and D is not considered.

2.7 Conclusion

This chapter examined the effect of "ambulance chasing" by lawyers. In the main model, a profit-driven lawyer controls an imperfectly informed victim's information to affect the victim's litigation decision in a signaling game with a perfectly informed defendant. The comparison of the signaling games with and without lawyers shows that although victims' lawyers may increase the number of victims who obtain compensation, they also induce more litigation in the state where the defendants are not liable. In addition, lawyer's fees reduce the net awards to victims who receive compensation, and constitute a net cost to victims who file cases against non-liable defendants. Furthermore, although lawyers may identify more liable cases, their fees discourage litigation. For a range of realistic parameters, a lawyer may help encourage more safety costs from a potential injurer when the victim on her own is likely to mistake a liable case for a non-liable case.

The models presented in this chapter can be applied to broader contexts. In my model, the lawyer affects the signaling equilibrium between the victim and the defendant by controlling the victim's information environment. Although the setup of the game is a tort case where an imperfectly informed victim interacts with a perfectly informed defendant in a signaling game, such a setting can easily be adapted to other civil cases, for example, contract or divorce cases, or even more generally, to situations in which an adviser solicits business from an imperfectly informed client who interacts with a perfectly informed opponent. For example, one could use the three-agent framework discussed here to analyze the effect of a financial adviser on a client who is about to begin an investment negotiation. The current model assumes the defendant has complete information and is purely strategic. However, one could consider the extension where a defendant is has some behavior traits, for example, when a defendant has some degree of trust-worthiness.

Chapter 3

Planner-Doer Game with Hidden Costs

Delegation is common in decision-making settings. Delegation usually comes with some costs when the planner needs to motivate doers to make appropriate choices. In general, delegation costs can be hidden, but they can be derived from observable choice primitives. This chapter provides such identifications using the planner's ex-ante preferences over finite menus. We then characterize a special case when delegation is either costless or impossible, that is, the delegation cost is either 0 or ∞. We also provide an algorithm to check whether ex-post choices conform to the delegation model with binary costs. Finally, we compare our model with related models in the literature. [1]

[1] This chapter is based on two joint working papers with my advisor, Igor Kopylov. The two papers are Revealed Delegation (Kopylov & Yang, 2020) and Delegation with Hidden Costs (Kopylov & Yang, 2022). All main results in this chapter are adapted from results in these two papers.

3.1 Introduction

Many models in the decision theory literature study endogenous constraints that cause preferences over menus and choices in menus to deviate from the predictions of the standard utility maximization assumption. These constraints can be self-control costs Gul and Pesendorfer (2001), inattention Masatlioglu, Nakajima, and Ozbay (2012), limited willpower Masatlioglu, Nakajima, and Ozdenoren (2020), hidden actions Chandrasekher (2017), and so on. In this chapter, we study another source of constraint, delegation, which is common in empirical settings.

Sunstein and Ulmann-Margalit (1999) distinguish between two general environments where decisions need to be delegated. In the intrapersonal case, a doer must delegate her upcoming choices to her *future self* who may succumb to spontaneous temptations and exhibit less patience towards delayed rewards. Such conflicts can explain dynamic inconsistences and various *commitment* strategies (e.g. Strotz (1956), Thaler and Shefrin (1981), Gul and Pesendorfer (2005)). Commitments can impose physical constraints on the feasible set, but also impose emotional or monetary penalties. For example, people may keep only healthy foods and drinks at home, use self-exclusion from casino gambling, make promises and vows, set deadlines, and so on. (see the review of Bryan, Karlan, and Nelson (2010)).

In the interpersonal case, *planners* delegate decisions to *doers* who do not share the same physical identity with the planner.[2] There is a vast literature (e.g., Laffont and Martimort (2002)) on the planner-doer problem with monetary incentives. In particular, such delegation is commonly studied in the theory of firms (Alonso & Matouschek, 2007; Halac & Yared, 2020; Holmstrom, 1980). Delegation with a non-monetary incentive is also a classic topic in the contract design literature (e.g. Alonso and Matouschek (2008)).

We illustrate preference over menus in delegation with hidden actions with the following

[2]Thaler and Shefrin Thaler and Shefrin (1981) propose the terms *planners* and *doers* in their early model of commitments and self-control. We use this terminology in interpersonal settings as well.

story.

Example 3.1. *A pharmaceutical company can use costly promotional efforts to influence doctors' prescription selections. Consider the following three drugs,* a, b*, and* c*. Drug* a *is new and so unknown, whereas* b *and* c *are well-known. Doctor 1 prefers* b *over* c*, whereas doctor 2 prefers* c *over* b*. Overall, doctor 1's ranking of the drugs is* $b \succ c \succ a$*, whereas doctor 2's ranking is* $c \succ b \succ a$*.*

Doctor 1's preference can be changed into $c \succ a \succ b$*, and doctor 2's preference can be changed into* $b \succ a \succ c$*, with a \$1000 advertisement on* a*. However, to further modify the doctors' preferences to* $a \succ b \succ c$ *and* $a \succ c \succ b$*, respectively, more promotion efforts are required, which would cost \$1000 more. So the total advertising cost to make* a *the first choice among* a, b *and* c *for the doctors is \$2000.*

Suppose that the profit of the pharmaceutical company for a, b *and* c *are \$3000, \$1200, and \$1500, respectively. Suppose the pharmaceutical company has the opportunity to bundle drug* a *with either* b, c *or both. If the bundle contains drug* b *or* c *but not both, the pharmaceutical company will want to promote drug* a *for \$1000 to maximize profit. On the other hand, promoting* a *when the bundle includes both drugs* b *and* c *outweighs the benefit. This limitation motivates the company's rankings over the bundles:*

$$\{a, c\} \succeq \{a, b\} \succ \{a, b, c\}. \tag{3.1.1}$$

Verification cost is a type of delegation cost incurred ex-post. Incomplete contract theory considers binary verification costs: a performance is either verifiable for free or is unverifiable. A verifiable term is legally valid; while an unverifiable term is legally void (see, e.g., Bernheim and Whinston (1998)). For example, a non-compete covenant - the promise by an employee not to compete with the employer's business after leaving - is unverifiable in court in California. However, it is verifiable in all other states. Fallick, Fleischman, and Rebitzer (2006) empirically finds that the unenforceability of non-compete covenants enhances the mobility of talents between competing firms. Such mobility leads to agglomeration economies operated in Silicon Valley. We illustrate the ex-post choice with the costly verification in the following example.

Example 3.2. *Let us imagine a manager in a technology company who wishes to assign a new hire to an appropriate position. Given Silicon Valley's job-hopping mentality, the new hire could depart at any time, and the manager takes verification costs for the non-compete agreement into account when assigning positions.*[3] *Assume that the company can only accommodate two generic positions, a and b, which correspond to product lines 1 and 2, respectively. Moreover, the new hire's competence is better suited to position a. As a result, the manager assigns the new hire to position a.*

Assume there are more specialized positions c and d in the industry for product lines 1 and 2, respectively. The entire product line associated with a specialized position will have access to more resources and clients, making it more expensive to have a departing employee in this product line giving up the opportunity to benefit from this insider knowledge. In other words, a specialized position increases the verification cost for the entire product line.

The new hire is not suitable for either c or d. As a result, if the company is able to accommodate positions a, b and c, the verification cost for a increases. The manager would assign the new employee to position b, which involves lower verification cost. If the company can accommodate a, b, c and d, the verification costs for both a, b increases, and the manager

[3] Severance payouts, which are regarded as a strategic complement to the non-compete provision, are often intentionally structured to be unenforceable and are paid in installments Sanga (2018). Such discretion is used to incentivize the non-compete agreement's performance. We can regard such severance payout as the ex-post verification cost for performing the non-compete clause.

assigns the new hire to a. Such choices give raise to choice function

$$c(ab) = a, c(abc) = b, c(abd) = a, c(abcd) = a. \tag{3.1.2}$$

To accommodate choice patterns (3.1.1) and (3.1.2), we have a new functional form

$$U(A) = \max_{x \in A}[u(x) - h(x, A)] \tag{3.1.3}$$

for any menu $A \in \mathcal{M}$, where $u : X \to \mathbb{R}$. $h(x, A) : X \times \mathcal{M} \to \mathbb{R}$ is a menu contingent cost function that satisfies some special properties. In Theorem 3.1 we use four axioms to characterize the representation (3.1.3). Theorem 3.2 is a special case where $h(x, A) = 0$ or ∞ for all $x \in A, A \subseteq Z$. This model is closely related to the planner-doer game in Chandrasekher (2017), and the path-independent filter model in Lleras, Masatlioglu, Nakajima, and Ozbay (2021). Theorem 3.3 discuses identification of delegation model with binary costs, 0 or ∞, in ex post choice data. Finally, we compare the delegation model with several related models.

3.1.1 Literature Review

Gul and Pesendorfer (2001) study temptation preferences with self-control costs in a lottery setting. Their model captures commitment and self-control and identifies the utility cost of self-control. Under such a preference, adding a tempting choice alternative to a menu requires costly self-control and thus reduces the utility of the menu. This insight is captured by the set betweenness (PSB) axiom:

$$A \succeq B \implies A \succeq A \cup B \succeq B.$$

Gul and Pesendorfer (2001) also allow for overwhelming temptation. When the temptation is too strong to resist, the decision-maker lexicographically maximizes temptation and commitment preference. Hence, the choice is made from only the most tempting alternatives. The willpower-constrained decision-making in Masatlioglu et al. (2020) is closely related to Gul and Pesendorfer (2001). In the willpower model, the choice is limited to the choice alternatives whose temptation utility is within a threshold from the most tempting choice alternative. This model captures the compromise effect:

$$C(A) \succeq C(A \cup B) \succeq B,$$

where $C : \mathcal{M} \to \mathcal{M}$ denotes the choices from a menu. Hence, the choice from the union of two menus is between the choices made from the two menus separately.

PSB captures one-dimensional temptation: there is one most tempting element that affects the decision maker's behavior. Dekel, Lipman, and Rustichini (2009) generalizes Gul and Pesendorfer (2001) and accommodate multi-dimensional temptations. Dekel et al. (2009) captures the case when (1) different temptations occur stochastically, (2) resisting more temptations is harder, or (3) uncertainty of whether a temptation would strike. This model is closely related to models that study the aggregation of state-dependent utility functions to capture preference for flexibility (Dekel, Lipman, & Rustichini, 2001; Kreps, 1979). The starting point is the finite additive expected utility (EU) representation:

$$V(x) = \sum_{i=1}^{I} \max_{\beta \in x} \omega_i(\beta) - \sum_{j=1}^{J} \max_{\beta \in x} v_j(\beta). \tag{3.1.4}$$

The temptation representation is characterized when (3.1.4) additionally satisfies two more axioms, Desire for Commitment (DFC) and Approximate Improvements are Chosen (AIC).

When an additive EU representation satisfies PSB, there is one positive state and multiple negative states. In this case, there exists a *no-uncertainty representation*, where $I = 1$ in (3.1.4). Dekel et al. (2009) shows that, given a finite additive EU representation, PSB fully characterizes the no-uncertainty representation. It is easy to see that a representation that obeys PSB also obeys DFC and AIC. The no-uncertainty representation is closely related to our costly delegation model in Theorem 3.1 and can accommodate the preference over menus in Example 3.1.1. However, our delegation model in Theorem 3.1 studies finite settings instead of lottery settings. Furthermore, we impose structures on the delegation costs.

Turn to the literature on ex-post choices. Under the standard assumption of utility maximization, ex post choices in menus obey the weak axiom of revealed preferences (WARP) (Samuelson, 1938):

$$x \in C(S), y \in S, \text{ then} x \notin C(T) \implies y \notin C(T).$$

WARP implies the pair-wise acyclicity of revealed preferences.

There is a large literature studying choices that violates WARP but satisfies its weakened version, weak WARP (WWARP):

$$\{x, y\} \subseteq T \subseteq S. \text{ If } x = C(x, y) = c(S), \text{ then } y \neq C(T).$$

WWARP is first proposed in the rational shortlist method (RSM) in Manzini and Mariotti (2007) to explain cyclical choices observed in experimental settings (Loomes, Starmer, and Sugden (1991); Roelofsma and Read (2000)). In RSM, two rationales P_1 and P_2 are applied sequentially to make a choice. The acyclic rationale P_1 determines a shortlist from which the asymmetric rationale P_2 chooses from.

Au and Kawai (2011) refine the RSM in Manzini and Mariotti (2007) and study the case when P_1 is transitive and P_2 is complete and transitive. Behaviorally, transitive-RSM is a special case of RSM which satisfies the No-Binary Chain Cycle Axiom (NBCC):

$$\text{there exists no binary chain } \{x_1, ..., x_n\} \text{ such that } x_1 = x_n.$$

Furthermore, RSM models have expansion property (Au & Kawai, 2011; Manzini & Mariotti, 2007). Manzini and Mariotti (2012) studies a variation of the RSM by restricting the rationales P_1 to be semi-orders. Other models that satisfies WWARP includes the rationalization model by Cherepanov, Feddersen, and Sandroni (2013) where the first state choice is determined by the psychological constraint function

$$\Gamma_{CFS}(S) = \{y \in S | \exists R_i \text{ such that } yR_ix \text{ for all } x \in S\},$$

where $\Gamma : \mathcal{M} \rightarrow \mathcal{M}$, $\Gamma(A) \subseteq A$.

The violation of WARP can also be explained by the limited consideration models of Lleras, Masatlioglu, Nakajima, and Ozbay (2017) and Masatlioglu et al. (2012). Lleras et al. (2017) studies the overwhelming choice model that represents choices according to a linear order on a competition filter. This model accommodates the choice overload effect and captures choice heuristics such as categorization, rationalization, narrowing down, and so on. The competition filter Γ obeys the following property

$$x \in S \subset T, x \in \Gamma(T) \implies x \in \Gamma(S).$$

Choice from a competition filter can be characterized by WARP-CO, which weakens WARP but implies WWARP.

Masatlioglu et al. (2012) studies the choices from the attention filter where not all feasible alternatives are considered. Attention to a choice alternative is revealed when there is choice reversal when the item is removed, and the attention filter is characterized by the following property:

$$\Gamma(S) = \Gamma(S \setminus x) \quad \text{for} \quad x \notin \Gamma(S).$$

This model captures heuristics such as top N, top on each criterion, and most popular category. The identification in the attention filter model is not unique. Attention model can violate WWARP and accommodate the attraction effect ($C(xy) = x, C(axy) = y, C(abxy) = x$), cyclical choice ($C\{x,y,z\} = x, C\{x,y\} = x, C\{y,z\} = y, C\{x,z\} = z$), and choose pairwise unchosen $C\{x,y,z\} = z, C\{x,y\} = x, C\{y,z\} = y, C\{x,z\} = x$). The revealed preference identified in the transitive-RSM model in Au and Kawai (2011) closely related to that in the attention filter model. When P_1 is transitive, the first stage choice satisfies the property of attention filter.

A limited consideration model that satisfies conditions of both the consideration filter and the attention filter is path-independent (PI) (Lleras et al., 2021). Indeed, the property of attention filter implies sen's property α(see Moulin, 1985b):

$$A \subseteq B \implies \Gamma(B) \cap A \subseteq \Gamma(A);$$

and the property of attention filter implies Aizerman and Malishevski (AM) (see Moulin, 1985b):

$$\Gamma(B) \subseteq A \subseteq B \implies \Gamma(A) \subseteq \Gamma(B).$$

Sens' property α and AM characterize a PI filter (Aizerman & Malishevski, 1981a; Moulin, 1985b). Lleras et al. (2021) can identify both the PI filter and the underlying preference uniquely.

Kopylov and Yang (2020) also characterize choice from a path-independent filter and further refine the model to analyze persuasion. In the proof of Theorem 3.1, we show that the maximization of a path-independent filter is a special case of the costly delegation model when the cost is either 0 or ∞. This chapter discusses this issue. This proof is based on results in Kopylov and Yang (2022).

Chandrasekher (2017) studies a planner-doer model to model preferences over formal commitments and informal constraints. The model is a two-stage problem. In the first stage, the planner makes a formal commitment by committing to a menu of second-stage choices. Then the planner takes an unobservable action (informal commitment) that exercises some options on the menu and defines a subjective feasible set. The characterization is with two axioms, CRW and Strong Reduction. CRW is related to AIC in Dekel et al. (2009). CRW restricts the delegation cost to zero. Strong Reduction is a strengthening of the axiom AM.

Theorem 1 in the hidden action model of Chandrasekher (2017) can be simplified as a corollary to the choice from a path-independent filter, as in Theorem 3.2. Indeed, CRW and AIC also imply Inclusion, which is the axiom Dominance in this chapter. The subjective feasible set in Chandrasekher (2017) is a PI filter.

3.2 Main Results

Consider the standard menu framework where choices are made sequentially at *ex ante* and *ex post* time periods.

Let $Z = \{x, y, z \dots\}$ be a finite set of alternatives that may become feasible ex post. Let $\mathcal{M} = \{A, B, C \dots\}$ be the set of all *menus*—non-empty finite subsets of Z. Interpret each menu $A \in \mathcal{M}$ as an action that, if taken ex ante makes the set $A \subset Z$ feasible ex post. Singletons $\{x\}$ are written as x.

Let $\mathcal{R}$ be the set of complete and transitive relations R on Z. Such relations are called *weak orders.* For any $R \in \mathcal{R}$, let P be its asymmetric part.

A weak order $R \in \mathbb{R}$ is called *total* if for all $x, y \in Z$, $xRyRx$ implies $x = y$. Let $\mathcal{T} \subset \mathcal{R}$

be the set of all total orders on Z.

For any order $R \in \mathcal{R}$, function $u : Z \to \mathbb{R}$, and menu $A \in \mathcal{M}$, let

$$u(A) = \max_{x \in A} u(x),$$
$$R(A) = \{x \in A : xRy \text{ for all } \ y \in A\}.$$

If R is total, then for each $A \in \mathcal{M}$, $R(A) \in Z$ is a singleton. By convention, let $R(\emptyset) = \emptyset$ and $u(\emptyset) = -\infty$.

Consider a *planner*[4] with a preference $\succeq$ over menus. Write its asymmetric and symmetric parts as $\succ$ and $\sim$ respectively.

Axiom 3.1 (Order)**.** $\succeq$ *is complete and transitive.*

Take any function $u \in \mathbb{R}^{\mathbb{Z}}$ that represents $\succeq$ on Z. Call u *commitment utility.*

Imagine that the planner must delegate ex-post choices to *doers*—her future selves or other individuals. In general, delegations can involve various unobservable incentives. For example, the planner can motivate her future self by mental commitments, promises, cues etc. In the interpersonal case, doers can be stimulated by direct monetary transfers and/or persuaded by suitable information disclosures. Again, it can be problematic to observe such incentives directly. Therefore, we model delegation strategies with *hidden costs.*

Let $\mathcal{D} = \{(x, A) \in Z \times \mathcal{M} : x \in A\}$ be the set of all pairs (x, A) where the option x is feasible in the menu A. For any $(x, A) \in \mathcal{D}$, interpret $h(x, A) \geq 0$ as the cost that the planner must incur to delegate x in A. Let $h(x, A) = +\infty$ when the doer is unwilling to choose x in A under any incentives that the planner can possibly provide. The costs $h(x, A)$ are hidden and hence, not taken as a primitive in our model. Instead, we use them to motivate axioms

[4] We model choices of both planners and doers. So the generic term *decision maker* would be confusing.

and representations for observable preferences.

Let $\mathcal{H}$ be the set of all cost functions $h : \mathcal{D} \to [0, +\infty]$. Say that $h \in \mathcal{H}$ is *selective* if for all menus $A, B \in \mathcal{M}$ and alternatives $x \in A$ and $y \in Z$,

(H1) $h(x, x) = 0$,

(H2) $h(x, A) \leq h(x, A \cup B)$,

(H3) $h(x, A) \geq \min\{h(x, y \cup A), h(y, y \cup A)\}$.

Condition H1 normalizes delegation costs to zero in singleton menus. Monotonicity condition H2 is plausible because the delegation of x in A can adapt the same incentives as in $A \cup B$ and hence, should not cost more than $h(x, A \cup B)$. Turn to H3. Keep the delegation strategy for x in menu $y \cup A$ the same as that in menu A. Since such a strategy is sufficient to exclude any choices alternatives in $A \setminus x$ in the set A, it is sufficient to exclude these choice alternatives the the set $A \cup y$ in which they become even harder to delegate.

Assume that the planner evaluates any menu A by delegating a choice $x \in A$ that has an optimal combination of her commitment utility $u(x)$ and delegation cost $h(x, A)$. By H1 and H3, each menu A must contain some $x \in A$ such that $h(x, A) < +\infty$ and hence, the planner can always focus only on elements with bounded delegation costs. Assume that the aggregation of $u(x)$ and $h(x, A)$ is monotonic—strictly increasing in $u(x)$ and decreasing in $h(x, A)$—but not necessarily additive. This assumption motivates several axioms for $\succeq$.

Axiom 3.2 (Positive Set-Betweenness (PSB))**.** *For all $A, B \in \mathcal{M}$,*

$$A \succeq B \implies A \succeq A \cup B.$$

Take any menus $A, B \in \mathcal{M}$. Let $x \in Z$ be the planner's optimal delegation in $A \cup B$.

Suppose that $x \in A$. By H2, $A \succeq A \cup B$ should hold because $h(x, A) \leq h(x, A \cup B)$. Similarly, if $x \in B$, then $B \succeq A \cup B$. PSB originally appears in Dekel et al. (2009)'s model of cumulative temptations.

Axiom 3.3 (Dominance)**.** *For all $y \in Z$ and $A \in \mathcal{M}$,*

$$y \succeq x \quad \text{for all } x \in A \quad \Rightarrow \quad y \cup A \succeq A.$$

Take any $A \in \mathcal{M}$, and let $x \in A$ be the planner's optimal delegation in A. By H3, either x or y can be delegated in $y \cup A$ at a cost that does not exceed $h(x, A)$. If $y \succeq x$, then $y \cup A \succeq A$ should hold.

For any $A \in \mathcal{M}$, an element $x \in A$ is called *costly* in A if $x \succ x \cup A_x$ where

$$A_x = \{y \in A : x \succ y\}.$$

Indeed, the ranking $x \succ x \cup A_x$ implies that it should be costly to delegate x in $x \cup A_x$ and a fortiori, in A.

This axiom implies that adding choices alternatives with higher commitment to the menu improves the menu.

Axiom 3.4 (Reduction)**.** *For any $A \in \mathcal{M}$ and $x, y \in Z$, if x and y are both costly in $y \cup A$, then x is costly in A.*

If $y \succeq x$, then Reduction is trivial because $A_x = (y \cup A)_x$. Let $x \succ y$. Then $(y \cup A)_x = y \cup A_x$ and $A_y \subset A_x$. As y is costly in $y \cup A$, then $h(y, y \cup A_y) > 0$ and by H2, $h(y, x \cup y \cup A_x) > 0$. As x is costly in $y \cup A$, then $h(x, x \cup y \cup A_x) > 0$. By H3, $h(x, x \cup A_x) > 0$. Thus the ranking $x \succ x \cup A_x$ should hold because x cannot be delegated for free in the menu $x \cup A_x$, and any other feasible option $y \neq x$ in $x \cup A_x$ is strictly worse than x for the planner.

Say that $U : \mathcal{M} \to \mathbb{R}$ *aggregates* a cost function $h \in \mathcal{H}$ if for all $A \in \mathcal{M}$,

$$U(A) = \max_{x \in A}[u(x) - h(x, A)] \tag{3.2.1}$$

where $u(x) = U(x)$ for all $x \in Z$. Say also that U is an aggregation of h, and h is aggregated by U. If $h \in \mathcal{H}$ is selective, then the aggregation formula (3.2.1) is well-defined for any $u \in \mathbb{R}^Z$. In this case, $U(x) = u(x)$ because $h(x, x) = 0$, and each value $U(A) \geq \min_{x \in A} u(x)$ is bounded because by H1 and H3, there is $x \in A$ such that $h(x, A) = 0$.

Theorem 3.1. *$\succeq$ satisfies Axioms 3.1–3.4 if and only if $\succeq$ has a utility representation $U : \mathcal{M} \to \mathbb{R}$ that aggregates some selective function $h \in \mathcal{H}$.*

The proof of Theorem 3.1 is in appendix C.1.

3.3 Informal Commitments

Due to information and resource constraints, the planner cannot make every decision on his own. He can only decide a menu of permissible actions, and the doer chooses from this menu. Hence, we observe decentralization of decision rights in a firm. Alonso and Matouschek (2007); Holmstrom (1980).

The model of *informal commitments* proposed by Chandrasekher (2017) captures such a situation. In the model, there is set $\Pi \subseteq \mathcal{M}$ of available commitments. The planner can freely impose any set $C \in \Pi$, such that $A \cap C \neq \emptyset$ as an informal (hidden) commitment on the doer's choice in a menu A. The the preference $\succeq$ is represented for all $A \in \mathcal{M}$ by

$$U(A) = \max_{C \in \Pi,\, x \in R^d(A \cap C)} u(x) \tag{3.3.1}$$

for some domain $\Pi \subset \mathcal{M}$ and weak order $R^d \in \mathbb{R}$, interpreted as the doer's preference.

When the delegation cost is either 0 or ∞, this model can be obtained from the Theorem 3.1. The following axiom imposes the delegation cost of a choice alternative to be either 0 or ∞.

Axiom 3.5 (Costless Verifications (CV))**.** *For all $A, B \in \mathcal{M}$ and $x \in A$,*

$$x \succ A \quad \textit{and} \quad A \subset B \quad \Rightarrow \quad B \sim B \setminus x.$$

Take any $A, B \in \mathcal{M}$ and $x \in Z$ such that $x \succ A$ and $A \subset B$. Then x cannot maximize the doer's order R^d in $A \cap C$ for any $C \in \Pi$. Otherwise, x could be delegated (verified) for free in A and $A \succeq x$ should hold. Thus the maximizers of R^d should be the same in $B \cap C$ and in $(B \setminus x) \cap C$. If the planner can delegate only such maximizers, then the indifference $B \sim B \setminus x$ should hold.

Theorem 3.2. *$\succeq$ satisfies Order, PSB, CV if and only if there is $u \in \mathbb{R}^Z$, $\Pi \subset \mathcal{M}$, and $R \in \mathcal{T}$ such that $\succeq$ is represented by* (3.3.1)*.*

This theorem corresponds to Chandrasekher (2017) main Theorem 1 and follows from Theorem 3.1 in this chapter. PSB and CV are arguably more transparent than Chandrasekher's counterparts.

3.4 Fitting Path Independent Model

In the proof of Theorem 3.2, we also established that (3.3.1) is equivalent to the model of choice from a path-independent filter (see Lemma 11):

$$U(A) = U(\cup_{R \in \Theta} R(A)) = \max_{R \in \Theta} u(R(A)). \tag{3.4.1}$$

This is the PI filter model characterized in Lleras et al. (2021). In this section, we discuss how to fit empirical ex-post choice data to the PI filter model.

Say that $\mathcal{D} \subset Z \times \mathcal{M}$ is a *dataset* if $a \in A$ for every $(a, A) \in \mathcal{D}$. Any such pair $(a, A) \in \mathcal{D}$ means that a is observed to be chosen in A. Let $M = |\mathcal{D}|$ be the number of such observations.

A ranking $R_0 \in \mathcal{R}$ is called *acceptable* for a dataset $\mathcal{D}$ if there is a set $\Theta \subset \mathcal{R}$ such that

$$a \in R_0(\Theta(A)) \quad \text{for all } (a, A) \in \mathcal{D}. \tag{3.4.2}$$

Thus $R_0 \in \mathcal{R}$ is acceptable if all observations in $\mathcal{D}$ are consistent with some delegation model where a hypothetical planner with the ranking R_0 delegates choices in menus A by selecting doers in the set $\Theta \subset \mathcal{R}$.

Let P_0 be the asymmetric component of R_0. For any menu $B \in \mathcal{M}$, let

$$\mathcal{N}(B) = \{b \in B : bP_0a \quad \text{for some } (a, A) \in \mathcal{D} \text{ such that } A \subset B\}$$

be the set of all elements $b \in B$ that are revealed to be non-delegable in some $A \subset B$ and hence, in B itself.

Theorem 3.3. *A ranking $R_0 \in \mathbb{R}$ is acceptable for a dataset $\mathcal{D} \subset Z \times \mathcal{M}$ if and only if for all $(a, A) \in \mathcal{D}$ and a menu $B \supset A$,*

$$B \neq \mathcal{N}(B) \cup [A \setminus a]. \tag{3.4.3}$$

Moreover, for any $R_0 \in \mathcal{R}$, it takes polynomial time $O(M^3)$ to establish whether R_0 is acceptable for $\mathcal{D}$ or not.

This result provides a criterion for acceptability of any given $R_0 \in \mathcal{R}$. This criterion can be checked in polynomial time via an algorithm that we discuss in the proof of Theorem 3.3 in the appendix.

3.5 Comparison with Other Models

Before we discuss several examples of acceptability, it is insightful to relax the definition (3.4.2) to adapt the models of inattention in Masatlioglu et al. (2012) and choice overload in Lleras et al. (2017). Say that $\varphi : \mathcal{M} \to \mathcal{M}$ is

- a *competition filter* if it satisfies Sen's α, $a \in R_0(\varphi(A))$ for all $(a, A) \in \mathcal{D}$,
- an *attention filter* if φ satisfies a strong form of Reduction[5]: for all $A, B \in \mathcal{M}$,

$$\varphi(B) \subset A \subset B \quad \Rightarrow \quad \varphi(A) = \varphi(B). \tag{3.5.1}$$

Say that R_0 is *a-acceptable* for a dataset $\mathcal{D}$ if there is an attention filter $\varphi : \mathcal{M} \to \mathcal{M}$ such that

$$a \in R_0(\varphi(A)) \quad \text{for all } (a, A) \in \mathcal{D}. \tag{3.5.2}$$

Say that R_0 is *c-acceptable* for a dataset $\mathcal{D}$ if (3.5.2) holds for some competition filter $\varphi : \mathcal{M} \to \mathcal{M}$.

These weaker notions of acceptability portray a decision maker with the ranking R_0 who pays attention only to elements in the filter φ.

[5] The attention property is implied by path independence.

Let $Z = \{a, b, c\}$, and consider a standard intransitivity cycle,

$$\mathcal{D}_1 = \{(a, \{a, b\}), (\{b, \{b, c\}\}), (\{c, \{a, c\})\}.$$

For this dataset $\mathcal{D}_1$, the total rankings $\langle acb]$, $\langle bac]$, and $\langle cba]$ are not acceptable. In particular, let $R_0 = \langle acb]$. Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Then $\mathcal{N}(B) = \{a, c\}$ and $B = \mathcal{N}(B) \cup [A \backslash a]$ which contradicts (3.4.3).

By contrast, any ranking R_0 is both a-acceptable and c-acceptable for $\mathcal{D}_1$. In fact, one can show a more general claim.

Say that $\mathcal{D}$ is *binary* if for all $(a, A) \in \mathcal{D}$, the menu $A = \{a, b\}$ has size two, and $(b, A) \notin \mathcal{D}$. In other words, binary datasets describe observations of a single choice in some two-elements menus.

Proposition 3.5.1. *Any ranking $R_0 \in \mathcal{R}$ is both a-acceptable and c-acceptable for any binary dataset $\mathcal{D}$.*

Proof. For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that bP_0a, let $\varphi_a(\{a, b\}) = \{a\}$. For all other menus A, let $\varphi_a(A) = A$. Then φ_a is an attention filter, and (3.5.2) holds.

For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that bP_0a, let $\varphi_c(\{a, b\}) = \{a\}$. For any observation $(a, \{a, b\}) \in \mathcal{D}$ such that aR_0b, let $\varphi_c(\{a, b\}) = \{a, b\}$. For all other menus A, let $\varphi_c(A)$ be the set of minimizers of the commitment utility u. Then u_c is a competition filter, and (3.5.2) holds. □

This proposition suggests roughly that our delegation model imposes a minimal structure on the filter φ such that the representation (3.5.2) has non-vacuous implications for choices in binary menus.

We illustrate this point with an experimental dataset found in Apesteguia and Ballester (2020), and which is publicly available on the both authors' websites. The data includes the choices of 87 individuals from all 36 binary menus for the following nine equiprobable lotteries:

Table 3.1: Lotteries

lottery	**payoffs**	**lottery**	**payoffs**	**lottery**	**payoffs**
l_1	(17)	l_2	$(50, 0)$	l_3	$(40, 5)$
l_4	$(30, 10)$	l_5	$(20, 15)$	l_6	$(50, 12, 0)$
l_7	$(40, 12, 5)$	l_8	$(30, 12, 10)$	l_9	$(20, 12, 15)$

Each individual's choices from the 36 binary menus are observed. We take the expected value rankings of the lotteries as R_0. Only 27 out of the 87 individuals conform to the delegation model. In contrast, all 87 individuals conform to the competition filter model and the attention filter model.

It is also surprising that the combination of competition and attention properties delivers path independence, but the separate use of these conditions in (3.5.2) makes the model vacuous for binary datasets.

Next, consider a more stringent intransitivity where $Z = \{a, b, c, d\}$, and

$$\mathcal{D}_2 = \{(a, \{a, b, c\}), (b, \{b, c, d\}), (c, \{c, d, a\}), \{d, \{d, a, b\}\}.$$

For this dataset, all total orders become unacceptable. Indeed, if $R_0 = \langle abcd]$, then (3.4.3) is violated for $B = \{a, b, c, d\}$ and $A = \{b, c, d\}$.

3.6 Conclusion

This chapter studies delegation with hidden costs, discusses a special case of informal commitments when the delegation costs are binary, and fits data for the delegation model with binary costs. We study more general issues in Kopylov and Yang (2022). To begin with, we study the equivalent functional forms to the hidden cost representation 3.2.1. We also study a more general version of Theorem 3.2, where informal commitments are costly. Finally, we discuss fitting choice data for general delegation costs. In Kopylov and Yang (2020), we study the delegation model with binary costs in a lottery setting and axiomatize the planner's preference over menus of lotteries when the planner can change the agents' beliefs about uncertain events by persuasion.

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Appendix A

Proofs for Chapter 1

A.1 Proof for Theorem 1.1

Several steps are required to prove the sufficiency of the axioms. First, I show that the types are quasi-linear. Following this discussion, I construct the RQUM $\pi \in \Pi$ and demonstrate that it is a discrete probability measure. Next, I show that π represents ρ on generic points. Finally, I prove the existence of tie-breaking rules in T for nongeneric points.

A.1.1 Proof for Sufficiency

A.1.1.1 Step 1: Types are quasi-linear

Essentially, Lemma 1 states that the types have quasi-linear preferences. In this subsection, I prove Lemma 1.

(1) Sufficiency (Axioms 1.1, 1.2 and 1.4 imply quasi-linearity).

In the first step, I construct the set of types on each coordinate i. I define $c^i \in \mathbb{R}_0^{n+1}$ to be such that $c^i_j \to \infty$ for all $j \in Z \setminus \{0\}, j \neq i$. By Axioms 1.1 and 1.4, $\rho_j(c^i) = 0$ for all $j \neq i, 0$. $\rho_i(x, c_{-i}^i)$ is piecewise constant in x and is discontinuous at finitely many points of $x \in \mathbb{R}$. I denote the set of discontinuity points of x to be $V_i = \{v_{i,1}, ..., v_{i,m_i}\}$, where $v_{i,1} < v_{i,2} < ... < v_{i,m_i}$.

$\rho_0(x, c_{-i}^i) = 1 - \rho_i(x, c_{-i}^i)$ is a nondecreasing step function, but it is not right-continuous as a discrete cumulative distribution function. Instead, the behavior at a discontinuity point is determined by a tie-breaking rule. However, we can construct $F^i : \mathbb{R} \to [0,1]$ from $\rho_i(x, c_{-i}^i)$ as follows:

$$F^i(x) = \begin{cases} 1 - \rho_i^-(x, c_{-i}^i) & \text{for } x \in V_i \\ 1 - \rho_i(x, c_{-i}^i) & \text{otherwise} \end{cases} = \begin{cases} \rho_0^+(x, c_{-i}^i) & \text{for } x \in V_i \\ \rho_0(x, c_{-i}^i) & \text{otherwise} \end{cases}.$$

where $\rho_i^-(x, c_{-i}^i) = \lim_{\varepsilon \to 0} \rho_i(x + \varepsilon, c_{-i}^i)$, and $\rho_0^+(x, c_{-i}^i) = \lim_{\varepsilon \to 0} \rho_0(x + \varepsilon, c_{-i}^i)$ by Axiom 1.1. By Axiom 1.4,

$$\lim_{x \to -\infty} F^i(x) = \lim_{x \to -\infty} 1 - \rho_i(x, c_{-i}^i) = 0 \quad \text{and} \quad \lim_{x \to \infty} F^i(x) = \lim_{x \to \infty} 1 - \rho_i(x, c_{-i}^i) = 1.$$

$F^i(x)$ is piecewise constant, nondecreasing, and right-continuous. Hence, F^i is a distribution function of the random variable V_i, where $F^i(x) = Prob(V_i \leq x)$, and

$$\begin{aligned} Prob(V_i = x) = Prob(V_i \leq x) - Prob(F_i < x) &= F^i(x) - \lim_{\varepsilon \to 0} F^i(x - \varepsilon) \\ &= \begin{cases} \rho_i^+(x, c_{-i}^i) - \rho_i(x, c_{-i}^i) = 0 & x \notin V_i \\ \rho_i^+(x, c_{-i}^i) - \rho_i^-(x, c_{-i}^i) = gap_i(x) & x \in V_i. \end{cases} \end{aligned}$$

The realizations of the discrete random variable V_i are the set of private values on good i. Different private values correspond to different marginal types.

Next, we want to show that the preferences of the types in V_i on the physical good-price pair $(i, x), i \in Z, x \in \mathbb{R}$ are indeed represented by quasi-linear utility functions. We define $\{\succeq^i\}$ to be a set of weak orders for the comparison between (i, x) and the status quo $(0, 0)$:

$$Prob((0,0) \succeq^i (i,x)) = Prob(V_i \leq x) = F^i(x).$$

Hence F^i is also the distribution of preferences in $\{\succeq^i\}$. $Prob((i,x) \sim^i (0,0)) = Prob(V_i = x) \in [0,1]$ is the probability of the types valuing good i at x, and by construction, it is positive for $x \in V_i$ and is 0 for all other $x \in \mathbb{R}$. Since $V_i = \{v_{i,1}, ..., v_{i,m_i}\}$ is finite, there is a finite set of rankings $\{\succeq^i_k\}_{k=1}^{m_i}$,

$$Prob((i, v_{i,k}) \sim^i_k (0,0)) = Prob(V_i = v_{i,k}), k = 1, ..., m_i.$$

So, from the above discussion, we understand that $(i,x) \succ^i_k (0,0)$ for $x < v_{i,k}$, $(i,x) \prec^i_k (0,0)$ for $x > v_{i,k}$, and $(i,x) \sim^i_k (0,0)$ for $x = v_{i,k}$.

Let $\mathbf{1} \in \mathbb{R}^{n+1}$ be such that $\mathbf{1} = (1, 1, ..., 1)$. By Axiom 1.1, for any constant vector $\alpha\mathbf{1}$ for $\alpha \in \mathbb{R}$,

$$F^i(x+\alpha) = \begin{cases} 1 - \rho_i^-(x + \alpha\mathbf{1}) & \text{for } x \in V_i, \\ 1 - \rho_i(x + \alpha\mathbf{1}) & \text{otherwise} \end{cases} = \begin{cases} 1 - \rho_i^-(x) & \text{for } x \in V_i, \\ 1 - \rho_i(x) & \text{otherwise} \end{cases} = F^i(x).$$

We define $Prob((0,\alpha) \succeq^i (i, x+\alpha)) = Prob(V_i + \alpha \leq x + \alpha) = F^i(x+\alpha)$. Therefore,

$$Prob((0,\alpha) \succeq^i (i, x+\alpha)) = F^i(x+\alpha) = F^i(x) = Prob((0,0) \succeq^i (i,x))$$

and

$$Prob((i, x+\alpha) \sim^i (0, \alpha)) = F^i(x) - \lim_{\varepsilon \to 0} F^i(x-\varepsilon)$$
$$= F^i(x+\alpha) - \lim_{\varepsilon \to 0} F^i(x+\alpha-\varepsilon) = Prob((i, x) \sim^i (0, 0)).$$

Hence, $Prob((i, x+\alpha) \sim^i (0, \alpha)) = Prob((i, x) \sim^i (0, 0)) > 0$ when $x \in V_i$, and

$$(i, v_{i,k}) \sim^i_k (0, 0) \iff (i, v_{i,k}+\alpha) \sim^i_k (0, \alpha) \iff (i, 0) \sim^i_k (0, -v_{i,k}), k = 1, ...m_i \tag{A.1.1}$$

by taking $\alpha = -v_{i,k}$.

Furthermore, $(i, x+\alpha) \succ^i_k (0, \alpha)$ when $x < v_{i,k}$, $(i, x+\alpha) \sim^i_k (0, \alpha)$ when $x = v_{i,k}$, and $(i, x+\alpha) \prec^i_k (0, \alpha)$ when $x > v_{i,k}$, where $\alpha \in \mathbb{R}$ is arbitrary. Therefore,

$$y < y' \iff (i, y) \sim^i_k (0, -v_{i,k}+y) \succ^i_k (i, y') \sim (0, -v_{i,k}+y') \tag{A.1.2}$$

by noticing $-v_{i,k} + y = -(v_{i,k} + y' - y) + y'$.

Let $U_k(i, x), x \in \mathbb{R}$ represent $\succ^i_k$. (A.1.2) suggests

$$U_k(0, -v_{i,k}+y) >^i_k U_k(0, -v_{i,k}+y') \iff y < y'.$$

This preference can be represented by $U_k(0, x) = -x$ for $x \in \mathbb{R}$. (A.1.1) suggest that

$$U_k(i, \alpha) = U_k(0, -v_{i,k}+\alpha) = v_{i,k} - \alpha, k = 1, ..., m_i.$$

Hence, for $V_i = \{v_{i,1}, ..., v_{i,m_i}\}$ and $Z = \{0, i\}$, U_k is quasi-linear. The same argument goes for any $j \in Z \setminus \{0\}$. The marginal distribution of the types for good i, $\pi_{\{0,i\}} \in \Delta(V_i)$, is according to the distribution function F^i.

(2) Necessity. This direction is trivial and is thus omitted.

A.1.1.2 Step 2: Show π constructed in (1.2.16) is a discrete probability measure

I have shown that the types are quasi-linear on each good. Thus, it makes sense to recover the type distribution as the distribution of private values on the goods. Next, I show that the construction of π in (1.2.16) is a discrete probability measure on its support, denoted as $\{0\} \times_{i=1}^{n} V_i$. In this case, π on the set $\{0, i\}$ is the marginal distribution for good $i \in Z$.

Notice that, by construction, $\pi(\emptyset) = 0$. We can take any $J \subseteq Z \setminus \{0\}$, denote $J = \{j_1, ..., j_{|J|}\}$, and $V_J = V_{j_1} \times ... \times V_{j_{|J|}}$. Let $\mathcal{S} = \{A \subseteq V_J : J \subseteq Z \setminus \{0\}\}$ to include the sets of private values of types on all subsets of goods. In fact, $\mathcal{S}$ is a finite subalgebra. For $J \subseteq J' \subseteq Z$, we can take any $v^* \in V_J$, $\{v^*\} = \{v \in V_{J'} : v_i = v_i^* \; \forall i \in J\}$. The additivity condition on $\mathcal{S}$ is that for $v^* \in V_J \in \mathcal{S}$, $J \subseteq J' \in Z$, then

$$\pi(v^*) = \sum_{v \in J'} \pi(v_{j'} : v_i = v_i^*, i \in J), \tag{A.1.3}$$

and that for $v_1, v_2 \in V_J, v_1 \neq v_2$,

$$\pi(\{v_1, v_2\}) = \pi(v_1) + \pi(v_2). \tag{A.1.4}$$

It is well known that if π satisfies additivity on $\mathcal{S}$, then π has an extension to $\sigma(\mathcal{S})$, the sigma-algebra generated by $\mathcal{S}$. It is also easy to check that $\sigma(\mathcal{S}) = \sigma(\{0\} \times V_{Z\setminus\{0\}})$. Furthermore, if $\pi(\{0\} \times V_{Z\setminus\{0\}}) = 1$, then π is a probability measure on $\{0\} \times V_{Z\setminus\{0\}}$. So, to show $\pi \in \Delta(\{0\} \times_{i=1}^{n} V_i)$, one just needs to show, first that π is additive on $\mathcal{S}$, and second, that

$\pi(\{0\} \times_{i=1}^{n} V_i) = 1$. The additivity condition (A.1.4) is satisfied considering the definition of π in (1.2.12). Lemma 2 implies that the additivity condition (A.1.3) is satisfied on $\mathcal{S}$.

Lemma 2. *Take $c \in \mathbb{R}^{n+1}$, such that $\rho_A(c) = \sum_{i \in A} \rho_i(c) = 1$ for some $A \subsetneq Z$. For any $j \notin A$,*

$$\pi(c) = \sum_{\alpha \in V_j} \pi(\alpha, c_{-j}). \tag{A.1.5}$$

Proof. For any $A \subseteq Z$, $A = \{i_{A_1}, ..., i_{A_{|A|}}\}$, $c_A = (c_{A_1}, ..., c_{A_{|A|}})$. $\pi(c) = \lim_{\varepsilon \to 0} E_{\varepsilon, A} = \{v \in V_A : v = c_A\}$, since any $j \in Z \setminus A$ is unavailable. When $c_j, j \notin A$ is modified to be $\alpha \in V_j$, we have $\rho_{A \cup \{j\}}(\alpha, c_{-j}) = 1$, and $\pi(\alpha, c_{-j}) = \lim_{\varepsilon \to 0} \pi(E_{\varepsilon, A \cup \{j\}}) = \pi\{v \in V_{A \cup \{j\}} : v = (c_A, \alpha)\}$. Therefore,

$$\begin{aligned}\pi((\alpha, c_{-j})|c) = \frac{\pi(\alpha, c_{-j})}{\pi(c)} &= \frac{\pi\{v \in V_{A \cup \{j\}} : v = (c_A, \alpha)\}}{\pi\{v \in V_A : v = c_A\}} = \pi\{\alpha \in V_j | c_{-j}\} \\ &= \rho_j^+(\alpha, c_{-j}) - \rho_j^-(\alpha, c_{-j}) = gap_j(\alpha, c_{-j}),\end{aligned}$$

where $gap_j(\alpha, c_{-j}) > 0 \iff \alpha \in V_j$, $V_j = \{\alpha_1, ..., \alpha_m\}$, $\alpha_1 < \alpha_2 < ... < \alpha_m$. Furthermore, $\rho_j^+(\alpha_1, c_{-j}) = 1, \rho_j^-(\alpha_m, c_{-j}) = 0$, and

$$\rho_j^+(\alpha_{k+1}, c_{-j}) = \rho_j^-(\alpha_k, c_{-j})$$

for $k = 1, ..., m-1$. Therefore,

$$\sum_{\alpha \in V_j} \frac{\pi(\alpha_k, c_{-k})}{\pi(c)} = \rho_j^+(\alpha_1, c_{-j}) - \rho_j^-(\alpha_m, c_{-j}) = 1.$$

Hence, (A.1.5) holds. □

Next, I show $\pi(\{0\} \times_{i=1}^n V_i) = 1$. Let $V_1 = \{\alpha_1, ..., \alpha_{m_1}\}$.

$$\pi(\{0\} \times_{i=1}^n V_i) = \sum_{c_1 \in V_1} ... \sum_{c_n \in V_n} \pi(0, c_1, ..., c_n) = \sum_{c_1 \in V_1} .. \sum_{c_{n-1} \in V_{n-1}} \pi(0, c_1, ..., c_{n-1}) =$$
$$= ... = \sum_{c_1 \in V_1} \pi(0, c_1) = \sum_{c_1 \in V_1} \rho_1^+(0, c_1) + \rho_0^+(0, c_1) - 1$$
$$= \sum_{c_1 \in V_1} \rho_1^+(0, c_1) - \rho_1^-(0, c_1) = \rho_1^+(0, \alpha_1) - \rho_1^-(0, \alpha_{m_1}) = 1.$$

Hence, $\pi \in \Delta(\{0\} \times_{i=1}^n V_i)$ is a probability distribution on the finite types. Next, in the subsection below, I show that π represents RCR ρ.

A.1.1.3 Step 3: Recover ρ from π

For any $c \in \mathbb{R}_0^{n+1}$, if $j \in M(v, c)$ for some $v \in \text{supp}(\pi)$, then $j \notin M(v, (c_j + \varepsilon, c_{-j}))$, and $j = M(v, (c_j - \varepsilon, c_{-j}))$. This observation links the measure π on the types with ties at c to the change in a single coordinate of ρ. I prove this intuition in Lemma 3.

I denote $\pi_A(c)$ to be the probability on all types $v \in \mathbb{R}_0^{n+1}$, such that its set of maximizers $M(v, c)$ includes A, i.e., $M(v, c) \supset A$. I also denote the set of types for which $i \in Z$ is a maximizer for $c \in \mathbb{R}^{n+1}$ as follows:

$$M_i(c) = \{v \in \text{supp}(\pi) : i \in M(v, c)\}. \tag{A.1.6}$$

Using the same definition of events $F_{\varepsilon,A}(c)$ and modifying $E_\varepsilon(c)$ so that it is defined for subsets $A \subseteq Z$:

$$E_{\varepsilon,A}(c) = \{v \in \mathbb{R}_0^{n+1} : v_i - c_i \text{ is within } \varepsilon \text{ from} \max_{j \in Z} v_j - c_j \ \forall i \in A \subseteq Z\};$$
$$\pi(E_{\varepsilon,A}) = \sum_{A' \subseteq A} (-1)^{|A'|+1} \pi(F_{\varepsilon,A'}(c)).$$

By the same argument as that found in construction (1.2.16),

$$\pi_A(c) = \pi(\cap_{i\in A} M_i(c)) = \lim_{\varepsilon\to 0}\pi(E_{\varepsilon,A}(c)) = \sum_{A'\subseteq A}(-1)^{|A'|+1}\rho^+_{A'}(c). \tag{A.1.7}$$

Denote

$$\tilde{\pi}_A(c) = \pi(v\in\mathbb{R}_0^{n+1} : M(v,c) = A).$$

Notice that $\{v\in\mathbb{R}_0^{n+1} : A'\subseteq M(v,c)\} = \cup_{A:A'\subseteq A\subseteq Z}\{v\in\mathbb{R}_0^{n+1} : M(v,c) = A\}$. By the inclusion-exclusion principle (Möbius inversion),

$$\tilde{\pi}_A(c) = \sum_{A'\subseteq Z:A\subseteq A'}(-1)^{|A'\setminus A|}\pi_{A'}(c), \tag{A.1.8}$$

Lemma 3. *For any $i\in Z$, $c\in\mathbb{R}^{n+1}$,*

$$gap_i(c) = \sum_{S\subseteq Z_{-i},|S|\geq 1}\tilde{\pi}_{S\cup\{i\}}(c). \tag{A.1.9}$$

Proof of Lemma 3. I prove Lemma 3 by demonstrating the following:

$$\sum_{S\subseteq Z_{-j},|S|\geq 1}\tilde{\pi}_{S\cup\{j\}}(c) \overset{(1)}{=} \sum_{S\subseteq Z_{-j},|S|\geq 1}(-1)^{|S|-1}\pi_{S\cup\{j\}}(c) \overset{(2)}{=} \rho^+_j(c) + \rho^+_{Z_{-j}}(c) - \rho^+_Z(c) \tag{A.1.10}$$

$$= \rho^+_j(c) - (1-\rho^+_{Z_{-j}}(c)) = gap_j(c). \tag{A.1.11}$$

First, I establish (1). From (A.1.8), the sum over $S\subseteq Z_{-j}$ of $\tilde{\pi}_{S\cup\{j\}}$ can be written as the sum over $\pi_{S'\cup\{j\}}, S\subseteq S'\subseteq Z_{-j}$. Alternatively, for any $S\subseteq Z$, the coefficient on $\pi_{S\cup\{j\}}$ will include contributions from $\tilde{\pi}_{S'\cup\{j\}}(c)$ for all $S'\subseteq S$. Fix $S\subseteq Z_{-j}$. The coefficent on

$\pi_{S\cup\{j\}}$ is:

$$\begin{aligned}
&\sum_{|S'|=1}^{|S|}(-1)^{|S\backslash S'|}\frac{\binom{n}{|S'|}\binom{n-|S'|}{|S|-|S'|}}{\binom{n}{|S|}} = (-1)^{|S|}\sum_{|S'|=1}^{|S|}(-1)^{|S'|}\frac{\binom{n}{|S|}\binom{|S|}{|S'|}}{\binom{n}{|S|}} \\
=&(-1)^{|S|}\sum_{|S'|=1}^{|S|}(-1)^{|S'|}\binom{|S|}{|S'|} = (-1)^{|S|}\left(\sum_{|S'|=0}^{|S|}(-1)^{|S'|}\binom{|S|}{|S'|} - (-1)^0\right) \\
=&(-1)^{|S|}\left((1-1)^{|S|}-1\right) = (-1)^{|S|-1}.
\end{aligned}$$

The term $\binom{n}{|S'|}$ is the number of sets with cardinality $|S'|$. The term $\binom{n-|S'|}{|S|-|S'|}$ is the number of sets with cardinality $|S|$ containing a given set with cardinality $|S'|$. The multiplication of the two terms accounts for the total number of times that sets of cardinality $|S'|$ are accounted for in the aggregate coefficient of sets of cardinality $|S|$. The term $\binom{n}{|S|}$ is the number of sets with cardinality $|S|$ formed from the set Z_{-j}. Thus, the fraction $\frac{\binom{n}{|S'|}\binom{n-|S'|}{|S|-|S'|}}{\binom{n}{|S|}}$ is the number of times that all $\tilde{\pi}_{S'\cup\{j\}}$ with $S' \subseteq S$, $|S'| = k, k = 1, ..., |S'|$ contributes to $\pi_{S\cup\{j\}}$ for a given $S \subseteq Z$ when written in the form of (A.1.10).

Next, I establish (2). By substituting (A.1.7) for π, $\sum_{S\subseteq Z_{-j},|S|\geq 1}(-1)^{|S|-1}\pi_{S\cup\{j\}}(c)$ can then be written in terms of $\rho_S^+(c)$. Notice that the terms that contributes to the coefficient on $\rho_j^+(c)$ come from $\pi_{S\cup\{j\}}, |S| \geq 1$, and the coefficient on $\rho_j^+(c)$ is calculated as

$$\binom{n}{1} - \binom{n}{2} + ... + (-1)^{n+1}\binom{n}{n} = -(1-1)^n + \binom{n}{0} = 1.$$

(A.1.7) also requires that $\rho_{S\cup\{j\}}^+(c)$ and $\rho_S^+(c)$ for $1 \leq |S| \leq n-1$ come from $\pi_{S'\cup\{j\}}$ for all $S \subseteq S'$. The coefficients on $\rho_S^+(c)$ and $\rho_{S\cup\{j\}}^+(c)$ are the same:

$$\begin{aligned}
\sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\frac{\binom{n}{|S'|}\binom{|S'|}{|S|}}{\binom{n}{|S|}} &= \sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\frac{\binom{n}{|S|}\binom{n-|S|}{|S'|-|S|}}{\binom{n}{|S|}} = \sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\binom{n-|S|}{|S'|-|S|} \\
&= \sum_{i=0}^{n-|S|}(-1)^{|S|-1}\binom{n-|S|}{i} = (-1)^{|S|-1}(1-1)^{n-|S|} = 0.
\end{aligned}$$

The term $\binom{n}{|S'|}$ is the number of all $\pi_{S'\cup\{j\}}$ with fixed cardinality $|S'|$. $\binom{|S'|}{|S|}$, $S \subseteq S'$ is the number of sets with cardinaility $|S|$ that can be obtained from sets with cardinaility $|S'|$. The multiplication gives the total number of $\rho^+_{S\cup\{j\}}(c)$ for a fixed cardinality $|S|$. $\binom{n}{|S|}$ gives the number of distinct $\rho^+_{S\cup\{j\}}(c)$ for fixed cardinaility $|S|$. So, the term $\frac{\binom{n}{|S'|}\binom{|S'|}{|S|}}{\binom{n}{|S|}}$ is the absolute value of coefficients on the term $\rho^+_{S\cup\{j\}}(c)$ for each S obtained from all $\pi_{S'\cup\{j\}}$ with $S \subseteq S' \subseteq Z$, $|S'|$ fixed and is in $\{|S|, ..., n\}$.

Note that $\rho^+_{Z_{-j}\cup\{j\}}(c) = \rho^+_Z(c)$ and $\rho^+_{Z_{-j}}(c)$ only come from the term $(-1)^{n+1}\pi_Z(c)$. The coefficient on $\rho^+_Z(c)$ is $(-1)^{n-1}(-1)^n = -1$, and the coefficient on $\rho^+_{Z_{-j}}(c)$ is $(-1)^{n-1}(-1)^{n-1} = 1$. Hence, (2) holds.

The rest of (A.1.10) comes from the observation that $\rho^+_Z(c) = 1$ and $\rho_j(c_j + \varepsilon, c_{-j}) = 1 - \rho^+_{Z_{-j}}(c)$ by Axiom 1.1 and that ρ is a probability measure. □

Lemma 3 shows that all ties at a nongeneric point $\tilde{c} \in \mathbb{R}^{n+1}_0$ can be broken by slightly reducing the cost at one coordinate i. Therefore, the probability measure on all the ties is equal to $gap_i(\tilde{c})$ constructed as (1.2.21). By the finiteness condition required in Axiom 1.4, there are no ties at $\lim_{\varepsilon\to 0}(\tilde{c}_i - \varepsilon, \tilde{c}_{-i})$. With this observation, I construct $\rho'(c)$ at any generic point $c \in \mathbb{R}^{n+1}_0$ from π.

Lemma 4. *One can construct $\rho'(c)$ with π, such that $\rho'(c) = \rho(c)$ for any generic $c \in \mathbb{R}^{n+1}$.*

Proof. Take a point $c_0 \in \mathbb{R}^{n+1}$, such that $\rho_0(c_0) = 1$. By Axiom 1.4, this can be achieved by taking the ith coordinate of c_0 to be very large for all $i \in Z \setminus \{0\}$. Let $\rho'(c_0) = \rho(c_0) = 1$.

Since the number of types is finite, the generic points are dense in $\mathbb{R}^{n+1}$. We find a grid-like path from c_0 to any generic point $c \in \mathbb{R}^{n+1}$, such that at any point on the path, the direction of change is along one of the coordinates. Furthermore, for any $\tilde{c}$ on the path, $|M(\tilde{c}, v)| \leq 2$ for all $v \in \text{supp}(\pi)$. If $M(\tilde{c}, v) = \{i, j\} \subseteq Z$ for some $v \in \text{supp}(\pi)$, then for all $v' \in \text{supp}(\pi)$ with $|M(\tilde{c}, v')| = 2$, $M(\tilde{c}, v') = \{i, j\}$. In other words, there is only one two-way tie, $\{i, j\}$, at $\tilde{c}$. Moreover, the path does not take a turn at points with a two-way tie. The density of generic points in $\mathbb{R}_0^{n+1}$ allows one to construct such paths between any two points with these specifications. An illustrate with $|Z| = 3$ is in Figure 1.2.

At any point on the path, we know the gap function from π by Lemma 3. WLOG, I call a point on the path c. Since at most two goods are tied on the path, we know how ρ' changes at each point. Thus, ρ' can be constructed along the path. I denote the following:

$$c_i^- = \lim_{\varepsilon \to 0} (c_i + \varepsilon, c_{-i}), \quad c_i^+ = \lim_{\varepsilon \to 0} (c_i - \varepsilon, c_{-i}).$$

The construction skips nongeneric points and thus ρ' is only defined on generic points. Let $\rho'(c)$ be the constructed choice function at c. If c is a generic point, then $\pi_{S \cup \{i\}} = 0$ for all $S \subseteq Z_{-i}$. Suppose c_i^+ is also on the path. Then, let

$$\rho'(c_i^+) = \rho'(c). \tag{A.1.12}$$

If c has a two-way tie where i and j are the maximizers, then both c_i^+ and c_i^- are on the path (or both c_j^+ and c_j^- are on the path), since there is no turn at c. c_i^+, c_i^- are generic points since the generic points are dense. WLOG, suppose $\rho'(c_i^-)$ has been constructed, and we want to construct $\rho'(c_i^+)$. At c, $\pi_{\{i,j\}} > 0$ and $\pi_{S \cup \{i\}} = 0$ for all $\emptyset \subseteq S \subseteq Z_{-i}, S \neq \{j\}$. We construct $\rho'(c^+)$ as follows:

$$\rho_i'(c_i^+) = \rho_i'(c_i^-) + \pi_{i,j}(c), \quad \rho_j'(c_i^+) = \rho_j'(c_i^-) - \pi_{i,j}(c), \quad \rho_k'(c_i^+) = \rho_k'(c_i^-) \ \forall \ k \neq i, j. \tag{A.1.13}$$

With constructions determined by (A.1.12) and (A.1.13), I show that if c is a generic point, then if $\rho(c) = \rho'(c) \implies \rho(c_i^-) = \rho'(c_i^-)$. If i, j are the only two maximizers for c, $\rho(c_i^-) = \rho'(c_i^-) \implies \rho'(c_i^+) = \rho'(c_i^+)$.

Case 1: Points with no ties. By Lemma 3, $gap_i(c) = 0$ for all $i \in Z$. Thus, $\rho(c) = \rho(c_i^+) = \rho(c_i^-)$. Axiom 1.2 implies that if $\rho(c) = \rho'(c)$, then $\rho(c_i^+) = \rho'(c_i^+)$.

Case 2: Points with a two-way tie. Suppose at c there is a two-way tie between i and j. So, $\pi_{\{i,j\}}(c) > 0$, $\pi_{S\cup\{i\}}(c) = 0$, for all other $\emptyset \subsetneq S \subseteq Z_{-i}$, $S \neq \{j\}$. By (A.1.8), $\tilde{\pi}_{\{i,j\}} = \pi_{\{i,j\}}$. By Lemma 3,

$$gap_i(c) = gap_j(c) = \pi_{\{i,j\}}(c), \quad gap_k(c) = 0 \;\forall\; k \in Z, k \neq i, j.$$

Therefore, by definition of the gap function, $\rho_i^+(c) = \rho_i^-(c) + gap_i(c)$, written equivalently as $\rho_i(c_i^+) = \rho_i(c_i^-) + gap_i(c)$.

Next, I show that $\rho_k(c_i^+) = \rho_k(c_i^-)$. By Axiom 1.2,

$$\rho_k(c_i - \varepsilon, c_j - \varepsilon, c_{-i,j}) \leq \rho_k(c_i^+) \leq \rho_k(c) \leq \rho_k(c_i^-) \leq \rho_k(c_i + \varepsilon, c_j + \varepsilon, c_{-i,j}), \quad \text{(A.1.14)}$$

for all $k \in Z_{-i,j}$.

Suppose for contradiction that $\rho_k(c_i^+) < \rho_k(c_i^-)$ for some $k \in Z_{-i,j}$. Then $\rho_k(c - \varepsilon, c_j - \varepsilon, c_{-i,j}) < \rho_k(c + \varepsilon, c_j + \varepsilon, c_{-i,j})$, and $\rho_{ij}^-(c) < \rho_{ij}^+(c)$. Since there is only one two-way tie at c, $\pi_{\{i,j\}}(c) = \pi_{\{i,j,k\}}(c)$. By (A.1.7), this equality implies

$$\rho_i^+(c) + \rho_j^+(c) - \rho_{ij}^+(c) = \rho_i^+(c) + \rho_j^+(c) + \rho_k^+(c) - \rho_{ij}^+(c) - \rho_{jk}^+(c) - \rho_{ik}^+(c) + \rho_{ijk}^+(c)$$
$$\implies \rho_k^+(c) - \rho_{jk}^+(c) - \rho_{ik}^+(c) + \rho_{ijk}^+(c) = 0$$

If $Z = \{i, j, k\}$, then the above equality implies

$$\rho_k^+(c) + \rho_i^-(c) + \rho_j^-(c) - 1 = 0$$

So, $\rho_i^-(c) + \rho_j^-(c) = 1 - \rho_k^+(c) = \rho_{ij}^-(c)$. Notice that $\rho_k^+(c) = 1 - \rho_i^-(c) - \rho_j^-(c)$, and that $\rho_{ij}^+(c) = 1 - \rho_k^-(c) \implies \rho_k^-(c) = 1 - \rho_{ij}^+(c)$. Therefore,

$$gap_k(c) = \rho_k^+(c) - \rho_k^-(c) = 1 - \rho_i^-(c) - \rho_j^-(c) - (1 - \rho_{ij}^+(c)) = \rho_{ij}^+(c) - \rho_i^-(c) - \rho_j^-(c).$$

Since by assumption $\rho_{ij}^-(c) < \rho_{ij}^+(c)$, $\rho_i^-(c) + \rho_j^-(c) < \rho_{ij}^+(c)$. So $gap_k(c) > 0$. This contradicts the fact that $gap_k(c) = 0$. Hence, we must have $\rho_{ij}^-(c) = \rho_{ij}^+(c)$.

$\rho_{ij}^+(c) = \rho_{ij}^-(c)$ implies that $\rho_{ij}^+(c) = \rho_{ij}^-(c) = \rho_{ij}(c)$, and thus, $\rho_k(c_i - \varepsilon, c_j - \varepsilon) = \rho_k(c_i + \varepsilon, c_j + \varepsilon)$. Considering (A.1.14), we have $\rho_k(c_i^-) = \rho_k(c_i^+) = \rho_k(c)$. Hence, $\rho_i(c_i^+) + \rho_j(c_i^+) = \rho_i(c_i^-) + \rho_j(c_i^-)$. Therefore,

$$\rho_j(c_i^+) = \rho_i(c_i^-) + \rho_j(c_i^-) - \rho_i(c_i^+) = \rho_j(c_i^-) - gap_j(c).$$

I have shown that in the actual observation, the RCR rule ρ is:

$$\rho_i(c_i^+) = \rho_i(c_i^-) + gap_i(c), \quad \rho_j(c^+) = \rho_j(c_i^-) - gap_j(c), \quad \rho_k(c_i^+) = \rho_k(c_i^-) \ \forall\ k \neq i, j.$$

This mirrors (A.1.13) exactly. Therefore, if $\rho(c_i^-) = \rho'(c_i^-)$, then $\rho(c_i^+) = \rho'(c_i^+)$ when c has only a two-way tie on the set $\{i, j\}$.

I have proven that on the grid-like path starting at c_0 considered here, $\rho'(c) = \rho(c)$ for any nongeneric point c on the path. Therefore, the Lemma is proved. □

A.1.1.4 Step 4: Existence of a tie-breaking rule

At a nongeneric point c, for some $v \in \text{supp}(\pi)$, $|M(v,c)| > 1$. I show that there is $t : \mathbb{R}_0^{n+1} \times \mathbb{R}^{n+1} \to \Delta(Z)$ for all $v \in \text{supp}(\pi)$, such that

$$\rho(c) = \sum_{v \in \text{supp}(v)} \pi(v) t(v,c).$$

Consider the case where $\rho_A(c) \in \mathbb{Q}$ for all $A \subseteq Z$. Construction (1.2.16) suggests that if $\rho_A(c) \in \mathbb{Q}$ for all $A \subseteq Z, c \in \mathbb{R}^{n+1}$, then $\pi(v) \in \mathbb{Q}$ for all $v \in \text{supp}(\pi)$. Let $\frac{1}{k} \in \mathcal{N}$ be a *unit*, as defined in Definition 1.1. So, $k\pi(v) \in \mathcal{N}$ for all $v \in \text{supp}(\pi)$, and $k\rho_A(c) \in \mathcal{N}$ for all $A \subseteq Z$. Each type $v \in \text{supp}(\pi)$ generates $k\pi(v)$ copies of a-units v', where $\pi(v') = \frac{1}{k}$. Let $\text{supp}(\pi) = \{v_1, ..., v_k\}$. Then the multi-set of a-units is

$$U_{multi} = \{{v'_1}^1 ... {v'_1}^{m(v'_1)}, {v'_2}^1, ... {v'_2}^{m(v'_2)}, ..., {v'_k}^1, ... {v'_k}^{m(v'_k)}, \},$$

where $m(v'_i) = k\pi(v_i)$. Each $\rho_i(c)$ generates $k\rho_i(c)$ copies of p-units i' with $\rho_{i'}(c) = \frac{1}{k}$. The multi-set of p-units is

$$P_{multi} = \{0'^1, 0'^{m(0')}, 1'^1, 1'^{m(1')}, 2'^1, 2'^{m(2')}, ..., n'^1, n'^{m(n')}\},$$

where $m(i') = k\rho_i(c), i = 0, 1..., n$.

For any $A \subseteq Z$, each a-unit $v' \in U_{multi}$ has edges to all p-units $i' \in P_{multi}$, such that $i \in M(v,c)$, $i \in A$. We call the edge a demand. For each a-unit $v' \in U_{multi}$, let $D_{v',c} = \{i' \in P_{multi} : i \in M(v,c)\}$ be the set of p-units that agent v' demands at cost c. For a multi-set

$$V'_A = \{v' \in U_{multi} : v \in \text{supp}(\pi), M(v,c) \subseteq A\}$$

of a-units, the demands/edges are all in A. If $i'^j \in D_{v',c}$ for $j \in \{1, ..., m(i')\}$, then $\{i'^k, k =$

$1, ..., m(i')\} \subseteq D_{v',c}$. So, we take

$$\{i' \in P_{multi} : i \in A \subseteq Z\}$$

when considering the demanded set.

Since $|P_{multi}| = \sum_{i=0}^{n} k\rho_i(c) = k$, and $|U_{multi}| = \sum_{i=1}^{k} k\pi(v_i) = k$, assigning the p-units to the a-units according to the demand is analogous to a problem of one-to-one matching. By Hall's marriage theorem (Hall, 1935), the goal is to show that Hall's marriage condition is satisfied. Hall's marriage condition requires that there are no overdemanded sets, where a set is overdemanded if the number of a-units demanding only items in this set is greater than the number of p-units in this set. In other words, for any $A \subseteq Z$, the number of p-units determined by A is at least as big as the number of a-units $|V'_A|$. Hence, the necessary and sufficient condition for the existence of one-to-one matching is:

$$|\{i' \in P_{multi} : i \ \in A\}| \geq |\{v' \in U_{multi} : v \in \text{supp}(\pi), M(v, c) \subseteq A\}|. \tag{A.1.15}$$

It is WLOG to consider V'_A instead of its subset that includes only subsets of a-units associated with v, since V'_A makes the right-hand side of condition (A.1.15) larger. Since $|\{i' : i \in A\}| = k\rho_A(c)$ and $|\{v' : v \in \text{supp}(\pi), M(v, c) \subseteq A\}| = k\pi\{v \in \text{supp}(\pi) : M(v, c) \subseteq A\}$, Lemma 5 implies that Hall's condition (A.1.15) can be satisfied.

Lemma 5. *For any nongeneric $c \in \mathbb{R}^{n+1}$,*

$$\rho_A(c) \geq \pi\{v \in supp(\pi) : M(v, c) \subseteq A\}.$$

Proof. Fix any $A \subseteq Z$. Take a generic point $c_A^- \in \mathbb{R}^{n+1}$, such that $\|c - c_A^-\| < \varepsilon$ for

$\varepsilon > 0, \varepsilon \to 0$, and $c_A^- \gg_A c$. Then by construction,

$$\rho_A(c_A^-) = \pi(v \in \text{supp}(\pi) : M(v, c_A^-) \subseteq A), \tag{A.1.16}$$

since π represents ρ on c_A^-. By Axiom 1.2,

$$\rho_A(c) \geq \rho_A(c_A^-) = \rho_A^-(c). \tag{A.1.17}$$

By definition, for any $v \in \text{supp}(\pi)$, $M(v, c_A^-) \subseteq A \implies M(v, c) \subseteq A$. Take any $v \in \mathbb{R}_0^{n+1}$ such that $M(v, c) \subseteq A$. So A contains the strict maximizer(s) of v at c. When $\varepsilon \to 0$, these elements still strictly maximizes v at c_A^-. Hence, $M(v, c_A^-) \subseteq A$, and therefore,

$$\{v \in \text{supp}(\pi) : M(v, c_A^-) \subseteq A\} = \{v \in \text{supp}(\pi) : M(v, c) \subseteq A\}.$$

Therefore,

$$\pi(\{v \in \text{supp}(\pi) : M(v, c) \subseteq A\}) = \pi(\{v \in \text{supp}(\pi) : M(v, c_A^-) \subseteq A\}) = \rho_A(c_A^-) \leq \rho_A(c).$$

□

Hence, there exists a one-to-one matching between the a-units and the p-units. If we denote (v', i') to be a matching, then $\{(v', i')^1, ..., (v', i')^{m(v', i')}\}$ is the multi-set of the matching (v', i'), and $m(v', i')$ is the number of this type of matching. Further, $m(v', i') \leq k\pi(v)$, since $m(v') = k\pi(v)$ and $m(v', i') \leq m(v')$. So, for any $v \in \text{supp}(\pi)$, the tie-breaking rule is as follows:

$$t_i(v, c) = \frac{m(v', i')}{m(v')} = \frac{m(v', i')}{k\pi(v)}.$$

$t = \{t_i(v, c), v \in \text{supp}(\pi), c \in \mathbb{R}^{n+1}\}$ is a tie-breaking rule for the RCR ρ when $\rho \in \mathbb{Q}$. Since $D_{v',c} = \{i' \in P_{multi} : i \in M(v, c)\} = \{i' \in P_{multi} : i \in M(v, c + \alpha\mathbf{1})\} = D_{v',c+\alpha\mathbf{1}}$, the matching problem at c and $c + \alpha\mathbf{1}$ are the same. Therefore, there exists quasi-linear tie-breaking rule.

Suppose $\rho_A(c) \notin \mathbb{Q}$ for some $A \subseteq Z$, $c \in \mathbb{R}^{n+1}$. We construct a sequence $\rho^j : \mathbb{R}^{n+1} \to \Delta(Z), j = 1, 2, ...,$ such that $\rho^j_A(c) \in \mathbb{Q}$ for all $A \subseteq Z, c \in \mathbb{R}^{n+1}$, $\rho^j \to \rho$. (1.2.16) constructs π^j from ρ^j, such that $\pi^j(v) \in \mathbb{Q}$ for all $v \in \text{supp}(\pi^j)$, where $\pi^j \to \pi$. We further require that $\text{supp}(\pi^j) = \text{supp}(\pi) = V$ for all π^j in the sequence. Such a sequence of ρ^j exists by the density of $\mathbb{R}$. Each (ρ^j, π^j) is a matching problem in the rationals, and from the discussion above, we know that there exists a t^j that can solve this problem in rationals. Fix $c \in \mathbb{R}^{n+1}$. Then $t \in (\Delta(Z))^{|V|}$. Since Z is finite, $(\Delta(Z))^{|V|}$ is compact. Then t^j is a point in the compact set $(\Delta(Z))^{|V|}$. By Bolzano-Wierstrauss theorem, there is a convergent subsequence, $t^{j_k} \to t^*$. This arguement holds for all $c \in \mathbb{R}^{n+1}$. So $(\rho^{j^k}, t^{j^k}) \to (\rho, t^*)$.

A.1.2 Proof for Necessity

Assume there exists $(\pi, t) \in \Pi \times T$, such that (1.2.4) holds. Axiom 1.1 holds by (1.2.3). Axiom 1.2 follows from (1.2.3) and the representation (1.2.4) because $M_i(c) = \{v \in \text{supp}(\pi) : v_i - c_i \geq v_j - c_j \ \forall j \in Z\}$ is a nonincreasing set function with c_i, and therefore, by definition, $t_i(v, c)$ and $\rho_i(v, c)$ are nonincreasing with c_i. The necessity of Axiom 1.3 is discussed in the construction of π in Section 2 (see (1.2.11)).

Next, I show Axiom 1.4. We take a fixed $v \in \text{supp}(\pi)$. Then,

$$t_A(v, c + \alpha\mathbf{1}_A) = \begin{cases} 1 & \max_{j \in A} v_j - c_j - \alpha > \max_{k \in Z \setminus A} v_k - c_k \\ x \in [0,1] & \max_{j \in A} v_j - c_j - \alpha = \max_{k \in Z \setminus A} v_k - c_k \\ 0 & \max_{j \in A} v_j - c_j - \alpha < \max_{k \in Z \setminus A} v_k - c_k \end{cases}$$

So, $t_A(v, c_\alpha\mathbf{1}_A)$ takes only three values for each $v \in \text{supp}(\pi)$, and thus, $\rho_A(c + \alpha\mathbf{1}_A)$ takes, at most, $3 \times \text{supp}(\pi)$ values. When $\alpha \to -\infty$, $t_A(c, v + \alpha\mathbf{1}_A) = 1$ for all $v \in \text{supp}(\pi)$, and thus,

$\rho_A(c+\alpha\mathbf{1}_A) = \sum_{v\in\text{supp}(\pi)} \pi(v) = 1$. When $\alpha \to \infty$, $t_A(c, v+\alpha\mathbf{1}_A) = 0$ for all $v \in \text{supp}(\pi)$, and therefore, $\rho_A(c+\alpha\mathbf{1}_A) = 0$. Hence, Axiom 1.4 holds.

A.2 Proofs for Theorems 2 and 3

A.2.1 Proof for Theorem 2

(1) Proof of necessity.

By Möbius inversion formula (Grabisch, 2016, p. 49),

$$\pi(v : M(v, c+\alpha\mathbf{1}_A) = A) = \sum_{A'\subseteq A} (-1)^{|A\setminus A'|} \pi(v \in \mathbb{R}_0^{n+1} : M(v, c+\alpha\mathbf{1}_{A'}) \subseteq A'). \tag{A.2.1}$$

Below, Lemma 6 shows that under uniform tie-breaking, only generic points matter in the inclusion-exclusion formula.

Lemma 6. *Fix v. If $0 < t_A(v, c+\alpha\mathbf{1}_A) < 1$, then*

$$\sum_{A'\subseteq A} (-1)^{|A\setminus A'|} t_{A'}(c+\alpha\mathbf{1}_{A'}) = 0.$$

Proof. $0 < t_A(v, c+\alpha\mathbf{1}_A) < 1$ implies $M(v, c+\alpha\mathbf{1}_A) = C \cup B$ where $B \subseteq Z \setminus A$, $C \subseteq A$. In this case, $t_{A'}(v, c+\alpha\mathbf{1}_{A'}) = 0$ if $C \not\subseteq A'$, and $t_{A'}(v, c+\alpha\mathbf{1}_{A'}) = \frac{|C|}{|C\cup B|}$ if $C \subseteq A' \subseteq A$. Therefore, WLOG, we only consider $A' = C \cup \tilde{A} \subseteq A$. Therefore,

$$\begin{aligned}\sum_{A'\subseteq A} (-1)^{|A\setminus A'} t_{A'}(c+\alpha\mathbf{1}_{A'}) &= \frac{|C|}{|C|+|B|} \sum_{\tilde{A}\subseteq(A\setminus C)} (-1)^{|(A\setminus C)\setminus\tilde{A}|} \\ =&\frac{|C|}{|C|+|B|} \left(\sum_{i=0}^{|A\setminus C|} \binom{|A\setminus C|}{i} (-1)^{|(A\setminus C)\setminus i|} \right) = \frac{|C|}{|C|+|B|}(1-1)^{|A\setminus C|} = 0.\end{aligned}$$

□

Hence, $\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}t_{A'}(c+\alpha\mathbf{1}_{A'}) > 0 \iff t_A(v, c+\alpha\mathbf{1}_A) = 1 \iff M(v, c+\alpha\mathbf{1}_A) \subseteq A$, and therefore $\pi(v \in \mathbb{R}_0^{n+1} : M(v, c+\alpha\mathbf{1}_{A'}) \subseteq A') = \rho_{A'}(c+\alpha\mathbf{1}_{A'})$. So,

$$\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\pi(v \in \mathbb{R}_0^{n+1} : M(v, c+\alpha\mathbf{1}_{A'}) \subseteq A') = \sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}). \tag{A.2.2}$$

By (A.2.1) and (A.2.2),

$$\pi(v : M(v, c+\mathbf{1}\alpha) = A) = \sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'})).$$

By definition, $\pi(v : M(v, c+\mathbf{1}\alpha) = A) \leq \pi(v : M(v, c) = A)$. Hence,

$$\begin{aligned}
\rho_i(c) &= \sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\pi\{v \in \mathbb{R}_0^{n+1} : M(v, c) = A\} \text{ by (1.3.1)} \\
&\geq \sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\pi\{v \in \mathbb{R}_0^{n+1} : M(v, c+\mathbf{1}\alpha) = A\} \\
&= \sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}).
\end{aligned}$$

Therefore, Axiom 1.5 is necessary for the U-RQUM representation.

(2) Proof for sufficiency.

By Axiom 1.5,

$$\rho_i(c) \geq \sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}). \tag{A.2.3}$$

So,

$$\sum_{i\in Z}\rho_i(c) \geq \sum_{i\in Z}\sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}) \tag{A.2.4}$$

By definition, the left-hand side of (A.2.4) is equal to 1, and

$$RHS = \sum_{A\subseteq Z}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}(c+\alpha\mathbf{1}_{A'}).$$

Notice that $\lim_{\alpha\to 0}\rho_A(c+\alpha\mathbf{1}_A) = \rho_A^-(c) = \pi\{v \in \mathbb{R}_0^{n+1} : M(v,c) \subseteq A\} = \sum_{A'\subseteq A}\pi\{v \in \mathbb{R}_0^{n+1} : M(v,c) = A'\}$. So, by Möbius inversion,

$$\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}^-(c) = \pi(v \in \mathbb{R}_0^{n+1} : M(v,c) = A).$$

Therefore,

$$\lim_{\alpha\to 0}\sum_{A\subseteq Z}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}^-(c) = \sum_{A\subseteq Z}\pi(v \in \mathbb{R}_0^{n+1} : M(v,c) = A) = 1.$$

This implies that when $\alpha \to 0$, the righ-hand side of (A.2.4) is also equal to 1. Hence when $\alpha \to 0$, equality holds in (A.2.4), and thus, equality holds in (A.2.3):

$$\rho_i(c) = \sum_{A\subseteq Z: i\in A}\frac{1}{|A|}\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\rho_{A'}^-(c).$$

This is precisely the representation (1.3.1).

A.2.2 Proof for Theorem 3

Axiom 3' implies Axiom 1.4. By Lemma 1, Axioms 1.1, 1.2, and 1.4 imply that each ranking can be represented by a quasi-linear type on the goods. For $i = Z\setminus\{0\}$, take V_i as constructed in step 1. By Axiom 3', $|V_i| = 1$ or 2 for all $i \in Z\setminus\{0\}$, WLOG, let $|V_i| = 2$ for all $i \in Z\setminus\{0\}$.

Then, we denote $V_i = \{u_i^1, u_i^2\}, u_i^1, u_i^2 \in \mathbb{R}, u_i^1 < u_i^2$.

Suppose $0 < a < 1, a \neq 0.5$. take $v^i \in \mathbb{R}^{n+1}$ to be such that $\rho_{\{0,i\}}(v^i) = 1$. WLOG, suppose $\rho_i^-(u_i^1, v_{-i}^i) = a$ for all $i \in A \subseteq \{1, ..., n\}$, and $\rho_i^-(u_i^1, v_{-i}^i) = 1 - a$ for all $i \in \{1, ..., n\} \setminus A$. Then, we can construct two types, v and v', such that $v_i = u_i^1$ for all $i \in A$, and $v_j = u_j^2$ for all $j \in \{1, ..., n\} \setminus A$, while $v'_i = u_i^2$ for all $i \in A$, and $v'_j = u_i^1$ for all $j \in \{1, ..., n\} \setminus A$. In this case, $\pi(v) = 1 - a, \pi(v') = a$.

When $a = 0.5$, take $v_1 = u_1^1, v'_1 = u_1^2$. If $\rho_0(-\varepsilon, u_1^1, u_2^1, \infty, ..., \infty) = 0.5$, then $v_2 = u_2^1$, $v'_2 = u_2^2$. If $\rho_0(-\varepsilon, u_1^1, u_2^1, -\infty, ..., -\infty) = 0$, then $v_2 = u_2^2, v'_2 = u_2^1$. Suppose we have constructed $v_1, ..., v_k$ and $v'_1, ..., v'_k$ for $k \geq 2$. If $\rho_0(-\varepsilon, v_1, ..., v_k, u_{k+1}^1, \infty, .., \infty) = 0.5$, and $v_{k+1} = u_{k+1}^1$ and $v'_{k+1} = u_{k+1}^2$. If $\rho_0(-\varepsilon, v_1, ..., v_k, u_{k+1}^1, \infty, .., \infty) = 0$ then $v_{k+1} = u_{k+1}^2$, $v'_{k+1} = u_{k+1}^1$. In this way, v and v' can be constructed using this process.

Appendix B

Proofs for Chapter 2

B.1 Baseline Model – Two-Agent Signaling Game

B.1.1 Settlement and Trial Choices

- Players: {Defendant (D), Victim (V), Nature(N)}
- Histories:

  (1) N decides the state to be ω_l or ω_{nl}. D observes the state realization.

  (2) V receives a signal $z = \{0, 1\}$ regarding the state.

  (3) V files a claim against D if $z = 1$; thus in cases filed $P(\omega_l) = p_s$ in equation (2.3.2).

(4) D offers a settlement of σ to V.

(5) V decides whether to accept or to reject σ.

- If V accepts, D transfers σ to V.
- If V rejects, a court trial occurs. D loses and transfers amount d to V if and only if liable.
- In a trial, the two parties also incur court costs c_v and c_d, for V and D, respectively.

B.1.2 Pooling Equilibrium

B.1.2.1 $\sigma = 0$ when $pd \leq c_v$

V always accepts $\sigma = 0$ in equilibrium because

$$p(d - c_v) + (1 - p)(-c_v) = pd - c_v \leq 0.$$

In such a case, D's and V's equilibrium payoffs are both 0.

B.1.2.2 $\sigma^* = pd - c_v$ when $\frac{c_v}{d} < p < \frac{c_v + c_d}{d} < 1$

- V accepts σ^* in equilibrium. V's equilibrium payoff is

$$\pi_V = pd - c_v.$$

V has no incentive to deviate because the settlement amount is the same as the expected payoff from litigation.

- V would go to court for a trial if D deviates and offers V 0:

$$p(d - c_v) + (1 - p)(-c_v) > 0.$$

- However, D's payoff is lower from a trial than that from a settlement. For a non-liable D:

$$pd - c_v < c_d$$

For a liable D, the expected payout from a trial would be $c_d + d > c_d > pd - c_v$.

- Therefore, D would not deviate.

B.1.2.3 No pooling equilibrium when $pd \geq c_v + c_d$

- When $pd \geq c_v + c_d$,

$$c_d < p(d - c_v) + (1 - p)(-c_v) < d + c_d.$$

- V will only accept offers $\sigma \geq p(d - c_v) + (1 - p)(c_v)$; however, D in $\Omega = \omega_{nl}$ would prefer to offer 0 and pay c_d in a trial rather than offering σ.

- D in $\Omega = \omega_l$ would prefer to offer σ rather than to offer 0 and incur costs $d + c_d$ from a trial.

B.1.3 Equilibria Selection: The Pooling Equilibrium $\sigma^* = pd - c_v$ When $\frac{c_v}{d} < p < \frac{c_v + c_d}{d}$ Do Not Survive the D1 Criterion

- D1 criterion: According to Cho and Kreps (1987), when there is a type t' tho wishes to defect and send message 0 whenever type t wishes to do so, then (t, m) is pruned from the game. Formally,

$$D_t = \left\{ \varphi \in MBR(T(m), m) : u^*(t) < \sum_r u(t, m, r)\varphi(r) \right\},$$
$$D_t^0 = \left\{ \varphi \in MBR(T(m), m) : u^*(t) = \sum_r u(t, m, r)\varphi(r). \right\}$$

- That is, if for some type t there exists a second type t' with $D_t \cup D_t^0 \subseteq D_{t'}$, then (t, m) may be pruned from the game.

- u^* is the expected payoff in equilibrium; φ is the receiver's mixed best response to m; and $\sum_r u(t, m, r)\varphi(r)$ is the sender's expected deviation payoff given the best response.

- Here, consider a liable and a non-liable D. They can both offer 0 as a settlement. Because $-c_v < 0$ and $d - c_v > 0$, V plays the mixed strategy of accepting or rejecting when the type is unknown. Let $\varphi = (1 - y, y)$ represent the probability of (accept, reject)

- For a non-liable D, $-pd + c_v < (1 - y) * 0 - yc_d \implies y \leq \frac{pd - c_v}{c_d}$.

- Therefore, $D_{nl} = [0, \frac{pd - c_v}{c_d})$; $D_{nl}^0 = [0, \frac{pd - c_v}{c_d}]$.

- Similarly, for a liable D, $D_l = [0, \frac{pd - c_v}{d + c_d})$; $D_l^0 = [0, \frac{pd - c_v}{d + c_d}]$.

- $D_l \cup D_l^0 \subseteq D_{nl}$. Therefore, a liable D is pruned for sending a settlement of 0. In other words, $(t, m) = (liable, 0)$ is ruled out.

- Thus, whenever V sees a settlement of 0, V believes that this is from a non-liable D, and will accept it. Therefore, a non-liable D will defect, and the pooling equilibrium will be eliminated.

B.1.4 Separating Equilibrium with Randomization When $c_v < d$ and $p \geq \frac{c_v}{d}$

- D offers 0 in ω_{nl} with probability 1. In ω_l, D offers 0 with probability x, and offers $\sigma^* = d - c_v$ with probability $1 - x$.

- Offer $\sigma \geq \sigma^*$ is accepted; any offer of $\sigma < \sigma^*$ is rejected with probability r.

- When offered 0, V is indifferent regarding the choice between accepting and rejecting:

$$\begin{aligned}&(\frac{xp}{xp + (1-p)})(d - c_v) + (1 - \frac{xp}{xp + (1-p)})(-c_v) = 0\\&\frac{xp}{xp + (1-p)} = \frac{c_v}{d}\\&\implies x = \frac{1-p}{p}\frac{c_v/d}{1 - c_v/d}.\end{aligned}$$

Liable D is indifferent regarding the choice between offering σ^* and offering 0:

$$\begin{aligned}&-(d - c_v) = r(-d - c_d) + (1 - r)0\\&\implies \quad r = \frac{1 - c_v/d}{1 + c_d/d}.\end{aligned} \tag{B.1.1}$$

- Non-liable D prefers to offer 0 over σ^* because $d - c_v > rc_d$.

- V's posterior belief is as follows: $\mu_s(\omega_l|\sigma^*) = 1$, $\mu_s(\omega_1|0) = \frac{xp}{xp+(1-p)}$

- The expected payoff for V is the following:

$$\pi_V = p(1-x)(d-c_v) + (1-p+xp)r(-c_v + \frac{xp}{xp+1-p}d).$$

- The expected payout for D is the following:

$$\pi_D = -p(1-x)(d-c_v) + r(1-p+px)(-c_d - \frac{xp}{xp+1-p}d).$$

- The trial rate among filed cases is the following: $(1-p+xp)r$.

- The restrictions on the parameters are the following:

$$\begin{aligned} 0 \leq x, r &\implies 0 \leq \frac{c_v}{d} \leq 1, \\ x \leq 1 &\implies p \geq \frac{c_v}{d}. \end{aligned}$$

B.1.5 Summary of Equilibrium in the Baseline Model

The equilibrium of the baseline model is as follows:

(1) The main case is obtained when $p_s > \frac{c_v}{d}$. In such a case, there is a *separating* equilibrium where,

   (i) if non-liable, D offers zero settlement

   (ii) if liable, D randomizes between two offers: zero with probability

$$x = \frac{1-p_s}{p_s}\frac{c_v}{d-c_v} = \frac{\beta_1}{1-\beta_0}\frac{1-p_0}{p_0}\frac{c_v}{d-c_v},$$

and positive settlement amount

$$\sigma^* = d - c_v$$

with probability $1 - x$;

(iii) V accepts positive offers σ^*, and rejects the zero offers with probability

$$r = \frac{d - c_v}{d + c_d}.$$

(2) When $p_s \leq \frac{c_v}{d}$, there is a pooling equilibrium where D offers zero to all Vs, and all Vs accept it. This situation is equivalent to V not filing a court case.

B.1.6 Probabilities of Going to Trial and Winning at Trial

Under the separating equilibrium (when $p_s > \frac{c_v}{d}$), a trial occurs when V rejects D's zero offers. Therefore, the probability of a court trial among the cases filed is

$$\begin{aligned}(1 - p_s + p_s x)r &= (1 - p_s)\frac{1}{1 + c_d/d} \\ &= \frac{\beta_1(1 - p_0)}{\beta_1(1 - p_0) + (1 - \beta_0)p_0}\frac{1}{1 + c_d/d}.\end{aligned}$$

V's probability of winning at trial is the proportion of ω_l cases among the cases that go to trial:

$$\frac{xp}{1 - p + xp} = c_v/d.$$

Because a trial reveals the true state, such a winning rate is the true proportion of liable cases among the cases that go to trial.

B.2 Three-Agent model When $z = 0$

- Players: { Defendant (D), Victim (V), Lawyer (L), Nature (N)}
- Histories:

  (1) N decides the state. The state is ω_l with probability p_0, and is ω_{nl} with probability $1 - p_0$. D and L observe the realization of the state.

  (2) Stage 0: V receives a signal $z = \{0, 1\}$ regarding the state. Both L and D observe z as well. L sends a signal $m = \{0, 1\}$ to V when $z = 0$. As a result, V files not only $z = 1$ cases but also some $z = 0$ cases.

  (3) D offers settlement σ to V.

  (4) V decides whether to accept or reject σ according to V's posterior belief. If V rejects σ, a trial occurs.

B.2.1 L' Persuasion Signal to V

Proof of Proposition 2.4.1

Proof for Proposition 2.4.1. For L to be credible we need to require that

$$\mu_s(\omega_l) = P(m = 1)P(\omega_l \mid m = 1) + P(m = 0)P(\omega_l \mid m = 0) = \mu_0$$

That is, V's posterior belief after receiving L's signal is the same as the true state (which is also the same as her prior belief).

For V to follow L's signal and file a case whenever $m = 1$, V's posterior belief after receiving $m = 1$ should be equal to or greater than μ_t, the threshold for filing a claim.

$$P(\omega_l \mid m = 1) = \frac{P(\omega_l, m = 1)}{P(m = 1)} \geq \mu_t. \tag{B.2.1}$$

Therefore, under L's signaling strategy, the proportion of case filings is $P(m = 1)$. L's objective is to bring about as many case filings as possible. In other words,

$$\max P(m = 1)$$

We denote L's signal as follows.

$$\begin{aligned} &P(m = 1 \mid \omega_l) = x, \qquad P(m = 0 \mid \omega_l) = 1 - x, \\ &P(m = 1 \mid \omega_{nl}) = y, \qquad P(m = 0 \mid \omega_{nl}) = 1 - y. \end{aligned} \tag{B.2.2}$$

Therefore,

$$P(m = 1) = P(m = 1 \mid \omega_l)P(\omega_l) + P(m = 1 \mid \omega_{nl})P(\omega_{nl}) = x\mu_0 + y(1 - \mu_0),$$

$$P(m = 0) = P(m = 0 \mid \omega_l)P(\omega_l) + P(m = 0 \mid \omega_{nl})P(\omega_{nl}) = (1 - x)\mu_0 + (1 - y)(1 - \mu_0),$$

$$P(\omega_l \mid m = 1) = \frac{P(\omega_l, m = 1)}{P(m = 1)} = \frac{x\mu_0}{x\mu_0 + y(1 - \mu_0)},$$
$$P(\omega_l \mid m = 0) = \frac{P(\omega_l, m = 0)}{P(m = 0)} = \frac{(1 - x)\mu_0}{(1 - x)\mu_0 + (1 - y)(1 - \mu_0)},$$
$$\mu_s(\omega_l) = P(\omega_l = 1, m = 1) + P(\omega_l, m = 0) = x\mu_0 + (1 - x)\mu_0 = \mu_0.$$

Thus, L must solve the following maximization problem:

$$\max_{x,y} x\mu_0 + y(1-\mu_0)$$
$$s.t. \frac{x\mu_0}{x\mu_0 + y(1-\mu_0)} \geq \mu_t$$

Notice that as x increase, the left hand side increases. Therefore, we can set $x = 1$. As y increases, the left hand side decreases. Therefore, the constraint is binding; that is,

$$\frac{x\mu_0}{x\mu_0 + y(1-\mu_0)} = \mu_t, \; x = 1 \implies y = \frac{\mu_0}{1-\mu_0}\frac{1-\mu_t}{\mu_t}.$$
$$\implies P(m=1) = \frac{\mu_0}{\mu_t}$$

Plug x, y back into (7), we obtain L's optimal signal, which is of the form in (5). The total number of claims filed is $P(m=1) = \frac{\mu_0}{\mu_t} > \mu_0$ if $\mu_0 < \mu_t$. □

Proof of Proposition 2.4.2 V's signal z is distributed as follows:

$$P(z=1) = (1-\beta_0)p_0 + \beta_1(1-p_0),$$
$$P(z=0) = \beta_0 p_0 + (1-p_0)(1-\beta_1).$$

Among $z = 0$ cases, the probability that D is liable (ω_l) is

$$p_s' = P(\omega_l|z=0) = \frac{\beta_0 p_0}{\beta_0 p_0 + (1-p_0)(1-\beta_1)} < p_s.$$

Proof of Proposition 2.4.2. L's signaling strategy: As determined in equation (2.3.3), when $z = 0$, V's prior probability that D is liable (ω_l) is

$$p_s' = P(\omega_l|z=0) = \frac{\beta_0 p_0}{\beta_0 p_0 + (1-p_0)(1-\beta_1)} < p_s$$

L's optimal signal is described in Proposition 2. In equilibrium, L solicits all liable cases and some non-liable cases until V's payoff from accepting the solicitation is the same as the payoff from rejecting the solicitation, which is 0. Modifying (5) in proposition 2.4.1 by replacing $\mu_0 = p'_s$ and $\mu_t = \frac{c_v+f+f_0}{\xi d}$, the lawyer's optimal signal in this game is as follows.

$$
\begin{aligned}
&P(m=1|\omega_l) = 1,\\
&P(m=0|\omega_l) = 0,\\
&P(m=1|\omega_{nl}) = \frac{p'_s}{1-p'_s}\frac{\xi d - c_v - f - f_0}{c_v+f+f_0} = \frac{\beta_0}{1-\beta_1}\frac{p_0}{1-p_0}\frac{\xi d - c_v - f - f_0}{c_v+f+f_0},\\
&P(m=0|\omega_{nl}) = 1 - P(m=1|\omega_{nl}).
\end{aligned}
$$

As a result, V's posterior belief after receiving L's signal is as follows.

$$
\begin{aligned}
&\mu'_s(\omega_l|m=1, z=0) = \frac{c_v+f+f_0}{\xi d} > \frac{c_v}{d},\\
&\mu'_s(\omega_l|m=0, z=0) = 0.
\end{aligned}
$$

□

L sends signal $m = \{0, 1\}$ to V if $z = 0$. Let L's strategy in $z = 0$ be

$$
\begin{aligned}
&P(m=1|\omega_l) = 1,\\
&P(m=0|\omega_l) = 0,\\
&P(m=1|\omega_{nl}) = k,\\
&P(m=0|\omega_{nl}) = 1-k.
\end{aligned}
$$

This is the optimal signal in proposition 2. Therefore,

$$
\begin{aligned}
&\mu'_s(\omega_l|m=1) = \frac{p'_s}{p'_s + k(1-p'_s)},\\
&\mu'_s(\omega_l|m=0) = 0.
\end{aligned}
$$

V files cases when $m = 1$. Denote the probability of D being liable in $z = 0$ cases filed as $\bar{p}$. Therefore,

$$\bar{p} = \mu_s'(\omega_l | m = 1) = \frac{p_s'}{p_s' + k(1 - p_s')}.$$

The amount cases filed is:

$$P(z = 0) * (p_s' + k(1 - p_s'))$$

Such signal a is credible because

$$p_s' = \mu_s'(\omega_l | m = 0)p(m = 0) + \mu_s'(\omega_l | m = 1)p(m = 1) \quad = 0 + \bar{p} * (p_s' + k(1 - p_s')) = p_s'.$$

B.2.2 Pooling Equilibrium in the Three-Agent model when $\xi d > c_v + f + f_0$

Suppose $\xi d \leq c_v + f + f_0$. In such a case, by accepting, V's payoff is less than 0. Thus, V does not take L's advice and does not file a case. Therefore, for a pooling equilibrium to exist, $\xi d \geq c_v + f + f_0$.

B.2.2.1 Case 1: $\sigma^* = f_0$ when $\xi d > c_v + f + f_0$ and $f_0 < c_d$

Suppose there is a pooling equilibrium where D offers $\xi d\bar{p} - c_v - f$ to all, and V accepts this offer. For V to accept L's advice and file a case, V's payoff from filing her case must be equal to or greater than not filing her case:

$$\xi d\bar{p} - c_v - f - f_0 \geq 0 \implies \xi d\bar{p} \geq c_v + f + f_0.$$

V's belief upon seeing the offer is still $\bar{p}$, and thus V has no incentive to deviate. To sustain such an equilibrium, a non-liable D must be willing to offer a settlement rather than go to a court trial. This requires

$$\xi d\bar{p} - c_v - f \leq c_d \implies \xi d\bar{p} \leq c_d + c_v + f.$$

Therefore, when $c_v + f + f_0 \leq \xi d\bar{p} \leq c_d + c_v + f$, there is a pooling equilibrium where D offers $\xi d\bar{p} - c_v - f$ to all, and V accepts this offer. This equilibrium requires $f_0 < c_d$.

L is paid f_0 when V files a case. L's payoff is

$$\pi_L = f_0(P(z = 0) * (p'_s + k(1 - p'_s))).$$

The only parameter that L controls in this equation is k. Thus, L wants to increase k. In other words, L wants to have the lowest $\bar{p} = \frac{p'_s}{p'_s + k(1-p'_s)}$. Therefore, $\bar{p} = \frac{c_f + f + f_0}{\xi d}$. Note that $\xi d\bar{p} > c_f + f + f_0$ is always satisfied. Thus,

$$\begin{aligned} \bar{p} = \frac{p'_s}{p'_s + k(1 - p'_s)} &= \frac{c_v + f + f_0}{\xi d} \\ \implies k = \frac{p'_s}{1 - p'_s} \frac{\xi d - c_v - f - f_0}{c_v + f + f_0}&. \end{aligned} \tag{B.2.3}$$

In equilibrium, when $\xi d > c_v + f + f_0$ and $f_0 < c_d$, D offers f_0 to V, and V accepts.

B.2.2.2 Case 2: No Pooling Equilibrium When $\xi d > c_v + f + f_0$ and $f_0 > c_d$

Suppose $\xi d\bar{p} > c_d + c_v + f$. This implies that $f_0 > c_d$. In such a case, a non-liable D would prefer a court trial over offering the settlement amount of $\xi d\bar{p} - c_v - f$. Thus, there is no pooling equilibrium.

B.2.3 Equilibrium Selection: Pooling Equilibrium When $f_0 < c_d$ Can Be Eliminated by the D1 Criterion

We apply the D1 criterion the same way as in Appendix A.4.

- Consider the situation where both liable and non-liable Ds offer 0 as a settlement. Because $-c_v - f < 0$ and $\xi d - c_v - f > 0$, V plays a mixed strategy. Let $\varphi = (1-y, y)$ be the probability of (accept, reject).
- For a non-liable D, $-f_0 < (1-y) * 0 - y * (c_d) \implies y < \frac{f_0}{c_d}$.
- For a liable D, $-f_0 < (1-y) * 0 - y * (c_d + d) \implies y < \frac{f_0}{c_D}$.
- Therefore, $D_{nl} = \left[0, \frac{f_0}{c_d}\right)$, $D_{nl}^0 = \left[0, \frac{f_0}{c_d}\right]$; $D_l = \left[0, \frac{f_0}{c_d+d}\right)$, $D_l^0 = \left[0, \frac{f_0}{c_d+d}\right]$.
- Because $D_l \cup D_l^0 \subseteq D_{nl}$, a liable D is pruned from sending a settlement of zero by the D1 criterion.
- Therefore, whenever V sees a settlement of zero, V believes it is from a non-liable, and will accept it because $-c_v - f < 0$. As a result, a non-liable D will defect and the pooling equilibrium will be eliminated.

B.2.4 Separating Equilibrium with Randomization When $\xi d \geq c_v + f + f_0$.

D's offer is:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{by liable D with probability 1-x} \\ 0 & \text{by non-liable D w.p. 1, by liable D w.p. x.} \end{cases}$$

V's action is:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{accepted by V,} \\ 0 & \text{rejected by V with probability r.} \end{cases}$$

In the settlement game, for V to randomize, V must be indifferent regarding the choice between rejecting and accepting 0:

$$\frac{x\bar{p}}{x\bar{p} + 1 - \bar{p}}(\xi d - c_v - f) + \frac{1 - \bar{p}}{x\bar{p} + 1 - \bar{p}}(-c_v - f) - f_0 = -f_0,$$
$$\implies x = \frac{1 - \bar{p}}{\bar{p}} \frac{c_v + f}{\xi d - c_v - f}.$$

For a liable D to randomize, the liable D must be indifferent regarding the choice between a settlement and a trial:

$$\xi d - c_v - f = 0 + r(d + c_d) \implies r = \frac{\xi d - c_v - f}{d + c_d}$$

For V to be willing to file a case in the first place, $\pi_V \geq 0$, this puts a restriction on $\bar{p}$:

$$\pi_V = (\xi d - c_v - f - f_0)\bar{p}(1 - x) + (-f_0)(1 - \bar{p} + x\bar{p})$$
$$= (\xi d - c_v - f)\bar{p}(1 - x) - f_0 \geq 0$$
$$\implies \bar{p} \geq \frac{f_0}{(\xi d - c_v - f)(1 - x)}.$$

We then plug in x,

$$
\begin{aligned}
\bar{p} \geq &\frac{\bar{p} f_0}{\xi d\bar{p} - c_v - f} \\
&\implies \bar{p}(\xi d\bar{p} - c_v - f - f_0) \geq 0 \\
&\implies \bar{p} \geq \frac{c_v + f + f_0}{\xi d}.
\end{aligned}
$$

L's payoff in such a game would be:

$$
\begin{aligned}
\pi_L &= P(z=0)(p_s' + k(1-p_s'))(f_0 + (1-\bar{p}+x\bar{p})r(f + \frac{x\bar{p}}{1-\bar{p}+x\bar{p}}(1-\xi)d)) \\
&= P(z=0)\left(p_s' f_0 + (1-p_s')\left(f_0 + \frac{\xi d}{d+c_d}\left(f + \frac{1-\xi}{\xi}(c_v+f)\right)\right)k\right).
\end{aligned}
\tag{B.2.4}
$$

Because π_L increases with k, L wants to increase k, which is the equivalent of decreasing $\bar{p}$ from equation (2.4.4). Therefore,

$$
\begin{aligned}
\bar{p} = &\frac{c_v + f + f_0}{\xi d} = \frac{p_s'}{p_s' + k(1-p_s')} \\
&\implies k = \frac{p_s'}{1-p_s'}\frac{\xi d - c_v - f - f_0}{c_v + f + f_0}.
\end{aligned}
\tag{B.2.5}
$$

We then plug in k to obtain x:

$$
\begin{aligned}
x = &\frac{1-\bar{p}}{\bar{p}}\frac{c_v + f}{\xi d - c_v - f} \\
= &\frac{\xi d - c_v - f - f_0}{\xi d - c_v - f}\frac{c_v + f}{c_v + f + f_0}.
\end{aligned}
$$

Thus, the probability of a trial among the cases filed because of L's solicitation is:

$$
\begin{aligned}
&P(z=0)(p_s' + k(1-p_s'))(1-\bar{p}+x\bar{p})r \\
&= P(z=0)p_s'\frac{\xi d}{d+c_d}\frac{\xi d - c_v - f - f_0}{c_v + f + f_0}.
\end{aligned}
$$

The probability of winning at trial is:

$$\frac{x\bar{p}}{1-\bar{p}+x\bar{p}} = \frac{c_v+f}{\xi d}.$$

B.2.5 Summary of Equilibrium in Three-Agent Model When $Z = 0$

In equilibrium, V's belief that D liable is $\bar{p} = \frac{c_v+f+f_0}{\xi d}$ in $z = 1$ cases. V hires L only when $\xi d - c_v - f - f_0 > 0$. There, there is a separating equilibrium with randomization. L's signal is as follows:

$$k = \frac{p_s'}{1-p_s'}\frac{\xi d - c_v - f - f_0}{c_v+f+f_0} = \frac{\beta_0}{1-\beta_1}\frac{p_0}{1-p_0}\frac{\xi d - c_v - f - f_0}{c_v+f+f_0}.$$

D's offer is as follows:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{by liable D with probability 1-x,} \\ 0 & \text{by non-liable D w.p. 1, by non liable D w.p. x.} \end{cases}$$

V's strategy is as follows:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{accepted by V,} \\ 0 & \text{rejected by V with probability r.} \end{cases}$$

where

$$x = \frac{\xi d - c_v - f - f_0}{\xi d - c_v - f}\frac{c_v+f}{c_v+f+f_0}, \quad r = \frac{\xi d - c_v - f}{d+c_d}.$$

B.2.6 Proof of Propositions 2.4.5 and 2.4.6

Proof of Proposition 2.4.5

Proof of Proposition 2.4.5. 1. When $z = 0$, V hires L only when $\xi d \geq c_v + f + f_0$. When $z = 1$, V hires L only when $\xi d p_s \geq c_v + f + f_0$. Ceteris paribus, such conditions are more likely to be satisfied when $f, f_0, 1 - \xi$ are low and when d is high.

2. When V is more likely to make a type II error, namely, when β_0 is higher, $P(z = 0) = \beta_0 p_0 + (1 - p_0)(1 - \beta_1)$ and p'_s are both higher. L corrects such a type II error when $z = 0$, and increases the total number of cases by $P(z = 0) p'_s \frac{\xi d}{c_v + f + f_0}$. Thus, a higher β_0 would mean that the lawyer's solicitation is more likely to be successful. When $z = 1$, V hires L when $\xi d p_s \geq c_v + f + f_0$, and thus, is more likely to hire L when p_s is high, namely, when V's signal is more precise.

□

Proof of Proposition 2.4.6

Proof of Proposition 2.4.6. 1. The increase in the number of cases filed follows from equation (2.4.4). No pooling equilibrium follows from the equilibrium characterizations found in propositions 4 and 5.

2. This conclusion follows from equilibrium trial wining rates proposition 1(3) and equations (2.4.6) and 2.5.1.

3. This conclusion follows from equilibrium characterizations in proposition 4 and 5.

□

B.3 Three Agent Model When $z = 1$ and $\xi dp_s \geq c_v + f + f_0$

When $z = 1$, L does not affect V's belief. Rather, L increases the trial winning rate in cases where D is liable. Suppose $\xi dp_s \leq c_v + f + f_0$. With such parameters, V's expected payoff for filing a case is less than 0. Therefore, V would not hire a lawyer, and this situation would reduce to the two-agent signaling game. V only hires L when $\xi dp_s \geq c_v + f + f_0$.

B.3.1 Pooling Equilibrium When $\xi dp_s \leq c_v + f + c_d$

Suppose there is a pooling equilibrium where D offers $\xi dp_s - c_v - f$ to all, and V accepts this offer. To sustain such an equilibrium, a non-liable D must prefer to settle rather than go to trial. f_0 is considered a sunk cost, and thus, does not affect the offer. Thus,

$$\xi dp_s - c_v - f \leq c_d.$$

Therefore, when $c_v + f + f_0 \leq \xi dp_s \leq c_v + f + c_d$ there is a pooling equilibrium where D offers $\xi dp_s - c_v - f$ to all, and V accepts.

When $\xi dp_s > c_v + f + f_0 + c_d$, a non-liable D would not offer such a settlement and would prefer to go to trial. hence, there is no pooling equilibrium.

B.3.2 Equilibrium Selection: D1 Criterion Prunes Pooling Equilibrium

The pooling equilibrium when $c_v + f + f_0 \leq \xi dp_s \leq c_v + f + f_0 + c_d$ can be pruned by the D1 criterion as in Appendices A.4 and B.3. D1 criterion states that when there is type t' wishes to defect and send message m whenever type t wishes to do so, then (t, m) is pruned from the game. In our case, whenever liable D wants to send 0, then non-liable D wants to send 0. Thus (liable, 0) is ruled out. Therefore, non-liable D would only offer 0. The pooling equilibrium does not survive the D1 criterion.

B.3.3 Separating Equilibrium with Randomization

In a separating equilibrium with randomization, D and V's strategies are as follows.

D's offer is:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{by liable D with probability 1-x,} \\ 0 & \text{by non-liable D w.p. 1, by non liable D w.p. x.} \end{cases}$$

V's action is:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{accepted by V,} \\ 0 & \text{rejected by V with probability r.} \end{cases}$$

V's must be indifferent regarding the choice between rejecting and accepting 0:

$$\frac{xp_s}{xp_s + 1 - p_s}(\xi d - c_v - f) + \frac{1 - p_s}{xp_s + 1 - p_s}(-c_v - f) - f_0 = -f_0$$
$$\implies x = \frac{1 - p_s}{p_s}\frac{c_v + f}{\xi d - c_v - f}.$$

For a liable D to randomize, the payoff from a settlement and from a trial would need to be the same:

$$\xi d - c_v - f = 0 + r(d + c_d)$$
$$\implies r = \frac{\xi d - c_v - f}{d + c_d}.$$

For V to be willing to file a case, $\pi_V \geq 0$:

$$\pi_V = (\xi d - c_v - f - f_0)p_s(1-x) + (-f_0)(1 - p_s + xp_s)$$
$$= (\xi d - c_v - f)p_s(1-x) - f_0 \geq 0$$
$$\implies p_s \geq \frac{f_0}{(\xi d - c_v - f)(1-x)}.$$

We then plug in x to restrict the parameters,

$$p_s \geq \frac{p_s f_0}{\xi d p_s - c_v - f}$$
$$\implies p_s(\xi d p_s - c_v - f - f_0) \geq 0$$
$$\implies \xi d p_s \geq c_v + f + f_0.$$

Such a condition is always satisfied in the assumption of this subsection.

The probability of a trial is

$$(1 - p_s + xp_s)r = (1 - p_s)\frac{\xi d}{d + c_d}.$$

The probability of winning at trial is:

$$\frac{xp_s}{1 - p_s + xp_s} = \frac{c_v + f}{\xi d}.$$

B.3.4 Summary of the Three-Agent Settlement Game When z=1

- V only hires L when $\xi d p_s \geq c_v + f + f_0$.
- There only exists a separating equilibrium:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{by liable D with probability 1-x,} \\ 0 & \text{by non-liable D w.p. 1, by non liable D w.p. x.} \end{cases}$$

V's action is the following:

$$\sigma^* = \begin{cases} \xi d - c_v - f & \text{accepted by V,} \\ 0 & \text{rejected by V with probability r,} \end{cases}$$

where

$$x = \frac{1-p_s}{p_s}\frac{c_v+f}{\xi d - c_v - f} = \frac{1-p_0}{p_0}\frac{1-\beta_1}{\beta_0}\frac{c_v+f}{\xi d - c_v - f}$$
$$r = \frac{\xi d - c_v - f}{d + c_d}.$$

Notice that in V's and D's strategy spaces, V's randomization strategy, r, and the trial winning rate are the same as those in the scenario of the lawyer's solicitation when $z = 0$ in Appendix B; while the trial probabilities and D's randomization strategies, x, in the two situations are different.

B.3.5 Calculation for numerical example in subsection 2.4.6.1

Without lawyers, V gets $z = 1$ in $100{*}0.7{+}900{*}0.1 = 160$ injuries. V files claims against

D in all 160 cases for which she thinks D is liable. Thus, $p_s = \frac{0.1*0.7}{0.7*0.1+0.1*0.9} = 0.4375$ by (2). According to propositions 1(2), because $p_s = 0.4375 > \frac{c_v}{d} = 0.05$, there is a separating equilibrium where V files 160 cases against D. By Proposition 1(2), the randomization of V and D are $x = \frac{1-0.4375}{0.4375}\frac{50}{950} \approx 0.068$, and $r = \frac{950}{1100} \approx 0.86$. Therefore, V obtains a settlement of \$950 in $70(1-x) \approx 65$ liable cases. There are $r(160-65) \approx 82$ trials, and V only win around $70-65 = 5$ of them. Thus, the trial winning rate is around 6%.

When V is represented by L, in the 160 injuries for which V gets $z = 1$, there are 70 liable cases. By Proposition 5(b), $x = \frac{1-0.4375}{0.4375}\frac{50+100}{700-50-100} \approx 0.35$, and $r = \frac{700-50-100}{1000+100} \approx 0.5$. Therefore, with the help of L, V gets a settlement of \$550 (\$700 − \$50 − \$100 = \$550) from D in $70(1-x) \approx 45$ of the 70 liable cases. Court trials occur in around $r*(160-45) \approx 57$ cases, among which, about $57 * \frac{70-45}{160-45} \approx 12$ are liable. Thus, V's trial winning rate is $\frac{12}{57} \approx 21$ %.

In the 840 injuries for which V gets $z = 0$, D is liable in $100 * 0.3 = 30$ of them. By (7), L can persuade V to file around $0.3 * 0.1 * \frac{700}{50+100+20} * 1000 \approx 124$ claims against D – among them, there are all the 30 liable cases and around 94 non-liable cases. In equilibrium, $x = \frac{700-50-100-20}{700-50-100} \times \frac{50+100}{50+100+20} \approx 0.85$, and $r = \frac{700-50-100}{1000+100} \approx 0.5$. Thus, D offers a settlement of $\$(700-50-100) = 550$ to V in about $30*(1-x) \approx 5$ liable cases, and offers \$0 in all the remaining cases. In the end, $(124-5)*r \approx 60$ cases go to trial, and V wins $60 * \frac{30-5}{124-5} \approx 13$ of them, resulting in a winning probability rate of 21%.

B.4 Extension 1: When L is Imperfectly Informed

L gets a noisy signal, s:

$P(s = 1|\omega_l) = 1 - \theta_0,$

$P(s = 0|\omega_l) = \theta_0$ (a false negative, or type II error),

$P(s = 1|\omega_{nl}) = \theta_1$ (a false positive, or a type I error),

$P(s = 0|\omega_{nl}) = 1 - \theta_1.$

Therefore, L's posterior belief after this signal is:

$$\mu_d = P(\omega_l \mid s = 1) = \frac{p_0(1-\theta_0)}{p_0(1-\theta_0) + (1-p_0)\theta_1},$$
$$\mu'_d = P(\omega_l \mid s = 0) = \frac{p_0\theta_0}{p_0\theta + (1-p_0)(1-\theta_1)}.$$

B.4.1 L's Persuasion Signaling Strategy

L's signaling strategy would be as follows:

$P(m = 1|s = 1) = 1$

$P(m = 0|s = 1) = 0$

$P(m = 1|s = 0) = k$

$P(m = 0|s = 0) = 1 - k$

In other words, L's strategy $m = \{0, 1\}$ only depends on s:

$$P(\omega, m \mid s) = P(\omega \mid s)P(m \mid s)$$

Therefore, the state ω and L's signal m to b are *mutually independent, conditional on the signal* s. Intuitively, L distinguishes state ω_l from state ω_{nl} only as well as his signal, s. If

L's signal is $s = 1$, he tells V that D is liable; and if L's signal is $s = 0$, with some probability k, he tells V that L is liable. L's signal has the following distribution:

$$P(s = 1) = P(s = 1 \mid \omega_l)P(\omega_l) + P(y = \mid \omega_{nl})P(\omega_{nl})$$
$$= (1 - \theta_0)p_0 + \theta_1(1 - p_0),$$
$$P(s = 0) = P(s = 0 \mid \omega_l)P(\omega_l) + P(y = 0 \mid \omega_{nl})P(\omega_{nl}) \quad = \theta_0 p_0 + (1 - p_0)(1 - \theta_1).$$

Therefore, the signal m that V receives from L has the following probabilities:

$$P(m = 1) = P(m = 1 \mid s = 1)P(s = 1) + P(m = 1 \mid s = 0)P(s = 0)$$
$$= P(s = 1) + kP(s = 0)$$
$$= (1 - \theta_0)p_0 + \theta_1(1 - p_0) + k[\theta_0 p_0 + (1 - p_0)(1 - \theta_1)],$$
$$P(m = 0) = 1 - P(m = 1)$$
$$= (1 - k)[1 - \theta_1 - p_0(1 - \theta_0 - \theta_1)].$$

Given L's signal quality, and the fact that m and ω are mutually independent conditional on s, we obtain the following joint probability distribution for the true state and L's signal to V, m, which is conditional on L's own signal s:

$$P(\omega_l, m=1 \mid s=1) = P(\omega_l \mid s=1)P(m=1 \mid s=1) = \mu_d = \frac{p_0(1-\theta_0)}{p_0(1-\theta_0)+(1-p_0)\theta_1},$$

$$P(\omega_l, m=0 \mid s=1) = P(\omega_l \mid s=1)P(m=0 \mid s=1) = 0,$$

$$P(\omega_{nl}, m=1 \mid s=1) = P(\omega_{nl} \mid s=1)P(m=1 \mid s=1) = 1-\mu_d = \frac{(1-p_0)\theta_1}{p_0(1-\theta_0)+(1-p_0)\theta_1},$$

$$P(\omega_{nl}, m=0 \mid s=1) = (\omega_{nl} \mid s=1)P(m=0 \mid s=1) = 0,$$

$$P(\omega_l, m=1 \mid s=0) = (\omega_l \mid s=0)P(m=1 \mid s=0) = k\mu_d' = \frac{kp_0\theta_0}{p_0\theta_0+(1-p_0)(1-\theta_1)},$$

$$P(\omega_l, m=0 \mid s=0) = (\omega_l \mid s=0)P(m=0 \mid s=0) = (1-k)\mu_d' = \frac{(1-k)p_0\theta_0}{p_0\theta_0+(1-p_0)(1-\theta_1)},$$

$$P(\omega_{nl}, m=1 \mid y=0) = (\omega_{nl} \mid s=0)P(m=1 \mid s=0) = k(1-\mu_d') = \frac{k(1-p_0)(1-\theta_1)}{p_0\theta_0+(1-p_0)(1-\theta_1)},$$

$$P(\omega_{nl}, m=0 \mid y=0) = (\omega_{nl} \mid s=0)P(m=0 \mid s=0) = (1-k)(1-\mu_d') = \frac{(1-k)(1-p_0)(1-\theta_1)}{p_0\theta_0+(1-p_0)(1-\theta_1)}.$$

By Bayes rule, the conditional probabilities of the true state on L's own signal s and L's signal to V, s, are as follows:

$$P(\omega_l \mid m=1, s=1) = P(\omega_l, m=1 \mid s=1)/P(m=1) = \frac{\mu_d}{P(m=1)},$$

$$P(\omega_l \mid m=0, s=1) = P(\omega_l, m=0 \mid s=1)/P(m=0) = 0,$$

$$P(\omega_{nl} \mid m=1, s=1) = P(\omega_{nl}, m=1 \mid s=1)/P(m=1) = \frac{1-\mu_d}{P(m=1)},$$

$$P(\omega_{nl} \mid m=0, s=1) = P(\omega_{nl}, m=0 \mid s=1)/P(m=0) = 0,$$

$$P(\omega_l \mid m=1, s=0) = P(\omega_l, m=1 \mid s=0)/P(m=1) = \frac{k\mu_d'}{P(m=1)},$$

$$P(\omega_l \mid m=0, s=0) = P(\omega_l, m=0 \mid s=0)/P(m=0) = \frac{(1-k)\mu_d'}{P(m=0)},$$

$$P(\omega_{nl} \mid m=1, s=0) = P(\omega_{nl}, m=1 \mid s=0)/P(m=1) = \frac{k(1-\mu_d')}{P(m=1)},$$

$$P(\omega_{nl} \mid m=0, s=0) = P(\omega_{nl}, m=0 \mid s=0)/P(m=0) = \frac{(1-k)(1-\mu_d')}{P(m=0)}.$$

Therefore, a signal m from L conveys the following information:

$$
\begin{aligned}
P(\omega_l \mid m=1) &= P(\omega_l \mid m=1, s=1)P(y=1) + P(\omega_l \mid m=1, s=0)P(s=0) \\
&= \frac{\mu_d}{P(m=1)}P(s=1) + \frac{k\mu_d'}{P(m=1)}P(s=0) \\
&= \frac{\mu_d P(y=1) + k\mu_d' P(y=0)}{P(s=1) + kP(y=0)} \\
&= \frac{p_0}{p_0(1-\theta_0) + (1-p_0)}\left[\frac{1-\theta_0}{1+k\frac{\theta_0 p_0 + (1-p_0)(1-\theta_1)}{(1-\theta_0)p_0 + \theta_1(1-p_0)}} + \frac{\theta_0}{1+\frac{1}{k}\frac{(1-\theta_0)p_0 + \theta_1(1-p_0)}{\theta_0 p_0 + (1-p_0)(1-\theta_1)}}\right],
\end{aligned}
$$

$$
\begin{aligned}
P(\omega_l \mid m=0) &= P(\omega_l \mid m=0, s=1)P(s=1) + P(\omega_l \mid m=0, s=0)P(s=0) \\
&= \frac{k\mu_d'}{P(m=0)}P(s=0) \\
&= \frac{k\mu_d'}{P(y=1) + kP(s=0)}P(s=0) \\
&= \frac{kp_0\theta_0}{p_0\theta_0 + (1-p_0)(1-\theta_1)} \frac{\theta_0 p_0 + (1-p_0)(1-\theta_1)}{(1-\theta_0)p_0 + \theta_1(1-p_0) + k[\theta_0 p_0 + (1-p_0)(1-\theta_1)]} \\
&= \frac{p_0}{p_0\theta_0 + (1-p_0)(1-\theta_1)} \frac{\theta_0}{1+\frac{1}{k}\frac{(1-\theta_0)p_0 + \theta_1(1-p_0)}{\theta_0 p_0 + (1-p_0)(1-\theta_1)}},
\end{aligned}
$$

$$P(\omega_{nl} \mid m=1) = 1 - P(\omega_l \mid m=1),$$

$$P(\omega_{nl} \mid m=0) = 1 - P(\omega_l \mid m=0).$$

Thus, we can see that $P(\omega_l \mid m=1)$ increases with k, whereas $P(\omega_l \mid m=0)$ increases with k only when $\theta_0 > 0.5$. (Denote $x = k\frac{\theta_0 p_0 + (1-p_0)(1-\theta_1)}{(1-\theta_0)p_0 + \theta_1(1-p_0)}$. Then $\frac{d}{dm}(\frac{1-\theta_0}{1+m} + \frac{\theta_0}{1+1/m}) = \frac{2\theta_0 - 1}{(m+1)^2}$.)

L's signal given his strategy and given the imperfect information, is the following:

$$
\begin{aligned}
P(m=1 \mid \omega_l) &= \frac{P(\omega_l \mid m=1)P(m=1)}{p_0} = \frac{\mu_d P(s=1) + k\mu_d' P(s=0)}{p_0} \\
&= \frac{(1-\theta_0)}{p_0(1-\theta_0)+(1-p_0)\theta_1}[(1-\theta_0)p_0 + \theta_1(1-p_0)] \\
&+ \frac{k\theta_0}{p_0\theta_0 + (1-p_0)(1-\theta_1)}[\theta_0 p_0 + (1-p_0)(1-\theta_1)] \\
&= 1-\theta_0 + k\theta_0, \\
P(m=1 \mid \omega_{nl}) &= \frac{P(\omega_{nl} \mid m=1)P(m=1)}{1-p_0} = \frac{P(m=1) - \mu_d P(s=1) - k\mu_d' P(s=0)}{1-p_0} \\
&= \frac{P(s=1) + kP(s=0) - \mu_d P(s=1) - k\mu_d' P(s=0)}{1-p_0} \\
&= \frac{P(s=1)(1-\mu_d) - kP(s=0)(1-\mu_d')}{1-p_0} \\
&= \frac{[(1-\theta_0)p_0 + \theta_1(1-p_0)]\left[\frac{(1-p_0)\theta_1}{p_0(1-\theta_0)+(1-p_0)\theta_1}\right]}{1-p_0} \\
&- \frac{[\theta_0 p_0 + (1-p_0)(1-\theta_1)]\left[\frac{k(1-p_0)(1-\theta_1)}{p_0\theta_0+(1-p_0)(1-\theta_1)}\right]}{1-p_0} \\
&= \theta_1 - k(1-\theta_1),
\end{aligned}
$$

$$
\begin{aligned}
P(m=0 \mid \omega_l) &= 1 - P(m=1 \mid \omega_l) = (1-k)\theta_0, \\
P(m=0 \mid \omega_{nl}) &= 1 - P(m=1 \mid \omega_{nl}) = (1+k)(1-\theta_1).
\end{aligned}
$$

From the above, we have now obtained L's persuasion signaling strategy as a function of his information. Next, we solve for k in the signaling game. The assumptions regarding the parameters in the models are as follows: $0 < \theta_1, \theta_0 < 0.5$, $k < \frac{\theta_1}{1-\theta_1} < 1$.

B.4.2 L's Solicitation When $z = 0$

Before receiving L's signal, V's prior belief that D is liable is p_s'.

$$
p_s' = P(\omega_l | z=0) = \frac{\beta_0 p_0}{\beta_0 p_0 + (1-p_0)(1-\beta_1)}.
$$

L can cause V's posterior belief to be the following:

$$\mu_v(\omega_l \mid m=1) = \frac{P(m=1 \mid \omega_l)p'_s}{P(m=1)} = \frac{(1-\theta_0+k\theta_0)p'_s}{(1-\theta_0)p_0+\theta_1(1-p_0)+k[\theta_0 p_0+(1-p_0)(1-\theta_1)]},$$
$$\mu_v(\omega_l \mid m=0) = \frac{P(m=0 \mid \omega_l)p'_s}{P(m=0)} = \frac{\theta_0 p'_s}{1-\theta_1-p_0(1-\theta_0-\theta_1)}.$$

We want to find out how μ_v changes with k. Thus, we take the following derivatives:

$$\frac{\partial \mu_v(\omega_l \mid m=1)}{\partial k} = \frac{(1-p_0)p'_s(\theta_0+\theta_1-1)}{(...)^2} < 0,$$
$$\frac{\partial \mu_v(\omega_l \mid m=0)}{\partial k} = 0.$$

If k increase (L sends more of $m=1$ when he receives $s=0$), V's posterior belief of D being liable after receiving $m=1$ decreases. However, V's posterior belief of D being liable after receiving $m=0$ is not affected by the change of k. Comparing this situation with the result found in Appendix B, we can see that k affects V's posterior belief in the same way.

V files a case when $m=1$. Thus, the number of cases filed is

$$P(m=1) = (1-\theta_0)p_0+\theta_1(1-p_0)+k[\theta_0 p_0+(1-p_0)(1-\theta_1)].$$

The probability that D is liable in the cases filed is the following:

$$\bar{p_v} = \mu_v(\omega_l \mid m=1) = \frac{(1-\theta_0+k\theta_0)p'_s}{(1-\theta_0)p_0+\theta_1(1-p_0)+k[\theta_0 p_0+(1-p_0)(1-\theta_1)]}$$
$$= \frac{(1-\theta_0+k\theta_0)}{(1-\theta_0)p_0+\theta_1(1-p_0)+k[\theta_0 p_0+(1-p_0)(1-\theta_1)]}\frac{\beta_0 p_0}{\beta_0 p_0+(1-p_0)(1-\beta_1)}$$

To solve for k, we identify the equilibrium in the game.

B.4.2.1 Pooling Equilibrium when $\xi d > c_v + f + f_0$

V will only hire L when $\xi d > c_v + f + f_0$. This is because when $\xi d \leq c_v + f + f_0$, D will offer 0 in a settlement, and the highest possible amount V would get from a trial is $\xi d - c_v - f - f_0 < 0$.

B.4.2.2 Separating Equilibrium with Randomization

When $\xi d > c_v + f + f_0$, there is a separating equilibrium in V's strategy:

$$\bar{p_v} = \frac{c_v + f + f_0}{\xi d}.$$

Therefore, V's and D's randomization strategies – x and r, respectively, – are the same as those found in section Appendix B. This also give the value of k as follows:

$$k = \frac{\xi d(1-\theta_0)p'_s - (c_v + f + f_0)[(1-\theta_0)p_0 + \theta_1(1-p_0)]}{(c_v + f + f_0)[\theta_0 p_0 + (1-p_0)(1-\theta_1)] - \xi d\theta_0 p'_s},$$

$$1 - k = \frac{c_v + f + f_0 - \xi d p'_s}{(c_v + f + f_0)[\theta_0 p_0 + (1-p_0)(1-\theta_1)] - \xi d\theta_0 p'_s}.$$

The wining rate from a trial is the same as that found in section section 3:

$$\frac{x\bar{p_v}}{1 - \bar{p_v} + x\bar{p_v}} = \frac{c_v + f}{\xi d}.$$

The number of cases filed increases by the following:

$$
\begin{aligned}
P(m=1)*P(z=0) &= \{(1-\theta_0)p_0+\theta_1(1-p_0)+k[\theta_0 p_0+(1-p_0)(1-\theta_1)]\}*P(z=0)\\
&= \frac{\xi d p_s'(1-p_0)(1-\theta_0-\theta_1)}{[\theta_0 p_0+(1-p_0)(1-\theta_1)](c_v+f+f_0)-\xi d\theta_0 p_s'}*P(z=0)
\end{aligned}
$$

When $\theta_1=\theta_0=0$, the number of cases filed is $\frac{\xi d p_s'}{c_v+f+f_0}$, which is the case when L is fully informed. The probability of a trial increases by the following:

$$
\begin{aligned}
&P(z=0)*P(m=1)*(1-\bar{p_v}+x\bar{p_v})r\\
&=P(z=0)*P(m=1)*\frac{\xi d-c_v-f-f_0}{d+c_d}.
\end{aligned}
$$

Thus, when L has imperfect information, the winning rate at trial is still $\frac{c_v+f}{\xi d}$. If L is more informed than V (L's signal s is less noisy), L can increase litigation. If L is almost perfectly informed, the equilibrium converges to the perfectly informed L situation discussed in section 3.

B.4.3 Equilibrium when $z=1$ is not Affected

When L receives an imperfect signal, the equilibrium when $z=1$ is not affected, as L does not need to solicit V. The assumption is that after V voluntarily hires L, L becomes fully informed during the discovery stage.

B.5 Extension 2: L's Solicitation When Altruistic

Consider the situation where L internalizes V's utility:

$$U_L = (1-\delta)\pi_L + \delta\pi_V, \quad \delta \in [0,1].$$

As before, according to Proposition 2, L's persuasion signal to V is:

$$P(m=1|\omega_l) = 1,$$

$$P(m=0|\omega_l) = 0,$$

$$P(m=1|\omega_{nl}) = k,$$

$$P(m=0|\omega_{nl} = 1-k, 0 \leq k \leq 1.$$

Under such a signaling strategy, and given that V's prior belief that D is liable is p'_s, V's posterior belief is $\bar{p} = \frac{p'_s}{p'_s + k(1-p'_s)}$.

We consider only the case where $\xi d \geq c_v + f + f_0$, and there is only a separating equilibrium:

1. If not liable, D offers zero settlement, and there is no litigation.
2. If liable, D randomizes between two offers: zero with probability

   $$x = \frac{1-\bar{p}}{\bar{p}} \frac{c_v + f}{\xi d - c_v - f},$$

   and a positive settlement amount,

   $$\sigma^* = \xi d - c_v - f$$

   with probability 1-x.

3. V accepts positive offer σ^*, and rejects zero offers with probability

$$r = \frac{\xi d - c_v - f}{d + c_d}.$$

Therefore, $p_{win} = \frac{x\bar{p}}{1-\bar{p}+x\bar{p}}$.

V's incentive compatibility requires the following:

$$\begin{aligned}
\pi_V &= (\xi d - c_v - f - f_0)\bar{p}(1-x) + (-f_0)(1-\bar{p}+x\bar{p}) \\
&= (\xi d - c_v - f)\bar{p}(1-x) - f_0 \geq 0 \\
&\implies \bar{p} \geq \frac{f_0}{(\xi d - c_v - f)(1-x)} \\
&\implies \bar{p} \geq \frac{c_v + f + f_0}{\xi d}.
\end{aligned}$$

Under such a signaling strategy, L's payoff in equilibrium is:

$$\begin{aligned}
\pi_L &= (p'_s + k(1-p'_s))(f_0 + (1-\bar{p}+x\bar{p})r(f + \frac{x\bar{p}}{1-\bar{p}+x\bar{p}}(1-\xi)d)) \\
&= p'_s f_0 + (1-p'_s)\left(f_0 + \frac{\xi d}{d+c_d}\left(f + \frac{1-\xi}{\xi}(c_v+f)\right)\right)k \\
&= p'_s f_0 + (1-p'_s)\left(f_0 + \frac{d}{d+c_d}\left(f + (1-\xi)c_v\right)\right)k.
\end{aligned}$$

V's payoff in equilibrium for each case filed is:

$$\begin{aligned}
\pi_v &= (\xi d - c_v - f - f_0)\bar{p}(1-x) + (1-\bar{p}+x\bar{p})(1-r)(-f_0) + \\
&\quad (1-\bar{p}+x\bar{p})r\left[\frac{x\bar{p}}{1-\bar{p}+x\bar{p}}(\xi d - c_v - f - f_0) + \frac{1-\bar{p}}{1-\bar{p}+x\bar{p}}(-c_v - f - f_0)\right] \\
&= \xi d\bar{p} - c_v - f - f_0 \\
&= \frac{p'_s \xi d}{p'_s + k(1-p'_s)} - c_v - f - f_0
\end{aligned}$$

Thus, V's expected payoff in equilibrium under L's signaling strategy is:

$$\begin{aligned}\pi_V &= (p'_s + k(1-p'_s))\pi_v \\ &= p'_s\xi d - (c_v + f + f_0)(p'_s + k(1-p'_s)).\end{aligned}$$

Therefore, L's utility is:

$$\begin{aligned}U_L &= (1-\delta)\pi_L + \delta\pi_V \\ &= (1-\delta)\left\{p'_s f_0 + (1-p'_s)\left(f_0 + \frac{d}{d+c_d}(f + (1-\xi)c_v)\right)k\right\} \\ &+ \delta[p'_s\xi d - (c_v + f + f_0)(p'_s + k(1-p'_s))].\end{aligned}$$

By the first order condition,

$$\begin{aligned}\frac{\partial U_L}{\partial k} &= (1-\delta)(1-p'_s)\left(f_0 + \frac{d}{d+c_d}(f + (1-\xi)c_v)\right) - \delta(c_v + f + f_0)(1-p'_s) > 0 \\ &\implies \delta < \delta* = \frac{1}{1+\kappa},\ \kappa = \frac{c_v + f + f_0}{f_0 + \frac{1}{1+c_d/d}(f + (1-\xi)c_v)}.\end{aligned}$$

Therefore, when $\delta < \delta^*$, L's optimal strategy is to choose the highest k possible. Because $\bar{p} = \frac{p'_s}{p'_s + k(1-p'_s)} \geq \frac{c_v+f+f_0}{\xi d}$, $k_{max} = \frac{p'_s}{1-p'_s}\frac{\xi d - c_v - f - f_0}{c_v + f + f_0}$. This is the same choice of k in section 3, when $\delta = 0$.

One the other hand, when $\delta \leq \delta^*$, L's optimal strategy is to choose the lowest k possible, namely, $k = 0$. In such a case, L always truthfully reports the state $x = 0, \bar{p} = 1, p_{win} = 0$. Furthermore, $\pi_V = p'_s(\xi d - c_v - f - f_0)$; $\pi_L = p'_s f_0$. This is the same as the equilibrium where there are two perfectly informed agents, D and V. V files cases when D is liable, and D offers a positive settlement whenever V files cases against him. Therefore there is no trial.

B.6 Extension 3: Persuasion in Litigation

As introduced in Section 5.3, α is the probability that V wins a liable cases at trial; and β is the probability that V wins a non-liable case at trial. J rules a case liable if and only if $\mu(\omega_l) \geq \frac{1}{1+\gamma}$.

B.6.1 Determining α and β

- J maximizes his expected utility. Thus, J will rule in favor of D when

$$Eu(D\ win) \geq Eu(V\ win)$$
$$\Longrightarrow \mu(\omega_{nl}) * u(D\ wins|\omega_{nl}) + \mu(\omega_l) * u(D\ wins|\omega_l)$$
$$\geq \mu(\omega_{nl}) * u(P\ wins|\omega_{nl}) + \mu(\omega_l) * u(P\ wins|\omega_l)$$
$$\Longrightarrow \mu(\omega_l) * (-\gamma) + (1 - \mu(\omega_l)) * 0 \geq \mu(\omega_l) * 0 + (1 - \mu(\omega_l)) * (-1)$$
$$\Longrightarrow \mu(\omega_l) \leq 1/(\gamma + 1).$$

- When J's prior belief $\mu_0(\omega_l) > \mu^*(\omega_l)$, D can send the optimal signal as in Proposition 2 to J, resulting in two beliefs: $\mu_s(\omega_l) = 1,\ \mu_s(\omega_l) = \mu^*(\omega_l)$.

  D loses in the former case, and wins in the latter.

- To be credible ,

$$\sum_{Supp(\tau)} \mu\tau(\mu) = \mu_0$$
$$\Longrightarrow y * 1 + (1 - y) * \mu^*(\omega_l) = \mu_0(\omega_l)$$
$$\Longrightarrow y = \frac{\mu_0(\omega_l) - \mu^*(\omega_l)}{1 - \mu^*(\omega_l)}.$$

  Here, y is the probability that D loses at trial. D wins at the threshold belief, and the number of cases that D wins $(1-y)$ is maximized.

- Further, D loses only when liable, and D wins some of the liable cases,as well as all non-liable cases. In other words, V loses all non-liable cases, and wins some liable cases.

- Therefore, when $\mu_0(\omega_l) > \mu^*(\omega_l) = \frac{1}{\gamma+1}$,

$$\begin{aligned} &\beta = 0, \\ &\alpha = \frac{y}{\mu_0(\omega_l)} = 1 - \frac{1-\mu_0(\omega_l)}{\gamma\mu_0(\omega_l)}. \end{aligned} \tag{B.6.1}$$

  α is determined in the separating equilibrium (see subsection B.2) to be $\frac{1+\gamma}{d/c_v+\gamma}$.

- $\mu_0(\omega_l)$ is the probability of D being liable in a litigated case. When $\mu_0(\omega_l) < \frac{1}{\gamma+1}$, D always wins, and $\alpha = \beta = 0$.

B.6.2 Separating Equilibrium with Randomization

B.6.2.1 Case 1: $0 \leq \frac{c_v}{d} \leq \alpha$, and $p \geq \frac{c_v/d}{\alpha}$; α is determined to be $\frac{1+\gamma}{d/c_v+\gamma}$

- D offers 0 in ω_{nl} with probability 1. In ω_l, D offers 0 with probability x, and offers $\sigma^* = \alpha d - c_v$ with probability $1-x$.

- Offer $\sigma \geq \sigma^*$ is accepted; any offer $\sigma < \sigma^*$ is rejected with probability r.

- V is indifferent regarding the choice between accepting and rejecting when she is offered

0:

$$\begin{aligned}
&(\frac{xp}{xp+(1-p)})(\alpha d - c_v) + (1 - \frac{xp}{xp+(1-p)})(0 - c_v) = 0 \\
&\frac{\alpha xp}{xp+(1-p)} = \frac{c_v}{d} \\
&\implies x = \frac{1-p_s}{p_s}\frac{c_v/d}{\alpha - c_v/d}.
\end{aligned}$$

Liable D is indifferent regarding the choice between offering σ^* and offering 0:

$$\begin{aligned}
&-(\alpha d - c_v) = r(-\alpha d - c_d) + (1-r)0 \\
&\implies \quad r = \frac{1 - c_v/\alpha d}{1 + c_d/\alpha d}.
\end{aligned} \tag{B.6.2}$$

- Non-liable D prefers the offer 0 over the offer of σ^* because:

$$\alpha d - c_v > r c_d$$

- V's posterior belief is the following: $\mu_s(\omega_l|\sigma^*) = 1$, $\mu_s(\omega_1|0) = \frac{xp}{xp+(1-p)}$.
- The prior $\mu_0(\omega_l)$ is determined by the cases that go to court trials: $\mu_0(w_l) = \frac{xp_s}{1-p_s+xp_s} = \frac{c_v/d}{\alpha}$.
- From the results of subsection F.1, $\alpha = 1 - \frac{1-\mu_0(\omega_l)}{\gamma\mu_0(\omega_l)}$. Therefore, $\alpha = \frac{1+\gamma}{d/c_v+\gamma} < 1$.
- Further, $\mu_0(w_l) = \frac{c_v/d}{\alpha} = \frac{1+\gamma c_v/d}{1+\gamma} > \mu^*(\omega_l) = \frac{1}{\gamma+1}$ holds.
- The trial rate is $(1-p+xp)r$, where
- $$x = \frac{1-p_s}{p_s}\frac{1/\gamma + c_v/d}{1-c_v/d}; \; r = \frac{\gamma(1-c_v/d)}{1+\gamma+c_d/c_v+\gamma c_d/d}. \tag{B.6.3}$$

- The restrictions on the parameters are the following:

$$0 \leq x, r \implies 0 \leq \frac{c_v}{d} \leq \alpha \implies c_v < d$$
$$x \leq 1 \implies p \geq \frac{c_v/d}{\alpha} \implies p \geq \frac{1+\gamma c_v/d}{1+\gamma}.$$

B.6.2.2 Case 2

If $c_v > d$ or $p < \frac{1+\gamma c_v/d}{1+\gamma}$, then there is no separating equilibrium. Rather, there is pooling equilibrium where the settlement offer is 0.

B.6.3 Pooling Equilibrium

B.6.3.1 $\sigma = 0$ when $\alpha \leq \frac{c_v}{d}$

This case is equivalent to the case of $c_v > d$, as discussed in F 2.2.

B.6.3.2 $\sigma = 0$ when $p \leq \frac{c_v/d}{\alpha}$ and $0 < \frac{c_v}{d} < \alpha$

This is equivalent to the case when $c_v < d$ and $p < \frac{1+\gamma c_v/d}{1+\gamma}$.

- V accepts the settlement offer of 0 in equilibrium because:

$$p(\alpha d - c_v) + (1-p)(-c_v) \leq 0.$$

- D's and V's equilibrium payoffs are both 0
- V's belief in equilibrium is $p(\omega_l) = p$
- D has no incentive to deviate in either state, as offering 0 is the dominant strategy in both states, ω_l and ω_{nl}.

B.6.3.3 $\sigma = \alpha pd - c_v$ when $\frac{c_v/d}{\alpha} < p < \frac{c_v/d + c_d/d}{\alpha}$ and $0 < \frac{c_v}{d} < \alpha$

- V accepts the settlement σ in equilibrium. V's equilibrium payoff is

  $$\pi_V = \alpha pd - c_v.$$

  V has no incentive to deviate, as the settlement amount would be the same as the expected payoff from a trial.

- V would go to trial if she is offered 0:

  $$p(\alpha d - c_v) + (1-p)(-c_v) > 0.$$

- However, D's payoff would be lower following a trial. For non-liable D:

  $$\alpha pd - c_v < c_d$$

  For liable D, the expected payout in a trial would be even lower.

- Therefore, D would not deviate from offering σ.

B.6.3.4 No pooling equilibria when $p \geq \frac{c_v/d + c_d/d}{\alpha}$ and $\alpha > \frac{c_v}{d}$

- When $p \geq \frac{c_v/d + c_d/d}{\alpha}$,

  $$c_d < p(\alpha d - c_v) + (1-p)(-c_v) < \alpha d + c_d.$$

- V will only accept offers $\sigma \geq p(\alpha - c_v) + (1-p)(-c_v)$; however, D in $\Omega = \omega_{nl}$ would prefer to offer 0 and pay c_d at trial rather than offering σ.

- D in $\Omega = \omega_l$ would prefer to offer σ rather than to offer 0 and incur costs $\alpha d + c_d$ at trial.

B.6.4 Equilibria Selection: Pooling Equilibria when $sigma = \alpha pd - c_v$ hen $\frac{c_v/d}{\alpha} < p < \frac{c_v/d + c_d/d}{\alpha}$ and $0 < \frac{c_v}{d} < \alpha$ Do Not Survive the D1 Criterion

The D1 creition in Cho and Kreps (1987) requires that when there is a type t' who wishes to defect and send message m whenever type t wishes to do so, then the t sends message m, (t, m) is pruned from the game. Formally,

$$D_t = \left\{ \varphi \in MBR(T(m), m) : u^*(t) < \sum_r u(t, m, r)\varphi(r) \right\},$$
$$D_t^0 = \left\{ \varphi \in MBR(T(m), m) : u^*(t) = \sum_r u(t, m, r)\varphi(r). \right\}$$

If for some type t there exists a second type t' with $D_t \cup D_t^0 \subseteq D_{t'}$, then (t, m) may be pruned from the game. Here,

- u^* is the expected payoff in equilibrium; φ is the receiver's mixed best response to m; and $\sum_r u(t, m, r)\varphi(r)$ is the sender's expected deviation payoff given the best response.

- Consider the case where both liable and non-liable D can offer 0 as the settlement. Because $-c_v < 0$ and $\alpha d - c_v > 0$, V plays the mixed strategy of accepting or rejecting when the type is unknown. Let $\varphi = (1-y, y)$ for the probability of (accept, reject)

- For non-liable D, $-\alpha pd + c_v < (1-y) * 0 - yc_d \implies y \leq \frac{\alpha pd - c_v}{c_d}$.

- Therefore, $D_{nl} = [0, \frac{\alpha pd - c_v}{c_d})$; $D_{nl}^0 = [0, \frac{\alpha pd - c_v}{c_d}]$.

- Similarly, for liable D, $D_l = [0, \frac{\alpha pd - c_v}{\alpha d + c_d})$; $D_l^0 = [0, \frac{\alpha pd - c_v}{\alpha d + c_d}]$.

- $D_l \cup D_l^0 \subseteq D_{nl}$. Therefore, liable D is pruned for sending as settlement of 0. That is, $(liable, 0)$ is ruled out.

- Thus, whenever V sees a settlement of 0, V believes that this is from a non-liable D, and she will accept it because $-c_v < 0$. Non-liable D will defect, and the pooling equilibrium will be eliminated.

B.6.5 Summary of Equilibrium in the Two-agent Model When D Persuades J

As obtained earlier, V's trial winning probability for a liable case is α, and for a non-liable case is β, where

$$\alpha = \frac{1+\gamma}{d/c_v + \gamma};\ \beta = 0.$$

The equilibria of the settlement game, when a trial is not true revealing, is as follows.

1. The separating Equilibrium when $p_s \geq \frac{1+\gamma c_v/d}{1+\gamma}$:

   (1) The main case is obtained when $p_s \geq \frac{1+\gamma c_v/d}{1+\gamma}$. In such a case, there is a separating equilibrium where

1 if not liable, D offers 0 settlement and there is no litigation;

2 if liable, D randomizes between two offers: zero with probability

$$x = \frac{1-p_s}{p_s}\frac{c_v/d-\beta}{\alpha-c_v/d} = \frac{1-p_s}{p_s},\frac{1/\gamma+c_v/d}{1-c_v/d},$$

and a positive settlement amount,

$$\sigma^* = \alpha d - c_v = \frac{1+\gamma}{d/c_v+\gamma}d - c_v..$$

with probability $1-x$;

3 V accepts positive offers σ^*, and rejects zero offers with probability

$$r = \frac{1-c_v/\alpha d}{1+c_v/\alpha d} = \frac{\gamma(1-c_v/d)}{1+\gamma+c_d/c_v+\gamma c_d/d}.$$

2. The pooling equilibrium when $p \leq \frac{1+\gamma c_v/d}{1+\gamma}$:

D offers zero settlement, and there is no litigation.

3. The probabilities of having a trial and winning at trial when $c_v < d$:

Under the separating equilibrium of the main case $p_s > \frac{1+\gamma c_v/d}{1+\gamma}$, a trial occurs when V rejects D's zero offers. Such a condition also implies $c_v < d$. The probability of a trial is

$$\begin{aligned}(1-p_s+p_s x)r &= (1-p_s)\frac{\alpha}{\alpha+c_d/d} = (1-p_s)\frac{1+\gamma}{1+\gamma+c_d/c_v+\gamma c_d/d}\\ &= (1-p_s)\frac{1}{1+\frac{c_d/c_v+\gamma c_d/d}{1+\gamma}} = \frac{\beta_1}{1-\beta_0}\frac{1-p_0}{p_0}\frac{1}{1+\frac{c_d/c_v+\gamma c_d/d}{1+\gamma}}.\end{aligned}$$

The probability of winning at trial is determined by the proportion of liable and non-liable cases among cases that go to trial:

$$\frac{xp_s}{1 - p_s + xp_s}\alpha = \frac{c_v}{d}.$$

B.7 Proof of Proposition 2.6.5

Proof. Using the optimal signal, D first sends a signal to cause J to believe $p(\omega_l) = 1$ for some ω_l cases; and to believe $p(\omega_l) = p^* - \varepsilon$, $\varepsilon > 0$ for all ω_{nl} cases and some ω_l cases. Let N denote the total number of cases, and $n'_{1,l}, n'_{1,nl}$ denote the number of liable and non-liable cases in round 1 after D's persuasion, respectively. This process satisfies the following: $p_0 = \frac{n'_{1,l}}{N} + \frac{n'_{1,nl}}{N}(p^* - \varepsilon)$.

L then sends a signal to affect J's belief regarding $p = p^* - \varepsilon$ cases, and causes $p(\omega_l) = 0$ for some ω_{nl} cases, and $p(\omega_l) = p^* + \delta$ ($\delta > 0$) for the remaining mixture of ω_l and ω_{nl} cases. Let $n_{1,l}, n_{1,nl}$ denote the number of liable and non-liable cases in round 1 after L's persuasion, respectively. This process is described as the following: $p^* - \varepsilon = 0 + \frac{n_{1,l}}{n'_{1,nl}}(p^* + \delta)$. Then the number of non-liable cases from the first round is $n_{1,l}$ from the equation, and the number of liable cases from round 1 is $n_{1,nl} = n'_{1,l} + n'_{1,l} - n_{1,nl}$. J's beliefs are $p(\omega_l) = 1$ for cases in $n'_{1,l}$ (i,e., case identified as liable after the first round of D's persuasion); $p(\omega_l) = p^* + \delta$ on $n_{1,l}$ cases (i.e., cases identified as liable after the first round L's persuasion); and $p(\omega_1 = 0)$ for $n'_{1,nl} - n_l$ cases (i.e., case identified as non-liable after the first round L's persuasion). Such beliefs are correct.

In the next round of persuasion, D and L persuade J on the $n_{1,l}$ cases where J's correct prior belief is $p(\omega_l) = p^* + \delta$. The process is the same as that in the first round: D sends a signal to help J recognize some ω_l cases and to have the belief of $p^* - \varepsilon$ in the remaining mixture. Then L sends a signal to help J recognize some ω_{nl} cases and to have belief $p^* + \delta$ in the remaining mixture.

As the process goes on, eventually, the true states for all cases are revealed. $\square$

Appendix C

Proofs for Chapter 3

C.1 Proof of Theorem 3.1

C.1.1 Sufficiency

Take any function $U : \mathcal{M} \to \mathbb{R}$. Let $u : Z \to \mathbb{R}$ be the restriction of U to Z. For any $(x, A) \in \mathcal{D}$, let $\mathcal{M}(x, A) = \{B \in \mathcal{M} : x \in B \subset A\}$. Accordingly,

$$e(x, A) = \max_{B \in \mathcal{M}(x,A)} [u(x) - U(B)] . \tag{C.1.1}$$

By definition, e satisfies H1 and H2, but in general, it can violate H3.

Let $E(x, A) \in \mathcal{M}(x, A)$ be a menu that has the smallest size among all maximizers in (C.1.1). Then

$$e(x, A) = u(x) - U(E(x, A)) \quad \text{and} \quad |E(x, A)| \leq |B| \tag{C.1.2}$$

for all $B \in \mathcal{M}(x, A)$ such that $e(x, A) = u(x) - U(B)$.

Let $\succeq$ be the preference that is represented by U on $\mathcal{M}$. Order is implied.

Lemma 7. *If $\succeq$ satisfies PSB, then U aggregates e. If U aggregates some $h \in \mathcal{H}$ that satisfies H1–H2, then $\succeq$ satisfies PSB, and $e(x, A) \leq h(x, A)$ for all $(x, A) \in \mathcal{D}$.*

Proof. Assume PSB. Take any $A \in \mathcal{M}$. Then

$$U(A) \geq \max_{x \in A}[u(x) - e(x, A)] \tag{C.1.3}$$

because for all $x \in A$, $e(x, A) \geq u(x) - U(A)$ by (C.1.1) with $B = A$. Suppose that (C.1.3) holds strictly. Then for all $x \in A$,

$$U(A) > u(x) - e(x, A) = u(x) - (u(x) - U(E(x, A))) = U(E(x, A)).$$

By PSB, $A \succ \bigcup_{x \in A} E(x, A) = A$. By contradiction, (C.1.3) holds as equality. Thus U aggregates e.

Conversely, suppose that U aggregates some $h \in \mathcal{H}$ that satisfies H1–H2. Take any $A, B \in \mathcal{M}$. As U aggregates h, then $U(A \cup B) = u(z) - h(z, A \cup B)$ for some $z \in A \cup B$. If $z \in A$, then by H2,

$$U(A) \geq u(z) - h(z, A) \geq u(z) - h(z, A \cup B) = U(A \cup B).$$

Similarly, if $z \in B$ then $U(B) \geq U(A \cup B)$. Thus PSB holds. Take any $x \in A$. Then for all $C \in \mathcal{M}(x, A)$,

$$U(C) \geq u(x) - h(x, C) \geq u(x) - h(x, A)$$

and hence, $e(x, A) \leq h(x, A)$. □

Lemma 7 characterizes all preferences $\succeq$ that can be represented by an aggregation of some monotonic cost function $h \in \mathcal{H}$. Moreover, the endogenous e is the minimal cost function that allows such aggregation for the given U.

Lemma 8. *If $\succeq$ satisfies Axioms 1–3, then for any $(x, A) \in \mathcal{D}$ and $B \in \mathcal{M}$,*

$$A_x \supset E(x, A) \setminus x \tag{C.1.4}$$

$$x \in B \text{ and } B_x = A_x \quad \Rightarrow \quad e(x, A) = e(x, B) \tag{C.1.5}$$

$$x \text{ is costly in } A \quad \Leftrightarrow \quad e(x, A) > 0. \tag{C.1.6}$$

Proof. Take any $(x, A) \in \mathcal{D}$. Recall that $A_x = \{y \in A : x \succ y\}$.

Take any $y \in E(x, A)$. Let y maximize u in $E(x, A)$. Suppose that $y \neq x$. Let $C = E(x, A) \setminus y$. As $C \in \mathcal{M}(x, A)$ is smaller than $E(x, A)$, then by (C.1.2),

$$u(x) - U(C) < e(x, A) = u(x) - U(E(x, A)).$$

Thus $C \succ E(x, A)$, which violates Dominance. Therefore, $y = x$ is the only maximizer of u in $E(x, A)$, that is, $A_x \supset E(x, A) \setminus x$.

Take any $B \in \mathcal{M}$ such that $x \in B$ and $B_x = A_x$. By (C.1.4), $E(x, B) \subset A$ and $E(x, A) \subset B$. Thus $e(x, A) = e(x, B)$.

Let x be costly in A, that is, $x \succ x \cup A_x$. Then

$$e(x, A) \geq u(x) - U(x \cup A_x) > 0.$$

Conversely, let $e(x, A) > 0$. Then $x \succ E(x, A)$. By PSB,

$$x \succ E(x, A) \cup \bigcup_{y \in A_x} y = E(x, A) \cup A_x.$$

By (C.1.4), $E(x, A) \cup A_x = x \cup A_x$. By definition, x is costly in A. □

A positive function $U : \mathcal{M} \to \mathbb{R}_{++}$ is called regular if for all $x \in X$ and $A \in \mathcal{M}$,

$$u(x) > U(A) \iff U(x) \geq 2U(A).$$

Lemma 9. *If $\succeq$ satisfies Axioms 1-4 and U is regular, then $e \in \mathcal{H}$ is selective and aggregated by U.*

Proof. Suppose that $\succeq$ satisfies Axioms 1–4, and is represented by a regular function $U : \mathcal{M} \to \Re_{++}$. By definition, for all $x \in \mathbb{Z}$ and $A \in \mathcal{M}$,

$$x \succ A \quad \Rightarrow \quad U(x) \geq 2U(A). \tag{C.1.7}$$

By Lemma 7, U aggregates e. Show that e is selective. H1 and H2 follow from definition (C.1.1). Prove H3. Take any $x, y \in \mathbb{Z}$ and $A \in \mathcal{M}$ such that $x \in A$. If $e(x, y \cup A) = 0$ or $e(y, y \cup A) = 0$, then H3 is trivial.

Let $e(x, y \cup A) > 0$ and $e(y, y \cup A) > 0$. By Lemma 8, both x and y are costly in $y \cup A$. By Reduction, x is costly in A. If $y \succeq x$, then by (C.1.4), $E(x, y \cup A) \subset A$. Thus

$$e(x, A) \geq u(x) - U(E(x, y \cup A)) = e(x, y \cup A).$$

Suppose that $x \succ y$. As $e(x, A) > 0$, then $x \succ E(x, A)$ and by (C.1.7),

$$e(x, A) = u(x) - U(E(x, A)) \geq \tfrac{1}{2}u(x). \tag{C.1.8}$$

As $x \succ y$, then by (C.1.7), $u(y) \leq \frac{1}{2}u(x)$. Therefore,

$$e(y, y \cup A) = u(y) - U(E(y, y \cup A)) < u(y) \leq \tfrac{1}{2}u(x) \leq e(x, A).$$

Thus the function e satisfies H3, and hence, e is selective. □

Suppose that $\succeq$ satisfies Axioms 1–4. As $\mathcal{M}$ is finite, then Order implies that $\succeq$ has a utility representation $V : \mathcal{M} \to \mathbb{N}$ with natural values. Let $U = 2^V$. Then U is regular. By Lemma 7, U aggregates e. By Lemma 9, e is selective.

C.1.2 Necessity

Suppose that $\succeq$ is represented for all $A \in \mathcal{M}$ by

$$U(A) = \max_{x \in A}[u(x) - h(x, A)] \tag{C.1.9}$$

where $u \in \mathbb{R}^Z$ and $h \in \mathcal{H}$ is a selective cost function. Show Axioms 1–4. Order is obvious. PSB follows from Lemma 7. Show Dominance. Take any $A \in \mathcal{M}$ and $y \in Z$. By (C.1.9), $U(A) = u(x) - h(x, A)$ for some $x \in A$. If $y \succeq x$, then $u(y) \geq u(x)$ and by H3,

$$U(y \cup A) \geq \max\{u(y) - h(y, y \cup A), u(x) - h(x, x \cup A)\} \geq u(x) - h(x, A) = u(A).$$

Thus Dominance holds. Show Reduction. Take any $A, B \in \mathcal{M}$, $x \in A$, $y \in Z$ such that both x and y are costly in $y \cup A$. If $y \succeq x$, then $A_x = (y \cup A)_x$ and Reduction is trivial. Let $x \succ y$. Then $(y \cup A)_x = y \cup A_x$. Assume that $h(x, x \cup A_x) = 0$. By H3, either $h(x, x \cup y \cup A_x) = 0$

or $h(y, x \cup y \cup A_x) = 0$. By H2 and (C.1.9),

$$h(x, x \cup y \cup A_x) = 0 \quad \Rightarrow \quad x \cup y \cup A_x \succeq x,$$
$$h(y, x \cup y \cup A_x) = 0 \quad \Rightarrow \quad h(y, y \cup A_y) = 0 \quad \Rightarrow \quad y \cup A_y \succeq y.$$

However, $x \succ s \cup y \cup A_x$ and $y \succ y \cup A_y$ because both x and y are costly in $y \cup A$, By contradiction, $h(x, x \cup A_x) > 0$. Thus $u(x) > U(x \cup A_x)$, and x is costly in A.

C.2 Proof of Theorem 3.2

Proof. Notice that if $\succeq$ satisfies CV, then it also satisfies Dominance and Reduction.

Show Dominance. If $y \succeq x$ for all $x \in A$, then by PSB, $y \succeq A'$ for all $A' \subseteq A$. Suppose $y \succ A'$ for some $A' \subseteq A$. Then CV implies that $A \cup y \sim A$. Suppose $y \sim A'$ for all $A' \subseteq A$. Suppose $A \succ y \cup A$. Then by assumption, $y \sim A \succ y \cup A$. By CV, $y \cup A \sim A$, contradiction. So $y \cup A \succeq A$.

Show Reduction. Suppose $x \succ y$. Then $(A \cup y)_x = A_x \cup y$, $(A \cup y)_y = A_y$, and $A_y \subseteq A_x$. By CV, $y \succ (A \cup y)_y = A_y$ implies $x \cup y \cup A_x \sim x \cup A_x$. So $x \succ x \cup (A \cup y)_x$ implies $x \succ x \cup A_x$. Suppose $y \succeq x$. Then $(A \cup y)_x = A_x$. So if $x \succ x \cup (A \cup y)_x$, then $x \succ x \cup A_x$.

Therefore, by Theorem 1, if $\succ$ satisfies Order, PSB, and CV, then it has utility representation $U : \mathcal{M} \to \mathbb{R}$ that aggregates some selective function $h \in \mathcal{H}$.

Next, we show that $e(x, A) = 0$ for all $(x, A) \in \mathcal{D}$, $e(x, A) = 0$. Consider the set

$$\varphi(A) = \{c \in A : c \cup A' \succeq c \text{ for all menus } A' \subseteq A\}.$$

Lemma 10. *$y \notin \varphi(A) \implies y \succ B' \cup y$ for all B' such that $B \subseteq B' \subseteq A_y$, where $B \subseteq A_y$.*

Proof of Lemma. Take $B \subseteq A$ to be the minimum subset such that $y \succ B \cup y$. Suppose there is $x \in B$ such that $x \succeq y$. Then $x \succeq y \succ y \cup B$. By CV, $y \cup B \sim y \cup B \setminus x$. Then $y \succ B \setminus x$, contradict with minimality of B. So $B \subseteq A_x$.

Since $y \succ z$ for $z \in A_x \setminus B$, then by PSB, $y \succ B \cup Z$. By induction, $y \succ B'$ for all B' such that $B \subseteq B' \subseteq A_x$. □

Let R^- represent $-u$. By Lemma 10, take $x = \max R^-(A)$, then $x \in \varphi(A)$. So $\varphi(A)$ is not empty.

Let $x \notin \varphi(A)$. Then $x \succ x \cup B$ for some $A' \subseteq A$. By H2, $u(x) > u(A')$. By Lemma 10, $x \succ A_x \cup x$. By CV, for all B such that $A_x \cup x \subseteq B \subseteq A$, $U(B \setminus x) = U(B)$. Therefore, $U(A) = U(A \setminus x)$. Repeat this process, we obtain $U(A) = U(\varphi(A))$. Therefore, $(x, A) \subseteq \mathcal{D} \iff x \in \varphi(A)$. For $x \notin \varphi(A)$, $h(x, A) = \infty$.

Let $y \in \varphi(A)$. Then $u(y) \leq U(B)$ for all $B \in \mathcal{M}(y, A)$. $\{y\} \in \mathcal{M}(y, A)$. Hence $e(y, A) = 0$ where $e(y, A)$ is defined as in (22).

Lemma 11. *φ satisfies*

1. *Sen's α: $A \subseteq B \implies \varphi(B) \cap A \subseteq \varphi(A)$;*
2. *Aizerman: $\varphi(B) \subseteq A \subseteq B \implies \varphi(A) \subseteq B$.*

Proof of Lemma 11. Sen's α follows from the definition of φ. Indeed, if $x \in \varphi(B) \cap A$, then $x \cup A' \succeq x$ for all $A' \subseteq B$. So $x \cup A' \succeq A'$ for all $A' \subseteq A \subseteq B$.

Show Aizerman. It is sufficient to show the case when $A = B \setminus x$. Suppose $y \notin \varphi(B)$. Then $y \succ y \cup B_y$ by Lemma 10. If $x \notin B_y$, then $y \notin \varphi(B \setminus x)$. Suppose $x \in B'$. Then $y \succ x$ and thus $y \notin B_x \subseteq B_y$. By assumption $x \notin \varphi(B)$; so $x \succ B_x \cup x$ by Lemma 10. By CV, $B_y \cup y \sim (B_y \setminus x) \cup y$. Since $y \succ B_y \cup y$, $y \succ (B_y \setminus x) \cup y$. Since $B_y \setminus x \subseteq B \setminus x$, $y \notin \varphi(B \setminus x)$.

We have proved that $y \notin \varphi(B) \implies y \notin \varphi(B \setminus x)$, that is, $\varphi(B \setminus x) \subseteq \varphi(B)$. $\square$

By Aizerman and Malishevski (1981) (see Theorem 5 in Moulin (1985b)), there is $\Theta \subseteq \mathbb{R}$ such that

$$\varphi(A) = \cup_{R \in \Theta} R(A).$$

Hence,

$$U(A) = u(\Theta(A)) = \cup_{R \in \Theta} R(A).$$

The above utility representation satisfies Order and PSB by Theorem 1. Show that it satisfies CV. Indeed, if $x \succ A$, then $x \notin R(A)$ for any $R \in \Theta$. Thence $\varphi(A) = \cup_{R \in \Theta} R(A) = \cup_{R \in \Theta} R(A \setminus x) = \varphi(A \setminus x)$. So CV holds.

Next we show that (21) holds. Take $R^d = R^-$ such that $x \succ y$ implies yR^-x. Let $R_+(x)$ be the upper contour set

$$R_+(x) = \{z \in Z : zRx\}.$$

Let $\Pi_+ = \{R_+(x) : R \in \Theta, x \in Z\} \subseteq \mathcal{M}$ to be determined by Θ, where Θ is determined by φ. Let $\Pi = \Pi_+ \cup Z$. Let

$$V(A) = \max_{C \in \Pi, A \cap C \neq \emptyset} u(R^-(A \cap C)).$$

For any $R \in \Theta$, $R(A) = A \cap R_+(R(A))$ where $R_+(R(A)) \in \Pi$. Then $V(A) \geq u(R(A))$, and therefore $V(A) \geq U(A)$. Take any $C \in \Pi$ such that $A \cap C \neq \emptyset$. As $C \in \Pi$, $C = R_+(x)$ for some $R \in \Theta$ and $x \in Z$. So $u(R^-(A \cap C)) \leq u(R(A))$. So $V(A) \leq \max_{R \in \Theta} u(R(A)) = U(A)$. So $V(A) = U(A)$. Hence (21) holds.

$\square$

C.3 Proof of Theorem 3.3

Proof. Show the necessity of condition (3.4.3). First we take any $(a, A) \in \mathcal{D}$. Because R_0 is *acceptable*, $a \in R_0(\Theta(A))$, where $\Theta(A) = \cup_{R\in\Theta} R(A)$, $\Theta \subseteq \mathcal{R}$. WLOG we can take $\Theta \subseteq \mathcal{T}$. Thus, there is $R_1 \in \Theta$, such that $a = R_1(A)$. If $A = B$, clearly, $a \notin B \setminus a \cup \mathcal{N}(B)$. We claim that for any $B \in \mathcal{M}$ such that $A \subsetneq B$, we have $R_1(B) \notin A \setminus a$. Indeed, as $A \subsetneq B$, $R_1(B)R_1a$ or $R_1(B) = a$. If $R_1(B) = a$, then clearly $R_1(B) \notin A \setminus a$. If $R_1(B)R_1a$ and $R_1(B) \in A \setminus a$, then $aR_1R_1(B)R_1a$. This contradicts the antisymmetric property of $R_1 \in \mathcal{T}$. Next, we show that $R_1(B) \notin \mathcal{N}(B)$. For the sake of contradiction, suppose $R_1(B) \in \mathcal{N}(B)$. By the definition of $\mathcal{N}(B)$, there is $(b', B') \in \mathcal{D}$, such that $\{R_1(B), b'\} \subseteq B' \subseteq B$ and $R_1(B)P_0b'$. This implies that $R_1(B) \notin \varphi(B')$. By the property α, $R_1(B) \notin \varphi(B)$. This contradicts $\varphi(B) = \Theta(B) = \cup_{R\in\Theta}(R(B))$.

Show the sufficiency of condition (3.4.3). Let $|\mathcal{D}| = M, |Z| = N$. We construct a set of linear orders $\Theta \subseteq \mathcal{T}$ and show that it makes R_0 acceptable. To constitute this Θ, we construct an R_i using Algorithm 1 for each $(a_i, A_i), i \in \{1, ..., M\}$, and take $\Theta = \cup_{i=1}^{M} R_i$. For each R_i, the algorithm stops at some set S_i, where no menu in $\mathcal{M}$ is a subset of S_i. Although S_i is not ranked by R_i, we can extend R_i on S_i in basically any fashion, since the data puts no restriction on the ranking of elements in S_i. Although $R_i \in \mathcal{T}$ is a linear order for all $i = 1, ..., M$, R_0 can be a weak order $R_0 \in \mathcal{R}$. Suppose there are two observations for one menu: (a_i, A_i) and (a_j, A_i). Then $R_i, R_j \in \Theta$ such that $R_i(A_i) = a_i$ and $R_j(A_i) = a_j$ implies that $a_i, a_j \in \Theta(A)$, and thus $a_i, a_j \in R_0(\Theta(A))$ is possible.

We next show that $\Theta = \cup_{i=1}^{M} R_i$ makes R_0 acceptable. We proceed in two steps. First, we demonstrate that for any $(a_i, A_i) \in R_i(A_i)$, $a_i = R_i(A_i)$. Next, we show that for any $m, i = 1, ..., M$, $R_m(A_i) \notin \mathcal{N}(A_i)$. This implies that $a_iR_0R_m(A_i)$ for all $R_m \in \Theta, m \neq i$. Therefore $a_i \in R_0(\Theta(A_i))$.

To show the two claims above, we introduce the following two notations for each $R_i, i =$

Algorithm 1: Start from any observation (a_i, A_i); find a ranking $R_i \in \mathcal{T}$ such that $a_i = R_i(A_i)$ and $R_i(A_m) \notin \mathcal{N}(A_m)$ for all $A_m \in \mathcal{M}$.

Result: R_i

Take one observation (a_i, A_i). Initialize $R_i = [\,]$, $B = Z$, $j = 1$, $a_0 = [\,]$, $a = a_i$, $A = A_i$;

while *There exists* $A' \in \mathcal{M}$ *such that* $A' \subsetneq B$ **do**

  define $\mathcal{N}(B) = \{b \in B : bP_0a'$ for some $(a', A') \in \mathcal{D},$ such that $A' \subseteq B\}$;

  if $B \setminus (\mathcal{N}(B) \cup A) \neq \emptyset$*;*

   then

    take any element in $B \setminus (\mathcal{N}(B) \cup A)$ and call it a_j;

    $B = B \setminus a_j$;

    R_i.append(a_j);

    $j = j + 1$;

  end

  Assign $a_j = a$;

  $B = B \setminus a_j$;

  R_i.append(a_j);

  $j = j + 1$;

  if *there exists* $A' \in \mathcal{M}$ *such that* $A' \subsetneq B$ **then**

    take any such A', and let $A = A'$, $a = a'$;

  end

end

Return($R_i = [a_1 a_2 .... a_j]$).

$1, ..., M$:

(1)Denote the elements in Z as $a_i^1, ..., a_i^N$, where a_i^k indicates that this element is ranked k^{th} by R_i.

(2)Denote $B_i^j = Z \setminus \cup_{l=1}^j a_i^j$.

Show that for any $(a_i, A_i) \in \mathcal{D}$, $a_i = R_i(A_i)$. Indeed, suppose that $a_i = a_i^k$. If $k = 1$, then we are done. If $k > 1$, then by Algorithm 1, $a_i^j \in Z \setminus (\mathcal{N}(Z) \cup A_i)$ for $j = 1, ..., k-1$. Therefore $a_i^j \notin A_i \setminus a_i$, which implies $A_i \setminus a_i \subset B_i^k$. The construction of R_i implies $a_i^k \gg_i B_i^k$, and thus $a_i \gg_i A_i \setminus a_i$. Therefore, $a_i \in R_i(A_i)$.

Show that for any $m, i = 1, ..., M$, $R_m(A_i) \notin \mathcal{N}(A_i)$. Since $m, i = 1, ..., M$ are arbitrary, if we show that $R_i(A_m) \notin \mathcal{N}(A_m)$ then we can conclude that $R_m(A_i) \notin \mathcal{N}(A_i)$. For simplicity in notation, we show $R_i(A_m) \notin \mathcal{N}(A_m)$. Take (a_i, A_i), $i \in \{1, ..., M\}$. By Algorithm 1, R_i ranks all elements in $Z \setminus S_i$. Let $|Z \setminus S_i| = r$. Suppose $R_i(A_m) = a_i^k$. If $k = 1$, then $R_i(A_m) = a_i^1 \in Z \setminus (\mathcal{N}(Z) \cup [A_i \setminus a_i])$. Because $\mathcal{N}(A_m) \subseteq \mathcal{N}(Z)$ by definition of $\mathcal{N}(\cdot)$, $R_i(A_m) \notin \mathcal{N}(A_m)$. Suppose $2 \leq k \leq r$. Then, $a_i^1, .., a_i^{k-1} \notin A_m$, and therefore, $A_m \subseteq B_i^{k-1}$. Take $B \supseteq B_i^{k-1}$ to be the smallest B in the algorithm that includes B_i^{k-1}. Then, by construction, $a_i^k \in B \setminus (\mathcal{N}(B)) \cup [A_j \setminus a_j])$ for some $j \in \{1, .., M\}$. Therefore, $a_i^k \notin \mathcal{N}(B)$. As $A_m \subseteq B_i^{k-1} \subseteq B$, by definition of $\mathcal{N}(\cdot)$, $\mathcal{N}(A_m) \subseteq \mathcal{N}(B_i^{k-1}) \subseteq \mathcal{N}(B)$. Therefore, $a_i^k \notin \mathcal{N}(A_m)$. The case $r < k \leq N$ is not possible. Indeed, a_i^k is the first element in A_m ranked by R_i. $k > r$ means that A_m is not ranked by R_i, and thus, $A_m \subseteq S_i$. This contradicts the fact that S_i has no subsets in $\mathcal{M}$. We thus have proved that $R_i(A_m) \notin \mathcal{N}(A_m)$ for any $i \in \{1, ..., M\}$.

Check time complexity. Algorithm 2 checks whether condition (3.4.3) holds for all $(a, A) \in \mathcal{D}$ and $B \in \mathcal{M}$, such that $A \subseteq B$. For the finite data $|\mathcal{D}| = M$, enumerate all the elements $\mathcal{D} = \{(a_1, A_1), ..., (a_M, A_M)\}$.

Algorithm 2: Check if $B \neq \mathcal{N}(B) \cup [A \setminus a]$ holds for (a, A), where $A \subseteq B \subseteq Z$. $\mathcal{M} \subseteq P(Z)$, $|\mathcal{M}| = M$, $|Z| = N$.

```
Result: Yes or no
Take any B ∈ M. index = 0;
for i = 1 : M do
    while index = 0 do
        if A_i ⊆ B;
        then
            if |B| = |N(B) ∪ [A_i \ a_i]|;
            then
                index = 1
            end
        end
    end
end
if index = 0 then
    Return(Yes);
else
    Return(No)
end
```

Notice that because $A \subseteq B$, $\mathcal{N}(B) \cup [A \setminus a] \subseteq B$. Thus, the equality for condition (3.4.3) holds only when B and $\mathcal{N}(B) \cup [A \setminus a]$ are the same size. Therefore, checking whether the equality holds takes only $O(1)$. Checking whether $A_i \subseteq B$ can be achieved with time complexity $O(M + N)$ by using a hash table. In worst case scenario, there are be M menus, $A_1, ..., A_M$, to check. Thus, in total, Algorithm 2 runs with time complexity $O(M(M+N))$. When the number of observations, M, is much greater than N, then the time complexity of Algorithm 2 is $O(M^2)$. The time complexity of Algorithm 2 is $O(M^2)$. To run Algorithm 2 on all M menus, the total time complexity is $O(M^3)$.

□