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Los Angeles

# Essays on the Economics of Innovation

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Economics

by

Minji Kang

2015

ABSTRACT OF THE DISSERTATION

Essays on the Economics of Innovation

by

Minji Kang

Doctor of Philosophy in Economics

University of California, Los Angeles, 2015

Professor Hugo Hopenhayn, Chair

This collection of essays considers three cases where two firms innovate in an imperfectly competitive market. In the first chapter, I introduce a novel model that represents the vertically differentiated industry, wherein the leader and the follower innovate continuously. I uncover the determinants of the innovation of firms, focusing on the ratio between the quality levels of their products, a key factor in deciding to innovate. Firms innovate to receive more profits from higher quality. Moreover, they choose the effort of innovation to widen the distance between the quality of two products. Consequently, firms tend to keep the distance of the quality constant. The second chapter extends the model in the first chapter, adding radical improvement to the quality by innovation, leapfrogging. I show that two firms are likely to innovate

actively with the possibility of leapfrogging. The momentum of leapfrogging is powerful when the quality of two firms is similar to each other and when the possible size of the jump from the innovation is larger. Finally, in the last chapter, I introduce the model with the dynamic innovation of firms, which produces complementary goods: the platform and the software. This model focuses on the software innovation induced by the innovation of the platform. I uncover the determinants of innovation for the platform firm and the software firm as well as the interdependence of the two firms' innovation choices.

The dissertation of Minji Kang is approved.

Sushil Bikhchandani

Simon Board

Connan Snider

Hugo Hopenhayn, Committee Chair

University of California, Los Angeles

2015

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# 1 Dynamic Competition in the Vertically Differentiated Market

## 1.1 Introduction

Do firms always want to have higher quality products when they compete within vertical differentiation? While a firm's innovation improves the products' quality, a determinant of the firm's profits, such profits are also affected by the quality of a rival's product. In a vertically differentiated market, firms' profits depend on the gap in quality between their products and those of other firms. This quality gap can be regarded as a measure of competition. When the gap is wide, products are much different from each other, so firms do not care very much about competition. On the other hand, when the gap is narrow and the two products are similar, the firms that produce them are in harsh competition. Therefore, a firm's incentives to innovate necessarily relate to how competitive the market is. In other words, the quality gap between firms' products highly affects firms' motivation to innovate.

In this paper, I study the dynamic innovation of two firms in a vertically differentiated market, comparing the different motivations of the market leader and the follower, which hinge on the quality levels of the two firms. In particular, this study captures two main phenomena of vertically differentiated industries. First, some industries, such as the CPU market, show a persistent gap between the leader and the follower in the quality of products. In detail, Eizenberg (2008) and Nosko (2010) have found that the demand for CPUs in PCs is largely segmented. Moreover, the degree of competition plays an important role in the decisions of firms in terms of innovation. Second, empirical studies of patenting, such as Geroski et al. (1996) and Cerfis and Orsenigo (2000), have also shown a persistent gap in patenting. Such studies have indicated that the market leader patents more than the follower. The number of patents can be interpreted in two ways: the number of patents is the measure of the innovation of quality, but the leading firm in the market tends to patent more to deter the innovation of the follower. In this study, patenting as an investment in deterrence illustrates the motivation for the leader's innovation.

By encompassing these phenomena in vertically differentiated markets, this study uncovers determinants of innovation that are derived from firms' strategic motivations to innovate. Firms generally innovate to achieve higher quality because, with higher quality, firms can charge higher prices. On the other hand, firms want to widen the gap between themselves and their competition. When firms compete via price, they have very small profits with similar quality products. Thus, by adjusting the quality of products, each firm wants to target consumer groups with a different marginal utility of quality. The leader is influenced in the same direction by these determinants, so it always innovates positively. Nonetheless, the determinants affect the follower in the opposite direction; thus, the follower innovates actively when the gap is large and is discouraged to innovate when the gap is small to avoid severe competition. As a result, the market is segmented in equilibrium. However, a segmented market with largely differentiated products, in turn, harms social welfare as it relates to the consumer's surplus because, when firms have products with more differentiation in quality, the prices of the products can be increased, and fewer consumers can buy products from either firm. Furthermore, the leader would be motivated to invest in a deterrence to widen the gap of quality further. Although the investment in deterrence does not increase the quality of goods, the profit of the leader will be increased due to the widened gap of quality by such a deterrence. The investment in deterrence is more harmful to social welfare because it widens the gap between the leader and the follower.

The model I studied analyzes the industry with vertically differentiated products from two firms whose profits depend on the quality of the leading firm's product and the quality ratio, where the quality ratio is defined as the quality of the follower divided by that of the leader. Firms decide on innovation efforts in continuous time, thereby continually affecting quality levels. As a result, firms' innovative efforts depend on the quality ratio of the market leader's product to the follower's product. Such continuity of innovation can show the dynamics of innovative behavior, as the quality ratio keeps changing in continuous time with the innovation of the leader and follower. When the gap between the leader and the follower is narrow, firms decide the quality of their products by innovating to widen the gap between leaders and followers. Firms want to widen the gap because they want to capture loyal consumers and avoid severe price competition from firms producing goods of a similar quality level. On the other hand, when the gap is wide enough, both firms mainly focus on increasing the quality of their products to receive higher profits. As a result, the leader always wants to innovate further, especially when the ratio gets higher. The follower also innovates further when the ratio is low enough, but the follower will not do so when the ratio gets too high.

My paper is informed by economic theories concerning persistent quality gaps in the industry. The hypothesis of "success breeds success" argues that successful innovation positively affects following innovations with cumulative technologies. In particular, Athey and Schmutzler (2001) have explored various market structures in a dynamic innovation model, employing the condition of "increasing dominance" of the leader in dynamic games in which the leader puts more effort into innovation than the follower. Athey and Schmutzler concluded that leaders tend to innovate more than followers. Nonetheless, the study was limited by explaining the role of the gap in quality between leader and follower in terms of firms' innovation. My model uncovers systematic relationships between the quality gap and innovation by focusing on the innovative behavior of the follower as well as that of the leader.

Moreover, studies have used a tug-of-war model wherein a dynamic tournament game with a fixed prize is proposed. Harris and Vickers (1993) is the seminary study of this model. Moscarini and Smith (2007) and Cao (2009) extended the model as continuous time versions. These models conclude that, when the leader is sufficiently ahead, the incentive for the leader to make a great effort drops. On the other hand, the follower is also discouraged if the distance from the leader is large enough. Thus, the strategic incentives to put in more effort also decrease for the follower. This "discouragement effect" is largely due to the fixed prize given at the end of the tournament. In contrast to these fixed-prize models, my model offers some distinctions. First, the firms receive flows of profit in continuous time rather than a fixed prize awarded after the tournament. Second, the profits of firms are not fixed but are highly dependent on the R&D efforts in which they have invested. Furthermore, the flows of profits are proportional to the quality of the leading firm's product and are functions of the ratio of product quality (i.e., follower's quality divided by leader's quality).

The paper proceeds as follows: Section 1.2 introduces the basic model and myopic firms' choices. Section 1.3 provides the dynamic game in continuous time. Section 1.4 extends the model from Section 1.3, dealing with intellectual properties. Section 1.5 concludes the paper.

## 1.2 Basic Set-up

Consider a market with two competing firms that offer vertically differentiated goods with levels of quality: $x_A > x_B$. Firm $A$ is the leader in the industry, while firm B is the follower. Marginal cost of production is zero for both goods.

There is a continuum of consumers that differ in their marginal utility for quality: $\theta\tilde{}F(\theta)$. A consumer's net utility is $U(\theta) = \theta q - p$, where $q$ is the quality of the good and $p$ is the price. For simplicity, assume $\theta$ is uniformly distributed on $[0,1]$. The consumers will buy from firm A if $\theta x_A - p_A \geq \theta x_B - p_B$, and from B if $\theta x_B - p_B \geq \theta x_A - p_A$ and $\theta x_B - p_B \geq 0$. These three inequalities and the assumption of uniform distribution make the demand for firm $A$: $s_A$ =(1-$\frac{p_A - p_B}{x_A - x_B}$) and for firm $B$, $s_B = (\frac{p_A - p_B}{x_A - x_B} - \frac{p_B}{x_B})$.

The price of $x_A$ and $x_B$, $p_A$ and $p_B$ is determined by the firms' profit maximization problem. The two firms maximize profits by choosing price $p_A$ and $p_B$ with the given quality for each firm's product.

$$\begin{aligned} p_A &\in \arg\max \pi_A = [1 - \frac{p_A - p_B}{x_A - x_B}]p_A \\ p_B &\in \arg\max \pi_B = [\frac{p_A - p_B}{x_A - x_B} - \frac{p_B}{x_B}]p_B \end{aligned}$$

When first order conditions are solved simultaneously, we get $p_A = \frac{2x_A(x_A - x_B)}{4x_A - x_B}$, and $p_B = \frac{x_B(x_A - x_B)}{4x_A - x_B}$. Profits per periods of the two firms are determined by the respective quality of the two firms' products.

$$\pi_A = \frac{(2x_A)^2(x_A - x_B)}{(4x_A - x_B)^2}, \pi_B = \frac{x_A x_B(x_A - x_B)}{(4x_A - x_B)^2}$$

$\pi_A$ is always increasing in $x_A$. On the other hand, $\frac{\partial \pi_B}{\partial x_B} = \frac{x_A^2(4x_A - 7x_B)}{(4x_A - x_B)^3}$ is not necessarily greater than zero. That is, firm B does not always want to innovate further; in particular, the sign of $\frac{\partial \pi_B}{\partial x_B}$ is highly dependent on the relative quality levels of the two firms' products.

Relying on this basic set-up, I will first analyze the firms' decisions about quality when they do not care about the future. Let $x_B = ux_A$ where $u(=\frac{x_B}{x_A})$ is the "follow-up" level, or the quality ratio between leader's and follower's products. . Then, $\pi_A$ and $\pi_B$ become multiplicably separable by the function of $u$ and $x_A$. The follow-up level is

$$\begin{aligned}\pi_A(x_A, u) &= \frac{(2x_A)^2(x_A - x_B)}{(4x_A - x_B)^2} = \frac{4x_A(1-u)}{(4-u)^2} = \hat{\pi}_A(u)x_A \\ \pi_B(x_A, u) &= \frac{x_A x_B(x_A - x_B)}{(4x_A - x_B)^2} = \frac{x_A u(1-u)}{(4-u)^2} = \hat{\pi}_B(u)x_A\end{aligned} \tag{1}$$

From the above equations, the following preliminary proposition emerges:

**Proposition 1** *The profits of leaders and followers in a given time period depend on the quality of the product of the leading firm and the ratio of the two products' quality levels.*

1. *The higher $x_A$, the higher the price of $x_A$ and the profits from $x_A$ are when the follow-up level (u) is fixed;* $p_A = \frac{2x_A(1-u)}{4-u}$ *and* $\pi_A = \frac{4x_A(1-u)}{(4-u)^2}$.

2. *The leading firm always receives a higher profit with a larger gap;* $\frac{\partial \pi_A(x_A,u)}{\partial u} < 0$ *for all $u$ and $x_A$.*

3. *The price and the profit of the follower increases in $x_A$ when $u$ is fixed:* $p_B = \frac{x_A u(1-u)}{(4-u)} > 0$, $\frac{x_A u(1-u)}{(4-u)^2} > 0$ *for all $u$.*

4. *The follower's profit is maximized at* $\bar{u}(=\frac{4}{7})$.

Proposition 1 implies that there are two different forces that determine the prices and profits of firms. First, firms like to have higher quality products when they have a fixed follow-up level u. A fixed follow-up level means another firm's product quality increases at the same rate. Second, they want to widen the quality gap. The two goods are less differentiated when u is high enough; at this point, both firms need to compete with each other because both firms would otherwise want to avoid severe price competition when the quality levels of their two products are similar. These two forces look apparent when we consider the leader's profit and price: the profit and price of the leader increase in $x_A$ and decrease in $u$ indicated by (1) and (2). Thus, for the leader, the two forces move in the same direction, so the leader always wants to improve the quality of its product if it has the chance. On the other hand, the two forces on firm B influence the profit and the price in different directions. When $u$ is small, the profit of the follower increases when $u$ increases as described in (4). In other words, the follower wants to follow the leader when the gap is wide. However, when the gap narrows, the competition of the two firms gets harsher, even extreme, since when u is close to 1, the profits of both firms are near zero. As a result, the profit of firm B is maximized at a specific follow- up level u.

## 1.3 Dynamic Game

The leader and the follower produce products continuously, facing a market of consumers at each instant for an infinite period with discount rate $\frac{1}{1+r} > 0$. Firm $i$

receives the flow of profit $\pi_i^t$ $(i = A, B)$ at time $t$, assuming zero marginal cost of production. The flows of profits are defined by equation (1), when product quality,$x_A$ and $x_B$, is given in each time period. Technology for innovation is as follows: the firm $i$ puts the innovation effort $\mu_i$ with the cost of innovation $c_i$; the effort of innovation $\mu_i$ positively affects the quality level $x_i$.

*States of Firms* Firms are described by the state $(x_A, x_B) \in (0, \infty) \times (0, \infty)$. This state describes the quality of the firm's product in a given time period. The value functions of each firm depend not only on its product's quality level, but also on that of the other firm. The evolution of product quality follows the stochastic process:

$$
\begin{aligned}
dx_A^t &= \mu_A dt + x_A^t \sigma_A dZ_t \\
dx_B^t &= \mu_B dt + x_A^t \sigma_B dZ_t
\end{aligned}
\tag{2}
$$

, where $Z_t^A$ and $Z_t^B$ are standard Brownian motion, which indicates idiosyncratic shock on firm $A$ and firm $B$. Firm A chooses $\mu_A$ and firm B chooses $\mu_B$ through R&D investment. The R&D costs of the firms are $c_A(\mu_A, x_t^A)$ and $c_B(\mu_B, x_t^A)$ which depend on the quality of the leading product.

In addition, without loss of generality, the firms' state variable can be also described by $(x_A, u) \in (0, \infty) \times (0, 1)$ where $u$ is $\frac{x_B}{x_A}$. Then, the evolution of state variable $u$ is:

$$du = (-\mu_A \frac{x_B}{x_A^2} + \mu_B \frac{1}{x_A})dt + (u^2\sigma_A^2 + \sigma_B^2)^{1/2} dZ_t \qquad (3)$$

, where $Z_t$ is standard Brownian motion.

*Timing* In each period the order of events is as follows:

1. Each firm faces state variable $(x_A, x_B)$ (or $(x_A, u)$). With the given state variable, firms choose the prices of products in order to maximize their utility as described in the previous section.

2. Given prices, consumers choose between $x_A$ and $x_B$. In other words, consumers determine whether they buy the products, and determine the profit of each firm.

3. Firms' innovation decisions are carried out. Product quality is updated by newly innovated technology.

*Equilibrium* I restrict attention to a Markov Perfect Equilibrium. A Markorv Perfect Equilibrium is a set of innovation strategies $X_A = \{\mu_A^t(x_A^t, x_B^t)\}_{t=0}^{\infty}$ and $X_B = \{\mu_B^t(x_A^t, x_B^t)\}_{t=0}^{\infty}$, when the evolution of state variables follows (2). At each instant, $\mu_A^t$ and $\mu_B^t$ depend only on the current state variables $(x_A^t, x_B^t)$ .

1. For all initial state $(x_A, x_B)$, $X_A$ maximizes the expected payoff given $X_B$.

$$V(x_A, x_B) = \max_{X_A} E_0[\lim_{T\to\infty} \int_0^T \{e^{-r\tau}\{\pi_A^t - c_A(\mu_A^t; x_t^A, x_t^B)\}dt \,|(x_A, x_B), X_B]$$

2. For all initial state $(x_A, x_B)$, $X_B$ maximizes the expected payoff given $X_A$.

$$W(x_A, x_B) = \max_{X_B} E_0[\lim_{T\to\infty} \int_0^T \{e^{-r\tau}\{\pi_B^t - c_B(\mu_B^t; x_t^A, x_t^B)\}dt \,|(x_A, x_B), X_A]$$

I impose the following assumptions on the cost function of innovation to ensure that the innovation choice of each firm is well defined.

**Assumption 1**

1. $c_A(\mu_A; x_A)$ is strictly increasing and convex in $\mu_A > 0$, and increasing in $x_A$. In addition, $c_A(\mu_A; x_A)$ is multiplicably separable in $x_A$ and the function of $g_A = \frac{\mu_A}{x_A}$. That is $c_A(\mu_A; x_A) = \hat{c}_A(g_A)x$, where $\hat{c}_A$ is convex and increasing in $g_A > 0$.
2. $c_B(\mu_B, x_A)$ is strictly convex in $\mu_B$, and increasing in $x_A$. It is multiplicably separable in $x_A$ and the function of $\tilde{g}_B = \frac{\mu_B}{x_A}$. That is $c_B(\mu_B, x_A) = \hat{c}_B(\tilde{g}_B)x_A$, where $\hat{c}_B$ is convex in $\tilde{g}_B$.
3. $\hat{c}(r) = \infty$

In this set of assumptions, I did not restrict firm B's cost as being defined by positive efforts. There is the possibility that the firm chooses to reduce its quality level with some cost. Furthermore, the growth rate of firm A is restricted to be below

$r$. This assumption is needed because the value function diverges when $g_A$ is greater than $r$.

Then, we can obtain the following Hamilton-Jacobi-Bellman(HJB) equations. Each firm's dynamic programs are:

$$\begin{aligned}
rV(x_A, x_B) &= \max_{\mu_A} \pi^A(x_A, x_B) - c_A(\mu_A, x_A) + \mu_A V_{x_A} + \mu_B V_{x_B} \\
&\quad + \frac{x_A^2 \sigma_A^2}{2} V_{x_A x_A} + \frac{x_A^2 \sigma_B^2}{2} V_{x_B x_B} \\
rW(x_A, x_B) &= \max_{\mu_B} \pi^B(x_A, x_B) - c_B(\mu_B, x_A) + \mu_A W_{x_A} + \mu_B W_{x_B} \\
&\quad + \frac{x_A^2 \sigma_A^2}{2} W_{x_A x_A} + \frac{x_A^2 \sigma_B^2}{2} W_{x_B x_B}
\end{aligned} \tag{4}$$

When the other firm's choice is given, at each moment the choice of each player deals with the trade-off between the current innovation cost $-c_i(\mu_i)$ and the effect of the current innovation on the firm's expected value $\mu_A V_{x_A}$ (or $\mu_B W_{x_B}$). The other firm's innovation choice also affects the firm's expected value, $\mu_B V_{x_B}$ (or $\mu_A W_{x_A}$). Each firm's value takes into account the uncertain evolution of states $x_A$ and $x_B$, $\frac{x_A^2 \sigma_A^2}{2} V_{x_A x_A} + \frac{x_A^2 \sigma_B^2}{2} V_{x_B x_B}$ (or $\frac{x_A^2 \sigma_A^2}{2} W_{x_A x_A} + \frac{x_A^2 \sigma_B^2}{2} W_{x_B x_B}$).

### 1.3.1 Normalized Game

Using assumption 1 and the fact that $\pi_A$ and $\pi_B$ are multiplicably separable by $x_A$ and the function of $u$, we can rescale the problem with one state variable instead of two.

**Proposition 2** *Let* $\hat{V}(u) = V/x_A$ *and* $\hat{W}(u) = W/x_A$. *Then the HJB equations defined in (3) can be normalized by the following equation:*

$$\begin{aligned} r\hat{V}(u) &= \max_{g_A} \hat{\pi}^A(u) - \hat{c}_A(g_A) + g_A(\hat{V}(u) - u\hat{V}'(u)) + \tilde{g}_B\hat{V}'(u) \\ &\quad + \frac{1}{2}(u^2\sigma_A^2 + \sigma_B^2)\hat{V}''(u) \\ r\hat{W}(u) &= \max_{\tilde{g}_B} \hat{\pi}^B(u) - \hat{c}_B(\tilde{g}_B) + g_A(\hat{W}(u) - u\hat{W}'(u)) + \tilde{g}_B\hat{W}'(u) \\ &\quad + \frac{1}{2}(u^2\sigma_A^2 + \sigma_B^2)\hat{W}''(u) \end{aligned} \tag{5}$$

*, where the state variable* $u$ *has the following process.*

$$du = (-g_A u + g_B)dt + (u^2\sigma_A^2 + \sigma_B^2 - 2u\sigma_A\sigma_B)^{1/2}dZ_t \tag{6}$$

The two firms' problems are now defined on the one dimensional state space of $u$. The HJB equations are concave in each decision variable with the provided assumption on the cost of effort. We can see the effect of the follow-up level with rescaled value functions, apart from the leading firm's quality level. The rescaled HJB equation shows the trade-off between innovation cost and the effect of the current innovation on rescaled value. More precisely, when the leader spends innovation cost $\hat{c}_A(g_A)$, the firm's value will grow by the growth rate since the innovation of the leader increases $x_A$, $g_A\hat{V}(u)$. The state $u$ evolves with the drift $(-g_A u + g_B)$, when firm $B$'s choice is given; thus, the value is changed, $(-g_A u + g_B)V'(u)$. The value of the firm takes account of the uncertainty of the state variable $u$, $\frac{1}{2}(u^2\sigma_A^2 + \sigma_B^2 - 2u\sigma_A\sigma_B)\hat{V}''(u)$. The follower faces similar a trade-off between innovation efforts and the changes in value as the state evolves.

The solutions of the two equations are well defined and result from the following first order conditions:

$$(\hat{V}(u) - u\hat{V}'(u)) - \hat{c}_A'(g_A) = 0 \quad (7)$$

$$\hat{W}'(u) - \hat{c}_B'(\tilde{g}_B) = 0 \quad (8)$$

### 1.3.2 Characterization of the Equilibrium

In this section, I analyze the characteristics of equilibrium in general and define the steady state when there are no shocks.

**Lemma 3** *The leader's value decreases in the follow-up level; the follower's value increases in $u$ when it is far behind the leader, but decreases when its product quality is close to the leader's product quality.*

1. $\hat{V}'(u) < 0$.
2. *There exists* $\bar{u}$. *such that for all* $u < \bar{u}$, $\hat{W}'(u) > 0$ *and for all* $u > \bar{u}$, $\hat{W}'(u) < 0$.

From the Lemma we can prove the following proposition.

**Proposition 4** *(Characterization of the Innovatiion Behavior of Two Firms)*

1. *The leader always invests a positive amount of innovation effort:* $g_A(u)$ *>0 for all* $u$*.*

2. *The follower invests in positive innovation when it is far behind the leader, but when it is close enough to the leader it reduces the quality of its product: there exists* $\bar{u}$*. such that for all* $u < \bar{u}$*,*$\tilde{g}_B > 0$ *and for all* $u > \bar{u}$*,* $\tilde{g}_B < 0$*.*

We can explain the motivation of innovation in two ways. First, firms want to have higher innovation when the ratio is constant. The leader always wants to have positive innovation since it always wants to keep the distance in quality high, thereby avoiding competition with the follower in this market. The follower can receive a higher price by improving the quality of products when the quality of the follower's products is far behind from that of the leader's. On the other hand, when the follower gets closer to the leader, the follower reduces the quality of the product in order to avoid severe price competition, even if it must pay for the cost of the reduction.

Without shocks on the state, if $u_0 < \bar{u}$, then $\tilde{g}_B(u_0^t) > 0$ for all $t > 0$ ;where $u_0^t$ is the state variable at $t > 0$, growing with optimal growth rate $g_A(u_0^t)$ and $\tilde{g}_B(u_0^t)$ with initial condition $u_0$. That is the region, where negative investment is made, is not reached. At $u = \bar{u}$, $\frac{du}{dt}\Big|_{u=\bar{u}} = -g_A(\bar{u})\bar{u} + \tilde{g}_B(\bar{u}) = -g_A(\bar{u})\bar{u} < 0$. Although $u_0^t$ becomes closer to $\bar{u}$,it never exeeds $\bar{u}$. Thus, $u_0^t < \bar{u}$ and $\tilde{g}_B(u_0^t) > 0$ for all $t > 0$.

We can define the steady state of the economy when there are no shocks on $u$.

.

**Definition 5** *(The steady state in the deterministic case) Suppose that* $\sigma_A = \sigma_B = 0$. *The steady state* $u_{ss}$ *is achieved when* $\frac{du}{dt} = -g_A u + \tilde{g}_B = 0$.

At the steady state the ratio of growth rates is constant, $\frac{\tilde{g}_B(u_{ss})}{g_A(u_{ss})} = u_{ss}$. The follower's innovation level is not negative, but the follower always innovates less than the leader. Denote $f(u) = -g_A u + \tilde{g}_B$ . If $g_A'(u) > 0$ and $\tilde{g}_B'(u) < 0$ then $f'(u) < 0$ for all $u$; thus, the stability and the uniqueness of the steady state is achieved. By the Implicit Function Theorem, $g_A'(u) = \frac{-u\hat{V}''(u)}{\tilde{c}_A''(g_A)}$ and $\tilde{g}_B'(u) = \frac{\hat{W}''(u)}{\tilde{c}_B''(\tilde{g}_B)}$. Thus, if $\hat{V}''(u) < 0$ and $\hat{W}''(u) < 0$, then we have a stable and unique steady state. In the next section, I will show the numerical example that has a stable and unique steady state, and further discuss the equilibrium.

### 1.3.3 Numerical Example of the Equilibrium

The competition of AMD and Intel in the CPU market is the classic example of a duopoly in a vertically differentiated market. Figure 1-1 represents the stok market value ratio, the value of AMD devided by that of the Intel. This stock market values are calculated by taking average of 5 years moving window. The ratio becomes higher in 2005-2010; however it becomes lower. [1] In other years, the intel tends to have far distance from AMD. One possible explanation for the large difference in the market values of the two firms is that AMD lacks the technological capability of Intel. Another

[1]Retrieved from http://finance.yahoo.com/

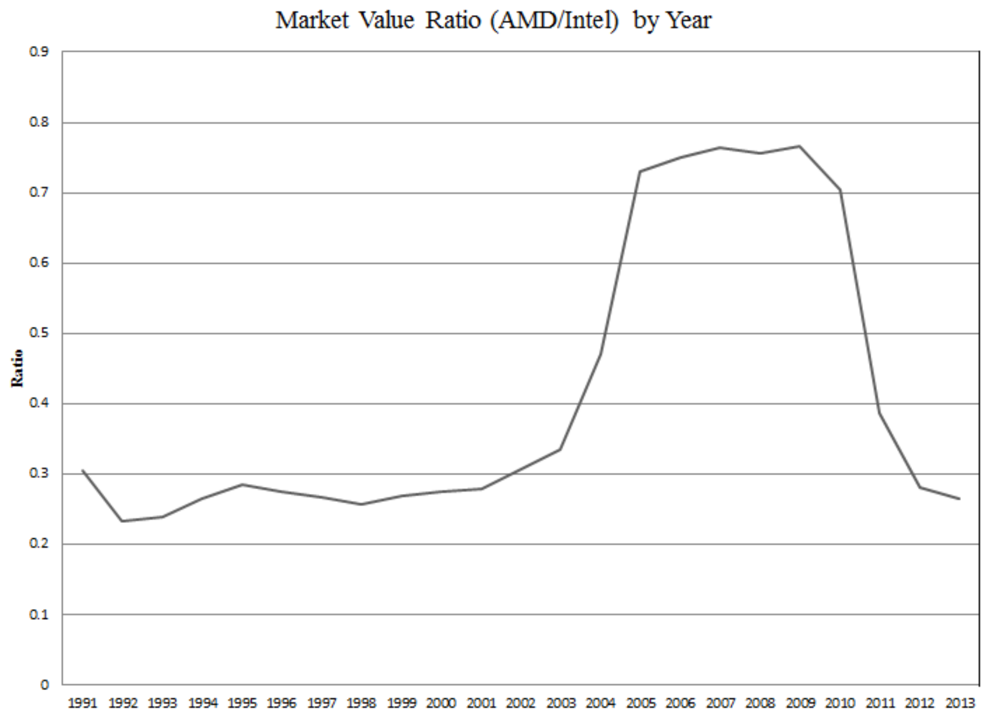


Figure 1: Figure1-1: Stock Market Value Ratio (AMD/Intel)

reason could be that AMD chose not to innovate for strategic reasons although it has the ability to innovate further. More precisely, AMD would not be able to compete with Intel's high-end CPUs; instead, it focused on lower-margin products. Nosko (2010) showed that competition is the most important factor in a firm's determining its quality level and innovation behavior. Therefore, AMD's choice not to follow Intel could be due to competition; instead AMD targets consumers who are sensitive to price, with low marginal utility on quality. In this section, I will demonstrate that this stable ratio between the market value of the duopolists can be explained by their innovation decisions based on competition.

I will show the numerical example of the equilibrium described in the previous section. The goal of the simulation is to show that the model suggested by this paper can explain the market behavior of duopolists when the quality is differentiated. First, I will demonstrate the phenomenon without the stochastic process and will show the steady state. Second, I will illustrate the stochastic process.

The strategy of the simulation is as follows:

1. I discretized the $u$ by $I$ discrete point such that $u_i$, $i = 1...I$. Thus, we can approximate $V(u_i)$ and $W(u_i)$. With approximated $V(u_i)$ and $W(u_i)$, I approximated $V'(u_i)$ and $W'(u_i)$ using following procedures:1) when $i < \frac{I}{2}$: $f'(u) \approx \frac{f(u_i - u_{i+1})}{\Delta u}$ (forward difference, 2) when $i > \frac{I}{2}$: $f'(u) = \frac{f(u_{i+1} - u_i)}{\Delta u}$ (backward difference), and 3)when $i = \frac{I}{2}$: $f'(x) = \frac{f(u_{i+1} - u_{i-1})}{2\Delta u}$ (central difference).

2. Second, starting from the initial guess of $V^0$ and $W^0$, I solve NFOC for given $V^n$, calculated $g_A$ and $g_B$, and update by the following approximation:

$$
\begin{aligned}
V^{n+1} &= V^n + \Delta((\hat{\pi}^A(u) - \hat{c}_A(g_A) + g_A(\hat{V}(u) - u\hat{V}'(u)) + \tilde{g}_B(\hat{V}'(u) - r\hat{V}^n) \\
W^{n+1} &= W^n + \Delta(\hat{\pi}^B(u) - \hat{c}_B(\tilde{g}_B) + g_A(\hat{W}(u) - u\hat{W}'(u)) + \tilde{g}_B\hat{W}'(u) - r\hat{W}^n)
\end{aligned}
$$

   If $|V^{n+1} - V^n|$ and $|W^{n+1} - W^n|$ are close to zero, I terminate the updating and use $V^n$ and $W^n$ as the value function.

3. I use a cost function and interest rate that satisfies the assumptions for the

contraction mapping as an example:

$r = 0.1$, $\hat{c}(g_A) = \begin{cases} 100g_A^2 & g_A < r \\ \infty & g_A \geq r \end{cases}$, and $\hat{c}_B(\tilde{g}_B) = 100\tilde{g}_B^2$. According to the FOC: $g_A = \frac{1}{200}(\hat{V}(u) - u\hat{V}'(u))$ and $\tilde{g}_B = \frac{1}{200}\hat{W}'(u)$.

**Deterministic Process**

First, I simulated the solution under the deterministic process without any adoption of technologies from the leader, when $\sigma_A = \sigma_B = 0$ . As discussed in Proposition 4, $g_A > 0$ for all $u$, and $\tilde{g}_B > 0$ when $u < \frac{4}{7}$ but $\tilde{g}_B < 0$ when $u > \frac{4}{7}$. Moreover, the leader's optimal policy increases in $u$ and the follower's optimal choice decreases in $u$. In other words, when two firms get closer they choose to widen the gap in order to avoid severe price competition with similar products. With computed $g_A$ and $\tilde{g}_B$, the steady state of this economy is specified by using $\frac{du}{dt} = 0$. When, $-g_A(i)u(i) + \tilde{g}_B(i)$ is close to zero, I can define $u(i)$ as $u_{ss} \approx 0.34$ As we can see from Figure 1-2, $\frac{du}{dt}$ is monotonically decreasing in $u$. Thus, the steady state is unique and stable.

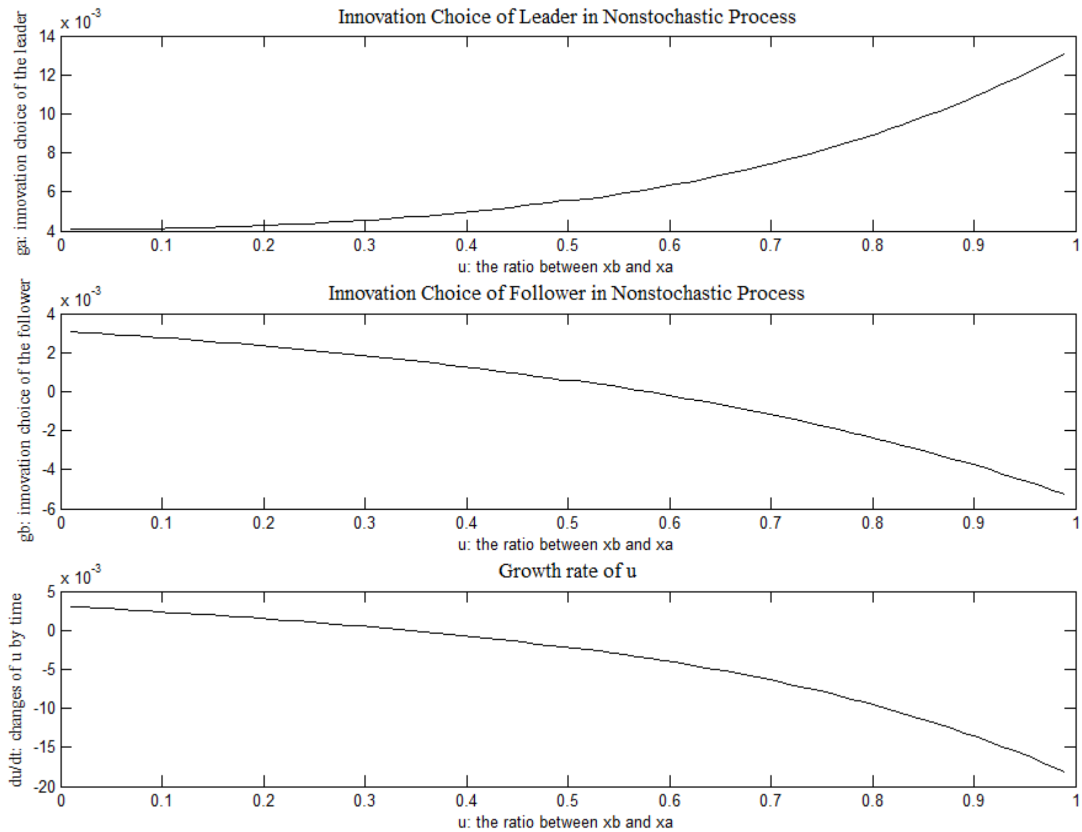


Figure 1-2: The Equilibrium Choices of Innovation

Figure 1-3 describes how the steady state of the economy is achieved starting from a different state. When the leader faces a large gap, or a small follow-up level (e.g. u=0.1216), it invests a positive amount to have a higher price. However, the motivation to move away from the follower is not very high. Thus, it chooses gradual changes in quality. The follower with a small follow-up level has two competing forces governing its investment decision. It wants to have higher quality to attract consumers willing to pay a higher price, while it also wants to get further away from the leader. The second force is not strong compared with the first effect. In sum, the follower wants to improve the quality of the product and thus the optimal growth rate is positive. As it improves the quality of the product, the distance narrows. On the other hand, when the firms face a smaller gap (e.g. u=0.8669), they are in a comparatively competitive environment. Therefore, the motivation to widen the gap gets stronger for each firm. As a result, the leader chooses a comparatively high growth rate, while the follower chooses to reduce its product quality. No matter which point the firms start from, they reach the steady state $u^{ss} = 0.34$. At the steady state, the optimal growth rates are $g_A^{ss} = g_A(u_{ss}) = 0.0047$ and $\tilde{g}_B^{ss} = \tilde{g}_B(u_{ss}) = 0.0016$. The ratio remains constant after the steady state is achieved.

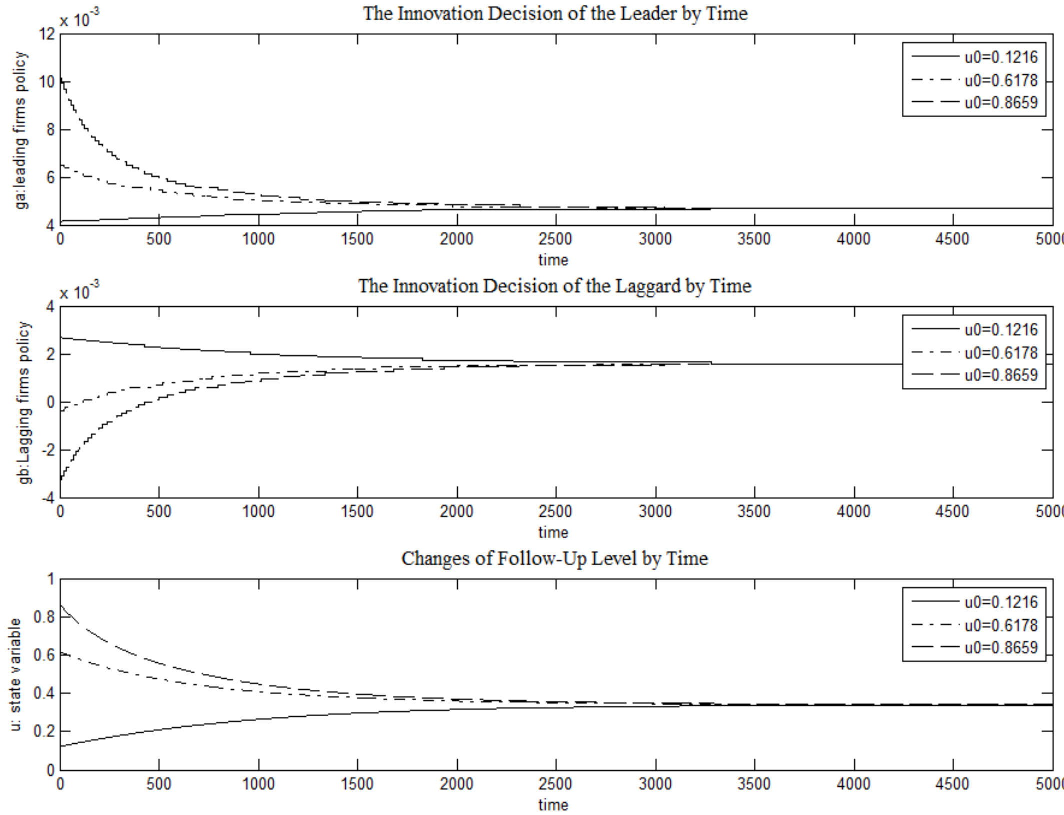


Figure 1-3: The Changes of the State Variable by Time in the Equilibrium

### Stochastic Process

When there are shocks on state in each period, the firms' decisions do not change much, as we can see in Figure 1-4. The parameters $\sigma = \sigma_A = \sigma_B = 0.1$ in this case. The implication of the decision according to the follow-up level remains the same as discussed in the deterministic process.

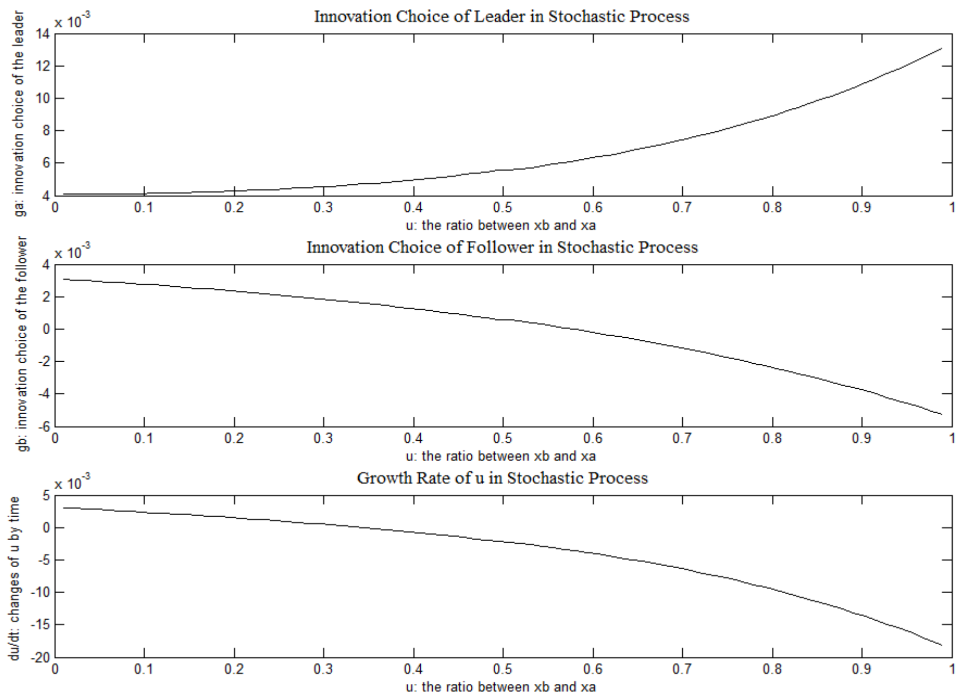


Figure 1-4 :The Equilibrium Choices of Innovation in Stochastic Process

Figure 1-5 shows the follow-up level by time with different volatility of stochastic process. In the stochastic process the steady state is never achieved. When the volatility is low($\sigma = 0.004$),the state variable roughly converges around a similar number as the steady state in the deterministic case.On the other hand, when the volatility is comparably high($\sigma = 0.1$), the state variable fluctuates a great deal, but it never diverges from the center. Suppose that there is a huge shock on the process, for instance, the leader gets an extremely high positive shock compared with the follower. Then, the state variable becomes low; so the leader chooses to expend little innovation effort, while the follower innovates eagerly. Consequently, the state variable recovers to the center as time goes by. In sum, although there is no steady state in this economy, firms tend to keep the distance similar over time.

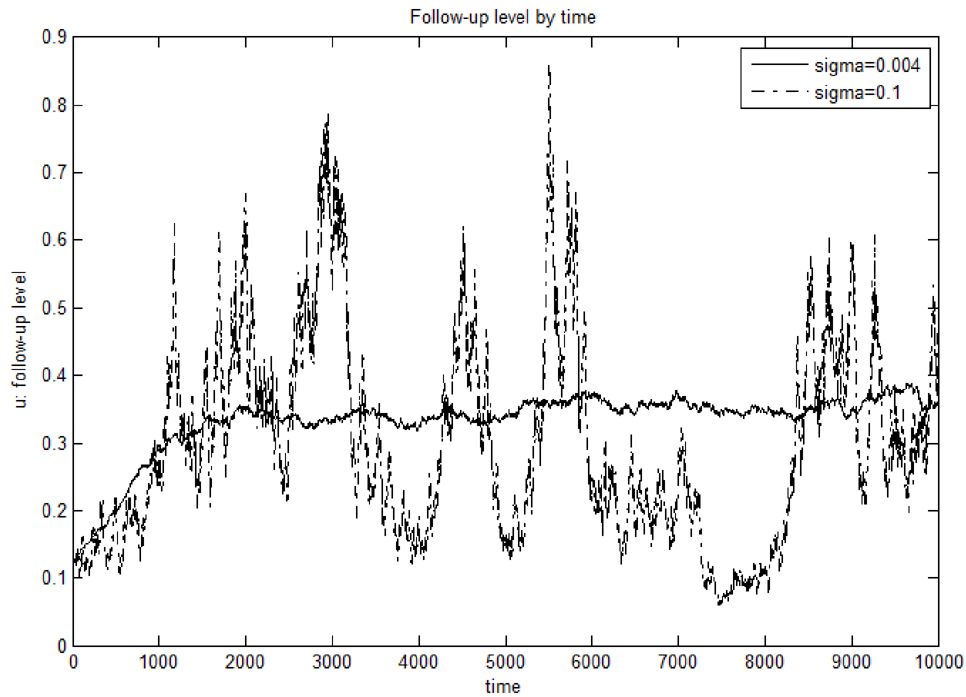


Figure 1-5: : Follow-up Level by Time with Different Variances of Innovation

## 1.4 Social Optimum

Aside from two firms' competition, I analyze the socially optimum level of innovation when there are no the follower. The social planner decides the innovation choices of firms, but does not affect the pricing decisions of the firm directly Let $cs_A$ and $cs_B$ be consumers' surplus in each sector of the market. Then, we can compute the flow of total welfare with the following equation:

$$w(x_A, x_B) = cs_A + cs_B + \pi_A + \pi_B = \int_{\theta \in S_A} \theta x_A d\theta + \int_{\theta \in S_B} \theta x_B d\theta$$

,where $S_A = [\frac{2-u}{4-u}, 1]$ and $S_B = [\frac{1-u}{4-u}, \frac{2-u}{4-u}]$. The consumers in the interval $S_A$ buy the product $x_A$, while the consumers in the interval $S_B$ buy the product $x_B$. The welfare function $w$ is mutiplicably separable by $x_A$ and the function of $u$.

$$w(x_A, x_B) = x_A \hat{w}(u) = x_A \frac{-2u^2 - u + 12}{2(4-u)^2}$$

Note that $\hat{w}(u)$ is always increasing in u. In other words, in capturing consumers' surplus, the social welfare in this economy increases when the quality of the follower's products approaches that of the leader. Suppose that the social planner tries to maximize the joint value function of the social welfare by choosing the growth rate of each product, $g_A$ and $\tilde{g}_B$.

$$\begin{aligned} rJ(u) &= \max_{g_A,\, \tilde{g}_B} \hat{w}(u) - \hat{c}_A(g_A) - \hat{c}_B(\tilde{g}_B) \\ &\quad + g_A(J(u) - u\ J'(u)) + \tilde{g}_B(J'(u)) \\ &\quad + \frac{1}{2}(u^2\sigma_A^2 + \sigma_B^2 - 2u\sigma_A\sigma_B)J''(u) \end{aligned}$$

Then, the FOC yield:

$$g_A^J(u) = \hat{c}_A'^{-1}(J(u) - u\ J'(u))$$

$$\tilde{g}_B^J(u) = \hat{c}_B'^{-1}(J'(u))$$

Using a method similar to that used for proving Lemma 1, $J'(u) > 0$ can be easily shown. Then, $\tilde{g}_B^J(u)$ is positive for all $u$, which means that it is always beneficial for social welfare to improve lagging products for social welfare. Nonetheless, the sign of $g_A^J(u)$ is vague and dependent on the relative sizes of $J(u)$ and $uJ'(u)$; $g_A^J(u)$ increases when $J(u)$gets larger because when the quality of the leader's products increases the social planner receives higher value as the follow-up level is fixed. On the other hand, if $uJ'(u)$ gets larger, $g_A^J(u)$ gets smaller because the growth of $x_A$ widens the gap.

Comparing $\tilde{g}_B$ in equilibrium and socially optimal $\tilde{g}_B^J$ shows us that the competition of two firms discourages the innovation of the follower even though the leader does not affect the follower directly. Social welfare capturing the consumer's surplus increases when the quality of both firms' products is higher. In other words, the decision of each firm as motivated by widening the gap would harm the social welfare. Therefore, regulation to protect the follower, such as subsidizing innovation or strengthening anti-trust law, is needed to encourage the innovation of the follower.

### 1.4.1 Numerical Example

As an exercise, I computed the simulated example of the social planner's optimal choice with the same cost function as in the previous numerical example. For simplicity, I consider only the deterministic case. As discussed earlier, $\tilde{g}_B^J(u)$ is always greater than zero even though the follower gets closer, because the social planner always receives higher value when the distance narrows. By the implicit function theorem $g_A^{J\prime}(u) = \frac{-uJ''(u)}{\hat{c}_A''}$ and $\tilde{g}_B^{J\prime}(u) = \frac{-J''(u)}{\hat{c}_B''}$increases in $u$. As we can infer from the plot of $\tilde{g}_B^J(u)$, $J'(u)$ is decreasing in $u$. Thus, $J''(u) < 0$, which leads to the conclusion that $g_A^J(u)$ increases in $u$ and $\tilde{g}_B^J(u)$ decreases in $u$. The steady state of $u$ is achieved around 0.73, which is much higher than that of two firms' game (see Figure 10). This condition is mainly because the innovation of firm B is discouraged in the dynamic game, as it cares about price competition with the leader. However, the social planner does not care about price competition but instead cares about capturing many consumers and providing higher quality products.

Figure 1-6 also shows the comparison between socially optimal innovations and innovations in equilibrium. In equilibrium, there are two motivations for innovation. First, firms want to have higher quality products since higher quality returns to the higher profit when the ratio is fixed. Second, they want to widen the gap due to competition in the vertically differentiated market. A social planner's choice does not consider the second motivation; it only considers the trade-off between innovation cost and higher welfare due to higher quality. The leader's optimal investment is higher than investment in equilibrium when the ratio of quality is low. However, when the ratio is high enough, equilibrium choice has the motivation to widen the gap while the optimal choice does not capture that motivation. Thus, a socially optimal level of innovation for the leader is lower than that of equilibrium level. In contrast, when we see the gap between the innovation of the socially optimal choice and that of the equilibrium, the gap becomes wider with the higher ratio of quality. The lagging firm in equilibrium with a high ratio of quality wants to lower its quality significantly to widen the gap.

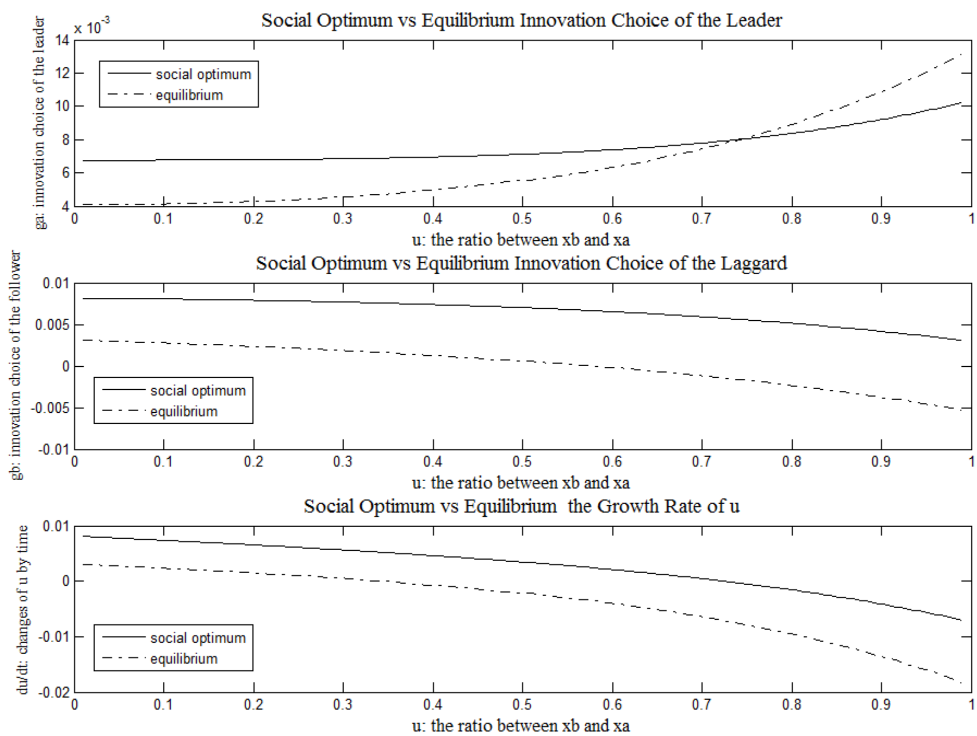


Figure 1-6: The Socially Optimal and Equilibrium Choices of Innovation

Figure 1-7 shows how firms' policies and the state variable $u$ approach the steady state. There are two forces that form the steady state of the economy. First, the social planner likes to have a narrow gap. Thus, the innovation of firm B is encouraged, and the innovation of firm A is discouraged in this scenario. Second, the leading firm still likes to have high $x_A$. These two forces push in opposite directions, forming the optimal choice and the steady state. Considering two forces might be more difficult in reality. If there is an antitrust policy which encourages the lagging firm's innovation, it would harm the innovation of the leader. As discussed in this example, there is a socially optimal steady state of the ratio. Therefore, the antitrust policy to encourage a follower's innovation should be implemented considering the optimal gap between two firms in dynamics.

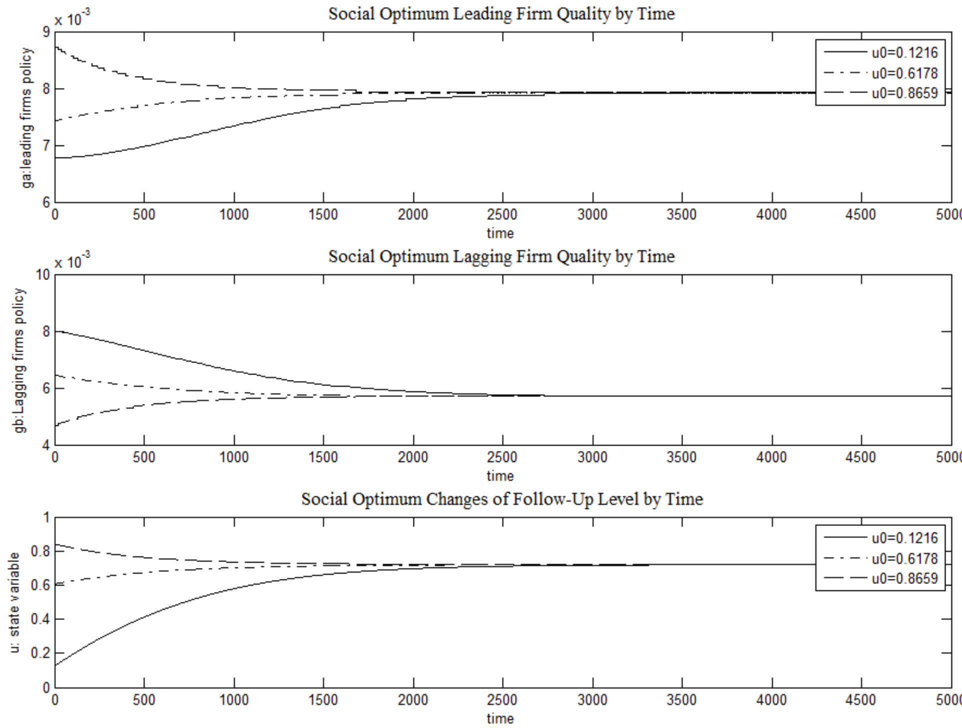


Figure 1-7: The Changes of the State Variable by Time in the Social Optimum

## 1.5 Intellectual Property of the Leader: Investment in Deterrence

One of the leader's motivations to invest is to widen the gap beyond simply gaining a higher price due to higher quality. In this case, the leader has the predatory motivation to invest to deter the innovation of the follower, even though the investment does not provide the leader with higher quality. Buying intellectual property is one example of predatory investment. Figure 1-8 shows that the number of Intel's new patents is much higher than AMD's. AMD acquired only 8 patents from 1980 to

2005.[2] [3] Intel acquired more than 430 patents on average in each year. The number of patents reflects the innovation effort to increase the quality of the product, but, aside from that, it required the firm's efforts, for instance in hiring a patent attorney, dealing with a patent examiner, and filing documents. The glaring disparity between the two firms regarding patenting is more extreme than the difference in their product quality. Therefore, Intel's patenting activity could be motivated by predatory reasons.

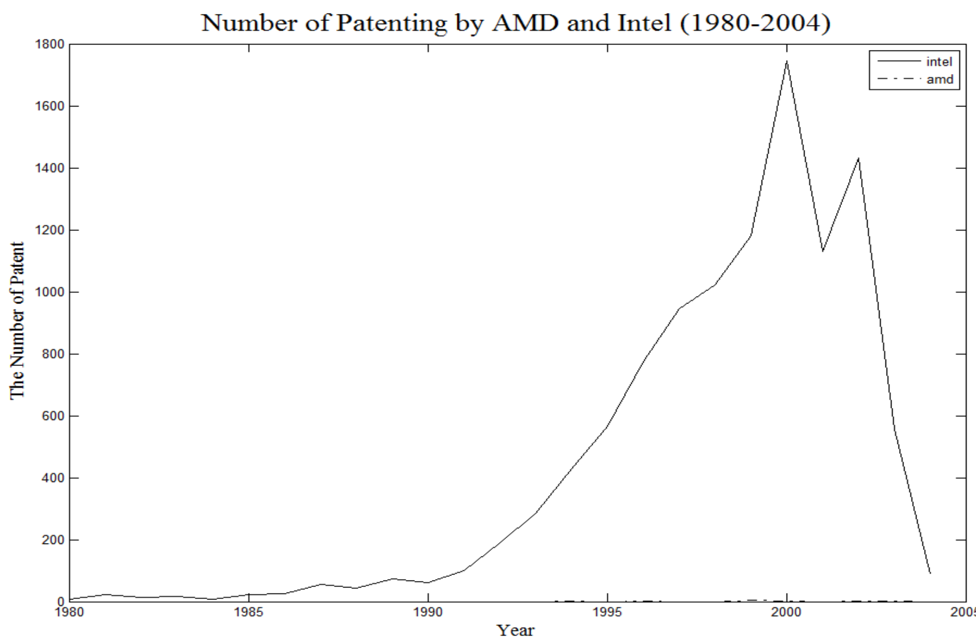


Figure 1-8: The Number of Patents of AMD and Intel by Year

[2] http://www.kauffman.org/comets/

Lynne G. Zucker and Michael R. Darby, COMETS Data Description, release 1.0, Los Angeles, CA: UCLA Center for International Science, Technology, and Cultural Policy, July 1, 2011.

[3] The patent data in 2003, 2004, and 2005 are incomplete, for there are patents applied for but not granted by the end of 2005.

Suppose that it is permitted to buy intellectual property $p$ which makes innovation difficult for the follower. However, this $p$ does not directly affect the growth of $x_A$. This section answers the question of what might happen if the leader invests in intellectual property due to predatory motivation without it directly helping its innovation. The price of intellectual property per $x_A$ is denoted by $\psi(p)$ which increases in $p$. This intellectual property can disturb the innovation of the follower by making it difficult to innovate. Thus, the cost of innovation of the follower $\hat{c}_B(\tilde{g}_B, p)$ is increasing and convex in $p$.

I restrict attention to the deterministic case for simplicity. Then the HJB equations of two firms are as follows:

$$\begin{aligned} r\hat{V}(u) &= \max_{g_A, p \geq 0} \hat{\pi}^A(u) - \hat{c}_A(g_A) - \psi(p) + g_A\hat{V}(u) + (-g_A u + \tilde{g}_B(u,p))\hat{V}'(u)) \\ r\hat{W}(u) &= \max_{\tilde{g}_B} \hat{\pi}^B(u) - \hat{c}_B(\tilde{g}_B, p) + g_A\hat{W}(u) + (-g_A(u) + \tilde{g}_B)\hat{W}'(u) \end{aligned}$$

Then, the FOC yields the following equations:

$$(\hat{V}(u) - u\hat{V}'(u)) - \hat{c}_A'(g_A) = 0$$

$$-\psi'(p) - \frac{\partial \hat{c}_B^{-1}}{\partial p}(\hat{W}'(u); p)\hat{V}'(u) = 0$$

$$\hat{W}'(u) - \hat{c}_B'(\tilde{g}_B; p) = 0$$

The argument used when I proved Lemma 1 is valid with this extended model. Thus, $\hat{V}'(u) < 0$ ; and there exists $\bar{u}$. such that for all $u < \bar{u}$, $\hat{W}'(u) > 0$ and for all $u > \bar{u}$, $\hat{W}'(u) < 0$. Then, the following proposition characterizes the equilibrium in this extended model.

**Proposition 6** *(Characterization of Equilibrium with Intellectual Property)*

1. *The leader's optimal growth rate is always positive:* $g_A > 0$.

2. *The choice of intellectual property depends on the cost of intellectual property and how* $p$ *affects the cost of* $\tilde{g}_B$*: when* $p$ *is expensive or easily deters the innovation of firm B, the optimal* $p$ *is low.*

3. *The follower exerts positive effort when it is far behind the leader, but when it is close enough to the leader it reduces the quality of its product: there exists* $\bar{u}$. *such that for all* $u < \bar{u}$,$\tilde{g}_B > 0$ *and for all* $u > \bar{u}$, $\tilde{g}_B < 0$.

4. *The follower's optimal choice decreases in* $p$. *Therefore, it decreases when the cost of intellectual property is high.*

5. *The steady state is achieved when* $\frac{du}{dt} = -g_A u + \tilde{g}_B = 0$. *With cheaper intellectual property, the steady state is formed with lower* $u$.

### 1.5.1 An Example of Investment in Deterrence

I now consider an example and provide the numerical solution in this section. The cost of innovation of the leading firm remains the same as in the previous section. The cost of the intellectual property is $\psi(p) = dp$, which is linear in $p$. for some $d$. Also the cost of the lagging firm's innovation is $c_B(\tilde{g}_B, p) = 100(\tilde{g}_B + p)^2$. According to the FOC: $\tilde{g}_B = \frac{1}{200}\hat{W}'(u) - p$, $p = \frac{-\hat{V}'(u)}{d}$. $d$ represents the marginal cost of the intellectual property. Figure 1-9 shows firms' investment choices and changes of $u$ by the state variable $u$ with different costs of patenting. As argued in Proposition 3, the leader's investment in the quality of the product is always positive and strongly increasing in u. Moreover, when the price of intellectual property rises, the optimal $p$ lowers. Since the follower's optimal choice decreases in $p$, it decreases when the cost of intellectual property is not high. The steady state of the economy is achieved ($\frac{du}{dt} = 0$) at a lower u compared with the basic model, for the innovation of the follower is disturbed; and it is achieved at a lower u when the marginal cost of patenting is lower.

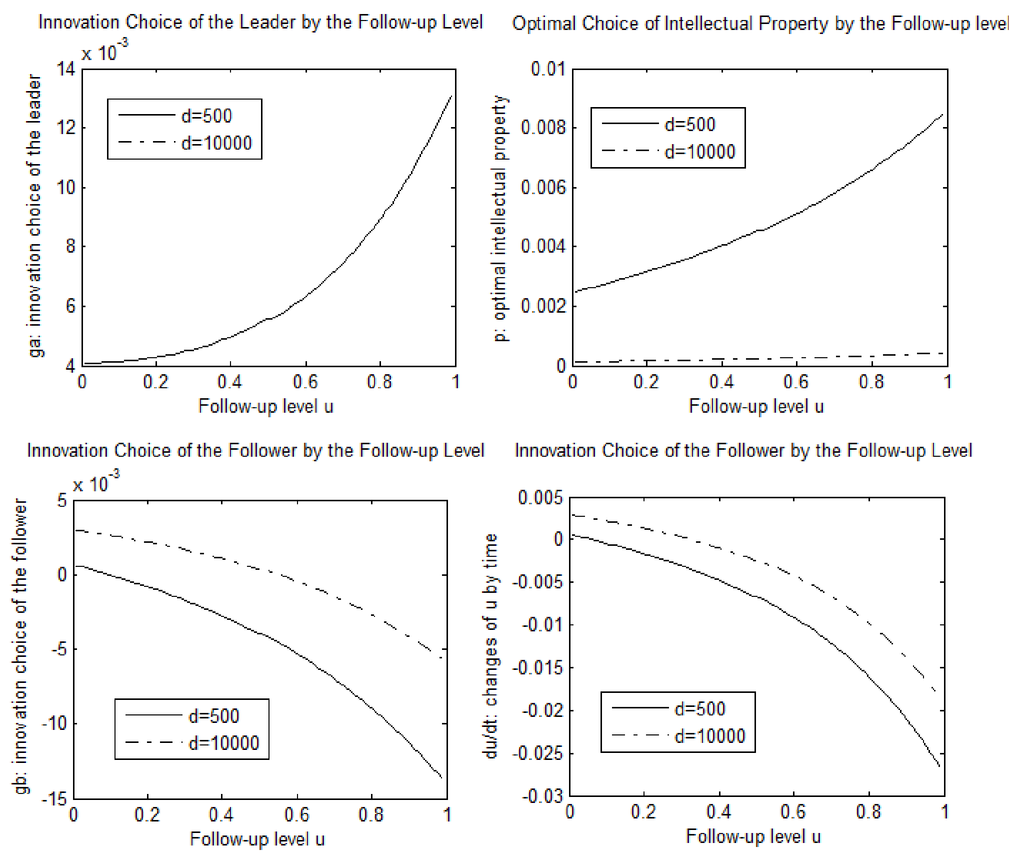


Figure 1-9: Firms' Choices with Different Marginal Costs of Patenting

| Table 1-1a | Innovation of Quality and Patenting in the Steady State | | |
|---|---|---|---|
| MC of Patenting | Steady State $u$ | $g_A$ and $g_B$ | Patenting per $x_A$ |
| $d = 500$ | 0.0658 | 0.0041 | 0.0027 |
| $d = 1,000$ | 0.1899 | 0.0043 | 0.0016 |
| $d = 10,000$ | 0.3263 | 0.0046 | 0.0002 |
| No Patenting | 0.3511 | 0.0047 | None |

| Table 1-2 | Social Welfare in the Steady State | | | |
|---|---|---|---|---|
| MC of Patenting | Consumer's Surplus | Leader's Surplus | Follower's Surplus | Total Welfare |
| $d = 500$ | 1.8398 | 0.7503 | 0.0397 | 2.6298 |
| $d = 1,000$ | 3.9988 | 0.5733 | 0.1059 | 4.6780 |
| $d = 10,000$ | 5.5649 | 0.3443 | 0.1629 | 6.0721 |
| No Patenting | 5.7687 | 0.3214 | 0.1672 | 6.2573 |

The steady state with different marginal costs of patenting is described in Table1-1. When the marginal cost is lower, the leader can buy more patents to deter the innovation of the follower and invest less in innovating the quality of its own product. The quality ratio $u$ in the steady state is lower with the lower cost of patenting. That is, the leader can widen the quality gap by buying the patent. As a result, growth of the economy is greatly disturbed. As can be seen from Table 1-1, the growth rate of $x_A$ and $x_B$ decreases as the marginal cost of investment in deterrence diminishes. Therefore the investment in deterrence harms the growth of the industry. Furthermore, the lower cost of deterrence generates a lower consumer surplus and total surplus of the economy, although it provides a higher surplus to the leader, as described in Table 1-2.

## 1.6 Concluding Remarks

The central ideas analyzed in this paper are the various motivations of firms' innovation in continuous time with vertically differentiated products. First, firms innovate to attract consumers with higher marginal utility of quality. Another motivation we considered in this paper is that firms choose the quality of their products in order to avoid intense competition. In other words, two firms want to keep their distance in order to attract customers with different marginal utility of quality. Two motivations thus force the leader in the industry to innovate actively. On the other hand, two motivations affect the follower's innovation in the opposite direction. That is, the follower innovates actively when it is far from the leader, but reduces the quality of its product when it is sufficiently close to the leader. As a result, firms keep the ratio of quality constant at the steady state; therefore, there is a wide gap between the product quality of the leader and that of the follower. Nonetheless, when a social planner decides on a firm's innovation choice at the steady state, the quality ratio is significantly higher than that of equilibrium. That is, with a more similar level between two products, more consumers can buy either good at a lower price.

To keep the quality distance further apart, the leader would want to deter the innovation of the follower. This motivation, which arises from the second motivation of innovation, can be called predatory investment. Focusing on the predatory motivation of the leader, I investigated the leader's investment in deterrence. That is, the leader may invest in order to disturb the innovation of the follower and to widen the quality gap. The leader tends to buy more intellectual property when the gap in product quality narrows. Since predatory investment widens the quality gap, it harms social welfare.

A policy maker could make the patent process more costly with a more expensive fee, a more complicated process, or a higher threshold to obtain a patent. Moreover, the government could provide subsidies to make follower innovation less costly. Then, the follower would innovate more than before; thus, the quality gap would narrow, i.e., the quality ratio would become higher. Therefore, with the government subsidizing the follower, the leader would innovate further, since the leader's innovation generally increases the quality ratio. Because these policies can accelerate the follower's innovation and narrow the quality gap, more consumers would be able to buy products from either firm at a lower price.

## 1.7 Appendix: Proofs

### 1.7.1 Normalization of the HJB equation

The HJB equations (4) can be rewritten as follows:

$$\begin{aligned} rV(x_A,u) &= \max_{\mu_A} x_A\hat{\pi}^A(u) - \hat{c}_A(g_A)x_A + g_A x_A(\hat{V}(u) - u\hat{V}'(u)) \\ &\quad +\tilde{g}_B x_A \hat{V}'(u) + \frac{x_A\sigma_A^2}{2}(uV''(u)) + \frac{x_A\sigma_B^2}{2}V''(u) \\ rW(x_A,u) &= \max_{\mu_B} x_A\hat{\pi}^B(u) - \hat{c}_B(\tilde{g}_B)x_A + g_A x_A(\hat{W}(u) - u\hat{W}'(u)) \\ &\quad +\tilde{g}_B x_A \hat{W}'(u) + \frac{x_A\sigma_A^2}{2}(uW''(u)) + \frac{x_A\sigma_B^2}{2}W''(u) \end{aligned}$$

By dividing both sides of each equation by $x_A$, we can get $r\hat{V}(u)$ and $r\hat{W}(u)$. Note that when we have a normalized equation that is only defined on the one state variable $u$ ,with given value of $x_A$, choosing $\mu_A$ or $\mu_B$ is equivalent to the choice of $g_A$ and $\tilde{g}_B$. Then WLOG the HJB equation described in (3) can be normalized by (4).

### 1.7.2 Characteristics of Equilibrium (Lemma 1)

I employed a method similar to that used in Lemma 2 of Board and Meyer-Ter-Vehn (2013).

(1)$\hat{V}'(u) < 0$.

Consider, $0 < u_1 < u_2 < 1$. I prove that $\hat{V}(u_1) > \hat{V}(u_2)$ by using $u_1$'s mimicking strategy. When the firm with $u_1$ mimics the optimal policy of $u_2$, it is suboptimal, but the firm $u_1$ still receives the higher flow of profit than the firm $u_2$.

Specifically, suppose that at $t = 0$ two firms have $u_1$ or $u_2$. Let $g^t_{A2}$ be the optimal choice of the firm starting with $u_2$ at time $t > 0$. Denote that $h$ is the history of states and optimal choices of two firms and their rivals. $u^t_2 = (u_2, h, g^t_{A2}; t > 0)$ is the state variable of the firm starting with $u_2$.$u^t_2$ grows with optimal growth rate $g^t_{A2}$ and $g^t_B(u^t_2)$ Assume the firm starts with $u_1$ and chooses $g^t_{A2}$ instead of his optimal choice $g^t_{A1}$ . $\hat{u}^t_1 = (u_1, h, g^t_{A2}; t > 0)$ will be the state variable of the firm $u_1$, when it chooses the mimicking strategy.$\hat{u}^t_1$ grows with growth rate $g^t_{A2}$ and $g^t_B(u^t_2)$which are the optimal policy with $u^t_2$ and Starting from $\hat{u}^t_1 < u^t_2$, there is possibility that at some $\tau$ it becomes $\hat{u}^\tau_1 = \hat{u}^\tau_2$ due to the choice of firm B($\tilde{g}_B$). Then, for $t \geq \tau$, $\tilde{g}_B(\hat{u}^t_1) = \tilde{g}_B(u^t_2)$, because $g_A(\hat{u}^t_1) = g_A(u^t_2) = g^t_{A2}$. Thus, $\hat{u}^t_1 < u^t_2$ when $t < \tau$ ; and $\hat{u}^t_1 = u^t_2$ when $t \geq \tau$. Since $\hat{\pi}_A$ decreases in $u$, $\hat{\pi}^t_A(\hat{u}^t_1) \geq \hat{\pi}^t_A(u^t_2)$, for all $t > 0$. In sum, we see that:

$$
\begin{aligned}
\hat{V}(u_1) &= E(\int_0^\infty e^{-rt}(\hat{\pi}^t_A(u^t_1) - \hat{c}_A(g^t_{A1}))dt) \geq E(\int_0^\infty e^{-rt}(\hat{\pi}^t_A(\hat{u}^t_1) - \hat{c}_A(g^t_{A2}))dt) \\
&> E(\int_0^\infty e^{-rt}(\hat{\pi}^t_A(u^t_2) - \hat{c}_A(g^t_{A2}))dt) = \hat{V}(u_2)
\end{aligned}
$$

The first inequality holds since $g^t_{A2}$ is suboptimal.

(2)There exists $\bar{u}$. such that for all $u < \bar{u}$, $\hat{W}'(u) > 0$ and for all $u > \bar{u}$, $\hat{W}'(u) < 0$.

Consider, $0 < u_2 < u_1 < \bar{u}$. I prove that $\hat{W}(u_1) > \hat{W}(u_2)$ by using $u_1$'s mimicking strategy. When the firm with $u_1$ mimics the optimal policy of $u_2$, it is suboptimal. Still, the firm $u_1$ still receives the higher flow of profit than the firm $u_2$.

In detail, suppose that at $t=0$ two firms have $u_1$ or $u_2$. Then, let $\tilde{g}^t_{B2}$ be the optimal choice of the firm starting with $u_2$ at time $t>0$. $u^t_2=(u_2,h,\tilde{g}^t_{B2};t>0)$ is the state variable of the firm starting with $u_2$. Assume that the firm starts with $u_1$ chooses $\hat{g}^t_B$ instead of his optimal choice $\tilde{g}^t_{B1}$, where $\hat{g}^t_B=\begin{cases}\tilde{g}^t_{B2} & u<\bar{u}\\ 0 & u=\bar{u}\end{cases}$. In this mimicking strategy when $u<\frac{4}{7}$, the firm adopts the policy of $u_2$; but, when $u$ becomes $\bar{u}$ it does not move, and hence stays at $u=\frac{4}{7}$. $\hat{u}^t_1=(u_1,h,\tilde{g}^t;t>0)$ is the state variable of the firm $u_1$, when it chooses the mimicking strategy. Remark that $u^t_2\le\hat{u}^t_1<\bar{u}$or $\hat{u}^t_1=\bar{u}$. As it is described in the proof of (1) there is a possibility that $u^\tau_2=\hat{u}^\tau_1$ at some $t=\tau$ due to the leader's choice $g_A$. After that, all strategic behavior of firms will be identical in both case of $u^t_2$ and $\hat{u}^t_1$, so $u^t_2=\hat{u}^t_1$ if $t\ge\tau$. Thus $\hat{\pi}^t_B(\hat{u}^t_1)\ge\hat{\pi}^t_B(u^t_2)$ for all $t>0$. It is shown that:

$$\begin{aligned}\hat{W}(u_1) &= E(\int_0^\infty e^{-rt}(\hat{\pi}^t_B(u^t_1)-\hat{c}_B(\tilde{g}^t_{B1}))dt)\ge E(\int_0^\infty e^{-rt}(\hat{\pi}^t_B(\hat{u}^t_1)-\hat{c}_B(\hat{g}^t_B))dt)\\ &> E(\int_0^\infty e^{-rt}(\hat{\pi}^t_B(u^t_2)-\hat{c}_B(\tilde{g}^t_{B2}))dt)=\hat{W}(u_2)\end{aligned}$$

The first inequality holds since $\hat{g}^t_B$ is suboptimal. $\hat{W}(u_3)>\hat{W}(u_4)$ if $\bar{u}<u_3<u_4<0$ can be easily shown using a similar method for $u_1$ and $u_2$.

# 2 Innovation Competition of Two Firms: Competition with Leapfrogging

## 2.1 Introduction

A firm's innovation is not always successful and does not create a fixed amount of achievement. Moreover, in some industries, firms tend to innovate with a time gap, naming a new version a new "generation," such as in the wireless telephone industry and operating systems of the personal computer industry. Another important feature of firms' innovation is that, in some industries, firms tend to compete with similar levels of quality. The competition between Android and iOS in wireless telephone operating systems is one example that shows neck-and-neck competition of Google and Apple. They innovate on the quality of the platform for the wireless telephone, and release significant breakthroughs intermittently.

In this chapter, I encompass and extend the model from the previous chapter—the dynamic innovation in the vertically differentiated market. In the previous chapter, I discussed two determinants of innovation: to increase the quality for more profit and to keep the distance of the quality, avoiding severe competition from identical products. Another possible determinant for the innovation of firms is taking or keeping the position as the leader, because, generally, the leader in the industry receives a higher profit than the follower. The continuous innovation model in the previous chapter is limited in that the follower never can be the leader in the industry, so it lacks the motivation to innovate when it is closer to the leader. Alternatively, this study also investigates innovations that yield a discrete jump via a Poisson arrival. In this model, both firms would innovate effortfully when the quality gap is close in order to become the leader in the market. In particular, for the follower, the momentum of widening the gap conflicts with the motivation to be the leader. However, if the size of the discrete jump to be achieved by the innovation is large, the force of jumping over the current leader dominates the force widening the gap.

This study encompasses phenomena shown in some empirical literature of leapfrogging. Schilling (2002) showed that, in the U.S. video game industry, a potential entrant can radically improve technology, thus successfully leapfrog the incumbent. Therefore, the leader might strategically defend its position. Moreover, Lee and Lim (2001) identified CDMA, D-RAM and automobile industries in Korea are examples of leapfrogging, showing that, in the final state of technological development of developing countries, where the quality of the follower(developing country) becomes similar to that of the leader (developed country), the follower tries to leapfrog by creating new product concepts and designs.

The distinct feature of this study is to set up dynamic innovation with leapfrogging and to analyze various the momentums of firms' innovation. Moreover, the systematic relationship of motivation of leapfrogging, widening the gap of quality to provide differentiated goods, and providing better quality of products for the higher profit are all examined regarding the quality ratio of two products and the size of possible innovation. Each momentum governs in a different state of development. When the quality is largely differentiated, each firm wants to increase the quality of its product for higher profit; in contrast, as the quality gap decreases, firms tend to widen the gap in quality, because it is more profitable to provide a more differentiated good in a vertically differentiated market. Nonetheless, as the quality gap becomes very narrow, the follower wants to leapfrog and the leader wants to defend its position as the leader. The momentum of leapfrogging becomes more powerful as the size of a discrete jump of innovation gets larger. Therefore, the size of a possible jump is a critical parameter in the analysis.

The plan of this paper is as follows. Section 2.2 sets up the basic model, retrieved from Chapter 1. In Section 2.3, a dynamic model with leapfrogging is described, and a numerical example that characterizes the equilibrium is provided. Section 2.4 represents the social planning model and compares it with the equilibrium. Finally, Section 2.5 concludes.

## 2.2 Basic Set-up

In this chapter, I employ the basic settings from chapter 1. Consider an industry with two firms which compete with levels of quality. Firm A produces $x_A$ and Firm B produces $x_B$. Firm A is the market leader in the industry, $x_A > x_B$; that is A produces better quality than B. There is no marginal cost of production for both goods.

A continuum of consumers have different marginal utility for quality $\theta$, which is uniformly distributed on $[0,1]$. A consumer's utility is defined as $\theta x_i - p_i$, where $p_i$ is the price of the product $x_i$. Then, there are three groups of consumers: the consumers will buy from firm A if $\theta x_A - p_A \geq \theta x_B - p_B$; the consumers will buy from B if $\theta x_B - p_B \geq \theta x_A - p_A$ and $\theta x_B - p_B \geq 0$; and the consumers will not buy any products from firm A and B if $\theta x_B - p_B \geq 0$. Then, the demands of each products are $s_A = 1 - \frac{p_A - p_B}{x_A - x_B}$ and $s_B = \frac{p_A - p_B}{x_A - x_B} - \frac{p_B}{x_B}$.

Each firm maximizes its profit $s_i p_i$ by choosing price $p_i$ with the given quality for each firm's product. Then, firms' profits per periods are represented as functions of the respective quality of two firms' products as follows:

$$\pi_A(x_A, x_B) = \frac{(2x_A)^2(x_A - x_B)}{(4x_A - x_B)^2} \tag{9}$$

$$\pi_A(x_A, x_B) = \frac{x_A x_B(x_A - x_B)}{(4x_A - x_B)^2} \tag{10}$$

Defining the follow-up level as $u = \frac{x_B}{x_A}$, $\pi_A$ and $\pi_B$ can be rewritten as the function of $u$ and $x_A$.

$$\pi_A(x_A, u) = \frac{(2x_A)^2(x_A - x_B)}{(4x_A - x_B)^2} = \frac{4x_A(1-u)}{(4-u)^2} = \hat{\pi}_A(u)x_A \tag{11}$$

$$\pi_B(x_A, u) = \frac{x_A x_B(x_A - x_B)}{(4x_A - x_B)^2} = \frac{x_A u(1-u)}{(4-u)^2} = \hat{\pi}_B(u)x_A \tag{12}$$

It is notable that $\pi_A$ and $\pi_B$ are multiplicably separable by $x_A$ and the function of

$u$. In other words, the quality of the leader's product and the follow-up level are crucial components for determining the profits and prices of the two firms. The increment of $x_A$ is profitable for the both firms, if $x_B$ increases at the same rate. However, as the follow-up level increases the profit of the leader decreases; the follower's profit increases in $u$ if $u$ is low enough, but decreases in $u$, if $u$ i high enough. Both firms profit are 0 when $u$ becomes 1; that is, when the levels of quality of firms are identical, due to the severe competition, they do not have positive profits. In sum, there are two forces determining profits of firms. First, firms' want to increase their quality, if the follow-up level is fixed, since both firms' profit increase in their quality. Second, they want to keep the gap of quality wide, because profits decrease as $u$ gets higher. These two forces affect the leader in the same direction; thus, the leader always want to have a higher quality of $x_A$. On the other hand, the two forces affect the follower in the opposite direction. Therefore, the follower wants to have the higher quality, when the follow-up level is low; in contrast, it wants to have the lower quality, if the follow-up level is high.

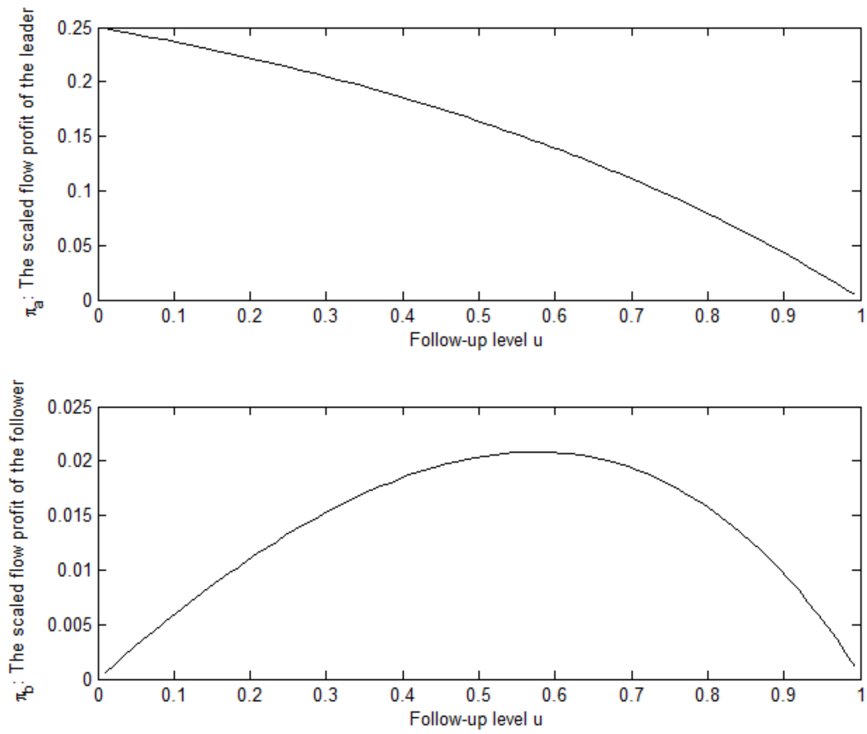


Figure 2-1: Normalized Profit by the Follow-up Level

Furthermore, both firms want to be a leader of the market. In particular, when the leader and the follower are reversed by the increment of $x_B$, the follower would like to have the higher quality if the follow-up level is high. More precisely, as can be seen from the Figure 2-1, the follower's maximum profit is 0.0208 when $u = 0.57$. However, if the follower becomes the leader, it would earn more than the maximum profit of the follower, although the quality of the new leader is a bit higher than the follower, $u > 1.05$. Therefore, the follower would like to be the leader, if it can sufficiently increase the quality.

## 2.3 Dynamic Game: Innovation with Leapfrogging

In this section, I consider the dynamic games of the leader and the follower, wherein they produce products continuously. The main focus of this section is on examining the follower's motivation to innovate, which depends on the possibility of being the leader. When there is no chance of being the leader, the innovation of the follower is greatly discouraged, as discussed in the previous sections. Nonetheless, if the follower can jump over the gap and become the new leader, then the follower can be highly motivated when the gap is narrow. Moreover, the leader also innovates aggressively when the gap is narrow since it is afraid of being caught by the follower. Then the motivation to innovate inherently depends on the size of the discrete changes made by the innovation.

The firms face a market of consumers at each instant for an infinite period with a discount rate $\frac{1}{1+r} < 1$. The state variables $x_A$ and $x_B$ are given in each time period. With the given state variables, firms maximize their profits by choosing the prices. Then, firms receive the flow profits as described in (9) and (10) respectively.

Firms' innovation make the quality of products change discretely. More precisely, the quality of products increases via Poisson arrival. The increment of quality when there is the size of Poisson arrival is randomly determined as $kx_A$, where the support of $k$ is $\varkappa = \{K\frac{n}{N} : 1 \leq n \leq N\}$, and where $K$ is the maximum value of $k$. The probability of the size of each increment is $P(k = \frac{n}{2N}) = \frac{1}{N}$ for all $n$. The higher $K$ indicates more possibilities of leapfrogging. The leader chooses innovation effort $\mu_A$

with the cost $C_A(\mu_A, x_A)$; the innovation effect $\mu_A$ determines the Poisson arrival rate of jump of the quality of $x_A$. The follower chooses innovation effort $\mu_B$ with the cost $C_B(\mu_B, x_A)$ which determines the arrival rate of jump of $x_B$. I impose the following assumptions on the cost of innovation.

**Assumption 3**

1. $C_A(\mu_A; x_A)$ is strictly increasing and convex in $\mu_A > 0$, and increasing in $x_A$. In addition, $C_A(\mu_A; x_A)$ is multiplicably separable in $x_A$ and the function of $\mu_A : C_A(\mu_A; x_A) = \hat{C}_A(\mu_A)x_A$

2. $C_B(\mu_B, x_A)$ is strictly convex in $\mu_B$, and increasing in $x_A$. It is multiplicably separable in $x_A$ and the function of $\mu_B$: $C_B(\mu_B; x_A) = \hat{C}_B(\mu_B)x_A$.

3. $\hat{C}_i(r) = \infty$, for $i = A, B$.

By this assumption, the HJB equations are well-defined as follows, wherein $I_K(x_A, x_B) =$

$I(x_A > x_B + kx_A) = I(k < \frac{x_A - x_B}{x_A})$ is the indicator that leapfrogging does not occur.

$$\begin{aligned} rV(x_A, x_B) &= \max_{\mu_A} \pi^A(x_A, x_B) - C_A(\mu_A, x_A) \\ &+\mu_A E_k[V(x_A + kx_A, x_B) - V(x_A, x_B)] \\ &+I_k(x_A, x_B)\mu_B E_k[V(x_A, x_B + kx_A) - V(x_A, x_B)] \\ &+(1 - I_k(x_A, x_B))\mu_B E_k[W(x_B + kx_A, x_A) - V(x_A, x_B)] \end{aligned} \tag{13}$$

$$\begin{aligned} rW(x_A, x_B) &= \max_{\mu_B} \pi^B(x_A, x_B) - C_B(\mu_B, x_A) \\ &+\mu_A E_k[W(x_A + kx_A, x_B) - W(x_A, x_B)] \\ &+I_k(x_A, x_B)\mu_B E_k[W(x_A, x_B + kx_A) - W(x_A, x_B)] \\ &+(1 - I_k(x_A, x_B))\mu_B E_k[V(x_B + kx_A, x_A) - W(x_A, x_B)] \end{aligned} \tag{14}$$

The innovation choice of each player deals with the trade-off between the cost of innovation that the firm spends in the current period, $C_A(\mu_A, x_A)$ (or $C_B(\mu_B, x_A)$) and the gains from the innovation. The leader's value increases through the innovation when the discrete changes of the quality occurs at the Poisson arrival rate $\mu_A$. Thus, the expected gains of the innovation of the leader is defined as $\mu_A E_k[V(x_A + kx_A, x_B) - V(x_A, x_B)]$. On the other hand, the follower's value depends on whether the leapfrogging occurs or not. If the leapfrogging does not occur ($x_B + kx_A < x_A$), the leader remains as the leader; thus, the expected gains of the innovation of the follower is defined as $\mu_B E_k[W(x_A, x_B + kx_A) - W(x_A, x_B)]$ in this case. If the leapfrogging occurs, the follower becomes the new leader of the industry.

In this case, the new state variable is $(x_B + kx_A, x_A)$ and the new leader, the previous follower, receives the value of $V(x_B + kx_A, x_A)$. Therefore, the expected gains of the innovation of the follower is $\mu_B E_k[V(x_B + kx_A, x_A) - W(x_A, x_B)]$.Furthermore, the innovation choice of one firm affects another firm's value. The leader's innovation affects the follower's value as $\mu_A E_k[W(x_A + kx_A, x_B) - W(x_A, x_B)]$, and the follower's innovation affects the value of the leader as well: $I_k(x_A, x_B)\mu_B E_k[V(x_A, x_B + kx_A) - V(x_A, x_B)] + (1 - I_k(x_A, x_B))\mu_B E_k[W(x_B + kx_A, x_A) - V(x_A, x_B)]$.

Using Assumption 3 and the fact that $\pi_A$ and $\pi_B$ are multiplicably separable by $x_A$ and the function of $u$ as discussed in the previous section, we rescale (13) and (14) as univariate equations.

**Proposition 7** *Let* $\hat{V}(u) = V/x_A$ *and* $\hat{W}(u) = W/x_A$*. Denote that* $I_k(u) = I(u+k < 1)$*, which is the indicator that leapfrogging does not occur. Then, the newly defined HJB equations can be normalized by the following equation:*

$$
\begin{aligned}
r\hat{V}(u) &= \max_{\mu_A} \hat{\pi}^A(u) - \hat{C}_A(\mu_A) \\
&\quad + \mu_A E_k[((1+k)\hat{V}(\frac{u}{1+k}) - \hat{V}(u))] \\
&\quad + \mu_B E_k[(I_k(u)\hat{V}(u+k) + (1-I(u))(u+k)\hat{W}(\frac{1}{u+k}) - \hat{V}(u))]
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
r\hat{W}(u) &= \max_{\mu_B} \hat{\pi}^B(u,\alpha) - \hat{C}_B(\mu_B) \\
&\quad + \mu_A E_k[((1+k)\hat{W}(\frac{u}{1+k})) - \hat{W}(u))] \\
&\quad + \mu_B E_k[(I(u)(\hat{W}(u+k) + (1-I(u))((u+k)\hat{V}(\frac{1}{u+k}) - \hat{W}(u))]
\end{aligned} \tag{16}
$$

(15) and (16) are now defined on one-dimensional state space of u. The rescaled HJB equations represents trade-offs between innovation efforts and the gains from the innovation in the rescaled value. By Assumption 3, the HJB equations are concave in each decision variables; thus, we can apply the first order conditions to get the optimal choices of innovation.

$$
\mu_A = \hat{C}'^{-1}_A(E_k[((1+k)\hat{V}(\frac{u}{1+k}) - \hat{V}(u))]) \tag{17}
$$

$$
\mu_B = \hat{C}'^{-1}_B(E_k[I(u)\hat{W}(u+k) + (1-I(u))(u+k)\hat{V}(\frac{1}{u+k}) - \hat{W}(u)]) \tag{18}
$$

The leader's innovation choice depends on the ratio of the quality changed by the innovation, $\frac{u}{1+k}$, and the increment of $x_A$ in the normalized term, $1+k$. On the other hand, the choice of the follower depends on whether leapfrogging occurs or not. If the leapfrogging does not occur, the innovation choice simply depends on how beneficial the increment of quality through innovation, $\hat{W}(u+k)-\hat{W}(u)$; in contrast, with the leapfrogging, the innovation choice also depends on the benefits of being the leader in the industry. Noting that $1-I(u)$ increases in the size of $K$, the maximum jump size, the follower has a lot of chances to become a new leader if $K$ is large. $K$ does not only affect the follower's innovation, but the leader's innovation as well. When the expected size of innovation is large, the leader is more motivated to innovate. Moreover, the possibility that the follower becomes the leader also affects the value function of the leader as can be seen from (15).

### 2.3.1 A Numerical Example

As an example, I consider different maximum sizes of the Poisson jump: $K = 0.1, \frac{1}{3}, \frac{2}{3}$ and 0.99. With the same discretization method described in section the Chapter 1, I computed firms' value functions for $u \in (0,1)$. I consider examples with the following cost functions and Poisson rate : $\hat{C}_A(\mu_A) = \begin{cases} 100\mu_A^2 & g_A < r \\ \infty & g_A \geq r \end{cases}$, $\hat{C}_B(\mu_B) = 100\mu_B^2$, $\lambda_A(\mu_A) = \mu_A$ and $\lambda_B(\mu_B) = \mu_B$. The interest rate $r$ is assumed as 0.1. Then, the first order conditions yield the following solutions:

$$
\begin{aligned}
\mu_A &= \frac{1}{200} E_k[(\hat{V}(\frac{u}{1+k})(1+k) - \hat{V}(u))] \\
\mu_B &= \frac{1}{200} E_k[(I(u)(\hat{W}(u+k) + (1-I(u))(u+k)\hat{V}(\frac{1}{u+k}) - \hat{W}(u)]
\end{aligned}
$$

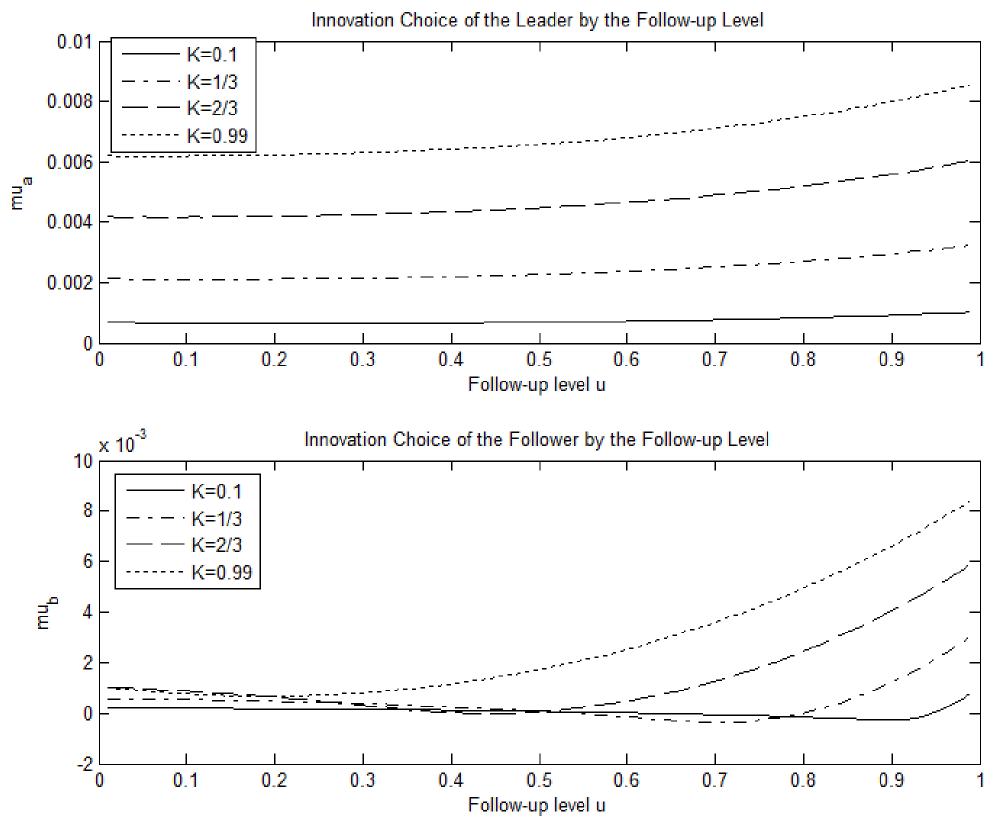


Figure 2-2: Firms' Innovation Choices with Different Sizes of Jumps

Figure 2-2 represents the numerical solutions of the dynamic program by follow-up level with different maximum jump sizes. As mentioned in Chapter 1, there are two forces of innovation: firms innovate to have higher quality and firms innovate to widen the quality gap. In addition, in this chapter, I discover another motivation for innovation: taking the position as the leader in the industry. This motivation grows as the size of the improvement made by innovation grows. Then, the follower is highly motivated to innovate when the quality ratio rises, expecting to be the leader in the industry. To keep the status as the leader, the leader invests aggressively when the size of the improvement made by innovation is large. Thus, the leader always innovates positively and tends to increase the innovation effort as the gap narrows. On the other hand, the follower has three phases of innovation choices in response to the quality gap. First, the follower innovates positively when the gap is wide, when it seeks a higher profit from higher product quality. At the intermediate point of the gap, the follower's innovation is passive or discouraged because the follower wants to avoid harsh price competition. Finally, when two firms are in neck-and-neck competition, the follower innovates actively, expecting to be the leader in the market. When the jump size is large, the third effect dominates the discouraging effect; thus, the follower innovates as actively as leader does.

A pseudo-steady state is defined in the following way. With different sizes of $K$, I calculate the follow-up level $u_t(K)$ by time, using the initial state $u_0 = 0.1216$. Taking out the first 5000 periods, the pseudo-steady state is calculated by average of follow-up level by time: $\tilde{u}_{ss}(K) = \frac{1}{5000}\sum_{t=5001}^{10000} u_t(K)$. Table 2-1 shows the pseudo-steady state by different levels of $K$. When the jump size is relatively small ($K = 0.1$), the pseudo-steady state is formed in the low level of follow-up level. In addition, as the jump size gets higher, the leader aggressively exerts innovation effort to avoid having the leader in the industry switch. The follower does not have to make much effort when the follow-up level is low because the discouragement effect is not diminished and the expectation to be the leader in the market is not that strong. Thus, the quality ratio in the pseudo-steady state becomes smaller, as the jump size grows due to the aggressive innovation of the leader and the passive effort of the follower. On the other hand, in the case where the maximum jump size is large ($K = 0.99$), the pseudo-steady-state is formed as 1.0980, which means the neck-and-neck competition Figure 2-3 shows how the state evolves by time when the jump size is 0.99. The state evolves easily to neck-and-neck competition. However, this state is not stable: as the leader is able to make large jumps by a series of innovation efforts, the state falls and the market could be segmented temporarily. Since the follower has the chance to make large jumps, the follow-up level recovers to the neck-and-neck state. Consequently, with the large jump size, both firms innovate very actively and the market leader of the industry keeps switching depending on the jumps that each firm makes.

| Table 2-1 | Pseudo-steady State in the Equilibrium |
|---|---|
| Maximum jump size | $\tilde{u}_{ss}(K)$ |
| $K = 0.1$ | 0.3765 |
| $K = \frac{1}{3}$ | 0.2170 |
| $K = \frac{2}{3}$ | 0.1667 |
| $K = 0.99$ | 1.0980 |

:

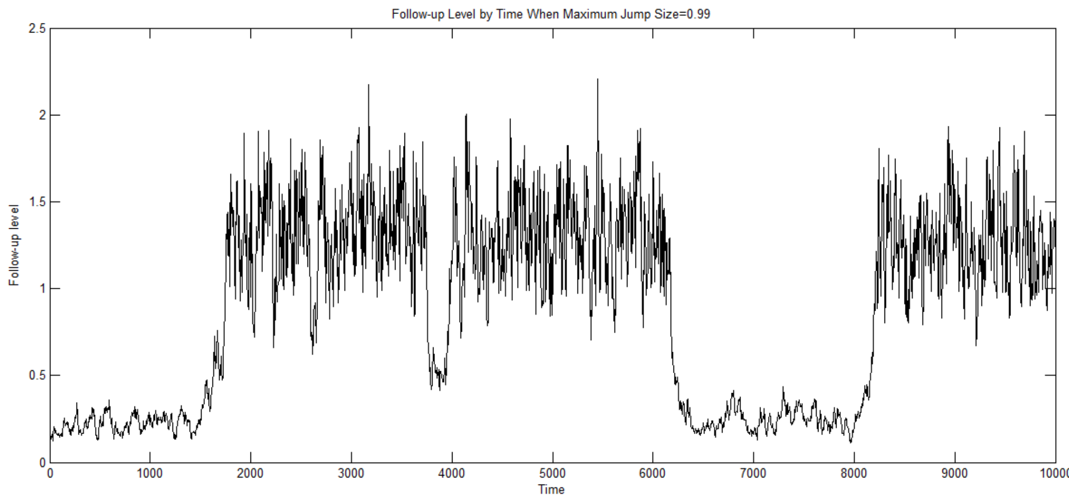


Figure 2-3: Follow-up Level by Time When Maximum Jump Size is 0.99

## 2.4 The Social Planning Problem

Consider an artificial economy, where the social planner decides innovation choices. The social planner considers not only the profits of the firms, but also consumers' surpluses who buys $x_A$ or $x_B$ in the market. Define $cs_A$ and $cs_B$ as consumers' surplus in each sector of the market. The consumers in the interval $S_A = [\frac{2-u}{4-u}, 1]$ buy the product $x_A$, while the consumers in the interval $S_B = [\frac{1-u}{4-u}, \frac{2-u}{4-u}]$ buy the product $x_B$. Then, the total social welfare of the economy at each instant can be computed as follows:

$$w(x_A, x_B) = cs_A + cs_B + \pi_A + \pi_B = \int_{\theta \in S_A} \theta x_A d\theta + \int_{\theta \in S_B} \theta x_B d\theta \qquad (19)$$

The welfare function $w$ is mutiplicably separable by $x_A$ and the function of $u$.

$$w(x_A, x_B) = x_A \hat{w}(u) = x_A \frac{-2u^2 - u + 12}{2(4-u)^2}, \text{ if } u < 1 \tag{20}$$

$$w(x_A, x_B) = x_B \hat{w}(\frac{1}{u}) = x_A \frac{-2\frac{1}{u}^2 - \frac{1}{u} + 12}{2(4-\frac{1}{u})^2}, \text{ if } u > 1 \tag{21}$$

The scaled welfare function, $\hat{w}(u)$, always increases in $u$ if $u < 1$, while decreases in $u$ if $u > 1$. The social welfare in this economy increases when the follower and the leader have similar levels of quality. The scaled HJB equation of the planner is as follows:

$$\begin{aligned} r\hat{J}(u) &= \max_{\mu_A,\, \mu_B} \hat{w}(u) - \hat{C}_A(\mu_A) - \hat{C}_B(\mu_B) \\ &\quad + \mu_A E_k[((1+k)\hat{J}(\frac{u}{1+k}) - \hat{J}(u))] \\ &\quad + \mu_B E_k[(I(u)\hat{J}(u+k) + (1-I(u))(u+k)\hat{J}(\frac{1}{u+k}) - \hat{J}(u))] \end{aligned} \tag{22}$$

Then, the FOCs yield:

$$\mu_A^J = \hat{C}_A'^{-1}(E_k[((1+k)\hat{J}(\frac{u}{1+k}) - \hat{J}(u))]) = 0 \tag{23}$$

$$\mu_B^J = \hat{C}_B'^{-1}(E_k[(I(u)\hat{J}(u+k) + (1-I(u))(u+k)\hat{J}(\frac{1}{u+k}) - \hat{J}(u))]) \tag{24}$$

(23) and (24) represents sthe social planner's choice. The social planner's choice of the leader's innovation is also dependent on the size of innovation that can produced by the innovation. If it is expected that the size is large, then the social planner invests more in the leader's innovation. The follower's innovation also depends on the size of innovation as well. In particular, $I(u)$ decreases in $K$; that is, the follower's probability of leapfrogging will increase in the maximum size of innovation. Therefore, the social planner would choose more innovation of the follower, if $K$ is large.

### 2.4.1 Numerical Example

As an exercise, I simulated an example of a social planner's optimal choice with the parameters in the previous numerical example.

Figure 2-4 shows the comparison between socially optimal innovations and innovations in equilibrium when the jump size is small ($K = 0.1$) and when the jump size is very large ($K = 0.99$). In both cases, both innovation choices are higher than the equilibrium choice, because the innovation of both firms can increase consumers' surpluses as well. In other words, the returns from the innovation are higher when we consider how much consumers would receive more through innovation. Moreover, the leader's innovation becomes significantly higher in the social planner's choice. The reason is that the social welfare is the highest when the ratio of the quality is around $u$. Thus, the leader should innovate aggressively, expecting large amounts of innovation of the follower; in particular, if the follower actively innovates with leapfrogging, the leader must innovate more to keep the ratio around 1.

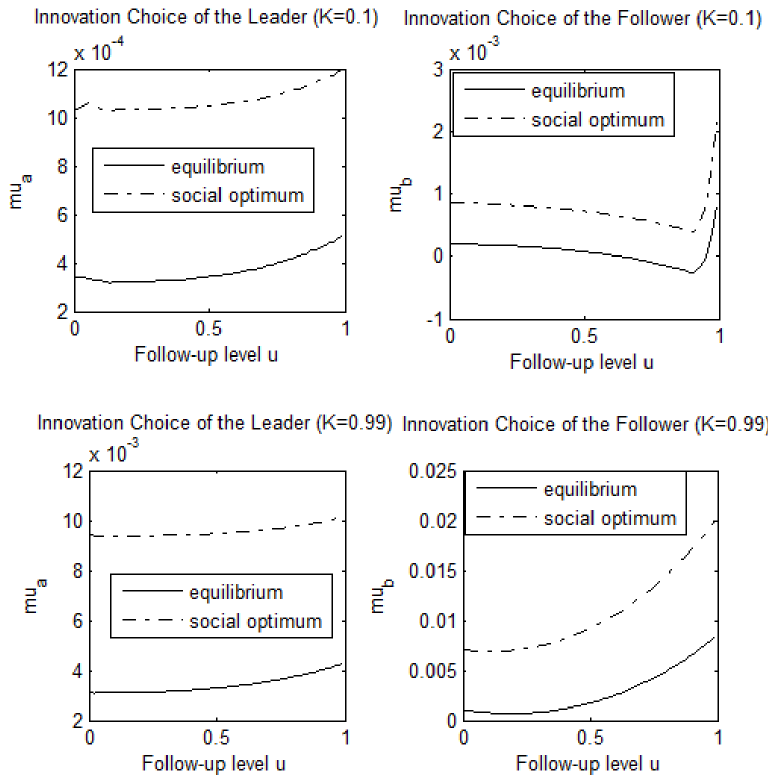


Figure 2-4: Firms' Innovation Choices in the Equilibrium and in the Social Planning

I calculated the pseudo-steady state of the social planner's problem as explained in the Section 2.3.1. Except the case that $K = 0.99$, the pseudo-steady state is higher in the social planning case. The primary reason is that social welfare always increases in the $u$. Therefore, the planner wants to keep $u$ higher than in the equilibrium. More precisely, the follower innovates very actively in the social planner's problem as can be seen from Figure 2-3, because $u$ increases as the quality of the follower increases. The pseudo-steady state is formed around 1, when the jump size is large. When the large jump of the competitor is expected, both firms would be very active in innovation; thus, they innovate keeping the state as neck-and-neck. It is notable that the pseudo-steady state in the equilibrium is similar to that of the social planner only when the jump size is very large. Consumers in this industry would have more welfare when jump size is very large, so the ratio of quality is around 1.

| Table 2-2 | Pseudo-steady State in the Equilibrium and in the Social Planning | |
|---|---|---|
| Maximum jump size | Pseudo-steady-state in the Equilibrium | Pseudo-steady-state of the Social Planning Problem |
| $K = 0.1$ | 0.3765 | 0.6013 |
| $K = \frac{1}{3}$ | 0.2170 | 0.6170 |
| $K = \frac{2}{3}$ | 0.1667 | 0.7657 |
| $K = 0.99$ | 1.0980 | 0.9993 |

In sum, the jump size affects the innovation choice of two firms in a decentralized economy and for the social planner. We can expect that firms innovate actively and increase social welfare through their innovation. However, the medium levels of the size of jump ($K = \frac{1}{3}$ or $K = \frac{2}{3}$) only decreases $u$, since it makes the leader innovate very actively, while the leapfrogging is not easy for the follower. In this case, the follower with the medium level of $u$, is discouraged from innovation, since the motivation that two firms tend to keep the distance far is more powerful than the leapfrogging motivation. Therefore, the medium size of the jump is harmful for the social optimum. On the other hand, when the jump size is very high ($K = 0.99$), the follower has the expectation of leapfrogging, even when the quality ratio is very low. Therefore, both firms innovate very actively, in particular when $u$ is high: the follower innovates actively to leapfrog, while the leader innovates actively to keep its status as the leader. Thus, the equilibrium choice with neck-and-neck competition, wherein $K = 0.99$, becomes close to the socially optimal choice.

## 2.5 Conclusion

The purpose of this study is to analyze various forces that affect firms' innovation choices in the vertically differentiated market. The distinct feature of this study is to examine the follower's leapfrogging and the defense of the leader aside from the determinants of innovation analyzed in Chapter 1. The leader wants to keep its status as the leader, while the follower wants to catch up and be the new leader in the industry with the neck-and-neck competition, as can be seen in various high-tech industries.

Therefore, I investigated the innovation choices of two firms when there is the possibility of leapfrogging. Firms' product quality increases discretely. When the size of a jump and the follow-up level are high enough, it is possible that the leader of the industry can change. This motivation forces two firms to innovate actively when the gap is narrow. Thus, it affects the follower in the opposite direction to the effect of the motivation of keeping a further distance. However, when the maximum possible jump is large, then the follower innovates further, expecting to be the leader of the industry. As a result, the possibility of higher leapfrogging always makes two firms innovate more competitively.

The notable finding is that the equilibrium ratio of quality, wherein the size of possible jump is very large, is similar to the socially optimum ratio. Thus, the policy maker can invest in the size of jump to encourage firms to innovate actively. An example of investment in size of jump is to establish a better communications network, which can drive the large jump of quality in the most of the IT industry.

## 2.6 Appendix: Proofs and Calculations

### 2.6.1 Normalization of value functions

Deviding both side of (13) and (14) by $x_A$, we get the following equations.

$$\begin{aligned}
\frac{rV(x_A,x_B)}{x_A} &= \max_{\mu_A} \hat{\pi}^A(u) - \hat{C}_A(\mu_A) \\
&+\mu_A E_k[\frac{V(x_A+kx_A,x_B)}{x_A} - \frac{V(x_A,x_B)}{x_A}] \\
&+I(x_A,x_B)\mu_B E_k[\frac{V(x_A,x_B+kx_A)}{x_A} - \frac{V(x_A,x_B)}{x_A}] \\
&+(1-I(x_A,x_B))\mu_B E_k[\frac{W(x_B+kx_A,x_A)}{x_A} - \frac{V(x_A,x_B)}{x_A}]
\end{aligned} \tag{25}$$

$$\begin{aligned}
\frac{rW(x_A,x_B)}{x_A} &= \max_{\mu_B} \hat{\pi}^B(x_A,x_B) - \hat{C}_B(\mu_B) \\
&+\mu_A E_k[\frac{W(x_A+kx_A,x_B)}{x_A} - \frac{W(x_A,x_B)}{x_A}] \\
&+I(x_A,x_B)\mu_B E_k[\frac{W(x_A,x_B+kx_A)}{x_A} - \frac{W(x_A,x_B)}{x_A}] \\
&+(1-I(x_A,x_B))\mu_B E_k[\frac{V(x_B+kx_A,x_A)}{x_A} - \frac{W(x_A,x_B)}{x_A}]
\end{aligned} \tag{26}$$

Let $\hat{V}(u) = V/x_A$ and $\hat{W}(u) = W/x_A$. Then, we get the following expression for each components in the equtions

$$
\begin{aligned}
\frac{V(x_A+kx_A,x_B)}{x_A} &= (1+k)\frac{V(x_A+kx_A,x_B)}{x_A+kx_A} = (1+k)\hat{V}(\frac{u}{1+k}) \\
\frac{V(x_A,x_B+kx_A)}{x_A} &= \hat{V}(u+k) \\
\frac{W(x_B+kx_A,x_A)}{x_A} &= (u+k)\frac{W(x_B+kx_A,x_A)}{x_B+kx_A} = (u+k)\hat{W}(\frac{1}{u+k}) \\
\frac{W(x_A+kx_A,x_B)}{x_A} &= (1+k)\frac{W(x_A+kx_A,x_B)}{x_A+kx_A} = (1+k)\hat{W}(\frac{u}{1+k}) \\
\frac{W(x_A,x_B+kx_A)}{x_A} &= \hat{W}(u+k) \\
\frac{V(x_B+kx_A,x_A)}{x_A} &= (u+k)\frac{V(x_B+kx_A,x_A)}{x_B+kx_A} = (u+k)\hat{V}(\frac{1}{u+k})
\end{aligned}
$$

Inserting these expressions in (25) and (26), we get the normalized HJB equations.

$$
\begin{aligned}
r\hat{V}(u) &= \max_{\mu_A} \hat{\pi}^A(u) - \hat{C}_A(\mu_A) \\
&+\mu_A E_k[((1+k)\hat{V}(\frac{u}{1+k}) - \hat{V}(u))] \\
&+\mu_B E_k[(I(u)\hat{V}(u+k) + (1-I(u))(u+k)\hat{W}(\frac{1}{u+k}) - \hat{V}(u))] \\
r\hat{W}(u) &= \max_{\mu_B} \hat{\pi}^B(u) - \hat{C}_B(\mu_B) \\
&+\mu_A E_k[((1+k)\hat{W}(\frac{u}{1+k})) - \hat{W}(u))] \\
&+\mu_B E_k[(I(u)(\hat{W}(u+k) + (1-I(u))((u+k)\hat{V}(\frac{1}{u+k}) - \hat{W}(u))]
\end{aligned}
$$

# 3 Dynamic Innovation of Complementary Goods

## 3.1 Introduction

When high-tech consumers buy a product under a platform/software system, they compare the availability of complementary products. For instance, a smartphone user might compare the number of apps available in the App Store to those in Google Play. In addition, he/she would compare the quality of the apps provided on each platform. Consumers would pay a great deal more for a platform that provided various high-quality apps. Under a platform/software system, firms' profits are interdependent on each other's quality because consumers in the market use combined products. In other words, a platform firm's profit derives from the utility consumers gain by using the software of the firm, while the software firm's profit depends on the quality of the platform.

The aim of this paper is to uncover the relationship between a platform firm's innovation and a software firm's innovation, wherein each firm increases the quality of its product, and to compare the innovative motivations of the two firms. In particular, this study encompasses phenomena provided in the empirical literature. Bayus (1987) and Gandal, Kende, and Rob (2000) examined the Compact Disc (CD) industry. They discovered that the growth in CD titles drove the growth in the CD player industry. Their studies analyzed the network effect on the industry based on the theory that there is feedback from software growth to platform industry from consumption externalities. Saloner and Shepard (1995) and Gandal, Greenstein, and Salant (1999) also studied the platform/software relationship industry based on network externalities. These studies showed development of software determined the value of a platform. Although these studies provided empirical evidences of positive feedback from software industries to platform industries, they did not clearly uncover the way in which development of platform industries affected the software industries. However, software development also depends highly on the quality of the platform. Consumers enjoy higher utility when they use software on the upgraded version of a platform. Representative examples are provided throughout the operating system industry. Bresnahan (2001) showed that the changes from Microsoft DOS to Windows and the release of improved versions of Windows allowed many software firms to increase the quality of their products. Moreover, the development of the 3G technology of the wireless telephone industry produced a slew of new apps. Gandal, Salant, and Waverman (2003) discussed the ways in which the wireless telephone industry developed with network effects under the platform/software system.

The model I studied used two firms that produce complementary products to analyze the industry. Firms decide on innovative efforts to increase the quality of products in continuous time. The distinctive features of the model are: a) consumers' utility is only dependent on the quality of software, assuming the practical utilities of consumers derive from usage of the software on the platform; b) the software's innovation cost becomes less expensive with the wider gap of quality between the platform and the software because the release of a newer version of the platform enables the practical quality of the software to increase; and c) the two firms receive profits derived from the joint production of new products and divide the joint profit in a fixed share ratio. Therefore, the gap in the quality of the platform and the software becomes a crucial variable in the model; the innovation of the platform tends to increase the gap in the quality, while the innovation of the software decreases the gap.

This study is remarkable in explaining the interdependent relationship between the innovation choices of the software and the platform and the motivations for such innovation based on the inter-dependent relationship. The study uncovers the primary determinants of the growth of the platform/software industry as follows. The first indication is that firms want higher-quality software to increase their flow profit. Thus, the software firm innovates to increase its quality, and the platform firm innovates to decrease the cost of software innovation, widening the gap between the platform and the software. The second involves firms wanting to keep the gap sufficiently wide. Narrow gaps increase the cost of software innovation, which decreases the profits of both firms. Third, firms want to smooth the cost of innovation. If they anticipate the need for a large increase in innovation, the firms tend to plan in advance of that innovation. A wider gap implies the expectation of a large degree of software innovation, which motivates the platform firms to innovate in order not to be outdistanced by the software.

This chapter proceeds as follows. Section 3.2 sets up the basic model. Section 3.3 introduces the dynamic game in continuous time and provides a closed-form solution. Section 3.4 describes the social planning problem. Section 3.5 provides a calculated example and compares the decentralized economy and the social planning model. Finally, section 3.6 concludes the paper

## 3.2 The Problem: Basic Setup and Assumption

Consider an industry with two firms that produce complementary products. One firm constructs a platform $x_t$; while another firm produces software, $y_t$. Only the combined goods of two products are sold in the market. Consumers cannot acquire the utilities before they use the software on the platform. Thus, the firms receive a joint profit $y_t$ which derives only from the usage of the software. For simplicity we can assume that firms receive profit according to the fixed sharing rule: a software firm gets $\alpha y_t$, and platform firm gets $(1-\alpha)y_t$ at time $t$.

There is no marginal cost of production, but the firms invest in innovation to increase the level of quality. The platform firm invests in constructing a new generation of platform, where $x_t \in \{0, 1, 2.......\}$. The investment of the platform firm comes into force as $\lambda$ that is the Poisson arrival rate of the new generation of platform; that is, $T\tilde{}\ \exp(\lambda)$, wherein $T$ denotes the time when the new generation of platform arrives. The cost of innovation $\psi(\lambda)$ increases and convexes in $\lambda$. On the other hand, the software firm invests in the innovation of the quality of the software, $y_t$. This innovation increases $\dot{y}_t = \frac{dy_t}{dt}$ with the cost of innovation $c(\dot{y}_t, x_t - y_t)$. $c(\dot{y}_t, x_t - y_t)$ is increasing and convexing in $\dot{y}_t$. However it decreases in $x_t - y_t$; in other words, when the quality gap between the platform and software decreases, innovating new version of software becomes comparatively easy.

The platform firm and the software firm innovate continuously and face a market of consumers at each juncture for an infinite period with a discount rate $\frac{1}{1+r} > 0$. The flow of profits is $(1-\alpha)y_t$ for the platform firm and $\alpha y_t$ for the software firm, according to the fixed sharing rule. Each firm faces the state variable $(x_t, y_t)$. Within a given state, the firms sell a combined product with the price $y_t$ and monopolize the market. Thus, the firms' innovative choices are complete with the state variable updated by the firms in continuous time.

**Definition 8** *A Markorv Perfect Equilibrium is a set of innovation strategies* $\Xi = \{\lambda_t(x_t, y_t)\}_{t=0}^{\infty}$ *and* $\Theta = \{\dot{y}_t(x_t, y_t)\}_{t=0}^{\infty}$.

1. *For all initial state* $(x, y)$, $\Xi$ *maximizes the expected payoff given* $\Theta$.

$$V(x,y) = \max_{\Xi} E_0[\lim_{T\to\infty} \int_0^T \{e^{-r\tau}\{(1-\alpha)y_t - \psi(\lambda_t)\}dt\,|(x,y),\Theta] \tag{27}$$

2. *For all initial state* $(x, y)$, $\Theta$ *maximizes the expected payoff given* $\Xi$.

$$W(x,y) = \max_{\Theta} E_0[\lim_{T\to\infty} \int_0^T \{e^{-r\tau}\{\alpha y_t - c(\dot{y}_t, x_t - y_t)\}dt\,|(x,y),\Xi] \tag{28}$$

I impose the following assumptions on the cost function of innovation to ensure that the innovation choice of each firm is well defined.

**Assumption 1**

1. $\psi(\lambda_t)$ strictly increases and convexes in $\lambda_t > 0$. In addition, $\psi(\lambda_t) = \frac{1}{2}a\lambda_t^2$.
2. $c(\dot{y}_t, x_t - y_t)$ strictly increases and convexes in $\dot{y}_t. > 0$ and decreasing in $x_t - y_t.c(\dot{y}_t, x_t - y_t) = \frac{1}{2(x_t - y_t)}by_t^2$

In the Assumption 1, the restriction on the firms' costs of innovation is in quadratic form without loss of generality. $a, b \in (0, \infty)$are arbitrary numbers that represent the difficulty of innovation.

The Hamilton-Jacobi-Bellman(HJB) equations are obtained as follow.

$$
\begin{aligned}
rV(x,y) &= \max_{\lambda}(1-\alpha)y + \lambda(V(x+1,y) - V(x,y)) + V_y(x,y)\dot{y}(x,y) - \psi(\lambda) && (29)\\
rW(x,y) &= \max_{\dot{y}} \alpha y + W_y(x,y)\dot{y} + \lambda(x,y)(W(x+1,y) - W(x,y)) && (30)\\
&\quad -c(\dot{y}, x-y)
\end{aligned}
$$

Hamilton-Jacobi-Bellman(HJB) equations ((29) and (30)) include the following components. The firm's value function contains the firm's flow profit $(1-\alpha)y$ (or $\alpha y$). At each moment, the innovation choice of each firm shows the trade-off between thecurrent innovation cost $\psi(\lambda)$ (or $c(\dot{y}, x-y)$) and the effect of the current innovation on the firm's value for infinite period $\lambda(V(x+1,y)-V(x,y))$ or $(W_y(x,y)\dot{y})$. The other firm's innovation choice also affects the firm's value$V_y(x,y)\dot{y}(x,y)(\lambda(x,y)(W(x+1,y)-W(x,y)))$.

## 3.3 Dynamic Game: Firms' Optimization

In this section, I study a firm's optimization problem. Discerning the intuitive relationship between innovation choice and state variables of (29) and (30) is not uncomplicated because they are multivariate. Therefore, I reformulate the HJB equation as a one-dimensional program that depends only on the gap between the platform and the software. Employing the first-order condition, we find the firms' optimization solution represented as the value functions. Value functions contract and have closed-forms. Thus, the gap between the platform and the software can represent the firms' optimal innovation choices.

The HJB equations, (29) and (30), are functions of $x$ and $y$, yet they can be decomposed by the flow profits and functions of the gap of the levels of quality of platform and software. If we denote the gap of quality as $z=x-y$, we can reformulate the HJB equations as univariate.

**Proposition 9** *(normalization of value functions) Let* $V(x,y)=\frac{(1-\alpha)y}{r}+\hat{V}(z)$ *and* $W(x,y)=\frac{\alpha y}{r}+\hat{W}(z)$*. Then, the HJB equations defined in (29) and (30) can be* *normalized by the following equation:*

$$r\hat{V}(z) = \max_{\lambda} \lambda(\hat{V}(z+1) - \hat{V}(z)) + (\frac{1-\alpha}{r} - \hat{V}'(z))\dot{y}(z) - \psi(\lambda) \quad (31)$$

$$r\hat{W}(z) = \max_{\dot{y}} \lambda(z)(\hat{W}(z+1) - \hat{W}(z)) + (\frac{\alpha}{r} - \hat{W}'(z))\dot{y} - c(\dot{y}, z) \quad (32)$$

We have defined the HJB equations the one-dimensional state space of z and concave in each decision variable with the provided assumption on the cost of effort. The normalized HJB equations show the trade-off between innovation cost and the returns of the innovation on the normalized value. The normalized HJB equations show the trade-off between innovation cost and the returns of the innovation on the normalized value. In detail, when the platform firm spends $\psi(\lambda)$ as the innovation cost, the gains from the current innovation for infinite period is $\lambda(\hat{V}(z+1) - \hat{V}(z))$. The additional gains in value function through developing a new generation platform determine the gain of innovation per unit of $\lambda$. Similarly, the cost of software innovation is $c(\dot{y}, z)$ to produce returns from the innovation $(\frac{\alpha}{r} - \hat{W}'(z))\dot{y}$. A growth of $y$ increases the profit per infinite period. Thus, the share of joint profit ($\alpha$) and the interest rate ($r$) affect the returns from the innovation. Moreover, the gains of value function ($-\hat{W}'(z)$) by the increment of $y$ (by the decrease of $z$) is another force of the software firm's innovation.

The solutions of the two equations are well defined and result from the following first order conditions:

$$\lambda(z) = \frac{1}{a}(\hat{V}(z+1) - \hat{V}(z)) \quad (33)$$

$$\dot{y}(z) = \frac{1}{b}(\frac{\alpha}{r} - \hat{W}'(z))z \quad (34)$$

As the difference between the value of the new generation of platform and the value of the current one becomes larger, the platform firm is more likely to invest in innovation, while it spends less on innovation as the cost of innovation increases. On the other hand, the software firm tends to innovate if the present value of the share of joint profits is larger. Also, as the effect of innovation of software on the value function ($-W\prime(z)$) becomes more significant, the software firm has greater motivation to innovate. The inclination is to innovate further as the cost of innovation becomes less expensive, has a bigger gap in quality, and a smaller difficulty level, $b$.

### 3.3.1 Closed-form Solution

Inserting (33) and (34) to (31) and (32) respectively offers the following expressions for the value functions.

$$r\hat{V}(z) = \frac{1}{2a}(\hat{V}(z+1)-\hat{V}(z))^2 + \frac{1}{b}(\frac{\alpha}{r}-\hat{W}'(z))(\frac{1-\alpha}{r}-\hat{V}'(z))z \quad (35)$$

$$r\hat{W}(z) = \frac{1}{a}(\hat{V}(z+1)-\hat{V}(z))(\hat{W}(z+1)-\hat{W}(z)) \quad (36)$$
$$+\frac{1}{2b}(\frac{\alpha}{r}-\hat{W}'(z))^2 z$$

Only the function of $z$ represents both sides of equation. Therefore, we can employ a Guess-and-Verify method to find the closed-form of the value function.

**Lemma 10** *HJB equations (31) and (32) have closed-form solutions that have following properties.*

1. $\hat{V}'(z) > 0$ *and* $\hat{V}''(z) > 0$

2. $\hat{W}'(z) < 0$

Then, we find the following Proposition, which describes the optimal choices of innovation for the platform firm and the software firm.

**Proposition 11** *(Characterization of Equilibrium)The optimal choices of innovation (33) and (34) have the following closed-form solution. As the distance between the software and platform increases, both firms have greater motivation to improve the* *quality of their products.*

$$\lambda(z) = \frac{-4\alpha}{3br} - \frac{1-\alpha}{ar} - \frac{3}{2}r - \frac{3br^3}{8\alpha} + (\frac{8\alpha}{3br} + r)z \tag{37}$$

$$\dot{y}(z) = \frac{4\alpha}{3br}z \tag{38}$$

(38) indicates that the innovation effort of software firm increases in z because it is comparatively easy to develop new software when the production of software is far behind that of the platform firm. Thus, developing a new generation of the platform boosts innovation in the software. In addition, two forces affect the decision for software innovation. Developing the quality of the software can increase the flow profit for an infinite period ($\frac{\alpha}{r}$). Moreover, the widening gap between the platform and the software is not profitable for the software firm itself.

The response of the software to the new generation of platform affects the innovation choice of the platform. Initially, the innovation of the platform firm motivates the innovation of the software firm because it lowers the cost of innovation. As the new generation of the platform develops, $y$ increases through (38). Thus, the flow profit of the firm increases due to the innovation. Moreover, the platform firm's innovation accelerates due to the new generation of the platform. More precisely, when the gap in quality is large, the software firm attempts to catch up quickly. However, when the gap narrows, the cost of software innovation becomes more expensive and discourages the software firm from innovation. In that case, the platform firm may invest more on the new generation. Expecting this phenomenon, the platform firm tends to invest more in advance to smooth the cost of innovation in advance before the software firm catches up. The larger $z$ means that the software firm innovates more actively. Thus, as $z$ increases, the platform firm innovate more to offset the software firm's efforts to catch up. Finally, if the firm does not develop another new generation of the platform, $\dot{y}$ decreases, as $y$ increases. Thus, this force negatively affects the innovation of the platform firm.

Parameters of the program, $\alpha$, $r$, $b$ and $a$, affect firms' choices of innovation directly or indirectly. The summary of signs of impact for each parameter follows.

**Proposition 12** *Parameters $a, b, r$ and $\alpha$ affect equilibrium solution (38) as follows.*

1. $\frac{\partial \dot{y}(z)}{\partial \alpha} > 0$ *for all* $z$.

2. $\frac{\partial \dot{y}(z)}{\partial r} < 0$ *for all* $z$.

3. $\frac{\partial \dot{y}(z)}{\partial b} < 0$ *for all* $z$.

Comparative statics for (38) are fairly straightforward. Software firms innovate more when the innovation creates more profit due to a higher share ratio ($\alpha$). However, the higher the interest rate, the less software firms innovate because they care more about the current cost of innovation than they do about future profits. Furthermore, the innovation investment can decrease in cost .

On the other hand, analyzing the impact of the parameters on platform firms' optimal innovation choices involves studying both the direct effects and indirect effects through the strategic aspect regarding the software firms' choices. Indirect effects arise from the relationship between $\lambda$ and $\dot{y}$. In other words, the innovation of the platform firm motivates greater innovation from the software firms.

**Proposition 13** *Parameters $a, b, r$ and $\alpha$ affect equilibrium solution (37) as follows.*

1. $\frac{\partial \lambda(z)}{\partial \alpha} > 0$ *if* $z > \frac{32a\alpha^2 - 24b\alpha^2 - 9abr^4}{64a\alpha^2}$.

2. $\frac{\partial \lambda(z)}{\partial r} < (>) 0$ *if* $z > \frac{(32a-24b)\alpha^2 + (24b - 36abr^2)\alpha - 27ab^2r^4}{64a\alpha^2 - 24abr^2\alpha}$ *and* $\alpha > (<) \frac{3br^2}{8}$ .

3. $\frac{\partial \lambda(z)}{\partial b} < 0$ *if* $z > \frac{1}{2} - \frac{9b^2r^4}{64\alpha^2 z}$

4. $\frac{\partial \lambda(z)}{\partial a} > 0$ *for all* $z$

Inequalities in the Proposition represent the strategic choices of firms in detail. For sufficiently large $z$, as the share of flow profit for the software firm increases, the platform firm innovates more. The larger share ratio means the returns on innovation investment is large, which motivates the software firms to innovate actively. Therefore, if innovation occurs in the quality of the software ($y$), the platform firm receives more profit. Furthermore, if $\alpha$ is sufficiently large, the platform firm tends to innovate less with the higher interest rate. In other words, when $\alpha$ is greater than $\frac{3br^2}{8}$, the platform firm receives fewer additional profits from increments of $y$, and the higher interest rate means that the firm cares more about the current period than about the expected future profits from the innovation. Thus, the platform firm may be reluctant to fund the cost of innovation because it is less profitable to motivate the software firm to innovate. In addition, the cost of innovation affects the choice of platform innovation. The higher the costs for the software firm, the fewer innovations of platform because the higher quality of platform is not effective in motivating the software firm to innovate due to the expense. On the other hand, the platform firm's innovation increases its cost of innovation. At first glance, it does not seem to match the direct inference. This is primarily because $\hat{V}''(z)$ is positive and increasing in $a$. In other words, the higher z accelerates more innovation of the platform due to the cost-smoothing motivation, as discussed with (37). With the higher cost of the platform's innovation, the cost-smoothing motivation becomes more powerful. Therefore, although the direct effect of the cost of innovation decreases the amount of innovation, the cost-smoothing motivation transcends the direct effect of the higher cost.

## 3.4 Social Planner's Problem

In this section, I describe an artificial economy wherein a social planner decides the innovation choices of the firms, optimizing the joint value function. The section compares the solution of the social planner's optimal choice to with the decentralized solution described in the previous section. The planner chooses $\lambda$ and $\dot{y}$, wherein the sum of the flow profit is $y$. The HJB equation for the social planner follows.

$$J(x,y) = \max_{\lambda,\dot{y}} y + \lambda(J(x+1,y) - J(x,y)) + J_y(x,y)\dot{y} - \psi(\lambda) - c(\dot{y}, x-y) \quad (39)$$

The HJB program can be normalized as a one dimensional equation by using the similar method.

$$r\hat{J}(z) = \max_{\lambda,\dot{y}} \lambda(\hat{J}(z+1) - \hat{J}(z)) + (\frac{1}{r} - \hat{J}'(z))\dot{y} - \psi(\lambda) - c(\dot{y}, x-y) \quad (40)$$

Therefore, the FOC yields:

$$\begin{aligned} \lambda(z) &= \frac{1}{a}(\hat{J}(z+1) - \hat{J}(z)) \quad (41) \\ \dot{y}(z) &= \frac{1}{b}(\frac{1}{r} - \hat{J}'(z))z \quad (42) \end{aligned}$$

Inserting (41) and (42) to (40), the following expression for the HJB equation is obtained. Using the Guess-and-Verify method produces a closed form solution.

$$r\hat{J}(z) = \frac{1}{2a}(\hat{J}(z+1) - \hat{J}(z))^2 + \frac{1}{2b}(\frac{1}{r} - \hat{J}'(z))z^2 \quad (43)$$

**Proposition 14** *(40) has a closed-form solution. Thus, the socially optimal choices of innovation have the following solution.*

$$\lambda^J(z) = \frac{1 + br^2 - \sqrt{2br^2 + b^2r^4}}{ar} \quad (44)$$

$$\dot{y}^J(z) = \frac{r - br^2 + \sqrt{2br^2 + b^2r^4}}{br} z \quad (45)$$

The important feature of socially optimal choice of platform is that it is chosen regardless of $z$, because it does not involve the strategic relationship with the choice of $\dot{y}$. In other words, platform firm's choice is made without considering the optimal response of the software firm. The cost smoothing effect disappears; thus the platform's innovation is not accelerated by the quality gap any more. It is also notable that both firms' choices decrease in the cost of software innovation. It is straightforward the reason that $\dot{y}^J(z)$ decreases in $b$. Platform's innovation also decreases in $b$, since the impact of increment of $z$ on $\dot{y}$ becomes weaker, if the innovation of the software becomes more costly. Furthermore, both innovation choices decrease in $r$; that is, firms reduce innovation efforts, as they care more about the current period than the future income.

The important feature of a socially optimal choice of platform is its selection regardless of $z$ because it does not involve a strategic relationship with the choice of $\dot{y}$. In other words, the platform firm chooses without considering the optimal response of the software firm. The cost-smoothing effect disappears; thus, the platform's innovation is no longer accelerated by the quality gap. It is also notable that both firms' choices decrease with the cost of software innovation. The reason that $\dot{y}^J(z)$ decreases in $b$ is evident. Platform innovation also decreases in $b$ because the impact of increment of $z$ on $y$ becomes weaker if the software innovation becomes more costly. Furthermore, both innovation choices decrease in $r$; that is, firms reduce innovation efforts because they care more about the current period than they do about future income.

## 3.5 A Simulated Example

In this section, I compute the closed-form solution in previous sections, compare the equilibrium and the social optimum, and discuss the implications. I employ the following examples of parameters and a state variable. Varying values of parameters confirms the comparative statics discussed in sections 3.3 and 3.4.

| Table 3-1 | Examples of Parameters |
|---|---|
| parameters and a state variable | values |
| $\alpha$ | 0.2, 0.5, or 0.8 |
| $r$ | 0.1 or 0.2 |
| $a$ | $5, 10$ or 15 |
| $b$ | $5, 10$ or 15 |
| $z$ | 0.2, 0.5, or 0.8 |

Inserting the listed parameters to (37), (38),(44) and (45) produces the results in Table 6-a, Table 6-b and Table 7.

Table 3–2-a and Table 3–2-b provide that the equilibrium solutions of $\lambda$ show the same characteristics as those from Proposition 6: the innovation of the platform tends to increase in $\alpha$, decrease in $r$, decrease in $b$, and slightly increase in $a$. Also, the social planning solution corresponds to the properties explained in Section 3.4.

| Table 3–2a | Platform Firm's innovation choice ($r = 0.1$) | | | | | |
|---|---|---|---|---|---|---|
| | Social Planner's Solution | | | Equilibrium Solution ($\alpha = 0.2$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| $a = 5$ | 1.4597 | 1.2835 | 1.1642 | 3.5406 | 1.1313 | 0.3219 |
| $a = 10$ | 0.7298 | 0.6417 | 0.5821 | 4.3406 | 1.9313 | 1.1219 |
| $a = 15$ | 0.4866 | 0.4278 | 0.3881 | 4.6073 | 2.1979 | 1.3885 |
| | Equilibrium Solution ($\alpha = 0.5$) | | | Equilibrium Solution ($\alpha = 0.8$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| $a = 5$ | 11.3463 | 5.3425 | 3.3388 | 19.1477 | 9.5453 | 6.3430 |
| $a = 10$ | 11.8463 | 5.8425 | 3.8388 | 19.3477 | 9.7453 | 6.5430 |
| $a = 15$ | 12.0129 | 6.0092 | 4.0054 | 19.4143 | 9.8120 | 6.6096 |

| Table 3–2b | Platform firm's innovation choice $\lambda$ ($r = 0.2$) | | | | | |
|---|---|---|---|---|---|---|
| | Social Planner's Solution | | | Equilibrium Solution ($\alpha = 0.2$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| $a = 5$ | 0.5367 | 0.4202 | 0.3510 | 2.2250 | 0.9500 | 0.4750 |
| $a = 10$ | 0.2683 | 0.2101 | 0.1755 | 2.6250 | 1.3500 | 0.8750 |
| $a = 15$ | 0.1789 | 0.1401 | 0.1170 | 2.7583 | 1.4833 | 1.0083 |
| | Equilibrium Solution ($\alpha = 0.5$) | | | Equilibrium Solution ($\alpha = 0.8$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| $a = 5$ | 6.1700 | 3.1400 | 2.1100 | 10.0813 | 5.2625 | 3.6438 |
| $a = 10$ | 6.4200 | 3.3900 | 2.3600 | 10.1813 | 5.3625 | 3.7438 |
| $a = 15$ | 6.5033 | 3.4733 | 2.4433 | 10.2146 | 5.3958 | 3.7771 |

There are some remarkable observations. First, equilibrium $\lambda$ seems to be more sensitive to the cost of software than does the social planning $\lambda$ because the platform firm does not want to be monitored by the software firm in a decentralized economy. The less expensive cost of innovation boosts innovation of the platform, not only because the increment of $y$ is profitable for the platform firm, but also because the firm wants to widen the gap of quality, expecting the future innovation cost of software to lower by doing so. Moreover, equilibrium $\lambda$ tends to be oversupplied, compared to the socially optimum $\lambda$. The same reason can apply to this observation as well; the platform firm wants to widen the gap. Finally, it is likely that $\lambda$ increases in $a$ less with the higher $r$ because the cost-smoothing motivation becomes weaker with the higher $r$.

As shown in Table 3-3, software's innovations illustrate the characteristics discussed in the previous sections. It is notable that the innovation of software increases in $\alpha$, but decreases in $r$. In addition, $\dot{y}$ tends to more sensitive to cost in the equilibrium. Therefore, the platform firm becomes more sensitive to $b$ as previously discussed. The remarkable findings from this comparison is that software innovation seems undersupplied, in particular when $\alpha$ is small. However, with the larger $\alpha$, $\dot{y}$ appears oversupplied. This indicates a specific value of $\alpha$ that can force the software firm to produce at an optimal level. Comparison of (38) and (45) indicates the socially optimal level of $\alpha$.

$$\alpha^{opt} = \frac{3(r - br^2 + \sqrt{2br^2 + b^2r^4})}{4} \tag{46}$$

$\alpha^{opt}$ decreases in $r$, which corresponds to the indication from the Table 3-3.

| Table 3-3 | Software Firm's Innovation Choice, $\dot{y}$ | | | | | |
|---|---|---|---|---|---|---|
| $r = 0.1$ | Social Planner's Solution | | | Equilibrium Solution ($\alpha = 0.2$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| | 3.7016 | 2.2913 | 1.7263 | 2.6667 | 1.3333 | 0.8889 |
| | Equilibrium Solution ($\alpha = 0.5$) | | | Equilibrium Solution ($\alpha = 0.8$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| | 6.6667 | 3.3333 | 2.2222 | 10.6667 | 5.3333 | 3.5556 |
| $r = 0.2$ | Social Planner's Solution | | | Equilibrium Solution ($\alpha = 0.2$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| | 3.3166 | 1.9495 | 1.4150 | 1.3333 | 0.6667 | 0.4444 |
| | Equilibrium Solution ($\alpha = 0.5$) | | | Equilibrium Solution ($\alpha = 0.8$) | | |
| | $b = 5$ | $b = 10$ | $b = 15$ | $b = 5$ | $b = 10$ | $b = 15$ |
| | 3.3333 | 1.6667 | 1.1111 | 5.3333 | 2.6667 | 1.7778 |

In summary, the simulated example corresponds with the characteristics of equilibrium and socially optimum solution as described in section 3.3 and 3.4. The remarkable finding is the platform firm usually oversupplies $\lambda$, but the software innovation seems undersupplied, in particular when $\alpha$ is small. This phenomenon is mainly because the platform firm and the software firm are oversensitive to the cost of the software innovation. As the gap in quality narrows, it is more costly for software innovation. Thus, there is motivation for firms to widen the gap of the quality in a decentralized economy

## 3.6 Conclusion

In this study, I primarily discussed the main determinants of innovation when firms produce complementary goods. The determinants of innovation predominantly relate to forces arising from the platform/ software relationship. The most remarkable force is that both firms want to increase the quality level of the software because the quality of software increases profits for both firms. The reason that the platform firms invest in innovation is to motivate the software firm to do the same. Nonetheless, firms want to keep the gap in quality. The cost of software innovation becomes less expensive when the firms widen the quality gap. Encompassing these forces, in equilibrium, the platform firm's innovation over produces, while software innovation tends to be less than socially optimum when the share of profit is small, but greater when the share of profit is large. Furthermore, the platform firm tends to smooth the cost of innovation. When the platform firm expects growth in software innovation, it may spend more to avoid the incremental cost of software innovation. In order to smooth the cost of innovation, it is more likely that the platform firm's innovation will increase the gap in quality.

One of the most important implications of this study is that the social planner, i.e. a policymaker, can find the level of share ratio beneficial for the economy, consequently requiring further study on ways to incentivize firms to innovate in accordance with the socially optimal level of the share ratio. The cost of software innovation requires careful investigation as well. Both firms are very sensitive to the cost of software innovation. Thus, there is a need to develop policies that subsidize the software firms rather than the platform firms.

## 3.7 Proofs and Calculations

### 3.7.1 Proofs for the Decentralized Problem

**Normalization of Value Functions** The HJB equations (29) and (30) can be decomposed by $V(x,y) = \frac{(1-\alpha)y}{r} + \hat{V}(z)$ and $W(x,y) = \frac{\alpha y}{r} + \hat{W}(z)$, because aside from the flow profit other components of the equations are only dependent on the distance of the quality $z$. Then, $V(x+1,y) - V(x,y) = \hat{V}(z+1) - \hat{V}(z)$, $V_y(x,y) = \frac{1-\alpha}{r} - \hat{V}'(z)$, $W(x+1,y) - W(x,y) = \hat{W}(z+1) - \hat{W}(z)$ and $W_y(x,y) = \frac{\alpha}{r} - \hat{W}'(z)$. Thus, the following normalized HJB equations are attained.

$$r\hat{V}(z) = \max_{\lambda} \lambda(\hat{V}(z+1) - \hat{V}(z)) + (\frac{1-\alpha}{r} - \hat{V}'(z))\dot{y}(z) - \psi(\lambda) \tag{47}$$

$$r\hat{W}(z) = \max_{\dot{y}} \lambda(z)(\hat{W}(z+1) - \hat{W}(z)) + (\frac{\alpha}{r} - \hat{W}'(z))\dot{y} - c(\dot{y}, z) \tag{48}$$

**Guess-and-Verify Method for Decentralized Probelm** From (35) and (36), we can infer that the $\hat{V}(z)$ is quadratic in $z$ and $\hat{W}(z)$ is linear, since otherwise they are conversing. Guess that $\hat{V}(z) = A + Bz + Cz^2$ and $\hat{W}(z) = D + Ez$, where $A$, $B$, $C$, $D$ and $E$ are undertermined coefficients. By calculating and differentiating our "guess" with respect to $z$, we obtain following equations.

$$\begin{aligned}
\hat{V}(z+1)-\hat{V}(z) &= (B+C)+2Cz \\
\hat{V}'(z) &= B+2Cz \\
\hat{W}(z+1)-\hat{W}(z) &= E \\
\hat{W}'(z) &= E
\end{aligned}$$

Inserting these equations to (35) and (36) yields,

$$\begin{aligned}
r\hat{V}(z) &= \frac{1}{2a}((B+C)+2Cz)^2+\frac{1}{b}(\frac{1-\alpha}{r}-B-2Cz)(\frac{\alpha}{r}-E)z \\
r\hat{W}(z) &= \frac{1}{a}((B+C)+2Cz)(E)+\frac{1}{2b}(\frac{\alpha}{r}-E)^2z
\end{aligned}$$

Substituting our guess to the LHS and rearrangint the RHS, we obtain

$$\begin{aligned}
& rA+rBz+rCz^2 \\
= \;& \frac{1}{2a}(B+C)^2+\frac{1}{b}(\frac{1-\alpha}{r}-B)(\frac{\alpha}{r}-E)z+\frac{2}{a}(B+C)Cz-\frac{1}{b}2C(\frac{\alpha}{r}-E)z^2+\frac{2}{a}C^2z^2 \\
& rD+rEz \\
= \;& \frac{1}{a}(B+C)E+(\frac{1}{a}2CE+\frac{1}{2b}(\frac{\alpha}{r}-E)^2)z
\end{aligned}$$

Then, we solve the match coefficient for $A$, $B$, $C$, $D$ and $E$. Consequently, we have the following closed-form solution

$$\begin{aligned}
A &= \frac{-64a\alpha^2 - 32\alpha b + 32\alpha^2 b - 48a\alpha br^2 - 9ab^2r^4}{24\alpha b^2 r^3} \\
&\quad + \frac{-64a\alpha^2 - 24\alpha b + 24\alpha^2 b - 48a\alpha br^2 - 9ab^2r^4}{48\alpha br} \\
&\quad + \frac{(-64a\alpha^2 - 24\alpha b + 24\alpha^2 b - 48a\alpha br^2 - 9ab^2r^4)^2}{2ar(24\alpha br)^2} \\
B &= \frac{-64a\alpha^2 - 24\alpha b + 24\alpha^2 b - 48a\alpha br^2 - 9ab^2r^4}{24\alpha br} \\
C &= (\frac{4\alpha}{3br} + \frac{r}{2})a \\
D &= \frac{32a\alpha^2 + 24\alpha b - 24\alpha^2 b + 36a\alpha br^2 + 9ab^2r^4}{72\alpha br^2} \\
E &= -\frac{\alpha}{3r}
\end{aligned}$$

### 3.7.2 Proofs for the Social Planning Problem

**Closed-Form Solution: Guess-and-Verify** From (43), guess that $\hat{J}(z) = K +$

$Lz$, where $K$ and $L$ are undetermined coefficients. Then, $\hat{J}(z+1) - \hat{J}(z) = L$ and $\hat{J}'(z) = L$. Rewriting (43) , we obtain:

$$rK + rLz = \frac{1}{2a}L^2 + \frac{1}{2b}(\frac{1}{r} - L)^2 z$$

Calculating the match coefficient for $L$ and $K$, the following closed-form solutions are attained.

$$L = \frac{1 + br^2 - \sqrt{2br^2 + b^2r^4}}{r}$$

$$K = \frac{1 + 4br^2 + 2b^2r^4}{2ar^3} + \frac{(1 + br^2)\sqrt{2b + b^2r^2}}{ar^2})$$

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