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### Authors

Huang, Hongdi

Nguyen, Van C

Vashaw, Kent B

et al.



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# QUANTUM-SYMMETRIC EQUIVALENCE IS A GRADED MORITA INVARIANT

HONGDI HUANG, VAN C. NGUYEN, KENT B. VASHAW, PADMINI VEERAPEN, AND XINGTING WANG

Abstract. We show that if two $m$-homogeneous algebras have Morita equivalent graded module categories, then they are quantum-symmetrically equivalent, that is, there is a monoidal equivalence between the categories of comodules for their associated universal quantum groups (in the sense of Manin) which sends one algebra to the other. As a consequence, any Zhang twist of an $m$-homogeneous algebra is a 2-cocycle twist by some 2-cocycle from its Manin's universal quantum group.

## 1. Introduction

Symmetry has been a central topic of study in mathematics for thousands of years. Symmetries of classical objects form a group; however, some quantum objects exhibit properties that cannot be captured by classical symmetries. This motivates the study of their quantum symmetries, which are better described by group-like objects known as quantum groups, whose representation categories provide examples of tensor categories (see e.g., [7]).

In his seminal work [13], Manin restored the "broken symmetry" of a quantized algebra by imposing some non-trivial relations on the coordinate ring of the general linear group. This led to the introduction of the now-called "Manin's universal quantum group".

**Definition 1.1.** [13, Lemma 6.6] *Let $A$ be any $\mathbb{Z}$-graded locally finite $\Bbbk$-algebra. The* right universal bialgebra $\underline{\mathrm{end}}^r(A)$ associated to $A$ *is the bialgebra that right coacts on $A$ preserving the grading of $A$ via $\rho : A \to A \otimes \underline{\mathrm{end}}^r(A)$ satisfying the following universal property: if $B$ is any bialgebra that right coacts on $A$ preserving the grading of $A$ via $\tau : A \to A \otimes B$, then there is a unique bialgebra map $f : \underline{\mathrm{end}}^r(A) \to B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[dr, "\tau"'] & A \otimes \underline{\mathrm{end}}^r(A) \arrow[d, "\mathrm{id} \otimes f"] \\ & A \otimes B \end{tikzcd} \tag{1.1}$$

*commutes. By replacing "bialgebra" with "Hopf algebra" in the above definition, we define the right universal quantum group $\underline{\mathrm{aut}}^r(A)$ to be the universal Hopf algebra right coacting on $A$.*

**Remark 1.2.** *One can also define a left-coacting version of Manin's universal quantum groups. All results in this paper can be proven analogously in that context.*

There is a current surge of interest in the study of universal quantum symmetries, see e.g., [1, 2, 5, 8, 9, 10, 11, 18, 22]. Notable results by Raedschelders and Van den Bergh in [18] showed that Manin's universal quantum groups of Koszul Artin-Schelter (AS) regular algebras with the same global dimensions have monoidally equivalent comodule categories. In [10], the authors together with Ure introduced quantum-symmetric equivalence to systematically study such algebras.

**Definition 1.3.** [10, Definition A] *Let $A$ and $B$ be two connected graded algebras finitely generated in degree one. We say $A$ and $B$ are* quantum-symmetrically equivalent *if there is a monoidal equivalence between the comodule categories of their associated universal quantum groups*

$$\mathrm{comod}(\underline{\mathrm{aut}}^r(A)) \overset{\otimes}{\cong} \mathrm{comod}(\underline{\mathrm{aut}}^r(B))$$

*in the sense of Manin, where this equivalence sends $A$ to $B$ as comodule algebras. We denote the quantum-symmetric equivalence class of $A$ by $QS(A)$, which consists of all connected graded algebras that are quantum-symmetrically equivalent to $A$.*

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For any connected graded algebra $A$ finitely generated in degree one, we aim to determine its $QS(A)$. The main findings in [10] demonstrate that all graded algebras in $QS(A)$ have various homological properties in common with $A$, and that the family of Koszul AS-regular algebras of a fixed global dimension forms a single quantum-symmetric equivalence class.

The purpose of this paper is to explore additional properties of $A$ that may help to identify characteristics of $QS(A)$ beyond the numerical and homological invariants explored in [10]. It is important to note that these numerical and homological invariants of $A$ are entirely determined by its graded module category $\mathrm{grmod}(A)$. Moreover, in [23], Zhang fully characterized the graded Morita equivalence between two $\mathbb{N}$-graded algebras by Zhang twists given by some twisting systems (see [19] for a generalization to $\mathbb{Z}$-graded algebras and [12] for a generalization to algebras in monoidal categories). Therefore, we pose a natural question: Does $\mathrm{grmod}(A)$ uniquely determine $QS(A)$? Our main finding in this paper answers this question positively in the case of all $m$-homogeneous algebras.

**Theorem 1.4.** *For any integer $m \geq 2$, let $A$ and $B$ be two $m$-homogeneous algebras. If $A$ and $B$ are graded Morita equivalent, then they are quantum-symmetrically equivalent.*

In particular, we show that a Zhang twist of an $m$-homogeneous algebra by a twisting system can be realized as a 2-cocycle twist by using its universal quantum group $\underline{\mathrm{aut}}^r(A)$. A base case of this result, when the twisting system is formed by the compositions of a single algebra automorphism, was achieved in [10, Theorem 2.3.3]; the present generalization to arbitrary twisting systems involves significant technical complications and applies in much greater generality (see, e.g., [21]).

**Conventions.** Throughout, let $\Bbbk$ be a base field with $\otimes$ taken over $\Bbbk$ unless stated otherwise. A $\mathbb{Z}$-graded algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is called *connected graded* if $A_i = 0$ for $i < 0$ and $A_0 = \Bbbk$. For any integer $m \geq 2$, an *$m$-homogeneous algebra* is a connected graded algebra $A := \Bbbk\langle A_1\rangle/(R)$ finitely generated in degree one, subject to $m$-homogeneous relations $R \subseteq A_1^{\otimes m}$. For any homogeneous element $a \in A$, we denote its degree by $|a|$. We use the Sweedler notation for the coproduct in a coalgebra $B$: for any $h \in B$, $\Delta(h) = \sum h_1 \otimes h_2 \in B \otimes B$. The category of right $B$-comodules is denoted by $\mathrm{comod}(B)$.

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## 2. Lifting twisting systems to universal bialgebras

For any $\mathbb{Z}$-graded algebra $A$, recall that a *twisting system* of $A$ consists of a collection $\tau := \{\tau_i : i \in \mathbb{Z}\}$ of $\mathbb{Z}$-graded bijective linear maps $\tau_i : A \to A$, satisfying any one of the following equivalent conditions (see [23, (2.1.1)-(2.1.4)]):

(1) $\tau_i(a\tau_j(b)) = \tau_i(a)\tau_{i+j}(b)$;
(2) $\tau_i(ab) = \tau_i(a)\tau_{i+j}\tau_j^{-1}(b)$;
(3) $\tau_i^{-1}(a\tau_{i+j}(b)) = \tau_i^{-1}(a)\tau_j(b)$;
(4) $\tau_i^{-1}(ab) = \tau_i^{-1}(a)\tau_j\tau_{i+j}^{-1}(b)$,

for homogeneous elements $a, b \in A$, where $a$ is of degree $j$ and $b$ is of any degree. By [23, Proposition 2.4], we may always assume the following additional two conditions:

(5) $\tau_i(1) = 1$ for any $i \in \mathbb{Z}$;
(6) $\tau_0 = \mathrm{id}_A$.

For any twisting system $\tau$ of $A$, the *right Zhang twist of $A$*, denoted by $A^\tau$, is the graded algebra such that $A^\tau = A$ as graded vector spaces with the twisted product $a \cdot_\tau b = a\tau_j(b)$, for homogeneous elements $a, b \in A$, where $a$ is of degree $j$ and $b$ is of any degree.

For an $m$-homogeneous algebra $A = \Bbbk\langle A_1\rangle/(R)$, we construct a twisting system of $A$ explicitly as follows. Let $\tau := \{\tau_i : A_1 \to A_1\}_{i\in\mathbb{Z}}$ be a collection of bijective linear maps on degree one (where $\tau_0 = \mathrm{id}$) with $\Bbbk$-linear inverses $\tau^{-1} := \{\tau_i^{-1} : A_1 \to A_1\}_{i\in\mathbb{Z}}$. We extend each $\tau_i$ and $\tau_i^{-1}$ (which we denote as $\tau_i$ and $\widetilde{\tau}_i$, respectively, by abuse of notation) to $\Bbbk\langle A_1\rangle$ inductively on the total degree of the element $ab$ by the rules:

$$\tau_i(1) = \widetilde{\tau}_i(1) = 1, \qquad \tau_i(ab) := \tau_i(a)\tau_{i+1}\widetilde{\tau}_1(b), \qquad \text{and} \qquad \widetilde{\tau}_i(ab) := \widetilde{\tau}_i(a)\tau_1\widetilde{\tau}_{i+1}(b), \tag{2.1}$$

for any $a \in A_1$ and $b$ is of any positive degree. In the following result, we use the rules in (2.1) to define a twisting system of $A$ by proving that $\tau_i$ and $\widetilde{\tau}_i$ indeed satisfy the twisting system axioms (with $\widetilde{\tau}_i$ being the inverse of $\tau_i$) if and only if they preserve the relation space $R$ of $A$.

**Proposition 2.1.** *Let $A = \Bbbk\langle A_1\rangle/(R)$ be an $m$-homogeneous algebra and $\tau_i$ and $\widetilde{\tau}_i$ be defined as in (2.1). If $\tau_i(R) = R$ for all $i \in \mathbb{Z}$, then $\tau_i$ and $\widetilde{\tau}_i$ are well-defined graded linear maps $A \to A$ that are inverse to each other. Moreover, $\tau = \{\tau_i : i \in \mathbb{Z}\}$ is a twisting system of $A$.*

*Proof.* By assumption, it is clear that $\tau_i$ is well-defined and bijective on all degrees up to and including $m$, and that $\widetilde{\tau}_i$ is well-defined on all degrees less than $m$, and is inverse to $\tau_i$ on degree 1 by definition. Furthermore, again by definition, $\tau$ and $\widetilde{\tau}$ satisfy the twisting system axioms and inverse twisting system axioms, respectively, on degrees $\leq 2$. We now show inductively on arbitrary degree $n$ that $\tau$ and $\widetilde{\tau}$ are well-defined, bijective, inverse to each other, and satisfy the (inverse) twisting system axioms.

We first show that $\tau$ satisfies the twisting systems axioms on the free algebra $\Bbbk\langle A_1\rangle$. Suppose that $a$ and $b$ are homogeneous monomial elements of degrees $j$ and $n-j$, respectively. Assume that $a = a_1a_2$ for $a_1$ of degree 1 and $a_2$ of degree $j-1$; note that elements of this form span $A_j$, since we assume that $A$ is generated in degree 1. Then for all $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\tau_i(ab) &= \tau_i(a_1a_2b)\\
&= \tau_i(a_1)\tau_{i+1}\widetilde{\tau}_1(a_2b)\\
&= \tau_i(a_1)\tau_{i+1}\big(\widetilde{\tau}_1(a_2)\tau_{j-1}\widetilde{\tau}_j(b)\big)\\
&= \tau_i(a_1)\tau_{i+1}\widetilde{\tau}_1(a_2)\tau_{i+j}\widetilde{\tau}_{j-1}\tau_{j-1}\widetilde{\tau}_j(b)\\
&= \tau_i(a_1)\tau_{i+1}\widetilde{\tau}_1(a_2)\tau_{i+j}\widetilde{\tau}_j(b)\\
&= \tau_i(a_1a_2)\tau_{i+j}\widetilde{\tau}_j(b)\\
&= \tau_i(a)\tau_{i+j}\widetilde{\tau}_j(b).
\end{aligned}$$

The second equality is from the definition of $\tau_i$ in (2.1). The third, fourth, and sixth equalities follow from the inductive hypothesis as $\tau$ and $\widetilde{\tau}$ satisfy the (inverse) twisting axioms up to degrees $< n$. Thus, $\tau$ satisfies the twisting system axioms. An analogous argument shows that $\widetilde{\tau}$ satisfies the inverse twisting system axioms. Moreover, we note that $\tau_i$ and $\widetilde{\tau}_i$ are inverse to one another on $\Bbbk\langle A_1\rangle$ by induction since

$$\widetilde{\tau}_i\tau_i(ab) = \widetilde{\tau}_i(\tau_i(a)\tau_{i+1}\widetilde{\tau}_1(b)) = \widetilde{\tau}_i\tau_i(a)\tau_1\widetilde{\tau}_{i+1}\tau_{i+1}\widetilde{\tau}_1(b) = ab,$$

for any $a \in A$ of degree 1 and $b \in A$ of degree $n-1$.

It remains to show that for any $i \in \mathbb{Z}$, $\tau_i$ preserves the homogeneous relation ideal $(R)$ of $A$ in $\Bbbk\langle A_1\rangle$. It is trivial for relations of degree $n \leq m$. An arbitrary relation of degree $n > m$ is a linear combination of terms of the form $ra$ and $ar$, where $a$ is an element of degree 1 in $A$ and $r$ is a relation of degree $n-1$. But note that $\tau_i(ar)$ is indeed a relation of $A$, since $\tau_i(ar) = \tau_i(a)\tau_{i+1}\widetilde{\tau}_1(r)$ by the twisting system axioms, and $\tau_{i+1}\widetilde{\tau}_1(r)$ is a relation of $A$ by the inductive hypothesis. Similarly, $\tau_i$ sends $ra$ to a relation of $A$, so $\tau_i$ preserves all homogeneous relations of degree $n$. This completes the proof. $\square$

Recall that the *Koszul dual* of an $m$-homogeneous algebra $A = \Bbbk\langle A_1\rangle/(R)$ is the $m$-homogeneous algebra

$$A^! := \Bbbk\langle A_1^*\rangle/(R^\perp),$$

where $A_1^*$ is the vector space dual of $A_1$ and $R^\perp \subseteq (A_1^*)^{\otimes m}$ is the subspace orthogonal to $R$ with respect to the natural evaluation $\langle -,-\rangle : A_1^* \times A_1 \to \Bbbk$.

Let $\tau = \{\tau_i : i \in \mathbb{Z}\}$ be a twisting system of $A$ with inverse twisting system $\{\tau_i^{-1} : i \in \mathbb{Z}\}$. We define the dual twisting system $\tau^! = \{\tau_i^! : i \in \mathbb{Z}\}$ together with the inverse dual twisting system $(\tau^!)^{-1} = \{(\tau_i^!)^{-1} : i \in \mathbb{Z}\}$ on the Koszul dual $A^!$ such that

$$\tau_i^!|_{A_1^!} := (\tau_i^{-1})^* = (\tau_i^*)^{-1} \qquad \text{and} \qquad (\tau_i^!)^{-1}|_{A_1^!} := \tau_i^*,$$

as linear maps $A_1^* \to A_1^*$. For $a \in A_1^!$ and $b \in A^!$ is of any positive degree, we define each $\tau_i^!$ and $(\tau_i^!)^{-1}$ inductively on the total degree of the element $ab$ as follows:

$$\tau_i^!(ab) = \tau_i^!(a)\tau_{i+1}^!(\tau_1^{-1})^!(b), \qquad \text{and} \qquad (\tau_i^!)^{-1}(ab) = (\tau_i^!)^{-1}(a)\tau_1^!(\tau_{i+1}^!)^{-1}(b). \tag{2.2}$$

Using Proposition 2.1, in the following we show that these maps give well-defined twisting systems of $A^!$.

**Proposition 2.2.** *Let $A, A^!$ and $\tau^{\pm 1}$, $(\tau^!)^{\pm 1}$ be defined as above. The collection of linear maps $\tau^!$, defined in* (2.2)*, forms a twisting system of $A^!$ with inverse $(\tau^!)^{-1}$.*

*Proof.* By Proposition 2.1, it is enough to show that $\tau_i^!(R^\perp) = R^\perp$. We first inductively show that

$$\langle \tau_i^!(f), a\rangle = \langle f, \tau_i^{-1}(a)\rangle \qquad \text{and} \qquad \langle (\tau_i^!)^{-1}(f), a\rangle = \langle f, \tau_i(a)\rangle \tag{2.3}$$

for $f \in (A_1^{\otimes n})^*$ and $a \in A_1^{\otimes n}$ for $n \geq 1$. The case $n = 1$ follows from the definition. Assume the inductive hypothesis, we now show (2.3) holds for $n+1$. Without loss of generality, let $f = yg$ and $a = xh$ for any $y \in A_1^*, g \in (A_1^*)^{\otimes n}$ and $x \in A_1, h \in (A_1)^{\otimes n}$. Then we have

$$\begin{aligned}
\langle \tau_i^!(f), a\rangle &= \langle \tau_i^!(y)\tau_{i+1}^!\tau_{i+1}^!(\tau_1^!)^{-1}(g), xh\rangle \\
&= \langle \tau_i^!(y), x\rangle\langle \tau_{i+1}^!(\tau_1^!)^{-1}(g), h\rangle \\
&= \langle y, \tau_i^{-1}(x)\rangle\langle g, \tau_1\tau_{i+1}^{-1}(h)\rangle \\
&= \langle yg, \tau_i^{-1}(x)\tau_1\tau_{i+1}^{-1}(h)\rangle \\
&= \langle f, \tau_i^{-1}(a)\rangle,
\end{aligned}$$

where the last equality follows from the fact that $\tau$ is a twisting system. By a straightforward induction, it similarly follows that $\langle (\tau_i^!)^{-1}(f), a\rangle = \langle f, \tau_i(a)\rangle$. So we have $\tau_i^!(R^\perp) = R^\perp \Leftrightarrow \langle \tau_i^!(R^\perp), R\rangle = 0 \Leftrightarrow \langle R^\perp, (\tau_i^{-1})(R)\rangle = 0 \Leftrightarrow \tau_i^{-1}(R) = R \Leftrightarrow \tau_i(R) = R$, which holds by assumption. It follows that $\tau^!$ is a twisting system of $A^!$. □

**Proposition 2.3.** *Let $A$ be an $m$-homogeneous algebra with a twisting system $\tau = \{\tau_i : i \in \mathbb{Z}\}$. Then $(A^!)^{\tau^!} = (A^\tau)^!$.*

*Proof.* Write $A = \Bbbk\langle A_1\rangle/(R)$ with $m$-homogeneous relations $R \subseteq A_1^{\otimes m}$. By Proposition 2.2, $\tau^! = \{\tau_i^! : i \in \mathbb{Z}\}$ is a twisting system of $A^!$. Similar to [14, Lemma 5.1.1], one can check that $A^\tau = \Bbbk\langle A_1\rangle/(R^\tau)$, where $R^\tau = (\mathrm{id} \otimes \tau_1^{-1} \otimes \tau_2^{-1} \otimes \cdots \otimes \tau_{m-1}^{-1})(R)$. Notice that

$$0 = \langle (R^\tau)^\perp, R^\tau\rangle = \langle (R^\tau)^\perp, (\mathrm{id} \otimes \tau_1^{-1} \otimes \tau_2^{-1} \otimes \cdots \otimes \tau_{m-1}^{-1})(R)\rangle = \langle (\mathrm{id} \otimes \tau_1^! \otimes \cdots \otimes \tau_{m-1}^!)(R^\tau)^\perp, R\rangle.$$

Hence, $R^\perp = (\mathrm{id} \otimes \tau_1^! \otimes \cdots \otimes \tau_{m-1}^!)(R^\tau)^\perp$ and so

$$(R^\perp)^{\tau^!} = (\mathrm{id} \otimes (\tau_1^{-1})^! \otimes \cdots \otimes (\tau_{m-1}^{-1})^!)(\mathrm{id} \otimes \tau_1^! \otimes \cdots \otimes \tau_{m-1}^!)(R^\tau)^\perp = (R^\tau)^\perp.$$

As a result, we have $(A^\tau)^! = (\Bbbk\langle A_1\rangle/R^\tau)^! = \Bbbk\langle A_1^*\rangle/((R^\tau)^\perp) = \Bbbk\langle A_1^*\rangle/((R^\perp)^{\tau^!}) = (A^!)^{\tau^!}$. □

Let $V, W$ be any two finite-dimensional vector spaces. For any integer $m \geq 1$, we denote the *s*huffle map

$$\mathrm{Sh}_{V,W,m} : V^{\otimes m} \otimes W^{\otimes m} \xrightarrow{\cong} (V \otimes W)^m$$

to be the map sending

$$v_1 \otimes v_2 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_m \mapsto v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes \ldots \otimes v_m \otimes w_m,$$

for any $v_i \in V$ and $w_j \in W$. When $V$, $W$, and $m$ are clear from context, we omit the subscripts and denote this map by Sh.

For two connected graded algebras $A = \Bbbk\langle A_1\rangle/(R(A))$ and $B = \Bbbk\langle B_1\rangle/(R(B))$ with $m$-homogeneous relations $R(A) \subseteq (A_1)^{\otimes m}$ and $R(B) \subseteq (B_1)^{\otimes m}$ respectively, we extend Manin's bullet product [13, §4.2] to $A$ and $B$ such that

$$A \bullet B := \frac{\Bbbk\langle A_1 \otimes B_1\rangle}{(\mathrm{Sh}(R(A) \otimes R(B)))},$$

where $\mathrm{Sh} : (A_1)^{\otimes m} \otimes (B_1)^{\otimes m} \to (A_1 \otimes B_1)^{\otimes m}$ is the shuffle map. When $B = A^! = \Bbbk\langle A_1^*\rangle/(R(A)^\perp)$ is the $m$-Koszul dual algebra of $A$, by the definition of the bullet product we see that $A \bullet A^!$ is a connected graded bialgebra with matrix comultiplication defined on the generators of $A_1 \otimes A_1^*$. In particular, choose a basis $\{x_1, \ldots, x_n\}$ for $A_1$ and let $\{x^1, \ldots, x^n\}$ be the dual basis for $(A^!)_1 = A_1^*$. Write $z_j^k = x_j \otimes x^k \in A_1 \otimes A_1^*$ as the generators for $A \bullet A^!$. Then the coalgebra structure of $A \bullet A^!$ is given by

$$\Delta(z_j^k) = \sum_{1 \leq i \leq n} z_i^k \otimes z_j^i, \qquad \text{and} \qquad \varepsilon(z_j^k) = \delta_{j,k}, \quad \text{for any } 1 \leq j, k \leq n.$$

The following result is a straightforward generalization of the quadratic case in [13], which describes Manin's universal bialgebra $\underline{\mathrm{end}}^r(A)$ and Manin's universal quantum group $\underline{\mathrm{aut}}^r(A)$ in terms of the bullet product of $A$ and its Koszul dual $A^!$.

**Lemma 2.4.** [10, Lemma 2.1.5] *Let $A$ be an $m$-homogeneous algebra and $A^!$ be its Koszul dual. We have:*
1. $\underline{\mathrm{end}}^r(A) \cong A \bullet A^!$*;*
2. $\underline{\mathrm{aut}}^r(A)$ *is the Hopf envelope of* $\underline{\mathrm{end}}^r(A)$*.*

We now show that the bullet product of two twisting systems of $A$ and of $B$ is indeed a twisting system of $A \bullet B$. As a consequence, we can extend any twisting system of $A$ to a twisting system of its universal bialgebra $\underline{\mathrm{end}}^r(A)$.

**Proposition 2.5.** *Let $A$ and $B$ be two $m$-homogeneous algebras. If $\tau = \{\tau_i : i \in \mathbb{Z}\}$ is a twisting system of $A$, and $\omega = \{\omega_i : i \in \mathbb{Z}\}$ is a twisting system of $B$, then there exists a twisting system $\tau \bullet \omega$ of the algebra $A\bullet B$, where $(\tau\bullet\omega)_i$ on the degree one space $(A\bullet B)_1 \cong A_1 \otimes B_1$ corresponds to the map $\tau_i \otimes \omega_i$. Furthermore, $(A \bullet B)^{\tau\bullet\omega} \cong A^\tau \bullet B^\omega$ as $m$-homogeneous algebras.*

*Proof.* We construct $\tau \bullet \omega$ by extending $\tau \bullet \omega$ to the free algebra $\Bbbk\langle A_1 \otimes B_1\rangle$ as in (2.1). We claim that

$$(\tau_i \bullet \omega_i)(\mathrm{Sh}(a \otimes b)) = \mathrm{Sh}(\tau_i(a) \otimes \omega_i(b))$$

for all $i \in \mathbb{Z}$, and $a \in A, b \in B$ are of the same degree $n$. It is trivial for $n = 0, 1$. By induction on $n$, suppose it holds for $n \geq 1$. We now show it holds for $n + 1$. Without loss of generality, we take $a = xa'$ and $b = yb'$ with $x \in A_1$, $a' \in (A_1)^{\otimes n}$ and $y \in B_1$, $b' \in (B_1)^{\otimes n}$. Then we have

$$\begin{aligned}(\tau_i \bullet \omega_i)(\mathrm{Sh}(a \otimes b)) &= (\tau_i \otimes \omega_i)(x \otimes y)\,\mathrm{Sh}(a' \otimes b')\\ &= (\tau_i \otimes \omega_i)(x \otimes y)(\tau_{i+1}\tau_1^{-1} \otimes \omega_{i+1}\omega_1^{-1})(\mathrm{Sh}(a' \otimes b'))\\ &= (\tau_i \otimes \omega_i)(x \otimes y)\,\mathrm{Sh}(\tau_{i+1}\tau_1^{-1}(a') \otimes \omega_{i+1}\omega_1^{-1}(b'))\\ &= \mathrm{Sh}(\tau_i(x)\tau_{i+1}\tau_1^{-1}(a') \otimes \omega_i(y)\omega_{i+1}\omega_1^{-1}(b'))\\ &= \mathrm{Sh}(\tau_i(a) \otimes \omega_i(b)).\end{aligned}$$

This proves our claim. Denote the degree $m$ relations of $A$ by $R$ and the degree $m$ relations of $B$ by $S$. In particular, we have

$$(\tau_i \bullet \omega_i)(\mathrm{Sh}(R \otimes S)) = \mathrm{Sh}((\tau_i(R) \otimes \omega_i(S)) = \mathrm{Sh}(R \otimes S).$$

According to Proposition 2.1, we know $\tau \bullet \omega$ is a well-defined twisting system of $A \bullet B$.

We now check the final claim (compare with [8, Lemma 3.1.1]). Denote by $R^\tau$ and $S^\omega$ the relation spaces of $A^\tau$ and $B^\omega$, respectively. Recall that we have $R^\tau = (\mathrm{id} \otimes\tau_1^{-1} \otimes \tau_2^{-1} \otimes \cdots \otimes \tau_{m-1}^{-1})(R)$, and $S^\omega$ can be presented likewise. Then the relations of $A^\tau \bullet B^\omega$ are precisely

$$\begin{aligned}\mathrm{Sh}(R^\tau \otimes S^\omega) &= \mathrm{Sh}((\mathrm{id} \otimes\tau_1^{-1} \otimes \tau_2^{-1} \otimes \cdots \otimes \tau_{m-1}^{-1})(R) \otimes (\mathrm{id} \otimes\omega_1^{-1} \otimes \omega_2^{-1} \otimes \cdots \otimes \omega_{m-1}^{-1})(S))\\ &= (\mathrm{id} \otimes \mathrm{id} \otimes\tau_1^{-1} \otimes \omega_1^{-1} \otimes \tau_2^{-1} \otimes \omega_2^{-1} \otimes \cdots \otimes \tau_{m-1}^{-1} \otimes \omega_{m-1}^{-1})(\mathrm{Sh}(R \otimes S)).\end{aligned}$$

The last equality gives the relations of $(A\bullet B)^{\tau\bullet\omega}$. Thus, $(A\bullet B)^{\tau\bullet\omega} \cong A^\tau\bullet B^\omega$ as $m$-homogeneous algebras. □

**Corollary 2.6.** *Let $A$ be an $m$-homogeneous algebra with a twisting system $\tau = \{\tau_i : i \in \mathbb{Z}\}$. Then $\tau \bullet \tau^!$ is a twisting system of $\underline{\mathrm{end}}^r(A)$, and $\underline{\mathrm{end}}^r(A)^{\tau\bullet\tau^!} \cong A^\tau \bullet (A^!)^{\tau^!} \cong A^\tau \bullet (A^\tau)^! \cong \underline{\mathrm{end}}^r(A^\tau)$ as graded algebras.*

*Proof.* This is a direct consequence of Proposition 2.5 by letting $B = A^!$ and applying Lemma 2.4(1) and Proposition 2.3. □

## 3. Systems of twisting functionals

Throughout this section, let $B$ be a bialgebra satisfying the twisting conditions below.

**Definition 3.1.** [10, Definition B] *A bialgebra $(B, M, u, \Delta, \varepsilon)$ satisfies the* twisting conditions *if*

- **(T1)** *as an algebra $B = \bigoplus_{n\in\mathbb{Z}} B_n$ is $\mathbb{Z}$-graded, and*
- **(T2)** *the comultiplication satisfies $\Delta(B_n) \subseteq B_n \otimes B_n$ for all $n \in \mathbb{Z}$.*

Recall that the space of linear functionals $\mathrm{Hom}_\Bbbk(B, \Bbbk)$ on $B$ has an algebra structure under the *convolution product* $*$ such that $f * g = (f \otimes g) \circ \Delta$ with unit $u \circ \varepsilon$.

**Lemma 3.2.** *Let $\alpha = \{\alpha_i : B \to \Bbbk : i \in \mathbb{Z}\}$ be a collection of linear functionals on a bialgebra $B$ such that each $\alpha_i$ is convolution invertible with inverse denoted by $\alpha_i^{-1}$. Then the following conditions are equivalent for any homogeneous elements $a, b \in B$, where $a$ is of degree $j$ and $b$ is of any degree:*

(1) $\sum \alpha_i(ab_1)\alpha_j(b_2) = \alpha_i(a)\alpha_{i+j}(b)$;
(2) $\alpha_i(ab) = \alpha_i(a)(\alpha_{i+j} * \alpha_j^{-1})(b)$;
(3) $\alpha_i^{-1}(ab) = \alpha_i^{-1}(a)(\alpha_j * \alpha_{i+j}^{-1})(b)$;
(4) $\sum \alpha_i^{-1}(ab_1)\alpha_{i+j}(b_2) = \alpha_i^{-1}(a)\alpha_j(b)$.

*Proof.* We use the properties of the counit $\varepsilon$ to show the equivalence below.
(1)⇒(2):

$$\begin{aligned}\alpha_i(ab) &= \sum \alpha_i(ab_1)\varepsilon(b_2) = \sum \alpha_i(ab_1)\alpha_j(b_2)\alpha_j^{-1}(b_3)\\ &= \sum \alpha_i(a)\alpha_{i+j}(b_1)\alpha_j^{-1}(b_2) = \alpha_i(a)(\alpha_{i+j} * \alpha_j^{-1})(b).\end{aligned}$$

(2)⇒(1):

$$\begin{aligned}\sum \alpha_i(ab_1)\alpha_j(b_2) &= \sum \alpha_i(a)(\alpha_{i+j} * \alpha_j^{-1})(b_1)\alpha_j(b_2)\\ &= \sum \alpha_i(a)\alpha_{i+j}(b_1)\alpha_j^{-1}(b_2)\alpha_j(b_3) = \alpha_i(a)\alpha_{i+j}(b).\end{aligned}$$

We can show that (3)⇔(4) similarly.
(2)⇒(3):

$$\begin{aligned}\alpha_i^{-1}(ab) &= \sum \alpha_i^{-1}(a_3)\alpha_i(a_2)\alpha_i^{-1}(a_1b_1)(\alpha_{i+j} * \alpha_j^{-1})(b_2)(\alpha_j * \alpha_{i+j}^{-1})(b_3)\\ &= \sum \alpha_i^{-1}(a_3)\alpha_i(a_2)(\alpha_{i+j} * \alpha_j^{-1})(b_2)\alpha_i^{-1}(a_1b_1)(\alpha_j * \alpha_{i+j}^{-1})(b_3)\\ &= \sum \alpha_i^{-1}(a_3)\alpha_i(a_2b_2)\alpha_i^{-1}(a_1b_1)(\alpha_j * \alpha_{i+j}^{-1})(b_3)\\ &= \alpha_i^{-1}(a)(\alpha_j * \alpha_{i+j}^{-1})(b).\end{aligned}$$

(3)⇒(2):

$$\begin{aligned}\alpha_i(ab) &= \sum \alpha_i(a_3)\alpha_i^{-1}(a_2)\alpha_i(a_1b_1)(\alpha_j * \alpha_{i+j}^{-1})(b_2)(\alpha_{i+j} * \alpha_j^{-1})(b_3)\\ &= \sum \alpha_i(a_3)\alpha_i^{-1}(a_2)(\alpha_j * \alpha_{i+j}^{-1})(b_2)\alpha_i(a_1b_1)(\alpha_{i+j} * \alpha_j^{-1})(b_3)\\ &= \sum \alpha_i(a_3)\alpha_i^{-1}(a_2b_2)\alpha_i(a_1b_1)(\alpha_{i+j} * \alpha_j^{-1})(b_3)\\ &= \alpha_i(a)(\alpha_{i+j} * \alpha_j^{-1})(b).\end{aligned}$$

□

**Definition 3.3.** *A collection of linear functionals $\alpha = \{\alpha_i : B \to \Bbbk\}_{i\in\mathbb{Z}}$ on a bialgebra $B$ is called a* system of twisting functionals *on $B$ if each $\alpha_i$ satisfies the following:*

(1) *$\alpha_i$ is convolution invertible with inverse $\alpha_i^{-1}$;*
(2) *$\sum \alpha_i(ab_1)\alpha_j(b_2) = \alpha_i(a)\alpha_{i+j}(b)$, for $a \in B$ is of degree $j$ and $b \in B$ is homogeneous of any degree;*
(3) *$\alpha_i(1) = 1$; and*
(4) *$\alpha_0 = \varepsilon$, the counit of $B$.*

Before we provide an example of a system of twisting functionals, we need the following notions. For any linear map $\pi : B \to \Bbbk$, we define a linear map $\Xi^l[\pi] : B \to B$ via

$$\Xi^l[\pi] = M \circ (\pi \otimes \mathrm{id}) \circ \Delta, \quad \text{that is,} \quad \Xi^l[\pi](b) = \sum \pi(b_1)b_2, \quad \text{for } b \in B,$$

where $M$ denotes the multiplication map. We call $\Xi^l[\pi]$ a *left linear winding map*, since it extends the notion of *left winding endomorphism* in [4, §2]. Similarly, the *right linear winding map* $\Xi^r[\pi]$ is defined by

$$\Xi^r[\pi] = M \circ (\mathrm{id} \otimes \pi) \circ \Delta, \quad \text{that is,} \quad \Xi^r[\pi](b) = \sum b_1\pi(b_2), \quad \text{for } b \in B.$$

If in addition $\pi : B \to \Bbbk$ is convolution invertible with inverse $\pi^{-1} : B \to \Bbbk$, one can check that the linear inverse of $\Xi^l[\pi]$ is $(\Xi^l[\pi])^{-1} = \Xi^l[\pi^{-1}]$, making $\Xi^l[\pi]$ a bijective linear winding map. Analogously, $\Xi^r[\pi]$ is also a bijective linear winding map with linear inverse $\Xi^r[\pi^{-1}]$.

**Example 3.4.** *Let $B$ be a Hopf algebra with antipode $S$. Let $\phi : B \to \Bbbk$ be any algebra map. The convolution inverse of $\phi$ is $\phi^{-1} = \phi \circ S$. Consider $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ where $\alpha_i = \phi * \cdots * \phi$ is the $i$th product of $\phi$ with itself with respect to the convolution product in $\mathrm{Hom}_{\Bbbk}(B, \Bbbk)$. It is straightforward to check that $\alpha$ is a system of twisting functionals on $B$. Moreover, the associated twisting system $\tau = \{\tau_i\}_{i\in\mathbb{Z}}$ where $\tau_i = \Xi^r(\alpha_i) = (\Xi^r(\phi))^i$ is the twisting system given by the right bijective linear winding map associated with $\phi$.*

Recall that the *Hopf envelope* of a bialgebra $B$ is the unique Hopf algebra $\mathcal{H}(B)$ together with a bialgebra map $\iota_B : B \to \mathcal{H}(B)$ satisfying the following universal property: for any bialgebra map $f : B \to K$ where $K$ is another Hopf algebra, there is a unique Hopf algebra map $g : \mathcal{H}(B) \to K$ such that $f = g \circ \iota_B$. It is proved in [8, Lemma 2.1.10] that if $B$ satisfies the twisting conditions in Definition 3.1 then so does $\mathcal{H}(B)$, and additionally $S(\mathcal{H}(B)_n) \subseteq \mathcal{H}(B)_{-n}$, for any $n \in \mathbb{Z}$.

Now, we construct explicitly the Hopf envelope $\mathcal{H}(B)$ as in [16, Theorem 2.6.3] and [17], which grew out of Takeuchi's construction for coalgebras [20]. Consider a presentation $B \cong \Bbbk\langle V\rangle/(R)$ as graded algebras, where $V$ is a subcoalgebra of $B$. We can extend the comultiplication $\Delta$ and counit $\varepsilon$ to the free algebra $\Bbbk\langle V\rangle$ as algebra maps, where $(R)$ is a homogeneous bi-ideal of $\Bbbk\langle V\rangle$. In this case, $B$ satisfies the twisting conditions. Denote infinitely many copies of the generating space $V$ as $\{V^{(k)} = V\}_{k\geq 0}$ and consider

$$T := \Bbbk\langle \oplus_{k\geq 0} V^{(k)}\rangle. \tag{3.1}$$

Let $S$ be the anti-algebra map on $T$ with $S(V^{(k)}) = V^{(k+1)}$ for any $k \geq 0$. Both algebra maps $\Delta : \Bbbk\langle V\rangle \to \Bbbk\langle V\rangle \otimes \Bbbk\langle V\rangle$ and $\varepsilon : \Bbbk\langle V\rangle \to \Bbbk$ extend uniquely to $T$ as algebra maps via identities $(S \otimes S) \circ \Delta = \Delta \circ S$ and $\varepsilon \circ S = \varepsilon$, which we still denote by $\Delta : T \to T \otimes T$ and $\varepsilon : T \to \Bbbk$. The Hopf envelope of $B$ has a presentation

$$\mathcal{H}(B) = T/W,$$

where the ideal $W$ is generated by

$$S^k(R), \quad (M \circ (\mathrm{id} \otimes S) \circ \Delta - u \circ \varepsilon)(V^{(k)}), \quad \text{and} \quad (M \circ (S \otimes \mathrm{id}) \circ \Delta - u \circ \varepsilon)(V^{(k)}), \quad \text{for all } k \geq 0. \tag{3.2}$$

One can check that $W$ is a Hopf ideal of $T$, and so the Hopf algebra structure maps $\Delta$, $\varepsilon$, and $S$ of $T$ give a Hopf algebra structure on $\mathcal{H}(B) = T/W$. Finally, the natural bialgebra map $\iota_B : B \to \mathcal{H}(B)$ is given by the natural embedding $\Bbbk\langle V\rangle \hookrightarrow T$ by identifying $V = V^{(0)}$.

Suppose $B = \Bbbk\langle V\rangle/(R)$ and $\alpha := \{\alpha_i : V \to \Bbbk\}_{i\in\mathbb{Z}}$ is a collection of linear functionals (with $\alpha_0 = \varepsilon$) on the subcoalgebra $V$ with convolution inverses $\alpha^{-1} := \{\alpha_i^{-1} : V \to \Bbbk\}_{i\in\mathbb{Z}}$. We extend each $\alpha_i$ and $\alpha_i^{-1}$ (which we denote again as $\alpha_i$ and $\alpha_i^{-1}$, by abuse of notation) to $\Bbbk\langle V\rangle$ inductively by the rules

$$\alpha_i(1) = \alpha_i^{-1}(1) = 1, \qquad \alpha_i(ab) := \alpha_i(a)(\alpha_{i+1} * \alpha_1^{-1})(b), \qquad \text{and} \qquad \alpha_i^{-1}(ab) := \alpha_i^{-1}(a)(\alpha_1 * \alpha_{i+1}^{-1})(b), \tag{3.3}$$

for any $a \in V$ and $b \in V^{\otimes n}$ for $n \geq 1$. We leave the proof of the following result to the reader as it is similar to the proof of Proposition 2.1.

**Proposition 3.5.** *Retain the above notation. If $\alpha_i(R) = 0$ for all $i \in \mathbb{Z}$, then $\alpha_i$ and $\alpha_i^{-1}$, defined in (3.3), are well-defined linear functionals on $B$ that are convolution inverse to each other. Moreover, $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ is a system of twisting functionals on $B$.*

When $B$ is a Hopf algebra, our next result shows how twisting functionals are valued at the antipodes.

**Lemma 3.6.** *Let $H$ be a Hopf algebra satisfying the twisting conditions. Let $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ be a system of twisting functionals on $H$ with convolution inverse $\alpha^{-1} = \{\alpha_i^{-1} : i \in \mathbb{Z}\}$. For any $i \in \mathbb{Z}$, any homogeneous element $a \in H$ of degree $j$, and any $k \geq 0$, we have:*

$$\alpha_i(S^k(a)) = \begin{cases} \alpha_i(a), & k \text{ is even} \\ (\alpha_{-j} * \alpha_{i-j}^{-1})(a), & k \text{ is odd} \end{cases} \quad \text{and} \quad \alpha_i^{-1}(S^k(a)) = \begin{cases} \alpha_i^{-1}(a), & k \text{ is even} \\ (\alpha_{i-j} * \alpha_{-j}^{-1})(a), & k \text{ is odd.} \end{cases} \tag{3.4}$$

*Proof.* For any $i \in \mathbb{Z}$, we proceed by induction on $k$. If $k = 0$, the statement is trivial. When $k = 1$, we have

$$\begin{aligned} \alpha_i(S(a)) &= \sum \alpha_i(S(a_1))(\alpha_{i-j} * \alpha_{-j}^{-1})(a_2)(\alpha_{-j} * \alpha_{i-j}^{-1})(a_3) \\ &= \alpha_i\left(\sum S(a_1)a_2\right)(\alpha_{-j} * \alpha_{i-j}^{-1})(a_3) \\ &= (\alpha_{-j} * \alpha_{i-j}^{-1})(a) \end{aligned}$$

and

$$\begin{aligned}\alpha_i^{-1}(S(a)) &= \sum \alpha_i^{-1}(S(a_1))(\alpha_{-j} * \alpha_{i-j}^{-1})(a_2)(\alpha_{i-j} * \alpha_{-j}^{-1})(a_3) \\ &= \alpha_i^{-1}\left(\sum S(a_1)a_2\right)(\alpha_{i-j} * \alpha_{-j}^{-1})(a_3) \\ &= (\alpha_{i-j} * \alpha_{-j}^{-1})(a).\end{aligned}$$

Inductively for $\alpha_i(S^{k+1}(a))$, we have

$$\alpha_i(S^{k+1}(a)) = \alpha_i S(S^k(a)) = (\alpha_{-j} * \alpha_{i-j}^{-1})(S^k(a)) = \sum \alpha_{-j}(S^k(a_1))\alpha_{i-j}^{-1}(S^k(a_2)) = (\alpha_{-j} * \alpha_{i-j}^{-1})(a),$$

for even $k$, and

$$\begin{aligned}\alpha_i(S^{k+1}(a)) = \alpha_i S(S^k(a)) = (\alpha_j * \alpha_{i+j}^{-1})(S^k(a)) &= \sum \alpha_j(S^k(a_2))\alpha_{i+j}^{-1}(S^k(a_1)) \\ &= \sum(\alpha_{-j} * \alpha_0^{-1})(a_2)(\alpha_i * \alpha_{-j}^{-1})(a_1) = \alpha_i(a),\end{aligned}$$

for odd $k$. Similarly, we can prove for $\alpha_i^{-1}(S^{k+1}(a))$. $\square$

**Proposition 3.7.** *Let $B$ be a bialgebra satisfying the twisting conditions. Then any system of twisting functionals on $B$ can be extended uniquely to a system of twisting functionals on its Hopf envelope $\mathcal{H}(B)$. Moreover, any system of twisting functionals on $\mathcal{H}(B)$ is obtained from some system of twisting functionals on $B$ in such a way.*

*Proof.* Let $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ be a system of twisting functionals on $B$. We use the presentation of $\mathcal{H}(B) = T/W$ based on $B = \Bbbk\langle V\rangle/(R)$ as discussed above.

First, we lift the system of twisting functionals $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ to the free bialgebra $\Bbbk\langle V\rangle$ in the following way. By formulas (3.3), we can extend the restrictions $\alpha_i|_V$ and $\alpha_i^{-1}|_V$ on the subcoalgebra $V$ to the free bialgebra $\Bbbk\langle V\rangle$. By abuse of notation, we still write them as $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ and $\alpha^{-1} = \{\alpha_i^{-1}\}_{i\in\mathbb{Z}}$. It is routine to check that $\alpha$ is a system of twisting functionals on $\Bbbk\langle V\rangle$ with convolution inverse $\alpha^{-1}$. Moreover, $\alpha_i(R) = \alpha_i^{-1}(R) = 0$ which factor through $B = \Bbbk\langle V\rangle/(R)$ giving back the original system of twisting functionals on $B$.

For simplicity, we write $V^{(k)} = S^k(V)$ in $T = \Bbbk\langle\oplus_{k\geq 0}V^{(k)}\rangle$. We now extend $\alpha$ and $\alpha^{-1}$ from $\Bbbk\langle V\rangle$ to $T$ by (3.4). Again, it is straightforward to check that $\alpha$ is a system of twisting functionals on $T$, with convolution inverse $\alpha^{-1}$, extending that on $\Bbbk\langle V\rangle$. By Proposition 3.5, it remains to show that $\alpha(W) = 0$, which would then yield a system of twisting functionals on $\mathcal{H}(B) = T/W$ extending that on $B$ via the natural bialgebra map $B \to \mathcal{H}(B)$. We will show that $\alpha$ and $\alpha^{-1}$ vanish on

$$S^{\ell_1}(V)\otimes\cdots\otimes S^{\ell_p}(V)\otimes S^k(R)\otimes S^{\ell_{p+1}}(V)\otimes\cdots\otimes S^{\ell_{p+q}}(V),$$
$$S^{\ell_1}(V)\otimes\cdots\otimes S^{\ell_p}(V)\otimes(M\circ(\mathrm{id}\otimes S)\circ\Delta - u\circ\varepsilon)(S^k(V))\otimes S^{\ell_{p+1}}(V)\otimes\cdots\otimes S^{\ell_{p+q}}(V),$$
$$S^{\ell_1}(V)\otimes\cdots\otimes S^{\ell_p}(V)\otimes(M\circ(S\otimes\mathrm{id})\circ\Delta - u\circ\varepsilon)(S^k(V))\otimes S^{\ell_{p+1}}(V)\otimes\cdots\otimes S^{\ell_{p+q}}(V),$$

by induction on $p+q$.

**Case 1:** Assume $p+q=0$. By (3.4), we have for any homogeneous element $r\in R$:

$$\alpha_i(S^k(r)) = \begin{cases}\alpha_i(r) & k \text{ is even}\\ (\alpha_{-|r|} * \alpha_{i-|r|}^{-1})(r) & k \text{ is odd,}\end{cases}$$

where $|r|$ denotes the degree of $r$. Since $\alpha_i(R) = \alpha_i^{-1}(R) = 0$ and $\Delta(R)\subseteq\Bbbk\langle V\rangle\otimes(R)+(R)\otimes\Bbbk\langle V\rangle$, one can check that $\alpha_i(S^k(R)) = 0$. A similar argument yields $\alpha_i^{-1}(S^k(R)) = 0$. Take any homogeneous element $a\in V$ of degree $j$, we have for $k$ even,

$$\begin{aligned}\alpha_i\left((M\circ(\mathrm{id}\otimes S)\circ\Delta - u\circ\varepsilon)(S^k(a)\right) &= \alpha_i\left(\sum S^k(a_1)S^{k+1}(a_2) - \varepsilon(a)\right) \\ &= \sum \alpha_i(S^k(a_1))(\alpha_{i+j} * \alpha_j^{-1})(S^{k+1}(a_2)) - \varepsilon(a) \\ &= \sum \alpha_i(a_1)(\alpha_{j-j} * \alpha_{i+j-j}^{-1})(a_2) - \varepsilon(a) \\ &= \sum \alpha_i(a_1)\alpha_i^{-1}(a_2) - \varepsilon(a) = 0,\end{aligned}$$

and for $k$ odd,

$$\begin{aligned}\alpha_i\left((M\circ(\mathrm{id}\otimes S)\circ\Delta-u\circ\varepsilon)(S^k(a))\right)&=\alpha_i\left(\sum S^k(a_2)S^{i+1}(a_1)-\varepsilon(a)\right)\\&=\sum\alpha_i(S^k(a_2))(\alpha_{i-j}*\alpha_{-j}^{-1})(S^{i+1}(a_1))-\varepsilon(a)\\&=\sum(\alpha_{-j}*\alpha_{i-j}^{-1})(a_2)(\alpha_{i-j}*\alpha_{-j}^{-1})(a_1)-\varepsilon(a)\\&=\sum\alpha_{-j}(a_2)\alpha_j^{-1}(a_1)-\varepsilon(a)=0.\end{aligned}$$

Similarly, we can show that $\alpha_i\left((M\circ(S\otimes\mathrm{id})\circ\Delta-u\circ\varepsilon)(S^k(V)\right)=0$ for $k\geq 0$ and also for $\alpha_i^{-1}$. This completes the $p+q=0$ case.

**Case 2:** Suppose $p+q>0$. Set

$$I_{p,q}=S^{\ell_1}(V)\otimes\cdots\otimes S^{\ell_p}(V)\otimes(M\circ(\mathrm{id}\otimes S)\circ\Delta-u\circ\varepsilon)(S^k(V))\otimes S^{\ell_{p+1}}(V)\otimes\cdots\otimes S^{\ell_{p+q}}(V).$$

We first claim that $I_{p,q}$ is a co-ideal in $T$, that is, $\Delta(I_{p,q})\subseteq T\otimes I_{p,q}+I_{p,q}\otimes T$. If $k$ is even, we have

$$\Delta((M\circ(\mathrm{id}\otimes S)\circ\Delta-u\circ\varepsilon)S^k(a))=\sum S^k(a_1)S^{k+1}(a_3)\otimes(M\circ(\mathrm{id}\otimes S)\circ\Delta-u\circ\varepsilon)S^k(a_2)\subseteq T\otimes I_{0,0}.$$

Then it is direct to check that $\Delta(I_{p,q})\subseteq T\otimes T_{p,q}$. The argument for $k$ is odd is the same. This proves our claim. Now let $p>0$. For any $a\in I_{p,q}$, without loss of generality, we can write $a=bc$ for some $b\in S^{\ell_1}(V), c\in I_{p-1,q}$. So, we can apply Lemma 3.2(2) to obtain that $\alpha_i(bc)=\sum\alpha_i(b)\alpha_{i+|b|}(c_1)\alpha_{|b|}^{-1}(c_2)=0$ since either $c_1$ or $c_2\in I_{p-1,q}$. The case for $q>0$ and $\alpha_i^{-1}$ can be argued analogously. Hence $\alpha_i(I_{p,q})=\alpha_i^{-1}(I_{p,q})=0$. By the same argument, we can show for

$$J_{p,q}=S^{\ell_1}(V)\otimes\cdots\otimes S^{\ell_p}(V)\otimes(M\circ(S\otimes\mathrm{id})\circ\Delta-u\circ\varepsilon)(S^k(V))\otimes S^{\ell_{p+1}}(V)\otimes\cdots\otimes S^{\ell_{p+q}}(V).$$

This concludes the inductive step. Finally, the uniqueness of the extension of $\alpha$ from $B$ to $\mathcal{H}(B)$ follows from Lemma 3.6. □

## 4. 2-cocycles via twisting system pairs

Throughout this section, let $B$ be a bialgebra satisfying the twisting conditions given in Definition 3.1. In this section, we introduce the notion of a twisting system pair of $B$, which we lift to that of its Hopf envelope $\mathcal{H}(B)$ and we use it to construct a certain 2-cocycle explicitly.

**Lemma 4.1.** *Let $B$ be a bialgebra satisfying the twisting conditions. Consider a collection of linear functionals $\alpha=\{\alpha_i:i\in\mathbb{Z}\}$ with convolution inverse $\{\alpha_i^{-1}:i\in\mathbb{Z}\}$ on $B$. The following are equivalent:*

(1) *The collection of maps $\alpha$ is a system of twisting functionals on $B$.*
(2) *The collection of maps $\tau=\{\tau_i:i\in\mathbb{Z}\}$ with $\tau_i=\Xi^r[\alpha_i]$ is a twisting system of $B$. In this case, the inverse twisting system $\tau^{-1}$ is given by $\tau_i^{-1}=\Xi^r[\alpha_i^{-1}]$.*
(3) *The collection of maps $\tau=\{\tau_i:i\in\mathbb{Z}\}$ with $\tau_i=\Xi^l[\alpha_i^{-1}]$ is a twisting system of $B$. In this case, the inverse twisting system $\tau^{-1}$ is given by $\tau_i^{-1}=\Xi^l[\alpha_i]$.*

*Proof.* (1)⇒(2): It is clear that for any $i\in\mathbb{Z}$, $\tau_i(a)=\Xi^r[\alpha_i](a)=\sum a_1\alpha_i(a_2)$ is a graded linear automorphism of $B$ with inverse $\tau_i^{-1}(a)=\Xi^r[\alpha_i^{-1}](a)=\sum a_1\alpha_i^{-1}(a_2)$, for any homogeneous $a\in B$ of degree $j$. Furthermore, we can compute that for $b\in B$ of any degree:

$$\tau_i(a\tau_j(b))=\tau_i\left(a\left(\sum b_1\alpha_j(b_2)\right)\right)=\sum a_1b_1\alpha_i(a_2b_2)\alpha_j(b_3)=\sum a_1b_1\alpha_i(a_2)\alpha_{i+j}(b_2)=\tau_i(a)\tau_{i+j}(b).$$

Moreover, we have $\tau_i(1)=1\alpha_i(1)=1$ and $\tau_0(a)=\sum a_1\alpha_0(a_2)=\sum a_1\varepsilon(a_2)=a$. So $\tau=\{\tau_i:i\in\mathbb{Z}\}$ is a twisting system of $B$.

(2)⇒(1): Suppose $\tau=\{\tau_i:i\in\mathbb{Z}\}$ is a twisting system of $B$. Then, we can compute that

$$\begin{aligned}\sum\alpha_i(ab_1)\alpha_j(b_2)&=\varepsilon\left(\sum a_1b_1\alpha_i(a_2b_2)\alpha_j(b_3)\right)=\varepsilon\left(\tau_i\left(a\left(\sum b_1\alpha_j(b_2)\right)\right)\right)=\varepsilon\left(\tau_i(a\tau_j(b))\right)\\&=\varepsilon\left(\tau_i(a)\tau_{i+j}(b)\right)=\varepsilon\left(\sum a_1\alpha_i(a_2)b_1\alpha_{i+j}(b_2)\right)=\alpha_i(a)\alpha_{i+j}(b).\end{aligned}$$

Also we have $\alpha_i(1)=1\alpha_i(1)=\tau_i(1)=1$ and $\alpha_0(a)=\varepsilon(a_1\alpha_0(a_2))=\varepsilon(\tau_0(a))=\varepsilon(a)$. Moreover, let $\beta_i=\varepsilon\circ\tau_i^{-1}$. Since $\Delta\circ\tau_i=(\mathrm{id}\otimes\tau_i)\circ\Delta$, one has $\Delta\circ\tau_i^{-1}=(\mathrm{id}\otimes\tau_i^{-1})\circ\Delta$. Then one can check that

$\tau_i^{-1} = \Xi^r[\beta_i]$. Hence $\tau_i \circ \tau_i^{-1} = \Xi^r[\alpha_i * \beta_i] = \mathrm{id}_B$ and $\tau_i^{-1} \circ \tau_i = \Xi^r[\beta_i * \alpha_i] = \mathrm{id}_B$, and so $\alpha_i * \beta_i = \beta_i * \alpha_i = u \circ \varepsilon$ and $\beta_i = \alpha_i^{-1}$. Thus, $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ is a system of twisting functionals of on $B$.

(1)⇔(3): This can be proved in a similar way. □

**Definition 4.2** (Twisting system pair)**.** *Let $(B, M, u, \Delta, \varepsilon)$ be a bialgebra satisfying the twisting conditions. A pair $(\tau, \mu)$ of twisting systems of $B$ is said to be a* twisting system pair *if for all $i \in \mathbb{Z}$:*

(**P1**) $\Delta \circ \tau_i = (\mathrm{id} \otimes \tau_i) \circ \Delta$ *and* $\Delta \circ \mu_i = (\mu_i \otimes \mathrm{id}) \circ \Delta$, *and*
(**P2**) $\varepsilon \circ (\tau_i \circ \mu_i) = \varepsilon$.

Using an argument similar to [8, Lemma 2.1.2], we show in Lemma 4.3 that for any twisting system pair $(\tau, \mu)$ of a bialgebra $B$, $\tau$ and $\mu$ are uniquely determined by each other as winding linear maps.

**Lemma 4.3.** *Let $B$ be a bialgebra satisfying the twisting conditions. For any twisting system pair $(\tau, \mu)$ of $B$, we have a system of twisting functionals $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ on $B$ such that $\tau = \{\tau_i = \Xi^r(\alpha_i) : i \in \mathbb{Z}\}$ and $\mu = \{\mu_i = \Xi^l(\alpha_i^{-1}) : i \in \mathbb{Z}\}$. Moreover, for any $i, j \in \mathbb{Z}$, we have the following properties:*

(**P3**) $\tau_i \circ \mu_j = \mu_j \circ \tau_i$, *and*
(**P4**) $(\tau_i \otimes \mu_i) \circ \Delta = \Delta$.

*Proof.* Let $\alpha_i = \varepsilon \circ \tau_i$ and $\alpha_i^{-1} = \varepsilon \circ \tau_i^{-1}$. Then we have

$$\tau_i(a) = \sum \tau_i(a)_1 \varepsilon(\tau_i(a)_2) \overset{(\mathbf{P1})}{=} \sum a_1 \varepsilon(\tau_i(a_2)) = \Xi^r[\varepsilon \circ \tau_i](a) = \Xi^r[\alpha_i](a).$$

Hence $\tau_i = \Xi^r[\alpha_i]$ and by Lemma 4.1, $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ is a system of twisting functionals on $B$. Since $\tau^{-1}$ satisfies (**P1**), we have $\tau_i^{-1} = \Xi^r[\alpha_i^{-1}]$. A straightforward computation shows that $\tau_i \circ \tau_i^{-1} = \Xi^r[\alpha_i * \alpha_i^{-1}] = \mathrm{id}_B$ and $\tau_i^{-1} \circ \tau_i = \Xi^r[\alpha_i^{-1} * \alpha_i] = \mathrm{id}_B$. This implies that $\alpha_i$ and $\alpha_i^{-1}$ are convolution inverse of each other. Similarly, we can show that $\mu_i = \Xi^l[\varepsilon \circ \mu_i]$. Condition (**P2**) implies that $\varepsilon = \varepsilon \circ (\tau_i \circ \mu_i) = (\varepsilon \circ \mu_i) * \alpha_i$. Hence we have $\varepsilon \circ \mu_i = \alpha_i^{-1}$ and $\mu_i = \Xi^l[\alpha_i^{-1}]$. Finally, for any $i, j \in \mathbb{Z}$, condition (**P3**) holds since

$$\begin{aligned}(\tau_i \circ \mu_j)(a) &= \tau_i\left(\Xi^l[\alpha_j^{-1}](a)\right) = \Xi^r[\alpha_i]\left(\sum \alpha_j^{-1}(a_1)a_2\right) = \sum \alpha_j^{-1}(a_1)a_2\alpha_i(a_3)\\ &= \Xi^l[\alpha_j^{-1}]\left(\sum a_1\alpha_i(a_2)\right) = \Xi^l[\alpha_j^{-1}]\left(\Xi^r[\alpha_i](a)\right) = (\mu_j \circ \tau_i)(a),\end{aligned}$$

and condition (**P4**) holds since

$$(\tau_i \otimes \mu_i)\Delta(a) = \sum \Xi^r[\alpha_i](a_1) \otimes \Xi^l[\alpha_i^{-1}(a_2)] = \sum a_1\alpha_i(a_2) \otimes \alpha_i^{-1}(a_3)a_4 = \sum a_1 \otimes a_2 = \Delta(a).$$

□

**Corollary 4.4.** *Let $B$ be a bialgebra satisfying the twisting conditions. Then any twisting system pair of a bialgebra $B$ can be extended uniquely to a twisting system pair of its Hopf envelope $\mathcal{H}(B)$. Moreover, any twisting system pair of $\mathcal{H}(B)$ is obtained from some twisting system pair of $B$ in such a way.*

*Proof.* This is a direct consequence of Lemma 4.3 and Proposition 3.7. □

Now, we consider any Hopf algebra $H$ satisfying the twisting conditions. A *right 2-cocycle* on $H$ is a convolution invertible linear map $\sigma : H \otimes H \to \Bbbk$ satisfying

$$\sum \sigma(x_1y_1, z)\sigma(x_2, y_2) = \sum \sigma(x, y_1z_1)\sigma(y_2, z_2) \qquad \text{and} \qquad \sigma(x, 1) = \sigma(1, x) = \varepsilon(x), \tag{4.1}$$

for all $x, y, z \in H$. The convolution inverse of $\sigma$, denoted by $\sigma^{-1}$, is a *left 2-cocycle* on $H$. Given a right 2-cocycle $\sigma$, let $H^\sigma$ denote the coalgebra $H$ endowed with the original unit and deformed product

$$x \cdot_\sigma y := \sum \sigma^{-1}(x_1, y_1)\, x_2y_2\, \sigma(x_3, y_3),$$

for any $x, y \in H$. In fact, $H^\sigma$ is a Hopf algebra with the deformed antipode $S^\sigma$ given in [6, Theorem 1.6]. We call $H^\sigma$ the *2-cocycle twist* of $H$ by $\sigma$. There is a monoidal equivalence

$$F : \mathrm{comod}(H) \overset{\otimes}{\cong} \mathrm{comod}(H^\sigma) \quad \text{sending} \quad U \mapsto F(U) =: U_\sigma.$$

We write $\otimes$ and $\otimes_\sigma$ for the tensor products in the corresponding right comodule categories. As a functor, $F$ is the identity functor since $H = H^\sigma$ as coalgebras. As a monoidal equivalence, $F$ is equipped with natural isomorphisms of $H^\sigma$-comodules:

$$\xi_{U,V} : F(U \otimes V) \xrightarrow{\sim} F(U) \otimes_\sigma F(V)$$

$$u \otimes v \ \mapsto \ \sum \sigma^{-1}(u_1, v_1)\, u_0 \otimes v_0,$$

compatible with the associativity, where the right coaction of $H$ on $U$ is given by $\rho : u \mapsto \sum u_0 \otimes u_1 \in U \otimes H$. In particular, $F$ sends a (connected graded) $H$-comodule algebra $A$ to the twisted (connected graded) $H^\sigma$-comodule algebra $F(A) = A_\sigma = A$ as vector spaces, with 2-cocycle twist multiplication $a \cdot_\sigma b = \sum a_0 b_0 \sigma(a_1, b_1)$, for any $a, b \in A$.

**Proposition 4.5.** *Let $H$ be a Hopf algebra satisfying the twisting conditions, and $(\tau, \mu)$ be a twisting system pair of $H$. Then $\tau \circ \mu = \{\tau_i \circ \mu_i : i \in \mathbb{Z}\}$ is a twisting system of $H$. Moreover, $H^{\tau\circ\mu} \cong H^\sigma$ as graded algebras, where the right 2-cocycle $\sigma : H \otimes H \to \Bbbk$ and its convolution inverse $\sigma^{-1}$ are given by*

$$\sigma(x, y) = \varepsilon(x)\varepsilon(\tau_{|x|}(y)) \qquad \text{and} \qquad \sigma^{-1}(x, y) = \varepsilon(x)\varepsilon(\mu_{|x|}(y))$$

*for any homogeneous elements $x, y \in H$ where $|x|, |y|$ denote the degrees of $x$ and $y$, respectively.*

*Proof.* We first show that $\tau \circ \mu$ is a twisting system. It is clear that $\nu := \{\nu_i = \tau_i \circ \mu_i : i \in \mathbb{Z}\}$ is a set of graded linear automorphisms with inverse $\nu^{-1} := \{\nu_i^{-1} = \mu_i^{-1} \circ \tau_i^{-1} : i \in \mathbb{Z}\}$ on $H$. By Lemma 4.3, we have

$$\tau_i = \Xi^r(\alpha_i), \qquad \tau_i^{-1} = \Xi^r(\alpha_i^{-1}), \qquad \mu_i = \Xi^l(\alpha_i^{-1}), \qquad \mu_i^{-1} = \Xi^l(\alpha_i), \tag{4.2}$$

for the system of twisting functionals $\alpha := \{\alpha_i = \varepsilon \circ \tau_i : i \in \mathbb{Z}\}$ on $H$. Let $x, y$ and $z$ be homogeneous elements in $H$. For any $i \in \mathbb{Z}$, $\nu$ is a twisting system of $H$ since

$$\begin{aligned} \nu_i(xy) &= \tau_i \circ \mu_i(xy) = (\tau_i \circ \mu_i(x)) \left(\tau_{i+|x|} \circ \tau_{|x|}^{-1} \circ \mu_{i+|x|} \circ \mu_{|x|}^{-1}(y)\right) \\ &\overset{\textbf{(P3)}}{=} (\tau_i \circ \mu_i(x)) \left(\tau_{i+|x|} \circ \mu_{i+|x|} \circ \mu_{|x|}^{-1} \circ \tau_{|x|}^{-1}(y)\right) = (\nu_i(x)) \left(\nu_{i+|x|} \circ \nu_{|x|}^{-1}(y)\right). \end{aligned}$$

We show next that $\sigma$ satisfies (4.1):

$$\begin{aligned} \sigma(x_1 y_1, z)\sigma(x_2, y_2) &= \sum \varepsilon(x_1 y_1)\alpha_{|x|+|y|}(z)\varepsilon(x_2)\alpha_{|x|}(y_2) = \sum \varepsilon(x)\alpha_{|x|}(y)\alpha_{|x|+|y|}(z) \\ &= \sum \varepsilon(x)\alpha_{|x|}(yz_1)\alpha_{|y|}(z_2) = \sum \varepsilon(x)\alpha_{|x|}(y_1 z_1)\varepsilon(y_2)\alpha_{|y|}(z_2) = \sum \sigma(x, y_1 z_1)\sigma(y_2, z_2), \end{aligned}$$

where the third equality follows from Lemma 3.2(1) and $\sigma(x, 1) = \varepsilon(x)\alpha_{|x|}(1) = \varepsilon(x) = \varepsilon(1)\alpha_0(x) = \sigma(1, x)$. Note that it is straightforward to check that $\sigma$ is convolution invertible with inverse $\sigma^{-1}(x, y) = \varepsilon(x)\alpha_{|x|}^{-1}(y) = \varepsilon(x)\varepsilon(\mu_{|x|}(y))$. Thus, $\sigma$ is a right 2-cocycle on $H$.

We now show that $H^{\tau\circ\mu} \cong H^\sigma$ as graded algebras via the identity map on vector spaces. By (4.2) and Lemma 4.3, we indeed have

$$\begin{aligned} x \cdot_\sigma y &= \sum \sigma^{-1}(x_1, y_1)x_2 y_2 \sigma(x_3, y_3) = \sum \varepsilon(x_1)\alpha_{|x|}^{-1}(y_1)x_2 y_2 \varepsilon(x_3)\alpha_{|x|}(y_3) \\ &= \sum x\alpha_{|x|}^{-1}(y_1)y_2\alpha_{|x|}(y_3) = x\mu_{|x|}\tau_{|x|}(y) = x\nu_{|x|}(y) = x \cdot_\nu y. \end{aligned}$$

Since $H^\sigma$ is a Hopf algebra, it implies that $H^{\tau\circ\mu}$ also has a Hopf algebra structure via the above identity isomorphism $\mathrm{id} : H^{\tau\circ\mu} \cong H^\sigma$. □

**Proposition 4.6.** *Let $B$ be a bialgebra satisfying the twisting conditions, $(\tau, \mu)$ be a twisting system pair of $B$, and $(\mathcal{H}(\tau), \mathcal{H}(\mu))$ be the induced twisting system pair of $\mathcal{H}(B)$ via Corollary 4.4. Then $\mathcal{H}(B^{\tau\circ\mu}) \cong \mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)}$ as Hopf algebras.*

*Proof.* Denote by $\tau^{-1}$ and $\mu^{-1}$ the inverse twisting systems of $\tau$ and $\mu$, respectively. Since $\tau^{-1}$ and $\mu^{-1}$ are twisting systems of $B^\tau$ and $B^\mu$ respectively, one can directly check that $(\tau^{-1}, \mu^{-1})$ is the twisting system pair of $B^{\tau\circ\mu}$ such that $B \cong (B^{\tau\circ\mu})^{\tau^{-1}\circ\mu^{-1}}$ as bialgebras. Similarly, we write $(\mathcal{H}(\tau)^{-1}, \mathcal{H}(\mu)^{-1}) = (\mathcal{H}(\tau^{-1}), \mathcal{H}(\mu^{-1}))$ as the unique extension of the twisting system pair $(\tau^{-1}, \mu^{-1})$ from $B^{\tau\circ\mu}$ to $\mathcal{H}(B^{\tau\circ\mu})$.

We denote by $\iota_B : B \to \mathcal{H}(B)$ and $\iota_{B^{\tau\circ\mu}} : B^{\tau\circ\mu} \to \mathcal{H}(B^{\tau\circ\mu})$ the corresponding bialgebra maps from bialgebras to their Hopf envelopes satisfying the required universal property.

By the universal property of the Hopf envelope, one has a unique Hopf algebra map $g : \mathcal{H}(B^{\tau\circ\mu}) \to \mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)}$ where the following diagram commutes:

$$\begin{tikzcd} B^{\tau\circ\mu} \arrow[r, "\iota_{B^{\tau\circ\mu}}"] \arrow[dr, "(\iota_B)^{\tau\circ\mu}"'] & \mathcal{H}(B^{\tau\circ\mu}) \arrow[d, "g"] \\ & \mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)}. \end{tikzcd}$$

Similarly, one has a unique Hopf algebra map $h : \mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)} \to \mathcal{H}(B^{\tau\circ\mu})$ making the diagram

$$\begin{tikzcd} B \arrow[r, "\iota_B"] \arrow[dr, "(\iota_{B^{\tau\circ\mu}})^{\mathcal{H}(\tau)^{-1}\circ\mathcal{H}(\mu)^{-1}}"'] & \mathcal{H}(B) \arrow[d, "h"] \\ & \mathcal{H}(B^{\tau\circ\mu})^{\mathcal{H}(\tau)^{-1}\circ\mathcal{H}(\mu)^{-1}} \end{tikzcd}$$

commute. By letting $l = h^{\tau\circ\mu}$, we have the following commutative diagram:

$$\begin{tikzcd} B^{\tau\circ\mu} \arrow[r, "(\iota_B)^{\tau\circ\mu}"] \arrow[dr, "(\iota_{B^{\tau\circ\mu}})"'] & \mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)} \arrow[d, "l"] \\ & \mathcal{H}(B^{\tau\circ\mu}). \end{tikzcd}$$

By the universal property of $\iota_B$ and $\iota_{B^{\tau\circ\mu}}$ again, one can show that $g\circ l$ and $l\circ g$ are identities on $\mathcal{H}(B)^{\mathcal{H}(\tau)\circ\mathcal{H}(\mu)}$ and $\mathcal{H}(B^{\tau\circ\mu})$, respectively. This completes our proof. $\square$

## 5. Proof of Theorem 1.4

Throughout this section, let $A$ be an $m$-homogeneous algebra and $A^!$ be its Koszul dual. Let $\tau$ be a twisting system of $A$ and $\tau^!$ be the dual twisting system of $A^!$, defined in Section 2. In the following results, we find a twisting pair of $A \bullet A^! \cong \underline{\mathrm{end}}^r(A)$ and lift it to give a Hopf algebra isomorphism between the universal quantum algebra of the Zhang twist $A^\tau$ and the 2-cocycle twist of the universal quantum algebra of $A$ (see Lemma 5.2). We then prove our main result, Theorem 1.4, which states that if two $m$-homogeneous algebras are graded Morita equivalent then they are quantum-symmetrically equivalent.

**Lemma 5.1.** *If $A$ is an $m$-homogeneous algebra with twisting system $\tau$, then $\tau \bullet \mathrm{id}$ and $\mathrm{id} \bullet \tau^!$ (defined in Section 2) form a twisting system pair of $A \bullet A^! \cong \underline{\mathrm{end}}^r(A)$. Moreover, we have the commutative diagrams:*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\tau"'] & A \otimes \underline{\mathrm{end}}^r(A) \arrow[d, "\mathrm{id}\otimes(\tau\bullet\mathrm{id})"] \\ A \arrow[r, "\rho"] & A \otimes \underline{\mathrm{end}}^r(A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A^! \arrow[r, "\rho^!"] \arrow[d, "\tau^!"'] & \underline{\mathrm{end}}^r(A) \otimes A^! \arrow[d, "(\mathrm{id}\bullet\tau^!)\otimes\mathrm{id}"] \\ A^! \arrow[r, "\rho^!"] & \underline{\mathrm{end}}^r(A) \otimes A^!. \end{tikzcd} \tag{5.1}$$

*Proof.* We know that both $\mu := \tau \bullet \mathrm{id}$ and $\xi := \mathrm{id} \bullet \tau^!$ are twisting systems of $\underline{\mathrm{end}}^r(A) = A \bullet A^!$, by Proposition 2.2 and Proposition 2.5. Suppose $\{x_1, ..., x_n\}$ is a basis of $A_1$; denote the dual basis of $A^!_1$ by $\{x^1, ..., x^n\}$. Recall that the coaction of $A \bullet A^!$ on $A$ sends

$$\rho : x_j \mapsto \sum x_k \otimes z^k_j,$$

where $z^k_j$ is the image of $x_j \otimes x^k$ in $A \bullet A^!$. Since each linear automorphism $\tau_i$ preserves degrees, we have some invertible scalar matrix $(\lambda^i_{jl})_{1\le j,l\le n}$ with inverse $(\phi^i_{jl})_{1\le j,l\le n}$ such that

$$\tau_i : x_j \mapsto \sum \lambda^i_{jl} x_l, \qquad \mu_i(z^k_j) = \sum_{1\le l\le n} \lambda^i_{jl} z^k_l, \qquad \text{and} \qquad \xi_i(z^k_j) = \sum_{1\le l\le n} z^l_j \phi^i_{lk}. \tag{5.2}$$

We show that (**P1**) and (**P2**) hold for $\mu$ and $\xi$ by induction on the degrees in $A \bullet A^!$. It is trivial for degree 0 and straightforward for degree 1 due to (5.2). Suppose (**P1**) and (**P2**) hold for all degrees $\le n$. Take any homogeneous elements $a, b$ in $A \bullet A^!$ with $a$ of degree $j$ and $b$ of degree $n+1-j$. Then one can check that

$$\begin{aligned}\Delta \circ \mu_i(ab) &= \Delta \circ (\mu_i(a)\mu_{i+j}\mu_j^{-1}(b)) = (\Delta \circ \mu_i)(a)(\Delta \circ \mu_{i+j}\mu_j^{-1})(b) \\ &= (\mathrm{id} \otimes \mu_i) \circ \Delta(a)(\mathrm{id} \otimes \mu_{i+j}\mu_j^{-1}) \circ \Delta(b) = \sum a_1 b_1 \otimes \mu_i(a_2)\mu_{i+j}\mu_j^{-1}(b_2) \\ &= \sum a_1 b_1 \otimes \mu_i(a_2 b_2) = (\mathrm{id} \otimes \mu_i) \circ \Delta(ab).\end{aligned}$$

So (**P1**) holds for $\mu$ and similarly for $\xi$. Now for (**P2**), we have

$$\begin{aligned}\varepsilon \circ (\mu_i \circ \xi_i)(ab) &= \varepsilon\mu_i\xi_i(a)\varepsilon(\mu_{i+j}\mu_j^{-1}\xi_{i+j}\xi_j^{-1})(b) = \varepsilon\mu_i\xi_i(a)\varepsilon(\mu_{i+j}\xi_{i+j}\mu_j^{-1}\xi_j^{-1})(b) \\ &= \varepsilon(a)\varepsilon(\mu_j^{-1}\xi_j^{-1})(b) = \varepsilon(a)\varepsilon(b) = \varepsilon(ab).\end{aligned}$$

Hence $(\mu, \xi)$ is a twisting system pair of $\underline{\mathrm{end}}^r(A)$.

For the diagrams in (5.1), we will show the first diagram is commutative. A similar argument can be applied to show the second diagram is commutative. One can check that

$$(\mathrm{id}_A \otimes \mu_i)\rho(x_j) = (\mathrm{id}_A \otimes \mu_i)\left(\sum x_k \otimes z_j^k\right) = \sum_{k,l} x_k \otimes \lambda_{jl}^i z_l^k = \rho\left(\sum_l \lambda_{jl}^i x_l\right) = \rho\tau_i(x_j).$$

Note that by a similar argument, we also have $(\mathrm{id}_A \otimes \mu_i^{-1})\rho = \rho\tau_i^{-1}$. Now by an inductive argument, we prove that the diagram commutes in degree $n$, supposing that for any degree $n-1$ element $a$, we have

$$\rho\tau_i(a) = (\mathrm{id}_A \otimes \mu_i)\rho \quad \text{and} \quad \rho\tau_i^{-1}(a) = (\mathrm{id}_A \otimes \mu_i^{-1})\rho.$$

Of course, it is enough to check on degree $n$ elements of the form $xa$, where $x \in A_1$ and $a \in A_{n-1}$, since we are assuming $A$ is generated in degree 1. Now we can check

$$\begin{aligned}\rho\tau_i(xa) &= \rho(\tau_i(x)\tau_{i+1}\tau_1^{-1}(a)) = \rho\tau_i(x)\rho\tau_{i+1}\tau_1^{-1}(a) \\ &= (\mathrm{id}_A \otimes \mu_i)\rho(x)(\mathrm{id}_A \otimes \mu_{i+1})(\mathrm{id}_A \otimes \mu_1^{-1})\rho(a) = (\mathrm{id}_A \otimes \mu_i)\rho(xa).\end{aligned}$$

The argument for $\rho\tau_i^{-1}$ is similar. By induction, the diagram commutes in all degrees. □

Let $\sigma$ be the 2-cocycle of $\underline{\mathrm{aut}}^r(A)$ corresponding to the twisting system pair $(\mathcal{H}(\tau \bullet \mathrm{id}), \mathcal{H}(\mathrm{id} \bullet \tau^!))$ in Corollary 4.4 and Proposition 4.5. We know that $\underline{\mathrm{aut}}^r(A)^{\mathcal{H}(\tau\bullet\mathrm{id})\circ\mathcal{H}(\mathrm{id}\bullet\tau)} \cong \underline{\mathrm{aut}}^r(A)^\sigma$. Moreover, by the universal property of the Hopf envelope and (5.1), the diagrams

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\tau"'] & A \otimes \underline{\mathrm{aut}}^r(A) \arrow[d, "\mathrm{id} \otimes \mathcal{H}(\tau\bullet\mathrm{id})"] \\ A \arrow[r, "\rho"] & A \otimes \underline{\mathrm{aut}}^r(A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A^! \arrow[r, "\rho^!"] \arrow[d, "\tau^!"'] & \underline{\mathrm{aut}}^r(A) \otimes A^! \arrow[d, "\mathcal{H}(\mathrm{id}\bullet\tau^!)\otimes\mathrm{id}"] \\ A^! \arrow[r, "\rho^!"] & \underline{\mathrm{aut}}^r(A) \otimes A^! \end{tikzcd} \tag{5.3}$$

commute. We use the next result to prove that quantum-symmetric equivalence is a graded Morita invariant.

**Lemma 5.2.** *Let $A$ be an $m$-homogeneous algebra and $\tau$ be a twisting system of $A$. We have an isomorphism of Hopf algebras $\underline{\mathrm{aut}}^r(A^\tau) \cong \underline{\mathrm{aut}}^r(A)^\sigma$, where $\sigma$ is the right 2-cocycle corresponding to the twisting system pair $(\mathcal{H}(\tau \bullet \mathrm{id}), \mathcal{H}(\mathrm{id} \bullet \tau^!))$ defined in Proposition 4.5.*

*Proof.* We check that

$$\underline{\mathrm{end}}^r(A^\tau) \cong A^\tau \bullet (A^\tau)^! \cong A^\tau \bullet (A^!)^{\tau^!} \cong (A \bullet A^!)^{\tau\bullet\tau^!} \cong \underline{\mathrm{end}}^r(A)^{\tau\bullet\tau^!},$$

where the second isomorphism follows by Proposition 2.3, and the third isomorphism follows from Proposition 2.5. Then we can show that

$$\underline{\mathrm{aut}}^r(A^\tau) \cong \mathcal{H}(\underline{\mathrm{end}}^r(A^\tau)) \cong \mathcal{H}(\underline{\mathrm{end}}^r(A)^{\tau\bullet\tau^!}) \cong \mathcal{H}(\underline{\mathrm{end}}^r(A))^{\mathcal{H}(\tau\bullet\tau^!)} \cong \underline{\mathrm{aut}}^r(A)^\sigma,$$

where the second isomorphism follows from our above computation, the third isomorphism follows from Proposition 4.6, and the fourth isomorphism follows from Proposition 4.5. □

**Proof of Theorem 1.4.** Let $A$ and $B$ be two $m$-homogeneous algebras that are graded Morita equivalent; we must show that they are quantum-symmetrically equivalent. Without loss of generality, by [23, Theorem 1.2], we can assume $B = A^\tau$ for some twisting system $\tau = \{\tau_i : i \in \mathbb{Z}\}$ of $A$. By Lemma 5.2, there exists a right 2-cocycle $\sigma$ on $\underline{\mathrm{aut}}^r(A)$ given by the twisting system pair $(\mathcal{H}(\tau \bullet \mathrm{id}), \mathcal{H}(\mathrm{id} \bullet \tau^!))$ such that $\underline{\mathrm{aut}}^r(A^\tau) \cong \underline{\mathrm{aut}}^r(A)^\sigma$. As a consequence, $\mathrm{comod}(\underline{\mathrm{aut}}^r(A))$ and $\mathrm{comod}(\underline{\mathrm{aut}}^r(A^\tau))$ are monoidally equivalent. Since $A$ is an $\underline{\mathrm{aut}}^r(A)$-comodule algebra, we can consider the corresponding $\underline{\mathrm{aut}}^r(A)^\sigma$-comodule algebra $A_\sigma$. It remains to show that there is an isomorphism $A_\sigma \cong A^\tau$ of $\underline{\mathrm{aut}}^r(A)^\sigma$-comodule algebras. The following computation concludes the proof: for any homogeneous elements $a, b \in A$,

$$\begin{aligned} a \cdot_\sigma b &= \sum a_0 b_0 \sigma(a_1, b_1) = \sum a_0 b_0 \varepsilon(a_1) \varepsilon(\mathcal{H}(\tau \bullet \mathrm{id})_{|a|}(b_1)) = a((\varepsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathcal{H}(\tau \bullet \mathrm{id})) \circ \rho)(b) \\ &= a(\varepsilon \otimes \mathrm{id})(\rho(\tau_{|a|}(b))) = a\tau_{|a|}(b) = a \cdot_\tau b, \end{aligned}$$

the fourth equality follows from (5.3). □

The following is now an immediate consequence of the main results proved in our paper.

**Corollary 5.3.** *Let $A$ be any $m$-homogeneous algebra and $H$ a Hopf algebra that right coacts on $A$ by preserving its grading. Then for any right 2-cocycle $\sigma$ on $H$, the following are equivalent.*

(i) *The 2-cocycle twist algebra $A_\sigma$ and $A$ are graded Morita equivalent.*
(ii) *There is a twisting system $\tau$ on $A$ such that $A_\sigma \cong A^\tau$ as graded algebras.*
(iii) *There is a 2-cocycle $\sigma'$ on $\underline{\mathrm{aut}}^r(A)$ given by some twisting system pair such that $A_\sigma \cong A_{\sigma'}$ as algebras.*

*Proof.* (i)⇔(ii) Note that by [8, Lemma 4.1.5], $A_\sigma$ is again an $m$-homogeneous algebra. So the equivalence directly follows from [23, Theorem 1.2].

(ii)⇒(iii) It is derived from the proof of Theorem 1.4, where the twisting system pair is given in Lemma 5.2.

(iii)⇒(ii) Without loss of generality, we can assume the 2-cocycle $\sigma$ is given by some twisting system $(f, g)$ on $\underline{\mathrm{aut}}^r(A)$. By Lemma 4.3, there is a system of twisting functionals $\{\alpha_i : i \in \mathbb{Z}\}$ on $\underline{\mathrm{aut}}^r(A)$ such that $f_i = \Xi^r[\alpha_i]$ and $g_i = \Xi^l[\alpha_i^{-1}]$. We define a collection of graded linear automorphisms $\tau = \{\tau_i : i \in \mathbb{Z}\}$ on $A$ via $\tau_i(a) = \sum a_0 \alpha_i(a_1)$ with linear inverse $\tau_i^{-1}(a) = \sum a_0 \alpha_i^{-1}(a_1)$. Similar to Lemma 4.1, one can easily check that $\tau$ is a twisting system on $A$. Note the 2-cocycle $\sigma$ on $\underline{\mathrm{aut}}^r(A)$ is given by $\sigma(x, y) = \varepsilon(x)\alpha_{|x|}(y)$ for any homogeneous elements $x, y \in \underline{\mathrm{aut}}^r(A)$. Therefore, we have

$$a \cdot_\sigma b = \sum a_0 b_0 \sigma(a_1, b_1) = \sum a b_0 \alpha_{|a|}(b_1) = a\tau_{|a|}(b) = a \cdot_\tau b$$

for any homogeneous elements $a, b \in A$. This proves the implication. □

**Remark 5.4.** *In* [3]*, Artin and Zhang introduced the concept of a noncommutative projective scheme* $\mathrm{Proj}(A)$*, which gives an analogue of the category of quasi-coherent sheaves for the noncommutative projective space associated to $A$. Since* $\mathrm{Proj}(A)$ *is a quotient of* $\mathrm{grmod}(A)$*, and we have proven that $QS(A)$ only depends on* $\mathrm{grmod}(A)$*, one might ask whether $QS(A)$ is actually an invariant of* $\mathrm{Proj}(A)$*. However, we point out that there are connected graded algebras whose noncommutative projective schemes are equivalent but are not quantum-symmetrically equivalent. For example, let $A$ be a polynomial algebra and $B = A^{\langle d \rangle}$ be the Veronese subalgebra, which always shares the same* Proj *with $A$ (see e.g.,* [15, Introduction] *for further details on the Veronese subalgebra). By* [10, Lemma 3.2.7]*, $A$ and $B$ are not quantum-symmetrically equivalent since $A$ has a finite global dimension, but $B$ does not when $d \geq 2$.*

We speculate that Theorem 1.4 holds in general for any two graded algebras that are finitely generated in degree one, without the $m$-homogeneous assumption. Since Theorem 1.4 implies that the tensor category $\mathrm{comod}(\underline{\mathrm{aut}}^r(A))$ depends only on $\mathrm{grmod}(A)$ rather than on $A$, we ask the following question.

**Question 5.5.** *For a connected graded algebra $A$ that is finitely generated in degree one, is there an intrinsic categorical construction for* $\mathrm{comod}(\underline{\mathrm{aut}}^r(A))$ *purely in terms of* $\mathrm{grmod}(A)$*?*

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(Huang) Department of Mathematics, Shanghai University, Shanghai, 200444, China
*Email address*: hdhuang@shu.edu.cn

(Nguyen) Department of Mathematics, United States Naval Academy, Annapolis, MD 21402, U.S.A.
*Email address*: vnguyen@usna.edu

(Vashaw) Department of Mathematics, University of California Los Angeles, Los Angeles, CA 90095, U.S.A.
*Email address*: kentvashaw@math.ucla.edu

(Veerapen) Department of Mathematics, Tennessee Tech University, Cookeville, TN 38505, U.S.A.
*Email address*: pveerapen@tntech.edu

(Wang) Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA
*Email address*: xingtingwang@math.lsu.edu