Title
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TENSOR PRODUCTS OF FAITHFUL MODULES

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Abstract. If k is a field, A and B k-algebras, M a faithful left A-module, and N a faithful left B-module, we recall the proof that the left $A \otimes_k B$-module $M \otimes_k N$ is again faithful. If k is a general commutative ring, we note some conditions on A, B, M and N that do, and others that do not, imply the same conclusion. Finally, we note a version of the main result that does not involve any algebra structures on A and B.

I needed Theorem 1 below, and eventually found a roundabout proof of it. Ken Goodearl found a simpler proof, which I simplified further to the argument given here. But it seemed implausible that such a result would not be classical, and I posted a query [1], to which Benjamin Steinberg responded, noting that Passman had proved the result in [4, Lemma 1.1]. His proof is virtually identical to one below.

In the mean time, I had made some observations on what is true when the base ring k is not a field, and on the "irrelevance" of the algebra structures of A and B, and added these to the write-up. So while I no longer expect to publish this note, I will arXiv it, and keep it online as an unpublished note, to make those observations available.

(Ironically, I eventually found an easier way to get the result for which I had needed Theorem 1.)

1. The main statement, and a generalization

Except where the contrary is stated, we understand algebras to be associative, but not necessarily unital.

For k a commutative ring, A and B k-algebras, M a left A-module, and N a left B-module, we recall the natural structure of left $A \otimes_k B$-module on $M \otimes_k N$: An element

(1) $\qquad f \;=\; \sum_{1 \leq i \leq n} a_i \otimes b_i \;\in\; A \otimes_k B, \qquad \text{where } a_i \in A,\ b_i \in B,$

acts on decomposable elements $u \otimes v$ $(u \in M,\ v \in N)$ of $M \otimes_k N$ by

(2) $\qquad u \otimes v \;\mapsto\; \sum_i a_i u \otimes b_i v.$

Since the right-hand side is bilinear in u and v, this map extends k-linearly to general elements of $M \otimes_k N$. The resulting action is easily shown to be compatible with the k-algebra structure of $A \otimes_k B$.

Theorem 1. *Let k be a field, A and B k-algebras, M a faithful left A-module, and N a faithful left B-module. Then the left $A \otimes_k B$-module $M \otimes_k N$ is also faithful.*

Proof (after K. Goodearl, D. Passman). Given nonzero $f \in A \otimes_k B$, we wish to show that it has nonzero action on $M \otimes_k N$. Clearly, we can choose an expression (1) for f such that the b_i are k-linearly independent. (We could simultaneously make the a_i k-linearly independent, but will not need to.) Since we have assumed (1) nonzero, not all of the a_i are zero; so as M is a faithful A-module, we can find $u \in M$ such that not all the $a_i u \in M$ are zero. Hence there exists a k-linear functional $\varphi : M \to k$ such that not all of the $\varphi(a_i u)$ are zero. Since the b_i are k-linearly independent, the element $\sum_i \varphi(a_i u)\, b_i \in B$ will thus be nonzero. So as N is a faithful B-module, we can choose $v \in N$ such that

(3) $\qquad (\sum_i \varphi(a_i u)\, b_i)\, v \;\neq\; 0 \quad \text{in } N.$

We claim that for the above choices of u and v, if we apply f to $u \otimes v \in M \otimes_k N$, the result, i.e., the right-hand side of (2), is nonzero. For if we apply to that element the map $\varphi \otimes \mathrm{id}_N : M \otimes_k N \to k \otimes_k N \cong N$, we get the nonzero element (3). Thus, as required, f has nonzero action on $M \otimes_k N$. ☐

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The above result assumes k a field. Succumbing to the temptation to examine what the method of proof can be made to give in the absence of that hypothesis, we record

Corollary 2 (to proof of Theorem 1)**.** *Let k be a commutative ring, A and B k-algebras, M a faithful left A-module, and N a faithful left B-module.*

Suppose, moreover, that elements of M can be separated by k-module homomorphisms $M \to k$, and that every finite subset of B belongs to a free k-submodule of B.

Then the left $A \otimes_k B$-module $M \otimes_k N$ is again faithful.

Proof. Exactly like the proof of Theorem 1. The added hypothesis on B is what we need to conclude that any element of $A \otimes_k B$ can be written in the form (1) with k-linearly independent b_i; the added hypothesis on M is what we need to construct φ. (Of course, the parenthetical comment in the proof of Theorem 1 about also making the a_i k-linearly independent does not go over.) □

Warren Dicks (personal communication) pointed out early on another proof of Theorem 1: The actions of A and B on M and N yield embeddings $A \to \mathrm{End}_k(M)$ and $B \to \mathrm{End}_k(N)$. Taking k-bases X and Y of M and N, one can regard the underlying vector spaces of $\mathrm{End}_k(M)$ and $\mathrm{End}_k(N)$ as M^X and N^Y. By two applications of [2, II, §3, №7, Cor. 3 to Prop. 7, p.AII.63], or one of [3, Theorem 2], one concludes that the natural map $M^X \otimes_k N^Y \to (M \otimes_k N)^{X \times Y}$ is an embedding, and deduces that the map $A \otimes_k B \to \mathrm{End}(M \otimes_k N)$ is an embedding, the desired conclusion.

2. Counterexamples to variant statements

The hypotheses of the above corollary are strikingly asymmetric in the pairs (A, M) and (B, N). We can, of course, get the same conclusion if we interchange the assumptions on these pairs. But what if we try to use one or the other hypothesis on both pairs; or concentrate both hypotheses on one of them?

It turns out that none of these modified hypotheses guarantees the stated conclusion. Here are three closely related constructions that give the relevant counterexamples. In all three examples, the k-algebras A and B are in fact commutative and unital.

Lemma 3. *Let C be a commutative principal ideal domain with infinitely many primes ideals, and let P_0, P_1 be two disjoint infinite sets of nonzero prime ideals of C. Then for k, A, M, B, N specified in each of the following three ways, the A-module M and the B-module N are faithful, and $A \otimes_k B$ is nonzero, but $M \otimes_k N$ is zero, hence not faithful. In each example, the variant of the hypotheses of Corollary 2 satisfied by that example is noted at the end of the description.*

(i) *Let $A = B = k = C$, and let $M = \bigoplus_{p \in P_0} k/p$ and $N = \bigoplus_{p \in P_1} k/p$. In this case, every finite subset of A or of B (trivially) lies in a free k-submodule of that algebra.*

In the remaining two examples, let C^+ be the commutative C-algebra obtained by adjoining to C an indeterminate y_p for each $p \in P_0 \cup P_1$, and imposing the relations $p\, y_p = \{0\}$ for each such p.

(ii) *Let $k = C^+$, let M be the ideal of k generated by the y_p for $p \in P_0$, let N be the ideal of k generated by the y_p for $p \in P_1$, let $A = k/N$, and let $B = k/M$. Note that since $MN = \{0\}$, we may regard M as an A-module and N as a B-module. In this case, M and N embed in k, hence k-module homomorphisms from each of those modules into k separate elements.*

(iii) *Let $k = A = C^+$, and let M be the ideal of A generated by all the y_p. (The partition of our infinite family of primes into P_0 and P_1 will not be used here.) On the other hand, let B be the field of fractions of C, made a k-algebra by first mapping k to C by sending the y_p to 0, then mapping C to its field of fractions; and let N be any nonzero B-vector-space. In this case, every finite subset of A (trivially) lies in a free k-submodule, and k-module homomorphisms into k clearly separate elements of M.*

Proof. The fact that P_0 and P_1 are infinite sets of primes in the commutative principal ideal domain C implies that each of those sets has zero intersection, hence that the M and N of (i) are faithful C-modules, equivalently, are a faithful A-module and a faithful B-module. That their tensor product is zero is clear.

Similar considerations show that in (ii), any element of A or B with nonzero constant term in C acts nontrivially on M, respectively, N. On the other hand, a nonzero element of A or B with zero constant term, i.e., a nonzero element u of the ideal M or N, will act nontrivially on the module M, respectively, N, because for some p in P_0, respectively P_1, the element u must involve a nonzero polynomial in y_p, which will have nonzero action on $y_p \in M$ or N as the case may be. (Since $y_p\, y_{p'} = \{0\}$ for $p \neq p'$, every element of M or N is a sum of one-variable polynomials in the various y_p with zero constant term.) Again, it is clear that $M \otimes_k N = \{0\}$. On the other hand, $A \otimes_k B = (k/M) \otimes_k (k/N) \cong k/(M+N) \cong C \neq \{0\}$.

In case (iii), M is faithful over A for the same reason as in (ii), while faithfulness of N over B is clear, as is the condition $M \otimes_k N = \{0\}$. On the other hand, since A admits a homomorphism into C, we have $A \otimes_k B \neq \{0\}$. □

We remark that in the above constructions, the condition that the principal ideal domain k have infinitely many primes can be weakened to say that it has at least two primes, p_0 and p_1, by replacing the modules $\bigoplus_{p\in P_0} k/p$ and $\bigoplus_{p\in P_1} k/p$ in (i) with $\bigoplus_{n>0} k/p_0^n$ and $\bigoplus_{n>0} k/p_1^n$, and making similar adjustments in (ii) and (iii). (In the last, only one prime is needed.) One just has to be a little more careful in the proofs.

In another direction, one may ask:

Question 4 (suggested by A. Ogus, personal communication)**.** *If we add to the hypotheses of Corollary 2 the condition that M be finitely generated as an A-module, and/or that N be finitely generated as a B-module, can some of the other hypotheses of that corollary be weakened, dropped, or modified* (*perhaps in some of the ways that Lemma 3 shows is* not *possible without such finite generation conditions*)*?*

3. A and B don't have to be algebras

The sharp-eyed reader may have noticed that the proof of Theorem 1 makes no use of the algebra structures of A and B. This led me to wonder whether the result was actually a special case of a statement that involved no such structure. As Theorem 5 below shows, the answer is, in a way, yes. But as the second proof of that theorem shows, one can equally regard Theorem 5 as a special case of Theorem 1.

Given a commutative ring k, and k-modules A, M_0 and M_1, let us define an *action* of A on (M_0, M_1) to mean a k-linear map $A \to \mathrm{Hom}_k(M_0, M_1)$, which will be written $(a, u) \mapsto au$ $(a \in A,\ u \in M_0,\ au \in M_1)$; and let us call such an action *faithful* if it is one-to-one as a map $A \to \mathrm{Hom}(M_0, M_1)$.

Given two actions, one of a k-module A on a pair (M_0, M_1) and the other of a k-module B on a pair (N_0, N_1), we see that an action of $A \otimes_k B$ on $(M_0 \otimes_k N_0,\ M_1 \otimes_k N_1)$ can be defined just as for algebras, with each element (1) acting by (2).

Theorem 5. *Let k be a field, and suppose we are given an action of a k-vector-space A on a pair (M_0, M_1) and an action of a k-vector-space B on a pair (N_0, N_1).*

Then if each of these actions is faithful, so is the induced action of the k-vector-space $A \otimes_k B$ on the pair $(M_0 \otimes_k N_0,\ M_1 \otimes_k N_1)$.

First proof. Exactly like proof of Theorem 1. (Note that u will be chosen from M_0, while φ will be a k-linear functional on M_1; and that v will be chosen from N_0 to make (3) hold in N_1.)

Second proof. Let us make A and B into k-algebras by giving them zero multiplication operations. Then we can make the vector space $M = M_0 \oplus M_1$ a left A-module using the action $a(u_0, u_1) = (0, a\,u_0)$, and similarly make $N = N_0 \oplus N_1$ a B-module. The faithfulness hypotheses on the given actions clearly make these modules faithful.

Hence by Theorem 1, $M \otimes_k N$ is a faithful $A \otimes_k B$-module. Now $M \otimes_k N$ is a fourfold direct sum $(M_0 \otimes_k N_0) \oplus (M_0 \otimes_k N_1) \oplus (M_1 \otimes_k N_0) \oplus (M_1 \otimes_k N_1)$; but the action of $A \otimes_k B$ annihilates all summands but the first, and has image in the last, so its faithfulness means that every nonzero element of $A \otimes_k B$ induces a nonzero map from $M_0 \otimes_k N_0$ to $M_1 \otimes_k N_1$, which is the desired conclusion. □

Of course, the analog of Corollary 2 holds for actions as in Theorem 5.

4. Remarks

Composing the proof of Theorem 5 from Theorem 1, and the proof of Theorem 1 from Theorem 5, we see that the general case of Theorem 1 follows from the zero-multiplication case of the same theorem. This is striking, since the main interest of the result is for algebras with nonzero multiplication.

One may ask why I made the convention that algebras are associative, if the algebra operations were not used in the theorem. The answer is that there is no natural definition of a module over a not-necessarily-associative algebra. (There is a definition of a module over a Lie algebra, based on the motivating relation between Lie algebras and associative algebras. But there is no natural definition of a Lie or associative structure on a tensor product of Lie algebras, so the result can't be used in that case.)

Which of Theorems 1 and 5 is the "nicer" result? I would say that Theorem 5 shows with less distraction what is going on, while Theorem 1 is likely to be more convenient for applications.

References

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