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# A Priori Bound on the Velocity in Axially Symmetric Navier-Stokes Equations

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Esteban Adan Navas

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## ABSTRACT OF THE DISSERTATION

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by

Esteban Adan Navas

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, December 2015
Dr. Qi S. Zhang , Chairperson

Relevant results and theory in the Axially Symmetric Navier-Stokes Equations are reviewed. Then we obtain pointwise, a priori bounds for the $r$, $\theta$ and $z$ components of the vorticity of axially symmetric solutions to the three-dimensional Navier-Stokes equations, which improves on an earlier bound in [1]. Finally, we show that, for any Leray-Hopf solution, $v$, we can use the $\theta$ component of vorticity to bound the velocity and derive

$$|v(x,t)| \leq \frac{C|\ln r|^{1/2}}{r^2}, \qquad 0 < r \leq 1/2,$$

where $r$ is the distance from the $z$ axis.

# Contents

# Chapter 1

# Introduction

In this dissertation we prove a priori bounds on the vorticity and velocity of solutions to the 3-dimensional axially symmetric Navier-Stokes Equations, which are a special case of the more general Navier-Stokes Equations (NSE). In cylindrical coordinates the axial symmetry translates to $\theta$-derivatives of the velocity being equal to zero. To begin with, we give a derivation of the (NSE) in cylindrical coordinates, from which the axially symmetric version easily follows. Definitions of the vorticity and stream functions are recorded.

In Chapter 3 we present the main theorem and review work published in 2008 by Chen, Strain, Tsai, and Yau [4], and a similar result with different approach by Koch, Nadirashvili, Seregin, and Sverak [12]. This includes regularity criteria for the axisymmetric (NSE), motivating our main results.

With the given results and framework, in Chapters 3 and 4 we introduce the setup and notation for the present results, including scaling of certain norms on hollow parabolic shells in $\mathbb{R}^3$. Using a refined cutoff function, we do our analysis first on the $\theta$-component of vorticity, which is used to control the $r$ and $z$-components of velocity. This estimate follows by a sharp embedding theorem proved by Kozono and Taniuchi [13] in 2000 and use of Poincare's Inequality on a ball, followed by a Nash-Moser iteration.

In Chapter 5 we work on the $r$ and $z$-components of voriticity. These estimates follow by a similar cutoff argument and grouping of the other two equations of vorticity; control of the $\theta$ component of vorticity yields control of these components of vorticity. Nash-Moser iteration is utilized again to prove similar a priori bounds.

Finally, the connection between vorticity and velocity is explored, showing how voriticity can be used to control the $L^{\infty}$-norm of velocity. This proves our main theorem.

# Chapter 2

# The Axially Symmetric Navier-Stokes Equations

## 2.1 The Navier-Stokes Equations

The vector form of the general NSE is the following:

$$\rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p + \nu \Delta v \tag{2.1}$$

$$\nabla \cdot v = 0$$

We take the density and viscosity, $\rho$ and $\nu$, of the fluid to have values of 1, for simplicity. By a solution to the NSE, we mean solving for the unknown functions $v$ and $p$, where $v$ is the velocity vector vield of the fluid, $v = (v_1(x,t), v_2(x,t), v_3(x,t)) : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$. And $p$ is the unknown scalar-valued pressure function $p(x,t)$. Initial data will be made explicit in the statement of the main theorem.

The operator $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is the gradient and the operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial}{\partial x_3^2}$ is the Laplacian; both operators as written are in rectangular coordinates. The second equation in the above system is the continuity equation, stating that the divergence of the velocity field must be 0.

Note that the vector form of the NSE actually represents a system of four equations: the first three arising from the first vector equation (one for each velocity field component $v_1$, $v_2$, and $v_3$), and the last one the continuity equation. Therefore we can write our system as:

$$\frac{\partial v_i}{\partial t} + v_1\frac{\partial v_i}{\partial x_1} + v_2\frac{\partial v_i}{\partial x_2} + v_3\frac{\partial v_i}{\partial x_3} = -\frac{\partial p}{\partial x_i} + \Delta v_i \quad 1 \leq i \leq 3$$
$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$$

## 2.2 Derivation of the Axially Symmetric Navier-Stokes Equations

The Axially Symmetric Navier-Stokes Equations (ASNSE) are a special case of the NSE. If we convert the NSE to cylindrical coordinates by letting $x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$ and $r = \sqrt{x_1^2 + x_2^2}$, then our unknown velocity function $v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ becomes:

$$v(x,t) = v_r(r,z,t)\mathbf{e}_r + v_\theta(r,z,t)\mathbf{e}_\theta + v_z(r,z,t)\mathbf{e}_z \quad ,$$

where

$$\mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0)$$

$$\mathbf{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0)$$

$$\mathbf{e}_z = (0, 0, 1)$$

are the cylindrical unit vectors. We can rewrite the NSE using this change of variables to cylindrical coordinates by utilizing the Chain Rule in several variables. That is, the Navier Stokes equations (using $v_i = \frac{\partial x_i}{\partial t}$ for each $i = 1, 2, 3$),

$$\frac{\partial v}{\partial t} + \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\partial v}{\partial x_1} + \frac{\mathrm{d}x_2}{\mathrm{d}t}\frac{\partial v}{\partial x_2} + \frac{\mathrm{d}x_3}{\mathrm{d}t}\frac{\partial v}{\partial x_3} = -\nabla p + \Delta v$$

becomes the following in cylindrical coordinates:

$$\frac{\partial v}{\partial t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial v}{\partial r} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial v}{\partial \theta} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial v}{\partial z} = -\nabla p + \Delta v$$

Therefore, using $v_r = \frac{\mathrm{d}r}{\mathrm{d}t}$, $v_\theta = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$, and $v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$, we have:

$$\frac{\partial v}{\partial t} + v_r\frac{\partial v}{\partial r} + \frac{v_\theta}{r}\frac{\partial v}{\partial \theta} + v_z\frac{\partial v}{\partial z} = -\nabla p + \Delta v$$

The rest of the derivation follows by computing the vector partial derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial r}$, $\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial z}$ in terms of the velocity $v = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$, then collecting all $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ terms. To compute these partial derivatives, we use the product rule for a scalar multiplying a vector (e.g. $v_r$ and $\mathbf{e}_r$), which gives:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial v_r}{\partial t}\mathbf{e}_r + \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta + \frac{\partial v_z}{\partial t}\mathbf{e}_z \\
v_r\frac{\partial v}{\partial r} &= v_r\frac{\partial v_r}{\partial r}\mathbf{e}_r + v_r\frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta + v_r\frac{\partial v_z}{\partial r}\mathbf{e}_z \\
\frac{v_\theta}{r}\frac{\partial v}{\partial \theta} &= \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\mathbf{e}_r + \frac{v_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)\mathbf{e}_\theta + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}\mathbf{e}_z \\
v_z\frac{\partial v}{\partial z} &= v_z\frac{\partial v_r}{\partial z}\mathbf{e}_r + v_z\frac{\partial v_\theta}{\partial z}\mathbf{e}_\theta + v_z\frac{\partial v_z}{\partial z}\mathbf{e}_z
\end{aligned}$$

On the right hand side of the equation we have $\Delta v = \Delta(v_r \mathbf{e}_r) + \Delta(v_\theta \mathbf{e}_\theta) + \Delta(v_z \mathbf{e}_z)$. These terms are expanded using the cylindrical gradient and Laplacian operators:

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$
$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Therefore,

$$-\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z$$
$$\Delta(v_r \mathbf{e}_r) = \left(\Delta - \frac{1}{r^2}\right) v_r \mathbf{e}_r + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\mathbf{e}_\theta$$
$$\Delta(v_\theta \mathbf{e}_\theta) = -\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\mathbf{e}_r + \left(\Delta - \frac{1}{r^2}\right) v_\theta \mathbf{e}_\theta$$
$$\Delta(v_z \mathbf{e}_z) = \Delta v_z \mathbf{e}_z$$

Since we assume our solutions are axially symmetric, they do not depend on $\theta$, hence every $\theta$-derivative term involving velocity and pressure goes to 0. Thus, writing $b = (v_r, 0, v_z)$, after summing we arrive at the Axially Symmetric Navier-Stokes equations:

$$\begin{aligned}
&\left(\Delta - \frac{1}{r^2}\right) v_r - (b \cdot \nabla) v_r + \frac{v_\theta^2}{r} - \frac{\partial p}{\partial r} - \frac{\partial v_r}{\partial t} = 0 \\
&\left(\Delta - \frac{1}{r^2}\right) v_\theta - (b \cdot \nabla) v_\theta + \frac{v_\theta v_r}{r} - \frac{\partial v_r}{\partial t} = 0 \\
&\Delta v_z - (b \cdot \nabla) v_z - \frac{\partial p}{\partial r} - \frac{\partial v_r}{\partial t} = 0 \\
&\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0
\end{aligned} \tag{2.2}$$

The last equation is the divergence-free condition written in cylindrical coordinates, notably lacking the $\theta$-derivative of velocity.

## 2.3 The Vorticity Equations for Axially Symmetric Solutions

Results on a related function in fluid dynamics, the voriticity, $\omega$, also yield results about the velocity, $v$. Therefore, the main theorem of this paper is stated in terms of the voriticity, and most of the analysis is done using the corresponding vorticity equations. If we compute $\omega = \operatorname{curl} v$ in the $(r, \theta, z)$ variables, for an axially symmetric velocity field $v$, then the vorticity $\omega = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$ can be rewritten as $\omega_r \mathbf{e}_r + \omega_\theta \mathbf{e}_\theta + \omega_z \mathbf{e}_z$, with

$$\omega_r = -\frac{\partial v_\theta}{\partial z}, \quad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}.$$

By taking the curl of the Navier-Stokes equaitons, we derive a new set of equations in the function $\omega = \operatorname{curl} v = \nabla \times v$, that does not depend on the pressure. The resulting general form of the vorticity equations, with density and viscosity equal to 1, is:

$$\frac{\partial \omega}{\partial t} - (v \cdot \nabla)\omega = (\omega \cdot \nabla) v - \Delta \omega$$

Expanding the notation slightly, we have

$$\frac{\partial \omega}{\partial t} - \left( \frac{\mathrm{d}x_1}{\mathrm{d}t} \frac{\partial \omega}{\partial x_1} + \frac{\mathrm{d}x_2}{\mathrm{d}t} \frac{\partial \omega}{\partial x_2} + \frac{\mathrm{d}x_3}{\mathrm{d}t} \frac{\partial \omega}{\partial x_3} \right) = \left( \omega_1 \frac{\partial v}{\partial x_1} + \omega_2 \frac{\partial v}{\partial x_2} + \omega_3 \frac{\partial v}{\partial x_3} \right) - \Delta \omega$$

This system of equations can be converted to cylindrical coordinates, by writing $v$ and $\omega$ in the cylindrical basis. Then the remaining details follow the same pattern as the ASNSE derivation; after explicitly computing all relevant partial derivatives using the product rule, and collecting all $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ terms, we derive the Vorticity Equations for

Axially Symmetric velocity fields:

$$\begin{aligned}
&\left(\Delta - \frac{1}{r^2}\right)\omega_r - (b\cdot\nabla)\omega_r + \omega_r\frac{\partial v_r}{\partial r} + \omega_z\frac{\partial v_r}{\partial z} - \frac{\partial \omega_r}{\partial t} = 0\\
&\left(\Delta - \frac{1}{r^2}\right)\omega_\theta - (b\cdot\nabla)\omega_\theta + 2\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial z} + \omega_\theta\frac{v_r}{r} - \frac{\partial \omega_\theta}{\partial t} = 0\\
&\Delta\omega_z - (b\cdot\nabla)\omega_z + \omega_z\frac{\partial v_z}{\partial z} + \omega_r\frac{\partial v_z}{\partial r} - \frac{\partial \omega_z}{\partial t} = 0
\end{aligned} \tag{2.3}$$

## 2.4 Equations for $\frac{\omega_\theta}{r}$ and $rv_\theta$

The equation we work on the most is a modified version of the second equation in (2.3). Writing $\Omega = \frac{\omega_\theta}{r}$, we have:

$$\Delta\Omega - (b\cdot\nabla)\Omega + \frac{2}{r}\frac{\partial\Omega}{\partial r} - \frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z} = 0, \;\; \nabla\cdot b = 0 \tag{2.4}$$

This equation follows by substituting $r\Omega$ for $\omega_\theta$ in the rotational equation for vorticity:

$$\left(\Delta - \frac{1}{r^2}\right)(r\Omega) - (b\cdot\nabla)(r\Omega) + \frac{2v_\theta}{r}\frac{\partial v_\theta}{\partial z} + \frac{v_r}{r}(r\Omega) - \frac{\partial(r\Omega)}{\partial t} = 0.$$

Using the product rule for the terms with spatial derivatives,

$$\begin{aligned}
\Delta(r\Omega) &= r\frac{\partial^2\Omega}{\partial r^2} + 3\frac{\partial\Omega}{\partial r} + r\frac{\partial^2\Omega}{\partial z^2},\\
(-b\cdot\nabla)(r\Omega) &= -v_r\Omega - r(b\cdot\nabla)\Omega,\\
-\frac{1}{r^2}(r\Omega) &= -\frac{\Omega}{r},\\
\frac{v_r}{r}(r\Omega) &= v_r\Omega,\\
-\frac{\partial}{\partial t}(r\Omega) &= -r\frac{\partial\Omega}{\partial t}.
\end{aligned}$$

We sum the above along with the inhomogeneous term $\frac{2v_\theta}{r}\frac{\partial v_\theta}{\partial z}$ to get:

$$r\frac{\partial^2\Omega}{\partial r^2} + \frac{\partial\Omega}{\partial r} + r\frac{\partial^2\Omega}{\partial z^2} - r(b\cdot\nabla)\Omega + 2\frac{\partial\Omega}{\partial r} - r\frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r}\frac{\partial v_\theta}{\partial z} = 0$$

Grouping terms and dividing by $r$ yields (2.3).

A similar equation for $rv_\theta$ can be derived using $\Gamma = rv_\theta$:

$$\Delta\Gamma - (b \cdot \nabla)\Gamma - \frac{2}{r}\frac{\partial\Gamma}{\partial r} - \frac{\partial\Gamma}{\partial t} = 0 \tag{2.5}$$

This equation differs from equation (2.2) by the opposite sign on the inhomogeneous term. We do not work directly with (2.5), though it can be used to prove, among other important results ([4], [5]), that $rv_\theta$ is uniformly bounded ([3], [20]).

# Chapter 3

# Background and Notation

## 3.1 Previous Results

In the 1960s, O.A. Ladyzhenskaya [14], Ukhovskii and Yudovich [23] proved that finite energy solutions to (2.2) are smooth for all time, under the no-swirl assumption $v_\theta = 0$. A simpler proof can be found in [19].

With nonzero swirl, long time smoothness of solutions to (2.2) is not known. It is known that any singularities can only be found along the $z$-axis, since by axisymmetry a singularity off of this axis would generate a circle of singularities. From the partial regularity theory, this could not happen because the Hausdorff measure of such a singular set is nonzero, contradicting the results in [2].

Recently in [4] and [5], C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau proved a lower bound on the possible blow-up rate of axisymmetric solutions. They proved that $v$ is bounded in $B_R \times [-T_0, 0]$ for any ball of radius $R > 0$, if $v$ satisfies the scaling invariant bound:

$$|v(x,t)| \leq \frac{C}{(r^2-t)^{1/2}} \qquad (x,t) \in \mathbb{R}^3 \times (-T_0, 0).$$

Under this assumption, which guarantees the first blow up time is no earlier than $t = 0$, they were able to show that $|v_\theta(t,r,z)| \leq Cr^{\alpha-1}$ for some small $\alpha > 0$, which is enough to prove $v$ is regular for every point on the $z$-axis.

H. Koch, N. Nadirashvili, G. Seregin, and V. Sverak published a similar result later in 2008, following as a corollary of Louiville theorems they had established for bounded ancient solutions of (2.1). Using the same notation as before, if $v$ is a weak solution to (2.2) and satisifies the bound:

$$|v(x,t)| \leq \frac{C}{r}, \qquad (x,t) \in \mathbb{R}^3 \times (0,T), \tag{3.1}$$

then $v$ is a bounded mild solution in $\mathbb{R}^3 \times (0,T)$. Hence, they show, $v$ is a smooth solution with pointwise bounds on all derivatives in $\mathbb{R}^3 \times (\tau, T)$ for any fixed $\tau > 0$. Seregin and Sverák later proved in [21] a local version of the results in [12]. They ruled out Type I singularities, defined as singularities that also satisfy the bound (3.1). More recent progress has been in ruling out singularities given an even less strict bound on $v(x,t)$ than (3.1). For instance, building on results in [6], in [16] the authors showed the bound $|rv_\theta(x,t)| \leq C|\ln r|^{-2}$, for any $0 < r \leq \delta_0$ and some $\delta_0 \in (0, 1/2)$, is enough to guarantee regularity. In [24] singularities were ruled out by relaxing the bound even further to $|rv_\theta(x,t)| \leq C|\ln r|^{-3/2}$.

Further results in this direction can be found in [17], where Z. Lei and Q. S. Zhang ruled out any singularities given $v_r\mathbf{e}_r + v_z\mathbf{e}_z \in L^\infty([0,T], \mathrm{BMO}^{-1})$. This extends the results in [4] and [12], since the condition $v_z \leq Cr^{-1}$ implies the axisymmetric stream function, $L_\theta$, is bounded, which by a main result in [17] is enough to prove regularity of solutions (see [15] for details).

The main focus of this paper is on establishing an a priori bound on solutions to (2.2). We establish the pointwise bound $|v(x,t)| \leq C\sqrt{\ln r}/r^2$, which in lieu of (3.1) brings us closer to proving regularity of solutions. A precise statement of the main theorem follows in the section.

## 3.2 Statement of the Main Theorem

**Theorem 3.1** *Suppose $v$ is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with initial data $v_0 = v(\cdot, -T) \in L^2(\mathbb{R}^3)$. Assume further $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$ and let $R = \min\{1/2, \sqrt{T/2}\}$.*

*Then for all $(x,t) \in \mathbb{R}^3 \times (-R^2, 0)$, the following bound holds:*

$$|v_r(x,t)| + |v_z(x,t)| \leq \frac{C\sqrt{|\ln r|}}{r^2}, \qquad r \in (0, R].$$

*Here $r$ is the distance from $x$ to the $z$-axis, and $C$ is a constant depending only on the initial data.*

The proof of the theorem is based on the following pointwise bound on the vorticity.

**Theorem 3.2** *Suppose $v$ is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with initial data $v_0 = v(\cdot, -T) \in L^2(\mathbb{R}^3)$, and $\omega$ is the vorticity. Assume further $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$, and let $R = \min\{1/2, \sqrt{T/2}\}$. Then the following a priori estimate holds.*

*There is a constant $C$, depending only on the initial data, such that the following holds for all $(x,t) \in \mathbb{R}^3 \times (-R^2, 0)$ with $r \in (0, R]$:*

$$|\omega_\theta(x,t)| \leq \frac{C \ln(1/r)}{r^{7/2}} \left[ \sup_{s\in[t-r^2,t]} \left( \int_{B(x,4r)} (v_r^2 + v_z^2)(y,s) dy \right)^{1/2} + r^{1/2}(\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1) \right]^2$$
$$\times \left[ \left( \int_{t-r^2}^t \int_{B(x,4r)} \omega_\theta^2(y,s) dy ds \right)^{1/2} + r^{1/2}(\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1) \right].$$

A similar a priori bound holds for the $r$- and $z$-components of vorticity.

**Theorem 3.3** *Suppose $v$ satisfies the conditions in Theorem 3.2 and $\omega$ is its vorticity. Let $R = \min\{1/2, \sqrt{T/2}\}$. Then the following a priori estimate holds.*

*There is a constant $C$, depending only on the initial data, such that the following holds for all $(x,t) \in \mathbb{R}^3 \times (-R^2, 0)$ with $r \in (0, R]$, and $\delta$, $\delta'$ any small numbers greater than zero:*

$$|\omega_r(x,t)| + |\omega_z(x,t)| \leq \frac{C \ln^2(1/r)}{r^{13/2}}$$
$$\times \left[ r^3 \ln(1/r) \sup_{s\in[t-r^2,t]} \left( \int_{B(x,4r)} (v_r^2 + v_z^2)(y,s) dy \right) + r^{7/2} \left( \int_{t-r^2}^t \int_{B(x,4r)} v^{10/3}(y,s) dy ds \right)^4 \right.$$
$$\left. + \left( \int_{t-r^2}^t \int_{B(x,4r)} \omega_\theta^2(y,s) dy ds \right)^4 + r^4 \right]$$
$$\times \left[ \left( \int_{t-r^2}^t \int_{B(x,4r)} \omega_r^2(y,s) dy ds \right)^{1/2} + \left( \int_{t-r^2}^t \int_{B(x,4r)} \omega_z^2(y,s) dy ds \right)^{1/2} \right].$$

Smoothness of solutions is assumed for technical simplicity. The first two theorems are proved in the recently accepted paper [15] by Z. Lei, Q. S. Zhang and the author, and there is some overlap here with their results. In Theorem 3.3 we prove a priori bounds on $\omega_r$ and $\omega_z$ using similar techniques, which expands on some of the findings in [15].

The latter two theorems are an improvement on J. Burke Loftus and Q. Zhang in [1], in which they used 3-dimensional Sobolev embedding and Nash-Moser iteration to prove $|\omega_\theta(x,t)| \leq C/r^5$. They also proved bounds on $\omega_r$ and $\omega_z$, though there was a significant loss of accuracy between these bounds and the bound on $\omega_\theta$. The gap has been improved by the results in this writing, but more work needs to be done to make the bounds on $\omega_r$ and $\omega_z$ as good as the bound on $\omega_\theta$.

Here we mention other related papers on the axially symmetric Navier-Stokes equations. J. Neustupa and M. Pokorny proved in [20] that regularity of either $v_r$ or $v_\theta$ implies regularity of the other two components of the velocity. Q. Jiu and Z. Xin proved regularity assuming smallness of zero-dimension scaled norms in [11]. D. Chae and J. Lee in [3] proved regularity assuming smallness of another particular zero-dimensional integral. A family of singular axially symmetric solutions with singular initial data was constructed in [22] by G. Tian and Z. Xin, whereas T. Hou and C. Li found a special class of globally smooth solutions in [8]. An extension was proved in [7] by T. Hou, Z. Lei and C. Li.

## 3.3 Outline of the Proof

The proof of the main result begins with the a priori bound on the rotational component of velocity: $r|v_\theta(\cdot,t)| \in L^\infty$. See Proposition 3.1. Using the equation for $\Omega = \omega_\theta/r$, we then prove Theorem 3.2 by first localizing equation (2.4) to scaled (blown up) dyadic balls away from the axis of symmetry. Using dimension reduction and the structure of the equations, we improve on [1] by estimating the oscillation of the angular stream function. Finally, after Moser's iteration, we re-scale back to small parabolic cylinders to finish the proof.

Theorem 3.3 is proved similarly to Theorem 3.2. The gap between these two estimates is removed with dimension reduction and bounding the velocity derivatives by the angular vorticity. Theorem 3.1 is proved at the end in Chapter 6 by using the localized Biot-Savart law. Since we are working with solutions that are axisymmetric, the $L^2$-integrals of velocity in small dyadic regions are shown to be smaller than usual. This combined with the a priori bound for $\omega_\theta$ implies the pointwise bound on $|v_r| + |v_z|$.

## 3.4 Notation and Scaling

Instead of the balls $B(x, 4r)$ in the statement of our theorem, our proof will be carried out over comparable cylindrical regions, allowing us to conveniently reduce our computations to a 2-dimensional setting. Specifically, let $(x, t) = (x_1, x_2, x_3, t)$ be the point in Theorem 3.2 and let $R > 0$, $S > 0$, $0 < A < B$ be constants. Denote

$$C_{AR,BR} = \{(x_1, x_2, x_3) : AR \le r \le BR, 0 \le \theta \le 2\pi, |x_3| \le BR\} \subset \mathbb{R}^3 \tag{3.2}$$

to be the hollowed out cylinder centered at the origin with inner radius $AR$, outer radius $BR$, and height $2BR$. Note that these shells become larger when $AR$ becomes smaller or when $BR$ becomes larger.

Denote $P_{AR,BR,SR}$ to be the parabolic region

$$P_{AR,BR,SR} = C_{AR,BR} \times (-S^2R^2, 0). \tag{3.3}$$

If $R = 1$ then we simply write $C_{A,B}$ and $P_{A,B,S}$. We prove our a priori bound for $\omega_\theta$ close to the $x_3$-axis, in the region $P_{2k,3k,\frac{3k}{4}}$ with $0 < k \le \min\{1/2, \sqrt{T/2}\}$. Using the scaling property of solutions to the Navier-Stokes equations, we shift from these small shells to the cube $P_{2,3,\frac{3}{4}}$.

Recall that the pair $(v(x,t), p(x,t))$ is a solution to (NS) if and only if for any $k > 0$ the re-scaled pair $(\widetilde{v}(x,t), \widetilde{p}(x,t))$ is also a solution, where $\widetilde{v}(x,t) = kv(kx, k^2t)$ and $\widetilde{p}(x,t) = k^2p(kx, k^2t)$. If $(x,t)$ is a point in the parabolic region $P_{2k,3k,\frac{3}{4}k}$, and we replace $(x,t)$ with $\tilde{x} = \frac{x}{k}$ and $\tilde{t} = \frac{t}{k^2}$, then the new point $(\tilde{x}, \tilde{t})$ is in $P_{2,3,\frac{3}{4}}$. Thus, if $(v,p)$ is a solution to the axially symmetric Navier-Stokes equations for $(x,t) \in P_{2k,3k,\frac{3k}{4}}$, then $(\widetilde{v}(\tilde{x},\tilde{t}), \widetilde{p}(\tilde{x},\tilde{t}))$ is a solution to the equations for $(\tilde{x},\tilde{t}) \in P_{2,3,\frac{3}{4}}$.

Blowing up to the region $P_{2,3,\frac{3}{4}}$ makes our computations much easier, however, doing so affects certain quantities. These include $r$ and the $L^\infty_t L^2_x$-norms of velocity and vorticity. Let $D$ be any domain in $\mathbb{R}^3$ and $kD = \{x : x = ky, y \in D\}$. Then the following changes must be considered when re-scaling back to $P_{2k,3k,\frac{3k}{4}}$:

$$r = \sqrt{x_1^2 + x_2^2}: \quad \widetilde{r} = \sqrt{\left(\frac{x_1}{k}\right)^2 + \left(\frac{x_2}{k}\right)^2} = \frac{r}{k}$$

$||v(x,t)||_{L^2(kD\times(-(kR)^2,0))}$ :

$$||\widetilde{v}(\tilde{x},\tilde{t})||_{L^2(D\times(-R^2,0))} = \left(\int_{-R^2}^0 \int_D |\widetilde{v}(\tilde{x},\tilde{t})|^2 d\tilde{x}d\tilde{t}\right)^{\frac{1}{2}}$$

$$= \left(\int_{-(kR)^2}^0 \int_{kD} |kv(x,t)|^2 \frac{1}{k^5} dxdt\right)^{\frac{1}{2}} = \frac{1}{k^{\frac{3}{2}}} ||v(x,t)||_{L^2(kD\times(-(kR)^2,0))}.$$

$b(x,t) = (v_r, 0, v_z)$ :

$$\widetilde{b}(x,t) = (kv_r(kx,k^2t), 0, kv_z(kx,k^2t)) = kb(kx,k^2t), \quad (x,t) \in P_{k,4k,k}$$

$$\Rightarrow \widetilde{b}(\tilde{x},\tilde{t}) = kb(x,t).$$

$||b(x,t)||_{L^\infty(-(kR)^2,0;L^2(kD))}$ :

$$||\widetilde{b}(\tilde{x},\tilde{t})||_{L^\infty(-R^2,0;L^2(D))} = \sup_{-R^2\le\tilde{t}<0} \left(\int_D |\widetilde{b}(\tilde{x},\tilde{t})|^2 d\tilde{x}\right)^{\frac{1}{2}}$$

$$\sup_{-(kR)^2\le t<0} \left(\int_{kD} |kb(x,t)|^2 \frac{1}{k^3} dx\right)^{\frac{1}{2}} = \frac{1}{k^{\frac{1}{2}}} ||b(x,t)||_{L^\infty(-(kR)^2,0;L^2(kD))}.$$

$\omega(x,t)$ : $\quad \widetilde{\omega}(x,t) = k^2\omega(kx,k^2t), \quad (x,t) \in P_{2,3,\frac{3}{4}} \quad \Rightarrow \quad \widetilde{\omega}(\tilde{x},\tilde{t}) = k^2\omega(x,t)$

$||\omega(x,t)||_{L^2(kD\times(-(kR)^2,0))}$ :

$$||\widetilde{\omega}(\tilde{x},\tilde{t})||_{L^2(kD\times(-(kR)^2,0))} = \left(\int_{-R^2}^0 \int_D |\widetilde{\omega}(\tilde{x},\tilde{t})|^2 d\tilde{x}d\tilde{t}\right)^{\frac{1}{2}}$$

$$\left(\int_{-(kR)^2}^0 \int_{kD} |k^2\omega(x,t)|^2 \frac{1}{k^5} dxdt\right)^{\frac{1}{2}} = \frac{1}{k^{\frac{1}{2}}} ||\omega(x,t)||_{L^2(kD\times(-(kR)^2,0))}.$$

Another re-scaled quantity we will need for the proof of Theorem 3.3 is:

$$||v(x,t)||_{L^{\frac{10}{3}}(kD\times(-(kR)^2,0))}:$$

$$||\widetilde{v}(\tilde{x},\tilde{t})||_{L^{\frac{10}{3}}(D\times(-R^2,0))} = \left(\int_{-R^2}^{0}\int_{D}|\widetilde{v}(\tilde{x},\tilde{t})|^{\frac{10}{3}}d\tilde{x}d\tilde{t}\right)^{\frac{3}{10}}$$

$$= \left(\int_{-(kR)^2}^{0}\int_{kD}|kv(x,t)|^{\frac{10}{3}}\frac{1}{k^5}dxdt\right)^{\frac{3}{10}} = \frac{1}{k^{\frac{1}{2}}}||v(x,t)||_{L^{\frac{10}{3}}(kD\times(-(kR)^2,0))}.$$

$\widetilde{\Gamma}(\tilde{x},\tilde{t}) = \tilde{r}\widetilde{v}_\theta(\tilde{x},\tilde{t})$ can be shown to be a solution to (2.5) and $\widetilde{\Omega}(\tilde{x},\tilde{t}) = \frac{\widetilde{\omega_\theta}(\tilde{x},\tilde{t})}{\tilde{r}}$ a solution to (2.3) in the variables $(\tilde{x},\tilde{t}) \in P_{2,3,\frac{3}{4}}$.

By the scaling invariance of $rv_\theta$, the boundedness of $rv_\theta$ in the following result guarantees $\tilde{r}\widetilde{v_\theta}$ is also uniformly bounded.

**Proposition 3.1** *([3] and [20]) Suppose $v$ is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations with initial data $v_0 \in L^2(\mathbb{R}^3)$. If $rv_{0,\theta} \in L^p(\mathbb{R}^3)$, then $rv_\theta \in L^\infty(0,T;L^p(\mathbb{R}^3))$. In particular, if $p=\infty$,*

$$|v_\theta(x,t)| \leq \frac{||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)}}{\sqrt{x_1^2+x_2^2}}.$$

A proof can be found in [3], Section 3, Proposition 1. It follows by multiplying equation (2.5) by $|\Gamma|^{p-2}\Gamma$, where $\Gamma = rv_\theta$, then integrating by parts and using Gronwall's inequality.

# Chapter 4

# A Priori Bound on $\omega_\theta$

The proof of this theorem follows by first using a refined cutoff function to establish an energy estimate on our scaled hollow parabolic cylinders, from which we can use Nash-Moser Iteration to get a local bound for $\omega_\theta$. We then re-scale back to the original parabolic cylinders, which gives us an upper bound on the growth of $\omega_\theta$ near the $z$-axis.

## 4.1 Choice of Cutoff Function and Energy Estimates

We first rewrite equation (2.4) in terms of the positive portion above $\Lambda = ||v_\theta||_{L^\infty(P_{1,4,1})}$:

$$\overline{\Omega}_+(x,t) = \begin{cases} \Omega(x,t) + \Lambda & \Omega(x,t) \geq 0, \\ \Lambda & \Omega(x,t) < 0. \end{cases}$$

By assumption $\Omega$ is smooth and $\overline{\Omega}_+$ is Lipshitz, and on integration by parts all boundary terms go to 0. Also $\overline{\Omega}_+(x,t) \geq \Lambda$ and all derivatives of $\overline{\Omega}_+$ are zero on the set where $\Omega(x,t) < 0$. For any $q > 1$, we have the equation for $\overline{\Omega}_+^q$ by direct computation:

$$\Delta\overline{\Omega}_+^q - (b\cdot\nabla)\overline{\Omega}_+^q + \frac{2}{r}\partial_r\overline{\Omega}_+^q - \partial_t\overline{\Omega}_+^q = -\frac{q\overline{\Omega}_+^{q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z} + q(q-1)\overline{\Omega}_+^{q-2}|\nabla\overline{\Omega}_+|^2. \tag{4.1}$$

For $\frac{5}{8} \le \sigma_2 < \sigma_1 \le 1$, define $P(\sigma_i) = C(\sigma_i) \times (-\sigma_i^2, 0)$ to be a hollowed out parabolic shell, with $i = 1, 2$ and $C(\sigma_i)$ defined by

$$C(\sigma_i) = \{(r, \theta, z) \,:\, (5 - 4\sigma_i) < r < 4\sigma_i,\; 0 \le \theta \le 2\pi,\; |z| < 4\sigma_i\}$$

We choose our cut-off function $\psi = \phi(y)\eta(s)$ satisfying

$$\text{supp}\, \phi \subset C(\sigma_1);\; \phi(y) = 1 \text{ for all } y \in C(\sigma_2);\; 0 \le \phi \le 1;$$

$$\frac{|\nabla \phi|}{\phi^{\delta}} \le \frac{c_1}{\sigma_1 - \sigma_2} \text{ for } \delta \in (0,1) \text{ to be chosen later in the proof;}$$

$$\text{supp}\, \eta \subset (-\sigma_1^2, 0];\; \eta(s) = 1,\; \text{for all } s \in [-\sigma_2^2, 0];\; 0 \le \eta \le 1;$$

$$|\eta'| \le \frac{c_2}{(\sigma_1 - \sigma_2)^2}.$$

Let $f = \overline{\Omega}_+^q$ and use $f\psi^2$ as a test function in (4.1) to get

$$\begin{aligned}
&\int_{P(\sigma_1)} (\Delta f - (b \cdot \nabla) f - \partial_s f + \frac{2}{r}\partial_r f) f\psi^2 dyds \\
&\quad = \int_{P(\sigma_1)} q(q-1)\overline{\Omega}_+^{q-2} |\nabla \overline{\Omega}_+|^2 f\psi^2 dyds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{q-1}}{r^2} \frac{\partial v_\theta^2}{\partial z} f\psi^2 dyds \\
&\quad = q(q-1) \int_{P(\sigma_1)} \overline{\Omega}_+^{-2} |\nabla \overline{\Omega}_+|^2 f^2\psi^2 dyds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2} \frac{\partial v_\theta^2}{\partial z} \psi^2 dyds \\
&\quad \ge - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2} \frac{\partial v_\theta^2}{\partial z} \psi^2 dyds
\end{aligned}$$

The inequality at the end follows from positivity of the integral in the previous line.

From here, we integrate the first term with $\Delta f = \nabla \cdot \nabla f$ by parts and rearrange the terms in the inequality to get:

$$\begin{aligned}
&\int_{P(\sigma_1)} \nabla(f\psi^2) \nabla f dyds \\
&\le \int_{P(\sigma_1)} \left( -b \cdot \nabla f (f\psi^2) - \partial_s f (f\psi^2) + \frac{2}{r^2}\partial_r f (f\psi^2) + \frac{q\overline{\Omega}_+^{2q-1}}{r^2} \frac{\partial v_\theta^2}{\partial z} \psi^2 \right) dyds. \qquad (4.2)
\end{aligned}$$

To get an energy inequality bounding the $L^2$-norm of the derivative $|\nabla(f\psi)|^2$, we use the product rule to write $|\nabla(f\psi)|^2 = |\nabla(f\psi)|^2 - |\nabla\psi|^2 f^2$. Therefore, replacing the left side of (4.2) with $|\nabla(f\psi)|^2$ and adding by $|\nabla\psi|^2$, we get

$$\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds \leq \int_{P(\sigma_1)} \Big( - b \cdot \nabla f(f\psi^2) - \partial_s f(f\psi^2) + \frac{2}{r}\partial_r f(f\psi^2) + \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 + |\nabla\psi|^2 f^2\Big) dyds$$

Using integration by parts on the term with the time derivative gives

$$\int_{P(\sigma_1)} -(\partial_s f) f\psi^2 dyds = -\frac{1}{2}\int_{P(\sigma_1)} \partial_s(f^2)\psi^2 dyds$$
$$= -\frac{1}{2}\left(\int_{C(\sigma_1)} f^2\psi^2(y,0)dy - \int_{C(\sigma_1)} f^2\psi^2(y,-\sigma_1^2)dy\right) - \frac{1}{2}\int_{P(\sigma_1)} \partial_s(\psi^2) f^2 dyds.$$

Since our choice of cutoff function $\psi$ has $\psi^2 = (\phi\eta)^2$, $\eta(0) = 1$, $\eta(-\sigma_1^2) = 0$, and $0 \leq \phi \leq 1$, we get

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
&\leq \int_{P(\sigma_1)} -b\cdot\nabla f(f\psi^2)dyds + \int_{P(\sigma_1)} (\eta\partial_s\eta + |\nabla\psi|^2) f^2 dyds \qquad (4.3)\\
&+ \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds + \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 dyds \\
&:= T_1 + T_2 + T_3 + T_4
\end{aligned}$$

## 4.2 Estimating $T_1$, the Drift Term

This term is bounded by following an argument in [25], which treated a parabolic equation with a similar drift term. We write our velocity $b$ in terms of its stream function and integrate by parts, then reduce the spatial dimension of $T_1$ in our hollow domains, since

$r$ is bounded by two constants. The benefit from reducing our dimension is gaining access to an embedding proved in [9], which allows us to prove a sharper 2-dimensional estimate on the drift term, compared to a priori estimates on vorticity previously established in [1].

Since div $b = 0$,

$$\begin{aligned} T_1 &= \int_{P(\sigma_1)} -b \cdot (\nabla f)(f\psi^2) dyds \\ &= \frac{1}{2} \int_{P(\sigma_1)} -b\psi^2 \cdot \nabla(f^2) dyds = \frac{1}{2} \int_{P(\sigma_1)} \text{div } (b\psi^2) f^2 dyds \\ &= \frac{1}{2} \int_{P(\sigma_1)} \text{div } b(\psi f)^2 dyds + \frac{1}{2} \int_{P(\sigma_1)} b \cdot \nabla(\psi^2) f^2 dyds \\ &= \int_{P(\sigma_1)} b \cdot (\nabla\psi)\psi f^2 dyds. \end{aligned}$$

Many of the computations will be carried out on the 2-dimensional shells

$$\begin{aligned} \overline{C}(\sigma_1) &= \{(r,z) | (r,\theta,z) \in C(\sigma_1)\}, \\ \overline{P}(\sigma_1) &= \{(r,z,s) | (r,\theta,z,s) \in P(\sigma_1)\}. \end{aligned} \tag{4.4}$$

Integration over these domains will be in the variable $\overline{y}$ defined by

$$\overline{y} = (r,z), \quad d\overline{y} = drdz, \quad \text{for} \quad dy = rdrdzd\theta.$$

Let $L_\theta$ be the angular component of the stream function in cylindrical coordinates, which can be used to rewrite the components of the velocity $b = v_r \mathbf{e}_r + v_z \mathbf{e}_z$ as

$$v_r = -\partial_z L_\theta, \quad v_z = \frac{1}{r} \partial_r (r L_\theta)$$

Let $a = a(t)$ be a function depending only on time, to be chosen later. Using integration by parts and the divergence free property of $b$, we have

$$\begin{aligned}
T_1 &= \int_{P(\sigma_1)} b\cdot(\nabla\psi)(\psi f^2)dyds \\
&= \int_{P(\sigma_1)} (v_r\partial_r\psi + v_z\partial_z\psi)(\psi f^2)dyds \\
&= 2\pi\int_{\overline{P}(\sigma_1)} \partial_r(rL_\theta - a)(\partial_z\psi)(\psi f^2)drdzds - 2\pi\int_{\overline{P}(\sigma_1)} \partial_z(rL_\theta - a)(\partial_r\psi)(\psi f^2)drdzds \\
&= -2\pi\int_{\overline{P}(\sigma_1)} (rL_\theta - a)\partial_r[(\partial_z\psi)(\psi f^2)]drdzds \\
&\quad + 2\pi\int_{\overline{P}(\sigma_1)} (rL_\theta - a)\partial_z[(\partial_r\psi)(\psi f^2)]drdzds. \qquad (4.5)
\end{aligned}$$

We will need an energy estimate on $\nabla(f\psi)$. However, this is not compatible with the outer $r$ or $z$-derivatives in the last line of (4.5). Therefore we multiply and divide by the cut-off function, $\psi$, then differentiate. Working on the first term, we have

$$\begin{aligned}
&\int_{\overline{P}(\sigma_1)} (rL_\theta - a)\partial_r[(\partial_z\psi)(\psi f^2)]drdzds \\
&\qquad = \int_{\overline{P}(\sigma_1)} (rL_\theta - a)\partial_r\left[(\partial_z\psi)\frac{(\psi f)^2}{\psi}\right]drdzds \\
&\qquad = \int_{\overline{P}(\sigma_1)} (rL_\theta - a)\left[\partial_r\partial_z\psi\frac{(\psi f)^2}{\psi} + \partial_z\psi\left(\frac{\psi\partial_r(\psi f)^2 - (\psi f)^2\partial_r\psi}{\psi^2}\right)\right]drdzds \\
&\qquad = \int_{\overline{P}(\sigma_1)} (rL_\theta - a)\left[\partial_r\partial_z\psi(\psi f^2) + 2f\partial_z\psi\partial_r(\psi f) - \partial_z\psi\partial_r\psi f^2\right]drdzds.
\end{aligned} \qquad (4.6)$$

A similar computation gives the following for the second term of (4.5):

$$\begin{aligned}
&\int_{\overline{P}(\sigma_1)} (rL_\theta - a)\partial_r[(\partial_z\psi)(\psi f^2)]drdzds \qquad (4.7) \\
&\qquad = \int_{\overline{P}(\sigma_1)} (rL_\theta - a)\left[\partial_z\partial_r\psi(\psi f^2) + 2f\partial_r\psi\partial_z(\psi f) - \partial_r\psi\partial_z\psi f^2\right]drdzds.
\end{aligned}$$

Hence, substituting (4.6) and (4.7) into (4.5) gives, along with properties of our cut-off function,

$$
\begin{aligned}
T_1 = & -2\pi \int_{\overline{P}(\sigma_1)} (rL_\theta - a)[\partial_r\partial_z\psi(\psi f^2) + 2f\partial_z\psi\partial_r(\psi f)]drdzds \\
& + 2\pi \int_{\overline{P}(\sigma_1)} (rL_\theta - a)[\partial_z\partial_r\psi(\psi f^2) + 2f\partial_r\psi\partial_z(\psi f)]drdzds \\
\leq & C \sup_{t\in(-\sigma_1^2,0)} ||rL_\theta - a||_{L^\infty(\overline{C}(\sigma_1))} \Big( \int_{\overline{P}(\sigma_1)} (|\partial_z\partial_r\psi| + |\partial_r\partial_z\psi|)\psi f^2 drdzds \\
& + c\int_{\overline{P}(\sigma_1)} |f||\overline{\nabla}^\perp\psi \cdot \overline{\nabla}(\psi f)|drdzds\Big) \\
\leq & C \sup_{t\in(-\sigma_1^2,0)} ||rL_\theta - a||_{L^\infty(\overline{C}(\sigma_1))} \Big( \int_{\overline{P}(\sigma_1)} f^2 drdzds \\
& + \frac{c}{(\sigma_1-\sigma_2)} \int_{\overline{P}(\sigma_1)} |f||\overline{\nabla}(\psi f)|drdzds\Big). \qquad (4.8)
\end{aligned}
$$

Here we used the Cauchy-Schwarz inequality and the notation $\overline{\nabla} = (\partial_r, \partial_z)$ for the 2-dimensional gradient, $\overline{\nabla}^\perp = (-\partial_z, \partial_r)$ for the rotation of $\overline{\nabla}$ with respect to the $r$ and $z$ variables.

Finally, we apply Young's Inequality to (4.8) to get

$$
\begin{aligned}
T_1 \leq & C \sup_{t\in(-\sigma_1^2,0)} ||rL_\theta - a||_{L^\infty(\overline{C}(\sigma_1))} \Big[ \int_{\overline{P}(\sigma_1)} f^2 drdzds \\
& + \frac{c}{(\sigma_1-\sigma_2)} \Big( \frac{\epsilon}{2} \int_{\overline{P}(\sigma_1)} f^2 drdzds + \frac{1}{2\epsilon} \int_{\overline{P}(\sigma_1)} |\overline{\nabla}(\psi f)|^2 drdzds\Big)\Big].
\end{aligned}
$$

Choosing $\epsilon = \dfrac{4c \sup_{t\in(-\sigma_1^2,0)} ||rL_\theta - a||_{L^\infty(\overline{C}(\sigma_1))}}{(\sigma_1-\sigma_2)}$, we thus obtain our first estimate on the drift term:

$$
\begin{aligned}
T_1 \leq & \frac{1}{8} \int_{\overline{P}(\sigma_1)} |\overline{\nabla}(\psi f)|^2 d\overline{y}ds \\
& + C\frac{\sup_{t\in(-\sigma_1^2,0)} \Big( ||rL_\theta - a(t)||^2_{L^\infty(\overline{C}(\sigma_1))} + 1\Big)}{(\sigma_1-\sigma_2)^2} \int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds. \qquad (4.9)
\end{aligned}
$$

## 4.3 Refining the Estimate on the Drift Term

An estimate for $||rL_\theta - a(t)||_{L^\infty(\overline{C}(\sigma_1))}$ can be derived using the embedding by Hou-Li in [9] and Poincaré's inequality.

Choose a 2 dimensional cut-off function $\phi = \phi(r,z) \in C_0^\infty(\mathbb{R}^2)$ such that $\phi = 1$ in $\overline{C}(\sigma_1)$, supp $\phi \in \overline{C}(9\sigma_1/8)$, $0 \le \phi \le 1$, and $|\overline{\nabla}\phi| + |\Delta_2 \phi| \le C$. Here $\Delta_2 = \partial_r^2 + \partial_z^2$ is the 2-dimensional Laplacian with respect to the $r$ and $z$ variables. Therefore, referring to embedding (1.10) of [9],

$$
\begin{aligned}
||rL_\theta - a(t)||_{L^\infty(\overline{C}(\sigma_1))} &\le ||(rL_\theta - a(t))\phi||_{L^\infty(\overline{C}(9\sigma_1/8))} \\
&\le C\Big(||\overline{\nabla}((rL_\theta - a(t))\phi)||_{L^2(\mathbb{R}^2)} + ||(rL_\theta - a(t))\phi||_{L^2(\mathbb{R}^2)} + 1\Big)\times \\
&\quad \Big[\log\left(||\Delta_2((rL_\theta - a(t))\phi)||_{L^2(\mathbb{R}^2)} + ||(rL_\theta - a(t))||_{L^2(\overline{C}(9\sigma_1/8))} + e\right)\Big]^{\frac{1}{2}}
\end{aligned}
$$

Choose $a(t)$ to be the average of $L_\theta(\cdot,t)$ on $\overline{C}(9\sigma_1/8)$ under the 2-dimensional volume element $drdz$. Then, applying the 2-dimensional Poincaré inequality, we get

$$
\begin{aligned}
||rL_\theta - a(t)||_{L^\infty(\overline{C}(\sigma_1))} &\le C(||\overline{\nabla}(rL_\theta)||_{L^2(\overline{C}(9\sigma_1/8))} + 1)\times \\
&\Big[\log(||\Delta_2((rL_\theta - a(t))\phi)||_{L^2(\mathbb{R}^2)} + C||\overline{\nabla}(rL_\theta)||_{L^2(\overline{C}(9\sigma_1/8))} + e)\Big]^{\frac{1}{2}}.
\end{aligned} \tag{4.10}
$$

A computation on the Laplacian term gives

$$
\begin{aligned}
\Delta_2((rL_\theta - a(t))\phi) &= \Delta_2(rL_\theta)\phi + 2\overline{\nabla}(rL_\theta)\cdot\overline{\nabla}\phi + (rL_\theta - a(t))\Delta_2\phi \\
&= \phi(\partial_r^2 + \partial_z^2)(rL_\theta) + 2\overline{\nabla}(rL_\theta)\cdot\overline{\nabla}\phi + (rL_\theta - a(t))\Delta_2\phi \\
&= \phi r\Big(\partial_r^2 L_\theta + \partial_z^2 L_\theta + \frac{2}{r}\partial_r L_\theta\Big) + 2\overline{\nabla}(rL_\theta)\cdot\overline{\nabla}\phi + (rL_\theta - a(t))\Delta_2\phi \\
&= \phi r\Big(\partial_r^2 L_\theta + \partial_z^2 L_\theta + \frac{1}{r}\partial_r L_\theta - \frac{1}{r^2}L_\theta\Big) + \phi\Big(\partial_r L_\theta + \frac{1}{r}L_\theta\Big) + 2\overline{\nabla}(rL_\theta)\cdot\overline{\nabla}\phi \\
&\qquad + (rL_\theta - a(t))\Delta_2\phi \\
&= -\phi r\omega_\theta + \phi v_z + 2\overline{\nabla}(rL_\theta)\cdot\overline{\nabla}\phi + (rL_\theta - a(t))\Delta_2\phi. \qquad (4.11)
\end{aligned}
$$

We used the fact that $\Psi L_\theta = \frac{\omega_\theta}{r}$, where $\Psi$ is the linear elliptic operator $\Psi = \frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}$.

At any point $(x,t)$, we also have

$$
|\overline{\nabla}(rL_\theta)|^2 = |\partial_r(rL_\theta)|^2 + r^2|\partial_z L_\theta|^2 = r^2\left[\left|\frac{1}{r}\partial_r(rL_\theta)\right|^2 + |\partial_z L_\theta|^2\right] = r^2|b|^2, \qquad (4.12)
$$

which, combined with (4.11), admits the bound

$$
|\Delta_2((rL_\theta - a(t))\phi)| \le |r\omega_\theta + v_z| + 2r|b| + C|rL_\theta - a(t)|.
$$

Using the 2-dimensional Poincaré inequality again, the Laplacian term in (4.10) has upper bound:

$$
\begin{aligned}
&||\Delta_2((rL_\theta - a(t))\phi)||_{L^2(\mathbb{R}^2)} \\
&\quad \le C||\omega_\theta||_{L^2(\overline{C}(9\sigma_1/8))} + C||b||_{L^2(\overline{C}(9\sigma_1/8))} + C||(rL_\theta - a(t))||_{L^2(\overline{C}(9\sigma_1/8))} \\
&\quad \le C||\omega_\theta||_{L^2(\overline{C}(9\sigma_1/8))} + C||b||_{L^2(\overline{C}(9\sigma_1/8))} + C||\overline{\nabla}(rL_\theta)||_{L^2(\overline{C}(9\sigma_1/8))} \\
&\quad \le C||\omega_\theta||_{L^2(\overline{C}(9\sigma_1/8))} + 2C||b||_{L^2(\overline{C}(9\sigma_1/8))}.
\end{aligned}
$$

Substituting this and (4.12) back into (4.10), we get

$$\begin{aligned} &||rL_\theta - a(t)||_{L^\infty(\overline{C}(\sigma_1))} \\ &\quad \le C(||v(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+1)\Big[\log(C||\omega_\theta(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+C||v(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+e)\Big]^{\frac{1}{2}}. \end{aligned}$$

Using this estimate in (4.9), the following bound has thus been established for the drift term:

$$\begin{aligned} T_1 &\le \frac{1}{8}\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(\psi f)|^2 d\overline{y}ds \\ &\quad + C\frac{\sup_{t\in(-\sigma_1^2,0)}\Big[\log(||\omega_\theta(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+||v(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+e)\Big]}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds. \end{aligned}$$

For simplicity of notation in the iteration, we will use $\overline{K}$ to mean:

$$\overline{K}=\overline{K}(v,\omega)\equiv \sup_{t\in(-\sigma_1^2,0)}\Big[\log(||\omega_\theta(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+||v(\cdot,t)||_{L^2(\overline{C}(9\sigma_1/8))}+e)\Big].$$

Therefore,

$$T_1 \le \frac{1}{8}\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(\psi f)|^2 d\overline{y}ds + C\frac{\overline{K}^2(v,\omega)}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds \tag{4.13}$$

## 4.4 Bounding the Integrals $T_2$ and $T_3$

The term $T_2 = \displaystyle\int_{P(\sigma_1)} (\eta\partial_s\eta + |\nabla\psi|^2) f^2 dyds$ is easily dealt with using properties of the cutoff function. Since

$$|\nabla\psi|^2 = |\eta\nabla\phi|^2 \le \left(\frac{|\nabla\phi|}{\phi^\delta}\right)^2 \le \frac{c_1^2}{(\sigma_1-\sigma_2)^2}$$

and

$$|\eta\partial_s\eta| \le |\partial_s\eta| \le \frac{c_2}{(\sigma_1-\sigma_2)^2},$$

we get

$$|T_2| \leq \frac{C}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds \leq \frac{C}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds. \tag{4.14}$$

For $T_3 = \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds$, we first rewrite $\partial_r f(f)$ as $\frac{1}{2}\partial_r(f^2)$, then integrate by parts. Since integration over $P(\sigma_1)$ avoids any possible singularities of solutions of the axially symmetric Navier Stokes equations (by staying away from the $z$-axis), all functions in the integrand are bounded and smooth. Therefore, the following computations are justified:

$$\begin{aligned}
T_3 &= \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds = \int_{P(\sigma_1)} \frac{1}{r}\partial_r(f^2)\psi^2 rdrd\theta dzds \\
&= \int_{P(\sigma_1)} \partial_r(f^2)\psi^2 drd\theta dzds = -\int_{P(\sigma_1)} \partial_r(\psi^2)f^2 drd\theta dzds \\
&= -\int_{P(\sigma_1)} \frac{2}{r}\mathbf{e}_r \cdot \nabla\psi(\psi f^2)dyds.
\end{aligned}$$

Taking absolute values and using the Cauchy-Schwarz inequality for vectors on the integrand gives

$$|T_3| \leq \int_{P(\sigma_1)} \frac{2}{r}|\nabla\psi|\psi f^2 dyds$$

Since $r$ is bounded by two constants in $P(\sigma_1)$, combined with properties of the cutoff function we thus arrive at a bound on $T_3$:

$$|T_3| \leq \frac{C}{(\sigma_1-\sigma_2)}\int_{P(\sigma_1)} f^2 dyds \leq \frac{C}{(\sigma_1-\sigma_2)}\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds. \tag{4.15}$$

## 4.5 Bounding $T_4$, the Inhomogenous Term

We finish bounding the integrals on the right of (4.3) by estimating $T_4$, which arose from the inhomogeneous term, $\frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z}$. Note that these inequalities have been derived for the function $f = \overline{\Omega}_+^q$, where $\overline{\Omega}_+$ was defined in section 4.1 by cutting off all parts of $\Omega(x,t)$ that were less than $\Lambda$:

$$\overline{\Omega}_+(x,t) = \begin{cases} \Omega(x,t) + \Lambda & \Omega(x,t) \geq 0, \\ \Lambda & \Omega(x,t) < 0. \end{cases}$$

Therefore, $\overline{\Omega}_+ \geq \Lambda$, where $\Lambda = ||v_\theta||_{L^\infty(P_{1,4,1})} \leq ||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)} < \infty$.

Using integration by parts yields

$$\begin{aligned} T_4 &= \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2} \frac{\partial v_\theta^2}{\partial z} \psi^2 dyds \\ &= -\int_{P(\sigma_1)} \frac{\partial}{\partial z} \Big( \frac{\overline{\Omega}_+^{2q} \psi^2}{\overline{\Omega}_+} \Big) \frac{q}{r^2} v_\theta^2 dyds \\ &= -\int_{P(\sigma_1)} \frac{\partial}{\partial z} (f\psi)^2 \frac{1}{\overline{\Omega}_+} \frac{q}{r^2} v_\theta^2 dyds + \int_{P(\sigma_1)} (\overline{\Omega}_+^q \psi)^2 \frac{1}{\overline{\Omega}_+^2} \frac{\partial \overline{\Omega}_+}{\partial z} \frac{q}{r^2} v_\theta^2 dyds \\ &= -\int_{P(\sigma_1)} \frac{\partial}{\partial z} (f\psi)^2 \frac{1}{\overline{\Omega}_+} \frac{q}{r^2} v_\theta^2 dyds \\ &\qquad + \frac{1}{2} \int_{P(\sigma_1)} \frac{1}{\overline{\Omega}_+} \Big[ \frac{\partial (\overline{\Omega}_+^{2q} \psi^2)}{\partial z} - \overline{\Omega}_+^{2q} \frac{\partial \psi^2}{\partial z} \Big] \frac{1}{r^2} v_\theta^2 dyds \\ &= -\int_{P(\sigma_1)} \frac{\partial}{\partial z} (f\psi)^2 \frac{1}{\overline{\Omega}_+} \frac{q-(1/2)}{r^2} v_\theta^2 dyds - \frac{1}{2} \int_{P(\sigma_1)} \frac{1}{\overline{\Omega}_+} \overline{\Omega}_+^{2q} \frac{\partial \psi^2}{\partial z} \frac{1}{r^2} v_\theta^2 dyds. \end{aligned}$$

Considering that $\frac{|v_\theta|}{\Lambda} \leq 1$, utilizing $\Lambda \leq \overline{\Omega}_+$, and $r = \sqrt{y_1^2 + y_2^2} \geq 1$ for all $y \in P(\sigma_1)$, we continue by fixing $\epsilon_3 > 0$. Apply Young's inequality with exponents both being 2 to get

$$\begin{aligned} |T_4| &\leq \int_{P(\sigma_1)} 2q|v_\theta||f|\psi \Big| \frac{\partial (f\psi)}{\partial z} \Big| dyds + \frac{c_3}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 |v_\theta| dyds \\ &\leq \int_{P(\sigma_1)} \Big| \frac{2q\Lambda}{(2\epsilon_3)^{\frac{1}{2}}} f\psi \Big| \times \Big| (2\epsilon_3)^{\frac{1}{2}} \frac{\partial (f\psi)}{\partial z} \Big| dyds + \frac{c_3 \Lambda}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 dyds \\ &\leq \frac{c_{12}\Lambda^2 q^2}{\epsilon_3} \int_{P(\sigma_1)} f^2 dyds + \epsilon_3 \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \frac{c_3 \Lambda}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 dyds. \end{aligned}$$

Thus

$$|T_4| \leq \frac{1}{4} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + C \left[ \Lambda^2 q^2 + \frac{\Lambda}{\sigma_1 - \sigma_2} \right] \int_{\overline{P}(\sigma_1)} f^2 d\overline{y} ds. \tag{4.16}$$

## 4.6 Combining Estimates $T_1 - T_4$ and Embedding Estimates

Combining the results from the previous two sections on the integrals $T_1 - T_4$, we substitute (4.13), (4.14), (4.15) and (4.16) into (4.3) to get

$$\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy$$
$$\leq \frac{3}{4}\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + C\left[\frac{C\overline{K}^2(v,w)}{(\sigma_1-\sigma_2)^2} + 1 + \Lambda^2 q^2 + \frac{1}{(\sigma_1-\sigma_2)^2}\right]\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds$$

We can rewrite this inequality, since $q > 1$ and $0 < \sigma_1 - \sigma_2 < 1$, as

$$\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(f\psi)|^2 d\overline{y}ds + \int_{\overline{C}(\sigma_1)} f^2(y,0)\phi^2(y)d\overline{y} \tag{4.17}$$
$$\leq \frac{Cq^2}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2(v,w)(C_{1,4}) + \Lambda^2 + 1\right)\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds.$$

The above energy estimate will be iterated using embeddings on BMO functions, which will now be derived.

Apply Lemma 1 in Section 2 of [13] to $(f\phi)^2$,

$$||(f\phi)^2||_{L^2(\overline{C}(\sigma_1))} \leq C||f\phi||_{L^2(\overline{C}(\sigma_1))}||f\phi||_{BMO(\overline{C}(\sigma_1))}, \tag{4.18}$$

and recall the definition of BMO:

$$||f\phi||_{BMO(\overline{C}(\sigma_1))} = \sup_{B(x,r)\subset\overline{C}(\sigma_1)}\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)} |f\phi - (f\phi)_{x,r}|d\overline{y}\right\}.$$

This supremum is taken over all 2-dimensional balls contained in $\overline{C}(\sigma_1)$, and the term $(f\phi)_{x,r}$ is the average of $f\phi$ over the ball $B(x,r)$. Applying Hölder's inequality and Poincare's

Inequality on the ball, we bound the inside of the BMO norm:

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}&|f\phi-(f\phi)_{B(x,r)}|d\overline{y}\\ &\le \frac{1}{|B(x,r)|}\left(\int_{B(x,r)}1d\overline{y}\right)^{\frac{1}{2}}\left(\int_{B(x,r)}|f\phi-(f\phi)_{x,r}|^2d\overline{y}\right)^{\frac{1}{2}}\\ &\le \frac{Cr}{|B(x,r)|^{\frac{1}{2}}}\left(\int_{B(x,r)}|\overline{\nabla}(f\phi)|^2d\overline{y}\right)^{\frac{1}{2}}\\ &\le C||\overline{\nabla}(f\phi)||_{L^2(\overline{C}(x,r))}\end{aligned}$$

Therefore, substituting this into (4.18) gives the embedding we will use to iterate (4.17):

$$||f\phi||_{L^4(\overline{C}(\sigma_1))}\le C||f\phi||^{\frac{1}{2}}_{L^2(\overline{C}(\sigma_1))}||\overline{\nabla}(f\phi)||^{\frac{1}{2}}_{L^2(\overline{C}(\sigma_1))}. \tag{4.19}$$

## 4.7 $L^2-L^\infty$ Estimate on $\omega_\theta$ using Moser's Iteration

Embedding (4.19) implies

$$\int_{\overline{C}(\sigma_1)}(f\phi)^4d\overline{y}\le C\int_{\overline{C}(\sigma_1)}(f\phi)^2\int_{\overline{C}(\sigma_1)}|\overline{\nabla}(f\phi)|^2d\overline{y}.$$

We multiply by $\eta^4(s)$ on both sides and integrate with respect to time to get

$$\begin{aligned}\int_{-\sigma_1^2}^0\int_{\overline{C}(\sigma_1)}&(f\psi)^4d\overline{y}ds\\ &\le C\int_{-\sigma_1^2}^0\int_{\overline{C}(\sigma_1)}(f\psi)^2d\overline{y}ds\int_{-\sigma_1^2}^0\int_{\overline{C}(\sigma_1)}|\overline{\nabla}(f\psi)|^2d\overline{y}ds\\ &\le C\sup_{-\sigma_1^2\le s\le 0}\left(\int_{\overline{C}(\sigma_1)}(f\psi)^2d\overline{y}\right)\int_{\overline{P}(\sigma_1)}|\overline{\nabla}(f\psi)|^2d\overline{y}ds\end{aligned} \tag{4.20}$$

Energy estimate (4.17) holds for all $s$ in the interval $-\sigma_1^2\le s<0$ as the upper limit of the time cut-off function, hence it controls both integrals in the above embedding. Therefore, substituting (4.17) into the embedding and using properties of the cut-off function gives

$$\int_{\overline{P}(\sigma_2)} f^4 d\overline{y}ds \leq C\left[\frac{q^2}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2+\Lambda^2+1\right)\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds\right]^2.$$

The term $\overline{K}$ is the shortened form of $\overline{K}(v,w)$. Since $f=\overline{\Omega}_+^q$, this shows

$$\int_{\overline{P}(\sigma_2)} \overline{\Omega}_+^{4q} d\overline{y}ds \leq C\left[\frac{q^2}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2+\Lambda^2+1\right)\int_{\overline{P}(\sigma_1)} \overline{\Omega}_+^{2q} d\overline{y}ds\right]^2. \tag{4.21}$$

For $i=0,1,2,\ldots$, in (4.21), take $q=2^i$ and replace $\sigma_1$ by $\sigma_i=1-\sum_{j=1}^{i}2^{-j-2}$ and $\sigma_2$ by $\sigma_{i+1}=1-\sum_{j=1}^{i+1}2^{-j-2}$. We set $\sum_{j=1}^{i}2^{-j-2}=0$ for the case $i=0$. Then (4.21) generalizes to

$$\left(\int_{\overline{P}(\sigma_{i+1})} \overline{\Omega}_+^{2^{i+2}} d\overline{y}ds\right)^{1/2} \leq c_1 c_2^{i+1} 2^{2i}\left(\overline{K}^2+\Lambda^2+1\right)\int_{\overline{P}(\sigma_i)} \overline{\Omega}_+^{2^{i+1}} d\overline{y}ds. \tag{4.22}$$

If we take the 1/2-th power of (4.22) then we get

$$\begin{aligned}\left(\int_{\overline{P}(\sigma_{i+1})} \overline{\Omega}_+^{2^{i+2}} d\overline{y}ds\right)^{1/2^2} &\leq c_1^{\frac{1}{2}} c_2^{\frac{i+1}{2}} 2^{\frac{2i}{2}}\left(\overline{K}^2+\Lambda^2+1\right)^{\frac{1}{2}}\left(\int_{\overline{P}(\sigma_i)} \overline{\Omega}_+^{2^{i+1}} d\overline{y}ds\right)^{\frac{1}{2}}\\ &\leq c_1^{\frac{1}{2}+1} c_2^{\left[\frac{i+1}{2}+i\right]} 2^{\left[\frac{2i}{2}+2(i-1)\right]}\left(\overline{K}^2+\Lambda^2+1\right)^{\frac{1}{2}+1}\int_{\overline{P}(\sigma_{i-1})} \overline{\Omega}_+^{2^{(i-1)+1}} d\overline{y}ds,\end{aligned} \tag{4.23}$$

where the second line is the result of applying (4.22) to the first line of (4.23). We continue this iterative process of taking the 1/2-th power of the above inequality, then applying (4.22), until we reach the $L^2$-norm of $\overline{\Omega}_+$ over $\overline{P}_{1,4,1}$ (enlarging the domain of integration if necessary):

$$\left(\int_{\overline{P}(\sigma_{i+1})} \overline{\Omega}_+^{2^{i+2}} d\overline{y}ds\right)^{\frac{1}{2^{i+1}}} \leq c_1^{\sum\frac{1}{2^j}} c_2^{\sum\frac{j+1}{2^{j-1}}} 2^{2\sum\frac{j-1}{2^{j-1}}}\left(\overline{K}^2+\Lambda^2+1\right)^{\sum\frac{1}{2^{j-1}}}\int_{\overline{P}_{1,4,1}} \overline{\Omega}_+^2 d\overline{y}ds.$$

Each sum in the powers of the constants are from $j=1$ to $j=i+1$. Letting $i\to\infty$, each resulting infinite series converges. Therefore,

$$\sup_{\overline{P}_{2,3,\frac{3}{4}}} \overline{\Omega}_+^2 \leq C\left(\overline{K}^2+\Lambda^2+1\right)^2\int_{\overline{P}_{1,4,1}} \overline{\Omega}_+^2 d\overline{y}ds.$$

We can repeat this argument on $\overline{\Omega}_- = \begin{cases} -\Omega + \Lambda & \Omega \leq 0 \\ \Lambda & \Omega > 0 \end{cases}$ to derive a similar $L^2 - L^\infty$ bound for $\overline{\Omega}_-$:

$$\sup_{\overline{P}2,3,\frac{3}{4}} \overline{\Omega}_-^2 \leq C\left(\overline{K}^2 + \Lambda^2 + 1\right)^2 \int_{\overline{P}_{1,4,1}} \overline{\Omega}_-^2 d\overline{y} ds.$$

Thus,

$$\sup_{\overline{P}2,3,\frac{3}{4}} \Omega^2 \leq C\left(\overline{K}^2 + \Lambda^2 + 1\right)^2 \int_{\overline{P}_{1,4,1}} \Omega^2 d\overline{y} ds.$$

In the region $\overline{P}_{1,4,1}$, since $r$ is bounded by two constants, the functions $\Omega = \omega_\theta / r$ and $\omega_\theta$ are equivalent. Therefore, the above estimate implies

$$\sup_{\overline{P}2,3,\frac{3}{4}} \omega_\theta^2 \leq C\left(\overline{K}^2 + \Lambda^2 + 1\right)^2 \int_{\overline{P}_{1,4,1}} (\omega_\theta^2 + \Lambda^2) d\overline{y} ds. \tag{4.24}$$

## 4.8 Re-scaling back to small Parabolic Cylinders; proof of Theorem 3.2

All of our computations so far were done on blown up parabolic cylinders, for the function $\widetilde{\omega}_\theta$. That is, bringing back the tilde notation, we have shown

$$\sup_{(\tilde{x},\tilde{t}) \in P_{2,3,\frac{3}{4}}} \widetilde{\omega}_\theta^2(\tilde{x},\tilde{t}) \leq C\left(\overline{K} + \widetilde{\Lambda} + 1\right)^4 \left( \int_{P_{\frac{1}{2},5,1}} \widetilde{\omega}_\theta^2(\tilde{x},\tilde{t}) d\tilde{x} d\tilde{t} + \widetilde{\Lambda}^2 \right). \tag{4.25}$$

Here $\tilde{x} = \frac{x}{k}$ and $\tilde{t} = \frac{t}{k^3}$, and we have enlarged the domain of integration for convenience. Furthermore,

$$\begin{aligned}
&\overline{K} = \overline{K}(\widetilde{v}, \widetilde{w}) \\
&\equiv \sup_{\tilde{t}\in(-1,0)} \Big[(||\widetilde{v}(\cdot,\tilde{t})||_{L^2(\overline{C}(9/8))} + 1)(\log^{1/2}(||\widetilde{\omega}_\theta(\cdot,\tilde{t})||_{L^2(\overline{C}(9/8))} + ||\widetilde{v}(\cdot,\tilde{t})||_{L^2(\overline{C}(9/8))} + e)\Big] \\
&\le \sup_{\tilde{t}\in(-1,0)} \Big[(||\widetilde{v}(\cdot,\tilde{t})||_{L^2(\overline{C}_{\frac{1}{2},5})} + 1)(\log^{1/2}(||\widetilde{\omega}_\theta(\cdot,\tilde{t})||_{L^2(\overline{C}_{\frac{1}{2},5})} + ||\widetilde{v}(\cdot,\tilde{t})||_{L^2(\overline{C}_{\frac{1}{2},5})} + e)\Big].
\end{aligned}$$

This inequality holds because $\overline{C}(9/8) \subset \overline{C}_{\frac{1}{2},5}$. Recall the effects of re-scaling on the $L^2_x L^\infty_t$ norms of velocity and vorticity:

$$||\widetilde{b}(\tilde{x},\tilde{t})||_{L^\infty(-1,0;L^2(C_{\frac{1}{2},5}))} = k^{-\frac{1}{2}}||b(x,t)||_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))},$$

$$||\widetilde{\omega}(\tilde{x},\tilde{t})||_{L^\infty(-1,0;L^2(C_{\frac{1}{2},5}))} = k^{\frac{1}{2}}||\omega(x,t)||_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))},$$

and

$$||\widetilde{\omega}(\tilde{x},\tilde{t})||_{L^2(P_{\frac{1}{2},5,1})} = k^{-\frac{1}{2}}||\omega(x,t)||_{L^2(P_{\frac{k}{2},5k,k})}.$$

Hence, returning to $C_{\frac{k}{2},5k}$ from $C_{\frac{1}{2},5}$, we have

$$\begin{aligned}
\overline{K} \le \sup_{k\in[-k^2,0]} \left(\frac{1}{k^{1/2}}||b||_{L^2(C_{\frac{k}{2},5k})} + 1\right) \log^{1/2}\Big(&k^{1/2}||\omega_\theta(\cdot,t)||_{L^2(C_{\frac{k}{2},5k})} \\
&+ k^{-1/2}||v(\cdot,t)||_{L^2(C_{\frac{k}{2},5k})} + e\Big)
\end{aligned}$$

In [1] it was proved that for any $x \in C_{\frac{k}{2},5k}$,

$$|\omega_\theta(x,t)| \le \frac{C}{k^5},$$

where $C$ depends only on the initial value. Therefore,

$$\overline{K} \le C(k^{-1/2}||b||_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))} + 1)\log^{1/2}(\frac{1}{k} + e) \tag{4.26}$$

The last term left to re-scale is $\Lambda$, which is scaling invariant, since by Proposition 3.1,

$$\widetilde{\Lambda} = \left( \sup_{P_{\frac{1}{2},5,1}} |\widetilde{v}_\theta(\widetilde{x}, \widetilde{t})| \right)$$

$$= ||rv_\theta(x, -T)||_{L^\infty(\mathbb{R}^3)} = ||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)}.$$

Substuting this and (4.26) into (4.25) gives, for $k \in (0, 1]$,

$$\begin{aligned}
&\sup_{(x,t)\in P_{2k,3k,\frac{3k}{4}}} k^4\omega_\theta^2(x,t) \\
&\quad \le C\Big( (k^{-1/2}\|b\|_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))} + 1) \log^{1/2}(\frac{1}{k} + e) + \|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1 \Big)^4 \\
&\qquad \times \Big( \int_{P_{\frac{k}{2},5k,k}} k^4\omega_\theta^2(x,t)\frac{1}{k^5}dxdt + \|rv_{0,\theta}\|^2_{L^\infty(\mathbb{R}^3)} \Big) \\
&\quad \le \frac{C}{k^3}\Big( \|b\|_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))} \log^{1/2}(\frac{1}{k} + e) + k^{1/2}\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + k^{1/2} \Big)^4 \\
&\qquad \times \Big( \|\omega_\theta\|^2_{L^2(P_{\frac{k}{2},5k,k})} + k\|rv_{0,\theta}\|^2_{L^\infty(\mathbb{R}^3)} \Big).
\end{aligned}$$

Thus, after dividing by $k^4$ and taking the square root, Theorem 3.2 is proved:

$$\begin{aligned}
&\|\omega_\theta(x,t)\|_{L^\infty(P_{2k,3k,\frac{3k}{4}})} \\
&\quad \le \frac{C}{k^{7/2}}\Big( \|b\|_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))} \log^{1/2}(\frac{1}{k} + e) + k^{1/2}\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + k^{1/2} \Big)^2 \\
&\qquad \times \left( \|\omega_\theta\|_{L^2(P_{\frac{k}{2},5k,k})} + \sqrt{k}\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} \right).
\end{aligned}$$

□

# Chapter 5

# A Priori Bounds on $\omega_r$ and $\omega_z$

We prove here similar pointwise estimates for the other components of vorticity in the cylindrical system, $\omega_r$ and $\omega_z$, which satisfy the equations:

$$\begin{cases} \Delta\omega_r - (b\cdot\nabla)\omega_r + \omega_r\left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2}\right) + \omega_z\frac{\partial v_r}{\partial z} - \frac{\partial \omega_r}{\partial t} = 0, \\ \Delta\omega_z - (b\cdot\nabla)\omega_z + \omega_z\frac{\partial v_z}{\partial z} + \omega_r\frac{\partial v_r}{\partial r} - \frac{\partial \omega_z}{\partial t} = 0. \end{cases} \tag{5.1}$$

Following the derivation in [1], we let $V$ be the matrix:

$$V = \begin{bmatrix} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} & \frac{\partial v_z}{\partial r} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Then the max norm of the matrix $V$ can be estimated using a lemma proved in the above paper:

**Lemma 5.1** *Let $v = v(x,t)$ be a divergence free, axisymmetric, smooth vector field in $Q_{1,4} = C_{1,4,1}\times[-T,T]$ for fixed $T > 0$. Then for all $q > 1$ there exists a constant $c = c(q) > 0$*

*such that:*

$$||\nabla v_r||_{L^q(Q_{2,3})} + \left|\left|\frac{v_r}{r}\right|\right|_{L^q(Q_{2,3})} + ||\nabla v_z||_{L^q(Q_{2,3})}$$
$$\leq c(||(curl\ v)_\theta||_{L^q(Q_{1,4})} + ||v||_{L^q(Q_{1,4})}).$$

## 5.1 Energy Estimates for $\omega_r$ and $\omega_z$

Once again we choose $\psi = \phi(y)\eta(s)$ to be the cut-off function satisfying:

$$\text{supp } \phi \subset C(\sigma_1);\ \phi(y) = 1 \text{ for all } y \in C(\sigma_2);\ \frac{|\nabla \phi|}{\phi^\delta} \leq \frac{c_1}{\sigma_1 - \sigma_2} \text{ for } \delta \in (0,1),\ 0 \leq \phi \leq 1;$$
$$\text{supp } \eta \subset (-\sigma_1^2, 0];\ \eta(s) = 1 \text{ for all } s \in [-\sigma_2^2, 0];\ |\eta'| \leq \frac{c_2}{(\sigma_1 - \sigma_2)^2};\ 0 \leq \eta \leq 1.$$

Taking $\omega_r^{2q-1}\psi^2$ as a test function in (5.1), we get:

$$\begin{aligned} 0 &= \int_{P(\sigma_1)} \left(\Delta\omega_r - b\cdot\nabla\omega_r + \omega_r\left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2}\right) + \omega_z\frac{\partial v_r}{\partial z} - \frac{\partial \omega_r}{\partial s}\right)\omega_r^{2q-1}\psi^2 dyds \\ &= \int_{P(\sigma_1)} \omega_r^{2q-1}\psi^2\Delta\omega_r dyds \\ &\quad - \int_{P(\sigma_1)} \frac{1}{q} b\cdot\nabla(\omega_r^q)(\omega_r^q\psi^2)dyds - \int_{P(\sigma_1)} \frac{1}{q}\partial_s(\omega_r^q)(\omega_r^q\psi^2)dyds \\ &\quad + \int_{P(\sigma_1)} \left(\frac{\partial v_r}{\partial_r} - \frac{1}{r^2}\right)(\omega_r^{2q}\psi^2) + \left(\frac{\partial v_r}{\partial z}\right)\omega_z\omega_r^{2q-1}\psi^2 dyds. \end{aligned}$$

We work the first term on the right hand side as we did in (4.2) for $\omega_\theta$, using integration by parts, direct calculations and algebraic manipulations:

$$
\begin{aligned}
\int_{P(\sigma_1)} \omega_r^{2q-1}\psi^2 \Delta\omega_r dyds &= -\int_{P(\sigma_1)} \nabla(\omega_r^{2q-1}\psi^2)\cdot\nabla\omega_r dyds \\
&= -\int_{P(\sigma_1)} (2q-1)(\omega_r^{2q-2}\nabla\omega_r)\cdot\nabla\omega_r\psi^2 + \omega_r^{2q-1}\nabla\omega_r\cdot\nabla(\psi^2) dyds \\
&= -\int_{P(\sigma_1)} (2q-1)(\omega_r^{q-1}\nabla\omega_r)\cdot(\omega_r^{q-1}\nabla\omega_r)\psi^2 + \nabla(\psi^2)\omega_r^q(\omega_r^{q-1}\nabla\omega_r) dyds \\
&= -\frac{2q-1}{q^2}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot\nabla(\omega_r^q)\psi^2 dyds - \frac{1}{q}\int_{P(\sigma_1)} \omega_r^q\nabla(\omega_r^q)\cdot\nabla(\psi^2) dyds \\
&\le -\frac{1}{q}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot(\nabla(\omega_r^q)\psi^2 + \nabla(\psi^2)\omega_r^q) dyds, \qquad \text{since } \frac{1}{q} < \frac{2q-1}{q^2} \\
&= -\frac{1}{q}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot\nabla(\omega_r^q\psi^2) dyds \\
&= -\frac{1}{q}\int_{P(\sigma_1)} (|\nabla(\omega_r^q\psi)|^2 - |\nabla\psi|^2\omega_r^{2q}) dyds,
\end{aligned}
$$

which implies:

$$
\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(\omega_r^q\psi)|^2 dyds \\
&\quad \le -\int_{P(\sigma_1)} b\cdot\nabla(\omega_r^q)(\omega_r^q\psi^2) dyds - \int_{P(\sigma_1)} \partial_s(\omega_r^q)(\omega_r^q\psi^2) dyds + \int_{P(\sigma_1)} |\nabla\psi|^2\omega_r^{2q} dyds \\
&\qquad + q\int_{P(\sigma_1)} \left[\left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2}\right)(\omega_r^{2q}\psi^2) + \left(\frac{\partial v_r}{\partial z}\right)\omega_z\omega_r^{2q-1}\psi^2\right] dyds. \qquad (5.2)
\end{aligned}
$$

Similarly, using $\omega_z^{2q-1}\psi^2$ as a test function in the equation for $\omega_z$ in (5.1),

$$
\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(\omega_z^q\psi)|^2 dyds \\
&\quad \le -\int_{P(\sigma_1)} b\cdot\nabla(\omega_z^q)(\omega_z^q\psi^2) dyds - \int_{P(\sigma_1)} \partial_s(\omega_z^q)(\omega_z^q\psi^2) dyds + \int_{P(\sigma_1)} |\nabla\psi|^2\omega_z^{2q} dyds \\
&\qquad q\int_{P(\sigma_1)} \left[\left(\frac{\partial v_z}{\partial z}\right)(\omega_z^{2q}\psi^2) + \left(\frac{\partial v_z}{\partial r}\right)\omega_r\omega_z^{2q-1}\psi^2\right] dyds. \qquad (5.3)
\end{aligned}
$$

We let $f = |\omega_r|^q + |\omega_z|^q$, and $|V|$ be the max norm of the matrix $V$, and add (5.2) and (5.3) to get:

$$\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds \le 2\int_{P(\sigma_1)} (-b\cdot\nabla f(f\psi^2) - \partial_s f(f\psi^2) + |\nabla\psi|^2 f^2 + qc|V|f^2\psi^2)dyds,$$

where we used the Cauchy-Schwartz inequality. Using the same steps as in (4.2), we get a similar estimate to (4.3):

$$\begin{aligned}\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\ \le -\int_{P(\sigma_1)} 2b\cdot\nabla f(f\psi^2)dyds + 2\int_{P(\sigma_1)}(\eta\partial_s\eta + |\nabla\psi|^2)f^2 dyds \\ + cq\int_{P(\sigma_1)} |V|f^2\psi^2 dyds \\ := T_1 + T_2 + T_3. \qquad (5.4)\end{aligned}$$

## 5.2 Bounding $T_3$

The $T_1$ and $T_2$ terms in (5.4) can be treated using the methods in Chapter 4. We therefore shift our focus to $T_3$. This integral was bounded in [1]:

$$T_3 \le \epsilon_2 ||(f\psi)^2||_{L^{\frac{5}{3}}(P(\sigma_1))} + q^4||V||^4_{L^{\frac{10}{3}}(P(\sigma_1))}\int_{P(\sigma_1)} f^2 dyds.$$

Using dimension reduction, enlarging the domain of the velocity derivative norm to $\overline{P}_{1,4,1}$, and bounding the $L^{5/3}$-norm of $(f\psi)^2$ by its $L^2$-norm, we have the following bound for $T_3$:

$$T_3 \le \epsilon_2 ||(f\psi)^2||_{L^2(\overline{P}(\sigma_1))} + cq^4||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})}\int_{P(\sigma_1)} f^2 d\overline{y}ds. \qquad (5.5)$$

## 5.3 Combining Estimates and Embedding

At this time we utilize the estimates for $T_1$ and $T_2$ (estimates 4.13 and 4.14, respectively), along with $T_3$ in the previous section. Energy estimate (5.4) then becomes:

$$\begin{aligned}
&\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(f\psi)|^2 d\overline{y}ds + \frac{1}{2}\int_{\overline{C}(\sigma_1)} f^2(y,0)\phi^2(y)d\overline{y} \\
&\quad \leq \frac{1}{8}\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(\psi f)|^2 d\overline{y}ds + 2\epsilon_2 ||(f\psi)^2||_{L^2(P(\sigma_1))} \\
&\qquad + C\left[\frac{\overline{K}^2(v,w)}{(\sigma_1-\sigma_2)^2} + \frac{1}{(\sigma_1-\sigma_2)^2} + q^4||V||^4_{L^{\frac{10}{3}}(\overline{P}(\sigma_1))}\right]\int_{\overline{P}(\sigma_1)} f^2\, d\overline{y}ds
\end{aligned}$$

After absorbing the first term on the right, we get:

$$\begin{aligned}
&\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(f\psi)|^2 d\overline{y}ds + \int_{\overline{C}(\sigma_1)} f^2(y,0)\phi^2(y)d\overline{y} \qquad (5.6)\\
&\quad \leq \frac{Cq}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2(v,w) + ||V||^4_{L^{\frac{10}{3}}(\overline{P}(\sigma_1))} + 1\right)\int_{\overline{P}(\sigma_1)} f^2\, d\overline{y}ds + 2\epsilon_2||(f\psi)^2||_{L^2(\overline{P}(\sigma_1))},
\end{aligned}$$

noting $\quad 0 < \sigma_1 - \sigma_2 < 1 \quad$ and $\quad q > 1.$

Now, recall (4.20) in Moser's iteration in Section 4.7, which follows from embedding (4.18), Poincare's inequality, and properties of the cut-off function:

$$\int_{-\sigma_1^2}^0 \int_{\overline{C}(\sigma_1)} (f\psi)^4 d\overline{y}ds \leq C \sup_{-\sigma_1^2\leq s\leq 0}\left(\int_{\overline{C}(\sigma_1)} (f\psi)^2 d\overline{y}ds\right)\int_{\overline{P}(\sigma_1)} |\overline{\nabla}(f\psi)|^2 d\overline{y}ds$$

As in Chapter 4, here we apply estimate (5.6), which controls both integrals in the embedding since it holds for every $s$ in the interval $-\sigma_1^2 \leq s < 0$:where $\tau = \sigma_1 - \sigma_2$

$$\begin{aligned}
\int_{\overline{P}(\sigma_1)} (f\psi)^4 d\overline{y}ds \leq \frac{Cq^2}{(\sigma_1-\sigma_2)^2}\Big[&\left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds \\
&+ 2\epsilon_2||(f\psi)^2||_{L^2(\overline{P}(\sigma_1))}\Big]^2
\end{aligned}$$

Taking the square root on both sides gives us an inequality bounding $||(f\psi)^2||_{L^2(\overline{P}(\sigma_1))}$ on the left. Therefore we choose $\epsilon_2 = \frac{1}{4}$ and absorb this $\epsilon_2$ term to the left. Upon removing the square root, we have the key to our iteration:

$$\int_{\overline{P}(\sigma_1)} (f\psi)^4 d\overline{y}ds \leq \frac{Cq^2}{(\sigma_1-\sigma_2)^2}\left[\left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)\int_{\overline{P}(\sigma_1)} f^2 d\overline{y}ds\right]^2 \tag{5.7}$$

## 5.4 $L^2 - L^\infty$ Estimate on $\omega_r$ and $\omega_z$ using Moser's Iteration

Recall that $f = |\omega_r|^q + |\omega_z|^q$. Making this substitution in (5.7) and using properties of the cut-off function, we get:

$$\int_{\overline{P}(\sigma_2)} (|\omega_r|^q + |\omega_z|^q)^4 d\overline{y}ds$$
$$\leq C\left[\frac{c_1 q^2}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)\int_{\overline{P}(\sigma_1)} (|\omega_r|^q + |\omega_z|^q)^2 d\overline{y}ds\right]^2$$

We define $h(x,t) = \max(|\omega_r|, |\omega_z|)$ and observe that since $h^q \leq |\omega_r|^q + |\omega_z|^q \leq 2h^q$, we can replace the integrands containing $f$ with $h$:

$$\int_{\overline{P}(\sigma_2)} h^{4q} d\overline{y}ds \leq C\left[\frac{q^2}{(\sigma_1-\sigma_2)^2}\left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)\int_{\overline{P}(\sigma_1)} h^{2q} d\overline{y}ds\right]^2. \tag{5.8}$$

The rest of this argument follows exactly the same as in section 4.7, but with $h$ in place of $\Omega_+$ and $\Omega_-$. For $i = 0, 1, 2, \ldots$ in (5.8), again we take $q = 2^i$ and replace $\sigma_1$ by $\sigma_i = 1 - \sum_{j=1}^{i} 2^{-i-j}$ and $\sigma_2$ by $\sigma_{i+1} = 1 - \sum_{j=1}^{i+1} 2^{-j-2}$. We take $\sigma_i = \sum_{j=1}^{i} 2^{-j-2}$ for the case $i = 0$. Then (5.8) generalizes to:

$$\left(\int_{\overline{P}(\sigma_{i+1})} h^{2^{i+2}} d\overline{y}ds\right)^{1/2} \leq c_1 c_2^{i+1} 2^{2i}\left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)\int_{\overline{P}(\sigma_i)} h^{2^{i+1}} d\overline{y}ds \tag{5.9}$$

If we take the 1/2-th power of (5.9) then we get

$$\left(\int_{\overline{P}(\sigma_{i+1})} h^{2^{i+2}} d\overline{y}ds\right)^{1/2^2} \leq c_1^{\frac{1}{2}} c_2^{\frac{i+2}{2}} 2^{\frac{2i}{2}} \left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)^{\frac{1}{2}} \left(\int_{\overline{P}(\sigma_i)} h^{2^{i+1}} d\overline{y}ds\right)^{\frac{1}{2}}$$
$$\leq c_1^{\frac{1}{2}+1} c_2^{\left[\frac{i+1}{2}+i\right]} 2^{\left[\frac{2i}{2}+2(i-1)\right]} \left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)^{\frac{1}{2}+1} \int_{\overline{P}(\sigma_{i-1})} h^{2^{(i-1)+1}} d\overline{y}ds, \tag{5.10}$$

where we applied (5.9) to the first line of (5.10). We repeat this process of taking the 1/2-th power of the above inequality, then applying (5.9), until we reach the $L^2$-norm of $h$ over $\overline{P}_{1,4,1}$:

$$\left(\int_{\overline{P}(\sigma_{i+1})} h^{2^{i+2}} d\overline{y}ds\right)^{\frac{1}{2^{i+1}}}$$
$$\leq c_1^{\sum \frac{1}{2^j}} c_2^{\sum \frac{j+1}{2^{j-1}}} 2^{2\sum \frac{j-1}{2^{j-1}}} \left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)^{\sum \frac{1}{2^{j-1}}} \int_{\overline{P}_{1,4,1}} h^2 d\overline{y}ds.$$

Note that all the sums in the exponents are from $j = 1$ to $j = i + 1$. Letting $i \to \infty$, the exponent series all converge. Therefore,

$$\sup_{\overline{P}_{2,3,\frac{3}{4}}} (\omega_r^2 + \omega_z^2) \leq C \left(\overline{K}^2 + ||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} + 1\right)^2 \left(\int_{\overline{P}_{1,4,1}} \omega_r^2 d\overline{y}ds + \int_{\overline{P}_{1,4,1}} \omega_z^2 d\overline{y}ds\right). \tag{5.11}$$

## 5.5 Controlling the Velocity Derivatives

Recall the definition of $V$:

$$V = \begin{bmatrix} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} & \frac{\partial v_z}{\partial r} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}.$$

The same argument in [1] is used to control $||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})}$, but with better accuracy due to the results from Chapter 4. By Lemma 5.1 in [1], with $\overline{P}_{1,4,1}$ as the domain on the left and $\overline{P}_{\frac{1}{2},5,1}$ on the right, we have:

$$||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} \le c\Big(||\omega_\theta||_{L^{\frac{10}{3}}(\overline{P}_{\frac{1}{2},5,1})} + ||v||_{L^{\frac{10}{3}}(\overline{P}_{\frac{1}{2},5,1})} + 1\Big)^4. \tag{5.12}$$

The $L^{10/3}$-norm of $\omega_\theta$ is controlled using (4.21), after choosing $q = 1$ and taking the $\frac{1}{4}$th power on each side:

$$||\overline{\Omega}_+||_{L^4(\overline{P}(\sigma_2))} \le C\frac{1}{(\sigma_1 - \sigma_2)^2}\left(\overline{K}^2 + \Lambda^2 + 1\right)^{\frac{1}{2}} ||\overline{\Omega}_+||_{L^2(\overline{P}(\sigma_1))}.$$

With $\overline{P}(\sigma_2) = \overline{P}_{1,4,1}$ and $\overline{P}(\sigma_1) = \overline{P}_{\frac{1}{2},5,1}$, we have estimates on $\overline{\Omega}_+$ and $\overline{\Omega}_-$:

$$||\overline{\Omega}_+||_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} \le c||\overline{\Omega}_+||_{L^4(\overline{P}_{1,4,1})} \le C\left(\overline{K} + \Lambda + 1\right)||\overline{\Omega}_+||_{L^2(\overline{P}_{\frac{1}{2},5,1})}$$

and

$$||\overline{\Omega}_-||_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} \le c||\overline{\Omega}_-||_{L^4(\overline{P}_{1,4,1})} \le C\left(\overline{K} + \Lambda + 1\right)||\overline{\Omega}_-||_{L^2(\overline{P}_{\frac{1}{2},5,1})}.$$

Combining these two estimates gives the same bound for $\Omega$. Writing $\Omega = \frac{\omega_\theta}{r}$, we have

$$||\omega_\theta||_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} \le C(\overline{K} + \Lambda + 1)||\omega_\theta||_{L^2(\overline{P}_{\frac{1}{2},5,1})},$$

where we used the equivalence of $\frac{\omega_\theta}{r}$ and $\omega_\theta$ in the blown up two-dimensional shells $\overline{P}_{1,4,1}$ and $\overline{P}_{\frac{1}{2},5,1}$ (with $r$ bounded by two constants). Thus, $||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})}$ satisfies the bound:

$$||V||^4_{L^{\frac{10}{3}}(\overline{P}_{1,4,1})} \le C\Big((\overline{K} + \Lambda + 1)^4||\omega_\theta||^4_{L^2(\overline{P}_{\frac{1}{2},5,1})} + ||v||^4_{L^{\frac{10}{3}}(\overline{P}_{\frac{1}{2},5,1})} + 1\Big)$$

The domain was enlarged proportionally to make the right-hand side more uniform. Substituting this back into (5.11) with the enlarged domain $\overline{P}_{\frac{1}{2},5,1}$ on the right, we have

$$\sup_{\overline{P}_{2,3,\frac{3}{4}}}(\omega_r^2 + \omega_z^2) \le A\left(\int_{\overline{P}_{\frac{1}{2},5,1}} \omega_r^2 d\overline{y}ds + \int_{\overline{P}_{\frac{1}{2},5,1}} \omega_z^2 d\overline{y}ds\right),$$

where

$$A = C\Big(\overline{K}^2 + \Big(\overline{K} + \Lambda + 1\Big)^4 ||\omega_\theta||^4_{L^2(\overline{P}_{\frac{1}{2},5,1})} + ||v||^4_{L^{\frac{10}{3}}(\overline{P}_{\frac{1}{2},5,1})} + 1\Big)^2.$$

## 5.6 Re-scaling; proof of Theorem 3.3

Recall our "tilde" notation and that what has actually been shown to this point is:

$$\sup_{(\tilde{x},\tilde{t})\in P_{2,3,\frac{3}{4}}} (\widetilde{\omega}_r^2 + \widetilde{\omega}_z^2)(\tilde{x},\tilde{t}) \leq \tilde{A}\left(\int_{P_{\frac{1}{2},5,1}} \widetilde{\omega}_r^2(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t} + \int_{P_{\frac{1}{2},5,1}} \widetilde{\omega}_z^2(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t}\right), \tag{5.13}$$

where $\tilde{x} = \frac{x}{k}$, $\tilde{t} = \frac{t}{k^2}$, $\widetilde{\omega}_r(\tilde{x},\tilde{t}) = k^2\omega_r(k\tilde{x}, k^2\tilde{t})$, $\widetilde{\omega}_z(\tilde{x},\tilde{t}) = k^2\omega_z(k\tilde{x}, k^2\tilde{t})$, and

$$\tilde{A} = C\Big(\overline{K}^2(\tilde{v},\widetilde{\omega}) + \Big(\overline{K}(\tilde{v},\widetilde{\omega}) + \tilde{\Lambda} + 1\Big)^4 ||\widetilde{\omega}_\theta||^4_{L^2(P_{\frac{k}{2},5k,k})} + ||\tilde{v}||^4_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} + 1\Big)^2.$$

From the scaling in Section 3.2 and estimate (4.26) on $\overline{K}(v,w)$, we have:

$$||\tilde{v}(\tilde{x},\tilde{t})||_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} = k^{-\frac{1}{2}}||v(x,t)||_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})},$$

$$||\widetilde{\omega}(\tilde{x},\tilde{t})||_{L^2(P_{\frac{k}{2},5k,k})} = k^{-\frac{1}{2}}||\omega(x,t)||_{L^2(P_{\frac{k}{2},5k,k})},$$

and

$$\overline{K}^2(\widetilde{v},\widetilde{w}) \leq C(k^{-1/2}||b||_{L^\infty(-k^2,0;L^2(C_{\frac{k}{2},5k}))} + 1)^2 \log(\frac{1}{k} + e).$$

We apply Theorem 3.2, for $(x,t) \in P_{\frac{k}{2},5k,k}$:

$$|\omega_\theta(x,t)| \leq Ck^{-\frac{7}{2}} \log(\frac{1}{k} + e),$$

where $C$ depends only on the initial value.

Therefore, rescaling and substituting estimates (4.26) and (**??**), we find that $\tilde{A}$ scales in the following way:

$$\tilde{A} \leq C\Big((k^{-1/2}||b||_{L^\infty(-k^2,0;L^2(P_{\frac{k}{2},5k,k}))} + 1)^2 \log(\frac{1}{k} + e)$$
$$+ \Big(k^{-1/2}\log^{\frac{1}{2}}(\frac{1}{k} + e) + \Lambda + 1\Big)^4 k^{-2}||\omega_\theta||^4_{L^2(P_{\frac{k}{2},5k,k})} + k^{-2}||v||^4_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} + 1\Big)^2.$$

Finally, after rescaling on (5.13) and substituting the estimate on $\tilde{A}$, we have, for any $k \in (0, 1]$,

$$\sup_{P_{2k,3k,\frac{3k}{4}}} k^4\left(\omega_r^2(x,t) + \omega_z^2(x,t)\right)$$
$$\leq C\Big((k^{-\frac{1}{2}}||b||_{L^\infty(-k^2,0;L^2(P_{\frac{k}{2},5k,k}))} + 1)^2 \log(\frac{1}{k} + e)$$
$$+ (k^{-\frac{1}{2}}\log^{\frac{1}{2}}(\frac{1}{k} + e) + \Lambda + 1)^4 k^{-2}||\omega_\theta||^4_{L^2(P_{\frac{k}{2},5k,k})} + k^{-\frac{1}{2}}||v||^4_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} + 1\Big)^2$$
$$\times\left(\int_{P_{\frac{k}{2},5k,k}} k^4\omega_r^2(x,t)\frac{1}{k^5}dxdt + \int_{P_{\frac{k}{2},5k,k}} k^4\omega_z^2(x,t)\frac{1}{k^5}dxdt\right)$$
$$\leq \frac{C}{k^9}\Big((k^{3/2}||b||_{L^\infty(-k^2,0;L^2(P_{\frac{k}{2},5k,k}))} + k^2)^2 \log(\frac{1}{k} + e)$$
$$+ (\log^{\frac{1}{2}}(\frac{1}{k} + e) + k^{1/2}\Lambda + k^{1/2})^4||\omega_\theta||^4_{L^2(P_{\frac{k}{2},5k,k})} + k^{7/2}||v||^4_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} + k^4\Big)^2$$
$$\times\left(||\omega_r||^2_{L^2(P_{\frac{k}{2},5k,k})} + ||\omega_\theta||^2_{L^2(P_{\frac{k}{2},5k,k})}\right).$$

Thus, after dividing by $k^4$ and taking the square root on both sides, we have proved the third part of the main theorem:

$$
\begin{aligned}
&||\omega_r(x,t)||_{L^\infty(\overline{P}_{2k,3k,\frac{3k}{4}})} + ||\omega_z(x,t)||_{L^\infty(\overline{P}_{2k,3k,\frac{3k}{4}})} \\
&\quad \le \frac{C}{k^{13/2}}\Big((k^{3/2}||b||_{L^\infty(-k^2,0;L^2(P_{\frac{k}{2},5k,k}))} + k^2)^2 \log(\frac{1}{k}+e) \\
&\qquad + (\log^{\frac{1}{2}}(\frac{1}{k}+e) + k^{1/2}\Lambda + k^{1/2})^4 ||\omega_\theta||^4_{L^2(P_{\frac{k}{2},5k,k})} + k^{7/2}||v||^4_{L^{\frac{10}{3}}(P_{\frac{k}{2},5k,k})} + k^4\Big)^2 \\
&\qquad\qquad \times \left(||\omega_r||^2_{L^2(P_{\frac{k}{2},5k,k})} + ||\omega_\theta||^2_{L^2(P_{\frac{k}{2},5k,k})}\right).
\end{aligned}
$$

□

# Chapter 6

# Vorticity to Velocity

Now that the local bound on $\omega_\theta$ has been established, in this chapter it is shown how the vorticity can control local growth of the velocity, $v = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$. Since $v_\theta$ a priori satisfies a good local estimate, we can work on just the $v_r$ and $v_z$ terms, which are used to define the $\theta$-component of vorticity, with $\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$. Recalling our notation $b = v_r e_r + v_z e_z$, we establish the following inequality for all $p \geq 1$:

$$\sup_{B_{r_0}(x)} |b| \leq C r_0^{-3/p} ||b||_{L^p(B_{2r_0}(x))} + C r_0 \sup_{B_{2r_0}(x)} |\omega_\theta| \tag{6.1}$$

## 6.1 Vorticity as the Laplacian of $b$

The cylindrical curl of $b$ is

$$\nabla \times b = \omega_\theta \mathbf{e}_\theta \tag{6.2}$$

Using the following identity for any smooth vector field: $\nabla(\nabla \cdot b) - \nabla \times (\nabla \times b) = \Delta b$, along with the divergence-free condition on $b$, we compute the curl of (6.2) to be:

$$\nabla \times (\nabla \times b) = -\Delta b$$

Therefore, replacing $\nabla \times b$ with $\omega_\theta \mathbf{e}_\theta$, we have established that

$$-\Delta b = \nabla \times (\omega_\theta \mathbf{e}_\theta) \tag{6.3}$$

## 6.2 Inversion of $-\Delta(\phi b)$ for a Cutoff Function $\phi$

To derive the local estimate on the $r$- and $z$-components of velocity, we use a cutoff argument on (6.3) by first inverting the Laplacian using a Green's function on $\mathbb{R}^3$, denoted by $\Gamma(x, y) = \frac{c_0}{|x-y|}$. The point $y = (y_1, y_2, y_3) = (\tilde{r}\cos\tilde{\theta}, \tilde{r}\sin\tilde{\theta}, \tilde{z})$ is any point in $\mathbb{R}^3$, while $x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$ is fixed.

Choose a smooth cutoff function $\phi$ on $\mathbb{R}^3$ with support contained in the ball $B_{2r_0} = B(x, 2r_0)$, $0 \leq \phi \leq 1$ in $B_{2r_0}$, $\phi \equiv 1$ in $B_{r_0}$, and having the following properties:

$$|\nabla \cdot \phi| \leq \frac{C}{r_0} \quad , \quad |\Delta \phi| \leq \frac{C}{r_0^2}$$

Then $\text{supp}(\nabla\phi) \subset B_{2r_0} \setminus B_{r_0}$. Computing the Laplacian gives:

$$\Delta(\phi b) = b\Delta\phi + 2\nabla\phi \cdot \nabla b + \phi\Delta b$$

We use (6.3) to replace $\Delta b$, and then solve for $\phi b$ by integration against the Green's function over $\mathbb{R}^3$:

$$\begin{aligned}\phi b &= \int_{\mathbb{R}^3} \Gamma(x,y)\Big(\Delta_y \phi\, b + 2\nabla_y \phi \cdot \nabla_y b - \phi \,\mathrm{curl}_y\,(\omega_\theta \mathbf{e}_\theta)\Big)\,dy \\ &= \int_{\mathbb{R}^3} \Gamma(x,y)\Delta_y \phi\, b\,dy + 2\int_{\mathbb{R}^3} \Gamma(x,y)\nabla_y \phi \cdot \nabla_y b\,dy - \int_{\mathbb{R}^3} \Gamma(x,y)\phi\,\mathrm{curl}_y(\,\omega_\theta \mathbf{e}_\theta)\,dy \\ &= I_1 + I_2 + I_3 \qquad (6.4)\end{aligned}$$

The $y$-dependence of $\phi$ and $b$ in the integrands is surpressed, and will remain so for most of the derivation of (6.1).

## 6.3 Bounding an Integral by $\sup_{B_{2r_0}(x)} |\omega_\theta|$

From here, we present detailed computations on bounding each integral in (6.4). Starting with the more complicated term, $I_3$, a quick computation of the cylindrical curl of $\omega_\theta \mathbf{e}_\theta$ gives, with the axial symmetry assumption:

$$\mathrm{curl}_y\,(\omega_\theta \mathbf{e}_\theta) = -\frac{\partial \omega_\theta}{\partial \tilde{z}}\mathbf{e}_{\tilde{r}} + \left(\frac{\omega_\theta}{\tilde{r}} + \frac{\partial \omega_\theta}{\partial \tilde{r}}\right)\mathbf{e}_{\tilde{z}}$$

Hence, $I_3$ becomes

$$I_3 = \int_{\mathbb{R}^3} \Gamma(x,y)\,\phi\,\frac{\partial \omega_\theta}{\partial \tilde{z}}\mathbf{e}_{\tilde{r}}\,dy - \int_{\mathbb{R}^3} \Gamma(x,y)\,\phi\,\frac{\omega_\theta}{\tilde{r}}\mathbf{e}_{\tilde{z}}\,dy + \int_{\mathbb{R}^3} \Gamma(x,y)\,\phi\,\frac{\partial \omega_\theta}{\partial \tilde{r}}\mathbf{e}_{\tilde{z}}\,dy, \qquad (6.5)$$

which expands, after integration by parts, to

$$
\begin{aligned}
I_3 = &- \int_{\mathbb{R}^3} \frac{\partial}{\partial \tilde{z}} \left( \Gamma(x,y)\, \phi\, \mathbf{e}_{\tilde{r}} \right) \omega_\theta \, dy - \int_{\mathbb{R}^3} \Gamma(x,y)\, \phi \, \frac{\omega_\theta}{\tilde{r}} \mathbf{e}_{\tilde{z}} \, dy \\
&+ \int_{\mathbb{R}^3} \frac{\partial}{\partial \tilde{r}} \left( \frac{\tilde{r}\, \phi(\tilde{r}, \tilde{\theta}, \tilde{z})}{\sqrt{(\tilde{r}\cos\tilde{\theta} - r\cos\theta)^2 + (\tilde{r}\sin\tilde{\theta} - r\sin\theta)^2 + (\tilde{z} - z)^2}} \, \mathbf{e}_{\tilde{z}} \right) \omega_\theta \, d\tilde{r} d\tilde{\theta} d\tilde{z}
\end{aligned}
$$

$$
\begin{aligned}
= &- \int_{\mathbb{R}^3} \left( \frac{\tilde{z} - z}{|x-y|^3}\, \phi + \frac{\partial_{\tilde{z}} \phi}{|x-y|} \right) \omega_\theta \mathbf{e}_{\tilde{r}} \, dy - \int_{\mathbb{R}^3} \Gamma(x,y)\, \phi \, \frac{\omega_\theta}{\tilde{r}} \mathbf{e}_{\tilde{z}} \, dy \\
&+ \int_{\mathbb{R}^3} \frac{|x-y|^2 (\phi + \tilde{r} \partial_{\tilde{r}} \phi) - \phi[(\tilde{x}_1 - x_1)\tilde{x}_1 + (\tilde{x}_2 - x_2)\tilde{x}_2]}{|x-y|^3} \frac{\omega_\theta}{\tilde{r}} \mathbf{e}_{\tilde{z}} \, dy
\end{aligned}
\tag{6.6}
$$

We simplify (6.6) by rearranging the cutoff function and $|x-y|^2$ term in the numerator of the third integral:

$$
\begin{aligned}
I_3 = &- \int_{\mathbb{R}^3} \left( \frac{\tilde{z} - z}{|x-y|^3}\, \phi + \frac{\partial_{\tilde{z}} \phi}{|x-y|} \right) \omega_\theta \mathbf{e}_{\tilde{r}} \, dy - \int_{\mathbb{R}^3} \Gamma(x,y)\, \phi \, \frac{\omega_\theta}{\tilde{r}} e_{\tilde{z}} \, dy \\
&- \int_{\mathbb{R}^3} \frac{(\tilde{x}_1^2 + \tilde{x}_2^2 - \tilde{x}_1 x_1 - \tilde{x}_2 x_2)}{|x-y|^3} \frac{\phi\, \omega_\theta}{\tilde{r}} \mathbf{e}_{\tilde{z}} \, dy + \int_{\mathbb{R}^3} \Gamma(x,y) \left( \frac{\phi}{\tilde{r}} + \partial_{\tilde{r}} \phi \right) \omega_\theta \mathbf{e}_{\tilde{z}} \, dy
\end{aligned}
$$

$$
\begin{aligned}
= &- \int_{\mathbb{R}^3} \left( \frac{\tilde{z} - z}{|x-y|^3}\, \phi + \frac{\partial_{\tilde{z}} \phi}{|x-y|} \right) \omega_\theta \mathbf{e}_{\tilde{r}} \, dy + \int_{B_{r_0}} \Gamma(x,y)\, \frac{1}{\tilde{r}} \partial_{\tilde{r}} (\tilde{r} \phi)\, \omega_\theta \mathbf{e}_{\tilde{z}} \, dy \\
&- \int_{\mathbb{R}^3} \frac{(\tilde{x}_1^2 + \tilde{x}_2^2 - \tilde{x}_1 x_1 - \tilde{x}_2 x_2)}{|x-y|^3} \frac{\phi\, \omega_\theta}{\tilde{r}} \mathbf{e}_{\tilde{z}} \, dy
\end{aligned}
\tag{6.7}
$$

The following two terms in (6.7): $\tilde{z} - z$ and $\tilde{x}_1^2 + \tilde{x}_2^2 - \tilde{x}_1 x_1 - \tilde{x}_2 x_2$, are both bounded by $|x-y|$ and $|x-y|^2$, respectively. Since the cylindrical basis vectors have norm 1, we can finally bound $I_3$:

$$|I_3| \leq \int_{\mathbb{R}^3} \left( \frac{|x-y|}{|x-y|^3}|\phi| + \frac{|\partial_{\tilde{z}}\phi|}{|x-y|} \right) |\omega_\theta| \, dy + \int_{\mathbb{R}^3} |\Gamma(x,y)| \left| \frac{1}{\tilde{r}} \partial_{\tilde{r}}(\tilde{r}\phi) \right| |\omega_\theta| \, dy$$
$$+ \int_{\mathbb{R}^3} \frac{|x-y|^2}{|x-y|^3} \frac{|\phi||\omega_\theta|}{\tilde{r}} \, dy$$

$$= \int_{\mathbb{R}^3} \left( \frac{|\phi|}{|x-y|^2} + \frac{|\partial_{\tilde{z}}\phi|}{|x-y|} \right) |\omega_\theta| \, dy + \int_{\mathbb{R}^3} |\Gamma(x,y)| \left| \frac{1}{\tilde{r}} \partial_{\tilde{r}}(\tilde{r}\phi) \right| |\omega_\theta| \, dy$$
$$+ \int_{\mathbb{R}^3} \frac{1}{|x-y|} \frac{|\phi||\omega_\theta|}{\tilde{r}} \, dy,$$

which becomes, after grouping together the cutoff terms:

$$\begin{aligned} |I_3| &\leq \int_{\mathbb{R}^3} \frac{|\phi|}{|x-y|^2} |\omega_\theta| \, dy + \int_{B_{r_0}} \left( |\partial_{\tilde{z}}\phi| + \left| \frac{1}{\tilde{r}} \partial_{\tilde{r}}(\tilde{r}\phi) \right| \right) |\Gamma(x,y)||\omega_\theta| \, dy \\ &= \int_{\mathbb{R}^3} \frac{|\phi|}{|x-y|^2} |\omega_\theta| \, dy + \int_{\mathbb{R}^3} \left( |\partial_{\tilde{z}}\phi| + \left| \frac{1}{\tilde{r}} \partial_{\tilde{r}}(\tilde{r}\phi) \right| \right) \frac{|\omega_\theta|}{|y-x|} \, dy \end{aligned} \tag{6.8}$$

We know that $|\partial_{\tilde{z}}\phi| + \left|\frac{1}{\tilde{r}}\partial_{\tilde{r}}(\tilde{r}\phi)\right| \leq |\nabla\phi|$. Moreoever, our cutoff function is supported in $B_{r_0}$, with $\nabla\phi$ supported in $B_{2r_0} \setminus B_{r_0}$ and $|\nabla\phi| \leq \frac{C}{r_0}$. We can therefore bound (6.8) by $\sup_{B_{2r_0}(x)} |\omega_\theta|$:

$$\begin{aligned} |I_3| &\leq \int_{B_{r_0}} \frac{|\omega_\theta(y)|}{|y-x|^2} \, dy + \int_{B_{r_0}} \frac{C}{r_0} \frac{|\omega_\theta(y)|}{|y-x|} \, dy \\ &\leq C \sup_{B_{2r_0}(x)} |\omega_\theta| \left( \int_{B_{2r_0}} \frac{1}{|y-x|^2} \, dy + \int_{Q_{9r_0/8}} \frac{1}{|y-x|} \, dy \right) \end{aligned}$$

Here we enlarged the domain of the second integral from $B_{r_0}$ to $Q_{9r_0/8} = Q(x, 9r_0/8)$, a cylinder of radius $9r_0/8$ and height $9r_0/4$, such that $B_{r_0} \subset Q_{9r_0/8} \subset B_{2r_0}$.

Now we expand each integral, using spherical coordinates for the ball and cylindrical coordinates for the cylinder, thus arriving at our final bound for $I_3$:

$$
\begin{aligned}
|I_3| &\le C \sup_{B_{2r_0}(x)} |\omega_\theta| \left( \int_0^{2\pi} \int_0^{\pi} \int_0^{2r_0} \frac{1}{\tilde{\rho}^2} \tilde{\rho}^2 \sin(\tilde{\phi}) d\tilde{\rho} d\tilde{\phi} d\tilde{\theta} + \int_0^{2\pi} \int_{-\frac{9r_0}{8}}^{\frac{9r_0}{8}} \int_0^{\frac{9r_0}{8}} \frac{1}{\tilde{r}} \tilde{r}\, d\tilde{r} d\tilde{z} d\tilde{\theta} \right) \\
&\le C r_0 \sup_{B_{2r_0}(x)} |\omega_\theta| \qquad (6.9)
\end{aligned}
$$

## 6.4 Bounding the remaining Integrals

For the terms $I_1$ and $I_2$, we integrate by parts and use properties of our cutoff function to bound the integrals in terms of the $L^p$-norm of $b$ in the ball $B_{2r_0}(x)$. The details are as follows, beginning with integration by parts:

$$
\begin{aligned}
I_1 + I_2 &= \int_{\mathbb{R}^3} \Gamma(x,y) \Delta_y \phi\, b\, dy + 2 \int_{\mathbb{R}^3} \Gamma(x,y) \nabla_y \phi \cdot \nabla_y b\, dy \\
&= \int_{\mathbb{R}^3} \Gamma(x,y) \Delta_y \phi\, b\, dy - 2 \int_{\mathbb{R}^3} \nabla_y \cdot (\Gamma(x,y) \nabla_y \phi) b\, dy.
\end{aligned}
$$

Surpressing the $y$-variable in the Laplacian and gradient symbols, we compute

$\nabla \cdot (\Gamma(x,y)\nabla\phi)$ and combine the resulting integrals:

$$
\begin{aligned}
I_1 + I_2 &= \int_{\mathbb{R}^3} \Gamma(x,y) \Delta\phi\, b\, dy - 2 \int_{\mathbb{R}^3} \nabla\Gamma(x,y) \cdot \nabla\phi\, b\, dy - 2 \int_{\mathbb{R}^3} \Gamma(x,y) \Delta\phi\, b\, dy \\
&= - \int_{\mathbb{R}^3} \Gamma(x,y) \Delta\phi\, b\, dy - 2 \int_{\mathbb{R}^3} \nabla\Gamma(x,y) \cdot \nabla\phi\, b\, dy
\end{aligned}
$$

Therefore, using properties of the cutoff function $\phi$, in particular $\nabla\phi$ having support in $B_{2r_0} \setminus B_{r_0}$, we get the bound

$$
\begin{aligned}
|I_1| + |I_2| &\le \int_{\mathbb{R}^3} |\Gamma(x,y)|\, |\Delta\phi|\, |b|\, dy + 2 \int_{\mathbb{R}^3} |\nabla\Gamma(x,y)|\, |\nabla\phi|\, |b|\, dy \\
&= \int_{B_{2r_0} \setminus B_{r_0}} |\Gamma(x,y)|\, |\Delta\phi|\, |b|\, dy + 2 \int_{B_{2r_0} \setminus B_{r_0}} |\nabla\Gamma(x,y)|\, |\nabla\phi|\, |b|\, dy \qquad (6.10)
\end{aligned}
$$

We have that $y \neq x$ in $B_{2r_0}(x) \setminus B_{r_0}(x)$. This fact, along with the previous calculations of $\partial\Gamma/\partial\tilde{r}$, $\partial\Gamma/\partial\tilde{z}$ and a similar calculation on $\frac{1}{\tilde{r}}\frac{\partial\Gamma}{\partial\tilde{\theta}}$, shows that $\Gamma(x, y)$ and $\nabla\Gamma(x, y)$ have the following bounds:

$$|\Gamma(x,y)| = \frac{c_0}{|y-x|} \leq \frac{C}{r_0} \qquad \text{and} \qquad |\nabla\Gamma(x,y)| \leq \frac{C}{|y-x|^2} \leq \frac{C}{r_0^2} \tag{6.11}$$

Combining (6.10) and (6.11) with properties of the cutoff function gives:

$$\begin{aligned} |I_1| + |I_2| &\leq \frac{C}{r_0^3} \int_{B_{2r_0} \setminus B_{r_0}} |b(y)|\, dy + \frac{C}{r_0^3} \int_{B_{2r_0} \setminus B_{r_0}} |b(y)|\, dy \\ &= \frac{C}{r_0^3} \int_{B_{2r_0} \setminus B_{r_0}} |b(y)|\, dy. \end{aligned}$$

Lastly, by Hölder's inequality,

$$\begin{aligned} |I_1| + |I_2| &\leq \frac{C}{r_0^3} \left( \int_{B_{2r_0} \setminus B_{r_0}} |b(y)|^p\, dy \right)^{\frac{1}{p}} \left( \int_{B_{2r_0} \setminus B_{r_0}} 1\, dy \right)^{1-\frac{1}{p}} \\ &= \frac{C}{r_0^3} (c r_0^3)^{1-\frac{1}{p}} ||b||_{L^p(B_{2r_0} \setminus B_{r_0})} \\ &\leq C r_0^{-\frac{3}{p}} ||b||_{L^p(B_{2r_0}(x))}. \end{aligned} \tag{6.12}$$

Putting (6.4), (6.9), and (6.12) all together, we have thus shown

$$\sup_{B_{r_0}(x)} |b| \leq C r_0^{-3/p} ||b||_{L^p(B_{2r_0}(x))} + C r_0 \sup_{B_{2r_0}(x)} |\omega_\theta| \tag{6.13}$$

# Chapter 7

# Proof of the Main Theorem

We pick a point $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ and let $r = |x'| = \sqrt{x_1^2 + x_2^2}$ be the distance from $x$ to the $x_3$-axis. Theorem 3.2 asserts that we can bound $\omega_\theta$ pointwise by

$$|\omega_\theta(x,t)| \leq \frac{C \ln(1/r)}{r^{7/2}}$$

When this is substituted into (6.13), with $p = 2$ we have

$$|b(x,t)| \leq C r_0^{-3/2} ||b||_{L^2(B_{2r_0}(x))} + C r_0 r^{-7/2} |\ln r| \tag{7.1}$$

Using our axial symmetry assumption, (7.1) actually has greater decay as $r \to 0$. The idea is that we are free to choose $r_0$, so we consider $r_0 < r \leq 1/2$ (since we only need to worry about the bound close to the $z$ axis) and choose $r_0$ to depend on $r$ in an optimal way.

## 7.1 Proof of Theorem 3.1

First we consider a torus at height $x_3$, with radius $r$, generated by rotating the ball $B(x, r_0)$ around the curve

$$(y_1, y_2, y_3) = \left\{ (y_1, y_2, y_3) \,\middle|\, \sqrt{y_1^2 + y_2^2} = r,\ y_3 = x_3 \right\}.$$

Dividing the circumference of this curve by the diameter of the balls $B(x, r_0)$, we can fit $2\pi r/(2r_0) \sim r/(2r_0)$ many disjoint balls of radius $r_0$ (rounding up to the nearest integer), whose union are contained in the torus.

Since the function $b$ is axially symmetric, the value of the integral of $|b|^2$ over each ball is the same, hence

$$\frac{r}{2r_0} \left( \int_{B_{r_0}(x)} |b(y)|^2 dy \right) \leq \int_{\mathbb{R}^3} |b(y)|^2 dy.$$

Writing this in terms of the $L^2$-norm, we have the bound

$$||b||_{L^2(B_{2r_0}(x))} \leq C \left( \frac{2r_0}{r} \right)^{\frac{1}{2}} ||b||_{L^2(\mathbb{R}^3)},$$

With this extra decay on $||b||_{L^2(B_{2r_0}(x))}$, we substitute into (7.1):

$$\begin{aligned} |b(x,t)| &\leq C r_0^{-3/2} ||b||_{L^2(B_{2r_0}(x))} + C r_0 r^{-7/2} |\ln r| \\ &\leq C \left( \frac{2r_0^{-2}}{r} \right)^{\frac{1}{2}} ||b||_{L^2(\mathbb{R}^3)} + C r_0 r^{-7/2} |\ln r| \end{aligned}$$

Setting the terms involving $r_0$ and $r$ equal to each other (or optimizing in the $r_0$ variable using single-variable calculus) suggests the best choice of $r_0$ to be $r_0 = r^{3/2} |\ln r|^{-1/2}$. When we substitute this choice of $r_0$, we finish the proof of Theorem 3.1:

$$\begin{aligned} |b(x,t)| &\leq C \left( \frac{2r^{-3} |\ln r|}{r} \right)^{\frac{1}{2}} ||b||_{L^2(B_{2r_0}(x))} + C r^{3/2} |\ln r|^{-1/2} r^{-7/2} |\ln r| \\ &= C r^{-2} |\ln r|^{1/2}. \end{aligned}$$

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