Title
Volume Comparison, Ricci Curvature, and Focal Radius

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Publication Date
2016

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Volume Comparison, Ricci Curvature, and Focal Radius

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

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December 2016

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Acknowledgments

I am grateful to my advisor, without whose help, I would not have been here.

To my family for all the support.

ABSTRACT OF THE DISSERTATION

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by

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In this paper, we seek to provide counter examples to two volume comparison lemmas found in [3] if we generalize their assumptions to a lower Ricci curvature bound. Second we seek to further understand Riemannian manifolds which contain embedded submanifolds of certain focal radius. First, in papers [2], [3] Grove and Petersen discussed the relationship between bounds on sectional curvature, radius, and volume and its effects on the topology of a closed Riemannian manifolds. They prove that for manifolds with a lower sectional curvature bound, an upper radius bound and almost maximal volume, one can give topological equivalence to either S^n or $\mathbb{R}P^n$. It was later proved to be diffeomorphic [9]. Also, Grove and Petersen showed that the limit space of a convergent sequence of manifolds with maximal volume has certain geometric properties. In the proofs of these theorems they use two powerful volume comparison lemmas. We discuss why the methods used in [3] cannot be extended in general to manifolds with a lower Ricci curvature bound by looking at some interesting counter examples. Second, in a paper by Guiljaro and Wilhelm [4] we see a relationship between closed embedded submanifolds of maximal

focal radius and topological type, and seek to understand the necessity of the submanifold being closed by providing counterexamples in which we consider Manifolds with embedded open submanifolds of certain focal radius which do not satisfy the conclusions of the results in [4].

Contents

List of Figures

Chapter 1

Introduction

1.1 Chapter 2

In Chapter 2 we begin with the necessary definitions and notations to help us understand convergence of manifolds with an upper radial bound, lower curvature bound and almost maximal volume. We begin with an exposition of Grove and Petersen's results for manifolds with radius bounded above, sectional curvature bounded below, and almost maximal volume, [3], and there use of the two volume comparison lemmas. In their proof they use a volume comparison theorem called the "Swiss Cheese" Lemma, and "Union of Ball" lemma. We see how these lemmas do not carry over in general to the class of manifolds with a lower Ricci curvature bound. We look at cases where the volume comparison lemmas work and cases in which they do not and determine certain obstructions to these lemmas being able to carry over to the Ricci curvature case.

1.2 Chapter 3

In Chapter 3 we define the notion of focal points and focal radius, and list the results shown by Guiljaro and Wilhelm, [4], on how manifolds with certain geometric obstructions that contain closed sub-manifolds with certain focal radii must be of a certain topological type. We see results for Manifolds with a lower sectional curvature bound as well as results for manifolds with intermediate Ricci curvature bounds. We then explore the notion of the necessity of having these sub-manifolds be closed and find counter examples to these theorems when we consider open sub-manifolds We also see how relaxing the curvature conditions to radial curvature conditions also does not give the desired results in [4].

Chapter 2

Curvature and Volume

2.1 Background

Given a metric space M we can define the radius of M, denoted Rad$M$, as:

$$\mathrm{Rad}M = \inf_p \sup_q dist(p, q).$$

We note that Rad$M \leq$ Diam$M \leq 2$Rad$M$. We can think of the radius of a metric space X as being the radius of the smallest metric ball that covers X. Let S_k^n denote the complete, simply connected, n-dimensional space form of constant curvature k, and $v_k^n(r)$ denote the volume of the n-dimensional r-ball in the constant curvature k metric. We will use the following notion of convergence.

Definition 1 (Gromov-Hausdorff Distance) *The Gromov Hausdorff, distance, denoted d_{GH}, satisfies $d_{GH}(X, Y) < \epsilon$ if and only if $d_Z^H(X, Y) < \epsilon$ for some metric on $Z = X \sqcup Y$ extending the ones on X and Y.*

Then, we can define Gromov-Hausdorff Convergence as follows:

Definition 2 (Gromov-Hausdorff Convergence) *For a collection of metric spaces X_i and a metric space Y, we say that $X_i \xrightarrow{GH} Y$ if $d_{GH}(X_i, Y) \to 0$.*

We note that the space of Riemannian Manifolds with Ric $\geq k(n-1)$ and Rad$M \leq r$ is pre-compact in the Gromov-Hausdorff Metric.

Recall that the segment domain at a point $p \in M$ of a complete Riemannian manifold is the closed, star-shaped subset of the tangent space centered at 0_p whose boundary is the tangent cut locus. In fact there are some interesting facts about seg_p.

Claim 3 *The segment domain of a product space $S_1^2 \times S_1^2$, is the product of the segment domains of each component.*

Proof. The segment domain for $p \in S_1^2$ is $D(0_p, \pi)$. We will show that the segment domain for $(p, q) \in S_1^2 \times S_1^2$ is $D(0_p, \pi) \times D(0_q, \pi)$.

Let $v \in T_p S^2$ $w \in T_q S^2$ be unit vectors, then

$$u = (v \cos\theta, w \sin\theta) \in T_{(p,q)} \left(S^2 \times S^2\right)$$

is a unit vector. Let:

$$\gamma_u(t) = \left(c_{v\cos\theta}(t), c_{w\sin\theta}(t)\right)$$

Where $c_{v\cos\theta}(t)$ and $c_{w\sin\theta}(t)$ are curves in S^2.

From what we know about S^2, the cut time of $c_{v\cos\theta}$ is $\frac{\pi}{\cos\theta}$ and the cut time of $c_{w\sin\theta}(t)$ is $\frac{\pi}{\sin\theta}$. The cut time of γ_u is

$$\leq \min\left\{\frac{\pi}{\cos\theta}, \frac{\pi}{\sin\theta}\right\}.$$

Indeed say the minimum is realized by $\frac{\pi}{\cos\theta}$ and $t_1 > \frac{\pi}{\cos\theta}$. Let γ be a segment of speed $\cos\theta$ in S^2 from p to $c_{v\cos\theta}(t_1)$. Then $(\gamma, c_{w\sin\theta}(t))$ is a curve in $S^2 \times S^2$ from (p,q) to $(\gamma(t_1), c_{w\sin\theta}(t_1)) = (c_{v\cos\theta}(t_1), c_{w\sin\theta}(t_1))$ that has length strictly shorter than $(c_{v\cos\theta}, c_{w\sin\theta})|_{[0,t_1]}$. On the other hand, if $t_1 < \min\left\{\frac{\pi}{\cos\theta}, \frac{\pi}{\sin\theta}\right\}$, then $c_{v\cos\theta}|_{[0,t_1]}$ and $c_{w\sin\theta}|_{[0,t_1]}$ are extend-able segments, so $(c_{v\cos\theta}, c_{w\sin\theta})|_{[0,t_1]}$ is an extend-able segment. For $0 \le \theta \le \pi/2$, the cut locus centered at (p,q) in $T_{(p,q)}\left(S^2 \times S^2\right)$ is exactly the boundary of the square formed by $D(0_p, \pi) \times D(0_q, \pi)$ ■

Now why is the product of segments a segment? Let J_{c_1} and J_{c_2} be non-trivial Jacobi fields along segments c_1 and c_2 parametrized from $[0, l_i]$ $i = 1, 2$. For $0 < t \le l_i$, $J_{c_i}(t)$ is non-zero along each segment. Moreover, there exists ϵ_1, $\epsilon_2 > 0$ so that $J_{c_i}(l_i + \epsilon_i)$ is non-zero. In the product manifold, we consider the curve (c_1, c_2) parametrized on $[0, l_1] \times [0, l_2]$. If the Jacobi field along the curve (c_1, c_2) is zero, then it must be zero in at least one factor, but this is a contradiction. There exists an $0 < \epsilon = min\{\epsilon_1, \epsilon_2\}$ so that The Jacobi field along the product curve is non-zero for $(0,0) < (s,t) < (l_1 + \epsilon, l_2 + \epsilon)$, so that the product curve is extend-able

Claim 4 *In general, for M and N, complete Riemannian manifolds. The segment domain of the product space $M \times N$ is equal to the product of the segment domain of M and the segment domain of N.*

Proof. Let $v \in T_pM$ $w \in T_qN$ be unit vectors, then $(vcos\theta, wsin\theta)$ is a unit vector in $T_pM \times T_qN$. Consider the two sets:

$$seg_p \times seg_q, \qquad seg_{(p,q)}$$

Where the first is the product of the segment domains at p and q, respectively. The latter is the segment domain of the product space at (p,q).

We can parametrize $seg_p \times seg_q$ as follows:

$$seg_p \times seg_q \equiv \{t(vcos\theta, wsin\theta) \mid v, \quad w \quad \text{unit vectors in } T_pM, T_qN,$$
$$\text{and } 0 \leq t < \ \min\{\frac{v_{cut}}{cos\theta}, \frac{w_{cut}}{sin\theta}\}\}$$

We can also write out what $seg_{(p,q)}$ as:

$$seg_{(p,q)} \equiv \{tu \mid u \text{ is a unit vector in } T_{(p,q)}M \times N, \text{ and } 0 \leq t < \ u_{cut}\}$$

Any unit vector $u \in T_{(p,q)}M \times N$ can be written as $u = (vcos\theta, wsin\theta)$ for $v \in T_pM$ $w \in T_qN$ unit vectors. We just need to show $u_{cut} = \min\{\frac{v_{cut}}{cos\theta}, \frac{w_{cut}}{sin\theta}\}$

Without loss of generality assume the minimum is realized by $\frac{v_{cut}}{cos\theta}$ and $u_{cut} > \frac{v_{cut}}{cos\theta}$. Then there exists a segment γ in M of strictly shorter length from p to $exp_p(u_{cut}v)$ so that the product curve $(\gamma, u_{cut}wsin\theta)$ has length shorter than $(u_{cut}vcos\theta, u_{cut}wsin\theta)$. So $u_{cut} \leq \min\{\frac{v_{cut}}{cos\theta}, \frac{w_{cut}}{sin\theta}\}$. If $u_{cut} < \min\{\frac{v_{cut}}{cos\theta}, \frac{w_{cut}}{sin\theta}\}$ then we have extend-able segments, so we have shown equality,and can write $seg_{(p,q)}$ as follows

$$seg_{(p,q)} \equiv \{t(vcos\theta, wsin\theta) \mid v, \quad w \quad \text{unit vectors in } T_pM, T_qN,$$
$$\text{and } 0 \leq t < \ \min\{\frac{v_{cut}}{cos\theta}, \frac{w_{cut}}{sin\theta}\}\}$$

Using this notation it is clear that $seg_p \times seg_q \equiv seg_{(p,q)}$. $\blacksquare$

In [2] they use a very general volume comparison lemma that can be altered to give us specific conclusions. We use some of the same notation as in the paper. So let M be a compact, connected n-dimensional Riemannian manifold with $\sec M \geq k$ and $\mathrm{Diam} M \leq D$ and fix a point $p \in M$ and a point $\bar{p} \in S_k^n$. For any subset $Q \subset M$ denote the collection of all minimal geodesics from p to all points $q \in Q$ parametrized on $[0,1]$ by $\Gamma_{pQ} = \cup_{q \in Q} \Gamma_{pq}$. Let $\dot{\Gamma}_{pQ} \in T_p M$ be the corresponding collection of initial velocity vectors. Now using the identification of $T_p M$ and $T_{\bar{p}} S_k^n$ we define $\bar{Q} = exp_{\bar{p}}(\dot{\Gamma}_{pQ}) \subset S_k^n$ and by construction we see that $\dot{\Gamma}_{pQ} = \dot{\Gamma}_{\bar{p}\bar{Q}} \subset T_{\bar{p}} S_k^n$. The lemma states:

Lemma 5 ([2]) *With the notation above, consider functions $F, G : Q \times [0,\infty) \to [0,\infty)$ and via $exp_p \circ \exp_{\bar{p}}^{-1}$ corresponding functions $\bar{F}, \bar{G} : \bar{Q} \times [0,\infty) \to [0,\infty)$ where $G(q,\cdot)$ is nondecreasing for all $q \in Q$. Then for any $R \leq D$, the sets*

$$H = \{x \in \bar{B}(p,R) | F(q, d(x,p)) \leq G(q, d(x,q)), q \in Q\}$$

,

$$\bar{H} = \{\bar{x} \in \bar{B}_k^n(\bar{p},R) | \bar{F}(\bar{q}, d(\bar{x},\bar{p})) \leq \bar{G}(\bar{q}, d(\bar{x},\bar{q})), \bar{q} \in \bar{Q}\}$$

are related by $\mathrm{vol} H \leq \mathrm{vol} \bar{H}$.

Proof. let $v \in \dot{\Gamma}_{pH}$ where $x = exp_p(v) \in H$. Let $\bar{x} = exp_{\bar{p}}(v)$. Then for a $\bar{q} = exp_{\bar{p}}(u)$ and $q = exp_p(u)$ where $u \in \dot{\Gamma}_{pQ} = \dot{\Gamma}_{\bar{p}\bar{Q}}$ we have the following inequalities:

$$\bar{F}(\bar{q}, d(\bar{x}, \bar{p})) \overset{1}{=} F(q, d(\bar{x}, \bar{p})) \overset{2}{=} F(q, d(x, p)) \overset{3}{\leq} G(q, d(x, q)) \overset{4}{\leq} G(q, d(\bar{x}, \bar{q})) \overset{5}{=} \bar{G}(\bar{q}, d(\bar{x}, \bar{q}))$$

Where 1 and 2 follow by construction and 3 follows by assumption since $x \in H$. Number 4 is true since we know by Toponogov distance comparison that $d(x, q) \leq d(\bar{x}, \bar{q})$, then using the hypothesis that G is nondecreasing. and 5 follows from construction. This shows us that $H \subset exp_p \circ exp_{\bar{p}}^{-1}(\bar{H})$, which gives us the volume inequality $\mathrm{vol}(H) \leq \mathrm{vol}(exp_p \circ exp_{\bar{p}}^{-1}(\bar{H})) \leq \mathrm{vol}(\bar{H})$, where the last inequality comes from the Rauch Comparison Theorem. $\blacksquare$

Letting F and G be explicit functions we get some interesting volume comparison results.

Example 6 *Here we look at some specific functions for F and G.*

1. *Let $F(q, t) = t$ and $G(q, t) = R$ for all $(q, t) \in Q \times [0, \infty)$. We see that*

   $H = \{x \in \bar{B}(p, R) | d(x, p) \leq R\} = \bar{B}(p, R)$

   $\bar{H} = \{\bar{x} \in \bar{B}_k^n(p, R) | d(\bar{x}, \bar{p}) \leq R\} = \bar{B}_k^n(\bar{p}, R)$

2. *Let $F(q, t) = R$ and $G(q, t) = t$ for all $(q, t) \in Q \times [0, \infty)$ and considering H to be a subset of M as opposed to $\bar{B}(p, R)$. We see that*

   $$H = \{x \in M | d(x, q) \geq R\} = M - \bar{B}(Q, R), \quad \bar{H} = \{\bar{x} \in S_k^n | d(\bar{x}, \bar{q}) \geq R\} = S_k^n - \bar{B}(\bar{Q}, R)$$

3. *Let $F(q, t) = G(q, t) = t$ for all $(q, t) \in Q \times [0, \infty)$ and again considering H to be a*

subset of M as opposed to $\bar{B}(p,R)$ We see that

$$H = \{x \in M | d(x,p) \le d(x,Q)\} = \textit{Half-space conatining } \{p\},$$

$$\bar{H} = \{\bar{x} \in S_k^n | d(\bar{p},\bar{x}) \le d(\bar{x},\bar{Q})\}$$

This theorem is particularly useful when we let Q be a discrete set of points. Specifically looking at the second example above, we get the notion of a "Swiss Cheese" volume comparison which we define below. Looking at the third example we get a volume comparison for the union of balls which we also will define below. In [3] they define what they call a "Swiss Cheese" and have the following volume comparison lemma

Definition 7 ("Swiss Cheese") *Let $Q \subset M$ and $d : Q \longrightarrow \mathbb{R}_+$ be a function. We define the "Swiss Cheese" relative to the ball $D(p,R)$ as*

$$K((Q,d);(p,R)) \equiv D(p,R) - \cup_{q \in Q} B(q, d(q))$$

. When $D(p,R) = M$ we use the notation

$$K(Q,d) \equiv M - \cup_{q \in Q} B(q, d(q)).$$

Before we state the lemma we use the following set up to put a constant curvature k metric on seg_p. Consider a linear isometry $i : T_pM \to T_{\bar{p}}S_k^n$ and the composition $T_pM \xrightarrow{i} T_{\bar{p}}S_k^n \xrightarrow{exp_{\bar{p}}} S_k^n$. Using the restriction $exp_{\bar{p}} \circ i|_{\mathrm{seg}_p}$ we get a diffeomorphism with S_k^n except when $k = 1$ M is isometric to S^n. So from now on we consider seg_p to have a constant

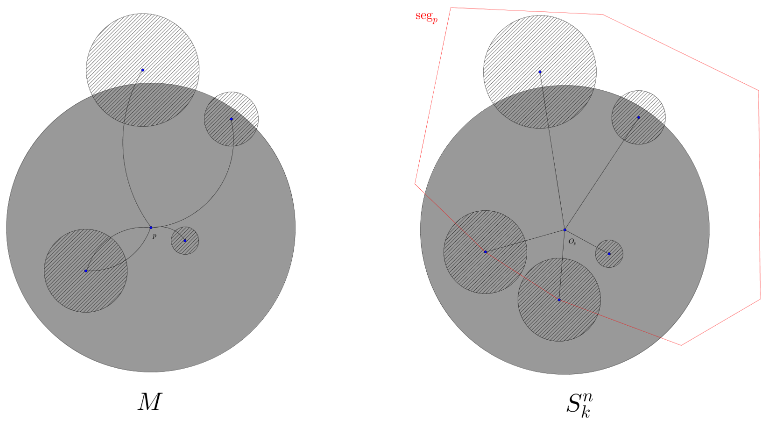

Figure 2.1: 'Swiss Cheese'

curvature k metric, and therefore we can view the exponential map $exp_p : \mathrm{seg}_p \to M$ as being distance non-increasing, by Toponogov's distance comparison.

Lemma 8 ("Swiss Cheese" Lemma) *[3]: Let M be an n-dimensional complete Riemannian manifold with $\sec M \geq k$. Let S_k^n be the n-dimensional space form of constant curvature k and $p \in M$. Identify seg_p with a closed subset of S_k^n. Then*

$$\mathrm{vol}(K(Q,d);(p,R)) \leq \mathrm{vol}(exp_p|_{\mathrm{seg}_p}^{-1} Q, d \circ exp_p);(0_p,R)).$$

Proof. Considering the "Swiss Cheese" as stated above and using the identification giving seg_p a constant curvature k metric we see that since exp_p is distance non increasing on the segment domain that:

$$K((Q,d),(p,R)) \subset exp_p(K((exp_p|_{\mathrm{seg}_p}^{-1} Q, d \circ exp_p);(0_p,R)))$$

And again since exp_p is distance non increasing on seg_p we see that

$$\mathrm{vol}(K(Q,d);(p,R)) \leq \mathrm{vol}(exp_p|_{\mathrm{seg}_p}^{-1} Q, d \circ exp_p);(0_p,R)).$$

$\blacksquare$

Lemma 9 ("Union of Balls" [3]) *Let M be a complete Riemannian Manifold with $\mathrm{sec} M \geq k$. Let $Q \subset M$, $d : Q \longrightarrow \mathbb{R}_+$. Define $D(Q,d) = \cup_{q \in Q} D(q, d(q))$. Then:*

$$vol D(Q,d) \leq vol D(I(Q), d \circ I^{-1})$$

provided that $I : Q \longrightarrow I(Q) \subset S_k^n$ is an isometry.

Proof. We note that if the balls are disjoint then the statement is clear by using Bishop's volume comparison. It suffices to show the case in which two balls overlap.

We know that for a ball $B(q,\rho)$ that $\mathrm{vol} B(q,\rho) \leq \mathrm{vol} B_k^n(\bar{q},\rho)$ and so to obtain the result we will show that

$$\mathrm{vol}\{B(p,R) - B(q,\rho)\} \leq \mathrm{vol}\{B_k^n(\bar{p},R) - B_k^n(\bar{q},\rho)\}$$

Once again, using the fact that $exp_p|_{\mathrm{seg}_p}$ is distance non-increasing we have that

$$B(p,R) - B(q,\rho) \subset exp_p(B_k^n(\bar{p},R) - B_k^n(\bar{q},\rho))$$

since by Hopf-Rinow $B(p,R) = exp_p(B((\bar{p}),R))$ and for $\bar{v} \in S_k^n$, $\mathrm{dist}(\bar{q},\bar{v}) \geq \mathrm{dist}(exp_p\bar{q}, \exp_p \bar{v})$ $= \mathrm{dist}(q, \exp_p \bar{v})$ tells us that if $exp_p\bar{v} \notin B(q,\rho)$ then $\bar{v} \notin B(\bar{q},\rho)$. Now that we have established the containment we use again that exp_p is distance non-increasing to get the volume result

$$\text{vol}\{B(p,R) - B(q,\rho)\} \leq \text{vol}\{B_k^n(\bar{p},R) - B_k^n(\bar{q},\rho)\}$$

■

Definition 10 *Let M be a complete Riemmanian manifold with* $\text{Rad}M \leq R$ *and* $\sec M \geq k$. *Then we say that M has maximal volume if* $\text{vol}M = v_k^n(R)$

2.2 The Sectional Curvature Case

We seek to how these volume comparisons transfer over to manifolds in the class $M_{k,r}^n$ = {n-dimensional complete Riemannian manifolds M with $\text{Ric}M \geq k(n-1)$ and $\text{Rad}M \leq r$}. We are primarily concerned with the volumes of small balls in this class of Riemannian manifolds. To understand this, we fist will discuss what happens when we have a stricter requirement, being, $\sec M \geq k$. This case is discussed in detail in [3]. For $k \in \mathbb{R}$, $n \geq 2$, consider closed Riemannian n-manifolds, M, with $RicM \geq k(n-1)$, and $radM \leq r$. Standard Volume Comparison gives you that

$$volM \leq v_k^n(r)$$

Equality occurring when $\sec M = 1$ and $r = \pi$ or $r = \pi/2$. Corresponding to S^n and $\mathbb{R}P^n$. There is actually another volume estimate obtained in the case where M is a complete closed

Riemannian n-manifold where $k > 0$ and $r > \frac{\pi}{2\sqrt{k}}$. Here we can say that

$$\text{vol}M \leq w_k^n(r) = \frac{r}{\pi/\sqrt{k}} v_k^n(\pi/\sqrt{k})$$

. We have the following results from [3]:

Theorem 11 *Fix a real number k, a positive r ($\leq \frac{\pi}{2\sqrt{k}}$ if k>0) and an integer n≥2. Then:*

(i)There is an $\epsilon = \epsilon(k, r, n) > 0$ such that any Riemannian n-manifold M with sec$M \geq K$, Rad$M \leq r$ and vol$M \geq v_k^n(r) - \epsilon$ is topologically either S^n or $\mathbb{R}P^n$.[Proved to be diffeomorphic to S^n or $\mathbb{R}P^n$ [9]] Moreover:

(ii)For every $\epsilon > 0$ there are Riemannian metrics on $M = S^n, \mathbb{R}P^n$ with sec$M \geq k$, Rad$M \leq r$ and vol$M \geq v_k^n(r) - \epsilon$.

Theorem 12 *Fix an integer $n \geq 2$, a positive k and $\pi/2\sqrt{k} < r \leq \pi/\sqrt{k}$. Then:*

(i_a)[Grove, Shiohama] Any Riemannian n-manifold M with sec$M \geq k$ and Rad$M > \frac{\pi}{2\sqrt{k}}$ is homeomorphic to S^n. (i_b) [Grove, Wilhelm] If we also assume vol$M \geq v_k^n(r) - \epsilon$, then we have that M is diffeomorphic to S^n. Moreover:

(ii) [Grove, Petersen] For every $\epsilon > 0$ there is a Riemannian metric on $M = S^n$ with sec$M \geq k$,Rad$M \leq r$ and vol$M \geq w_k^n(r) - \epsilon$.

This volume estimates above are optimal, but not realized by Riemannian manifolds except in the two cases of S^n and $\mathbb{R}P^n$. There are some important singular manifolds which do have maximal volume. We will call these spaces the "cross-cap", "purse", and the "lemon."

- **Curvature k cross-caps**

  let $k \in \mathbb{R}$,$r > 0$ ($\leq \frac{\pi}{2\sqrt{k}}$ if k >0) and consider the closed r-ball in S_k^n, denoted $D_k^n(r)$.

Let the map $A : D_k^n(r) \longrightarrow D_k^n(r)$ be reflection across the center and consider the space $C_{k,r}^n = D_k^n(r)/u \sim A(u)$ for $u \in \partial D_k^n(r)$

- **Curvature k purses**

  Let k and r be the same as the previous example and this time consider reflection R of $D_k^n(r)$ across a totally geodesic hyperplane, H, through the center. Let $P_{k,r}^n = D_k^n(r)/v \sim R(v)$ for $v \in \partial D_k^n(r)$.

- **Curvature k lemons**

  Let $k > 0$ and $\frac{\pi}{2\sqrt{k}} < r \leq \frac{\pi}{\sqrt{k}}$ and consider a totally geodesic S_k^{n-2}, two totally geodesic D_k^{n-1}'s with common boundary S_k^{n-2} and let $0 \leq \theta \leq \pi$ be the angle formed between the two D_k^{n-1}'s. Call the region formed between these two D_k^{n-1}'s $L_{k,r}^n(\theta)$, and consider $S_k^n \setminus L_{k,r}^n(\theta)$. Let R be the totally geodesic S^{n-1} that splits $L_{k,r}^n(\theta)$ in half and consider a map H that is identification across the hyperplane S^{n-1}.

  Essentially, we are taking a lune out of a sphere and pinching the remaining sphere shut, obtaining singularities across the pinched region and the corners of the pinch. This manifold by construction will have maximal volume and be an Alexandrov space with curvature bounded below by k.

In all these examples we have the geometric requirements and volume equal to $v_k^n(r)$ for the Cross-cap and the Purse, and equal to $w_k^n(r)$ for the Lemon. These examples are, however, not Riemannian manifolds, but we can consider smooth perturbations of these spaces to "smooth out" these singularities and get a Riemannian manifold with volume optimized by $v_k^n(r)$ or $w_k^n(r)$ respectively by embedding these space in an S_k^{n+1} and taking the boundary of a convex neighborhood. We then have the following result from [3] regarding convergence and almost maximal volume:

Theorem 13 ([3]) *Fix $n \geq 2$, $k \in \mathbb{R}$ $r >$ 0 and let $\{M_i\}$ be a sequence of closed Riemannian n-manifolds with sec$M_i \geq k$ and Rad$M_i \leq r$.*

(a) Suppose $\{$vol$M_i\}$ converges to $v_k^n(r)$, where $r \leq \frac{\pi}{2\sqrt{k}}$ if k>0. Then a subsequence of $\{M_i\}$ converges either to $C^n_{k,r}$ or $P^n_{k,r}$ in the Gromov-Hausdorff Topology.

(b) For $k >$0 and $\pi/2\sqrt{k} < r \leq \pi/\sqrt{k}$ suppose $\{$vol$M_i\}$ converges to $w_k^n(r)$. Then $\{M_i\}$ converges to $L^n_{k,2r}$ in the Gromov-Haudorff topology.

Here we will give a proof of Theorem 12 as given in [3], since we will be using similar techniques to prove the main theorem below. We start out with the following facts and use three lemmas to aid our proof.

Let $X = \lim M_i$. We seek to show that X is isometric to either $C^n_k(r)$ or $P^n_k(r)$. For each M_i choose a $p_i \in M_i$ that realizes the radius of M_i (i.e. $D(p_i, \mathrm{Rad}M_i) = M_i$). Then by standard volume comparison we know that $\mathrm{vol}M_i \leq v^n_k(\mathrm{Rad}M_i) \leq v^n_k(r)$ and that $r = \lim \mathrm{Rad}M_i = \mathrm{Rad}X$. We can also assume that p_i converges to $p \in X$ where p realizes the radius of X and that both $\mathrm{seg}_{p_i} \to \mathrm{seg}_p \subset S^n_k$ and $exp_{p_i} \to \big(exp_p : \mathrm{seg}_p \to X\big)$. We now prove the following lemmas which will establish our desired result.

Lemma 14 ([3]) *The exponential map exp_p described above satisfies the following:*

1. $\mathrm{seg}_p = D(\bar{p}, r) \subset S^n_k$

2. *$exp_p : B(\bar{p}, r) \to X$ is injective (The exponential map is injective on the interior of the segment domain)*

3. *$exp_p : D(\bar{u}, \varepsilon) \to X$ is an isometry whenever $D(\bar{u}, 2\varepsilon) \subset D(\bar{p}, r)$*

4. $exp_p : \partial D(\bar{p}, r) \to X$ is at most two-to-one.

Proof. To prove (1), we see that

$$\mathrm{vol}M_i \le \mathrm{vol}(\mathrm{seg}_{p_i}) \le \mathrm{vol}D(\bar{p}, r) = v_k^n(r)$$

by standard volume comparison and since $\mathrm{vol}M_i \to v_k^n(r)$ then $\mathrm{vol}(\mathrm{seg}_{p_i}) \to v_k^n(r)$. Since we have that $\mathrm{seg}_{p_i} \to \mathrm{seg}_p$ and $\mathrm{seg}_{p_i} \subset D(\bar{p}, r)$ then $\mathrm{seg}_p \subset D(\bar{p}, r)$ then $D(\bar{p}, r) \setminus \mathrm{seg}_p$ must have no interior points and must be empty.

To prove (2) we now use the "Swiss Cheese" lemma. Assume that $\bar{u}, \bar{v} \in int\mathrm{seg}_p$ with $\bar{u} \neq \bar{v}$, and $exp_p(\bar{u}) = exp_p(\bar{v})$. For $\varepsilon > 0$ choose two disjoint ε balls around each $\bar{u}$ and $\bar{v}$. Let $\bar{u}_i$ and $\bar{v}_i$ be sequences in seg_{p_i} with $\lim \bar{u}_i = \bar{u}$ and $\lim \bar{v}_i = \bar{v}$ and by our assumptions above $\lim exp_{p_i}(\bar{u}_i) = \lim exp_{p_i}(\bar{v}_i)$. We consider the following "Swiss Cheese":

$$M_i = K(exp_{p_i}(\bar{u}_i), exp_{p_i}(\bar{v}_i), \varepsilon) \cup (D(exp_{p_i}(\bar{u}_i), \varepsilon) \cup D(exp_{p_i}(\bar{v}_i), \varepsilon))$$

Then using the "Swiss Cheese" lemma and the Union of balls version as well we see that

$$\lim \mathrm{vol}M_i \le (v_k^n(r) - 2v_k^n(\varepsilon)) + v_k^n(\varepsilon)$$

, which is a contradiction to the maximal volume assumption. This proves (2). We prove (3) and (4) in a similar way using contradiction. For (3), let$\bar{u}, \bar{v} \in \mathrm{int}(\mathrm{seg}_p)$ with $d(\bar{u}, \bar{v}) = 2c > 0$. and assume that the open balls $B(\bar{u}, c), B(\bar{v}, c) \subset \mathrm{int}(\mathrm{seg}_p)$. We then seek to prove the claim that $d(exp_p(\bar{u}), exp_p(\bar{v})) = d(\bar{u}, \bar{v})$. Assume it is not true and for $\delta > 0$,

$$d(exp_p(\bar{u}), exp_p(\bar{v})) = d(\bar{u}, \bar{v}) - \delta$$

. Let $\bar{u}_i, \bar{v}_i \in \text{seg}_{p_i}$ with their limits being $\bar{u}, \bar{v}$ respectively and $\lim exp_{p_i}(\bar{u}_i) = exp_p(\bar{u})$, $\lim exp_{p_i}(\bar{v}_i) = exp_p(\bar{v})$. Using the same construction as above for M_i but replacing ε by c we get

$$\lim \text{vol} M_i \leq (v_k^n(r) - 2v_k^n(c)) + \text{vol}(D(\bar{q}_1, c) \cup D(\bar{q}_2, c))$$

where $\bar{q}_1, \bar{q}_2 \in S_k^n$ with $d(\bar{q}_1, \bar{q}_2) = 2c - \delta$. This contradicts maximal volume and proves (3). Again to prove (4) we assume for $q \in X \setminus B(p, r)$ we have three distinct points $\bar{u}, \bar{v}, \bar{w} \in exp_p^{-1}(q) \subset \partial\text{seg}_p$ Using the same methods we consider disjoint ε balls around each pre-image, $D(\bar{u}, \varepsilon), D(\bar{v}, \varepsilon), D(\bar{w}, \varepsilon)$ and pre-limits, in the M_i's approaching these points. From the fact that $M_i = K(q, \varepsilon) \cup D(q, \varepsilon)$ we get

$$\lim \text{vol} M_i \leq (v_k^n(r) - 3\bar{v}_k^n(\varepsilon)) + v_k^n(\varepsilon)$$

where the $\bar{v}_k^n(\varepsilon) = \text{vol} D(\bar{p}, r) \cap D(\bar{q}, \varepsilon)$ for $\bar{q} \in \partial\text{seg}_p$. Then for ε sufficiently small we have a contradiction, since n, r, k are fixed. This proves (4) and concludes the proof of the lemma.

■

We define the following relation R on the boundary of the segment domain. $\bar{u} \sim \bar{v}$ iff $exp_p(\bar{u}) = exp_p(\bar{v})$ for $\bar{u}, \bar{v} \in \partial\text{seg}_p$.

Lemma 15 ([3]) *Give the (n-1)-sphere, $\partial D(\bar{p}, r) = \partial\text{seg}_p$ the constant curvature Riemannian metric induced from $D(\bar{p}, r) = \text{int}(\text{seg}_p) \subset S_k^n$*

1. *The relation R defines an isometric involution on $\partial\mathrm{seg}_p$.*

2. *$exp_p : \mathrm{seg}_p \to X$ induces an isometry between the inner metric spaces seg_p/R and X.*

Proof. Any path in seg_p can be uniformly approximated by paths in $\mathrm{int}(\mathrm{seg}_p)$ and we have just seen by the previous Lemma that $exp_p : \mathrm{seg}_p \to X$ preserves lengths and is injective on the interior of the segment domain. So we need only prove the first part of the lemma. On $\partial\mathrm{seg}_p$, define $R(\bar{u}) = \bar{u}$ if $exp_p^{-1} \circ exp_p(\bar{u}) = \bar{u}$ and $R(\bar{u}) = \bar{v} \neq \bar{u}$ if $exp_p(\bar{u}) = exp_p(\bar{v})$ (since exp_p is at most two-to-one, this is well defined). Then the map $R : \partial\mathrm{seg}_p \to \partial\mathrm{seg}_p$ is continuous, since any point of discontinuity would lead to a splitting (bifurcating) geodesic. It is now clear that R is an involution that preserves lengths of paths since R is distance non-increasing and R^2=id. So R is an isometry. ■

Let D^n be the unit Euclidean disc centered at the origin and consider for each $0 \leq m \leq n$ representing $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ and the linear involution R_m for each $0 \leq m \leq n$ determined by $R_m|_{\mathbb{R}^n} = id.$ and $R_m|\mathbb{R}^{n-m} = -id.$

Lemma 16 ([3]) *The identification space $X_m^n = D^n/u \sim R_m u$ for $u \in \partial D^n$ is homeomorphic to the m-th suspension $\sum^m \mathbb{R}P^{n-m}$. In particular, X_m^n has the homology of a manifold if and only if $m = 0$ or $n-1$, corresponding to $\mathbb{R}P^n$ or S^n.*

Proof. The first part is clear if one views $D^n = \sum^m D^{n-m}$ and $\partial D^n = \sum^m \partial D^{n-m}$, with $R_m = \sum^m(-\mathrm{id}|_{\partial D^{n-m}})$. Since we see that $X_0^n \cong \mathbb{R}P^n, X_{n-1}^n \cong S^n$ and $X_n^n \cong D^n$ then we only need to show that X_m^n does not have the homology of a manifold for $1 \leq m \leq n-2$. In these cases, we see that

$$H_*(X_m^n) \cong H_{*-m}(\mathbb{R}P^{n-m}).$$

So in particular we have that X^n_m does not satisfy Poincar duality with $\mathbb{Z}_2$ coefficients. ∎

Putting these Lemmas together we have proved Theorem [13]. Now consider the following classes of manifolds:

$$\begin{array}{ccccc} \sec M \geq k & & \mathrm{Ric} M \geq k(n-1) & & \mathrm{Ric} M \geq k(n-1) \\ \mathrm{Rad} M \leq R & \subset & \sec M \geq k', (k' < k) & \subset & \mathrm{Rad} M \leq R \\ & & \mathrm{Rad} M \leq R & & \end{array}$$

We seek to understand why these volume comparisons do not hold in general but do hold in certain geometric configurations.

2.3 The Ricci Curvature Case

We first must mention Bishop-Gromov's relative volume comparison theorem.

Theorem 17 (Relative Volume Comparison) *Let M^n be a smooth be a smooth Riemannian manifold with $\mathrm{Ric} M \geq k(n-1)$ and let S^n_k denote the smooth simply connected n-dimensional space form of constant curvature k, and $v^n_k(r)$ be the volume of an r-ball in the space form. Then*

$$\frac{\mathrm{vol} B(p,r)}{v^n_k(r)} \searrow r$$

is a non-increasing function of r whose limit is 1 as $r \to 0$

Corollary 18 *Let M^n, S^n_k, and $v^n_k(r)$ be as above. Then for any star convex domain D centered at $p \in M$ and its lift $\bar{D} \subset S^n_k$*

$$\mathrm{vol} D \leq \mathrm{vol} \bar{D}$$

We also note the following result which is an exercise in [8].

Lemma 19 *[8] If a complete Riemannian manifold M with* $\mathrm{Ric}M \geq k(n-1)$ *contains a ball $B(p,R) \subset M$ that has maximal volume (i.e* $\mathrm{vol}(B(p,R)) = v_k^n(R)$*, then $B(p,R)$ has constant curvature.*

Before we prove this lemma we will need to cite a definition and another lemma, lemma 1.6, from [4]

Definition 20 *Let γ be a unit speed geodesic in a complete Riemannian n-manifold M, and let $\mathcal{J}$ be the vector space of normal Jacobi fields along γ. We define the Riccati operator, S, along γ, to be the map*

$$S_t : \mathcal{J} \to \mathcal{J}$$

$$S_t(J_1(t)) = J_1'(t)$$

If we consider Λ to be the subspace of $\mathcal{J}$ on which the Riccati operator is self adjoint, then for the set of times t so that

$$\{J(t)|J \in \Lambda\} = \mathrm{span}\{\gamma'(t)\}^{\perp}$$

then we get a well defined Riccati operator

$$S_t : \mathrm{span}\{\gamma'(t)\}^{\perp} \to \mathrm{span}\{\gamma'(t)\}^{\perp}$$

$$S_t(v) = J_v'(t)$$

where J_v is the unique Jacobi field in Λ with $J_v(t) = v$.

Lemma 21 *For $t \in [t_0, t_{max})$, Let $\hat{S}(t), \hat{R}(t) : V \to V$ be symmetric endomorphisms on a k-dimensional vector space V so that*

$$\dot{\hat{S}} + \hat{S}^2 + \hat{R} = 0.$$

Let $\tilde{\lambda}(t)$ be the eigenvalue of the Riccati operator of $\tilde{\Lambda}$ at time t, where $\tilde{\Lambda}$ is a Lagrangian family of normal Jacobi fields along a geodesic $\tilde{\gamma}$ in the space form.
Let $\tilde{\lambda}$ have no singularities on (t_0, t_{max}), and suppose that

$$\mathrm{tr}(\hat{S})(t_0) \leq k \cdot \tilde{\lambda}(t_0), \text{ and}$$

$$\mathrm{tr}(\hat{R})(t) \geq k \cdot \kappa$$

for all $t \in [t_0, t_{max})$. Then

$$\hat{S} \equiv \tilde{\lambda} \cdot \mathrm{id} \text{ and } \hat{R} = \kappa \cdot \mathrm{id},$$

if any of the following hold:

1. *If $\lim_{t \to t_{max}^-} \tilde{\lambda}(t) = -\infty$, then the statements about $\hat{S}$ and $\hat{R}$ are true on (t_0, t_{max}).*

2. *If $t_{max} = \infty, \kappa = 0 = \tilde{\lambda}(t_0)$, then the statements about $\hat{S}$ and $\hat{R}$ are true on $[t_0, \infty)$.*

3. *If $\mathrm{tr}(\hat{S})(t_1) = k \cdot \tilde{\lambda}(t_1)$ for some $t_1 \in (t_0, t_{max}]$, then the statements about $\hat{S}$ and $\hat{R}$ are true on (t_0, t_{max}).*

Proof of Lemma 19. Let $p \in M$ and γ_v be a geodesic leaving p at time $t = 0$ and in the direction of v. Since $B(p, R)$ has maximal volume then by Bishop-Gromov volume comparison for time $t = t_1$, with $dist(\gamma_v(t_1), p) = R$ the volume is maximal and $\det|(dexp_p)_{tv}|_{t=t_1}$ is maximal. If $\{\{J_i\}_1^{n-1}, \frac{\partial}{\partial t}\}$ is the frame along this geodesic γ_v where the J_i are the Jacobi fields with $J_i(0) = 0$, then we have that at $t = t_1$ that

$$\det|J_1, J_2, ..., J_{n-1}, \frac{\partial}{\partial t}|_{t=t_1} = \det|\overline{J_1}, \overline{J_2}, ..., \overline{J_{n-1}}, \frac{\partial}{\partial t}|_{t=t_1}$$

where the $\overline{J_i}$ are the Jacobi fields along $\overline{\gamma}_{\overline{v}}$ in the space form S_k^n. By identifying the tangent space T_pM with the constant curvature k space form, S_k^n we obtain these Jacobi fields through an isometry $i : T_pS_k^n \longrightarrow T_pM$ where $i(\overline{v}) = v$. By Bishop-Gromov, we know that

$$\frac{d}{dt}\left(\frac{\det|J_1, J_2, ..., J_{n-1}, \frac{\partial}{\partial t}|}{\det|\overline{J_1}, \overline{J_2}, ..., \overline{J_{n-1}}, \frac{\partial}{\partial t}|}\right)^2 < 0$$

along radial geodesics up until first conjugate point. Using this derivative we can see that:

$$\sum_{i=1}^{n-1} < J_i', J_i > \leq \sum_{i=1}^{n-1} < \bar{J}_i', \bar{J}_i >$$

Let S denote the Ricatti Operator on M and $\bar{S}$ denote the Riccati Operator on the space form S_k^n, then we get the inequality:

$$\mathrm{tr}(S) \leq \mathrm{tr}(\bar{S}).$$

Because of the maximal volume assumptions we have that for time $t = 0$ and $t = t_1$

$$\mathrm{tr}(S)(0) = \mathrm{tr}(\bar{S})(0) = (n-1) \cdot \tilde{\lambda}(0)$$

$$\mathrm{tr}(S)(t_1) = \mathrm{tr}(\bar{S})(t_1) = (n-1) \cdot \tilde{\lambda}(t_1)$$

Then by number (3) in Lemma 19 We have that

$$\hat{S} \equiv \tilde{\lambda} \cdot \mathrm{id} \ \ \textit{and} \ \ \hat{R} = \kappa \cdot \mathrm{id},$$

on (t_0, t_{max}).

This tells us that we have constant radial curvature on $B(p, R)$ which implies constant curvature on $B(p, R)$ ■

Remark 22 (Almost disjoint Balls/ Almost the same ball) *Let M be a Riemannian manifold with* $\mathrm{Ric}M \geq k(n-1)$ *and* $\mathrm{rad}M < R$. *Let $p, q \in M$ and consider the two balls $B(p, r), B(q, r) \subset M$ with $0 < r < R/2$. Let $\varepsilon_r > 0$ with $\varepsilon_r << r$. Consider the two cases where $d(p, q) = \varepsilon_r$ and $d(p, q) = 2r - \varepsilon_r$. We look at the volume of the union of these two balls,* $\mathrm{vol}B(p, r) \cup B(q, r)$. *In the first case, where $d(p, q) = \varepsilon_r$, we are looking at the volume of a union of two balls that are almost overlapping. In the second case, where $d(p, q) = 2r - \varepsilon_r$, we are looking at the volume of a union of two balls that are almost disjoint. In both cases, the union is less than that in the space form,* $\mathrm{vol}B(p, r) \cup B(q, r) \leq \mathrm{vol}B(p\bar{p}, r) \cup B(\bar{q}, r)$. *By Lemma [19], and Bishop-Gromov's relative volume comparison that* $\mathrm{vol}B(p, r) < \mathrm{vol}B(\bar{p}, r)$, *unless M has constant curvature on $B(p, r)$. Since the inequality is strict, by wriggling the ball a slight amount, giving us the*

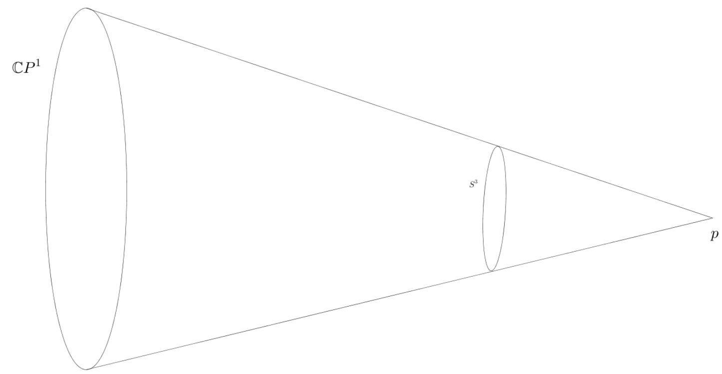

Figure 2.2: $\mathbb{C}P^2$

case where $d(p,q) = \varepsilon_r$*, then* $\mathrm{vol}B(p,r) \cup B(q,r) \leq \mathrm{vol}B(p\bar{p},r) \cup B(\bar{q},r)$*. In the same way if two balls are disjoint, then the volume of their union is a strict inequality and therefore, wriggling the balls so that they "kiss" will still give an inequality, giving us the case where* $d(p,q) = 2r - \varepsilon_r$*, and* $\mathrm{vol}B(p,r) \cup B(q,r) \leq \mathrm{vol}B(\bar{p},r) \cup B(\bar{q},r)$*.*

2.3.1 Counter-examples with a Lower Ricci Bound

Counter-Example to "Swiss-Cheese" Using $\mathbb{C}P^2$

We will now see why the "Swiss Cheese" Lemma [8] in general does not hold given a lower Ricci curvature bound by looking at $\mathbb{C}P^2$. We first note the following lemma.

Lemma 23 *Let* $p \in S^4_2$ *and consider the* S^3 *at distance* $\pi/2$ *from* p*. Consider the sub-sphere* $S^1 \subset S^3 \subset S^4_2$ *and the* S^1 *at maximal distance within the* S^3*, denoted by* $S^1_{\max}$*. Then the spherical suspension of* $S^1_{\max}$ *is the normal cut locus for the original* S^1*.*

Proof. Consider S^4_2 as the warped product $dt^2 + \left(\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\right)^2 ds_3^2$ and set $\psi(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$. If we think of our circle $S^1 \subset S(p,\pi/2) \equiv S^3 \subset S^4_2$ then there is an S^1 at maximal distance, denote by $S^1_{\max}$, within this metric S^3 at a distance $\frac{\pi}{2\sqrt{2}}\sin\sqrt{2}\pi/2$. We note that this distance is exactly half the Riemannian diameter of the metric $S^3 \subset S^4_2$ at distance $\pi/2$ from p. We look at geodesics leaving our $S^1 \subset S^3 \subset S^4_2$ orthogonally. To verify the normal cut locus of our S^1 is indeed $S^1_{\max}$, we look at the set of focal points for geodesics leaving our S^1 orthogonally and also check for crossing geodesics. To check for crossing geodesics consider the spherical sub-suspension of $S^1_{\max} \subset S^3 \subset S^4_2$. The spherical suspension gives us a totally geodesic S^2 inside of S^4_2. Take a geodesic in S^4_2 leaving S^1 orthogonally, then it sits in a spherical suspension of a geodesic in S^3. Taking two different points in S^3 they are in different geodesics between the north and south pole of S^4_2 and these geodesics do not meet before the north and south pole. These geodesics are in spherical suspensions of geodesics in S^3 and geodesics in S^3 do not meet before the times listed above. So if the geodesics leave S^1 orthogonally they will not meet again until the spherical suspension of $S^1_{\max}$. Putting these two facts together we have that geodesics leaving our S^1 orthogonally will not meet again until the spherical suspension of $S^1_{\max}$. To see that there are no focal points before the spherical suspension of $S^1_{\max}$, consider the normal Jacobi fields given by variations of geodesics leaving $S^1 \subset S^3 \subset S^4_2$ orthogonally at time $t = 0$. They are of the form $\psi(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)E(t)$, where $E(t)$ is a parallel field, and $f(t) = \frac{1}{\sin\frac{\sqrt{2}}{2}\pi}\sin(\sqrt{2}(-t+\frac{\pi}{2}))E(t)$. The second field is obtained by translating the Jacobi field $\psi(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)E(t)$ from p by $\pi/2$ and taking a negative, so that it is increasing in the direction back up to p. We need to scale appropriately so that at $t = 0$ we get

$f(0) = 1$ and the derivative at $f'(0)$ agrees with the shape operator for the metric sphere at $\frac{\pi}{2}$, which for S^4_2 is $\sqrt{2}\cot(\sqrt{2}t)$. The resulting field is the one above. The normal cut locus of S^1 inside of S^3 is S^1_{max}. If we look at the first focal cut times of any geodesic leaving S^1 orthogonally they do not occur before time $t = -(\frac{\pi}{\sqrt{2}} - \frac{\pi}{2})$ and $t = \frac{\pi}{\sqrt{2}}$, where these times are the first zeros of $f(t)$ and $\psi(t)$, respectively, and there are no zeros before the above times, so that there is not a focal point before these times, and that the parallel fields $f(t)$ and $\psi(t)$ are orthogonal. With these two facts together we have indeed shown that the spherical suspension of S^1_{max} is the normal cut locus to our S^1. ■

Consider $\mathbb{C}P^2$ with the normalized Fubini-Study Metric so that $1 \leq \sec \leq 4$ and $\text{Ric} \equiv 6$. We consider the space form S^4_2 with warping function $f(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$; giving $\sec \equiv 2$ and $\text{Ric} \equiv 6$. We seek to construct an example in which the "Swiss Cheese" comparison does not hold.

Let $p \in \mathbb{C}P^2$ and consider the ball $B(p, \pi/2)$. Then $\text{vol}B(p, \pi/2) = \text{vol}(\mathbb{C}P^2) = \pi^2/2$. If we consider the lifted ball $\bar{B}(\bar{p}, \pi/2) \subset S^4_2$, to calculate the volume we use:

$$\text{vol}(\bar{B}(\bar{p}, \pi/2)) = \text{vol}(S^3)\int_0^{\pi/2} \frac{1}{2\sqrt{2}}\sin^3(\sqrt{2}t)dt = \frac{\pi^2}{3} - \frac{\pi^2\cos(\frac{\sqrt{2}}{2}\pi)}{2} + \frac{\pi^2\cos^3(\frac{\sqrt{2}}{2}\pi)}{6}$$

By Bishop-Gromov, we know that $\text{vol}\mathbb{C}P^2 \leq \text{vol}(\bar{B}(\bar{p}, \pi/2))$ and in this case the difference is:

$$\text{vol}(\bar{B}(\bar{p}, \pi/2)) - \text{vol}\mathbb{C}P^2 < 0.978547$$

Now consider a point $q \in \mathbb{C}P^2$ with $dist(p,q) = \pi/2$ and the r-ball around that point.

$$\mathrm{vol}(B(q,r)) = \mathrm{vol}(S^3)\int_0^r \sin^2(t)\cdot\frac{1}{2}\sin(2t)dt$$

Using a double angle identity and substitution we get that

$$\mathrm{vol}(B(q,r)) = \frac{\pi^2}{2}\sin^4(r).$$

If we consider now the lifted $exp_p^{-1}(q) \subset S_2^4$ we get a great circle in the metric sphere, $S(\bar{p},\pi/2)$. Using the warping function we can calculate the circumference of this circle as $2\pi(\frac{1}{\sqrt{2}}\sin(\frac{\sqrt{2}}{2}\pi))$.

To calculate the volume of the r-tube about the metric sphere, we use the following:

$$\begin{aligned}
\mathrm{vol}(\bar{B}_{exp_p^{-1}(q)}(r)) &= \mathrm{vol}(S^2)(\text{circ. of circle})\int_0^r \frac{1}{2}\sin^2(\sqrt{2}t)\frac{1}{\sin(\frac{\sqrt{2}}{2}\pi)}\sin(\sqrt{2}(-t+\frac{\pi}{2}))dt \\
&= \mathrm{vol}(S^2)(2\pi(\frac{1}{\sqrt{2}}\sin(\frac{\sqrt{2}}{2}\pi)))\int_0^r \frac{\sin^2(\sqrt{2}t)}{2\sin(\frac{\sqrt{2}}{2}\pi)}\sin(\sqrt{2}(-t+\frac{\pi}{2}))dt \\
&= -\frac{1}{6}\pi^2(8\cos(\frac{\pi}{\sqrt{2}})+\cos(\frac{\pi-6r}{\sqrt{2}})-6\cos(\frac{\pi-2r}{\sqrt{2}})-3\cos(\frac{\pi+2r}{\sqrt{2}}))
\end{aligned}$$

This gives the volume of the r-tube. We note that the integral makes sense only up to the first cut point of the normal geodesics leaving the circle inside the metric sphere. If we consider a geodesic leaving orthogonally from the circle by lemma [23] we have that the first cut time does not occur before time $\pi/\sqrt{2}-\pi/2$, which is the distance to the south pole. (the antipodal point of p) Also, the only portion of the r-tube intersecting the ball $\bar{B}(\bar{p},\pi/2)$ is the directions towards $\bar{p}$ and the volume of that side of the r-tube is greater than the directions moving away from $\bar{p}$ since the values of the normal Jacobi fields are greater in that direction. So a lower bound for the portion of the r-tube that intersects the ball can be found by taking the above integral and multiplying by 1/2 to get

$$-\frac{1}{12}\pi^2(8\cos(\frac{\pi}{\sqrt{2}}) + \cos(\frac{\pi-6r}{\sqrt{2}}) - 6\cos(\frac{\pi-2r}{\sqrt{2}}) - 3\cos(\frac{\pi+2r}{\sqrt{2}}))$$

The volume here of the portion of the r-tube is greater than the volume of the r-ball in $\mathbb{C}P^2$ for small values of r between 0 and $\pi/\sqrt{2}-\pi/2$ and in fact when $0.55 < r_0 < \pi/\sqrt{2}-\pi/2 \approx 0.65$

$$\frac{1}{2}\mathrm{vol}(\bar{B}_{exp_p^{-1}(q)}(r_0)) - \mathrm{vol}(B(q,r_0)) = \frac{1}{2}\mathrm{vol}((\mathrm{r}_0 - \mathrm{tube})) - \mathrm{vol}(B(q,r_0)) > 0.983132$$

This is a lower bound on the difference between these two volumes, and the difference is greater than the difference $\mathrm{vol}(\bar{B}(\bar{p},\pi/2)) - \mathrm{vol}\mathbb{C}P^2 < 0.978547$. So even though the volume of the original ball in $\mathbb{C}P^2$ is smaller than in the space form S_2^4 we have taken more volume away in a lifted ball at maximal distance in S_2^4 than the ball taken away in $\mathbb{C}P^2$ and it exceeds the difference lacking in the original ball. Therefore we have

$$\begin{aligned}\mathrm{vol}(\bar{B}(\bar{p},\pi/2)) \setminus \mathrm{vol}(\bar{B}_{exp_p^{-1}(q)}(r_0)) &= \mathrm{vol}(\bar{B}(\bar{p},\pi/2)) \setminus \mathrm{vol}((\mathrm{r}_0 - \mathrm{tube}))) \\ &< \mathrm{vol}(B(p,\pi/2)) \setminus \mathrm{vol}(\bar{B}_q(r_0))\end{aligned}$$

Which is a counterexample to the “Swiss Cheese” lemma assuming only a lower Ricci curvature bound.

Union of Balls

Recall the union of balls lemma, which states:

Lemma 24 *Let M be a complete Riemannian Manifold with $secM \geq k$. Let $Q \subset M$, $d : Q \longrightarrow \mathbb{R}_+$. Define $D(Q, d) = \cup_{q \in Q} D(q, d(q))$. Then:*

$$volD(Q, d) \leq volD(I(Q), d \circ I^{-1})$$

provided that $I : Q \longrightarrow I(Q) \subset S_k^n$ is an isometry.

Remark 25 *We note that since Q is an arbitrary subset of M, we can take Q to be a continuous portion of a geodesic $\gamma(t) \subset M$, and also the radius function, d, can be held constant so that $d(\gamma(t)) = r$. Thus giving us that r-tubular neighborhoods of geodesics fall under the prescribed criteria in* [24].

We use the following Theorem from a paper by Alfred Gray in which he discusses the volume of small geodesic balls, [1]. We cite theorem 3.1 in [1].

Let M be an analytic Riemannian manifold and let $r > 0$ be small enough so that the exponential map is defined on a ball of radius r in each tangent space. Let $v_p(r)$ be the volume of an r-ball in M, and using the notation in the paper let

$$\tau(R) = \sum_{i=1}^{n} R_{ii}, \qquad ||R||^2 = \sum_{i,j,k,l=1}^{n} R_{ijkl}^2$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R_{i,j}^2 \qquad \Delta R = \sum_{i=1}^{n} \nabla_{ii}^2 \tau(R)$$

Let $v^{\mathbb{R}^n}(r)$ be the volume of a r ball in Euclidean space. Then theorem 3.1 in [1] states:

$$v_p(r) = v^{\mathbb{R}^n}(r)\left(1 - \frac{\tau(R)}{6(n+2)}r^2 + \frac{-3||R||^2 + 8||\rho(R)||^2 + 5\tau(R)^2 - 18\Delta R}{360(n+2)(n+4)}r^4 + O(r^6)\right)_p$$

Lemma 26 *For small $r, \varepsilon > 0$, r-tubes around short time geodesics $\gamma : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{C}P^n$, have smaller volume than the corresponding r-tube in the space form $S^{2n}_{\frac{2n+2}{2n-1}}$, where $\mathbb{C}P^n$ has the standard Fubini-Study metric.*

Proof. We break up into three cases. $n = 1, n > 2$ and $n = 2$.

n=1

This case is clear as when $n = 1$ the Ricci curvature is the same as the sectional curvature so the union of balls holds in the Ricci curvature comparison space as it is the same as the sectional curvature space form.

n=2

We start our by noting that this case is not resolved. We have to be slightly more careful as the volume of hyper-surfaces are larger in the manifold than in the space form. Consider $\mathbb{C}P^2$ with the normalized Fubini-Study Metric so that $1 \leq \sec \leq 4$ and $\mathrm{Ric} \equiv 6$. We consider the space form S^4_2 with warping function $f(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$; giving $\sec \equiv 2$ and $\mathrm{Ric} \equiv 6$. Let $\gamma(t)$ be a short time geodesic, $-\varepsilon \leq t \leq \varepsilon$ in $\mathbb{C}P^2$ and consider its isometric embedding into S^4_2. We will let $\gamma(t) = Q$ be the subset of $\mathbb{C}P^2$. Define $d : Q \to \mathbb{R}_+$ to be our radius function which will be constant and equal to a small value $0 < r < \varepsilon$ for $-\varepsilon \leq t \leq \varepsilon$ so that we are looking at the volume of a small r-tube around this geodesic for time $-\varepsilon \leq t \leq \varepsilon$. To do this we will consider the volume of the hyper-discs normal to the geodesic at time t_0.

Let $\gamma'(t)$ be the tangent vectors along this geodesic and $\{u_1, v_1, w_4\}$ be a frame along $\gamma(t)$ for the normal space. Where the subscript denotes the curvature of the plane spanned by $\gamma'(t)$ and the frame vector. If we consider the normal space and a hyper disc in this space of radius r, we use Gray's Theorem in [1] to compare these volumes. For the hyperplane in $\mathbb{C}P^2$ we have:

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R_{ijkl}^2 = 18$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R_{i,j}^2 = 54$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 12$$

$$\Delta R = \sum_{i=1}^{n} \nabla_{ii}^2 \tau(R) = 0$$

If we look now in the comparison space. All curvature planes are constant and equal to 2 and we get:

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R_{ijkl}^2 = 12$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R_{i,j}^2 = 48$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 12$$

$$\Delta R = \sum_{i=1}^{n} \nabla_{ii}^2 \tau(R) = 0$$

Now using the volume formula we have:

$$v_p(r) = v^{\mathbb{R}^3}(r)\left(1 - \frac{\tau(R)}{6(n+2)}r^2 + \frac{-3||R||^2 + 8||\rho(R)||^2 + 5\tau(R)^2 - 18\Delta R}{360(n+2)(n+4)}r^4 + O(r^6)\right)_{\gamma_{t_0}}$$

Since in both spaces the scalar curvature of the hyper-surface is the same the second order terms are the same. However the fourth order terms are different and it comes down to comparing $-3||R||^2 + 8||\rho(R)||^2$. The fourth order coefficient in $\mathbb{C}P^2$ is $\frac{438}{12{,}600}$ and the fourth order term in S^4_2 is $\frac{408}{12{,}600}$. In $\mathbb{C}P^2$ this value is larger so that the volume of the hyper-disc in $\mathbb{C}P^2$ is larger than the volume of the same hyper-disc in S^4_2 at time t_0, the difference being $v^{\mathbb{R}^3}(r)(\frac{30}{12{,}600}r^4 + O(r^6))$. Let H_{t_0} and $\bar{H}_{t_0}$ represent the hyper-discs in $\mathbb{C}P^2$ and S^4_2 respectively. Since the hyper-surfaces are not equidistant in $\mathbb{C}P^2$ we need to see how they vary along this geodesic. Let $L(s)$ be the Jacobi field along a geodesic leaving γ orthogonally at time t_0 so that $L(t_0) = \gamma'(t_0)$ and $J(s)$ be a normal Jacobi field to γ so that $J(t_0) = 0$ and $J'(t_0) \perp \gamma'(t_0) = L(t_0)$. Consider the Taylor expansion of $< J, L >$ evaluated at $s = t_0$. Then the coefficients are

$$\frac{d}{ds} < J, L > |_{s=t_0} =< J', L > |_{s=t_0} + < J, L' > |_{s=t_0} = 0$$

$$\frac{d^2}{ds^2} < J, L > |_{s=t_0} = [< J'', L > +2 < J', L' > + < J, L'' >]|_{s=t_0} = 0$$

$$\frac{d^3}{ds^3} < J, L > |_{s=t_0} =< J''', L > +3 < J', L'' > |_{s=t_0} \neq 0$$

So that $< J, L >= O(s^3)$. Now normalizing these fields in $\mathbb{C}P^n$ we have that $\frac{<J,L>}{||J||} = O(s^2)$. Now given a fixed direction for our normal geodesic to γ let θ_s be the angle between J and L as we move along this normal geodesic to γ, then $\cos(\theta_s) = O(s^2)$ which implies $\sin^2(\theta_s) =$

$1-\cos^2(\theta_s) = 1-O(s^4)$. This gives us that $\sin(\theta_s) = \sqrt{\sin^2(\theta_s)} = \sqrt{1-\cos^2(\theta_s)} = \sqrt{1-O(s^4)} = 1-O(s^4)$; which is the projection onto the normal space along the geodesic where $\sin(\theta_s)|_{s=t_0} = 1$. Now in general θ_s also depends on the direction of your normal geodesic so we have dependency $\theta_{s,\varphi}$ where φ is our variable direction in S^2. If $J_1(s), J_2(s)$ are normal Jacobi fields to γ at t_0 then we have the volume of the r-tube without the end caps can be found by integrating:

$$\int_{-\varepsilon}^{\varepsilon}\int_{S^2}\int_0^r |J_1\wedge J_2\wedge L| ds d\varphi dt = \int_{-\varepsilon}^{\varepsilon}\int_{S^2}\int_0^r |J_1\wedge J_2|\sin(\theta_{s,\varphi}) ds d\varphi dt$$

If we look at the same set up in the space form $S^{2n}_{\frac{2n+2}{2n-1}}$, then the order of $<\bar{J},\bar{L}>$ is $\geq O(s^4)$since in the set up above both terms in the third derivative are still zero, and therefore, $\sin(\theta_s)$ has order $\geq 1-O(s^6)$. If we look at the volume in the space form, because of the constant curvature we have that θ is only a function of s and we have the volume of the r-tube less the end caps is:

$$\int_{-\varepsilon}^{\varepsilon}\int_{S^2}\int_0^r |\bar{J}_1\wedge \bar{J}_2\wedge \bar{L}| ds d\varphi dt = \int_{-\varepsilon}^{\varepsilon}\int_{S^2}\int_0^r |\bar{J}_1\wedge \bar{J}_2|\sin(\theta_s) ds d\varphi dt$$

So even though using Gray's formulas we have that the volume of the hyper-surface is greater in $\mathbb{C}P^2$ than it is in S^4_2, we also have that the hyper-surfaces do not stay orthogonal to the geodesics and the order of that deviation is greater than that of the space form. So there is quite a bit more subtle analysis needed.

n>2

If we generalize the concept to $\mathbb{C}P^n$, we have

$$\int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |J_1 \wedge ... \wedge J_{2n-2} \wedge L| ds d\varphi dt = \int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |J_1 \wedge ... \wedge J_{2n-2}| \sin(\theta_{s,\varphi}) ds d\varphi dt$$

and in the space form $S^{2n}_{\frac{2n+2}{2n-1}}$

$$\int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |\bar{J}_1 \wedge ... \wedge \bar{J}_{2n-2} \wedge \bar{L}| ds d\varphi dt = \int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |\bar{J}_1 \wedge ... \wedge \bar{J}_{2n-2}| \sin(\theta_s) ds d\varphi dt$$

Now we use Gray's formulas to find the volume of the hyper-discs in these manifolds.

$$v_p(r) = v^{\mathbb{R}^{n-1}}(r)\left(1 - \frac{\tau(R)}{6(n+2)}r^2 + \frac{-3||R||^2 + 8||\rho(R)||^2 + 5\tau(R)^2 - 18\Delta R}{360(n+2)(n+4)}r^4 + O(r^6)\right)_{\gamma_{t_0}}$$

Since each space has the same constant Ricci curvature, the hyper-surfaces will have the same scalar curvature, and since the scalar curvature is constant the Laplacian will be 0. So again we want to look at how these volumes differ in each space. As $n \to \infty$ the space form curvature approaches 1. For the hyperplane in $\mathbb{C}P^n$ we have:

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R^2_{ijkl} = 2n^2 + 12n - 14$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R^2_{i,j} = (2n-2)^2[1 + \frac{(2n+1)^2}{2n-2}]$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 4n^2 - 4$$

$$\Delta R = \sum_{i=1}^{n} \nabla^2_{ii}\tau(R) = 0$$

and the hyperplane in $S^{2n}_{\frac{2n+2}{2n-1}}$ we have:

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R^2_{ijkl} = \frac{(2n+2)^2}{2n-1}(n-1)$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R^2_{i,j} = (2n-2)^2 \frac{(2n+2)^2}{2n-1}$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 4n^2 - 4$$

$$\Delta R = \sum_{i=1}^{n} \nabla^2_{ii} \tau(R) = 0$$

Substituting into the volume equation we get, for $\mathbb{C}P^n$,

$$v_p(r) = v^{\mathbb{R}^{n-1}}(r) \Bigg(1 - \frac{4n^2-4}{6(n+2)} r^2 + \frac{-3(2n^2+12n-14) + 8((2n-2)^2[1+\frac{(2n+1)^2}{2n-2}]) + 5(4n^2-4)^2}{360(n+2)(n+4)} r^4 + O(r^6) \Bigg)_{\gamma_{t_0}}$$

and in $S^{2n}_{\frac{2n+2}{2n-1}}$,

$$\bar{v}_p(r) = v^{\mathbb{R}^{n-1}}(r) \Bigg(1 - \frac{4n^2-4}{6(n+2)} r^2 + \frac{-3(\frac{(2n+2)^2}{2n-1}(n-1)) + 8((2n-2)^2\frac{(2n+2)^2}{2n-1}) + 5(4n^2-4)^2}{360(n+2)(n+4)} r^4 + O(r^6) \Bigg)_{\gamma_{t_0}}$$

If we consider the difference between the to volumes we have

$$v_p(r) - \bar{v}_p(r) = v^{\mathbb{R}^{n-1}}(r)\left(\frac{-3(9n-9-\frac{9(n-1)}{2n-1})+8(9-\frac{9}{2n-1})}{360(2n+1)(2n+3)}r^4 + O(r^6)\right)$$

and simplifying further we see that

$$v_p(r) - \bar{v}_p(r) = v^{\mathbb{R}^{n-1}}(r)\left(\frac{-(n-1)(3n-11)}{20(2n-1)(2n+1)(2n+3)}r^4 + O(r^6)\right)$$

We have for $n > 2$ the hyper-surface volume is smaller than the volume in the space form. Also, using the same justification as the $n = 2$ case we have that $\sin(\theta_{\varphi,s}) = 1 - O(s^4)$ in $\mathbb{C}P^n$ while $\sin(\theta_s) \geq 1 - O(s^6)$ in $S^{2n}_{\frac{2n+2}{2n-1}}$. So for small s $\sin(\theta_{\varphi,s}) > \sin(\theta_s) \geq 1 - O(s^6)$. Now going back to the integral for the volume of the r-tube less the end caps in $\mathbb{C}P^n$,

$$\int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |J_1 \wedge ... \wedge J_{2n-2}| \sin(\theta_{s,\varphi}) ds d\varphi dt$$

$$< \int_{-\varepsilon}^{\varepsilon}\int_{S^{2n-2}}\int_0^r |\bar{J}_1 \wedge ... \wedge \bar{J}_{2n-2}| \sin(\theta_s) ds d\varphi dt$$

Putting the end caps of the tube together you are comparing the volume of an r-ball in $\mathbb{C}P^n$ with the volume of an r-ball in $S^{2n}_{\frac{2n+2}{2n-1}}$. So by standard volume comparison that volume is also smaller. So together we have the for $n > 2$ and for small r the volume of an r-tube about a short time geodesic is smaller in $\mathbb{C}P^n$ than in the space form. $\blacksquare$

The $\mathbb{H}P^n$ case

We will now briefly take a look at Gray's formulas to consider hyper-surfaces now in $\mathbb{H}P^n$. In the $\mathbb{H}P^n$ case we are dealing with constant Ricci curvature $4n+2$ and comparison space $S^{4n}_{\frac{4n+2}{4n-1}}$. If we again, like the $\mathbb{C}P^n$ case, consider volumes of hyper-balls orthogonal to a short time geodesic, we use Gray's formulas to get

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R^2_{ijkl} = 8n^2 + 24n - 14$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R^2_{i,j} = (4n-2)^2[1 + \frac{(4n+1)^2}{4n-2}]$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 16n^2 - 4$$

$$\Delta R = \sum_{i=1}^{n} \nabla^2_{ii}\tau(R) = 0$$

and the hyperplane in $S^{4n}_{\frac{4n+2}{4n-1}}$ we have:

$$||R||^2 = \sum_{i,j,k,l=1}^{n} R^2_{ijkl} = \frac{(4n+2)^2}{4n-1}(2n-1)$$

$$||\rho(R)||^2 = \sum_{i,j=1}^{n} R^2_{i,j} = (4n-2)^2\frac{(4n+2)^2}{4n-1}$$

$$\tau(R) = \sum_{i=1}^{n} R_{ii} = 16n^2 - 4$$

$$\Delta R = \sum_{i=1}^{n} \nabla^2_{ii}\tau(R) = 0$$

Since again, each space has the same constant Ricci curvature, the hyper-surfaces will have the same scalar curvature, and since the scalar curvature is constant the Laplacian will be 0. Again here we experience the same phenomenon where in the Ricci tensor terms cancel in the cubic, quadratic and linear terms, while the curvature tensor terms cancel only in the quadratic terms. Using Gray's formulas for the volume approximation for the hyper-balls, we again see that in $\mathbb{H}P^2$ the volume of the hyper-ball is larger than the space form hyper-volume and for $n > 2$ the hyper-ball volume is smaller than that of the space form.

Remark 27 *Based on the similarity of the calculations in both $\mathbb{C}P^n$ and $\mathbb{H}P^n$, i believe it is safe to assume the same thing happens in the octonionic case, $\mathbb{O}P^n$, but we omit the calculations.*

Counter-example to Union of Balls

Theorem 28 *Given a complete, smooth Riemannian n-manifold M, consider the space of metrics, $\mathcal{M}(k) = \{g|\mathrm{Ric}_g \geq k(n-1)\}$ on M with the C^1 topology. Then within this space of metrics on M, let $\mathcal{B} \subset \mathcal{M}(k)$ be the sub-class of metrics in which there is a collection of balls whose union has a larger volume than the corresponding union in the space form, S_k^n. Then $\mathcal{B}$ is dense in $\mathcal{M}$ in the C^1 topology.*

Lemma 29 *Given $g \in \mathcal{B}$, to obtain a counter-example to the 'Union of Balls' inequality it is enough to find a geodesic in M so that along the geodesic the scalar curvature of the hyper-surfaces orthogonal to the geodesic are strictly smaller than that of the space form S_k^n*

Theorem 30 *Given a closed Riemmanian manifold (M, g), and a geodesic $\gamma : (-\eta, \eta) \longrightarrow M$, there is a $D > 0$ with the following property. Given $K, \varepsilon > 0$, there is a conformal* *change $\tilde{g}$ and a neighborhood U of the unit normal space to $\gamma(-\eta,\eta)$ so that:*

1. *For $t \in (-\eta,\eta)$ the scalar curvature of $\exp_{\gamma(t)}\{v \in TM | v \perp \acute{\gamma}(t)\}$ is $\leq -2K(n-2)(n-1)$ along γ.*

2. *$sec_{\tilde{g}} \geq -K - D$.*

3. $$\left|\left(\tilde{R}-R\right)(e_1,e_2,...e_n)\right| < \varepsilon$$

   except if up to symmetries of $\tilde{R}-R,(e_1,e_2,...e_n)$ corresponds to the sectional curvature of a plane with a vector in U.

Theorem 31 *Given a closed Riemmanian manifold (M,g), a distance function $f : U \subset M \longrightarrow (-2c,2c)$ and $\hat{K},\varepsilon > 0$, there is a Riemannian metric $\tilde{g}$ on M and an $\eta > 0$ with the following properties.*

1. *For all $t \in (-\eta,\eta)$, $\widetilde{\sec}|_{f^{-1}(t)}(\mathrm{grad} f,\cdot) \geq \hat{K}$.*

2. *For all planes P tangent to M*

$$\widetilde{\sec}(P) \geq \sec(P) - \varepsilon.$$

Moreover, η only depends on $\mathrm{Hess}_f|_{f^{-1}(-c,c)}$ and $\min\sec\left(f^{-1}(-c,c), g\right)$.

In this example we use constructions from [10], and [7] to increase the scalar curvature of hyper-surfaces to a geodesics while controlling the Ricci curvature along the geodesic to maintain a lower Ricci curvature bound in the manifold and giving us a counter-example to the volume of a Union of balls.

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq k(n-1)$. We will consider a conformal change of the metric on a tubular neighborhood of a short time geodesic in M. Let γ be a geodesic in M, with $\gamma : [-\varepsilon, \varepsilon] \to M$ and $\gamma(0) = p \in M$. Let $dist(\gamma, \cdot)$ be the distance from the geodesic and $inj\,(\gamma)$ be the normal injectivity radius. Letting Ω be an open subset of the tubular neighborhood $B(\gamma, \frac{inj\gamma}{2})$, $X = \mathrm{grad}(dist(\gamma, \cdot))$. Using the same notation as [10], we give $\nu\,(\gamma)$, the normal bundle of γ, the Sasaki metric. That is, the foot point map $\nu\,(\gamma) \longrightarrow \gamma$ is a Riemannian submersion, the metric on the vertical distribution comes from g, and the horizontal distribution, $\tilde{\mathcal{H}}$, is determined by normal parallel transport along γ. Let

$$\tilde{X} \oplus \tilde{\mathcal{V}}$$

be the orthogonal decomposition of the vertical distribution of $\nu\,(\gamma) \longrightarrow \gamma$, where $\tilde{X}$ is the radial, unit field from the 0–section, $\nu_0\,(\gamma)$, and $\tilde{\mathcal{V}}$ is the orthogonal complement of $\tilde{X}$. Set

$$\begin{aligned}
\mathcal{H} &\equiv d\exp_\gamma^\perp\left(\tilde{\mathcal{H}}\right), \\
\mathcal{V} &\equiv d\exp_\gamma^\perp\left(\tilde{\mathcal{V}}\right), \\
X &= d\exp_\gamma^\perp\left(\tilde{X}\right),
\end{aligned}$$

where $\exp_\gamma^\perp : \nu\,(\gamma) \longrightarrow M$ is the normal exponential map.

Note that $X \oplus \mathcal{V}$ is the tangent space to the fibers of the closest point map $Pr : \Omega \setminus \gamma \longrightarrow \gamma$, and on $\Omega \setminus \gamma$,

$$X = \mathrm{grad}\,(dist\,(\gamma, \cdot)))\,.$$

The distribution $\mathcal{H}$ need not be orthogonal to $X \oplus \mathcal{V}$; however, in [10] they show that it is asymptotically orthogonal to $X \oplus \mathcal{V}$ near γ, and hence is very close to $\overline{\mathcal{H}}$, the distribution that is orthogonal to $span\left\{X, \mathcal{V}\right\}$. Our conformal factor will have the form e^{2f}, where $f = \rho \circ \text{dist}\left(\gamma, \cdot\right)$, $\rho : [0, \infty) \longrightarrow \mathbb{R}$ is C^{∞}, satisfies $\rho|_{\left(\frac{inj(\gamma)}{2}, \infty\right)} \equiv 0$, which we further define later. We set $\tilde{g} = e^{2f} g$.

$$
\begin{aligned}
f' &\equiv \rho' \circ \text{dist}\left(\gamma, \cdot\right), \\
\text{grad} f &= f' X, \\
f'' &\equiv \rho'' \circ \text{dist}\left(\gamma, \cdot\right).
\end{aligned}
$$

We also recall the curvature tensor equations of a conformal change as in [11],

$$
\begin{aligned}
e^{-2f} \tilde{R}\left(V, Y, Z, U\right) &= R\left(V, Y, Z, U\right) - g\left(V, U\right) \text{Hess}_f\left(Y, Z\right) - g\left(Y, Z\right) \text{Hess}_f\left(V, U\right) \\
&\quad + g\left(V, Z\right) \text{Hess}_f\left(Y, U\right) + g\left(Y, U\right) \text{Hess}_f\left(V, Z\right) \\
&\quad + g\left(V, U\right) D_Y f D_Z f + g\left(Y, Z\right) D_V f D_U f \\
&\quad - g\left(Y, U\right) D_V f D_Z f - g\left(V, Z\right) D_Y f D_U f \\
&\quad - g\left(Y, Z\right) g\left(V, U\right) \left|\text{grad} f\right|^2 + g\left(V, Z\right) g\left(Y, U\right) \left|\text{grad} f\right|^2.
\end{aligned}
$$

To see that γ is still a geodesic we use formulas from [8]. If $\tilde{\nabla}$ represents the covariant derivative of the conformal metric, and X is tangent to our geodesic, then

$$
\tilde{\nabla}_X X = \nabla_X X + (D_X f) X + (D_X f) X - g(X, X) \nabla f
$$

and the last three terms on the right hand side are 0 since f is constant along γ and the gradient vanishes along the geodesic. The main result we seek to prove is an analog to a non-compact conformal change theorem as in [10]. We establish that $\tilde{g}, \tilde{\mathrm{R}}, \widetilde{\sec}, \widetilde{\mathrm{Ric}}$ will represent the metric and curvatures associated with the conformal change, while $g, \mathrm{R}, \sec, \mathrm{Ric}$ will represent the original metric and curvatures on the manifold.

Theorem 32 *Let (M, g), γ, X, and $\mathcal{V}$ be as above, and let $\mathcal{C}_1$, $\mathcal{C}_3$ be any compact subset of the image of γ with $\mathcal{C}_1 \subset \mathrm{Int}(\mathcal{C}_3)$. There is a D, $0 < D < \infty$ so that for any $\varepsilon > 0, K \in \mathbb{R}$ there are numbers σ_1, σ_3 with $0 < \sigma_1 < \sigma_3 < \frac{inj(\mathcal{C}_3)}{2}$, and a metric $\tilde{g} = e^{2f} g$ with the following properties.*

1. *Setting $\Omega_1 \equiv \exp_\gamma^\perp B\left(\nu_0(\gamma)|_{\mathcal{C}_1}, \sigma_1\right)$ and $\Omega_3 \equiv \exp_\gamma^\perp B\left(\nu_0(\gamma)|_{\mathcal{C}_3}, \sigma_3\right)$, the metrics $\tilde{g}$ and g coincide on $M \setminus \Omega_3$.*

2. *For all $Z \in T\Omega_1$ and all $V \in \mathrm{span}\{X, \mathcal{V}\}$*

$$K - D < \widetilde{\sec}(V, Z)|_{\Omega_1} < K.$$

3. *If $\{E_1, \ldots, E_n\}$ is a local orthonormal frame for Ω_3 with $X = E_1$ and span$\{E_2, \ldots, E_r\} = \mathcal{V}$ for $2 \leq r \leq n$, then*

$$\left| \tilde{R}(E_i, E_j, E_k, E_l) - R(E_i, E_j, E_k, E_l) \right| < \varepsilon,$$

except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \mathrm{span}\{X\} \cup \mathcal{V}$.

4.

$$\widetilde{\sec}(V,W) < \sec(V,W) + \varepsilon$$

for all $V, W \in TM$.

We first seek to prove the following adaptation of the *key lemma* as in [10].

Lemma 33 *There is a* $D > 0$ *so that for any* $\varepsilon > 0, K \in \mathbb{R}$ *there is a* $\delta > 0$, *and a* $\sigma_1 \in \left(0, \frac{inj\gamma}{2}\right)$ *so that the following holds. Suppose that for all* $Z \in T\Omega$, *for all* $V \in \mathrm{span}\{X, \mathcal{V}\}$, *and for some* $\sigma_3 \in \left(\sigma_1, \frac{inj\gamma}{2}\right)$,

$$\begin{array}{rl} (K+1) & |V|^2 |Z|^2 - D \leq \\ R(Z,V,V,Z) & |_{B(\gamma,\sigma_1)} - f''|_{B(\gamma,\sigma_1)} |Z|^2 \left|V^{\mathrm{span}\{X\}}\right|^2 - \dfrac{f'}{\mathrm{dist}(\gamma,\cdot)}|_{B(\gamma,\sigma_1)} \left|V^{\mathcal{V}}\right|^2 |Z|^2 \\ \leq & (K+1)|V|^2 |Z|^2 \end{array}$$

$$\begin{aligned} f' &\geq 0, \\ f''|_{B(\gamma,\sigma_1)} &\geq 0, \end{aligned}$$

$$\begin{aligned} |f| + |f'| &< \delta, \\ f'' &> -\delta, \\ f|_{M \setminus B(\gamma,\sigma_3)} &\equiv 0. \end{aligned}$$

Then

1.

$$K - D < \widetilde{\sec}\,(V, Z)\,|_{B(\gamma,\sigma_1)} < K.$$

2. *If* $\{E_1, \ldots, E_n\}$ *is a local orthonormal frame for* $B(\gamma, \sigma_3)$ *with* $X = E_1$ *and* *span*$\{E_2, \ldots, E_r\} = \mathcal{V}$ *for* $2 \leq r \leq n$*, then*

$$\left|\tilde{R}\,(E_i, E_j, E_k, E_l) - R\,(E_i, E_j, E_k, E_l)\right| < \varepsilon,$$

except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \operatorname{span}\{X\} \cup \mathcal{V}$.

3. *For all* $Z, W \in TM$.

$$\widetilde{\sec}\,(Z, W) < \sec\,(Z, W) + \varepsilon.$$

Because of our assumptions about f in lemma [33] we have the curvature tensor under a conformal change formula simplifies to

$$\begin{aligned} e^{-2f}\tilde{R}\,(V, Y, Z, U) &= R\,(V, Y, Z, U) - g\,(V, U)\,\mathrm{Hess}_f\,(Y, Z) - g\,(Y, Z)\,\mathrm{Hess}_f\,(V, U) \\ &\quad + g\,(V, Z)\,\mathrm{Hess}_f\,(Y, U) + g\,(Y, U)\,\mathrm{Hess}_f\,(V, Z) \\ &\quad \pm O\left(\delta^2\right)|V|\,|Y|\,|Z|\,|U|\,. \end{aligned}$$

Before we prove this lemma we need to understand some Hessian estimates which we cite from [10]

Lemma 34 *On* $\Omega \setminus \gamma$

1. $\mathrm{Hess}_f(X, X) = f''$.

2. *For* $Y \in \mathrm{span}\{X, \mathcal{V}, \mathcal{H}\}$ *and* $Z \in \mathcal{H}$ *and* $\delta > 0$

$$|\mathrm{Hess}_f(Y, Z)| < O(\delta)\,|Y|\,|Z|\,.$$

3. *For* $Y \in \mathcal{V}$ *and* $Z \in \mathrm{span}\{X, \mathcal{V}\}$

$$\left|\mathrm{Hess}_f(Y, Z) - \frac{f'}{\mathrm{dist}(\gamma, \cdot)} g(Y, Z)\right| \leq \delta O(\mathrm{dist}(\gamma, \cdot))\,|Y|\,|Z|$$

We also note the following corollary from [10], which comes from the lemma above and the fact that we can bound the inner product of Jacobi fields along geodesics leaving γ orthogonally.[[10] Proposition 2.8].

Corollary 35 *For* $\bar{Y} \in \mathrm{span}\left\{X, \mathcal{V}, \overline{\mathcal{H}}\right\}$*,* $\bar{Z} \in \overline{\mathcal{H}}$*, with foot-point in* $\Omega \setminus \gamma$ *sufficiently close to* γ*, and for* δ *as in Lemma 33*

$$\left|\mathrm{Hess}_f\left(\bar{Y}, \bar{Z}\right)\right| < O(\delta)\left|\bar{Y}\right|\left|\bar{Z}\right|.$$

proof of Lemma 33. From the conformal curvature equation we have for $Y \perp U$

$$\begin{aligned} e^{-2f}\tilde{R}(U, Y, Y, U) \;\leq\;& R(U, Y, Y, U) \\ & - g(U, U)\,\mathrm{Hess}_f(Y, Y) - g(Y, Y)\,\mathrm{Hess}_f(U, U) + \left|O\left(\delta^2\right)\right| |Y|^2\,|U|^2\,. \end{aligned}$$

and

$$
\begin{aligned}
e^{-2f}\tilde{R}\left(U,Y,Y,U\right) \quad \geq \quad & R\left(U,Y,Y,U\right) \\
& -g\left(U,U\right)\mathrm{Hess}_f\left(Y,Y\right)-g\left(Y,Y\right)\mathrm{Hess}_f\left(U,U\right)-\left|O\left(\delta^2\right)\right|\left|Y\right|^2\left|U\right|^2.
\end{aligned}
$$

Combining this with Lemma 34 we have for $Z \in T\Omega$ and all $V \in \mathrm{span}\left\{X, \mathcal{V}\right\}$ with $Z \perp V$

$$
\begin{aligned}
e^{-2f}\tilde{R}\left(Z,V,V,Z\right)|_{B(\gamma,\sigma_1)} \quad \leq \quad & R\left(Z,V,V,Z\right)|_{B(\gamma,\sigma_1)} \\
& -f''|_{B(\gamma,\sigma_1)}\left(\left|Z\right|^2\left|V^{\mathrm{span}\{X\}}\right|^2+\left|Z^{\mathrm{span}\{X\}}\right|^2\left|V\right|^2\right) \\
& -\frac{f'}{\mathrm{dist}\left(\gamma,\cdot\right)}|_{B(\gamma,\sigma_1)}\left(\left|V^{\mathcal{V}}\right|^2\left|Z\right|^2+\left|Z^{\mathcal{V}}\right|^2\left|V\right|^2\right) \\
& +\left|O\left(\delta\right)\right|\left|Z\right|^2\left|V\right|^2
\end{aligned}
$$

and on the other side

$$
\begin{aligned}
e^{-2f}\tilde{R}\left(Z,V,V,Z\right)|_{B(\gamma,\sigma_1)} \quad \geq \quad & R\left(Z,V,V,Z\right)|_{B(\gamma,\sigma_1)} \\
& -f''|_{B(\gamma,\sigma_1)}\left(\left|Z\right|^2\left|V^{\mathrm{span}\{X\}}\right|^2+\left|Z^{\mathrm{span}\{X\}}\right|^2\left|V\right|^2\right) \\
& -\frac{f'}{\mathrm{dist}\left(\gamma,\cdot\right)}|_{B(\gamma,\sigma_1)}\left(\left|V^{\mathcal{V}}\right|^2\left|Z\right|^2+\left|Z^{\mathcal{V}}\right|^2\left|V\right|^2\right) \\
& -\left|O\left(\delta\right)\right|\left|Z\right|^2\left|V\right|^2
\end{aligned}
$$

provided σ_1 is sufficiently small. Since we assumed that

$$\begin{array}{cc} (K+1) & |V|^2 |Z|^2 - D \leq \\ R(Z,V,V,Z) & |_{B(\gamma,\sigma_1)} - f''|_{B(\gamma,\sigma_1)} |Z|^2 \left| V^{\mathrm{span}\{X\}} \right|^2 - \frac{f'}{\mathrm{dist}(\gamma,\cdot)} |_{B(\gamma,\sigma_1)} \left| V^{\mathcal{V}} \right|^2 |Z|^2 \\ \leq & (K+1) |V|^2 |Z|^2 \end{array}$$

$f'|_{B(\gamma,\sigma_1)} \geq 0$, $f''|_{B(\gamma,\sigma_1)} \geq 0$, and $|f| < \delta$ we obtain

$$K - D < \widetilde{\sec}(V,Z) |_{B(\gamma,\sigma_1)} < K,$$

provided δ is sufficiently small.

Now consider, not necessarily distinct, orthonormal vectors $E, Y, Z, U \in \mathrm{span}\{X\} \cup \mathcal{V} \cup \bar{\mathcal{H}}$. Then

$$\begin{aligned} e^{-2f} \tilde{R}(E,Y,Z,U) &= R(E,Y,Z,U) \\ &\quad - g(E,U) \mathrm{Hess}_f(Y,Z) - g(Y,Z) \mathrm{Hess}_f(E,U) \\ &\quad + g(E,Z) \mathrm{Hess}_f(Y,U) + g(Y,U) \mathrm{Hess}_f(E,Z) \\ &\quad \pm O\left(\delta^2\right) |E| |Y| |Z| |U| . \end{aligned}$$

If we further assume that $R(E,Y,Z,U)$ does not correspond, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \mathrm{span}\{X\} \cup \mathcal{V}$, it then follows from Lemma 34 and Corollary 35 that all four Hessian terms are bounded from above by $O(\delta)$. So

$$e^{-2f}\tilde{R}(E,Y,Z,U) = R(E,Y,Z,U) \pm O(\delta)|E||Y||Z||U|.$$

We then get the second result of Lemma 33 by choosing δ to be sufficiently small.

On $M \setminus B(\gamma,\sigma_3)$ the third result of Lemma 33 follows from the hypothesis that $f|_{M\setminus B(\gamma,\sigma_3)} \equiv 0$. We get the third result on $B(\gamma,\sigma_3)$ by combining the second result of Lemma 33 and the first inequality stated in the proof above with Lemma 34, Corollary 35, and the hypothesis that $|f'| < \delta$ and $f'' > -\delta$.

■

We are now ready to prove an analog of Theorem 2.13 in [10] and we adopt the same notation from [10] which we state here.

Let (M,g) be a compact Riemannian n–manifold. Let γ be a geodesic in (M,g). Let $\mathcal{C}_1$ be a compact subset of γ. Let $inj(\mathcal{C}_1)$ be the injectivity radius of the normal bundle $\nu(\gamma)|_{\mathcal{C}_1}$. Let $\nu_0(\gamma)|_{\mathcal{C}_1}$ be the image of the zero section of $\nu(\gamma)|_{\mathcal{C}_1} \longrightarrow \mathcal{C}_1$. Let

$$\Omega \equiv \exp_\gamma^\perp\left(B\left(\nu_0(\gamma)|_{\mathcal{C}_1}, \frac{inj(\mathcal{C}_1)}{2}\right)\right),$$

and let

$$X \oplus \mathcal{V} \oplus \mathcal{H}$$

be the splitting of $T\Omega$ given in the beginning of the section.

For the remainder of our set up we consider a function $\rho : \mathbb{R} \to \mathbb{R}$, a $\tilde{D} > 0$ with $\tilde{D} < \max \sec_g - \min \sec_g$ and $0 < \sigma_1 < \sigma_2 < \sigma_3 < \infty$ such that

1. All derivatives of ρ of odd order at 0 are equal to 0.

2. $-\rho''(t)|_{[0,\sigma_1]} + \max \sec_g < K + 1.$

3. $(K+1) - \tilde{D} < -\rho''(t)|_{[0,\sigma_1]} + \min \sec_g .$

4. $\rho''(t)|_{[0,\sigma_2]} \geq 0$, $\rho'(t) \geq 0$.

5. $-\delta \leq \rho''|_{(\sigma_2,\infty)} < 0.$

6. $|\rho'| + |\rho| < \delta.$

7. $\rho|_{[\sigma_3,\infty)} \equiv 0.$

Lemma 36 *Let (M,g), γ, $\mathcal{C}_1$, X, and $\mathcal{V}$ be as above, and let $\mathcal{C}_3$ be any compact subset of the image of γ with $\mathcal{C}_1 \subset \mathrm{Int}(\mathcal{C}_3)$ and ρ as above. Let $\tilde{f} = \rho \circ dist(\cdot,\gamma)$. Then for $\delta > 0$ there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ so that $f = \varphi \cdot \tilde{f}$ satisfies*

$$|grad(f)| < O(\delta).$$

and

$$\mathrm{Hess}_f(V,W) = \varphi \mathrm{Hess}_{\tilde{f}}(V,W) + O(\delta)|V||W|.$$

Proof. Let $\mathcal{C}_2$ and $\mathcal{C}_4$ be compact subsets of γ with $\mathcal{C}_1 \subset \mathrm{Int}(\mathcal{C}_2)$, $\mathcal{C}_2 \subset \mathrm{Int}(\mathcal{C}_3)$, and $\mathcal{C}_3 \subset \mathrm{Int}(\mathcal{C}_4)$. Let $inj(\mathcal{C}_4)$ be the injectivity radius of the normal bundle $\nu(\gamma)|_{\mathcal{C}_4}$. Let $\bar{\varphi} : \gamma \longrightarrow [0,1]$ be C^∞ and satisfy

$$\bar{\varphi} = \begin{cases} 1 & \text{on } \mathcal{C}_2 \\ 0 & \gamma \setminus \mathcal{C}_3 \end{cases}.$$

Given $\sigma_4 \in \left(0, \frac{inj(\mathcal{C}_4)}{2}\right)$, extend $\bar{\varphi}$, by exponentiation, to a function φ, defined on $\exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_4}, \sigma_4\right)$ by setting

$$\varphi\left(x\right) = \bar{\varphi}\left(\text{footpoint}\left(\left(\exp_\gamma^\perp\right)^{-1}\left(x\right)\right)\right).$$

Our conformal factor is e^{2f}, where

$$f\left(x\right) \equiv \begin{cases} \left(\rho \circ \text{dist}\left(\gamma, x\right)\right) \cdot \varphi\left(x\right) & \text{for } x \in \exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_4}, \sigma_4\right) \\ 0 & \text{for } x \in \ M \setminus \exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_3}, \sigma_3\right) \end{cases}$$

and ρ is as above. Since $\left(\rho \circ \text{dist}\left(\gamma, x\right)\right) \cdot \varphi\left(x\right)$ is 0 on $\left(\exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_4}, \sigma_4\right)\right) \setminus \left(\exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_3}, \sigma_3\right)\right)$, f is a well defined C^∞ function.

Setting $\tilde{f} \equiv \rho \circ \text{dist}\left(\gamma, \cdot\right)$, we have that on $\exp_\gamma^\perp B\left(\nu_0\left(\gamma\right)|_{\mathcal{C}_4}, \sigma_4\right)$,

$$f = \varphi \cdot \tilde{f},$$

$$\text{grad}\left(f\right) = \varphi \text{grad}\left(\tilde{f}\right) + \tilde{f} \text{grad}\left(\varphi\right).$$

Since $\left|\tilde{f}\right|, \left|\text{grad}\left(\tilde{f}\right)\right| < \delta$ and $|\varphi| \leq 1$, we see from the above equation for the gradient that if δ is sufficiently small compared to $|\text{grad}\varphi|$, then

$$|\text{grad}\left(f\right)| < O\left(\delta\right).$$

Since $\operatorname{grad}(f) = \varphi \operatorname{grad}\left(\tilde{f}\right) + \tilde{f}\operatorname{grad}(\varphi)$,

$$\begin{aligned}
\operatorname{Hess}_f(V,W) &= g\left(\nabla_V\left(\varphi\operatorname{grad}\left(\tilde{f}\right) + \tilde{f}\operatorname{grad}(\varphi)\right), W\right) \\
&= (D_V\varphi)\, g\left(\operatorname{grad}\left(\tilde{f}\right), W\right) + \varphi g\left(\nabla_V\left(\operatorname{grad}\left(\tilde{f}\right)\right), W\right) \\
&\quad + \left(D_V\tilde{f}\right) g\left(\operatorname{grad}(\varphi), W\right) + \tilde{f} g\left(\nabla_V \operatorname{grad}(\varphi), W\right) \\
&= (D_V\varphi)\, D_W\tilde{f} + \varphi\operatorname{Hess}_{\tilde{f}}(V,W) \\
&\quad + \left(D_V\tilde{f}\right) D_W\varphi + \tilde{f}\operatorname{Hess}_\varphi(V,W).
\end{aligned}$$

Using $\left|\tilde{f}\right|, \left|\operatorname{grad}\left(\tilde{f}\right)\right| < \delta$ and choosing δ small compared to both $|\operatorname{grad}\varphi|$ and $|\operatorname{Hess}_\varphi|$ gives us

$$\operatorname{Hess}_f(V,W) = \varphi\operatorname{Hess}_{\tilde{f}}(V,W) + O(\delta)\,|V|\,|W|.$$

■

proof of Theorem 32. Given $\varepsilon > 0, K \in \mathbb{R}$ choose δ and σ_1 as in lemma [33]. Let σ_2, σ_3, and σ_4 be such that $\sigma_1 < \sigma_2 << \sigma_3 < \sigma_4 < \min\left\{\frac{inj(\gamma)}{2}, \frac{1}{4}\right\}$, and let $\rho : [0,\infty) \longrightarrow \mathbb{R}$ satisfy the conditions as in Lemma [36]. Let $f = \rho \circ \operatorname{dist}(\gamma, \cdot)$, Condition 1 on ρ gives us that our conformal factor e^{2f} is a smooth function on M. Following the same method of the proof in [10]

$$\begin{aligned}
-\rho'(t)\,|_{[0,\sigma_1]} &< (K + 1 - \max\sec_g)\, t, \text{ so} \\
-\frac{\rho'(t)\,|_{[0,\sigma_1]}}{t} + \max\sec_g &< (K+1).
\end{aligned}$$

For $V \in \operatorname{span}\{X, \mathcal{V}\}$, write $V = V^{\operatorname{span}\{X\}} + V^{\mathcal{V}}$. Then Condition 2 gives

$$-\rho''(t)|_{[0,\sigma_1]} \left|V^{\operatorname{span}\{X\}}\right|^2 + \max \sec_g \left|V^{\operatorname{span}\{X\}}\right|^2 < (K+1)\left|V^{\operatorname{span}\{X\}}\right|^2,$$

and the first inequality gives

$$-\frac{\rho'(t)|_{[0,\sigma_1]}}{t}\left|V^{\mathcal{V}}\right|^2 + \max \sec_g \left|V^{\mathcal{V}}\right|^2 < (K+1)\left|V^{\mathcal{V}}\right|^2.$$

Adding the previous two inequalities we get

$$-\rho''(t)|_{[0,\sigma_1]} \left|V^{\operatorname{span}\{X\}}\right|^2 - \frac{\rho'(t)|_{[0,\sigma_1]}}{t}\left|V^{\mathcal{V}}\right|^2 + \max \sec_g |V|^2 < (K+1)|V|^2.$$

Let $t = \operatorname{dist}(\gamma, \cdot)$, then $f' \equiv \rho'(t)$ and $f'' \equiv \rho''(t)$. Making these substitutions, multiplying both sides by $|Z|^2$, and using $R(Z,V,V,Z)|_{B(\gamma,\sigma_1)} \leq \max \sec_g |V|^2 |Z|^2$ gives

$$\begin{aligned} R(Z,V,V,Z)|_{B(\gamma,\sigma_1)} - f''|_{B(\gamma,\sigma_1)} |Z|^2 \left|V^{\operatorname{span}\{X\}}\right|^2 - \tfrac{f'}{\operatorname{dist}(\gamma,\cdot)}|_{B(\gamma,\sigma_1)} \left|V^{\mathcal{V}}\right|^2 |Z|^2 \\ \leq (K+1)|V|^2 |Z|^2. \end{aligned}$$

On the other side we have

$$\begin{aligned} -\rho'(t)|_{[0,\sigma_1]} &> \left((K+1) - \tilde{D} - \min \sec_g\right) t, \text{ so} \\ -\frac{\rho'(t)|_{[0,\sigma_1]}}{t} + \min \sec_g &> (K+1) - \tilde{D}. \end{aligned}$$

Using the same argument as above we eventually obtain

$$-\rho''(t)\,|_{[0,\sigma_1]}\left|V^{\mathrm{span}\{X\}}\right|^2-\frac{\rho'(t)\,|_{[0,\sigma_1]}}{t}\left|V^{\mathcal{V}}\right|^2+\min\sec_g|V|^2>\left(K+1-\tilde{D}\right)|V|^2.$$

using the same substitution as above and the fact that

$R(Z,V,V,Z)\,|_{B(\gamma,\sigma_1)}\geq\min\sec_g|V|^2|Z|^2$

$$\begin{aligned}R(Z,V,V,Z)\,|_{B(\gamma,\sigma_1)}-f''|_{B(\gamma,\sigma_1)}|Z|^2\left|V^{\mathrm{span}\{X\}}\right|^2-\tfrac{f'}{\mathrm{dist}(\gamma,\cdot)}|_{B(\gamma,\sigma_1)}\left|V^{\mathcal{V}}\right|^2|Z|^2\\ \geq\left(K+1-\tilde{D}\right)|V|^2|Z|^2.\end{aligned}$$

Both sides of this inequality establish the first inequality of [33]. The other hypotheses of [33] follow from the properties of ρ (numbered 3–6, above). We then apply [33] to obtain the curvature bounds of the theorem. Inequality of the gradient and the equation for the Hessian in lemma [36] allow us to argue, as above, to obtain the curvature estimates as stated in the theorem. ■

proof of Theorem [30.] Let (M,g) be a closed Riemannian manifold and $\gamma:(-\eta,\eta)\to M$ be a geodesic. Let $X,\mathcal{V},\mathcal{C}_1,\mathcal{C}_3,D,\varepsilon,\Omega_1,\Omega_3$ be as in Theorem [32], with U being Ω_1 and $K>0$. Then we can construct a metric $\tilde{g}$ so that for all vectors $\exp_{\gamma(t)}\{v\in TM|v\perp\acute{\gamma}(t)\}$ we have

$$-K-D<\widetilde{\sec}(v,\gamma')<-K$$

This proves part 2. Given $v,w\perp\gamma'(t)$ we have that

$$-2(K-D)<\widetilde{\sec}(v,w)<-2K.$$

So considering the scalar curvature of the hyper-surface orthogonal to γ we have that along γ the scalar curvature is $< -2K(n-2)(n-1)$. Part 3 of the Theorem comes form part 3 of Theorem [32]. ■

One other important tool we need comes from Lemma 6 in [7]. We use the method of orthogonal partial conformal change as presented in [7]. We make similar assumptions to theirs and they are For the remainder of this section we assume the following.

- M is complete.
- $\Omega \subset M$ is pre-compact and open.
- Let $h : \Omega \longrightarrow \mathbb{R}$ be smooth and the distance function from a closed sub-manifold $N \subset \partial\Omega$.
- Set $X \equiv \nabla h$, and define

$$\mathrm{II}\,(U,V) = -\frac{1}{2}\,(L_X g)\,(U,V) = -g\,(\nabla_U X, V) = -g\,(T\,(U)\,,V)$$

- Suppose that everywhere on Ω, X is tangent N.
- Let $\mathcal{D}$ be the distribution on Ω that is perpendicular to grad$h$
- Let

$$\varphi \equiv \varphi \circ h$$

  where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ is smooth and assume that $|\varphi - 1|$ is supported in Ω.

We use the following lemma from their paper, [7],

Lemma 37 *If $U, V, W, Z \perp X$ are vectors in T_pM with $p \in \Omega$, then*

1. If, in addition, $U, V, W, Z \perp \mathcal{D}$, then

$$\tilde{R}(U,V,W,Z) - R(U,V,W,Z) = O(|\varphi - 1|)\,|U|_g\,|V|_g\,|W|_g\,|Z|_g\,.$$

2. In general,

$$\begin{aligned}
\tilde{R}(U,V,W,Z) - R(U,V,W,Z) &= O\left(|\varphi - 1|\,(1 + |\mathrm{II}|) + |D_X\varphi|^2\right)|U|_g\,|V|_g\,|W|_g\,|Z|_g \\
&+ \varphi\,(D_X\varphi)\left(g\left(V^{\mathcal{D}}, W^{\mathcal{D}}\right)\mathrm{II}\,(U,Z)\right. \\
&+ \left. g\left(U^{\mathcal{D}}, Z^{\mathcal{D}}\right)\mathrm{II}\,(V,W)\right) \\
&- \varphi\,(D_X\varphi)\left(g\left(V^{\mathcal{D}}, Z^{\mathcal{D}}\right)\mathrm{II}\,(U,W)\right. \\
&+ \left. g\left(U^{\mathcal{D}}, W^{\mathcal{D}}\right)\mathrm{II}\,(W,Z)\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{R}(U,V,W,X) - R(U,V,W,X) &= O(|D_X\varphi| + |\varphi - 1|)\,|U|_g\,|V|_g\,|W|_g \\
&\quad + O(|\varphi - 1|)\,|\mathrm{II}|\,|U|_g\,|V|_g\,|W|_g
\end{aligned}$$

$$\begin{aligned}
\tilde{R}(U,X,X,U) - R(U,X,X,U) &= -\varphi\,(D_X D_X\varphi)\,g\left(U^{\mathcal{D}}, U^{\mathcal{D}}\right) \\
&\quad + O(|\varphi - 1| + |D_X\varphi|)\,|U|_g^2 + O(|\varphi - 1|)\,|\mathrm{II}|\,|U|_g^2
\end{aligned}$$

proof of Theorem 31. Let (M,g) be a closed Riemannian manifold and $f : U \subset M \to (-2c, 2c)$ be a smooth distance function. Using the construction for Lemma 6 in [7],Lemma

[37]. Choosing $\rho : \mathbb{R} \to \mathbb{R}_+$ with $|\rho - 1|$ supported in U, ρ being C^1 small and letting $\rho = \rho \circ f$ with $\rho'' >> 0$ inside $(-\eta, \eta)$. Then from Lemma [37], part 2, The curvature difference after our change in metric to $\tilde{g}$ is small except for the last difference in which it depends on how large ρ'' is.

$\tilde{R}(U, X, X, U) - R(U, X, X, U) =$

$$-\varphi (D_X D_X \varphi) g \left(U^{\mathcal{D}}, U^{\mathcal{D}}\right) + O\left(|\varphi - 1| + |D_X \varphi|\right) |U|_g^2 + O\left(|\varphi - 1|\right) |\mathrm{II}| \, |U|_g^2$$

The last difference gives us

$$\widetilde{\sec}(\mathrm{grad} f, \cdot) - \sec(\mathrm{grad} f, \cdot).$$

So given $\hat{K} > 0$ then we can choose ρ'' so that

$$\widetilde{\sec}(\mathrm{grad} f, \cdot) > \hat{K}.$$

From the other two differences in part 2 of Lemma [37], we have that the difference is small and given $\varepsilon > 0$, the η we choose depends on $\mathrm{Hess}_f | f^{-1}(-c, c)$ and and $\min \sec(f^{-1}(-c, c), g)$.

■

With the tools above we are ready to construct our counter example.

Proof of Lemma 29. Recall that in the 'Union of Balls' Lemma, [24], Q is an arbitrary subset of M, we can take Q to be a continuous portion of a geodesic $\gamma(t) \subset M$, and also the radius function, d, can be held constant so that $d(\gamma(t)) = r$. Thus giving us that r-tubular neighborhoods of geodesics fall under the prescribed criteria in [24]. Now applying Gray's formulas for the volume of small balls we have

$$v_p(r) = v^{\mathbb{R}^n}(r)\left(1 - \frac{\widetilde{\tau(R)}}{6(n+2)}r^2 + O(r^4)\right)_{\gamma(0)}.$$

If we consider the volume of hyper-balls orthogonal to our geodesic, then if the scalar curvature of the hyper-surface is smaller than that of the space form then the volume of the hyper-ball will be larger in the manifold than in the space form. In considering the end-caps of our r-tube, we note that it has a higher order volume difference so it is enough to consider the hyper-balls orthogonal to our geodesic. ■

proof of Theorem 28. Let (M, g) be a smooth, complete, closed Riemannian n-manifold with Ric$M \geq L$. Let γ be a geodesic and consider its restriction to a short time, $\gamma : [-\eta, \eta] \to M$. By Theorem [30] there is a $D > 0$ so that given a $K, \varepsilon > 0$ there is a conformal change $\tilde{g}$ and a neighborhood U of the unit normal space to $\gamma(-\eta, \eta)$ so that the scalar curvature of the orthogonal hyper-surfaces along γ are $\leq -2K(n-2)(n-1)$ while still keeping the sectional curvatures bounded below and not changing the curvature tensor by much anywhere else. However, with the new metric $\tilde{g}$ as prescribed in Theorem [30 we have changed our lower Ricci curvature bound, considerably lowering it for our example. Looking at what the Ricci curvature is, let w be orthogonal to $\gamma(0)$ and v tangent to $\gamma(0)$. Let $\{e_1, ..., e_{n-2}, v, w\}$ be an orthonormal frame and consider the $0-2$ Ricci tensor

$$\text{Ric}(v, w) = \sum_{i=1}^{n-2} R(e_i, v, w, e_i)$$

If $e_i = v$ or w then the term is 0, and for the rest of the vectors, they are orthogonal to v and w, and so the change $|\widetilde{\mathrm{Ric}}(v,w) - \mathrm{Ric}(v,w)|$ is as small as we like. So tracking the significant change in the lower Ricci curvature bound comes down to understanding $|\widetilde{\mathrm{Ric}} - \mathrm{Ric}|(v,v)$ and $|\widetilde{\mathrm{Ric}} - \mathrm{Ric}|(w,w)$, for the $K \in \mathbb{R}$ we are given. Since we also have that $\widetilde{\sec}(V,Z) > -K - D$ we get both an upper and lower bound on the Ricci curvature so as to prevent the curvature from blowing up to $-\infty$. These bounds are

$$(K - D)(n-1) < \widetilde{\mathrm{Ric}}(v,v) < K(n-1)$$

and

$$2(K-D)(n-2) + K < \widetilde{\mathrm{Ric}}(w,w) < 2K(n-2) + K$$

So now the trick is to bring back up the lower Ricci curvature bound while not changing the curvatures orthogonal to the image of γ, thus keeping the scalar curvature of the orthogonal hyper-plane very small.

To do this we use the method outlined in Theorem [31]. Pick a point $x \in M$ that is on the image of γ and is far away from $\gamma(-\eta)$ and $\gamma(\eta)$. Let f be our smooth distance function from x. Since x is a fixed distance from $\gamma([-\eta,\eta])$ we have that the second fundamental form, II, is bounded. This bound will determine the $\hat{\eta}$ for which we bring up the curvature along tangent planes to the geodesic in $\gamma(-\hat{\eta},\hat{\eta})$. So we choose x the appropriate distance away so that given a $\hat{K}, \hat{\varepsilon} > 0$ we get an $\hat{\eta}$ so that $\widetilde{\sec}|_{f^{-1}(t)}(\mathrm{grad} f, \cdot) \geq \hat{K}$ and $\gamma(-\eta,\eta) \subset \gamma(-\hat{\eta},\hat{\eta})$, while not changing the other tangent planes by more than $\hat{\varepsilon}$. For our specific example we will want $\hat{K} > \frac{(2(K-D)(n-2)+K)+L}{n-1}$. This brings back up the lower Ricci curvature bound so that $\widetilde{\text{Ric}}M \geq L$, while not changing the sectional curvatures orthogonal γ and keeping the scalar curvature very small.

Using the above construction we have taken $(M, g) \in \mathcal{M}(\frac{L}{n-1})$ and constructed a metric $\tilde{g}$, with $|g - \tilde{g}|_{C^1}$ small so that the scalar curvature, which we denote as $\widetilde{\tau(R)}$, of the hyper-surface orthogonal to γ within U at the geodesic is smaller than that of the space form S^n_L, and in fact can be chosen through this construction to be as disparate as we see fit. Now using Gray's volume formulas,

$$v_p(r) = v^{\mathbb{R}^n}(r)\left(1 - \frac{\widetilde{\tau(R)}}{6(n+2)}r^2 + O(r^4)\right)_{\gamma(0)}$$

we see that for r sufficiently chosen the volume of the hyper-surface ball at $\gamma(0)$ is much larger than that of the space form S^n_L since the scalar curvature is smaller. So for small tubular neighborhoods of the image of γ within U, the volume of the r-tube without it's end cap is greater than that of the volume in S^n_L. Since the end caps have a higher order volume difference to begin with it is enough to consider the tubular region without these caps. Using this fact and Remark [25] we have a counter example to the inequality in Lemma [24]. ■

Chapter 3

Maximal Focal Radius

3.1 Background

In a paper by Guijarro and Wilhelm [4], they show that the focal radius of any submanifold N of positive dimension in a manifold M with sectional curvature greater than or equal to 1 does not exceed $\frac{\pi}{2}$. In the case of equality, they show that N is totally geodesic in M and the universal cover of M is isometric to a sphere or a projective space with their standard metrics, provided N is closed. In their proof, the entire curvature tensor on the submanifold is used and we will see that relaxing this to only the radial curvatures of the submanifold being greater than 1 will not be enough to prove the result. Also, there are results in regards to closed submanifolds having infinite focal radius. We will explore the requirement of N being closed and the sectional curvature requirement to obtain some interesting counter examples when relaxing these conditions. This result will also apply to the other Theorems proved in Guijarro and Wilhelm's paper. First, we start with the definition of focal point and focal radius.

Definition 38 *Let $N \subset M$ be a submanifold of a Riemannian manifold M. A point $q \in M$ is a* ***focal point*** *of $N \subset M$ iff it is a critical point of $exp^{\perp}$.*

Definition 39 *The* ***focal radius*** *of a submanifold M is the infimum of focal distances over all points in N.*

Using the definition as in [4],

Definition 40 *A Riemannian manifold M has k^{th}* ***intermediate Ricci curvature*** *$\geq l$, denoted $\mathrm{Ric}_k \geq l$, if for any orthonormal $k+1$-frame, $\{v, E_1, E_2, ..., E_k\}$ the sectional curvature sum, $\sum_{i=1}^{k} \sec(v, E_i)$ is greater than l.*

3.2 Theorems and Results

Theorem 41 ([4]) *Let M be a complete Riemannian n-manifold with sectional curvature ≥ 1; and let N be any closed, embedded submanifold of M with $\dim(N) \geq 1$ and focal radius $\frac{\pi}{2}$. Then the universal cover of M is isometric to the sphere or a projective space with the standard metrics.*

Theorem 42 *There are smooth Riemannian n-manifolds that contain closed embedded submanifolds with radial curvatures ≥ 1 and focal radius $\frac{\pi}{2}$ and whose universal covers are not isometric to the sphere or projective space.*

Theorem 43 ([4]) *Let M be a complete Riemannian n-manifold with $Ric_k \geq 0$, and let N be any closed submanifold of M with $dim N \geq k$ and infinite focal radius.*

1. *N is totally geodesic*

2. *The normal bundle* $\nu(N)$ *with the pull back metric* $(exp_N^\perp)^*(g)$ *is a complete manifold with* $Ric_k \geq 0$.

3. $exp_N^\perp : (\nu(N), (exp_N^\perp)^*) \longrightarrow (M, g)$ *is a Riemannian cover.*

4. *The zero section* N_0 *is totally geodesic in* $(\nu(N), (exp_N^\perp)^*(g))$

5. *the projection* $\pi : (\nu(N), (exp_N^\perp)^*(g)) \longrightarrow N$ *is a Riemannian submersion.*

6. *If* $c : I \longrightarrow N$ *is a unit speed geodesic in* N *and* V *is a parallel normal field along* c*, then*

$$\Phi : I \times \mathbb{R} \longrightarrow M, \qquad \Phi(s,t) = exp_{c(s)}^\perp(tV(s))$$

is a totally geodesic immersion whose image has constant curvature 0.

7. *All radial sectional curvatures from* N *are nonnegative. That is, for* $\gamma(t) = exp_N^\perp(tv)$ *with* $v \in \nu(N)$*, the curvature of any plane containing* $\gamma'(t)$ *is nonnegative.*

8. *For all* $r > 0$*, the intrinsic metric on the* r*-sphere,* $S(N,r)$*, around* N *in* M *has* $Ric_k \geq 0$.

Theorem 44 *There are smooth Riemannian n-manifolds* M *with* $Ric_k \geq 0$*, and open submanifolds* N *in* M *with infinite focal Radius that do not satisfy all criteria of Theorem [43].*

3.2.1 Counterexamples

The Pill

Our first example we will construct we shall call the "Pill". Consider two hemispheres of the standard sphere S^2 and for $\varepsilon > 0$ glue a small curvature 0 cylinder of height $\varepsilon/2$ and radius 1 along their boundary and identify the boundaries of the cylinders by the identity map, thus creating a "Pill" shape. Consider the Jacobi fields along radial geodesics leaving the North Pole to be :

$$\begin{cases} J(t) = sin(t) & 0 \leq t \leq \frac{\pi}{2} \\ J(t) = 1 & \frac{\pi}{2} < t < \frac{\pi}{2} + \varepsilon \\ J(t) = sin(t - \varepsilon) & \frac{\pi}{2} + \varepsilon < t < \pi + \varepsilon \end{cases}$$

However, this example is only C^1. In general, to construct our example, let f be C^∞ on $\mathbb{R}$ and have the following conditions. For $\varepsilon > 0$,

$f'' < 0$ for $0 \leq t \leq \frac{\pi}{2}$ and $\frac{\pi}{2} + \varepsilon \leq t \leq \pi + \varepsilon$

f is constant on $\frac{\pi}{2} \leq t \leq \frac{\pi}{2} + \varepsilon$

$f'(0) = 1$ $f'(\pi + \varepsilon) = -1$

$f^{(2n)}(0) = f^{(2n)}(\pi + \varepsilon) \equiv 0$

Then the metric $dt^2 + f(t)^2 d\theta^2$ will be a smooth manifold with sec$M \geq 0$. Let our open submanifold with infinite focal radius be a radial geodesic between times $\pi/2$ and $\pi/2 + \varepsilon$. Condition (2) does not hold, condition (3) is only a cover onto its image, thus the necessity that N be closed.

We can also consider a similar construction by considering the metric $dt^2+f(t)^2d\theta_1^2+g(t)^2d\theta_2^2$

for smooth functions f and g, where:

$$\begin{cases} f'' < 0, 0 \leq t \leq \frac{\pi}{2} & g \text{ constant } 0 \leq t \leq \frac{\pi}{2} + \varepsilon \\ f \text{ constant } \frac{\pi}{2} \leq t \leq \pi & g'' < 0, \frac{\pi}{2} + \varepsilon \leq t \leq \pi + \varepsilon \\ f(0) = 0 & g(\pi + \varepsilon) = 0 \\ f'(0) = 1 & g'(\pi + \varepsilon) = -1 \\ f^{(2n)} \equiv 0 & g^{(2n)} \equiv 0 \end{cases}$$

Here again we consider a radial geodesic between times $\pi/2$ and $\pi/2 + \varepsilon$. The geodesics normal focal radius again will be infinite and the same conditions fail from Theorem [43] as above.

Before we go on to the next counter example, we have the following definition

Definition 45 *Let X be an Alexandrov space. Let (S, g) be a Riemannian manifold. Let $dist^X$ be the distance of X, and let $dist^S$ be the distance on S induced by g. An embedding $i : (S, g) \hookrightarrow X$ is smooth and isometric if the following hold.*

1. There is a neighborhood U of the diagonal, $\Delta(S) \subset S \times S$, so that $dist^X$ is smooth on $U - \Delta S$

2. For every $\varepsilon > 0$ there is $\delta > 0$ so that

$$|D_v i^*(dist^X(\cdot, \cdot)) - D_v dist^S(\cdot, \cdot)| < \varepsilon$$

for all unit $V \in T(B(\Delta S, \delta) - \Delta S)$. Here D_v is the directional derivative operator associated to V, and $B(\Delta S, \delta)$ is the δ-neighborhood of the diagonal in $S \times S$ with respect to $dist^S$.

The Cube

The next example we consider is by looking at the surface of a cube $[0,1]^3 \subset \mathbb{R}^3$ This is an Alexandrov space with singularities at the corner points. The intrinsic metric on the cube is (apart from the corner points) locally isometric to $\mathbb{R}^2$. However, if we take a point p close to an edge on one face and another point q close to p but on the adjacent face of the cube, then the directional derivative of the distance function as described in part 2 of [45] approaches 0 in the cube, but $\sqrt{2}$ in $\mathbb{R}^3$ so we have some smoothing to do to construct a counter example. Let p be a point on the center of one of the faces and consider a direction v from p such that the segment γ_v starting at p hits the edge of the cube perpendicularly. Follow this path around the cube until you again hit p to get a closed subspace N of the cube. Consider a small open subset $U \subset N$ containing p. Then U has infinite focal radius. If we can somehow smooth the corners of this cube we have constructed an example of why closed is necessary for Theorem 2. To smooth the corners of the cube we consider a smoothing of $|x|$. Let f be a smooth function such that $f(0) = 0$ and f agrees with $|x|$ outside $(-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$. Take this curve and rotate it about the y-axis to get our desired smoothing of the corners.

Warped Projective Space

Here we will look at how Theorem [41] is affected when we look at only radial curvatures being greater than or equal to 1. Consider $\mathbb{C}P^2$ with the standard quarter pinched metric so that $1 \leq \sec \leq 4$. Using the same notation as in [8], we write the metric as:

$$dt^2 + \sin^2(t)g + (\frac{1}{2}\sin(2t))^2 h$$

Where h is the metric along the hopf fiber and g is the metric on hopf$^\perp$. We will change the metric along the hopf fiber by multiplying by a function, $f(t, \varphi)$, where φ can be thought of as a distance function on $\mathbb{C}P^1$ where the domain of φ is between $[0, \pi/2]$, and in a neighborhood around $t = 0$ and $t = \pi/2$, f is constant and equal to 1. We constrain our function by making $|f(t, \varphi) - 1|_{C^2} < \varepsilon$. Where ε is chosen based on our neighborhood of the endpoints where f is constant and the maximum of $\cot(2t)$ outside those neighborhoods. We then get the metric:

$$dt^2 + \sin^2(t)g + (f(t, \varphi)\frac{1}{2}\sin(2t))^2 h$$

If we let X be unit tangent to hopf$^\perp$ and Y be a unit tangent vector to our hopf fiber and look at the radial curvatures from a point p we get that they are of the form:

$$\begin{aligned}
\sec(\partial/\partial t, X) = -\frac{[\sin(t)]''}{\sin(t)} &= 1 \\
\sec(\partial/\partial t, Y) = -\frac{[1/2 f(t,\varphi)\sin(2t)]''}{1/2 f(t,\varphi)\sin(2t)} &= 4 - 4\frac{\frac{d}{dt}f(t,\varphi)}{f(t,\varphi)}\cot(2t) - \frac{\frac{d^2}{dt^2}f(t,\varphi)}{f(t,\varphi)}
\end{aligned}$$

Since f is constant on a neighborhood of 0 and $\pi/2$ we have that at 0 and $\pi/2$ that $f'(t)|_{t=0, t=\pi/2} = 0$. Since f is C^2 close to 1 with the appropriate choice of ε we can maintain control of the radial curvatures in the hopf fiber direction so that $\sec(\partial/\partial t, Y) \geq 1$.Since in this construction we only care about radial curvatures we note that X and Y are eigenvectors of the Jacobi operator so that the curvatures of any other direction is a spherical combination of X and Y. By our construction $\mathbb{C}P^2$ is not isometric to it's universal cover with the product metric as stated at the beginning of this section. The universal cover of $\mathbb{C}P^2$ of course being itself, $\mathbb{C}P^2$. In fact if we look at our special point p and consider the $\mathbb{C}P^1$ at maximal distance to be our sub-manifold. Then $\mathbb{C}P^1$ is an example of a closed sub-manifold in $\mathbb{C}P^2$ which has focal radius exactly $\pi/2$ but is not isometric to itself as it's own universal cover. This shows us that having only radial curvatures greater than 1 is not enough to reach the conclusion of Theorem [41]. Also we note that any open subset of $\mathbb{C}P^1$ will have focal radius $\pi/2$.

Chapter 4

Conclusions

In conclusion we have seen why we can not in general extend the volume comparison lemmas in [3] to a lower Ricci curvature bound, and therefore use them to conclude general results about maximal volume and a lower Ricci curvature bound. While These volume comparison lemmas do not hold in general it is interesting to note that for very small balls and either very slight intersection or almost complete overlap the union of balls does hold. Also, how considering an infinite union of balls along a geodesic leads us to counter examples in the union of balls example, but that in certain manifolds, like $\mathbb{C}P^2$, it does hold. In our counterexample of 'Union of Balls' construction we noted that the given a complete Riemannian manifold M^n, and all metrics g which give Ric$M \geq k(n-1)$. The the space of metrics $g_{\text{counter}} \subset g$ which contain a counter example to the 'Union of Balls' is dense in C^1 topology. So while this tells us that given a Riemmanian manifold with Ricci bounded below there is a counter example, given a random configuration of balls, the chance that it is a counter example is actually very small. It would be interesting to determine the exact geometric obstructions needed in a lower Ricci curvature bound case in order for these volume comparison lemmas to hold. It seems from this discourse that requiring the manifold to be Einstein is a start. While we have also made progress in obstructions for maximal focal radius and topological type, we still seek to find an example of a manifold M where all sectional curvatures are bounded below by 1 and an open submanifold N which has focal radius $\pi/2$ and M is not isometric to the sphere of projective space.

In future work, I hope to continue to study manifolds with a lower Ricci curvature bound and comparison geometry. Also, further study applications of focal radius to topological type.

Bibliography

[1] Alfred Gray. The volume of a small geodesic ball of a riemmanian manifold. *Michigan Math. J.*, 20:329–344, 1973.

[2] K. Grove and P. Petersen. Manifolds near the boundary of existence. *J. Differential Geometry*, 33:379–394, 1991.

[3] K. Grove and P. Petersen. Volume comparison a la alexandrov. *Acta Math*, 169:131–151, 1992.

[4] L. Guijarro and F. Wilhelm. Submetries, souls, and second fundamental forms. *Preprint*, 2016.

[5] G. Perelman. Elementary morse theory on alexandrov spaces. *St. petersburgh Math. Journ.*, 5:207–214, 1994.

[6] P. Petersen and F. Wilhelm. Some principles for deforming non-negative curvature. *preprint*, 2010.

[7] P. Petersen and F. Wilhelm. An exotic sphere with positive sectional curvature, 2016.

[8] Peter Petersen. *Riemannian Geometry*. Springer, 3 edition, 2016.

[9] M. Pro, C. Sill and F. Wilhelm. The diffeomorphism type of manifolds with almost maximal volume. *Communications in Analysis and Geometry*, to appear.

[10] C. Searle and F. Wilhelm. How to lift positive ricci curvature. *Geometry and Topology*, 19, 2013.

[11] G. Walschap. *Metric Structures in Differential Geometry*. Springer-Verlag, 2004.