Title
Competitive Equilibrium Without Transitivity, Monotonicity, or Free Disposal

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Competitive Equilibrium Without Transitivity, Monotonicity or Local Non-Satiation

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There are several well-known proofs of the existence of competitive equilibrium for convex economies. Here additional theorems are offered which are of interest both for their greater generality and the relative simplicity of their proofs. The central notions of the equilibrium proof are introduced in the first section. The ideas of this section have been found to provide a flexible and powerful basis for generalizations to the treatment of non-convex preferences (Bergstrom (1973)) and of Lindahl equilibrium (Bergstrom (1971)). Since the structure of proof in these papers is somewhat obscured by ancillary conditions peculiar to the special problems studied, it is useful to illustrate the essential workings in a traditional setting where the logical structure is more clearly revealed and where possible generalizations may be more easily apparent to the reader. The main substantive difference between the assumptions used in this section and those of the usual theorems is that preferences are not assumed to be monotonic nor are commodities assumed to be freely disposable.

In the second section the assumptions of transitivity and completeness of preferences are greatly weakened as is the assumption of local non-satiation. This allows us to admit a broad class of preferences which were previously excluded from equilibrium theory. For example, the theorem applies to some economies in which consumer preference is characterized by an inability to perceive differences between "similar" commodity bundles.

In the third section explicit attention is paid to the demarcation of "pathological" preferences which are encompassed by the equilibrium theory. The final section extends the usual welfare theory to the broader class of economies considered here.

It will be apparent to those who have read Debreu (1962) that the proofs of this paper are crucially influenced by ideas introduced there. The possibility of equilibrium theory without transitive complete preferences has been previously explored by Rader (1972), Hildenbrand, Zamir and Schmeidler (1971), and Sonnenschein (1971). To these writings as well, this paper owes a large intellectual debt.

SECTION I

Results will be stated for an exchange economy. These results can be extended in a natural way to a productive economy using either the method of Debreu (1959) or Rader (1963). Since nothing new regarding the theory of production is introduced here we thus avoid superfluous notational superstructure and allow the reader the option of grafting on his own favorite production theory.

In the economy to be considered, there is a set M consisting of m consumers. There are n commodities. Each consumer $i\epsilon M$ has a consumption set $C_i \subset E^n$, an initial endowment $w_i \epsilon E^n$ and a preference relation R_i defined on C_i. Define the strict preference relation P_i so that xP_iy if and only if xR_iy and not yR_ix. Define $w \equiv \sum_M w_i$. For every $x\epsilon C_i$ define the sets $R_i(x) \equiv \{y|yR_i x\}$, $R_i^{-1}(x) \equiv \{y|xR_iy\}$, $P_i(x) \equiv \{y|yP_ix\}$ and $P_i^{-1}(x) \equiv \{y|xP_iy\}$. Where $p\epsilon E^n$ define the budget set $B_i(p) \equiv C_i \cap \{x|px \leq pw_i\}$ and the demand set $D_i(p) \equiv B_i(p) \cap \{x|P_i(x) \cap B_i(p) = \emptyset\}$. Let $D(p) \equiv \sum_M D_i(p) - w$. A <u>competitive equilibrium price</u> is a vector $\bar{p}\epsilon E^n$ such that $0\epsilon D(\bar{p})$. [1/]

The existence theory of this section will employ the following assumptions.

A.1 - For all $i\epsilon M$, C_i is a closed convex subset of the non-negative orthant in E^n and $w_i \epsilon C_i$.

A.2 - For all $i\epsilon M$, R_i is a complete pre-ordering.

A.3 - For all $i\epsilon M$, and all $x\epsilon C_i$, $R_i(x)$ is a closed convex set.

A.4 - For all $i \epsilon M$ if $x_i \epsilon C_i$ and $x_i \leqq w$, then in every open neighborhood of x_i there is x_i' such that $x_i' \; P_i \; x_i$,

A.5 - For all $i \epsilon M$, and all $x \epsilon C_i$, $R_i^{-1}(x)$ is a closed set.

A.6 - a. $w \epsilon$ Interior $\sum_M C_i$.

b. For all $i \epsilon M$ if $(x_1, \ldots, x_m) \epsilon \underset{M}{X} R_i(w_i)$ and $\sum_M x_i = w$, then there exists $(x_1', \ldots, x_m') \; \epsilon \underset{M}{X} C_i$ such that $\sum_M x_i' = w$ and for all $j \neq i$, $x_j' \; P_j \; x_j$.

The main result of this section is the following.

Theorem 1 - _If assumptions A.1-A.6 are satisfied, there exists a competitive equilibrium_.

The first five assumptions are commonplace in general equilibrium theory. Assumption A.6 -a, requires that the economy be sufficiently rich that any small alteration in total holdings leaves enough resources so that for some feasible allocation of resources each consumer receives a commodity bundle which belongs to his consumption set. Assumption A.6-b. requires "conflict of interest" in the sense that for any consumer i and for any feasible allocation which all consumers like as well as the initial allocation, there is another feasible allocation which is better for all consumers other than i (but possibly worse for i). The idea is simply that whenever all consumers are as well off as in the initial allocation, it is possible that any consumer i could feasibly be further exploited by the consumers other than i.

The most familiar technique in proofs of the existence of competitive equilibrium is to map the simplex $\{p \epsilon E^n | p \geq 0, \sum_{i=1}^{n} p_i = 1\}$ into excess demands and to apply Kakutani's fixed point theorem to a closely related mapping. Where some commodities may be undesirable and where there is not free disposal, this device is not adequate since candidates for possible equilibrium price vectors may include some negative prices. A natural procedure might appear to be to map from the unit sphere, $\{p|\ |p| = 1\}$. However, the sphere is not homeomorphic to a convex set. Hence the Kakutani theorem cannot be directly applied. We pursue an alternate strategy of mapping from the unit ball $S^n \equiv \{p|\ |p| \leq 1\}$. The set S^n is, of course, convex. The difficulty is that S^n contains the vector 0 and in the application of fixed point theory to mappings which are similar to excess demand correspondences, one is likely to find that a fixed point at 0 has an economically degenerate interpretation. What is done is to continuously distort the excess demand mappings in such a way that fixed points occur only where $|p| = 1$. We are thus able to eliminate all vestiges of the monotonicity assumption.

The distorted excess demand correspondence is contructed as follows. Let $\beta_i(p) \equiv C_i \cap \{x_i | px_i \leq pw_i + \frac{1-|p|}{m}\}$ and let $\mathring{\beta}_i(p) \equiv C_i \cap \{x_i | px_i < pw_i + \frac{1-|p|}{m}\}$.

Let $F_i(p) \equiv \beta_i(p) \cap \{x | P_i(x) \cap \mathring{\beta}_i(p) = \emptyset\}$.

Let $F(p) \equiv \sum_M F_i(p) - w$.

<u>Lemma 1</u> - If assumptions A.1-A.3 are satisfied, and if, also, C_i is a compact set for all $i \in M$, then for some $\bar{p} \in E^n$, $0 \in F(\bar{p})$.

This lemma falls short of a proof of the existence of a competitive equilibrium price vector on two accounts. Since we do not know that $|\bar{p}| = 1$, we cannot assert that $\beta_i(\bar{p}) = B_i(\bar{p})$. Even if this is the case, the definition of $F(\bar{p})$ does not guarantee that $F(\bar{p}) = D(\bar{p})$. Here the situation is analogous to that with Debreu's proof of the existence of a quasi-equilibrium. Lemma 2 removes the requirement that C_i be a compact set and also ensures that $|\bar{p}| = 1$.

<u>Lemma 2</u> - If assumptions A.1 - A.4 are satisfied, then there exists $\bar{p} \in S^n$ such that $|\bar{p}| = 1$ and $0 \in F(\bar{p})$.

Now assumptions A.5 and A.6 can be used to show that where $\bar{p}$ satisfies lemma 2, it must be that $D(\bar{p}) = F(\bar{p})$. But then the existence of competitive equilibrium as asserted in Theorem 1 is immediate.

We now write down formal proofs of lemmas 1 and 2 and of theorem 1.

Proof of Lemma 1:

Define the correspondence M with domain $\sum_M C_i - w$ so that $M(z) = \{\frac{z}{|z|}\}$ if $z \neq 0$ and $M(z) = S^n \equiv \{p \epsilon E^n \,|\, |p| \leq 1\}$ if $z = 0$. Define the correspondence H which maps $S^n \times \sum_M C_i - w$ into its subsets in such a way that $H(p,z) \equiv M(z) \times F(p)$. The mapping H clearly maps a compact, convex domain into its subsets. It is an easy matter to verify that the correspondences M and F are upper semi-continuous and have non-empty, compact, convex image sets.[2/] These properties are inherited by the correspondence H. We can apply Kakutani's fixed point theorem to assert the existence of $(\bar{p},\bar{z}) \epsilon H(\bar{p},\bar{z})$. It will be demonstrated that $\bar{z} = 0$.

Suppose $\bar{z} \neq 0$. Then since $\bar{p} \epsilon M(\bar{z})$, $\bar{p} = \frac{\bar{z}}{|\bar{z}|}$. Hence $|\bar{p}| = 1$. Also, $\bar{p}\bar{z} = \frac{\bar{z}\bar{z}}{|\bar{z}|} = |\bar{z}| > 0$. Since $\bar{z} \epsilon F(\bar{p})$, $\bar{z} = \sum_M \bar{x}_i - w$ where $\bar{x}_i \epsilon F_i(\bar{p})$ for all $i \epsilon M$. Since $|\bar{p}| = 1$ and $\bar{x}_i \epsilon F_i(\bar{p})$, it must be that $\bar{p}\,\bar{x}_i \leq \bar{p} w_i$ for all $i \epsilon M$. But then $\bar{p}\,\bar{z} = \bar{p} \sum_M \bar{x}_i - \bar{p} \sum_M w_i \leq 0$. This contradicts our earlier assertion that $\bar{p}\,\bar{z} > 0$. We must conclude that $\bar{z} = 0$. Since $\bar{z} \epsilon F(\bar{p})$, the lemma is proved.

Q.E.D.

Proof of Lemma 2:

Consider a sequence of hypercubes $\{Z^n\}$ in E^n such that for n = 1, 2, ..., $Z^n \equiv \{x \epsilon E^n | 0 \leq x \leq 2^n w\}$. For any n, let $C_i^n \equiv C_i \cap Z^n$. An economy in which consumption sets are the truncated sets C_i^n satisfies the conditions of lemma 1.

Thus if we define $F_i^n(p)$ in the same way as $F_i(p)$ except that the sets C_i are replaced by the sets C_i^n, and if we define $F^n(p) \equiv \sum_M F_i^n(p) - w$, then lemma 1 tells us that for all positive integers n, there exists p^n such that $0 \epsilon F^n(p^n)$. Thus for every n, there exists an allocation $(x_1^n, \ldots, x_m^n)$ such that $x_i^n \epsilon F_i^n(p^n)$ for all $i \epsilon M$, and such that $\sum_M x_i^n = w$. Since each C_i^n is contained in the non-negative orthant, it must be that for all $i \epsilon M$, $0 \leqq x_i^n \leqq w$ for all positive integers n. Thus the sequence $\{(x_1^n, \ldots, x_m^n, p^n)\}$ is contained in a compact set. This sequence must, then, have a subsequence convergent to some $(\bar{x}_1, \ldots, \bar{x}_m, \bar{p}) \epsilon \underset{M}{X} C_i \times S^n$. Clearly, $\sum_M \bar{x}_i = w$. It is also not difficult to verify that $\bar{x}_i \epsilon F_i(\bar{p})$ for all $i \epsilon M$.[3/] Hence $0 \epsilon F(\bar{p})$.

It remains to be shown that $|\bar{p}| = 1$. Since preferences are locally non-satiated, it follows that $\bar{p}\bar{x}_i = \bar{p}w_i + \frac{1 - |\bar{p}|}{m}$ for all $i \epsilon M$. Therefore, $\bar{p} \sum_M \bar{x}_i = \bar{p} w + 1 - |\bar{p}|$. But since $\sum_M \bar{x}_i = w$, it then follows that $1 - |\bar{p}| = 0$.

Q.E.D.

<u>Proof of Theorem 1</u> -

From Lemma 2 it is immediate that there exists $\bar{p} \epsilon E^n$ such that $|\bar{p}| = 1$ and $(\bar{x}_1, \ldots, \bar{x}_m) \epsilon \underset{M}{X} C_i$ such that:

(i) For all $i \epsilon M$, $p\bar{x}_i \leqq pw_i$ and if $x_i P_i \bar{x}_i$ then $\bar{p}\, x_i \geqq \bar{p}\bar{x}_i$.

(ii) $\sum_M \bar{x}_i = w$.

The vector $\bar{p}$ will be a competitive equilibrium price vector if (i) can be strengthened to assert that for all $i \epsilon M$, if $x_i P_i \bar{x}_i$, then $\bar{p}\, x_i > \bar{p}\, \bar{w}_i$. According to a well-known result of Debreu,(1959), this will be true for consumer i under our assumptions A.1 - A.5 if $\bar{p}\, \bar{x}_i > \min_{x \epsilon C_i} \bar{p}x$. We demonstrate that this inequality holds for all consumers.

From assumption A.6.a it follows that for some $k \in M$, $\bar{p}\bar{x}_k > \min_{x \in C_k} \bar{p}\, x$.

Suppose that $\bar{p}\,\bar{x}_j = \min_{x \in C_j} p\, x$ for some $j \in M$. According to A.6.b there exists an allocation $(\hat{x}_1, \ldots, \hat{x}_m) \in \underset{M}{X}\, C_i$ such that $\sum_M \hat{x}_i = w$ and such that $\hat{x}_i\, P_i\, \bar{x}_i$ for all $i \in M$, $i \neq j$. But then $\bar{p}\,\hat{x}_i \geqq \bar{p}\,\bar{x}_i$ for all $i \in M$, $i \neq j$. and $\bar{p}\,\hat{x}_k > \bar{p}\,\bar{x}_k$. Since $\hat{x}_j \in C_j$ it must be that $\bar{p}\,\hat{x}_j \geqq \min_{x \in C_j} \bar{p}x = \bar{p}\,\bar{x}_j$. Thus $\bar{p} \sum_M \hat{x}_j > \bar{p} \sum_M \bar{x}_j = \bar{p}w$.

But this is impossible since $\sum_M \hat{x}_j = w$. We must conclude that $\bar{p}\,\bar{x}_i > \min_{x \in C_i} \bar{p}x$ for all $i \in M$. But then from the result of Debreu cited in the previous paragraph theorem 1 is immediate.

Q.E.D.

SECTION II

Here it is convenient to replace the symmetric relation R_i by the asymmetric "strict preference" relation P_i as the fundamental preference relation of our axiom system. The axioms of Section I will be replaced by the following axiom system.

B.1 - Same as A.1

B.2 - For all $i \in M$, if X is a compact convex subset of $E^n \wedge C_i$ then there

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that this inequality holds for all consumers.

From assumption A.6.a it follows that for some $k \in M$, $\bar{p}\bar{x}_k > \min_{x \in C_k} \bar{p}\, x$.

Suppose that $\bar{p}\,\bar{x}_j = \min_{x \in C_j} p\, x$ for some $j \in M$. According to A.6.b there exists an allocation $(\hat{x}_1, \ldots, \hat{x}_m) \in \underset{M}{X}\, C_i$ such that $\sum_M \hat{x}_i = w$ and such that $\hat{x}_i\, P_i\, \bar{x}_i$ for all $i \in M$, $i \neq j$. But then $\bar{p}\,\hat{x}_i \geqq \bar{p}\,\bar{x}_i$ for all $i \in M$, $i \neq j$. and $\bar{p}\,\hat{x}_k > \bar{p}\,\bar{x}_k$.

B.6.b - For all $j \in M$, if $(x_1, \ldots, x_m) \in \underset{M}{X}[P_i^{-1}(w_i)]^c$ and if $\sum_M x_i = w$, then for each $i \neq j$ there exists a vector $z_i \in C_i$ such that $w_j - \sum_{i \neq j} z_i \in C_j$ and for every $i \neq j$, $x_i + k_i z_i P_i x_i$ for some $k_i > 0$.

The main result of this section is the following.

<u>Theorem 2</u> - <u>If assumptions B.1 - B.5 are satisfied, then there exists a competitive equilibrium</u>.

It is not difficult to show that the axioms B.1-B.5 are implied by the axiom system A.1-A.5. In section 3 it will be demonstrated that a rich class of preferences which are excluded by axioms A.1-A.5 are allowed by B.1-B.5.

More fundamental assumptions which imply B.2 are examined in section 3. The reader who wishes to assess the strength of this assumption is referred to theorems 3 and 4 below. Assumption B.3 is similar to A.3. In fact, where preferences are complete and P_i is defined as in section 1, $C_i \cap [P_i^{-1}(x)]^c = R(x)$. Assuming closedness of $C_i \cap [P_j^{-1}(x)]^c$ is equivalent to assuming that $P_i^{-1}(x)$ is open. The assumption that this set is convex seems to have neither more nor less to recommend it than the common assumption that $R_i(x)$ is convex.

Where the relation P_i is transitive, assumption B.4 is equivalent to the assumption that for no consumer is there a "bliss point" which is feasible with existing resources. Where P_i is not necessarily transitive, it turns out that we need a slight strengthening of the no bliss point assumption. Assumption B.4 is considerably weaker than the assumption of local non-satiation. In particular, B.4 does not exclude the possibility that all "sufficiently small" differences between commodity bundles be imperceptible.

Assumption B.5 is satisfied if $P_i(x)$ is an open set for all $x \in C_i$. This slight weakening of the continuity assumption turns out to be important when preferences are viewed as induced preferences on trades where some commodities may be used by the consumers to produce other commodities. See Rader (1963).

Assumption B.6 is weaker than A.6. Our motive for choosing this weaker assumption is as follows. Axioms B.1-B.5 allow the possibility that individuals have "thick indifference curves". Where there are many consumers and thick indifference curves, it may be that some consumer does not have sufficient resources to raise everyone else's level of preference. All that is required by B.6 is that any consumer is able to distribute some portion of his initial holdings among all other consumers in such a way that some positive multiple of the bundle received by each of the others would improve his welfare.

The structure of proof used for theorem 2 is basically that of the previous section. Our steps must be a bit more cautious and our path more roundabout because we assume neither **transitivity** of preference nor local non-satiation. The main need for **transitivity** in the previous proof was to ensure that the preference relation takes at least one maximal element on any compact budget set. This requirement is here satisfied directly by assumption B.2. Dispensing with local non-satiation is somewhat more difficult. In particular, we used this assumption in the proof of lemma 2 to show that $|\bar{p}| = 1$. Debreu (1962) has shown a way of proving the existence of equilibrium without local non-satiation. His method, however, requires transitivity. The most verbose portion of our proof involves a modification of Debreu's technique **to avoid assuming** transitivity. To this end, we will replace the correspondences F_i and F of the previous section by the correspondences $\bar{G}_i$ and $\bar{G}$ defined as follows.

Let $G_i(p) = \{x | x \in F_i(p) \text{ and } px \geqq px' \text{ for all } x' \in F_i(p)\}$. Let $\bar{G}_i$ be the correspondence whose graph is the closure of the graph of G_i. Let

$\bar{G}(p) = \sum_M \bar{G}_i(p) - w$. The correspondence G_i thus selects the most expensive elements of $F_i(p)$ for each p. The correspondence $\bar{G}_i$ is a "smoothed" version of G_i.

En route to a proof of theorem 2, we prove lemmas 3 and 4 which closely parallel lemmas 1 and 2.

Lemma 3 - If B.1-B.3 are satisfied, and if C_i is a compact set for all $i \varepsilon M$ then there exists $\bar{p} \varepsilon E^n$ such that $0 \varepsilon \bar{G}(\bar{p})$.

Proof:

Assumptions B.2, B.3, compactness of C_i and convexity of the budget sets imply that the image sets $\bar{G}_i(p)$ and $\bar{G}(p)$ are all non-empty, convex, compact sets.4/ Upper semi-continuity of the correspondences $\bar{G}_i$ and $\bar{G}$ is immediate from their definition. It is easily verified that if $x_i \varepsilon \bar{G}_i(p)$ then $px_i \leqq pw_i + \frac{1 - |p|}{m}$ and hence that if $|p| = 1$ and $x_i \varepsilon \bar{G}_i(p)$, then $p x_i \leqq p w_i$. With these remarks in mind, it is easy to see that the proof of lemma 1 can be borrowed in its entirety. Each step of the argument remains valid when the correspondences F_i and F are supplanted by $\bar{G}_i$ and $\bar{G}$ respectively.5/

Q.E.D.

Lemma 4 - If B.1 - B.4 are satisfied, then there exists $\bar{p} \varepsilon E^n$ such that $|\bar{p}| = 1$ and $0 \varepsilon F(\bar{p})$.

Proof of lemma 4:

Appealing to lemma 3 and mimicking the argument of the first paragraph of the proof of lemma 2, we can assure ourselves of the existence of a sequence $\{(x_1^n, \ldots, x_m^n, p^n)\}$ converging to $(\bar{x}_1, \ldots, \bar{x}_m, \bar{p})$ where for all $i \varepsilon M$, and all integers n, $x_i^n \varepsilon \bar{G}_i^n(p^n)$ and $\sum_M x_i^n = w$. (Here G_i^n is defined so that $G_i^n(p) \equiv \{x | x \varepsilon F_i^n(p)$ and $px \geqq px'$ if $x' \varepsilon F_i^n(p)\}$). Since $G_i^n(p) \subset F_i^n(p)$ and since F_i^n is an upper semi-continuous correspondence, it follows that $\bar{G}_i^n(p) \subset F_i^n(p)$.

Hence for all $i \varepsilon M$, $x_i^n \varepsilon F_i^n(p^n)$ for all integers n. It then follows that $\bar{x}_i \varepsilon F_i(\bar{p})$ for all $i \varepsilon M$.[6] Since for all integers n, $\sum_M x_i^n = w$, it must be that $\sum_M \bar{x}_i = w$.

All that remains to be done is to show that $|\bar{p}| = 1$. We will do this by first showing that for all $i \varepsilon M$, $\bar{p}\,\bar{x}_i = \bar{p}\,w_i + \frac{1 - |\bar{p}|}{m}$. When this is done the demonstration that $|\bar{p}| = 1$ is the same as that in the last paragraph of the proof of lemma 2.

Matters are expedited by the following result.

Remark 1 - Let $\mathring{C}_i^n$ be the interior of C_i^n (relative to the non-negative orthant of E^n.) If $x \varepsilon \;\bar{G}_i^n(p) \bigcap \mathring{C}_i^n$; then $px = pw_i + \frac{1 - |\bar{p}|}{m}$.

Proof of Remark:

We will first show that the statement of the remark is true when $\bar{G}_i^n$ is replaced by G_i^n. Let $\bar{x} \varepsilon G_i^n(p) \bigcap \mathring{C}_i^n$. According to B.4, there exists $\hat{x} \varepsilon C_i$ such that $\hat{x}\, P\, \bar{x}$ and such that $P(\hat{x}) \subset P(\bar{x})$. Since $G_i(p) \subset F_i(p)$, $\bar{x} \varepsilon F_i(p)$. Therefore $p\,\hat{x} \geq p\,w_i + \frac{1 - |p|}{m}$. Since $P(\hat{x}) \subset P(\bar{x})$ and $\bar{x} \varepsilon F_i(p)$, it follows that $P(\hat{x}) \bigcap \mathring{\beta}^n(p) = \emptyset$. (where $\mathring{\beta}^n(p) \equiv \{x \varepsilon C_i^n \mid p\,x < p\,w_i + \frac{1 - |p|}{m}\}$.) Noting that the statement $P(x) \bigcap A = \emptyset$ is equivalent to the statement $x \varepsilon \bigcap_{x \varepsilon A} [P^{-1}(x)]^c$, we observe that $\hat{x}$ and $\bar{x}$ both belong to the set $\bigcap_{x \varepsilon \mathring{\beta}^n(p)} [P^{-1}(x)]^c$. But this set is convex since by B.3, $[P^{-1}(x)]^c$ is a convex set for all $x \varepsilon C_i$. Therefore if $0 < \lambda < 1$, then $x_\lambda \equiv \lambda\hat{x} + (1-\lambda)\,\bar{x} \;\varepsilon \bigcap_{x \varepsilon \mathring{\beta}(p)} [P^{-1}(x)]^c$. Suppose that $p\,\bar{x} < p\,w + \frac{1 - |p|}{m}$. Then for sufficiently small positive λ, $p\,\bar{x} < p\,x_\lambda \leq p\,w_i + \frac{1 - |p|}{m}$ and $x_\lambda \;\varepsilon\; C_i^n$. But since $x_\lambda \;\varepsilon \bigcap_{x \varepsilon \mathring{\beta}^n(p)} [P^{-1}(x)]^c$ it must be that $P(x_\lambda) \bigcap \mathring{\beta}^n(p) = \emptyset$ and hence that $x_\lambda \;\varepsilon\; F_i^n(p)$. Since $px_\lambda > p\bar{x}$, this contradicts the assumption that $\bar{x} \in G_i(p)$. We must therefore conclude that $p\,\bar{x} = p\,w_i + \frac{1 - |p|}{m}$, whenever $\bar{x} \in G_i^n(p) \cap \mathring{C}_i^n$.

Suppose $\bar{x} \in \bar{G}_i^n(p) \cap \mathring{C}_i^n$. According to the definition of $\bar{G}_i^n$, there exists a sequence $\{(x^q,p^q)\} \to (\bar{x},p)$ such that $x^q \in G_i^n(p^q)$ for all integers q. Since $x^q \to \bar{x}$ and $\bar{x} \in \mathring{C}_i^n$ it must be that $x^q \in \mathring{C}_i^n$ for all q sufficiently large. Therefore $x^q \in G_i^n(p^q) \cap \mathring{C}_i^n$ for all q sufficiently large. From the result of the previous paragraph it follows that $p^q x^q = p^q\,w_i + \frac{1 - |p^q|}{m}$ for all q sufficiently large. But then it must be that $p\,\bar{x} = p\,w_i + \frac{1 - |p|}{m}$. This proves the remark.

Q.E.D.

We now complete the proof of lemma 4. Consider the allocation and price vectors $(x_1^n, \ldots, x_m^n, p^n)$ of the first paragraph of the proof. Since $\sum_M x_i^n = w$, it must be that for all $i \in M$, $x_i^n \leqq w$. From the construction of the sets C_i^n, it is clear that $x_i^n \in \mathring{C}_i^n$ for all $n \geqq 1$. Since, also, $x_i^n \in \bar{G}_i^n(p)$ for all $n \geqq 1$, remark 1 ensures that $p^n\,x_i^n = p_n\,w_i + \frac{1 - |p^n|}{m}$ for all $n \geqq 1$. But then it must be that $\bar{p}\,\bar{x}_i = \bar{p}\,w_i + \frac{1 - |\bar{p}|}{m}$ for all $i \in M$. As was previously remarked, one can then show that $|\bar{p}| = 1$ simply by repeating the argument of the last paragraph of lemma 2. This completes the proof of lemma 4.

Q.E.D.

<u>Proof of theorem 2</u> -

Much as is the case in proving theorem 1 from lemma 2, we need here to show that where $\bar{p}$ satisfies lemma 3, $F_i(\bar{p}) = D_i(\bar{p})$ for all $i \in M$. As is the case in the earlier proof, it can be shown that this will be the case whenever $\bar{p}\,\bar{x}_i > \min_{x \in C_i} \bar{p}\,x$ for all $i \in M$. It is this fact which we must prove here.

From B.6-a, it follows that for some $k \in M$, $\bar{p}\,\bar{x}_k > \min_{x \in C_k} \bar{p}\,x$. Suppose that $\bar{p}\,\bar{x}_j = \min_{x \in C_j} \bar{p}\,\bar{x}$ for some $j \in M$. From B.6-b, it follows that there exists a vector $(z_1, \ldots, z_m)$ such that $\sum_M z_i = 0$, $\bar{x}_j + z_j \in C_j$ and such that for all $i \neq j$, $\bar{x}_i + k_i z_i \, P_i \, \bar{x}_i$ for some $k_i > 0$. Since $\bar{x}_i \in F_i(\bar{p})$, for all $i \in M$, it follows that $\bar{p}\,z_i \geqq 0$ for all $i \in M$ and that for consumer k, $\bar{p}\,z_k > 0$. But $\sum_M z_i = 0$ and hence $\bar{p} \sum_M z_i = 0$. Therefore $\bar{p}\,z_j < 0$. But $\bar{x}_j + z_j \in C_j$. Therefore $\bar{p}\,\bar{x}_j > \min_{x \in C_j} \bar{p}\,x_j$. We must conclude that for all $i \in M$, $\bar{p}\,x_i > \min_{x \in C_i} \bar{p}\,x_i$.

It is then an easy matter to show that $F_i(\bar{p}) \subset D_i(\bar{p})$ for all $i \in M$ and hence that the price $\bar{p}$ found in lemma 3 is a competitive equilibrium price.[7/]

Q.E.D.

SECTION III

The existence result of theorem 2 requires that the relations P_i take maximal elements on compact convex sets. Here we offer two separate results which produce this condition as a consequence of more fundamental assumptions on preferences.

Theorem 3 (Sonnenschein)

Let R be a complete relation defined on a compact convex set X. If for all $x \in X$, R (x) is a closed set and P(x) is a convex set (where $P(x) = R(x) \cap [R^{-1}(x)]^c$) then there exists $\bar{x} \in X$ such that $\bar{x}$ Rx for all $x \in X$.

This remarkable result is proved in elegant fashion by Sonnenschein (1971). More directly pertinent to the discussion of this paper is the following corollary.

Corollary 1 - Let P be an asymmetric relation defined on a compact convex set X. If for all $x \in X$, $[P^{-1}(x)]^c$ is a closed set and $P(x)$ is a convex set, then for some $\bar{x} \in X$, $P(\bar{x}) \cap X = \emptyset$.

Proof of Corollary:

Define the relation R so that $x R y$ if and only if $x \in X$, $y \in X$ and not $y P x$. Since P is asymmetric, R must be complete. Also, $R(x) = [P^{-1}(x)]^c$ and $P(x) = R(x) \cap [R^{-1}(x)]^c$ for all $x \in X$. From theorem 3 it follows that for some $\bar{x} \in X$, $\bar{x} R x$ for all $x \in X$. But this implies that $P(\bar{x}) \cap X = \emptyset$.

Q.E.D.

An alternative theorem which guarantees that maximal elements are chosen on compact sets requires the assumption of acyclicity of preferences. A relation P defined on a set X is said to be acyclic if there is no finite set $\{x_1, \ldots, x_n\} \subset X$ such that for $i = 1, \ldots, n-1$, $x_i P x_{i+1}$ and such that $x_n P x_1$.

Theorem 4 - Let X be a compact set and let P be an acyclic relation such that for all $x \in X$, $P^{-1}(x)$ is an open set (relative to X). Then for some $\bar{x} \in X$, $P(\bar{x}) \cap X = \emptyset$.

Sen (1970) demonstrates that an acyclic relation takes maximal elements on finite sets. Here we extend this result to compact sets.[8/] Key to the proof are the following standard results of topology and set theory respectively. Both can be found in Kelley (1955).

Finite Intersection Principle

Let $\mathcal{J}$ be a collection of compact sets. If the intersection of every finite subcollection of $\mathcal{J}$ is non-empty, then the intersection of all sets in $\mathcal{J}$ is non-empty.

Kuratowski's Lemma

Every totally ordered subset of a partially ordered set is contained in a maximal totally ordered subset.

The following lemma expedites our proof.

Lemma 5

Let X be a compact set and let P be an asymmetric, transitive relation on x such that for all $x \varepsilon X$, P^{-1} (x) is an open set. Then for some $\bar{x} \varepsilon X$, $P(\bar{x}) \cap X = \emptyset$.

Proof of lemma 5:

Since P is transitive, P is a partial order. According to Kuratowski's lemma, there exists a maximal totally ordered subset of X. Call this set A. Let B be an arbitrary finite subset of A. For all $x \varepsilon X$, define $S(x) \doteq X \cap [P^{-1}(x)]^c$. Since B is finite and totally ordered, there exists $\hat{x} \varepsilon B$ such that $\hat{x}\ P\ x$ if $x \varepsilon B$ and $x \neq \hat{x}$. Since P is asymmetric, $\hat{x} \varepsilon \bigcap_{x \varepsilon B} S(x)$.

Since P^{-1} (x) is open, S (x) is closed and hence compact for all $x \varepsilon X$. It follows from the finite intersection principle that there exists $\bar{x} \varepsilon \bigcap_{x \varepsilon A} S(x)$.

Hence if $x \varepsilon A$, $x \notin P(\bar{x})$. Suppose that $x \varepsilon X \cap A^c$. If $x\ P\ \bar{x}$, then $A \cup \{x\}$ is a totally ordered subset of X. But this is impossible since A is a maximal totally ordered subset. It follows that $P(\bar{x}) \cap X = \emptyset$.

Q.E.D.

Proof of theorem 4:

Define the relation Q so that x Q y if and only if there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $x\ P\ x_1$, $x_i\ P\ x_{i+1}$ for $i = 1, \ldots, n-1$ and $x_n\ Py$. It is easy to see that Q is a transitive relation and also that $Q^{-1}(x)$

is open for all $x \epsilon X$. Since P is assumed to be acyclic, Q must be asymmetric. Applying lemma 5, we find that there exists $\bar{x} \epsilon X$ such that $Q(\bar{x}) \cap X = \emptyset$. But $P(\bar{x}) \subset Q(\bar{x})$. Hence $P(\bar{x}) \cap X = \emptyset$.

Q.E.D.

Theorem 3 allows us to replace assumption B.2 of theorem 2 by the following:

B.2' - For all $i \epsilon M$, P_i is an asymmetric relation and $P_i(x)$ is a convex set for all $x \epsilon C_i$.

Theorem 5 - If assumptions B.1, B.2', and B.3-B.6 are satisfied there exists a competitive equilibrium.

Proof: Theorem 3 tells us that assumptions B.2' and B.3 imply B.2. Theorem 5 is then immediate from theorem 2.

Q.E.D.

Theorem 4 can be applied to replace assumption B.2 as follows.

B.2". For all $i \epsilon M$, P_i is an acyclic relation.

Theorem 6 - If assumptions B.1, B.2" and B.3-B.6 are satisfied, then there exists a competitive equilibrium.

Proof: Assumption B.3 requires that $[P_i^{-1}(x)]^c$ be closed in C_i, hence $P_i^{-1}(x)$ is open in C_i. Thus theorem 4 tells us that B.2" and B.3 imply B.2. Theorem 6 is then immediate from theorem 2.

Q.E.D.

In order to demonstrate that theorem 6 is a non-trivial extension of theorem 1 we consider the following class of examples. Let C_i be the non-negative orthant in E^n. Let u_i and ℓ_i be continuous real valued functions such that for all $x \varepsilon C_i$, $u_i(x) \geq \ell_i(x)$. Define the relation P_i so that $x \, P_i \, y$ if and only if $\ell_i(x) > u_i(y)$. A preference relation which can be thus represented is called an interval order by Fishburn (1970b).[9/] Notice that for $x \varepsilon C_i$, $P_i(x) = \{y \mid \ell_i(y) > u_i(x)\}$ and $[P_i^{-1}(x)]^c = \{y \mid u_i(y) \geq \ell_i(x)\}$. Where u_i is a continuous quasi-concave function which is not bounded in C_i, it can be verified that the relation P_i satisfies the assumptions on preferences needed for theorem 6. However, unless ℓ_i is also quasi-concave, $P_i(x)$ is not a convex set and thus theorem 5 does not apply.

More explictly, an example of a preference relation satisfying the assumptions of theorem 6 but not those of theorem 5 would be the relation P defined on the non-negative orthant of E^2 as in the previous paragraph where $u_i(x_1, x_2) = x_1 + x_2$ and $\ell_i(x_1, x_2) = [x_1^2 + x_2^2]^{1/2}$. Other rather interesting examples can be found. In particular, the assumptions of theorem 5 would be satisfied for any u_i and ℓ_i such that $u_i(x_1, \ldots, x_n) = [\Sigma \, x_i^{\,r}]^{1/r}$,

$\ell_i(x_1, \ldots, x_n) = [\Sigma \, x_i^{\,s}]^{1/s}$ and $0 < r < s$ and $r \leq 1$. A proof that $\ell_i(x) \leq u_i(x)$

can be found in Hardy, Littlewood and Polya (1952), page 28. Another class of examples which satisfy the assumptions of both theorems 5 and 6 are those where $u(x) = [\Sigma\lambda_i x_i^{s}]^{1/s}$, $\ell(x) = [\Sigma\lambda_i x_i^{r}]^{1/r}$ where $\lambda_i \geq 0$ for each i, $\Sigma \lambda_i = 1$, and $r \leq s \leq 1$. Hardy, Littlewood, and Polya (1952, page 26) prove that in this case $\ell(x) \leq u(x)$ for all $x \geq 0$.

At present, I do not have an example of preferences which are not acyclic but satisfy the conditions of theorem 5. Sonnenschein (1971) offers an example of preferences which satisfy our assumption B.2' and for which there are cycles. However, in his example there is a bliss point.

SECTION IV

The usual definition of Pareto optimality states that a Pareto optimal allocation is a feasible allocation such that no feasible allocation exists which is "at least as good" for all consumers and better for some consumer. Where, as in the second section of this paper, "strict preference" is taken as the basic preference notion, there remains ambiguity in this formulation until the relation "at least as good as" is formally defined. Perhaps the most obvious possibility is to define the relations R_i so that for x and y in C_i, xR_iy if and only if not yP_ix. Thus x is "indifferent to" y if neither yP_ix nor xP_iy. We may, however, feel uneasy about treating a pair of commodity bundles which are not comparable by the relation P_i as indifferent for welfare economic purposes. Where the assumptions on preference relations are as weak as in section 2, it is possible to provide alternative distinct notions of "indifference" which may have at least as much appeal for welfare economics. For example, it would be possible to define indifference so that x and y are indifferent if and only if $P(x) = P(y)$. Alternatively, we could define x and y to be indifferent if $P^{-1}(x) = P^{-1}(y)$. Yet another possibility would be to require that $P(x) = P(y)$ and $P^{-1}(x) = P^{-1}(y)$.[10/] For each of these definitions of "indifference" an alternative theory of Pareto optimality could be constructed.

For any of the above treatments of Pareto optimality, there is an uncomfortable difficulty for the study of equivalence between Pareto optima and competitive equilibria. The trouble is that unless local non-satiation is assumed, competitive equilibrium need not be Pareto optimal. It is useful to employ another definition of optimality. In particular, an allocation x will be said to be a weak Pareto optimum if it is feasible and if there is no feasible allocation which all consumers prefer to x.

Using this definition of optimality allows one to avoid the question of what things are "indifferent". It also works out conveniently that any competitive equilibrium is a weak Pareto optimum even if there is local satiation. The converse theorem does not fare quite so well, but it is true that a substitute result can be found which seems adequate for many purposes.

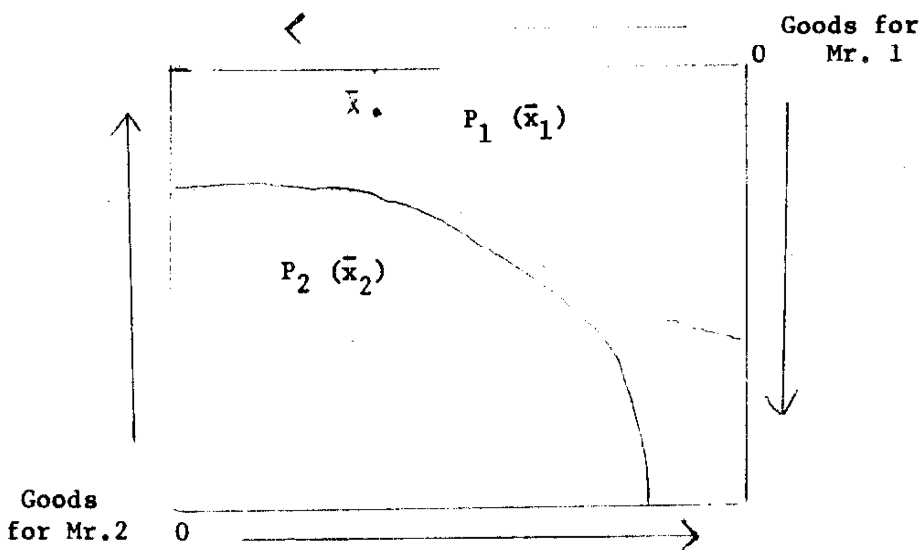
<u>Theorem 7</u> - <u>A competitive equilibrium is a weak Pareto optimum.</u>

Proof: Let $(\bar{x}_1, \ldots, \bar{x}_m, \bar{p})$ be a competitive equilibrium allocation and price. Suppose $x_i \; P_i \; \bar{x}_i$ for all $i \in M$. Then $\bar{p}\, x_i > \bar{p}\, \bar{x}_i$ for all $i \in M$ and hence $\bar{p} \sum_M x_i > \bar{p} \sum_M \bar{x}_i$. But this is impossible if $\sum_M x_i = w = \sum_M \bar{x}_i$.

Q.E.D.

Without local non-satiation it is fairly easy to display a weak Pareto optimum which is not a competitive equilibrium. This is illustrated in Figure 1.

FIGURE 1



Allocations are represented by the usual conventions of the Edgeworth box. Since there is not local non-satiation, the allocation $\bar{x}$ can be exterior to both the sets P_1 $(\bar{x}_1)$ and P_2 $(\bar{x}_2)$. Since these two sets are disjoint, $\bar{x}$ is a weak Pareto optimum. Although price vectors can be found which separate P_1 $(\bar{x}_1)$ and P_2 $(\bar{x})$, any such price vector must pass below the point $\bar{x}$. Thus at such prices consumer 1 cannot afford $(\bar{x}_1^1, \bar{x}_2^1)$.

The following is an immediate corollary of our theorem 2.

Corollary - Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ be an allocation such that $\sum_M \bar{x}_i = w$. Suppose that the assumptions of theorem 2 hold when the initial wealth distribution is such that $w_i = \bar{x}_i$ for all $i \in M$. Then there exists a competitive equilibrium price and allocation $\hat{p}$ and $(\hat{x}_1, \ldots, \hat{x}_m)$ such that there is no consumer i for whom

$\bar{x}_i \; P_i \; \hat{x}_i$.

Proof: Consider the economy in which initial allocations are such that $w_i = \bar{x}_i$ for all $i \in M$. For that economy let $\hat{p}$ and $(\hat{x}_1, \ldots, \hat{x}_m)$ be a competitive equilibrium price and allocation. Since for all $i \varepsilon M$, $x_i \varepsilon B_i(\hat{p})$, it can not be that $\bar{x}_i \, P_i \, \hat{x}_i$.

Q.E.D.

In particular if $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ is a Pareto optimal allocation, there exists a competitive equilibrium which no consumer finds inferior to $\bar{x}$. Thus, if the purpose of the "second optimality theorem of welfare economics" is simply to assert that the competitive mechanism is neutral with respect to the "distribution of pleasure," this result serves the same purpose quite adequately.

FOOTNOTES

1/

The definition of competitive equilibrium can be slightly generalized at small cost by generalizing the wealth functions. In particular, let $W(p) = (W_1(p),\ldots,W_m(p))$ be a function from E^n to E^m such that for all $p \in E^n$, $\sum_{i=1}^{m} W_i(p) = pw$. Where the budget sets are defined to be $B_i(p) = C_i \cap \{x | px \leqq W_i(p)\}$, a vector $\bar{p} \in E^n$ such that $0 \in D(\bar{p})$ can be called a competitive equilibrium price relative to the wealth distribution function W. If W(p) is assumed to be a continuous function such that for all $i \in M$ and all $p \in E^n$, there exists $x \in C_i$ such that $px \leqq W_i(p)$, then all of the existence proofs of this paper extend almost trivially to the more general notion of equilibrium.

2/

The proof that F_i is upper semi-continuous is as follows. Let $p^n \to p$, $x^n \to x$ and $x^n \in F_i(p^n)$ for all integers n. Then for all n, $p^n x^n \leqq p^n w_i + \frac{1 - |p^n|}{m}$ and hence $px \leqq pw_i + \frac{1 - |p|}{m}$. Since C_i is closed, it must be that $x \in \beta_i(p)$. Suppose that $x' \in \mathring{\beta}_i(p)$. Then $x' \in C_i$ and $px' < pw_i + \frac{1 - |p|}{m}$. Since $x^n \in F_i(p^n)$, it follows that for all sufficiently large n, $x^n \in R_i(x')$. But $R_i(x')$ is closed and $x^n \to x$. Therefore $x \, R_i \, x'$ if $x' \in \mathring{\beta}_i(p)$. It follows that $x \in F_i(p)$ and hence that F_i is upper semi-continuous.

Q.E.D.

3/

Suppose $x_i \in \mathring{\beta}_i(\bar{p})$. Then for all n sufficiently large, $x_i \in C_i^n \cap \{x | p^n x < p^n w_i\}$. Since $x_i^n \in F_i^n(p^n)$, it must be that $x_i \notin P_i(x_i^n)$. Hence $x_i^n \in R_i(x_i)$. Since $x_i^n \to \bar{x}_i$ and $R_i(x_i)$ is closed, $\bar{x}_i \in R_i(x_i)$. Therefore $P(\bar{x}_i) \cap \mathring{\beta}_i(\bar{p}) = \phi$. A straightforward continuity argument shows that $\bar{x}_i \in \beta_i(\bar{p})$. It follows that $\bar{x}_i \in F_i(\bar{p})$.

4/

Assumption B.2 ensures that $G_i(p)$ is non-empty. Convexity of $[P_i^{-1}(x)]^c$ ensures that $G_i(p)$ is convex. Closedness of $[P_i^{-1}(x)]^c$ and compactness of C_i guarantee that $G_i(p)$ is compact.

5/

It should be noticed that at each step in the argument where $R_i(x_i)$ appears, this set must here be replaced by $[P_i^{-1}(x)]^c$.

6/

The argument is essentially the same as that outlined in footnote 3 where $R_i(x_i)$ is replaced by $[P_i^{-1}(x_i)]^c$.

7/

Suppose $\bar{x}_i \in F_i(\bar{p})$, $\bar{p}\, x_i \leq \bar{p}w_i$ and $x_i P_i \bar{x}_i$. Let $\mathring{x}_i \in C_i$ such that $\bar{p}\, \mathring{x}_i < \bar{p}\bar{x}_i \leq \bar{p}\, w_i$. According to Assumption B.5, there exists λ such that $0 < \lambda < 1$ and $\lambda\, \mathring{x}_i + (1-\lambda)\, x_i\, P_i\, \bar{x}_i$. But $\bar{p}(\lambda\mathring{x}_i + (1-\lambda)\, x_i) < \bar{p}\, w_i$. This is impossible since $P(\bar{x}_i) \cap \mathring{\beta}(\bar{p}) = \phi$. It follows that $P(\bar{x}_i) \cap \beta(\bar{p}) = \phi$ and hence that $\bar{x}_i \in D_i(\bar{p})$.

8/

Fishburn (1970a) offers an alternative treatment of acyclic preference relations (which he calls suborders). He shows that with certain monotonicity and Archimedean assumptions, there exists an upper semi-continuous function u such that x Py implies $u(x) > u(y)$ where P is acyclic. Since the function u is upper semi-continuous, it takes a maximum on compact subsets of its domain. Where $\bar{x}$ maximizes u(x) on a compact set X, $P(\bar{x}) \cap X = \phi$. Hence, Fishburn's conditions on preferences also imply our assumption B.2. Our theorem 4 allows us to avoid monotonicity and substitute the topological assumption, $P^{-1}(x)$ is open, for the Archimedean continuity conditions of Fishburn.

9/

Fishburn (1970b) presents an elegant system of axioms which are necessary and sufficient for preferences to be representable in this way.

10/

Herzberger (1973) discusses such alternative notions of indifference.

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