Title
Extended von Neumann Dimension For Representations of Groups and Equivalence Relations

Author
Hayes, Benjamin Richard

Publication Date
2014

Peer reviewed|Thesis/dissertation

University of California
Los Angeles

Extended von Neumann Dimension
For Representations of Groups and Equivalence Relations

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Benjamin Richard Hayes

2014

Abstract of the Dissertation

Extended von Neumann Dimension For Representations of Groups and Equivalence Relations

by

Benjamin Richard Hayes

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Dimitri Shlyakhtenko, Chair

This thesis is on two related research problems, and is divided into 2 parts:

Part 1: Let Γ be a countable discrete sofic group, we given an entropic formula for the von Neumann dimension of a Hilbert space representation of Γ contained in a multiple of the left regular representation. We use our formula to extend von Neumann to *any* uniformly bounded representation of Γ on a separable Banach space. We give computations for the left regular representable representation of Γ on ℓ^p, as well actions on noncommutative L^p-spaces and ℓ^p-Betti numbers of free groups. We prove some general results about the properties of this invariant, including that the extended von Neumann dimension is always zero when the group is infinite and the representation is finite-dimensional.

Part 2: We work on an analogous problem for representations of a sofic, discrete, measure-preserving equivalence relation. Again, we are able to find an entropic formula for von Neumann dimension of a Hilbert space representation of a sofic, discrete, measure-preserving equivalence relation $\mathcal{R}$. Again, this allows us to extend von Neumann dimension to actions of $\mathcal{R}$ on a Banach space. Following techniques of Gaboriau in [12], we are able to define the L^p-Betti numbers of (finitely presented) equivalence relations. We also indicate

how this gives a potential way to solve the cost versus L^2-Betti number problem as posed by Gaboriau.

The dissertation of Benjamin Richard Hayes is approved.

Delroy Baugh

Terence Tao

Sorin Popa

Dimitri Shlyakhtenko, Committee Chair

University of California, Los Angeles

2014

To Steve, Lyda, Margaret and Hilary, as well as the memories of Matthew, Marilyn and Jim.

Table of Contents

Acknowledgments

I am indebted to my advisor Dimitri Shlyakhtenko for his tremendous support, for his encouragement as well as incredible intuition on these problems, introducing me to several mathematicians, and giving me several opportunities to travel. I will always owe him a debt for how much he furthered my mathematical career. I am indebted to Yoann Dabrowski for suggesting the problem of defining extended von Neumann dimension for sofic groups, as well as the suggestion to read Gournay's work. These suggestions inspired most of the thesis.

Much of this thesis benefited from the opportunity to talk at seminars: I would like to thank Natasha Blitivic, Nicolas Monod, Jesse Peterson, and Dan Voiculescu for opportunities to speak at Indiana University, École Polytechnique Federal de Lausanne, Vanderbilt and Berkeley respectively. Their hospitality is greatly appreciated, as well as providing great opportunities to improve my work. I would like to further thank Nicolas Monod for inviting me for a second visit to École Polytechnique Federal de Lausanne where I benefited greatly from discussions with him, Gabor Elek and Henrik Densing Petersen. Many people have contributed interesting discussions during visits which furthered this research: in particular I thank Dietmar Bisch, Lewis Bowen, Damien Gaboriau, Vaughan Jones, Kate Juschenko, David Kerr, Hanfeng Li, Jesse Peterson, Sorin Popa, Yehuda Shalom, Thomas Sinclair, Andreas Thom, Stefaan Vaes, and Alain Valette in this regard. I would like to additionally thank Kate Juschenko and Jesse Peterson for not only interesting mathematical discussions, but also being good mentors and giving useful career advice.

Additional thanks go to those who made comments on earlier version of the papers comprising this work: in particular Lewis Bowen, Damien Gaboriau, and Hanfeng Li.

I would like to thank Douglas Lind for letting me work on a research project as an undergrad. Though the research was on quite different things than this thesis, the research experience was extremely useful and led to my interest in functional analysis. Additionally, I am thankful to him for various interesting mathematical conversations on essentially any field of mathematics.

Special thanks to all the members of my family. I thank Margaret, Steve and Lyda for supporting me through everything and encouraging me to do what I love. I thank Hilary and Matthew for discouraging me from taking myself too seriously and having an unchangeable affect on my life.

Last, but certainly not least, I thank Stephanie Lewkiewicz for incredible moral support, improving my work ethic, lightening my mood, numerous inside jokes and making me a better person in every way. She has helped me through some of the most stressful times of my life. Additionally I thank her numerous colleagues for their helpful mathematical advice.

Vita

2009	B.S. (Mathematics), University of Washington.
2009-2011	Teaching Assistant Department of Mathematics, UCLA
2012	Graduate Student Instructor, Department of Mathematics, UCLA
2013-2014	Dissertation Year Fellowship, UCLA

Publications

Ben Hayes, *An l^p-version of von Neumann dimension for Banach space representations of sofic groups*, Journal of Functional Analysis, **266** (2), 989-1040

Ben Hayes,*An l^p-Version of von Neumann dimension for Banach space representation of sofic groups II* (submitted) 2014

Ben Hayes, *Metric mean dimension for algebraic Actions of sofic groups* (submitted) 2014

CHAPTER 1

Introduction

The thesis is on extending a quantity called *von Neumann dimension* associated to certain Hilbert space representations of groups and equivalence relations to more general representations on *Banach* spaces.

The original definition of von Neumann dimension is due to Murray and von Neumann and heavily depends upon Hilbert space structure. For example, one starts with a unitary representation of a countable discrete group Γ which is a subspace of $\ell^2(\Gamma \times \mathbb{N})$ with the left translation action. Then, one takes the orthogonal projection onto this subspace and notices that it lands in a certain operator algebra with a trace, and then takes the trace of the projection. This is natural from linear algebra, as it straightforward to verify that the dimension of a subspace is the trace of the projection onto this subspace. More generally one can replace Γ with a *tracial von Neumann algebra* and this is what generalizes the theory to equivalence relations, measure spaces, etc.

The peculiar aspect of this dimension is that it typically takes on all values in $[0, \infty]$ instead of just integer values. For example, this is the true in the group case described above if the group is infinite. Moreover, for abelian groups the theory is relatively simple: by Fourier analysis invariant subspaces of $\ell^2(\Gamma)$ correspond to measurable subsets of the dual group, and the dimension is just the measure of the correspond set. A similar theory works for any abelian von Neumann algebra. This theory of dimension allows one to define ℓ^2-Betti numbers of groups or equivalence relations, and these numbers have tremendous applications in group theory, orbit equivalence and ergodic theory, as well as operator algebras itself.

With the incredible success of von Neumann dimension, it is reasonable to wonder if one can extend the theory to more general actions on Banach spaces. However, the definition of von Neumann dimension highly relies on Hilbert space structure: the existence of projections, the structure of operator algebras on Hilbert spaces, and properties of traces on these algebras. It is not clear how one could remove this structure. A possible approach was suggested by Gromov in [15]. For this, it turns out to be useful to view von Neumann dimension in a different way. Namely, one can view von Neumann dimension as being analogous to entropy. For example, we have a canonical inclusion

$$\ell^2(\Gamma) \subseteq \mathbb{C}^\Gamma,$$

and we view $\mathbb{C}^\Gamma$ as a Bernoulli shift. Unfortunately, $\mathbb{C}^\Gamma$ does not have the structure of a compact space or a nice probability space structure that usually allows one to analyze Bernoulli shifts. However, the spaces $\ell^p(\Gamma)$ clearly have nice analytic structure. Since classification for Bernoulli shifts is done by entropy, we expect invariants for $\Gamma \curvearrowright \ell^p(\Gamma)$ to have an entropic flavor.

This is not just a vague heuristic: Voiculescu in [27] discovered an entropic formula for representations of *amenable* groups analogous to entropy of an action of an amenable group on a topological space. His definition allows one to relate entropy for actions on certain non-commutative spaces (i.e. C^*-algebras) to von Neumann dimension. Following up on comments of Gromov, Antoine Gournay in [13] discovered a different entropic formula for von Neumann dimension, but this time with the aim of extending von Neumann dimension to actions on ℓ^p-spaces instead of Hilbert spaces. These results make clear the relationship between entropy and von Neumann dimension in the case of amenable groups.

Quite recently, the theory of entropy of actions on a group on a topological space or measure space has been extended to the class of *sofic* groups in the work of L. Bowen [2] and Kerr-Li [18]. The class of sofic groups is much larger than the class of amenable groups: it contains all residually amenable (in fact, residually sofic) groups, locally sofic groups, and is closed under free products with amalgamation over amenable subgroups. Given our analogy between entropy and dimension it is reasonable to expect one to be able express von

Neumann dimension and entropy for sofic groups, and not just amenable groups. This is the main content of this thesis, as well as exploring what happens when one drops the Hilbert space structure of the representation. In particular, this leads us to define ℓ^p-Betti numbers for sofic groups, as well as sofic equivalence relations. Further, the ℓ^p-Betti numbers give a potential approach to the cost versus ℓ^2-Betti number, a significant and important problem in orbit equivalence theory.

The thesis is divided into several parts. I have tried as much as possible to keep the thesis accessible to a general audience. Thus the first chapter contains some preliminaries on the less standard material: sofic groups, von Neumann algebras, and equivalence relations. Assuming the reader takes a few things for granted, I have given a self-contained construction of the classical von Neumann dimension (which isn't even technically needed for most of the thesis). The preliminaries are actually a relatively small amount of material, and so I hope that readers familiar with functional analysis (e.g. locally convex spaces and introductory C^*-algebra theory), will be able to read most of the text.

Interested readers may wish to decide which of the material in the thesis they want to skip. In particular, the section on noncommutative L^p-spaces and any material requiring measure-preserving equivalence relations is probably the most technical. A reader only knowing basic functional analysis can read the section on extended von Neumann dimension for groups, provided they roughly understand the construction of the usual von Neumann dimension.

Because it comprises such a small part of the text, I have delegated preliminaries on noncommutative L^p-spaces to the appendix. The material there is essentially a comprehensive introduction to the theory of noncommutative L^p-spaces. In particular, the (nonobvious) fact that the noncommutative L^p-norms are norms is proved in a fairly short manner and in a way that can be generalized to other noncommutative spaces analogous to those appearing with classical analysis, e.g. noncommutative Lorentz spaces. The techniques can also be used to prove that symmetrically normed ideals are in fact normed ideals in a short manner.

CHAPTER 2

Preliminaries

2.1 Von Neumann Algebras

2.1.1 Basic Definitions

In this section, we discuss the concept of a von Neumann algebra, this is a certain algebra of operators on a Hilbert space. It turns out to be quite natural to think of a von Neumann algebra as a "noncommutative measure space". The commutative von Neumann algebras will correspond to measure spaces, and the intuition for many techniques in von Neumann algebra theory come from measure theory.

Definition 2.1.1. Let $\mathcal{H}$ be a Hilbert space. The weak operator topology on $B(\mathcal{H})$ is the locally convex topology defined by the family of pseudonorms $\rho_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle|$, for $\xi, \eta \in \mathcal{H}$. Equivalently, the weak operator topology has the basis of open sets $U_{T,E,F,\varepsilon}$ index by $T \in B(\mathcal{H})$ and finite subsets $E, F \subseteq \mathcal{H}$ and $\varepsilon > 0$

$$U_{T,E,F,\varepsilon} = \bigcap_{\xi\in E,\eta\in F} \{S \in B(\mathcal{H}) : |\langle S\xi, \eta\rangle - \langle T\xi, \eta\rangle| < \varepsilon\}.$$

The *strong operator topology* on $\mathcal{H}$ is the locally convex topology on $\mathcal{H}$ defined by the family of pseudonorms $\rho_\xi(T) = \|T\xi\|$ for $\xi \in \mathcal{H}$. Equivalently the strong operator topology has the following basis of open sets $U_{T,E,\varepsilon}$ indexed by $T \in B(\mathcal{H}), E \subset \mathcal{H}$ finite and $\varepsilon > 0$

$$U_{T,E,\varepsilon} = \bigcap_{\xi\in E,\eta\in F} \{S \in B(\mathcal{H}) : \|S\xi - T\xi\| < \varepsilon\}.$$

These topologies have the following descriptions in terms of nets: if we have a net $T_i \in$

$B(\mathcal{H})$ then $T_i \to T$ in the weak operator topology if and only if for all $\xi, \eta \in \mathcal{H}$,

$$\langle T_i \xi, \eta \rangle \to \langle T\xi, \eta \rangle$$

similarly, $T_i \to T$ in the strong operator topology if and only if for all $\xi \in \mathcal{H}$ we have

$$\|T_i \xi - T\xi\| \to 0.$$

We collect some basic facts about these topologies.

Lemma 2.1.2. *Let $\mathcal{H}$ be a Hilbert space.*

(i): Let $K \subseteq B(\mathcal{H})$ be convex, then $\overline{K}^{\text{SOT}} = \overline{K}^{\text{WOT}}$.

(ii): Let $C \subseteq B(\mathcal{H})$ be a norm bounded set, then $\overline{C}^{\text{WOT}}$ is compact in the weak operator topology.

Proof. (i): It is clear that $\overline{K}^{\text{SOT}} \subseteq \overline{K}^{\text{WOT}}$. For the reverse inclusion, let $T \in \overline{K}^{WOT}$, let $\xi_1, \cdots, \xi_n \in \mathcal{H}$ and $\varepsilon > 0$. Let

$$\Xi = \{((T - S)\xi_1, \cdots, (T - S)\xi_n) : S \in K\}.$$

Then Ξ is a convex subset of $\mathcal{H}^{\oplus n}$, and since $T \in \overline{K}^{\text{WOT}}$, we have $0 \in \overline{\Xi}^{weak}$. As the weak and norm topologies always have the same closed convex sets (see [4] Theorem V.1.4), we know that

$$0 \in \overline{\Xi}^{\|\cdot\|}.$$

Thus, there is some $S \in K$ so that

$$\|(T - S)\xi_j\| < \varepsilon$$

for $j = 1, \cdots, n$, as $\varepsilon > 0, \xi_1, \ldots, \xi_n$ are arbitrary we have $T \in \overline{K}^{\text{SOT}}$.

(ii): As $\{T \in B(\mathcal{H}) : \|T\| \leq R\}$ is closed in the weak operator topology, it suffices by scaling to show that

$$\{T \in B(\mathcal{H}) : \|T\| \leq 1\}$$

is weak operator topology compact. Thinking of

$$\mathcal{H}^{\mathcal{H}}$$

as all functions $\mathcal{H} \to \mathcal{H}$ we have the inclusion

$$\{T \in B(\mathcal{H}) : \|T\| \leq 1\} \subseteq \prod_{\xi \in \mathcal{H}} \{\eta \in \mathcal{H} : \|\eta\| \leq \|\xi\|\}.$$

Call the right hand side F. If we give F the product of the weak topology on $\mathcal{H}$, then we know that F is compact by Tychonoff's theorem. Further the subset

$$\{T \in B(\mathcal{H}) : \|T\| \leq 1\},$$

corresponds to all linear functions in F. This is easily seen to be a closed subset of F, and thus

$$\{T \in B(\mathcal{H}) : \|T\| \leq 1\}$$

is weak operator topology compact.

□

Definition 2.1.3. A *von Neumann algebra* is a subalgebra $M \subseteq B(\mathcal{H})$ which is closed under taking adjoints and the weak operator topology and contains the identity of $B(\mathcal{H})$.

For $X \subseteq B(\mathcal{H})$ we use $X' = \{S \in B(\mathcal{H}) : TS = ST \text{ for all } T \in X\}$, this is called the commutant of X. Note that X' is a von Neumann algebra with the same identity as $B(\mathcal{H})$. It follows that $X'', X''', ...$ are all von Neumann algebras with the same identity as $B(\mathcal{H})$. We would like to prove the double commutant Theorem, which connects commutants to von Neumann algebras. We first need to collect the following facts. For a closed linear subspace $V \subseteq \mathcal{H}$, we use P_V for the orthogonal projection onto V. For a Hilbert space $\mathcal{H}$, we use $\ell^2(\mathbb{N}, \mathcal{H})$ for all functions $f \colon \mathbb{N} \to \mathcal{H}$ such that

$$\sum_{n=1}^{\infty} \|f(n)\|^2 < \infty,$$

the inner product

$$\langle f,g\rangle = \sum_{n=1}^{\infty}\langle f(n),g(n)\rangle$$

turns $\ell^2(\mathbb{N},\mathcal{H})$ into a Hilbert space. We clearly have a similar notion of $\ell^2(k,\mathcal{H})$. For notation, we set $\ell^2(\infty,\mathcal{H}) = \ell^2(\mathbb{N},\mathcal{H})$. For $n \in \mathbb{N}\cup\{\infty\}$, Consider the operators $V_k\colon \mathcal{H} \to \ell^2(n,\mathcal{H})$, for $k \in \mathbb{N}, k \leq n$ defined by $(V_k\xi)(j) = \delta_{j=k}\xi$. For an operator $T \in B(\ell^2(n,\mathcal{H})$, we let $T_{kl} = V_k^* T V_l$. We can think of T as the matrix T_{kl}, for example

$$(T^*)_{kl} = (T_{lk})^*,$$

$$(TS)_{kl} = \sum_{r\in\mathbb{N}:r\leq n} T_{kr}S_{rl}$$

with the sum converging the strong operator topology when $n = \infty$. For a unital von Neumann subalgebra $M \subseteq B(\mathcal{H})$, we let

$$M\overline{\otimes}B(\ell^2(n)) = \{T \in B(\ell^2(n,\mathcal{H})) : T_{kl} \in M \text{ for all } k,l \in \mathbb{N}, k,l \leq n\}$$

$$M\overline{\otimes}1_{\ell^n} = \{T \in B(\ell^2(n),\mathcal{H})) : \text{there is a } x \in M \text{ with } T_{kl} = \delta_{k=l}x\}.$$

Proposition 2.1.4. *Let $X \subseteq B(\mathcal{H})$ contain the identity and be closed under adjoints.*

(i) A closed linear subspace $V \subseteq \mathcal{H}$ is invariant under all the operators in X if and only if $P_V \in X'$.

(ii) For any von Neumann algebra $M \subseteq B(\mathcal{H})$ with

$$(M\overline{\otimes}B(\ell^2(n)))' = M'\overline{\otimes}1_{\ell^2(n)}$$

$$(M\overline{\otimes}1_{\ell^2(n)})' = M'\overline{\otimes}B(\ell^2(n)).$$

Proof. (i): First suppose that $P_V \in X'$. Then for $T \in X, v \in V$ we have

$$P_V(Tv) = T(P_V v) = T(v)$$

so $Tv \in V$. Conversely suppose that V is X-invariant. We first claim that $V^{\perp}$ is X-invariant. For this, suppose that $\xi \in V^{\perp}, v \in V$, then

$$\langle T\xi, v\rangle = \langle \xi, T^*v\rangle$$

as $T^* \in X$ by assumption, we know that $T^*v \in V$, thus

$$\langle \xi, T^*v \rangle = 0$$

so $T\xi \in V^\perp$. Now for $\xi \in \mathcal{H}$, let

$$\xi = \xi_1 + \xi_2$$

with $\xi_1 \in V, \xi_2 \in V^\perp$ so

$$T\xi = T\xi_1 + T\xi_2.$$

As we already showed that $V, V^\perp$ are T-invariant we have that

$$P_V(T\xi) = T\xi_1 = TP_V(\xi)$$

so T commutes with P_V.

(ii): This is a direct computation. □

We also use $W^*(X)$ for the smallest von Neumann subalgebra of $B(\mathcal{H})$.

Theorem 2.1.5 (Double Commutant Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $X \subseteq B(\mathcal{H})$ be a set which contains the identity and is closed under adjoints. Then,*

$$X'' = W^*(X).$$

Proof. It is easy to see that our hypothesis implies that X' is a von Neumann algebra with the same identity as $B(\mathcal{H})$. Also X' is closed under adjoints and contains the identity, thus X'' is a von Neumann algebra with the same identity as $B(\mathcal{H})$, and clearly contains X. Thus

$$X'' \supseteq W^*(X).$$

For the reversion inclusion, let $T \in X''$. Let A be the subalgebra of $B(\mathcal{H})$ generated by X. Then $X' = A'$, so $X'' = A''$, and as A is a $*$-algebra, we know

$$W^*(X) = \overline{A}^{\text{SOT}}.$$

Let $\xi_1, \cdots, \xi_n \in \mathcal{H}$, and $\varepsilon > 0$. Let

$$K = \overline{\{(a\xi_1, \cdots, a\xi_n) : a \in A\}}.$$

Then K is invariant under $W^*(X)\overline{\otimes}1_{\ell^2(n)}$, so by the preceding proposition $P_{\mathcal{K}} \in W^*(X)\overline{\otimes}1_{\ell^2(n)}$. Applying the preceding proposition again, we see that $P_{\mathcal{K}}$ commutes with

$$X''\overline{\otimes}1_{\ell^2(n)}.$$

Thus

$$P_K((T\xi_1, \cdots, T\xi_n)) = T(P_K(\xi_1, \cdots, \xi_n)) = (T\xi_1, \cdots, T\xi_n),$$

the last equality following from the fact that A is unital. Thus there is some $S \in A$ so that

$$\sum_{j=1}^{n} \|(S - T)\xi_j\|^2 < \varepsilon.$$

As $\xi_1, \cdots, \xi_n$ are arbitrary we find that $S \in \overline{A}^{\text{SOT}} = W^*(X)$.

□

2.1.2 Abelian von Neumann Algebras

Since it will greatly help with our intuition, we have decided to single out the case of abelian von Neumann algebras. We shall see that roughly they correspond to measure spaces. Let us first prove the following proposition.

Proposition 2.1.6. *Let (X, μ) be a σ-finite measure space. View $L^\infty(X, \mu) \subseteq B(L^2(X, \mu))$ via multiplication operators. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$, in particular $L^\infty(X, \mu)$ is a von Neumann algebra.*

Proof. Let us first assume that μ is a probability measure. It is clear that $L^\infty(X, \mu) \subseteq L^\infty(X, \mu)'$, suppose that $T \in L^\infty(X, \mu)'$. Set

$$f = T(1),$$

a priori $f \in L^2(X,\mu)$, but we claim that $f \in L^\infty(X,\mu)$ with $\|f\|_\infty \leq \|T\|$. For all $g \in L^\infty(X,\mu)$, we have

$$\|gf\|_2 = \|gT(1)\|_2 = \|T(g)\|_2 \leq \|g\|_2\|T\|.$$

Suppose that $\varepsilon > 0$, and $\mu(\{x \in X : |f(x)| \geq \|T\| + \varepsilon\}) > 0$. Let $f = \alpha|f|$, with α a measurable function and $|\alpha| = 1$ almost everywhere. Set

$$g = \overline{\alpha}\chi_{\{x\in X:|f(x)|\geq\|T\|+\varepsilon\}},$$

then

$$(\|T\|+\varepsilon)\mu(\{x \in X : |f(x)| \geq \|T\|+\varepsilon\})^{1/2} \leq \|fg\|_2 \leq \|g\|_2\|T\| = \mu(\{x \in X : |f(x)| \geq \|T\|+\varepsilon\})\|g\|_2.$$

This is a contradiction, so

$$\|f\|_\infty \leq \|T\|.$$

Let us now handle the σ-finite case. We may find a $\phi \in L^1(X,\mu)$ such that $0 < \phi(x) < \infty$ for almost every x, and

$$\int \phi(x)\,d\mu(x) = 1.$$

Set

$$\nu = \phi\,d\mu.$$

Define $U\colon L^2(X,\mu) \to L^2(X,\nu), V : L^2(X,\nu) \to L^2(X,\mu)$ by

$$U(f) = f\phi^{-1/2}, V(f) = f\phi^{1/2},$$

then U, V are isometries inverse to each other, and so U is a unitary. For $f \in L^\infty(X,\mu) = L^\infty(X,\nu), \xi \in L^2(X,\mu)$ we have

$$U(f\xi) = fU(\xi).$$

As ν, is a probability measure we have by the first case $L^\infty(X,\nu)' = L^\infty(X,\nu)$. Pulling this back via U we find that

$$L^\infty(X,\mu) = L^\infty(X,\mu)'.$$

□

We now prove a converse of this in the separable case.

Theorem 2.1.7. *Let $\mathcal{H}$ be a separable Hilbert space, and let $M \subseteq B(\mathcal{H})$ be an abelian von Neumann algebra. Then, there is a compact metrizable space X, a Borel probability measure μ on X, a sequence $(f_j)_{j=1}^{\infty}$ in $L^1(X,\mu)$, and a unitary*

$$U\colon \mathcal{H} \to \bigoplus_{j=1}^{\infty} L^2(X, f_j\, d\mu),$$

so that if we define

$$\rho\colon L^{\infty}(X,\mu) \to B\left(\bigoplus_{j=1}^{\infty} L^2(X, f_j\, d\mu\right)$$

by

$$\rho(f)(\xi_j)_{j=1}^{\infty} = (f\xi_j)_{j=1}^{\infty},$$

then

$$UMU^* = \rho(L^{\infty}(X,\mu)).$$

Further ρ can be chosen to be an isometry.

Proof. By Zorn's Lemma and separability, we may find a countable set J, and a maximal family $(\xi_j)_{j\in J}$ of vectors in $\mathcal{H}$ such that $\|\xi_j\| = 1$, and $\overline{M\xi_j} \perp \overline{M\xi_k}$ for $j \neq k$ in J. By maximality,

$$\mathcal{H} = \bigoplus_{j\in J} \overline{M\xi_j}.$$

Choose $A \subseteq M$ a unital separable C^*-subalgebra of M with $1_A = 1_M$ and so that

$$M = \overline{A}^{\mathrm{SOT}}.$$

Let X be the Gelfand spectrum of A, and $\Phi\colon A \to C(X)$ the Gelfand isomorphism (see [4] Theorem VIII.2.1) . As A is separable, we know that X is a compact metrizable space. By the Riesz Representation Theorem, we may find Borel probability measures $\mu_j, j \in J$ on X so that

$$\langle a\xi_j, \xi_j\rangle = \int_X \Phi(a)\, d\mu_j, \text{ for } a \in A, j \in J.$$

Because J is countable, we may find positive numbers $b_j, j \in J$ so that

$$1 = \sum_{j \in J} b_j,$$

set

$$\mu = \sum_{j \in J} b_j \mu_j.$$

By Radon-Nikodym, we find a $f_j \in L^1(X, \mu)$ so that

$$d\mu_j = f_j \, d\mu.$$

Define unitaries $U_j \colon \overline{M\xi_j} \to L^2(X, \mu_j)$ by

$$U_j(a\xi_j) = \Phi(a), \;\; a \in A$$

it is easy to see that the above extends uniquely to a unitary operator. Set $U = \bigoplus_{j \in J} U_j$. We claim that these $U, (f_j)_{j \in J}, \mu$ do the trick.

Note that ρ as defined in the statement of the theorem is an isometry in this case. Indeed, as in the preceding proposition one sees that

$$\begin{aligned} \|\rho(f)\| &= \inf\{\alpha \in [0, \infty) : \mu_j(\{x \in X : |f(x)| > \alpha\}) = 0 \text{ for all } j\} \\ &= \inf\{\alpha \in [0, \infty) : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\} = \|f\|_\infty. \end{aligned}$$

To show that $UMU^* = \rho(L^\infty(X, \mu))$ we first prove that $\rho(L^\infty(X, \mu))$ is a von Neumann algebra.

Suppose $T \in \overline{\rho(L^\infty(X, \mu))}^{\text{SOT}}$. Let $A_j = \{x \in X : f_j(x) \neq 0\}$, define

$$\rho_j \colon L^\infty(X, \mu) \to B(L^2(X, f_j \, d\mu)),$$

by

$$\rho_j(f)\xi = f\xi.$$

Note that $\rho_j(L^\infty(X, \mu)) = L^\infty(A_j, \mu)$.

It is straightforward to show that there are $T_j \in \overline{\rho_j(L^\infty(X,\mu))}^{\text{SOT}}$ such that

$$T = \bigoplus_{j\in J} T_j.$$

As $L^\infty(A_j,\mu)$ is a von Neumann algebra by the preceding proposition, we can find $f_j \in L^\infty(A_j,\mu)$ so that

$$T = \bigoplus_{j\in J} \rho_j(f_j).$$

Fix $j,k \in J$. We claim that $f_j(x) = f_k(x)$ for almost every $x \in A_j \cap A_k$. Define $S \in B\left(\bigoplus_{j\in J} L^2(X, f_j\, d\mu)\right)$ by

$$(S\eta)_\alpha = 0 \text{ for } \alpha \in J \setminus \{j,k\},$$

$$(S\eta)_j = \chi_{A_j\cap A_k}\eta_k,$$

$$(S\eta)_k = \chi_{A_k\cap A_k}\eta_j.$$

Then S commutes with $\rho(L^\infty(X,\mu))$, and so S commutes with T. Let $\eta \in \bigoplus_{j=1}^\infty L^2(A_j, f_j\, d\mu)$ be defined by $\eta_\alpha = \chi_{\{j,k\}}(\alpha)1$. As

$$(ST\eta)_k = f_j\chi_{A_j\cap A_k},$$

$$(TS)_j = f_k\chi_{A_j\cap A_k},$$

so

$$f_j\chi_{A_j\cap A_k} = f_k\chi_{A_j\cap A_k}$$

almost everywhere. As J is countable, and $\|f_j\chi_{A_j}\|_\infty \le \|T\|$, we may find a $f \in L^\infty(X,\mu)$ so that $f\chi_{A_j} = f_j\chi_{A_j}$ almost everywhere. Then, $T = \rho(f)$, so $\rho(L^\infty(X,\mu))$ is a von Neumann algebra.

By construction,

$$UAU^* \subseteq \rho(L^\infty(X,\mu)).$$

Since $\rho(L^\infty(X,\mu))$ is a von Neumann algebra, and A is strong operator topology dense in M,

$$UMU^* \subseteq \rho(L^\infty(X,\mu)).$$

Conversely, given $f \in L^\infty(X,\mu)$, choose $f_n \in C(X)$ so that $\|f_n\|_\infty \leq \|f\|_\infty$ and $f_n \to f$ almost everywhere. If $a_n \in A$ is such that $\Phi(a_n) = f_n$, we have

$$Ua_nU^* = \rho(f_n) \to \rho(f),$$

in the strong operator topology. Thus

$$\rho(L^\infty(X,\mu)) \subseteq UMU^*.$$

□

2.1.3 Tracial von Neumann Algebras

Here we define the notion of a tracial von Neumann algebra. Tracial von Neumann algebras will be the von Neumann algebras we will use to extend the usual dimension theory from linear algebra. For terminology, we call a bounded linear map $T\colon M \to N$ between von Neumann algebras *normal* if

$$T\big|_{\{x\in M:\|x\|\leq 1\}}$$

is weak operator topology continuous. For future use, we note the following equivalent conditions for a linear functional to be normal.

Proposition 2.1.8. *Let M be a von Neumann algebra, and $\phi \in M^*$. The following are equivalent.*

(i) ϕ is normal,

(ii) $\ker(\phi) \cap \{x \in M : \|x\| \leq 1\}$ *is weak operator topology closed,*

(iii) $\ker(\phi) \cap \{x \in M : \|x\| \leq 1\}$ *is strong operator topology closed,*

(iv) $\phi\big|_{\{x\in M:\|x\|\leq 1\}}$ *is strong operator topology continuous.*

Proof. We have that (ii) and (iii) are equivalent since the weak operator topology and the strong operator topology have the same closed convex sets. The implications (i) implies (i) implies (ii), and (iii) implies (iv) are clear.For (ii) implies (i), suppose that x_i is a net with

$\|x_i\| \leq 1$, and $x_i \to x$ in the strong operator topology. If $\phi = 0$, the claim is zero. Otherwise, choose $a \in M$ with $\phi(a) = 1$. Since $\phi \in M^*$, we have that $|\phi(x_i)|$ is bounded. Let $x_{i(\alpha)}$ be a subnet of x_i and $t \in \mathbb{C}$ with $\phi(x_{i(\alpha)}) \to t$. Set

$$y_\alpha = \frac{x_{i(\alpha)} - a\phi(x_{i(\alpha)})}{1 + \|a\||\phi(x_{i(\alpha)})|},$$

then $\|y_\alpha\| \leq 1$, $y_\alpha \in \ker(\phi)$, and

$$y_\alpha \to \frac{x - at}{1 + \|a\|t},$$

in the weak operator topology. Thus by assumption,

$$\frac{x - at}{1 + \|a|t} \in \ker(\phi)$$

so

$$\phi(x) = t.$$

We thus find that every subnet of $\phi(x_i)$ converges to $\phi(x)$. As $|\phi(x_i)|$ is bounded, this implies $\phi(x_i) \to \phi(x)$. The implication (iii) implies (iv) is done in the same way.

□

Definition 2.1.9. A *tracial von Neumann algebra* is a pair (M, τ) where $\tau \in M^*$ satisfies

1: $\tau(1) = 1$,

2: $\tau(x^*x) \geq 0$, *with equality if and only if* $x = 0$,

3: $\tau(xy) = \mathrm{Tr} \otimes \tau(yx)$, *for all* $x, y \in M_n(L(\Gamma))$,

4: τ *is normal.*

Given a tracial von Neumann algebra, we define the following inner product on M :

$$\langle x, y \rangle = \tau(y^*x).$$

We let $L^2(M,\tau)$ be the Hilbert space completion of M with respect to this inner product. We define a $*$-representation $\lambda\colon M \to B(L^2(M,\tau))$, and a $*$-anti-representation $\rho\colon M \to B(L^2(M,\tau))$ by

$$\lambda(x)y = xy \text{ for } x, y \in M$$

$$\rho(x)y = yx\text{for } x, y \in M.$$

We need to check that this is well-defined, i.e, that $\lambda(x), \rho(x)$ are $L^2 - L^2$ bounded. But for $y \in M$ we have

$$(xy)^*(xy) = y^*x^*xy \leq \|x\|^2 y^*y,$$

so

$$\|xy\|_2 \leq \|x\|\|y\|_2.$$

Also,

$$\|yx\|_2^2 = \tau(x^*y^*yx) = \tau(yxx^*y^*) \leq \|x\|^2\tau(yy^*) = \|x\|^2\tau(y^*y) = \|x\|^2\|y\|_2^2.$$

Thus

$$\|\lambda(x)\| \leq \|x\|,$$

$$\|\rho(x)\| \leq \|y\|.$$

So λ, ρ extend uniquely to $*$-representations and $*$-anti-representations of M. This will turn out to be a natural way to view a tracial von Neumann algebra, and in fact more natural than whatever Hilbert space M was originally represented on. Additionally define $J\colon L^2(M,\tau) \to L^2(M,\tau)$ densely by

$$J(x) = x^*,$$

for $x \in M$. By traciality we see that $\|J(x)\|_2 = \|x\|_2$ for $x \in M$, so J extends uniquely to a conjugate linear isometry.

We collect a few basic facts here. We recall that if $X \subseteq B(\mathcal{H})$, then

$$X' = \{T \in B(\mathcal{H}) : ST = TS \text{ for all } S \in X\}.$$

Proposition 2.1.10. *Let (M,τ) be a tracial von Neumann algebra, and let λ,ρ be the maps constructed above.*

(i): λ,ρ are injective and in particular are isometric.

(ii): λ,ρ are normal.

(iii):

$$\lambda(M)\subseteq\rho(M)'$$

$$\rho(M)\subseteq\lambda(M)'.$$

(iv): We have $J^2=\mathrm{Id}$. For $\xi,\eta\in L^2(M,\tau)$ we have $\langle\xi,\eta\rangle=\langle J\eta,J\xi\rangle$. Additionally $J(\lambda(x)\xi)=\rho(x^*)J\xi, J(\rho(x)\xi)=\lambda(x^*)J\xi)$ for $x\in M,\xi\in L^2(M,\tau)$.

(v): If $T\in\lambda(M)'$, then there is a unique $\xi\in L^2(M,\tau)$ so that $T(x)=\lambda(x)\xi$ and $T^*(x)=\lambda(x)J\xi$ for $x\in M$. Similarly, if $T\in\rho(M)'$, then there is a unique $\xi\in L^2(M,\tau)$ so that $T(x)=\rho(x)\xi,T^*(x)=\rho(x)J\xi$ for all $x\in M$.

Proof. (i): The "in particular" part follows from the fact that an injective $*$-homomorphisms between C^*-algebras is isometric (see [4] Theorem VIII.4.8, for the statement for ρ we are using the C^*-algebra M^{op}). To see that λ is injective note that $\lambda(x)1=x$. Thus $\lambda(x)=0$ implies that

$$\tau(x^*x)=\|x\|_2=\|\lambda(x)1\|_2=0.$$

(ii): Suppose that $\|x_i\|\leq 1, x_i\to x$ in the weak operator topology. To show that $\lambda(x_i)\to\lambda(x)$ converges in the weak operator topology, it suffices to by density of M to show that when $a,b\in M,\|a\|,\|b\|\leq 1$, we have

$$\langle\lambda(x_i)a,b\rangle\to\langle\lambda(x)a,b\rangle.$$

The left hand-side of the above is

$$\tau(b^*x_ia)$$

since

$$\|b^*x_ia\|\leq 1,$$

$$b^*x_ia \to b^*xa, \text{ in the weak operator topology}$$

normality of τ implies that

$$\tau(b^*x_ia) \to \tau(b^*xa) = \langle \lambda(x)b, a\rangle,$$

the proof for ρ is similar.

(iii): We clearly have

$$\lambda(x)\rho(y)a = \rho(y)\lambda(x)a$$

for $x, y, a \in M$, the claim now follows by density.

(iv): The identities in question can be all checked directly on M, and the general claims follow by density and continuity.

(v): Suppose $T \in \lambda(M)'$, set $\xi = T(1)$, then for $x \in M$,

$$T(x) = T(\lambda(x)1) = \lambda(x)T(1) = \lambda(x)\xi.$$

To show that $T^*(x) = \lambda(x)J\xi$, it suffices to show that $T^*(1) = J\xi$. For $x \in M$ we have

$$\langle T^*(1), x\rangle = \langle 1, \lambda(x)\xi\rangle = \langle x^*, \xi\rangle = \langle J\xi, x\rangle,$$

thus $T^*(1) = J\xi$. $\square$

Theorem 2.1.11. *Let (M, τ) be a tracial von Neumann algebra. Let λ, ρ be the representation and anti-representation on $L^2(M, \tau)$ corresponding to τ. Then,*

$$\lambda(M)' = \rho(M)$$

$$\rho(M)' = \lambda(M).$$

Proof. By the double commutant theorem it suffices to show one of the equalities. By the preceding proposition, we have

$$\rho(M) \subseteq \lambda(M)'.$$

For the reverse inclusion, let $T \in \lambda(M)'$. To show $T \in \rho(M)$ it suffices, by the double commutant theorem, to show T commutes with any $S \in \rho(M)'$. By the preceding proposition, we may find a ξ, η so that

$$T(x) = \lambda(x)\xi, T^*(x) = \lambda(x)J\xi,$$

$$S(x) = \rho(x)\eta, S^*(x) = \rho(x)J\eta$$

for any $x \in M$. Then, for any $x, y \in M$,

$$\langle TS(x), y\rangle = \langle \rho(x)\eta, \lambda(y)J\xi\rangle = \langle \lambda(y^*)\rho(x)\eta, J\xi\rangle = \langle \lambda(y^*)\eta, \rho(x^*)J\xi\rangle,$$

where in the last equality we use that $\lambda(M)$ commutes with $\rho(M)$. Applying part (iv) of the preceding proposition we have

$$\langle TS(x), y\rangle = \langle \lambda(x)\xi, \rho(y)J\eta\rangle = \langle T(x), S^*(y)\rangle = \langle ST(x), y\rangle$$

so $TS = ST$. Thus $T \in \rho(M)'' = \rho(M)$.

□

2.1.4 Definition of von Neumann Dimension

In this section we define von Neumann Dimension for normal representations of tracial von Neumann algebras. The idea is to follow the usual linear algebra formula, "dimension is the trace of a projection." We need to extend the trace to $M\overline{\otimes}B(\ell^2(n))$ for $n \in \mathbb{N} \cup \{\infty\}$.

Definition 2.1.12. Let (M, τ) be a tracial von Neumann algebra, and $n \in \mathbb{N} \cup \{\infty\}$. Let $T \in M\overline{\otimes}B(\ell^2(n))$ be a positive operator, set

$$\operatorname{Tr} \otimes \tau(T) = \sum_{1 \leq i \leq n, i \in \mathbb{N}} \tau(T_{ii}).$$

If $n < \infty$, we consider $\operatorname{Tr} \otimes \tau(T)$ to be defined by the same formula for all $T \in M\overline{\otimes}B(\ell^2(n))$ (and not just positive T)

Note that if $n = \infty$, the sum is of nonnegative terms, and this is always defined, but may be infinite.

Proposition 2.1.13. *Let (M,τ) be a tracial von Neumann algebra, and $n \in \mathbb{N}\cup\{\infty\}$, then we have the following.*

$$\operatorname{Tr}\otimes\tau(1) = n,$$

$$\operatorname{Tr}\otimes\tau(x^*x) = \operatorname{Tr}\otimes\tau(xx^*), \text{ for all } x \in M\overline{\otimes}B(\ell^2(n)),$$

$$\operatorname{Tr}\otimes\tau(x^*x) \geq 0 \text{ for all } x \in M\overline{\otimes}B(\ell^2(n)), \text{ with equality if and only if } x = 0,$$

Further we have the following semi-continuity: if $x^{(i)} \in M\overline{\otimes}B(\ell^2(n))$ and $0 \leq x^{(i)} \leq 1$, and $x^{(i)} \to x$ in the weak operator topology, then

$$\operatorname{Tr}\otimes\tau(x) \leq \liminf_i \operatorname{Tr}\otimes\tau(x^{(i)}),$$

further if $n < \infty$, then

$$\operatorname{Tr}\otimes\tau\big|_{\{x\in M:\|x\|\leq 1\}}$$

is weak operator topology continuous.

Proof. The first statement is clear. For the second, note that

$$(x^*x)_{ii} = \sum_{1\leq k\leq n, k\in\mathbb{N}} (x^*)_{ik}x_{ki} = \sum_{1\leq k\leq n, k\in\mathbb{N}} x_{ki}^*x_{ki},$$

(with the sum converging in the weak operator topology if $n = \infty$), hence

$$\tau((x^*x)_{ii}) = \sum_{1\leq k\leq n, k\in\mathbb{N}} \tau(x_{ki}^*x_{ki}).$$

Thus

$$\operatorname{Tr}\otimes\tau(x^*x) = \sum_{1\leq i\leq n, i\in\mathbb{N}}\ \sum_{1\leq k\leq n, k\in\mathbb{N}} \tau(x_{ki}^*x_{ki}) \geq 0$$

since $\tau(x_{ki}^*x_{ki}) \geq 0$, further the above sum equals zero if and only if $\tau(x_{ki}^*x_{ki}) = 0$ for all k, i which is true if and only if $x_{ki} = 0$ for all k, i. Thus

$$\operatorname{Tr}\otimes\tau(x^*x) \geq 0$$

with equality if and only if $x = 0$.

Since $\tau(x_{ki}^* x_{ki}) \geq 0$, we may interchange the sums to see that

$$\operatorname{Tr} \otimes \tau(x^* x) = \sum_{1 \leq k \leq n, k \in \mathbb{N}} \sum_{1 \leq i \leq n, i \in \mathbb{N}} \tau(x_{ki}^* x_{ki}) = \sum_{1 \leq k \leq n, k \in \mathbb{N}} \sum_{1 \leq i \leq n, i \in \mathbb{N}} \tau(x_{ki} x_{ki}^*)$$

using traciality. As

$$\tau(x_k x_k^*) = \sum_{1 \leq i \leq n, i \in \mathbb{N}} \tau(x_{ki} x_{ki}^*),$$

we have that

$$\operatorname{Tr} \otimes \tau(x^* x) = \operatorname{Tr} \otimes \tau(x x^*).$$

It is clear if $n < \infty$, then $\operatorname{Tr} \otimes \tau\big|_{\{x \in M : \|x\| \leq 1\}}$ is weak operator topology continuous. Suppose $x^{(i)} \in M \overline{\otimes} B(\ell^2(n))$, with $0 \leq x^{(i)} \leq 1$ and $x^{(i)} \to x$ in the weak operator topology. Then, $\tau(x_{jj}^{(i)}) \to \tau(x)$, for all $j \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$,

$$\sum_{j=1}^{k} \tau(x_{jj}) = \lim_i \sum_{j=1}^{k} \tau(x_{jj}^{(i)}) \leq \liminf_i \operatorname{Tr} \otimes \tau(x^{(i)}).$$

Taking the supremum over k completes the proof.

□

Let (M, τ) be a tracial von Neumann algebra. By Theorem 2.1.11, we have a trace τ' on $\lambda(M)' = \rho(M)$, by

$$\tau'(\rho(x)) = \tau(x).$$

Suppose $n \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{H} \subseteq \ell^2(n, L^2(M, \tau))$ is invariant under the diagonal action of M, then by Proposition 2.1.4, we know that $P_{\mathcal{H}} \in \rho(M) \overline{\otimes} B(\ell^2(n))$. Thus, we may define the *von Neumann dimension of* $\mathcal{H}$ by

$$\dim_{(M,\tau)}(\mathcal{H}) = \operatorname{Tr} \otimes \tau'(P_{\mathcal{H}}),$$

if τ is implicit we will often drop it. The next proposition collects some of the basic facts about von Neumann dimension.

Proposition 2.1.14. *Let* (M, τ) *be a tracial von Neumann algebra. Let* $m, n \in \mathbb{N} \cup \{\infty\}$

(i): If $\mathcal{H} \subseteq \mathcal{K} \subseteq \ell^2(n, L^2(M, \tau))$*, we have*

$$\dim_M(\mathcal{H}) \leq \dim_M(\mathcal{K}).$$

(ii): If $\mathcal{H}, \mathcal{K} \subseteq \ell^2(n, L^2(M, \tau))$ *are* M*-invariant for some* $n \in \mathbb{N} \cup \{\infty\}$*, and there is a bounded linear* M*-equivariant map* $\mathcal{H} \to \mathcal{K}$ *with dense image, then* $\dim_M(\mathcal{K}) \leq \dim_M(\mathcal{K})$*. In particular, if* $\mathcal{H} \cong \mathcal{K}$ *as Hilbert* M*-modules, then* $\dim_M(\mathcal{K}) = \dim_M(\mathcal{H})$*.*

(iii): If $\mathcal{H} \subseteq \ell^2(n, L^2(M, \tau))$*,* $\mathcal{K} \subseteq \ell^2(m, L^2(M, \tau))$*, then regarding* $\mathcal{H} \oplus \mathcal{K} \subseteq \ell^2(n + m, L^2(M, \tau))$ *we have*

$$\dim_M(\mathcal{H} \oplus \mathcal{K}) = \dim_M(\mathcal{H}) + \dim_M(\mathcal{K}).$$

(iv): If $\mathcal{H}_k \subseteq \ell^2(n, L^2(M, \tau))$*, are an increasing sequence of closed* M*-invariant linear subspaces, then*

$$\dim_M\left(\overline{\bigcup_k \mathcal{H}_k}\right) = \sup_k \dim_M(\mathcal{H}_k).$$

(v): If $\mathcal{H}_k \subseteq \ell^2(m, L^2(M, \tau))$*, are a decreasing sequence of closed* M*-invariant linear subspaces, and* $\dim_M(\mathcal{H}_1) < \infty$*, then*

$$\dim_M\left(\bigcap_{n=1}^{\infty} \mathcal{H}_k\right) = \inf_n \dim_M(\mathcal{H}_k).$$

Proof. (i) Obvious from the fact that

$$P_{\mathcal{K}} \leq P_{\mathcal{H}}.$$

(ii) Let $T = U|T|$ be the polar decomposition of T (see [4] Theorem VIII 3.11). We leave it as an exercise to verify (by the Spectral Theorem) that

$$U = WOT - \lim_{\varepsilon \to 0} T(|T| + \varepsilon)^{-1},$$

hence we have that $U \in \rho(M)\overline{\otimes}B(\ell^2(n))$. Since T has dense image,

$$U^*U = P_{\ker(T)^\perp} \leq P_{\mathcal{H}},$$

$$UU^* = P_{\mathcal{K}}.$$

Thus,

$$\dim_M(\mathcal{H}) \geq \mathrm{Tr} \otimes \tau(U^*U) = \mathrm{Tr} \otimes \tau(UU^*) \geq \dim_M(\mathcal{K}).$$

(iii): For this, we note that if $\mathcal{H}$ is any Hilbert M-module, and $\mathcal{H}$ is isomorphic to a subspace $\mathcal{K}$ of $\ell^2(\mathbb{N}, L^2(M, \tau))$, then part (i) implies that we can define

$$\dim_M(\mathcal{H}) = \dim_M(\mathcal{K}),$$

and this is independent of the choice of $\mathcal{K}$. Thus we can consider von Neumann dimension to be unambiguously defined for Hilbert M-modules embeddable in $\ell^2(\mathbb{N}, L^2(M, \tau))$. This in particular applies to $\mathcal{H} \oplus \mathcal{K}$ if m or n is infinite. With these comments part (ii) follows from the formula

$$P_{\mathcal{H}} \oplus P_{\mathcal{K}} = P_{\mathcal{H}} + P_{\mathcal{K}}.$$

(iv): Set

$$\mathcal{H} = \overline{\bigcup_k \mathcal{H}_k}.$$

By part (i),

$$\dim_M(\mathcal{H}) \geq \sup_k \dim_M(\mathcal{H}_k).$$

As

$$P_{\mathcal{H}_k} \to P_{\mathcal{H}}$$

in the strong operator topology, we know by the preceding proposition that

$$\dim_M(\mathcal{H}) \leq \liminf_{k \to \infty} \dim_M(\mathcal{H}_k) = \sup_k \dim_M(\mathcal{H}_k).$$

(v): Set $\mathcal{K}_k = \mathcal{H}_1 \cap (\mathcal{H}_k)^\perp$. Then $\mathcal{K}_k$ are increasing, and if we set

$$\mathcal{K} = \overline{\bigcup_k \mathcal{K}_k},$$

we have

$$\mathcal{K} = \mathcal{H} \cap \left(\bigcap_k \mathcal{H}_k \right)^\perp.$$

Part (iii) implies that

$$\dim_M(\mathcal{K}) = \dim_M(\mathcal{H}_1) - \dim_M\left(\bigcap_k \mathcal{H}_k\right),$$

$$\dim_M(\mathcal{K}_k) = \dim_M(\mathcal{H}_1) - \dim_M(\mathcal{H}_k),$$

now part (v) follows from part (iv).

□

As explained in part (ii) of the preceding proposition, if $\mathcal{H}$ is a Hilbert M-module embeddable in $\ell^2(\mathbb{N}, L^2(M,\tau))$ then we unambiguously define $\dim_M(\mathcal{H})$, by choosing a M-invariant closed linear subspace $\mathcal{K} \subseteq \ell^2(\mathbb{N}, L^2(M,\tau))$ isomorphic to $\mathcal{H}$ and setting

$$\dim_M(\mathcal{H}) = \dim_M(\mathcal{K}).$$

Thus we often just assume that $\mathcal{H}$ is embeddable into $\ell^2(\mathbb{N}, L^2(M,\tau))$ without using a specific embedding.

Proposition 2.1.15. *Let (M,τ) be a tracial von Neumann algebra.*

(i): Let

$$0 \longrightarrow \mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3 \longrightarrow 0$$

be a weakly exact sequence (i.e. T is injective, $\overline{\operatorname{im}(T)} = \ker(S), \overline{im(S)} = \mathcal{H}_3$) of Hilbert M-modules embeddable into $\ell^2(\mathbb{N}, L^2(M,\tau))$. Then

$$\dim_M(\mathcal{H}_2) = \dim_M(\mathcal{H}_1) + \dim_M(\mathcal{H}_3).$$

(ii): Let $\mathcal{H}, \mathcal{K}$ be Hilbert M-modules embeddable into $\ell^2(\mathbb{N}, L^2(M,\tau))$, and $T\colon \mathcal{H} \to \mathcal{K}$ be a M-equivariant bounded linear bijection. Then

$$\dim_M(\overline{\operatorname{im}(T)}) = \dim_M(\mathcal{H}) - \dim_M(\ker(T)).$$

Proof. (i): Let $S = U_1|S|, T = U_2|S|$, as in the previous proposition we have that U_1, U_2 are M-equivariant maps. We have that U_1 induces a unitary M-equivariant isomorphism

$(\ker(S))^\perp \cong \mathcal{H}_3$, and U_2 a unitary M-equivariant isomorphism $\ker(S) = \overline{\mathrm{im}(T)} \cong \mathcal{H}_1$. Thus by the preceding proposition,

$$\dim_M(\mathcal{H}_2) = \dim_M(\ker(S)) + \dim_M((\ker(S))^\perp) = \dim_M(\mathcal{H}_1) + \dim_M(\mathcal{H}_3).$$

(ii): Apply (i) to the weakly exact sequence

$$0 \longrightarrow \ker(T) \longrightarrow \mathcal{H} \xrightarrow{T} \overline{\mathrm{im}(T)} \longrightarrow 0.$$

□

2.1.5 Group von Neumann Algebras

Let Γ be a countable discrete group, define the left regular representation and the right anti-regular representations of Γ by

$$\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma)),$$

$$\rho\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$$

by

$$(\lambda(g)f)(h) = f(g^{-1}h),$$

$$(\rho(g)f)(h) = f(hg^{-1}).$$

Let $L(\Gamma)$ be the von Neumann algebra generated by $\lambda(\Gamma)$, and $R(\Gamma)$ be the von Neumann algebra generated by $\rho(\Gamma)$. Define

$$\tau\colon L(\Gamma) \to \mathbb{C},$$

by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle.$$

Theorem 2.1.16. *The pair $(L(\Gamma), \tau)$ is a tracial von Neumann algebra, additionally $L(\Gamma)' = R(\Gamma)$. Let*

$$\mathcal{L} = \{\xi \in \ell^2(\Gamma) : \xi * f \in \ell^2(\Gamma) \textit{ for all } f \in \ell^2(\Gamma).\},$$

$$\mathcal{R} = \{\xi \in \ell^2(\Gamma) : f * \xi \in \ell^2(\Gamma) \textit{ for all } f \in \ell^2(\Gamma).\}.$$

For $\xi \in \mathcal{L}, \eta \in \mathcal{R}$ *and* $f \in \ell^2(\Gamma)$ *define*

$$\lambda(\xi)f = \xi * f,$$

$$\rho(\eta)f = f * \eta.$$

Then $\lambda(\xi), \rho(\eta)$ *are bounded. Further*

$$L(\Gamma) = \{\lambda(\xi) : \xi \in \mathcal{L}\},$$

$$R(\Gamma) = \{\rho(\xi) : \xi \in \mathcal{R}\},$$

and the map

$$\mathcal{L} \to L(\Gamma)$$

defined by

$$\xi \to \lambda(\xi)$$

is a bijection with $\|\xi\|_2 = \|\lambda(\xi)\|_2$.

Proof. It is clear that $\tau \in L(\Gamma)^*$ and is weak operator topology continuous. It is also direct to check that

$$\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)) \tag{2.1}$$

for $g, h \in \Gamma$. For $x, y \in L(\Gamma)$, find nets x_i, y_j where each x_i, y_j are in $\mathrm{Span}\{\lambda(g) : g \in \Gamma\}$ and $x_i \to x, y_j \to y$ in the weak operator topology. By weak operator topology continuity,

$$\tau(xy) = \lim_j \lim_i \tau(x_i y_j).$$

By (2.1), we know that

$$\tau(x_i y_j) = \tau(y_j x_i).$$

By weak operator topology continuity again,

$$\lim_j \lim_i \tau(y_j x_i) = \lim_j \tau(y_j x) = \tau(yx).$$

Thus $\tau(yx) = \tau(xy)$. As $\tau(x^*x) = \|x\delta_e\|_2^2$, it is clear that $\tau(x^*x) \geq 0$. Suppose that $\tau(x^*x) = 0$. Since $\rho(\Gamma) \subseteq \lambda(\Gamma)'$, it is not hard to argue by taking weak limits of linear combinations of $\lambda(g)$, that $\rho(\Gamma) \subseteq L(\Gamma)'$. Thus

$$\|x\delta_g\|_2 = \|x\rho(g)\delta_e\|_2 = \|\rho(g)x\delta_e\|_2 = 0,$$

as $x\delta_e = 0$ by assumption. Since $\mathrm{Span}\{\delta_g : g \in \Gamma\}$ is dense in $\ell^2(\Gamma)$, we find that $x = 0$.

Since

$$\tau(\lambda(h)^{-1}\lambda(g)) = \langle \delta_g, \delta_h \rangle$$

we have a $L(\Gamma)$-equivariant unitary isomorphism

$$U\colon L^2(L(\Gamma), \tau) \to \ell^2(\Gamma)$$

defined by

$$U(\lambda(g)) = \delta_g.$$

If we identify $L^2(L(\Gamma), \tau)$ with $\ell^2(\Gamma)$ via U, then $\rho(\lambda(g))$ becomes $\rho(g)$. Thus

$$L(\Gamma)' = R(\Gamma)$$

by Theorem 2.1.11.

For the last claim, the fact that $\lambda(\xi), \rho(\eta)$ are bounded for $\xi \in \mathcal{L}, \eta \in \mathcal{R}$ follows from the closed graph theorem. For $x \in L(\Gamma)$, set $\xi = x(\delta_e)$. Then

$$x(\delta_g) = x(\rho(g)\delta_e) = \rho(g)x(\delta_e) = \xi * \delta_g.$$

Hence $x(f) = \xi * f$ for all $f \in c_c(\Gamma)$. If $f \in \ell^2(\Gamma)$, choose $f_n \in c_c(\Gamma)$ with

$$\|f - f_n\|_2 \to 0.$$

By Fatou's Lemma,

$$\|\xi * f\|_2 \leq \liminf_{n\to\infty} \|\xi * f_n\|_2 \leq \liminf_{n\to\infty} \|x\|\|f_n\|_2 = \|x\|\|f\|_2.$$

Thus $\xi \in \mathcal{L}$, and

$$\lambda(\xi)f = \lim_{n\to\infty} \lambda(\xi)f_n = \lim_{n\to\infty} x(f_n) = x(f),$$

so $x = \lambda(\xi)$. Further

$$\|\xi\|_2^2 = \|x\delta_e\|_2^2 = \|x\|_2^2.$$

Thus it remains to show that $\lambda(\xi) \in L(\Gamma)$ for all $\xi \in \mathcal{L}$. By the double commutant theorem, it suffices to show that $\lambda(\xi)$ commutes with $R(\Gamma)$. Since $R(\Gamma)$ is generated by $\rho(\Gamma)$, it is enough to show that $\lambda(\xi)$ commutes with $\rho(g)$. But this is clear: $\lambda(\xi)$ is left convolution by ξ, and $\rho(g)$ is right-convolution by δ_g.

□

Because of the above Theorem, if $\mathcal{H}$ is a unitary representation of Γ which is contained in $\ell^2(\mathbb{N}, \ell^2(\Gamma))$ we can define $\dim_{L(\Gamma)}(\mathcal{H}) \in [0,\infty]$ which is an isomorphism invariant with the following properties:

1: $\dim_{L(\Gamma)}(\ell^2(\Gamma)) = 1$,

2: $\dim_{L(\Gamma)}(\mathcal{H} \oplus \mathcal{K}) = \dim_{L(\Gamma)}(\mathcal{H}) + \dim_{L(\Gamma)}(\mathcal{K})$,

3: $\dim_{L(\Gamma)}(\mathcal{H}) \leq \dim_{L(\Gamma)}(\mathcal{K})$ is there is a Γ-equivariant bounded linear map $\mathcal{K} \to \mathcal{H}$ with dense image.

4: $\dim_{L(\Gamma)}(\mathcal{H}) = \sup_n \dim_{L(\Gamma)}(\mathcal{H}_n)$ if $\mathcal{H} = \overline{\bigcup_n \mathcal{H}_n}$ and $\mathcal{H}_n$ are increasing,

5: $\dim_{L(\Gamma)}\left(\bigcup_{n=1}^{\infty} \mathcal{H}_n\right) = \inf_n \dim_{L(\Gamma)}(\mathcal{H}_n)$ if $\mathcal{H}_n$ is a decreasing sequence of closed Γ-invariant subspaces, and $\dim_{L(\Gamma)}(\mathcal{H}_1) < \infty$.

2.1.6 Equivalence Relations and their von Neumann Algebras

Ergodic theory may roughly be stated as the study of group actions on measure spaces. The consideration of a measure-preserving transformation of a probability space is quite natural from probability and statistical mechanics. Considering such a transformation is equivalent to studying the action of the integers on a probability space, and from a mathematical point of view it is quite natural to generalize this to arbitrary group actions. Additionally, many interesting properties of groups may be expressed in terms of their actions. For instance, in the appendix it is proved that an amenable group may be characterized as one for which any action on a compact metrizable space has an invariant measure. Other properties such as Property (T) or the Haagerup property may also be expressed this way.

It turns out to be useful to view the action itself as an analogue of a group. The way to do this is to consider the orbit equivalence relation. By analyzing symmetries of this relation a surprising algebraic structure is developed. Further, many properties of a group are simplified in this way. For example, a group may have that all of its elements have infinite order and be finitely generated, but not finite. Similarly, there are non-amenable groups which do not contain free subgroups. However, if we view groups from the point of view of equivalence relations these complications disappear: every equivalence relation contains a "copy" of $\mathbb{Z}$ (a precise version of this is Theorem B.2.6), and from recent work of Gaboriau-Lyons (see [5]) the Bernoulli action of a non-amenable group always "contains" a copy of the free group. Additionally, we know from the appendix that every amenable group is $\mathbb{Z}$ from the point of view of equivalence relations. This often allows us to reduce properties of amenable groups to $\mathbb{Z}$. Lastly, equivalence relations have connections to operator algebras as they can be axiomatized by certain maximal abelian subalgebras of finite von Neumann algebras (see [10]).

We need some standard notions from descriptive set theory. A *Polish space* is a topological space X which is separable and has a compatible complete metric (we do *not* wish to define a Polish space to be a complete separable metric space, by our definition the irrationals are a Polish space but no sane person would ever call them a complete metric space, similarly open subsets of Polish spaces are Polish and this is blatantly false for complete metric spaces). A set X equipped with a σ-algebra of subsets $\mathcal{B}$ is said to be a standard Borel space if it is isomorphic (as a measurable space) to a Polish space with its algebra of

Borel sets. We will abuse terminology and call $\mathcal{B}$ the algebra of Borel sets. We will in fact commit the greater sin of typically not referencing $\mathcal{B}$ and simply saying that a set is Borel.

Definition 2.1.17. A *discrete, measure-preserving equivalence relation* is a triple $(\mathcal{R}, X, \mu)$ where X is a standard Borel space, μ is a Borel probability measure on X, $\mathcal{R}$ is a Borel subset of $X \times X$ so that

1: *The relation $x \sim y$ if and only if $(x, y) \in \mathcal{R}$ is an equivalence relation,*

2: *for almost every $x \in X$, $\mathcal{O}_x = \{y : (x, y) \in \mathcal{R}\}$ is countable*

3: *for every Borel $B \subseteq \mathcal{R}$,*

$$\int_X |\{y : (x, y) \in B\}| \, d\mu(x) = \int_X |\{x : (x, y) \in B\}| \, d\mu(y).$$

We call the above common quantity $\overline{\mu}(B)$, it follows that $\overline{\mu}$ is a measure on $\mathcal{R}$. If we fix a standard probability space (X, μ) a Borel $\mathcal{R} \subseteq X \times X$ so that $(\mathcal{R}, X, \mu)$ is a discrete, measure-preserving equivalence relation will be called a *discrete, measure-preserving measure-preserving equivalence relation on* (X, μ).

Note that equivalence relations have a nice "localization" property that is absent in discrete groups. Namely, if $A \subseteq X$ is measurable, we have a new equivalence relation $(\mathcal{R}_A, A, \frac{\mu}{\mu(A)})$ where

$$\mathcal{R}_A = \{(x, y) \in A \times A : (x, y) \in \mathcal{R}\}.$$

We call $\mathcal{R}_A$ the compression of $\mathcal{R}$ by A.

Let $\mathcal{R}_i, i = 1, 2$ be a discrete, measure-preserving equivalence relation on $(X_i, \mu_i), i = 1, 2$. We say that $\mathcal{R}_1$ is isomorphic to $\mathcal{R}_2$ if there is a bimeasurable bijection

$$\theta \colon X_1 \to X_2$$

with $\theta_* \mu_1 = \mu_2$ and

$$\theta(\mathcal{O}_x) = \mathcal{O}_{\theta(x)},$$

for almost every $x \in X$. A *partial morphism* on $\mathcal{R}$ is a bimeasurable bijection $\phi\colon \operatorname{dom}(\phi) \to \operatorname{ran}(\phi)$, where $\operatorname{dom}(\phi), \operatorname{ran}(\phi)$ are measurable subsets of X, such that $(x, \phi(x)) \in \mathcal{R}$ for almost every $x \in \operatorname{dom}(\phi)$. We let $[[\mathcal{R}]]$ be the set of partial morphisms of $\mathcal{R}$, we identify $\phi, \psi \in [[\mathcal{R}]]$ if $\mu(\operatorname{dom}(\phi)\Delta \operatorname{dom}(\psi)) = 0$, and $\phi(x) = \psi(x)$ for almost every $x \in X$. We let $[\mathcal{R}]$ be the set of $\phi \in [[\mathcal{R}]]$ so that $\mu(\operatorname{dom}(\phi)) = .1$ For $A \subseteq X$ measurable, we let Id_A be partial morphisms with $\operatorname{dom}(\operatorname{Id}_A) = A, \operatorname{ran}(\operatorname{Id}_A) = A$, and $\operatorname{Id}_A(x) = x$ for all $x \in A$.

The space of partial morphisms admits some useful algebraic structure. For $\phi, \psi \in [[\mathcal{R}]]$ we define $\phi\psi \in [[\mathcal{R}]]$ by

$$\operatorname{dom}(\phi\psi) = \psi^{-1}(\operatorname{dom}(\phi)) \cap \operatorname{dom}(\psi),$$

and $\phi\psi(x) = \phi(\psi(x))$ for $x \in \operatorname{dom}(\phi\psi)$. For $\phi \in [[\mathcal{R}]]$, we let ϕ^{-1} be the element of $[[\mathcal{R}]]$ with $\operatorname{dom}(\phi^{-1}) = \operatorname{ran}(\phi), \operatorname{ran}(\phi^{-1}) = \operatorname{dom}(\phi)$ and is the inverse to ϕ on $\operatorname{ran}(\phi)$. We let

$$\operatorname{graph}(\phi) = \{(x, \phi(x)) : x \in \operatorname{dom}(\phi)\}.$$

If $\phi, \psi \in [[\mathcal{R}]]$ and $\overline{\mu}(\operatorname{graph}(\phi) \cap \operatorname{graph}(\psi)) = 0$, we let $\phi + \psi \in [[\mathcal{R}]]$ be defined by

$$\operatorname{dom}(\phi+\psi) = \{x \in \operatorname{dom}(\phi)\setminus\operatorname{dom}(\psi) : \phi(x) \notin \operatorname{ran}(\psi)\} \cup \{x \in \operatorname{dom}(\psi)\setminus\operatorname{dom}(\phi) : \psi(x) \notin \operatorname{ran}(\phi)\},$$

and

$$(\phi + \psi)(x) = \begin{cases} \phi(x), \text{if } x \in \operatorname{dom}(\phi + \psi) \cap \operatorname{dom}(\phi) \\ \psi(x), \text{if } x \in \operatorname{dom}(\phi + \psi) \cap \operatorname{dom}(\psi) \end{cases}.$$

Lastly, we define a distance on $[[\mathcal{R}]]$ by

$$d_{[[\mathcal{R}]]}(\phi, \psi)^2 = \mu(\operatorname{dom}(\phi)\Delta \operatorname{dom}(\psi)) + 2\mu(\{x \in \operatorname{dom}(\phi) \cap \operatorname{dom}(\psi) : \phi(x) \neq \psi(x)\}).$$

We discuss one example. Let X be a standard Borel space and μ a Borel probability measure on X. Suppose that Γ is a countable discrete group and $\Gamma \curvearrowright (X, \mu)$ by measure-preserving transformations. We can then define the *orbit equivalence relation* of Γ by

$$\mathcal{R}_{\Gamma \curvearrowright (X,\mu)} = \{(x, gx) : g \in \Gamma\}.$$

It is easy to check that this is a discrete-measure preserving equivalence relation on (X, μ). We will call this the orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$. If the action is *free*, i.e. for all $g \in \Gamma \setminus \{e\}$,

$$\mu(\{x \in X : gx = x\}) = 0,$$

then we expect properties of the equivalence relations to reflect properties of the group, (e.g. see the appendix on amenable equivalence relations and groups). We say that two countable discrete groups Γ and Λ are *orbit equivalent* if they admit free actions on standard diffuse probability spaces whose corresponding orbit equivalence relations are isomorphic.

It will be useful to have the following measurable selection principle first proved by von Neumann. Recall that a subset of a standard Borel space is said to be *analytic* if it is the image of a Borel subset of a standard Borel space under a Borel map. It is known that such sets are universally measurable (i.e. measurable with respect to every Borel probability measure) see [22] Theorem 4.3.1.

Theorem 2.1.18 (Measurable Selection Principle)**.** *Let X, Y be Polish spaces, and $A \subseteq X \times Y$ analytic. Let $\pi\colon X \times Y \to X$ be the projection onto the first factor. Then there is a universally measurable function $\phi\colon \pi(A) \to Y$ so that $\pi \circ \phi = \operatorname{Id}$.*

We leave it is an exercise to the reader to prove the following from the measurable selection principle.

Corollary 2.1.19. *Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving equivalence relation. Then there is a countable $(\phi_j)_{j \in J}$ of elements of $[[\mathcal{R}]]$ with disjoint graphs and such that*

$$\overline{\mu}\left(\mathcal{R} \setminus \bigcup_{j \in J} \operatorname{graph}(\phi_j)\right) = 0.$$

For later use, we would like to discuss when an equivalence relation can act on a Banach space.

Definition 2.1.20. Let V be a Banach space, and $(\mathcal{R}, X, \mu)$ a discrete, measure-preserving equivalence relation. A *representation* of $[[\mathcal{R}]]$ on V is a map $\pi\colon [[\mathcal{R}]] \to B(V)$ so that

$$\pi(\operatorname{Id}_X) = \operatorname{Id}_{B(V)},$$

$$\pi(\phi\psi) = \pi(\phi)\pi(\psi) \textit{ for } \phi, \psi \in [[\mathcal{R}]]$$

$$\pi(\phi + \psi) = \pi(\phi) + \pi(\psi) \textit{ if } \phi, \psi \in [[\mathcal{R}]] \textit{ and } \overline{\mu}(\mathrm{graph}(\psi) \cap \mathrm{graph}(\phi)) = 0$$

$$\pi(\phi_n)v \to \pi(\phi)v \textit{ for all } v \in V, \textit{ if } d_{[[\mathcal{R}]]}(\phi_n, \phi) \to 0.$$

$$\pi(\phi) = 0 \textit{ if } \mu(\mathrm{dom}(\phi)) = 0.$$

We say the representation is *uniformly bounded* if there is a $C > 0$ so that $\|\pi(\phi)\| \leq C$ for all $\phi \in [[\mathcal{R}]]$. If V is a Hilbert space, we say the action is *unitary* if $\pi(\phi^{-1}) = \pi(\phi)^*$ for all $\phi \in [[\mathcal{R}]]$.

Since we are identifying two elements of $[[\mathcal{R}]]$ if they differ on set of measure zero, implicit in the above definition is that $\pi(\phi) = \pi(\psi)$ if they differ on a set of measure zero. Also, we will frequently drop π and write ϕv instead of $\pi(\phi)v$. Here is a natural example: for $\phi \in [[R]]$, and $1 \leq p \leq \infty$, and $\xi \in L^p(\mathcal{R}, \mu)$, we define

$$(\phi\xi)(x, y) = \chi_{\mathrm{ran}(\phi)}\xi(\phi^{-1}(x), y).$$

We note that for $1 \leq p \leq \infty$, we have a natural way for $L^\infty(X, \mu)$ to act on $L^p(\mathcal{R}, \overline{\mu})$ by

$$(gf)(x, y) = g(x)f(x, y),\ f \in L^p(\mathcal{R}, \overline{\mu}), g \in L^\infty(X, \mu).$$

This gives to the von Neumann algebra of $\mathcal{R}$.

Definition 2.1.21. Let $\mathcal{R}$ be a measure-preserving equivalence relation on the standard probability space (X, μ). We let $L(\mathcal{R}) = W^*(\{\phi : \phi \in [[\mathcal{R}]]\})$, (under the above action of $\mathcal{R}$ on $L^2(\mathcal{R}, \overline{\mu})$) and we define $\tau \colon L(\mathcal{R}) \to \mathbb{C}$ by

$$\tau(x) = \langle x\chi_\Delta, \chi_\Delta \rangle,$$

where $\Delta = \{(x, x) : x \in X\}$.

We also define an anti-representation

$$\rho\colon [[\mathcal{R}]] \to B(L^2(R, \overline{\mu}))$$

by

$$(\rho(\phi)f)(x, y) = \chi_{\mathrm{dom}(\phi)}(y)f(x, \phi(y)).$$

Theorem 2.1.22. *(i): The pair $(L(\mathcal{R}), \tau)$ is a tracial von Neumann algebra.*

(ii): $L(\mathcal{R})' = W^*(\rho([[\mathcal{R}]]))$.

(iii): We have a canonical inclusion $L^\infty(X, \mu) \subseteq L(\mathcal{R})$ defined densely by $\chi_A \to \mathrm{Id}_A$.

(iv): If we set $\mathcal{N} = \{u \in \mathcal{U}(L(\mathcal{R})) : uL^\infty(X, \mu)u^* = L^\infty(X, \mu)\}$, then $L(\mathcal{R}) = W^*(N)$.

Proof. (i) As in the group case, the linearity and weak operator topology continuity are clear. For $\phi \in [[\mathcal{R}]]$, we have that

$$\begin{aligned}\tau(\phi) &= \mu(\{x \in \mathrm{ran}(\phi) : \phi^{-1}(x) = x\}) \\ &= \mu(\{x \in \mathrm{dom}(\phi) : \phi(x) = x\}),\end{aligned}$$

where in the last line we use the measure-preserving transformation $x \mapsto \phi(x)$.

Thus

$$\begin{aligned}\tau(\phi\psi) &= \mu(\{x \in \mathrm{dom}(\phi\psi) : \phi\psi(x) = x\}) \\ &= \mu(\{x \in \mathrm{dom}(\psi) \cap \psi^{-1}(\mathrm{dom}(\phi) \cap \mathrm{ran}(\psi)) : \phi\psi(x) = x\}) \\ &= \mu(\{x \in \mathrm{dom}(\psi) \cap \psi^{-1}(\mathrm{dom}(\phi) \cap \mathrm{ran}(\psi)) : \psi(x) = \phi^{-1}(x)\}).\end{aligned}$$

Now apply the measure-preserving transformation $x \mapsto \psi(x)$, we see that

$$\tau(\phi\psi) = \mu(\{x \in \mathrm{ran}(\psi) \cap \mathrm{dom}(\phi) : \phi(x) = \psi^{-1}(x)\}).$$

It is not hard to show that

$$\{x \in \mathrm{ran}(\psi) \cap \mathrm{dom}(\phi) : \phi(x) = \psi^{-1}(x)\} = \{x \in \mathrm{dom}(\psi\phi) : \psi\phi(x) = x\}).$$

Thus

$$\tau(\phi\psi) = \mu(\{x \in \mathrm{dom}(\psi\phi) : \psi\phi(x) = x\}) = \tau(\psi\phi).$$

Playing the same tricks as in the group case with weak convergence shows that

$$\tau(xy) = \tau(yx)$$

for $y, x \in L(\mathcal{R})$.

We have that

$$\tau(x^*x) = \|x\chi_\Delta\|_2^2 \geq 0.$$

Note that $\rho([[\mathcal{R}]])$ commutes with every v_ϕ and thus with $L(\mathcal{R})$. A direct computation shows that $\rho(\phi)\chi_\Delta = \chi_{G(\phi^{-1})}$, where $G(\phi^{-1})$ is the graph of ϕ^{-1}. Thus if $\tau(x^*x) = 0$, we find that

$$\|x\chi_{G(\phi^{-1})}\|_2 = \|\rho(\phi)x\chi_\Delta\|_2 = 0,$$

as $x\chi_\Delta = 0$. Since $\mathcal{R}$ can be written as the union (up to sets of measure zero) of graphs of partial morphisms, we have that

$$\overline{\operatorname{Span}\{\chi_{G(\phi)} : \phi \in [[\mathcal{R}]]\}} = L^2(\mathcal{R}, \overline{\mu}).$$

Hence, $x = 0$. Thus $(L(\mathcal{R}), \tau)$ is a tracial von Neumann algebra.

(iii) For this, define $\Psi\colon L^\infty(X, \mu) \to B(L^2(\mathcal{R}, \overline{\mu}))$ by

$$(\Psi(f)\xi)(x, y) = f(x)\xi(x, y).$$

Regarding $L^\infty(X, \mu)$ as represented on $L^2(X, \mu)$ we have that Ψ is weak operator topology continuous, and $\Psi(\chi_A) = v_{\mathrm{Id}_A}$. Weak operator topology continuity implies that $\Psi(L^\infty(X, \mu)) \subseteq L(\mathcal{R})$. Additionally, it is straightforward to check $\|\Psi(f)\| = \|f\|_\infty$.

(iv) This follows from the fact that every partial morphism has an extension to an element in the full group.

□

Again, as in the group case, the above theorem implies that we have a dimension theory for representations of $\mathcal{R}$ contained in $\ell^2(\mathbb{N}, L^2(\mathcal{R}, \overline{\mu}))$.

For later use, we mention another example of representations. Let (X, μ) be a standard probability space and $\Gamma \curvearrowright (X, \mu)$ a free measure-preserving action. Define the *Zimmer cocycle*

$$\theta\colon \mathcal{R} \to \Gamma$$

by

$$\theta(x,y)y = x.$$

Given a Banach space V and a representation $\rho\colon \Gamma \to B(V)$, for $1 \leq p \leq \infty$, we define a representation $\pi\colon [[R_{\Gamma \curvearrowright (X,\mu)}]] \to B(L^p(X,\mu,V))$ by

$$(\pi(\phi)\xi)(x) = \chi_{\mathrm{ran}(\phi)}(x)\rho(\theta(x,\phi(x)))\xi(\phi^{-1}x).$$

For $x,y,z \in X$ we have

$$\theta(x,z) = \theta(x,y)\theta(y,z),$$

and from this it is not hard to show that ρ is a representation.

2.1.7 Basic Properties of Equivalence Relations

Though it will be slightly disjoint from the rest of the material in this chapter, since we have just introduced equivalence relations we would like to mention some of their basic properties. These properties will be used frequently in Chapter 4. We first note the following version of the Ergodic Decomposition.

Theorem 2.1.23 ([26] Theorem 4.2)**.** *Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving equivalence relation. Then, there is a standard measure space (Y,ν) a measurable map $\pi\colon X \to Y$ with $\pi_*\mu = \nu$, and probability measures $(\mu_y)_{y\in Y}$ on X with the following properties.*

1: For $\overline{\mu}$-almost every $(x,y) \in \mathcal{R}$ we have $\pi(x) = \pi(y)$,

2: for almost every $y \in Y$, $\mu_y(\pi^{-1}(\{y\})) = 1$,

3: for all $B \subseteq X$ measurable, the map $y \mapsto \mu_y(B)$ is ν-measurable,and

$$\int_Y \mu_y(B)\, d\mu(y) = \mu(B),$$

4: for almost every $y \in Y$, the equivalence relation $\mathcal{R}_y = \{(p,q) : \pi(p) = \pi(q) = y, (p,q) \in \mathcal{R}\}$ is a well-defined measurable equivalence relation on $(\pi^{-1}(\{y\}), \mu_y)$ and is ergodic.

The relations $(\mathcal{R}_y, \pi^{-1}(\{y\}), \mu_y)$ are typically called the "ergodic components" of $(\mathcal{R}, X, \mu)$. The next property has as particular a consequence that, from the point of view of equivalence relations, every infinite group "measurably contains" a copy of $\mathbb{Z}/n\mathbb{Z}$. This is one instance in which equivalence relations can be used to fix the complicated subgroup structure of a group.

Proposition 2.1.24. *Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving, equivalence relation and suppose that $\mathcal{O}_x$ is infinite for almost every $x \in X$. Then for every $n \in \mathbb{N}$, there is a free, measure-preserving action $\mathbb{Z}/n\mathbb{Z} \curvearrowright (X, \mu)$ so that*

$$\mathcal{R}_{\mathbb{Z}/n\mathbb{Z} \curvearrowright (X,\mu)} \subseteq \mathcal{R}.$$

Proof. Note that almost every ergodic component has infinite orbits almost everywhere. Thus we may as well assume that $\mathcal{R}$ is ergodic. We leave it is an exercise to the reader to use the measurable selection theorem to show that if $A, B \subseteq X$ have equal measure and are disjoint then there is a $\phi \in [[\mathcal{R}]]$ with $\operatorname{dom}(\phi) = A, \operatorname{ran}(\phi) = B$ (ignoring sets of measure zero). Since $\mathcal{R}$ has infinite orbits almost everywhere, we must have that (X, μ) is diffuse. Thus, we may find disjoint measurable subsets $A_1, \dots, A_n$ in X so that

$$\mu(A_j) = \frac{1}{n} \text{ for } 1 \leq j \leq n.$$

Thus our preceding remarks imply that we can find $\phi_1 \in [[\mathcal{R}]], 1 \leq j \leq n$, with $\operatorname{dom}(\phi_j) = A_j, \operatorname{ran}(\phi_j) = A_{j+1}$ for $1 \leq j \leq n-1$, and $\operatorname{dom}(\phi_n) = A_n, \operatorname{ran}(\phi_n) = A_1$. Define $\alpha \colon X \to X$ by

$$\alpha\big|_{A_j} = \phi_j,$$

it is straightforward to check that α induces a free action of $\mathbb{Z}/n\mathbb{Z}$. By definition,

$$\mathcal{R}_{\mathbb{Z}/n\mathbb{Z} \curvearrowright (X,\mu)} \subseteq \mathcal{R}.$$

□

2.2 Sofic Groups and Equivalence Relations

Definition 2.2.1. Let Γ be a countable discrete group. A *sofic approximation of* Γ is a sequence $\Sigma = (\sigma_i \colon \Gamma \to S_{d_i})$ of functions (not assumed to be homomorphisms) such that

1: $d_i \to \infty$

2: $u_{d_i}(\{j : (\sigma_i(g)\sigma_i(h))(j) = \sigma_i(gh)(j)\}) \to 1$, for all $g, h \in \Gamma$

3: $u_{d_i}(\{j : \sigma_i(g)(j) \neq \sigma_i(h)(j)\}) \to 1$, for all $g, h \in \Gamma, g \neq h$.

We say that Γ is *sofic* if it has a sofic approximation.

We could remove the condition $d_i \to \infty$, and still have the same definition of a sofic group. However, in order for the definition of topological entropy to be an invariant we need $d_i \to \infty$. The condition $d_i \to \infty$ is also implied if Γ is infinite, which will be the main case we are interested in anyway. It is known that the class of sofic groups contain all amenable groups, all residually sofic groups, all locally sofic groups, all linear groups and is closed under free products with amalgamation over amenable subgroups. For more see [9],[8],[6]. Let us mention another condition related to soficity for a group. On $M_n(\mathbb{C})$ we define

$$\operatorname{tr}(A) = \frac{1}{n}\sum_{j=1}^{n} A_{jj},$$

this is the canonical tracial state on $M_n(\mathbb{C})$. We use $\|A\|_2$ for the L^2 norm with respect to this trace.

Definition 2.2.2. Let Γ be a countable discrete group. An *embedding sequence of* Γ is a sequence $\Sigma = (\sigma_i \colon \Gamma \to U(d_i))$ so that

1: $d_i \to \infty$

2: $\|\sigma_i(g)\sigma_i(h) - \sigma_i(gh)\|_2 \to 0$ for all $g, h \in \Gamma$

3: $\operatorname{tr}(\sigma_i(g)^{-1}\sigma_i(h)) \to 0$, for all $g, h \in \Gamma, g \neq h$.

We say that Γ is *$\mathcal{R}^\omega$-embeddable* if it has an embedding sequence.

The terminology comes from the Connes Embedding problem, as one can show that $L(\Gamma)$ embeds into a tracial ultrapower of $\mathcal{R}$ if and only if it has an embedding sequence. Viewing $S_n \subseteq U(n)$ and using that

$$\|\sigma - \tau\|_2^2 = \frac{|\{j : \sigma(j) = \tau(j)\}|}{n},$$

for $\sigma, \tau \in S_n$ we see that every sofic group is $\mathcal{R}^\omega$-embeddable.

Following our philosophy that one should try to study a group by its action on measure-spaces it makes sense to study *sofic* equivalence relations. For notation, we use $[[\mathcal{R}_n]]$ for the equivalence relation on $(\{1, \dots, n\}, u_n)$ defined by declaring all points to be equivalent.

Definition 2.2.3. Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving equivalence relation. A *sofic approximation of* $[[\mathcal{R}]]$, is a sequence of functions $\sigma_i \colon [[\mathcal{R}]] \to [[\mathcal{R}_{d_i}]]$ such that

1: $d_i \to \infty$,

2: $d_{[[\mathcal{R}_n]]}(\sigma_i(\phi\psi), \sigma_i(\phi)\sigma_i(\psi)) \to 0$ *for* $\phi, \psi \in [[\mathcal{R}]]$,

3: for all $A \subseteq X$ measurable, there is a $A_i \subseteq \{1, \dots, d_i\}$ so that $\sigma_i(\mathrm{Id}_A) = \mathrm{Id}_{A_i}$,

4: for all $\phi \in [[\mathcal{R}]]$, $\frac{|\{j \in \mathrm{dom}(\sigma_i(\phi)) : \sigma_i(\phi)(j) = j\}|}{d_i} \to \mu(\{x \in \mathrm{dom}(\phi) : \phi(x) = x\})$.

We say that $\mathcal{R}$ is *sofic* if it has a sofic approximation.

This definition is due to Elek and Lippner in [7], and they proved many important properties of sofic equivalence relations. We will need such finite approximations to define extended von Neumann dimension. The point is that since S_n has a natural action on $\ell^p(n)$, we can think of these maps as giving an "almost action" of our group or equivalence relation on $\ell^p(d_i)$. Following the spirit of Lewis Bowen, David Kerr and Hanfeng Li, as well as ideas of Voiculescu we will show that von Neumann dimension of a unitary representation $\Gamma \curvearrowright \mathcal{H}$ can be computed as a normalized limit of the "size" of a space of almost equivariant maps

$\mathcal{H} \to \ell^2(d_i)$. Since the Hilbert space structure places no role, we will be able to remove it and consider almost equivariant maps $X \to \ell^p(d_i)$, when X is a Banach space and $\Gamma \curvearrowright X$. We proceed to express the main properties that go into this fact, but we will work in a more general situation than just groups and equivalence relations.

For the next definition we need some terminology. Fix a set E, the universal $\mathbb{C}$-algebra generated by elements $(X_a)_{a\in E}, (X_a^*)_{a\in E}$ will be called the algebra of $*$-polynomials in n noncommuting variables and will be denoted by

$$\mathbb{C}^*\langle X_a : a \in E\rangle.$$

Elements of this algebra will be called $*$-polynomials. The algebra $\mathbb{C}^*\langle X_a : a \in E\rangle$ has a unique conjugate linear involution $P \mapsto P^*$ for $P \in \mathbb{C}^*\langle X_a : a \in E\rangle$ which maps X_a to X_a^* and such that $(PQ)^* = Q^*P^*$, for $P, Q \in \mathbb{C}^*\langle X_a : a \in E\rangle$. Given $(x_a)_{a\in E}$ in a von Neumann algebra M there is a unique homomorphism $\phi\colon \mathbb{C}^*\langle X_a : a \in E\rangle \to M$ mapping X_a to x_a and such that

$$\phi(P^*) = \phi(P)^*, \text{ for } P \in \mathbb{C}^*\langle X_a : a \in E\rangle.$$

We let $P(x_a : a \in E)$ denote $\phi(P)$.

Definition 2.2.4. Let (M, τ) be a tracial von Neumann algebra, and $E \subseteq M$. An *embedding sequence* for E is a sequence E is a sequence of functions $\sigma_i\colon E \to M_{d_i}(\mathbb{C})$ such that

1: $\sup_i \|\sigma_i(x)\|_\infty < \infty$ for all $x \in E$

2: for all $x_1, \dots, x_n \in E$ and all $*$-polynomials P in n non-commuting variables, $\operatorname{tr}(P(\sigma_i(x_1), \dots, \sigma_i(x_n))) \to \tau(P(x_1, \dots, x_n))$.

Note that sofic approximations and embedding sequences for equivalence relations and groups are embedding sequences in the sense of the above definition viewing $\Gamma \subseteq L(\Gamma)$, $[[\mathcal{R}]] \subseteq L(\mathcal{R})$. For the proof of the next lemma, we need the following definition.

Definition 2.2.5. Let (M, τ) be a tracial von Neumann algebra, and $a \in M_n(M)$ a normal element. The *spectral measure with respect to τ of a* is the measure μ_a on the spectrum of a defined by $\mu_a(E) = \operatorname{Tr}\otimes\tau(\chi_E(a))$ for all Borel subsets E of the spectrum of a.

Lemma 2.2.6. *Let (M, τ) be a tracial von Neumann algebra, and let $E \subseteq M$ be a subset so that $M = W^*(x)$. Then, every embedding sequence of $(\sigma_i\colon E \to M_{d_i}(\mathbb{C}))$ extends to one of M.*

Proof. Let $A = \{P(\alpha : \alpha \in E) : P \in \mathbb{C}^*\langle X_\alpha : \alpha \in E\rangle\}$. We first extend σ to A. For each $a \in A$, choose a $P_a \in \mathbb{C}^*\langle X_\alpha : \alpha \in E\rangle$ with $a = P_a(\alpha : \alpha \in E)$. For $a \in A$, set

$$\sigma_i(a) = P_a(\sigma_i(\alpha) : \alpha \in E)$$

Using that we may "compose" $*$-polynomials, it is not hard to see that σ_i is an embedding sequence for A. Thus, we may assume that $E = A$. We first prove some preliminary claims.

Claim 1: Let $a \in A$ we claim that If $(T_{a,i})_{a \in A}$ are in $M_{d_i}(\mathbb{C})$ and

$$\|T_{a,i} - \sigma_i(a)\|_2 \to 0,$$

$$\sup_i \|T_{a,i}\|_\infty < \infty,$$

then the map $\widetilde{\sigma}_I\colon A \to M_{d_i}(\mathbb{C})$ defined by

$$\widetilde{\sigma}_i(a) = T_{a,i}$$

is an embedding sequence for A.

The claim follows from the inequality

$$\|TS - XY\|_2 \leq \|T\|_\infty \|S - Y\|_2 + \|S\|_\infty \|T - X\|_2$$

for elements T, S, X, Y in a tracial von Neumann algebra.

Claim 2: For all $a \in A$:

$$\mu_{\sigma_i(a)^*\sigma_i(a)} \to \mu_{a^*a}.$$

For this, it is trivial from the definition of embedding sequence that if P is a polynomial then

$$\int P\, d\mu_{\sigma_i(a)^*\sigma_i(a)} \to \int P\, d\mu_{a^*a}.$$

The general claim follows from the fact that the measures $\mu_{\sigma_i(a)^*\sigma_i(a)}$ have uniformly bounded supports and the Weierstrass approximation theorem.

Claim 3: For all $a \in A$, there are $a_i \in M_{d_i}(\mathbb{C})$ so that

$$\|a_i\|_\infty \leq \|a\|_\infty,$$

and

$$\|a_i - \sigma_i(a)\|_2 \to 0.$$

Let $\sigma_i(a) = u_i|\sigma_i(a)|$ be the polar decomposition. Let $\phi \in C_c([0,\infty))$ be a continuous function with $\phi(t) = t$ for $t \in [0, \|a\|_\infty]$, and $|\phi(t)| \leq \|a\|_\infty$ for all $t \in [0,\infty)$. Set

$$a_i = u_i\phi(|\sigma_i(a)|).$$

Then,

$$\begin{aligned} \||a_i - \sigma_i(a)\|_2^2 \leq \|\sigma_i(a) - \phi(|\sigma_i(a)|)\|_2^2 &= \int_{[0,\infty)} |\phi(t^{1/2}) - t^{1/2}|^2 \, d\mu_{\sigma_i(a)^*\sigma_i(a)}(t) \\ &\to \int_{[0,\infty)} |\phi(t^{1/2}) - t^{1/2}|^2 \, d\mu_{a^*a}(t) \\ &= 0, \end{aligned}$$

by Claim 2, the fact that $\phi(t) = t$ on $[0, \|a\|_\infty]$, and the fact that μ_{a^*a} is supported on $[0, \|a\|_\infty^2]$.

We now prove the Lemma. By Claim 1 and Claim 3, we may assume that

$$\|\sigma_i(a)\|_\infty \leq \|a\|_\infty$$

for all $a \in A$. Let $x \in M \setminus A$. By Kaplansky's Density Theorem, we may find a sequence $a_{n,x} \in A$ so that

$$\|a_{n,x}\|_\infty \leq \|x\|_\infty$$

and

$$\|a_{n,x} - x\|_2 < 2^{-n}.$$

Choose an increasing sequence of integers i_n so that if $i \geq i_n$, then for al1 $\leq j, k \leq n$,

$$|\|\sigma_i(a_{j,x}) - \sigma_i(a_{k,x})\|_2 - \|a_{j,x} - a_{k,x}\|_2| < 2^{-n}.$$

Set $\sigma_i(x) = a_{n,x}$ where $i_n \leq i < i_{n+1}$, and define $\sigma_i(x)$ arbitrarily for $i < i_1$. If $i \geq i_n$, and $k \geq n$ is such that $i_k \leq i < i_{k+1}$, then

$$\|\sigma_i(x) - \sigma_i(a_{n,x})\|_2 \leq \|\sigma_i(a_{k,x}) - a_{n,x}\|_2 \leq 2^{-k} + \|a_{k,x} - a_{n,x}\|_2.$$

Hence

$$\limsup_{i\to\infty} \|\sigma_i(x) - \sigma_i(a_{n,x})\|_2 \leq \|x - a_{n,x}\|_2.$$

For $x, y \in M$,

$$\begin{aligned}\|\sigma_i(x)\sigma_i(y) - \sigma_i(xy)\|_2 \leq{}& \|x\|_\infty\|\sigma_i(y) - \sigma_i(a_{n,y})\|_2 + \|\sigma_i(xy) - \sigma_i(a_{n,xy})\|_2 \\ &+ \|\sigma_i(a_{n,y})\|_\infty\|\sigma_i(x) - \sigma_i(a_{n,x})\|_2 \\ &+ \|\sigma_i(a_{n,x})\sigma_i(a_{n,y}) - \sigma_i(a_{n,xy})\|_2.\end{aligned}$$

Letting $i \to \infty$,

$$\begin{aligned}\|\sigma_i(x)\sigma_i(y) - \sigma_i(xy)\|_2 \leq{}& \|x\|_\infty\|y - a_{n,y}\|_2 + \|xy - a_{n,xy}\|_2 \\ &+ \|y\|_\infty\|x - a_{n,x}\|_2 + \|a_{n,x}a_{n,y} - a_{n,xy}\|_2\end{aligned}$$

Letting $n \to \infty$ completes the proof.

$\square$

CHAPTER 3

Extended von Neumann Dimension for Sofic Groups

We now proceed with the first major part of the thesis: the definition of extended von Neumann dimension for actions of sofic groups on Banach spaces. Let us recall some history.

Voiculescu in [27] and Gournay in [13] noticed that for *amenable* groups Γ, we can compute von Neumann dimension as a limit of normalized approximate dimensions of $F_n\Omega$, with F_n a Følner sequence, and $\Omega \subseteq H$. This formula is analogous to the definition of entropy for actions of an amenable group on a compact metrizable space or measure space. Gournay noted that a formula for von Neumann dimension similar to Voiculescu's makes senses for subspaces of $\ell^p(\Gamma, V)$, with Γ amenable. Using this, he defined an isomorphism invariant for subspaces of $\ell^p(\Gamma, V)$ agreeing with von Neumann dimension in the case $p = 2$. In particular, Gournay shows that if Γ is amenable, and there is an injective Γ-equivariant linear map of finite type (see [13] for the definition) with closed image from $\ell^p(\Gamma, V) \to \ell^p(\Gamma, W)$ then $\dim V \leq \dim W$.

Recently, in [2],[18] a theory of entropy for actions of a *sofic* group on a probability space or a compact metrizable space has been developed. Using this theory, it was shown for sofic groups Γ that probability measure preserving Bernoulli actions $\Gamma \curvearrowright (X, \mu)^\Gamma, \Gamma \curvearrowright (Y, \nu)$ are not isomorphic if the entropy of (X, μ) does not equal the entropy of (Y, ν) and that Bernoulli actions $\Gamma \curvearrowright X^\Gamma, \Gamma \curvearrowright Y^\Gamma$ are not isomorphic as actions on compact metrizable spaces if $|X| \neq |Y|$ (when X and Y are finite). We can think of the action of Γ on $\ell^p(\Gamma, V)$ as analogous to a Bernoulli action, since both actions are given by translating functions on the group. Combining ideas of Kerr and Li [18] and Voiculescu in [27], we define an isomorphism invariant

$$\dim_{\Sigma,\ell^p}(Y,\Gamma)$$

for a uniformly bounded action of a sofic group on a separable Banach space Y. This definition of dimension has the following properties:

Property 1: $\dim_{\Sigma,\ell^p}(Y,\Gamma) \leq \dim_{\Sigma,\ell^p}(X,\Gamma)$ if there is an equivariant bounded linear map $X \to Y$ with dense image,

Property 2: $\dim_{\Sigma,\ell^p}(V,\Gamma) \leq \dim_{\Sigma,\ell^p}(W,\Gamma) + \dim_{\Sigma,\ell^p}(V/W,\Gamma)$, if $W \subseteq V$ is a closed Γ-invariant subspace,

Property 3: $\dim_{\Sigma,\ell^p}(Y \oplus W,\Gamma) \geq \dim_{\Sigma,\ell^p}(Y,\Gamma) + \underline{\dim}_{\Sigma,\ell^p}(W,\Gamma)$ for $2 \leq p < \infty$, where $\underline{\dim}$ is a "lower dimension," and is also an invariant,

Property 4: $\dim_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma) = \underline{\dim}_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma) = \dim(V)$ for $1 \leq p \leq 2$,

Property 5: $\dim_{\Sigma,\ell^p}(X,\Gamma) \geq \dim_{L(\Gamma)}(\overline{X}^{\|\cdot\|_2})$, when $X \subseteq \ell^p(\mathbb{N},\ell^p(\Gamma))$ and $1 \leq p \leq 2$.

We also note that for defining $\dim_{\ell^p}(Y,\Gamma)$, little about soficity of Γ is used, and we can more generally define our invariants associated to a sequence of maps $\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i)$ where V_i are finite-dimensional Banach spaces.

In particular, we can show that $\dim_{\Sigma,\ell^2}(Y,\Gamma)$ can be defined for $\mathcal{R}^\omega$-embeddable groups Γ. Because unitaries also act isometrically on the space of Schatten p-class operators, we can also define an invariant

$$\dim_{\Sigma,S^p}(Y,\Gamma),$$

S^p dimension has properties analogous to ℓ^p dimension.

Property 1: $\dim_{\Sigma,S^p}(Y,\Gamma) \leq \dim_{\Sigma,S^p}(X,\Gamma)$ if there is a Γ-equivariant bounded linear bijection $X \to Y$,

Property 2: $\dim_{\Sigma,S^p}(V,\Gamma) \leq \dim_{\Sigma,S^p}(W,\Gamma) + \dim_{\Sigma,S^p}(V/W,\Gamma)$, if $W \subseteq V$ is a closed Γ-invariant subspace, Property 3: $\dim_{\Sigma,S^p}(Y \oplus W, \Gamma) \geq \dim_{\Sigma,S^p}(Y, \Gamma) + \underline{\dim}_{\Sigma,S^p}(W, \Gamma)$ for $2 \leq p < \infty$,

Property 4: $\underline{\dim}_{\Sigma,S^p}(\ell^p(\Gamma, V), \Gamma) = \dim(V)$ for $1 \leq p \leq 2$,

Property 5: $\underline{\dim}_{\Sigma,S^p}(W, \Gamma) \geq \dim_{L(\Gamma)}(\overline{W}^{\|\cdot\|_2})$ if $W \subseteq \ell^p(\mathbb{N}, \ell^p(\Gamma))$ is a nonzero closed invariant subspace and $1 \leq p \leq 2$,

Property 6: $\underline{\dim}_{\Sigma,\ell^2}(H, \Gamma) = \dim_{\Sigma,\ell^2}(H, \Gamma) = \dim_{L(\Gamma)} H$ if $H \subseteq \ell^2(\mathbb{N}, \ell^2(\Gamma))$ is Γ invariant.

Property 7: $\dim_{\Sigma,\ell^p}(X, \Gamma) = 0$, if X is a finite-dimensional Banach space.

In particular $\ell^p(\Gamma, V)$ is not isomorphic to $\ell^p(\Gamma, W)$ as a representation of Γ, if Γ is $\mathcal{R}^\omega$-embeddable and $1 \leq p < \infty$. This extends a result of [13] from amenable groups to $\mathcal{R}^\omega$-embeddable groups, and answers a question of Gromov (see [15] page 353) in the case of $\mathcal{R}^\omega$-embeddable groups. Lastly, we shall also define and compute ℓ^p-Betti numbers of free groups, as well as dimensions for actions of Γ on noncommutative L^p-space.

3.1 Definition of the Invariants

Definition 3.1.1. Let X be a Banach space. An action Γ on X by is said to be *uniformly bounded* if there is a constant $C > 0$ such that

$$\|sx\| \leq C\|x\| \text{ for all } x \in X, s \in \Gamma.$$

We say that a sequence $S = (x_j)_{j=1}^\infty$ in X is *dynamically generating,* if S is bounded and $\mathrm{Span}\{sx_j : s \in \Gamma, j \in \mathbb{N}\}$ is dense.

If X is a Banach space we shall write $\mathrm{Isom}(X)$ for the group of all linear isometries from X to itself.

Definition 3.1.2. Let V be a vector space with a pseudonorm ρ. If $A \subseteq V$, a linear subspace $W \subseteq V$ is said to *ε-contain* A, denoted $A \subseteq_\varepsilon W$, if for every $v \in A$, there is a $w \in W$ such that $\rho(v-w) < \varepsilon$. We let $d_\varepsilon(A,\rho)$ be the minimal dimension of a subspace which ε-contains A.

Definition 3.1.3. A *dimension triple* is a triple $(X,\Gamma,\Sigma = (\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i)))$, where X is a separable Banach space, Γ is a countable discrete group with a uniformly bounded action on X, each V_i is finite-dimensional, and the σ_i are functions with no structure assumed on them.

Definition 3.1.4. Let $(X,\Gamma,\Sigma = (\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i)))$ be a dimension triple. Fix $S = (x_j)_{j=1}^\infty$ a dynamically generating sequence in X. For $e \in E \subseteq \Gamma$ finite, $l \in \mathbb{N}$ let

$$X_{E,l} = \mathrm{Span}\{sx_j : s \in E^l, 1 \leq j \leq l\}.$$

If $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $C,\delta > 0$, let $\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)_C$ be the set of all linear maps $T\colon X_{F,m} \to V_i$ such that $\|T\| \leq C$ and

$$\|T(s_1\cdots s_k x_j) - \sigma_i(s_1)\cdots\sigma_i(s_k)T(x_j)\| < \delta$$

if $1 \leq j,k \leq m, s_1,\ldots,s_k \in F$. If $C = 1$ we shall use $\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)$ instead of $\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)_1$.

We shall frequently deal with inducing pseudonorms on $\ell^\infty(\mathbb{N},V)$ from pseudonorms on $\ell^\infty(\mathbb{N})$. For this, we use the following notation: if ρ is a pseudonorm on $\ell^\infty(\mathbb{N})$ and V is a Banach space, we let ρ_V be the pseudonorm on $\ell^\infty(\mathbb{N},V)$ defined by $\rho_V(f) = \rho(j \mapsto \|f(j)\|)$.

Definition 3.1.5. Let Σ, S be as in the proceeding definition and let ρ be a pseudonorm on $\ell^\infty(\mathbb{N})$. Let $\alpha_S\colon B(X_{F,m},V_i) \to \ell^\infty(\mathbb{N},V_i)$ be given by $\alpha_S(T)(j) = \chi_{\{k\leq m\}}(j)T(x_j)$. We let

$$\widehat{d_\varepsilon}(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho) = d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)),\rho_{V_i})$$

define the *dimension of S with respect to ρ* by

$$f.\dim_\Sigma(S,F,m,\delta,\varepsilon,\rho) = \limsup_{i\to\infty} \frac{1}{\dim V_i}\widehat{d_\varepsilon}(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho),$$

$$f.\dim_\Sigma(S,\varepsilon,\rho) = \limsup_{\substack{e\in F\subseteq\Gamma \text{ finite} \\ m\in\mathbb{N} \\ \delta>0}} f.\dim_\Sigma(S,F,m,\delta,\varepsilon,\rho)$$

$$f.\dim_\Sigma(S,\rho) = \sup_{\varepsilon>0} f.\dim_\Sigma(S,\varepsilon,\rho),$$

where the pairs (F, m, δ) are ordered as follows $(F, m, \delta) \leq (F', m', \delta')$ if $F \subseteq F', m \leq m', \delta \geq \delta'$. We also use

$$\underline{f.\dim}_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \liminf_{i\to\infty} \frac{1}{\dim V_i} \widehat{d}_{\varepsilon}(\mathrm{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i), \rho),$$

$$\underline{f.\dim}_{\Sigma}(S, \varepsilon, \rho) = \liminf_{\substack{e\in F\subseteq\Gamma \text{ finite} \\ m\in\mathbb{N} \\ \delta>0}} f.\dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho)$$

$$\underline{f.\dim}_{\Sigma}(S, \rho) = \sup_{\varepsilon>0} f.\dim_{\Sigma}(S, \varepsilon, \rho).$$

In section 3.2 we will show that

$$f.\dim_{\Sigma}(S, \rho) = \sup_{\varepsilon>0} \liminf_{(F,m,\delta)} \limsup_{i\to\infty} \frac{1}{\dim V_i} \widehat{d}_{\varepsilon}(\mathrm{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i), \rho),$$

$$\underline{f.\dim}_{\Sigma}(S, \rho) = \sup_{\varepsilon>0} \limsup_{(F,m,\delta)} \liminf_{i\to\infty} \frac{1}{\dim V_i} \widehat{d}_{\varepsilon}(\mathrm{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i), \rho).$$

We introduce two other versions of dimension, which will be used to prove that the above notion of dimension does not depend on the generating sequence.

Definition 3.1.6. Let X be a separable Banach space, we say that X has the *C-bounded approximation property* if there is a sequence $\theta_n \colon X \to X$ of finite rank maps such that $\|\theta_n\| \leq C$ and

$$\|\theta_n(x) - x\| \to 0, \text{ for all } x \in X.$$

We say that X has the *bounded approximation property* if it has the C-bounded approximation property for some $C > 0$.

Definition 3.1.7. Let X be a separable Banach space with a uniformly bounded action of a countable discrete group Γ. Let $q \colon Y \to X$ be a bounded linear surjective map, where Y is a separable Banach space with the bounded approximation property. A *q-dynamical filtration* is a pair $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, (Y_{E,l})_{e\in E\subseteq\Gamma \text{ finite}, l\in\mathbb{N}})$ where $a_{sj} \in Y$, $Y_{E,l} \subseteq Y$ is a finite dimensional linear subspace such that

1: $\sup_{(s,j)} \|a_{sj}\| < \infty$,

2: $q(a_{sj}) = sq(a_{ej})$,

3: $(q(a_{ej}))_{j=1}^{\infty}$ *is dynamically generating,*

4: $Y_{E,l} \subseteq Y_{E',l'}$ *if* $E \subseteq E', l \leq l'$

5: $\ker(q) = \overline{\bigcup_{(E,l)} Y_{E,l} \cap \ker(q)}$,

6: $Y_{E,l} = \mathrm{Span}\{a_{sj} : s \in E^l, 1 \leq j \leq l\} + \ker(q) \cap Y_{E,l}$.

Note that if X has the bounded approximation property and $Y = X$ with q the identity, then a dynamical filtration simply corresponds to a choice of a dynamically generating sequence. In general, if $S = (x_j)_{j=1}^{\infty}$ is a dynamically generating sequence, then there is always a q-dynamical filtration $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l})$ such that $q(a_{ej}) = x_j$. Simply choose a_{sj} such that $\|a_{sj}\| \leq C\|x_j\|$ and $q(a_{sj}) = sx_j$ for some $C > 0$. If $(y_j)_{j=1}^{\infty}$ is a dense sequence in $\ker(q)$, we can set

$$Y_{E,l} = \mathrm{Span}\{a_{sj} : (s,j) \in E^l \times \{1,\dots,l\}\} + \sum_{j=1}^{l} \mathbb{C}y_j.$$

We can always find a Banach space Y with the bounded approximation property and a quotient map $q\colon Y \to X$, in fact it is a standard exercise that we can choose $Y = \ell^1(\mathbb{N})$.

Definition 3.1.8. A *quotient dimension tuple* is a tuple $(Y, q, X, \Gamma, \sigma_i\colon \Gamma \to \mathrm{Isom}(V_i))$ where (X, Γ, σ_i) is a dimension triple, Y is a separable Banach space with the bounded approximation property and $q\colon Y \to X$ is a bounded linear surjection.

Definition 3.1.9. Let $(Y, q, X, \Gamma, \sigma_i\colon \Gamma \to \mathrm{Isom}(V_i))$ be a quotient dimension triple, and let $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l})$ be a q-dynamical filtration. For $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}, \delta, C > 0$

we let $\mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C$ be the set of all bounded linear maps $T\colon Y \to V_i$ such that $\|T\| \leq C$ and

$$\|T(a_{s_1\cdots s_k j}) - \sigma_i(s_1)\cdots\sigma_i(s_k)T(a_{ej})\| < \delta$$

$$\left\| T\big|_{\ker(q)\cap Y_{F,l}} \right\| < \delta.$$

As before, if $C = 1$ we will use $\mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)$ instead of $\mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C$.

Again, in the case X has the bounded approximation property, we are simply looking at almost equivariant maps from Γ to V_i, and this is similar in spirit to the definition of topological entropy in [18]. In the general case, note that genuine equivariant maps from X to V_i would correspond to maps on Y which vanish on the kernel of q, and so that

$$T(a_{s_1\cdots s_k j}) = \sigma_i(s_1)\cdots\sigma_i(s_k)T(a_{ej}),$$

so we are still looking at almost equivariant maps on X, in a certain sense.

Definition 3.1.10. Fix a pseudonorm ρ on $\ell^\infty(\mathbb{N})$, let $(Y, q, X, \Gamma, \Sigma = (\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i)))$ be a quotient dimension tuple, and $\mathcal{F}$ a q-dynamical filtration. Let $\alpha_{\mathcal{F}}\colon B(Y, V_i) \to \ell^\infty(\mathbb{N}, V_i)$ be given by $\alpha_{\mathcal{F}}(\phi) = (\phi(a_{ej}))_{j=1}^\infty$ we again use $\widehat{d_\varepsilon}(A, \rho) = d_\varepsilon(\alpha_{\mathcal{F}}(A), \rho_{V_i})$. We define the dimension of $\mathcal{F}$ with respect to ρ, Σ as follows:

$$f.\dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho) = \limsup_{i\to\infty} \frac{1}{\dim V_i} \widehat{d_\varepsilon}(\mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho),$$

$$f.\dim_\Sigma(\mathcal{F}, \varepsilon, \rho) = \inf_{\substack{e\in F\subseteq\Gamma \text{ finite} \\ m\in\mathbb{N} \\ \delta>0}} f.\dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho),$$

$$f.\dim_\Sigma(\mathcal{F}, \rho) = \sup_{\varepsilon>0} f.\dim_\Sigma(\mathcal{F}, \varepsilon, \rho).$$

Note that unlike $f.\dim_\Sigma(S, F, m, \delta, \varepsilon, \rho)$ we know that $f.\dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho)$ is smaller when we enlarge F and m and shrink δ, thus the infimum is a limit and there are no issues between equality of limit suprema and limit infima for this definition.

Definition 3.1.11. Let Y, X be Banach spaces, and let ρ be a pseudonorm on $B(X,Y)$. For $\varepsilon > 0, 0 < M \leq \infty$, and $A, C \subseteq B(X,Y)$, the set C is said to (ε, M) *contain* A if for every $T \in A$, there is a $S \in C$ such that $\|S\| \leq M$ and $\rho(S-T) < \varepsilon$. In this case we shall write $A \subseteq_{\varepsilon,M} C$. We let $d_{\varepsilon,M}(A,\rho)$ be the smallest dimension of a linear subspace which (ε, M) contains A.

Definition 3.1.12. Let $(Y, q, X, \Gamma, \sigma_i\colon \Gamma \to \mathrm{Isom}(V_i))$ be a quotient dimension tuple. Let $\mathcal{F} = (a_{sj}, Y_{F,l})$ be a q-dynamical filtration. Fix a sequence of pseudonorms of ρ_i on $B(Y, V_i)$ and $0 < M \leq \infty$, set

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, F, m, \delta, \varepsilon, \rho_i) = \limsup_{i\to\infty} \frac{1}{\dim V_i} d_{\varepsilon,M}(\operatorname{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho_i),$$

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \varepsilon, \rho_i) = \inf_{\substack{e\in F\subseteq\Gamma \text{ finite}\\ m\in\mathbb{N}\\ \delta>0}} \operatorname{opdim}_{\Sigma,M}(\mathcal{F}, F, m, \delta, \varepsilon, \rho),$$

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho_i) = \sup_{\varepsilon>0} \operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \varepsilon, \rho).$$

As before, we shall use

$$\underline{\operatorname{opdim}}_{\Sigma,M}(\mathcal{F}, \rho_i), \underline{f.\dim}_\Sigma(\mathcal{F}, \rho)$$

for the same definitions as above, but replacing the limit supremum with the limit infimum.

By scaling,

$$\inf_{0<M<\infty} \operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho_i), \operatorname{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_i), f.\dim_\Sigma(S, \rho), f.\dim_\Sigma(\mathcal{F}, \rho)$$

remain the same when we replace $\operatorname{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)$, $\operatorname{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$, by $\operatorname{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C$, $\operatorname{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_C$, for C a fixed constant. This will be useful in several proofs.

Note that if ρ is a pseudonorm on $\ell^\infty(\mathbb{N})$, then we get a pseudonorm $\rho_{\mathcal{F},i}$ on $B(Y, V_i)$ by

$$\rho_{\mathcal{F},i}(T) = \rho(j \mapsto \|T(a_{ej})\|).$$

Further, for $0 < M \leq \infty$

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho_{\mathcal{F},i}) \geq f.\dim_\Sigma(\mathcal{F}, \rho).$$

Definition 3.1.13. A *product norm* ρ is a norm on $\ell^\infty(\mathbb{N})$ such that

1 : ρ induces a topology stronger than the product topology,

2 : ρ induces a topology which agrees with the product topology on $\{f \in \ell^\infty(\mathbb{N}) : \|f\|_\infty \leq 1\}$.

Typical examples are the ℓ^p-norms:

$$\rho(f)^p = \sum_{j=1}^{\infty} \frac{1}{2^j}|f(j)|^p.$$

We shall show that there is constant $M > 0$, depending only on Y, so that if $\mathcal{F}, \mathcal{F}'$ are dynamical filtrations of q and S is a dynamically generating sequence, then for any two product norms ρ, ρ',

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho'_{\mathcal{F},i}) = \operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho_{\mathcal{F},i}) = f.\dim_\Sigma(\mathcal{F}, \rho) =$$

$$f.\dim_\Sigma(\mathcal{F}', \rho) = \dim_\Sigma(S, \rho).$$

and the same with dim replaced by $\underline{\dim}$. In particular all these dimension only depend of the action of Γ on X, and give an isomorphism invariant. When we show all these equalities we let

$$\dim_\Sigma(X, \Gamma)$$

denote any of these common numbers.

The equality between these dimensions is easier to understand in the case when X has the bounded approximation property. When X has the bounded approximation property, we can take $Y = X, q = \operatorname{Id}$ and then the equality

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F}, \rho_{\mathcal{F},i}) = f.\dim_\Sigma(S, \rho),$$

says the data of *local* almost equivariant maps on X is the same as the data of *global* almost equivariant maps on X. This is essentially because if we take $\theta_{E,l} \colon X \to X_{E,l}$ which tend pointwise to the identity, then any almost equivariant map on $X_{E,l}$ gives an almost equivariant map on X by composing with $\theta_{E,l}$.

Since the maps $\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i)$ are not assumed to have any structure, this invariant is uninteresting unless the maps σ_i model the action of Γ on X in some manner. Thus we note that if Γ is a sofic group, then the maps $\sigma_i\colon \Gamma \to S_{d_i}$ model at least the group Γ in a reasonable manner.

Because S_n acts naturally on $\ell^p(n)$ we get an induced sequence of maps $\sigma_i\colon \Gamma \to \mathrm{Isom}(\ell^p(d_i))$ and the above invariant measures how closely the action of Γ on X is modeled by these maps. When Γ is sofic, and $\Sigma = (\sigma_i\colon \Gamma \to S_{d_i})$ is a sofic approximation and $\Sigma^{(p)} = (\sigma_i\colon \Gamma \to \mathrm{Isom}(\ell^p(d_i)))$ are the maps induced by the action of S_n on $\ell^p(n)$, we let

$$\dim_{\Sigma,\ell^p}(X,\Gamma) = \dim_{\Sigma^{(p)}}(X,\Gamma)$$

$$\underline{\dim}_{\Sigma,\ell^p}(X,\Gamma) = \underline{\dim}_{\Sigma^{(p)}}(X,\Gamma).$$

Similarly, if Γ is $\mathcal{R}^\omega$-embeddable, and $\sigma_i\colon \Gamma \to U(d_i)$ is a embedding sequence, then since $U(d_i)$ is the isometry group of $\ell^2(d_i)$ we shall let

$$\dim_{\Sigma,\ell^2}(X,\Gamma) = \dim_{\Sigma}(X,\Gamma)$$

$$\underline{\dim}_{\Sigma,\ell^2}(X,\Gamma) = \underline{\dim}_{\Sigma}(X,\Gamma).$$

Just as S_n acts on commutative ℓ^p-Spaces, we have two natural actions of $U(n)$ on non-commutative L^p-spaces. Let $S^p(n)$ be $M_n(\mathbb{C})$ with the norm

$$\|A\|_{S^p} = \mathrm{Tr}(|A|^p)$$

where $|A| = (A^*A)^{1/2}$. Then $U(n)$ acts isometrically on $S^p(n)$ by conjugation and by left multiplication. We shall use

$$\dim_{\Sigma,S^p,\mathrm{conj}}(X,\Gamma)$$

for our dimension defined above, thinking of σ_i as a map into $\mathrm{Isom}(S^p(n))$ by conjugation and

$$\dim_{\Sigma,S^p,\mathrm{mult}}(X,\Gamma)$$

thinking of σ_i as a map into $\mathrm{Isom}(S^p(n))$ by left multiplication.

One of our main applications will be showing that when Γ is $\mathcal{R}^\omega$-embeddable

$$\underline{\dim}_{\Sigma,S^p,\mathrm{conj}}(\ell^p(\Gamma)^{\oplus n},\Gamma)=\dim_{\Sigma,S^p,\mathrm{conj}}(\ell^p(\Gamma)^{\oplus n},\Gamma)=n,$$

if $1\le p\le 2$, and

$$\underline{\dim}_{\Sigma,\ell^p}(\ell^p(\Gamma)^{\oplus n},\Gamma)=\dim_{\Sigma,\ell^p}(\ell^p(\Gamma)^{\oplus n},\Gamma)=n,$$

if $1\le p\le 2$, In particular the representations $\ell^p(\Gamma)^{\oplus n}$ are not isomorphic for different values of n, if Γ is $\mathcal{R}^\omega$-embeddable.

3.2 Invariance of the Definitions

In this section we show that our various notions of dimension agree. Here is the main strategy of the proof. First we show that there is an $M>0$, independent of $\mathcal{F}$ so that

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho_{\mathcal{F},i})=f.\dim_\Sigma(\mathcal{F},\rho),$$

the constant M comes from the constant in the definition of bounded approximation property. A compactness argument shows that

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho_{\mathcal{F},i})$$

does not depend on the choice of pseudonorm. We then show that

$$\mathrm{opdim}_{\Sigma,\infty}(\mathcal{F},\rho_{\mathcal{F},i})$$

does not depend on the choice of $\mathcal{F}$, this is easier than trying to show that

$$f.\dim_\Sigma(S,\rho)$$

does not depend on the choice of S. This is because the maps used to define

$$\mathrm{opdim}_{\Sigma,\infty}(\mathcal{F},\rho_{\mathcal{F},i})$$

all have the same domain, which makes it easy to switch from one generating set to another, since we can use that generators for $\mathcal{F}$ have to be close to linear combinations of generators for $\mathcal{F}'$. Then we show that

$$f.\dim_{\Sigma}(\mathcal{F},\rho)=f.\dim_{\Sigma}(S,\rho),$$

this will reduce to showing that if we are given an almost equivariant map $\phi\colon Y\to V_i$ which is small on the kernel of q, then there is a $T\colon X'\to V$ with $X'\subseteq X$ finite dimensional such that $T\circ q$ is close to ϕ on a prescribed finite set.

First we need a simple fact about spaces with the bounded approximation property.

Proposition 3.2.1. *Let Y be a separable Banach space with the C-bounded approximation property, and let I be a countable directed set. Let $(Y_\alpha)_{\alpha\in I}$ be an increasing net of subspaces of Y such that*

$$Y=\overline{\bigcup_\alpha Y_\alpha}.$$

Then there are finite-rank maps $\theta_\alpha\colon Y\to Y_\alpha$ such that $\|\theta_\alpha\|\le C$ and

$$\lim_\alpha\|\theta_\alpha(y)-y\|=0$$

for all $y\in Y$.

Proof. Fix $y_1,\dots,y_k\in Y$ and $\varepsilon>0$. Then there is a finite rank $\theta\colon Y\to Y$ such that

$$\|\theta(y_j)-y_j\|<\varepsilon,$$

$$\|\theta\|\le C.$$

Write

$$\theta=\sum_{j=1}^n\phi_j\otimes x_j$$

with $\phi_j\in Y^*$ and $x_j\in Y$. If α is sufficiently large, then we can find $x_j'\in Y_\alpha$ close enough to x_j so that if we let

$$\theta_0=\sum_{j=1}^n\phi_j\otimes x_j',$$

$$\widetilde{\theta} = \begin{cases} \theta_0 & \text{if } \|\theta_0\| \leq C \\ C\frac{\theta_0}{\|\theta_0\|} & \text{otherwise} \end{cases}$$

then

$$\|\widetilde{\theta}(y_j) - y_j\| < 2\varepsilon.$$

Now let $(y_j)_{j=1}^{\infty}$ be a dense sequence in Y, and let

$$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots$$

with $\alpha_j \in I$ be such that for all $\beta \in I$, there is a j such that $\beta \leq \alpha_j$. By the preceding paragraph, we can inductively construct an increasing sequence n_k of integers and finite-rank maps

$$\theta_k \colon Y \to Y_{\alpha_{n_k}}$$

such that

$$\|\theta_k\| \leq C$$

$$\|\theta_k(y_j) - y_j\| \leq 2^{-k} \text{ if } j \leq k.$$

Set $\theta_\alpha = \theta_{\alpha_{n_k}}$ if k is the largest integer such that α_{n_k} is not bigger than α. Let $\theta_\alpha = 0$ if $\alpha < \alpha_1$. Then θ_α has the desired properties. □

Lemma 3.2.2. *Let $(Y, q, X, \Gamma, \Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)))$ be a quotient dimension tuple. Let $\mathcal{F} = ((a_{sj})_{(s,j) \in \Gamma \times \mathbb{N}}, Y_{F,l})$ be a q-dynamical filtration and ρ a product norm, and let $C > 0$ be such that Y has the C-bounded approximation property. Fix $M > C$. Then for any $V \subseteq Y$ finite-dimensional, and $\kappa > 0$, there is a $F \subseteq \Gamma$ finite $m \in \mathbb{N}$, $\delta, \varepsilon > 0$ and linear maps*

$$L_i \colon \ell^\infty(\mathbb{N}, V_i) \to B(Y, V_i)$$

so that if $\phi \in \mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), f \in \ell^\infty(\mathbb{N}, V_i)$ satisfy $\rho_{V_i}(\alpha_{\mathcal{F}}(\phi) - f) < \varepsilon$, then

$$\|L_i(f)\| \leq M,$$

$$\left\|L_i(f)\big|_V - \phi\big|_V\right\| < \kappa.$$

Proof. Note that for every V finite-dimensional there are a $E \subseteq \Gamma$ finite, $l \in \mathbb{N}$, such that

$$\max_{\substack{v \in V \\ \|v\|=1}} \inf_{\substack{w \in Y_{E,l} \\ \|w\|=1}} \|v - w\| < \kappa,$$

so we may assume that $V = Y_{E,l}$ for some E, l.

Fix $\eta > 0$ to be determined later. By the preceding proposition let $\theta_{F,k} \colon Y \to Y_{F,k}$ be such that

$$\|\theta_{F,k}\| \leq C,$$

$$\lim_{(F,k)} \|\theta_{F,k}(y) - y\| = 0 \text{ for all } y \in Y.$$

Choose F, m sufficiently large such that

$$\|\theta_{F,m}\big|_{Y_{E,l}} - \operatorname{Id}\big|_{Y_{E,l}}\| \leq \eta.$$

Let $\mathcal{B}_{F,m} \subseteq F^m \times \{1, \dots, m\}$ be such that $\{q(a_{sj}) : (s,j) \in \mathcal{B}_{F,m}\}$ is a basis for $X_{F,m} = \operatorname{Span}\{q(a_{sj}) : (s,j) \in F^m \times \{1, \dots, m\}\}$. Define

$$\widetilde{L}_i \colon \ell^\infty(\mathbb{N}, V_i) \to B(X_{F,m}, V_i)$$

by

$$\widetilde{L}_i(f)(q(a_{sj})) = \sigma_i(s) f(j) \text{ for } (s,j) \in \mathcal{B}_{F,m}.$$

We claim that if $\delta > 0, \varepsilon' > 0$ are sufficiently small, $\phi \in \operatorname{Hom}_\Gamma(\mathcal{F}, F^m, m, \delta, \sigma_i)$ and $f \in \ell^\infty(\mathbb{N}, V_i)$ satisfy

$$\rho_{V_i}(f - \alpha_{\mathcal{F}}(\phi)) < \varepsilon',$$

then

$$\|\widetilde{L}_i(f) \circ q\big|_{Y_{F,m}} - \phi\big|_{Y_{F,m}}\| \leq \eta. \tag{3.1}$$

By finite-dimensionality, there is a $D(F,m) > 0$ such that if $v \in \ker(q) \cap Y_{F,m}, (d_{tr}) \in \mathbb{C}^{\mathcal{B}_{F,m}}$, then

$$\sup(\|v\|, |d_{tr}|) \leq D(F,m) \left\| v + \sum_{(t,r) \in \mathcal{B}_{F,m}} d_{tr} a_{tr} \right\|.$$

Thus if $x = v + \sum_{(t,r)\in\mathcal{B}_{F,m}} d_{tr}a_{tr}$ with $v \in \ker(q) \cap Y_{F,m}$ has $\|x\| = 1$, then

$$\|\widetilde{L}_i(f)(q(x)) - \phi(x)\| \leq D(F,m)\delta + D(F,m) \sum_{(t,r)\in\mathcal{B}_{F,m}} \|\phi(a_{tr}) - \sigma_i(t)f(r)\|$$

$$\leq D(F,m)\delta + D(F,m)|F|^m m\delta + \sum_{(t,r)\in\mathcal{B}_{F,m}} \|\phi(a_{er}) - f(r)\|,$$

if $\delta < \frac{\eta}{2D(F,m)(1+|F|^m m)}$, and $\varepsilon' > 0$ is small enough so that $\rho(g) < \varepsilon'$ implies

$$\sum_{(t,r)\in\mathcal{B}_{F,m}} |g(r)| < \frac{\eta}{2},$$

then our claim holds.

So assume that $\delta, \varepsilon' > 0$ are small enough so that (3.1) holds, and set $L_i(f) = \widetilde{L}_i(f) \circ q\big|_{Y_{F,m}} \circ \theta_{F,m}$. Then

$$\|L_i(f)\| \leq C(1+\eta)$$

and for ϕ, f as above and $y \in Y_{E,l}$

$$\|L_i(f)(y) - \phi(y)\| \leq (1+\eta)\|\theta_{F,m}(y) - y\| + \|\widetilde{L}_i(f) \circ q(y) - \phi(y)\| \leq (2+\eta)\eta\|y\|.$$

So we force η to be small enough so that $(2+\eta)\eta < \kappa, C(1+\eta) < M$.

□

Lemma 3.2.3. *Let $(Y, q, X, \Gamma, \Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)))$ be a quotient dimension tuple.*

Let $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l})$ be a q-dynamical filtration, and ρ a product norm, suppose that Y has the C-bounded approximation property.

(a) If $\infty \geq M > C$, then

$$f.\dim_\Sigma(\mathcal{F},\rho) = \mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho),$$

$$\underline{f.\dim}_\Sigma(\mathcal{F},\rho) = \underline{\mathrm{opdim}}_{\Sigma,M}(\mathcal{F},\rho).$$

(b) If ρ' is another product norm then for all $0 < M < \infty$,

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho_{\mathcal{F},i}) = \mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho'_{\mathcal{F},i}),$$

$$\underline{\mathrm{opdim}}_{\Sigma,M}(\mathcal{F},\rho_{\mathcal{F},i}) = \underline{\mathrm{opdim}}_{\Sigma,M}(\mathcal{F},\rho'_{\mathcal{F},i}).$$

Proof. (a) First note that

$$\operatorname{opdim}_{\Sigma,M}(\mathcal{F},\rho) \geq \operatorname{opdim}_{\Sigma,\infty}(\mathcal{F},\rho) \geq f.\dim_{\Sigma}(\mathcal{F},\rho)$$

so it suffices to handle the case that $M < \infty$.

Let $A > 0$ be such that

$$\|a_{sj}\| \leq A \text{ for all } (s,j) \in \Gamma \times \mathbb{N}$$

Take $1 > \varepsilon > 0$. Let k be such that if $f \in \ell^{\infty}(\mathbb{N})$, and $\|f\|_{\infty} \leq 1$, and f is supported on $\{n : n \geq k\}$, then $\rho(f) < \varepsilon$. Since ρ induces a topology weaker than the norm topology, we can find an $\varepsilon > \kappa > 0$ such that

$$\rho(f) < \varepsilon$$

if

$$\|f\|_{\infty} \leq \kappa.$$

By Lemma 3.2.2, let $e \in F \subseteq \Gamma$ be finite, $m \in \mathbb{N}$, $\varepsilon > \varepsilon' > 0, \kappa > \delta > 0$ and $L_i \colon \ell^{\infty}(\mathbb{N}, V_i) \to B(Y, V_i)$ be such that if $\phi \in \operatorname{Hom}_{\Gamma}(\mathcal{F}, F, m, \delta, \sigma_i)$ and $f \in \ell^{\infty}(\mathbb{N}, V_i)$ has $\rho_{V_i}(\alpha_{\mathcal{F}}(\phi) - f) < \varepsilon'$, then

$$\left\| L_i(f)\big|_{Y_{\{e\},k}} - \phi\big|_{Y_{\{e\},k}} \right\| < \kappa,$$

$$\|L_i(f)\| \leq M.$$

Then if ϕ, f are as above we have

$$\rho_{\mathcal{F},i}(\phi - L_i(f)) \leq (M+1)A\varepsilon + \rho(\chi_{l \leq k}(j)(\|\phi(a_{ej}) - L_i(f)(a_{ej})\|)_{j=1}^{\infty})$$

and for $j \leq k$

$$\|\phi(a_{ej}) - L_i(f)(a_{ej})\| \leq A(M+1)\kappa.$$

Thus

$$\rho_{\mathcal{F},i}(\phi - L_i(f)) \leq (M+1)(A+1)\varepsilon.$$

This implies that

$$d_{((M+1)(A+1)\varepsilon,M}(\mathrm{Hom}_{\Gamma}(\mathcal{F},F',m',\delta',\sigma_i),\rho_{\mathcal{F},i}) \leq \widehat{d}_{\varepsilon'}(\mathrm{Hom}_{\Gamma}(\mathcal{F},F',m',\delta',\sigma_i),\rho_{\mathcal{F},i})$$

for all $F' \supseteq F, m' \geq m$, and all $\delta' < \delta$. This completes the proof.

(b) This is a simple consequence of the compactness of the $\|\cdot\|_\infty$ unit ball of $\ell^\infty(\mathbb{N})$ in the product topology.

□

Lemma 3.2.4. *Let* $(Y,q,X,\Gamma,\sigma_i\colon \Gamma \to \mathrm{Isom}(V_i))$ *be a quotient dimension tuple. Let* $\mathcal{F},\mathcal{F}'$ *be two* q*-dynamical filtrations. If* ρ_i *is any fixed sequence of pseudonorms on* $B(Y,V_i)$*, then for all* $0 < M \leq \infty$,

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F},\rho_i) = \mathrm{opdim}_{\Sigma,M}(\mathcal{F}',\rho_i),$$

$$\underline{\mathrm{opdim}}_{\Sigma,M}(\mathcal{F},\rho_i) = \underline{\mathrm{opdim}}_{\Sigma,M}(\mathcal{F}',\rho_i),$$

Proof. Let $\mathcal{F}' = ((a'_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y'_{E,l})$, $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l})$. We do the proof for opdim_{Σ}, the other case is proved in the same manner. Let $C > 0$ be such that $\|sx\| \leq C\|x\|$ for all $s \in \Gamma, x \in X$ and such that $\|a_{sj}\|, \|a'_{sj}\| \leq C$. Fix $F \subseteq \Gamma$ finite, and $m \in \mathbb{N}, \delta > 0$. Fix $\eta > 0$ which will depend upon F, m, δ in a manner to be determined later.

Choose $E \subseteq \Gamma$ finite $l \in \mathbb{N}$, such that for $1 \leq j \leq m, s \in F^m$ there are $c_{j,t,k}$ with $(t,k) \in E \times \{1,\ldots,l\}$ and $v_{sj} \in Y'_{E,l} \cap \ker(q)$ such that

$$\left\| a_{sj} - v_{sj} - \sum_{(t,k)\in E\times\{1,\ldots,l\}} c_{j,t,k} a'_{stk} \right\| < \eta,$$

and so that for every $w \in Y_{F,m} \cap \ker(q)$ there is a $v \in Y'_{E,l} \cap \ker(q)$ such that $\|v - w\| \leq \eta\|w\|$. Let $A(\eta) = \sup(|c_{j,t,k}|, \sup\|v_{sj}\|)$

Set $m' = 2\max(m,l) + 1$, $F' = [(F \cup F^{-1} \cup \{e\})(E \cup E^{-1} \cup \{e\})]^{2m'+1}$, we claim that we can choose $\delta' > 0, \eta > 0$ small so that

$$\mathrm{Hom}_{\Gamma}(\mathcal{F}',F',m',\delta',\sigma_i) \subseteq \mathrm{Hom}_{\Gamma}(\mathcal{F},F,m,\delta,\sigma_i).$$

If $T \in \mathrm{Hom}_\Gamma(\mathcal{F}', F', m', \delta', \sigma_i)$, $1 \leq j, r \leq m$, and $s_1, \ldots, s_r \in F$ then

$$\|T(a_{s_1 \cdots s_r j}) - \sigma_i(s_1) \cdots \sigma_i(s_r) T(a_{ej})\| \leq$$

$$2\eta + \|T(v_{sj})\| + \|\sigma_i(s_1) \cdots \sigma_i(s_r) T(v_{ej})\| +$$

$$\left\| \sum_{(t,k) \in E \times \{1, \ldots, l\}} c_{j,t,k} [T(a'_{s_1 \cdots s_r tk}) - \sigma_i(s_1) \cdots \sigma_i(s_r) T(a'_{tk})] \right\| \leq$$

$$2\eta + \delta' A(\eta) + \delta' A(\eta) + 2|E| l A(\eta) \delta'.$$

By choosing $\eta < \delta/2$, and then choosing δ' very small we can make the above expression less than δ. If we also force $\delta' < \delta/2$ our choice of η implies that

$$\|T(w)\| \leq \delta \|w\|$$

for T as above and $w \in Y_{F,m} \cap \ker(q)$. This completes the proof. $\square$

Because of the above lemma, the only difficulty in proving that $\mathrm{opdim}_\Sigma(\mathcal{F}, \rho_{\mathcal{F},i})$ does not depend on the choice of $\mathcal{F}$ is switching the pseudonorm from $\rho_{\mathcal{F},i}$ to $\rho_{\mathcal{F}',i}$. Because of this we will investigate how the dimension changes when we switch pseudonorms.

Definition 3.2.5. Let $(Y, q, X, \Gamma, \Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)))$ be a quotient dimension tuple, and fix a q-dynamical filtration $\mathcal{F}$. If ρ_i, q_i are pseudonorms on $B(Y, V_i)$ we say that ρ_i is $(\mathcal{F}, \Sigma)$-*weaker than* q_i and write $\rho_i \preceq_{\mathcal{F},\Sigma} q_i$ if the following holds. For every $\varepsilon > 0$, there are $F \subseteq \Gamma$ finite, $\delta, \varepsilon' > 0$, $m, i_0 \in \mathbb{N}$, and linear maps $L_i \colon B(Y, V_i) \to B(Y, V_i)$ for $i \geq i_0$ such that if $\phi \in \mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)$ and $\psi \in B(Y, V_i)$ satisfy $q_i(\phi - \psi) < \varepsilon'$, then $\rho_i(\phi - L_i(\psi)) < \varepsilon$. We say that ρ_i *is* $(\mathcal{F}, \Sigma)$ *equivalent to* q_i, and write $\rho_i \sim_{\mathcal{F},\Sigma} q_i$, if $\rho_i \preceq_{\mathcal{F},\Sigma} q_i$ and $q_i \preceq_{\mathcal{F},\Sigma} \rho_i$.

Lemma 3.2.6. *Let $(Y, X, q, \Gamma, \Sigma)$ be a quotient dimension tuple and $\mathcal{F}$ a q-dynamical filtration.*

(a) If ρ_i, q_i are pseudonorms with $\rho_i \preceq_{\mathcal{F},\Sigma} q_i$, then

$$\mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_i) \leq \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}, q_i),$$

$$\underline{\mathrm{opdim}}_{\Sigma,\infty}(\mathcal{F}, \rho_i) \leq \underline{\mathrm{opdim}}_{\Sigma,\infty}(\mathcal{F}, q_i).$$

(b) Let $\mathcal{F}' = ((a'_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y'_{E,l})$, $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l})$ be q-dynamical filtrations. Let ρ be any product norm. Define a pseudonorm on $B(Y, V_i)$ by $\rho_{\mathcal{F},i}(\phi) = \rho((\|\phi(a_{ej})\|)_{j=1}^{\infty})$, and similarly define $\rho_{\mathcal{F}',i}$. Then

$$\rho_{\mathcal{F}',i} \preceq_{\mathcal{F},\Sigma} \rho_{\mathcal{F},i}.$$

Proof. Let $\Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i))$.

(a) This follows directly follow the definitions.

(b) Let $C > 0$ be such that Y has the C-bounded approximation property and

$$\|a_{sj}\| \leq C$$

$$\|a'_{sj}\| \leq C$$

Choose $m \in \mathbb{N}$ such that $\rho(f) < \varepsilon$ if $\|f\|_\infty \leq 1$ and f is supported on $\{n : n \geq m\}$, and let $\kappa > 0$ be such that $\rho(f) < \varepsilon$ if $\|f\|_\infty \leq \kappa$.

By Lemma 3.2.2 choose $F' \supseteq F$ finite $m \leq m' \in \mathbb{N}$, and $\delta, \varepsilon > 0$ and

$$\widetilde{L}_i \colon \ell^\infty(\mathbb{N}, V_i) \to B(Y, V_i)$$

so that if $f \in \ell^\infty(\mathbb{N}, V_i)$ and $\phi \in \mathrm{Hom}_\Gamma(\mathcal{F}, F', m', \delta, \sigma_i)$ has $\rho_{V_i}(\alpha_{\mathcal{F}}(\phi) - f) < \varepsilon'$ then

$$\left\| \widetilde{L}_i(f)\big|_{Y'_{\{e\},m}} - \phi\big|_{Y'_{\{e\},m}} \right\| < \kappa,$$

$$\|\widetilde{L}_i(f)\| \leq 2C.$$

Let $L_i \colon B(Y, V_i) \to B(Y, V_i)$ be given by $L_i(\psi) = \widetilde{L}_i(\alpha_{\mathcal{F}}(\psi))$.

Suppose $\phi \in \mathrm{Hom}_\Gamma(\mathcal{F}, F', m', \delta', \sigma_i)$ and $\psi \in B(Y, V_i)$ satisfy $\rho_{\mathcal{F},i}(\phi - \psi) < \varepsilon'$. Then, for $1 \leq j \leq m$ we have

$$\|\phi(a'_{ej}) - L_i(\psi)(a'_{ej})\| \leq C\kappa.$$

Our choice of m, κ then imply that $\rho_{\mathcal{F}',i}(\phi - L_i(\psi)) < 2C(C+1)\varepsilon$. This completes the proof.

□

Corollary 3.2.7. *Let $(Y, q, X, \Gamma, \sigma_i\colon \Gamma \to \mathrm{Isom}(V_i))$ be a quotient dimension tuple. Let ρ, ρ' be two product norms. For any two q-dynamical filtrations $\mathcal{F}, \mathcal{F}'$ we have*

$$\mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_{\mathcal{F},i}) = \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}', \rho_{\mathcal{F}',i}) = \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}', \rho'_{\mathcal{F}',i}).$$

$$\underline{\mathrm{opdim}}_{\Sigma,\infty}(\mathcal{F}, \rho_{\mathcal{F},i}) = \underline{\mathrm{opdim}}_{\Sigma,\infty}(\mathcal{F}', \rho_{\mathcal{F}',i}) = \underline{\mathrm{opdim}}_{\Sigma}(\mathcal{F}', \rho'_{\mathcal{F}',i}).$$

Proof. Combining Lemmas 3.2.3, 3.2.6, and 3.2.4 we have

$$\mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}', \rho'_{\mathcal{F}',i}) = \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}', \rho_{\mathcal{F}',i}) \leq \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_{\mathcal{F},i}).$$

The opposite inequality follows by symmetry.

□

Because of the preceding corollary $f.\dim_{\Sigma}(\mathcal{F}, \rho)$ only depends on the action of Γ and the quotient map $q\colon Y \to X$. Thus we can define

$$\dim_{\Sigma}(q, \Gamma) = \mathrm{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_{\mathcal{F},i}) = f.\dim_{\Sigma}(\mathcal{F}, \rho)$$

where $\mathcal{F}$ is any q-dynamical filtration and ρ is any product norm.

We now proceed to show that $\dim_{\Sigma,\infty}(q, \Gamma)$ does not depend on q, as stated before the idea is to prove that

$$\dim_{\Sigma}(q, \Gamma) = f.\dim_{\Sigma}(S, \rho)$$

where S is any dynamically generating sequence for X.

For this, we will prove that we can approximate maps T on Y which almost vanish on the kernel of q, by maps on X. For the proof, we need the construction of ultraproducts of Banach spaces.

Let X_n be a sequence of Banach spaces and $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ a free ultrafilter. We define the ultraproduct of the X_n, written $\prod^{\omega} X_n$ by

$$\prod^{\omega} X_n = \{(x_n)_{n=1}^{\infty} : x_n \in X_n, \sup_n \|x_n\| < \infty\}/\{(x_n)_{n=1}^{\infty} : x_n \in X_n, \lim_{n\to\omega} \|x_n\| = 0\}.$$

We use $(x_n)_{n\to\omega}$ for the image of $(x_n)_{n=1}^\infty$ under the canonical quotient map to

$$\prod^\omega X_n.$$

If a set $A \subseteq \mathbb{N}$ is in ω, we will say that A is ω-large.

Lemma 3.2.8. *Let X, Y be Banach spaces with X and $q\colon Y \to X$ a bounded linear surjective map. Let $F \subseteq X$ be finite and Z a finite-dimensional subspace of Y with $q(F) \subseteq Z$. Let $C > 0$ be such that for all $x \in X$, there is a $y \in Y$ with $\|y\| \le C\|x\|$ such that $q(y) = x$, and fix $A > C$. Let I be a countable directed set, and $(Y_\alpha)_{\alpha\in I}$ a net of subspaces of Y such that $Y_\alpha \subseteq Y_\beta$ if $\alpha \le \beta$, and*

$$q(Y_\alpha) \supseteq Z,$$

$$\ker(q) = \overline{\bigcup_\alpha Y_\alpha \cap \ker(q)},$$

$$F \subseteq \bigcup_\alpha Y_\alpha.$$

Then for all $\varepsilon > 0$, there are a $\delta > 0$ and α_0 with the following property. If $\alpha \ge \alpha_0$ and W is a Banach space with $T\colon Y_\alpha \to W$ a linear contraction such that

$$\left\| T\big|_{\ker(q)\cap Y_\alpha} \right\| \le \delta,$$

then there is a $S\colon Z \to W$ such that $\|S\| \le A$ and

$$\|T(x) - S \circ q(x)\| \le \varepsilon,$$

for all $x \in F$.

Proof. Note that our assumptions imply

$$Y = \overline{\bigcup_\alpha Y_\alpha}.$$

Fix a countable increasing sequence α_n in I, such that for every $\beta \in I$ there is an n such that $\beta \le \alpha_n$. Assume also that $F \subseteq Y_{\alpha_1}$. Since I is directed, if the claim is false, then we can find an $\varepsilon > 0$ and an increasing sequence β_n with $\beta_n \geq \alpha_n$ and a $T_n\colon Y_{\beta_n} \to W_n$ such that $\|T_n\| \leq 1$,

$$\left\| T_n\big|_{\ker(q)\cap Y_{\beta_n}} \right\| \leq 2^{-n},$$

and for every $S\colon X \to W_n$ with $\|S\| \leq A$,

$$\|T_n(x) - S \circ q(x)\| \geq \varepsilon, \text{ for some } x \in F.$$

Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and let

$$W = \prod^{\omega} W_n.$$

Define

$$T\colon \bigcup_n Y_{\beta_n} \to W$$

by

$$T(x) = (T_n(x))_{n\to\omega},$$

note that for any k, the map T_n is defined on Y_{β_k} for $n \geq k$, so T is well-defined. Also

$$\|T(x)\| \leq \|x\|$$

$$T(x) = 0 \text{ on } \textstyle\bigcup_n Y_{\beta_n} \cap \ker(q).$$

Our density assumptions imply that T extends uniquely to a bounded linear map, still denoted T, from Y to W, which vanishes on the kernel of q. Thus there is $S\colon Z \to W$ such that $T = S \circ q$, and our hypothesis on C implies that $\|S\| \leq C$.

Since Z is finite dimensional, we can find $S_n\colon X \to W_n$ such that $S(x) = (S_n(x))_{n\to\omega}$. Compactness of the unit sphere of Z and a simple diagonal argument show that

$$C \geq \|S\| = \lim_{n\to\omega} \|S_n\|.$$

Thus $B = \{n : \|S_n\| < A\}$ is an ω-large set, and by hypothesis

$$B = \bigcup_{x\in F} \{n \in B : \|T_n(x) - S_n(q(x))\| \geq \varepsilon\}.$$

Since B is ω-large, there is some $x \in F$ such that

$$\{n \in B : \|T_n(x) - S_n(q(x))\| \geq \varepsilon\}$$

is ω-large. But then $T(x) \neq S \circ q(x)$, a contradiction.

$\square$

Lemma 3.2.9. *Let $(Y, q, X, \Gamma, \Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)))$ be a quotient dimension tuple. Fix a dynamically generating sequence S in X, and ρ a product norm. Then*

$$\dim_\Sigma(q, \Gamma) = f.\dim_\Sigma(S, \rho).$$

$$\underline{\dim}_\Sigma(q, \Gamma) = \underline{f.\dim}_\Sigma(S, \rho).$$

Proof. We will only do the proof for $\dim$.

Let $S = (x_j)_{j=1}^\infty$ and let $\mathcal{F} = ((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l})$ be a dynamical filtration such that $q(a_{ej}) = x_j$. Let $C > 0$ be such that

$$\sup_{(s,j)} \|a_{sj}\| \leq C$$

$$\sup_j \|x_j\| \leq C$$

$$\|q\| \leq C,$$

$$\text{for every } x \in X, \text{ there is a } y \in Y \text{ such that } q(y) = x \text{ and } \|y\| \leq C\|x\|,$$

and so that Y has the C-bounded approximation property. By Proposition, 3.2.1, we may find $\theta_{E,l} \colon Y \to Y_{E,l}$ such that $\|\theta_{E,l}\| \leq C$ and

$$\lim_{(E,l)} \|\theta_{E,l}(y) - y\| = 0 \text{ for all } y \in Y.$$

We first show that

$$\dim_\Sigma(q, \Gamma) \geq f.\dim_\Sigma(S, \rho).$$

For this, fix $\varepsilon > 0$, and choose $r \in \mathbb{N}$ such that

$$\rho(f) < \varepsilon, \text{ if } f \text{ is supported on } \{n : n \geq r\} \text{ and } \|f\|_\infty \leq 1,$$

as before choose $\varepsilon \geq \kappa > 0$ such that if $\|f\|_\infty \leq \kappa$, then

$$\rho(f) < \varepsilon.$$

Let $e \in E \subseteq \Gamma$ finite and $l \in \mathbb{N}$ be such that if $E \subseteq F \subseteq \Gamma$ is finite, and $k \geq l$ then

$$\|\theta_{F,k}(a_{ej}) - a_{ej}\| < \kappa$$

for $1 \leq j \leq r$.

Now fix $E \subseteq F \subseteq \Gamma$ finite, $l \leq m \in \mathbb{N}, \delta > 0$. We claim that we can find $F \subseteq F' \subseteq \Gamma$ finite $m \leq m'$ in $\mathbb{N}$, $\delta > \delta' > 0$ such that

$$\mathrm{Hom}_\Gamma(S, F', m', \delta', \sigma_i) \circ q\big|_{Y_{F',m'}} \circ \theta_{F',m'} \subseteq \mathrm{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}.$$

For $T \in \mathrm{Hom}_\Gamma(S, F', m', \delta', \sigma_i)$, for $1 \leq j, k \leq m$ and $s_1, \ldots, s_k \in F$,

$$\begin{aligned}
&\|T \circ q \circ \theta_{F',m'}(a_{s_1 \cdots s_k j}) - \sigma_i(s_1) \cdots \sigma_i(s_k) T \circ q \circ \theta_{F',m'}(a_{ej})\| \\
&\leq C\|\theta_{F',m'}(a_{s_1 \cdots s_k j}) - a_{s_1 \cdots s_k j}\| + C\|\theta_{F',m'}(a_{ej}) - a_{ej}\| \\
&+ \|T(s_1 \cdots s_k x_j) - \sigma_i(s_1) \cdots \sigma_i(s_k) T(x_j)\| \\
&< C\|\theta_{F',m'}(a_{s_1 \cdots s_k j}) - a_{s_1 \cdots s_k j}\| + C\|\theta_{F',m'}(a_{ej}) - a_{ej}\| \\
&+ \delta'.
\end{aligned}$$

Also for $y \in \ker(q) \cap Y_{F,m}$ we have

$$\|T \circ q \circ \theta_{F',m'}(y)\| \leq C\|\theta_{F',m'}(y) - y\|.$$

So it suffices to choose $\delta' < \min(\delta, \kappa)$ and then $F' \supseteq F, m' \geq \max(m, l, r)$ such that

$$C\|\theta_{F',m'}(a_{s_1 \cdots s_k j}) - a_{s_1 \cdots s_k j}\| + C\|\theta_{F',m'}(a_{ej}) - a_{ej}\| < \delta - \delta',$$

$$C\left\|\theta_{F',m'}\big|_{Y_{F,m}} - \mathrm{Id}\big|_{Y_{F,m}}\right\| < \delta.$$

for $1 \leq j, k \leq m$ and $s_1, \ldots, s_k \in F$.

Suppose that δ', F', m' are so chosen. If $T \in \operatorname{Hom}_\Gamma(S, F', m', \delta', \sigma_i)$ and $\phi = T \circ q\big|_{Y_{F',m'}} \circ \theta_{F',m'}$ then,

$$\rho_{V_i}(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi)) \leq C(C^2+1)\varepsilon + \rho_{V_i}(\chi_{\{j: j \leq r\}}(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi)))$$

and if $j \leq r$,

$$\|\alpha_S(T)(j) - \alpha_{\mathcal{F}}(\phi)(j)\| = \|T(x_j) - T \circ q \circ \theta_{F,l}(a_{ej})\| \leq C\kappa + \|T(x_j) - T \circ q(a_{ej})\| = C\kappa.$$

Thus

$$\rho_{V_i}(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi)) \leq (C^2 + C + 1)\varepsilon.$$

Therefore

$$\widehat{d}_{(C^2+C+2)\varepsilon}(\operatorname{Hom}_\Gamma(S, F', m', \delta', \sigma_i), \rho) \leq \widehat{d}_\varepsilon(\operatorname{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}, \rho).$$

Since F', m' can be made arbitrary large and δ' arbitrarily small, this implies

$$f.\dim_\Sigma(S, \rho, (C^2 + 2C + 1)\varepsilon) \leq \limsup_i \frac{1}{\dim V_i} \widehat{d}_\varepsilon(\operatorname{Hom}_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}, \rho),$$

taking the limit supremum over (F, m, δ) and then the supremum over $\varepsilon > 0$,

$$f.\dim_\Sigma(S, \rho) \leq f.\dim_\Sigma(q, \Gamma).$$

For the opposite inequality, fix $1 > \varepsilon > 0$ and let r, κ, E, l be as before. Fix $E \subseteq F \subseteq \Gamma$ finite, $m \geq \max(r, l)$ and $\delta < \min(\kappa, \varepsilon)$.

By Lemma 3.2.8 we can find $\delta' < \delta$, and $F \subseteq F' \subseteq \Gamma$ finite and $m \leq m' \in \mathbb{N}$ such that if W is a Banach space and

$$T \colon Y_{F',m'} \to W$$

has

$$\|T\| \leq 1,$$

$$\|T\big|_{\ker(q) \cap Y_{F',m'}}\| \leq \delta',$$

then there is a $\phi \colon X_{F,m} \to W$ such that

$$\|T(a_{s_1 \cdots s_k j}) - \phi(s_1 \cdots s_k x_j)\| \leq \delta, \text{ for } 1 \leq j, k \leq m, s_1, \ldots, s_k \in F$$

and $\|\phi\| \leq 2C$.

Fix $T \in \mathrm{Hom}_\Gamma(\mathcal{F}, F', m', \delta', \sigma_i)$, and choose $\phi\colon X_{F,m} \to V_i$ such that $\|\phi\| \leq 2C$ and

$$\|T(a_{s_1\cdots s_k j}) - \phi \circ q(a_{s_1\cdots s_k j})\| \leq \delta, \text{ for } 1 \leq j, k \leq m, s_1, \dots, s_k \in F.$$

Thus for $1 \leq j, k \leq m$ and $s_1, \dots, s_k \in F$ we have

$$\begin{aligned}\|\phi(s_1 \cdots s_k x_j) - \sigma_i(s_1) \cdots \sigma_i(s_k)\phi(x_j)\| &\leq 2\delta \\ &\quad + \|T(a_{s_1\cdots s_k j}) - \sigma_i(s_1)\cdots\sigma_i(s_k)T(a_{ej})\| \\ &< 2\delta + \delta' \\ &< 3\delta.\end{aligned}$$

Thus $\phi \in \mathrm{Hom}_\Gamma(S, F, m, 3\delta, \sigma_i)_{2C}$. Furthermore, for $1 \leq j \leq r$

$$\|\alpha_S(T)(j) - \alpha_{\mathcal{F}}(\phi)(j)\| = \|T(a_{ej}) - \phi \circ q(a_{ej})\| \leq \kappa,$$

so

$$\rho_{V_i}(\alpha_{\mathcal{F}}(T) - \alpha_S(\phi)) \leq \varepsilon + (2C^2 + C)\varepsilon = (2C^2 + C + 1)\varepsilon.$$

Thus

$$f.\dim_\Sigma(\mathcal{F}, (2C^2 + C + 2)\varepsilon, \rho) \leq \limsup_i \frac{1}{\dim V_i} \widehat{d_\varepsilon}\left(\mathrm{Hom}_\Gamma(S, F, m, 3\delta, \sigma_i)_{2C}, \rho\right),$$

and since $F, m, \delta, \varepsilon$ are arbitrary this completes the proof.

□

Because of the preceding Lemma and Corollary 3.2.7, we know that

$$f.\dim_\Sigma(S, \rho), \dim_\Sigma(q, \Gamma)$$

only depend upon the action of Γ on X, and are equal. Because of this we will use

$$\dim_\Sigma(X, \Gamma) = f.\dim_\Sigma(S, \rho) = \dim_\Sigma(q, \Gamma)$$

for any dynamically generating sequence S, and any bounded linear surjective map $q\colon Y \to X$, where Y has the bounded approximation property. We similarly define $\underline{\dim}_\Sigma(X, \Gamma)$.

We now prove a lemma which allows us to treat the limit supremum over (F,m,δ) in the definition of $f.\dim_\Sigma(S,\rho)$ as a limit.

Lemma 3.2.10. *Let $(X,\Gamma,\Sigma=(\sigma_i\colon\Gamma\to\mathrm{Isom}(V_i))$ be a dimension triple, fix a dynamically generating sequence S in X and ρ a product norm. Then*

$$f.\dim_\Sigma(S,\rho)=\sup_{\varepsilon>0}\liminf_{(F,m,\delta)}\limsup_i \frac{1}{\dim V_i}\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho),$$

$$\underline{f.\dim}_\Sigma(S,\rho)=\sup_{\varepsilon>0}\limsup_{(F,m,\delta)}\liminf_i \frac{1}{\dim V_i}\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho).$$

Proof. Let $S=(x_j)_{j=1}^\infty$. We do the proof for dim only, the proof for $\underline{\dim}$ is the same. Fix $\varepsilon>0$ and choose $k\in\mathbb{N}$ such that if $\|f\|_\infty\le 1+\sup_{j\in\mathbb{N}}\|x_j\|$ and f is supported on $\{n:n\ge k\}$, then $\rho(f)<\varepsilon$. It suffices to show that

$$f.\dim_\Sigma(S,\rho)\le\sup_{\varepsilon}\liminf_{(F,m,\delta)}\limsup_i \frac{1}{\dim V_i}\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho).$$

Fix $F\subseteq\Gamma$ finite $m\ge k,\delta>0$. Then for any $F\subseteq F'\subseteq\Gamma$ finite, $m'\ge m,\delta'<\delta$ and $\psi\in\mathrm{Hom}_\Gamma(S,F',m',\delta',\sigma_i)$ we have $\psi\in\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)$.

Furthermore if $f,g\in\ell^\infty(\mathbb{N},V_i)$ are defined by

$$f(j)=\chi_{\{n\le m\}}(j)\psi(x_j),g(j)=\chi_{\{n\le m'\}}\psi(x_j)$$

then

$$\rho(j\mapsto\|f(j)-g(j)\|)<\varepsilon.$$

Thus

$$\widehat{d}_{2\varepsilon}(\mathrm{Hom}_\Gamma(S,F',m',\delta',\sigma_i),\rho)\le\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho).$$

Therefore

$$f.\dim_\Sigma(S,2\varepsilon,\rho)\le\limsup_i\frac{1}{\dim V_i}\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho).$$

Since F,m,δ were arbitrary

$$f.\dim_\Sigma(S,2\varepsilon,\rho)\le\liminf_{(F,m,\delta)}\limsup_i\frac{1}{\dim V_i}\widehat{d}_\varepsilon(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i),\rho),$$

and taking the supremum over $\varepsilon>0$ completes the proof.

□

3.3 Main Properties of $\dim_\Sigma(X,\Gamma)$

The first property that we prove is that dimension is decreasing under surjective maps, as in the usual case of finite-dimensional vector spaces.

Proposition 3.3.1. *Let* $(Y,\Gamma,\Sigma=(\sigma_i\colon\Gamma\to\mathrm{Isom}(V_i))),(X,\Gamma,\Sigma)$ *be two dimension triples. Suppose that there is a* Γ*-equivariant bounded linear map* $T\colon Y\to X$*, with dense image. Then*

$$\dim_\Sigma(X,\Gamma)\le\dim_\Sigma(Y,\Gamma).$$

$$\underline{\dim}_\Sigma(X,\Gamma)\le\underline{\dim}_\Sigma(Y,\Gamma).$$

Proof. Let $S'=(y_j)_{j=1}^\infty$ be a dynamically generating sequence for Y. Let $S=(T(x_j))_{j=1}^\infty$, then S is dynamically generating for X. Then

$$\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)\circ T\subseteq\mathrm{Hom}_\Gamma(S',F,m,\delta,\sigma_i)_{\|T\|},$$

and

$$\alpha_{S'}(\phi\circ T)=\alpha_S(\phi),$$

so the proposition follows.

□

We next show that dimension is subadditive under exact sequences. It turns out to be strong of a condition to require that dimension be additive under exact sequences. As noted in [13] if $\dim_{\Sigma,\ell^p}$ is additive under exact sequences and

$$\dim_{\Sigma,\ell^p}(\ell^p(\Gamma)^{\oplus n},\Gamma)=n,$$

then we can write the Euler characteristic of a group as an alternating sum of dimensions of ℓ^p cohomology spaces. But torsion-free cocompact lattices in $SO(4,1)$ have positive Euler characteristic and their ℓ^p cohomology vanishes when p is sufficiently large, so this would give a contradiction.

Proposition 3.3.2. *Let $(V, \Gamma, \Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)))$ be a dimension triple. Let $W \subseteq V$ be a closed Γ-invariant subspace. Then*

$$\dim_\Sigma(V, \Gamma) \leq \dim_\Sigma(V/W, \Gamma) + \dim_\Sigma(W, \Gamma),$$

$$\underline{\dim}_\Sigma(V, \Gamma) \leq \underline{\dim}_\Sigma(V/W, \Gamma) + \dim_\Sigma(W, \Gamma),$$

$$\underline{\dim}_\Sigma(V^{\oplus n}, \Gamma) \leq n\underline{\dim}_\Sigma(V, \Gamma).$$

Proof. Let $S_2 = (w_j)_{j=1}^\infty$ be a dynamically generating sequence for W, and let $S_1 = (a_j)_{j=1}^\infty$ be a dynamically generating sequence for V/W. Let $x_j \in V$, be such that $x_j + W = a_j$, and $\|x_j\| \leq 2\|a_j\|$. Let S be the sequence

$$x_1, w_1, x_2, w_2, \cdots .$$

We shall use the product norm on $\ell^\infty(\mathbb{N})$ given by

$$\rho_1(f) = \sum_{j=1}^\infty \frac{1}{2^j}|f(j)|,$$

$$\rho_2(f) = \sum_{j=1}^\infty \frac{1}{2^j}|f(2j)| + \sum_{j=1}^\infty \frac{1}{2^j}|f(2j-1)|.$$

Let $\varepsilon > 0$, and choose m such that $2^{-m} < \varepsilon$. Let $e \in F_1 \subseteq \Gamma$ be finite,$m \leq m_1 \in \mathbb{N}$, and $\delta_1 > 0$. Let $\eta > 0$ to be determined later. By Lemma 3.2.8, we can find a $\delta_1 > \delta > 0$, a $F_1 \subseteq E \subseteq \Gamma$ finite, and a $m \leq k \in \mathbb{N}$, so that if X is a Banach space, and

$$T \colon V_{E,2k} \to X$$

has $\|T\| \leq 2$, and

$$\|T\big|_{W \cap V_{E,2k}}\| \leq \delta,$$

then there is a $\phi \colon (V/W)_{F_1,m_1} \to X$ with $\|\phi\| \leq 3$, and

$$\|\phi(s_1 \cdots s_k a_j) - T(s_1 \cdots s_k x_j)\| < \delta_1,$$

for all $1 \leq j, k \leq m_1$, and $s_1, \ldots, s_k \in F_1$.

By finite-dimensionality, we can find a finite set $F' \supseteq E, m' \geq 2k$, and a $0 < \delta' < \delta_1$, so that if $T\colon V_{F',m'} \to X$, satisfies

$$\|T(s_1 \cdots s_k x_j)\| < \delta'$$

for all $1 \leq j, k \leq m'$, and $s_1, \ldots, s_k \in F'$, then

$$\|T\big|_{W \cap V_{E,2k}}\| \leq \delta.$$

Define

$$R\colon \operatorname{Hom}_\Gamma(S, F', 2m', \delta', \sigma_i) \to \operatorname{Hom}_\Gamma(S_2, F', m', \delta', \sigma_i)$$

by

$$R(T) = T\big|_{W_{F',m'}}.$$

Find

$$\Theta\colon \operatorname{im}(R) \to \operatorname{Hom}_\Gamma(S, F', 2m', \delta', \sigma_i)$$

so that $R \circ \Theta = \operatorname{Id}$.

Then

$$(T - \theta(R(T))(s_1 \cdots s_k w_j) = 0,$$

for all $1 \leq j, k \leq m'$, and $s_1, \ldots, s_k \in F'$. Thus by assumption, we can find a

$$\phi\colon (V/W)_{F_1,m_1} \to V_i,$$

so that $\|\phi\| \leq 3$, and

$$\|\phi(s_1 \cdots s_k a_j) - (T - \theta(R(T)))(s_1 \cdots s_k x_j)\| < \delta_1,$$

for all $1 \leq j, k \leq m_1$, $s_1, \ldots, s_k \in F_1$, in particular,

$$\|\phi(a_j) - (T - \theta(R(T)))(x_j)\| < \delta_1,$$

for $1 \leq j \leq m$.

Thus whenever $1 \leq j, k \leq m_1$, $s_1, \ldots, s_k \in F_1$,

$$\|\phi(s_1 \cdots s_k a_j) - \sigma_i(s_1) \cdots \sigma_i(s_k)\phi(a_j)\| \leq 2\delta_1 + 2\delta' < 4\delta_1.$$

Now suppose that

$$\alpha_{S_2}(\mathrm{Hom}_\Gamma(S_2,F_1,m_1,\delta_1,\sigma_i))\subseteq_{\varepsilon,\rho_{1,V_i}} G,$$

$$\alpha_{S_1}(\mathrm{Hom}_\Gamma(S_1,F,m,4\delta_1,\sigma_i)_3)\subseteq_{\varepsilon,\rho_{1,V_i}} F.$$

Let $E\subseteq\ell^\infty(\mathbb{N},V_i)$ be the subspace consisting of all h so that there are $f\in F,g\in G$ so that

$$h(2k)=g(k),h(2k-1)=f(k).$$

Then $\dim(E)=\dim(F)+\dim(G)$. It easy to see that

$$\alpha_S(\mathrm{Hom}_\Gamma(S,F',m',\delta',\sigma_i))\subseteq_{3\varepsilon+\delta_1,\rho_{2,V_i}} E.$$

So if $\delta_1<\varepsilon$, we find that

$$\alpha_S(\mathrm{Hom}_\Gamma(S,F_1,m_1,\delta',\sigma_i))\subseteq_{3\varepsilon} E.$$

From this the first two inequalities follow.

The last inequality is easier and its proof will only be sketched. Let $S=(x_j)_{j=1}^\infty$ be a dynamically generating sequence for X, and $y_j=x_q\otimes e_r$ if $j=nq+r$, with $1\leq r\leq n$, and $x_q\otimes e_r$ is the element of $X^{\oplus n}$ which is zero in all coordinates except for the r^{th}, where it is x_q. If $F\subseteq\Gamma$ is finite $m\in\mathbb{N},\delta>0$, then

$$\mathrm{Hom}_\Gamma(S,F,nm,\delta,\sigma_i)\subseteq\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)^{\oplus n}.$$

The rest of the proof proceeds as above. □

We note here that subadditivity is not true for *weakly* exact sequences, that is sequences

$$0\longrightarrow X\longrightarrow Y\longrightarrow Z\longrightarrow 0,$$

where $X\to Y$ is injective, $\overline{\mathrm{im}(X)}=\ker(Y\to Z)$, and the image of Y is dense in Z. In fact, using $\mathbb{F}_n$ for the free group on n letters $a_1,\dots,a_n$, it is known that the map

$$\partial\colon \ell^1(\mathbb{F}_n)^{\oplus n}\to\ell^1(\mathbb{F}_n),$$

given by

$$\partial(f_1,\dots,f_n)(x) = \sum_{j=1}^{n} f_j(x) - \sum_{j=1}^{n} f_j(xa_j^{-1})$$

has dense image and is injective. We will show in section 3.8 that

$$\underline{\dim}_{\Sigma,\ell^1}(\ell^1(\mathbb{F}_n)^{\oplus n},\mathbb{F}_n) = \dim_{\Sigma,\ell^1}(\ell^1(\mathbb{F}_n)^{\oplus n},\mathbb{F}_n) = n,$$

$$\underline{\dim}_{\Sigma,\ell^1}(\ell^1(\mathbb{F}_n),\mathbb{F}_n) = \dim_{\Sigma,\ell^1}(\ell^1(\mathbb{F}_n),\mathbb{F}_n) = 1,$$

this gives a counterexample to subadditivity under weakly exact sequences. This also gives a counterexample to monotonicity under injective maps, though once should note in this case that the map defined above does not have closed image.

For $2 \leq p \leq \infty$, we have a lower bound for direct sums, whose proof requires a few more lemmas.

Lemma 3.3.3 ([27], Lemma 8.5)**.** *Let H_1, H_2 be Hilbert spaces and let $H = H_1 \oplus H_2$ and let $\Omega_j \subseteq H_j$ and suppose $C_1, C_2 > 0$ are such that $C_1 \leq \|\xi\| \leq C_2$, for all $\xi \in \Omega_j$. If $0 < \delta < C_1$, then*

$$d_{C_2^{-1}\delta}(\Omega_1 \oplus 0 \cup 0 \oplus \Omega_2) \geq d_{C_1^{-1}\sqrt{5\delta}}(\Omega_1) + d_{C_1^{-1}\sqrt{5\delta}}(\Omega_2).$$

Proof. By replacing Ω_j with

$$\left\{\frac{\xi}{\|\xi\|} : \xi \in \Omega_j\right\}$$

we may assume $C_1 = C_2 = 1$. Let P_i be the projection onto each H_i, and set $\Omega = (\Omega_1 \oplus 0) \cup (0 \oplus \Omega_2)$. Suppose that V is a subspace such that $\Omega \subseteq_\delta V$, and let Q be the projection onto V and $T = QP_1Q\big|_V$. Define

$$\Omega_1' = Q(\Omega_1 \oplus 0), \Omega_2' = Q(0 \oplus \Omega_2).$$

For $\xi \in \Omega$ we have

$$\|(1-Q)\xi\| \leq \delta$$

thus for $\xi \in \Omega_1 \oplus \{0\}$

$$\langle TQ\xi, Q\xi\rangle = \langle QP_1Q\xi, Q\xi\rangle = \|P_1Q\xi\|^2 \geq (\|\xi\| - \|P_1(1-Q)\xi\|)^2 \geq (1-\delta)^2.$$

So if $T = \int_{[0,1]} t\, dE(t)$ we have with $\eta = Q\xi$

$$(\sqrt{1-\delta^2} - \delta)^2 \leq \left\langle \left(1 - \frac{1}{2}E\left([0,1/2]\right)\right)\eta, \eta\right\rangle \leq 1 - \frac{1}{2}\left\|E([0,1/2])\eta\right\|^2.$$

Thus

$$\|E([0,1/2])\eta\|^2 \leq 2(1-(1-\delta)^2) \leq 4\delta$$

i.e.

$$\|\eta - E((1/2,1])\eta\|^2 \leq 4\delta.$$

Thus

$$\Omega_1' \subseteq_{2\sqrt{\delta}} E((1/2,1])V.$$

Similarly, because $QP_2Q\big|_V = 1 - T$ we have

$$\Omega_2' \subseteq_{2\sqrt{\delta}} E([0,1/2])V.$$

For any projection P' and any $x \in H$ we have $\|x - P'x\|^2 = \|x\|^2 - \|P'x\|^2$. So for all $\xi \in \Omega_1 \oplus 0$ we have since, $QE((1/2,1]) = E((1/2,1])$ (and $E((1/2,1]Q = E((1/2,1])$ by taking adjoints), that

$$\|\xi - E((1/2,1])Q\xi\|^2 = \|\xi - E((1/2,1])\xi\|^2 = \|\xi\|^2 - \|E((1/2,1])\xi\|^2 =$$

$$\|\xi\|^2 - \|Q\xi\|^2 + \|Q\xi\|^2 - \|E((1/2,1])\xi\|^2 =$$

$$\|\xi - Q\xi\|^2 + \|Q\xi - E((1/2,1])Q\xi\|^2 \leq \delta^2 + 4\delta < 5\delta.$$

Thus with a similar proof for Ω_2 we have

$$\Omega_1 \oplus 0 \subseteq_{\sqrt{5\delta}} E((1/2,1])V$$

$$0 \oplus \Omega_2 \subseteq_{\sqrt{5\delta}} E([0,1/2))V$$

since

$$V = E([0,1/2])V \oplus E\left((1/2,1]\right)V$$

the desired claim follows.

□

Lemma 3.3.4. *Let (X, Γ, Σ) be a dimension triple. Let S be a dynamically generating sequence in X, and ρ a product norm such that $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$. Set*

$$\rho^{(N)}(f) = \rho(\chi_{j \leq N} f).$$

Then

$$f.\dim_\Sigma(S, \rho) = \lim_{N \to \infty} f.\dim_\Sigma(S, \rho^{(N)}),$$

$$\underline{f.\dim}_\Sigma(S, \rho) = \lim_{N \to \infty} \underline{f.\dim}_\Sigma(S, \rho^{(N)}).$$

Proof. Let $\Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i))$. Let $S = (x_j)_{j=1}^\infty$, $C = \sup_j \|x_j\|$.

Since $\rho^{(N)} \leq \rho$, for any $\varepsilon > 0$

$$f.\dim_\Sigma(S, \varepsilon, \rho^{(N)}) \leq f.\dim_\Sigma(S, \varepsilon, \rho) \leq f.\dim_\Sigma(S, \rho),$$

thus

$$\limsup_{n \to \infty} f.\dim_\Sigma(S, \rho^{(n)}) \leq f.\dim_\Sigma(S, \rho).$$

For the opposite inequality, fix $\varepsilon > 0$. and choose N such that $\rho(f) < \varepsilon$ if $f \in \ell^\infty(\mathbb{N}, V_i)$ is supported on $\{k : k \geq N\}$ and $\|f\|_\infty \leq C$. Thus for $T \in B(X, V_i)$, and $f \in \ell^\infty(\mathbb{N}, V_i)$ with $\|T\| \leq 1$, and $n \geq N$ we have

$$|\rho_{V_i}(\alpha_S(T) - \chi_{\{j \leq N\}}) - (\rho_{V_i}^{(n)}(\alpha_S(T) - \chi_{\{j \leq N\}} f))| \leq |\rho_{V_i}(\chi_{\{k > n\}} \alpha_S(T))| \leq \varepsilon.$$

Thus for $n \geq N$,

$$f.\dim_\Sigma(S, 2\varepsilon, \rho) \leq f.\dim_\Sigma(S, \varepsilon, \rho^{(n)}) \leq f.\dim_\Sigma(S, \rho^{(n)}),$$

so

$$f.\dim_\Sigma(S, 2\varepsilon, \rho) \leq \liminf_{n \to \infty} f.\dim_\Sigma(S, \rho^{(n)}).$$

□

For the next lemma, we recall the notion of the volume ratio of a finite-dimensional Banach space. Let X be an n-dimensional real Banach space, which we will identify with

$\mathbb{R}^n$ with a certain norm. By an *ellipsoid* in $\mathbb{R}^n$ we mean a set which is the unit ball for some Hilbert space norm on $\mathbb{R}^n$. Let $B \subseteq \mathbb{R}^n$ be the unit ball of X. We define the volume ratio of B, denoted $\mathrm{vr}(B)$ by

$$\mathrm{vr}(B) = \inf \left(\frac{\mathrm{vol}(B)}{\mathrm{vol}(D)} \right)^{1/n},$$

where the infimum runs over all ellipsoids $D \subseteq B$. It is know that for any unit ball B of a Banach space norm on $\mathbb{R}^n$, there is an ellipsoid $D^{\max}$ such that $D^{\max} \subseteq B$, and $D^{\max}$ has the largest volume of all such ellipsoids. So we have

$$\mathrm{vr}(B) = \left(\frac{\mathrm{vol}(B)}{\mathrm{vol}(D^{\max})} \right)^{1/n}.$$

The main property we will need to know about volume ratio is the following theorem.

Theorem 3.3.5 (Theorem 6.1,[21])**.** *Let $B \subseteq \mathbb{R}^n$ be the unit ball for a norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $D \subseteq B$ be an ellipsoid. Set*

$$A = \left(\frac{\mathrm{vol}(B)}{\mathrm{vol}(D)} \right)^{1/n}.$$

Let $|\cdot|$ be a norm such that D is the unit ball of $(\mathbb{R}^n, |\cdot|)$, in particular $\|\cdot\| \leq |\cdot|$. Then for all $k = 1, \dots, n-1$ there is a subspace $F \subseteq \mathbb{R}^n$ such that $\dim F = k$ and for every $x \in F$

$$|x| \leq (4\pi A)^{\frac{n}{n-k}} \|x\|. \tag{3.2}$$

Further if we let G_{nk} be the Grassmanian manifold of k-dimensional subspaces of $\mathbb{R}^n$, then

$$\mathbb{P}(\{F \in G_{nk} : \text{ for all } x \in F, \text{ equation } (3.2) \text{ holds}\}) > 1 - 2^{-n},$$

for the unique $O(n)$-invariant probability measure on G_{nk}.

What we will actually use is the following corollary.

Corollary 3.3.6. *Let $B \subseteq \mathbb{R}^n$ be the unit ball for a norm $\|\cdot\|$ on $\mathbb{R}^n$, and let B^o be its polar. Let $D \subseteq B^o$ be an ellipsoid. Set*

$$A = \left(\frac{\mathrm{vol}(B^o)}{\mathrm{vol}(D^o)} \right)^{1/n}.$$

Let $|\cdot|$ be a norm such that D is the unit ball of $(\mathbb{R}^n, |\cdot|)$, in particular $|\cdot| \leq \|\cdot\|$. Then for all $k = 1, \dots, n-1$ there is a subspace $F \subseteq \mathbb{R}^n$ such that $\dim F = k$ and for every $x \in \mathbb{R}^n/F^\perp$

$$\|x\|_{(\mathbb{R}^n/F^\perp, \|\cdot\|)} \leq (4\pi A)^{\frac{n}{n-k}} |x|_{(\mathbb{R}^n/F^\perp, |\cdot|)}, \tag{3.3}$$

where we use $\|\cdot\|_{(\mathbb{R}^n/F^\perp,\|\cdot\|)}$ for the quotient norm induced by $\|\cdot\|$ and similarly for $|\cdot|$. Further,

$$\mathbb{P}(\{F \in G_{nk} \colon \text{ for all } x \in F, \text{ equation (3.3) holds}\}) > 1 - 2^{-n}.$$

Proof. This is precisely the dual of the above theorem. □

Here is the main application of the above corollary to dimension theory.

Theorem 3.3.7. *Let Γ be a countable group with a uniformly bounded action on separable Banach spaces X, Y. Let $\Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i))$ with $\dim V_i < \infty$. Suppose that V_i is the complexification of a real Banach space V_i' such that*

$$\sup_i \mathrm{vr}((V_i')^*) < \infty,$$

and there are constants $C_1, C_2 > 0$ so that

$$C_1(\|x\|_{V_i'} + \|y\|_{V_i'}) \leq \|x + iy\| \leq C_2(\|x\|_{V_i'} + \|y\|_{V_i'}),$$

for all $x, y \in V_i$. Then the following inequalities hold,

$$\underline{\dim}_\Sigma(X \oplus Y, \Gamma) \geq \underline{\dim}_\Sigma(X, \Gamma) + \underline{\dim}_{\Sigma, X_i}(Y, \Gamma),$$

$$\dim_\Sigma(Y_1 \oplus Y_2, \Gamma) \geq \dim_\Sigma(X, \Gamma) + \underline{\dim}_\Sigma(Y, \Gamma),$$

$$\dim_\Sigma(Y^{\oplus n}, \Gamma) \geq n \dim_\Sigma(Y, \Gamma),$$

Proof. We will do the proof for dim only, the proof of the other claims are the same. Let $S = (x_n)_{n=1}^\infty, T = (y_n)_{n=1}^\infty$ be dynamically generating sequences, enumerate $S \oplus \{0\} \cup \{0\} \oplus T$ by $x_1, y_1, x_2, y_2, \dots,$ and fix integers k, m. By Lemma 3.3.4, it suffices to show that for fixed $m, k \in \mathbb{N}$, and for the pseudonorms ρ, ρ_1, ρ_2 on $\ell^\infty(\mathbb{N})$ given by

$$\rho(f) = \left(\sum_{j=1}^{m+k} |f(j)|^2 \right)^{1/2},$$

$$\rho_1(f) = \left(\sum_{j=1}^{m} |f(j)|^2\right)^{1/2},$$

$$\rho_2(f) = \left(\sum_{j=1}^{k} |f(j)|^2\right)^{1/2},$$

we have

$$f.\dim_\Sigma(S \oplus 0 \cup 0 \oplus T, \rho) \geq f.\underline{\dim}_\Sigma(S, \rho_1) + f.\dim_\Sigma(T, \rho_2).$$

Fix $\kappa, \varepsilon > 0$ and fix $\eta > 0$ which will depend upon κ, ε in a manner to be determined later. By Corollary 3.3.6 there is a constant A, which depends only on κ, C_1, C_2 Hilbert space norms $|\cdot|_i$ on X_i, and finite dimensional complex subspaces $F_i \subseteq V_i^*$ of complex dimension $\lfloor(1-\kappa)(\dim V_i)\rfloor$ such that

$$\frac{1}{A}|x|_i \leq \|x\| \leq \|x\| \leq A|x|_i$$

for all $x \in V_i/F_i^{\perp}$. Here, as in the Corollary 3.3.6, we abuse notation by using $\|x\|$ for the norm on $X_i/F_i^{\perp}$ induced by $\|\cdot\|$, and similarly for $|\cdot|$.

For $m' \geq m \in \mathbb{N}, \delta > 0$ and $F \subseteq \Gamma$ finite we have

$$\mathrm{Hom}_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus \mathrm{Hom}_\Gamma(T, F, 2m', \delta, \sigma_i)_2 \subseteq \mathrm{Hom}_\Gamma((S \oplus \{0\}) \cup (\{0\} \oplus T), F, m', 2\delta, \sigma_i).$$

Thus

$$\widehat{d_\eta}\left(\mathrm{Hom}_\Gamma((S \oplus \{0\}) \cup (\{0\} \oplus T), F, 2m', 2\delta, \sigma_i)_2, \rho\right) \geq$$

$$\widehat{d_\eta}\left(\mathrm{Hom}_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus \mathrm{Hom}_\Gamma(T, F, 2m', \delta, \sigma_i)_2, \rho\right).$$

Let

$$K_1 = \{(T(x_1), \ldots, T(x_m)) : T \in \mathrm{Hom}_\Gamma(S, F, 2m', \delta, \sigma_i)\}$$

$$K_2 = \{(S(y_1), \ldots, S(y_k)) : S \in \mathrm{Hom}_\Gamma(S, F, 2m', \delta, \sigma_i)\}.$$

Then, by definition,

$$\widehat{d_\eta}\left(\mathrm{Hom}_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus \mathrm{Hom}_\Gamma(T, F, 2m', \delta, \sigma_i), \rho\right) =$$

$$d_\eta\left(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right)$$

where we use the ℓ^2-direct sum.

Let $\pi_i\colon V_i \to V_i/F_i^{\perp}$ be the quotient map and let

$$G_j = \pi_i^{\oplus l}(K_j),$$

where $l = m$ if $j = 1$, and $l = k$ if $j = 2$.

Then

$$d_\eta\left(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right) \geq d_\eta\left(G_1 \oplus G_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right) \geq$$

$$d_{A\eta}\left(G_1 \oplus G_2, |\cdot|_i^{\oplus m} \oplus |\cdot|_i^{\oplus k}\right).$$

Set

$$B_i = \left\{x \in G_i : lA \geq |x| \geq A\frac{\varepsilon}{4}\right\},$$

where $l = m$ if $i = 1$, and $l = k$ if $i = 2$.

Then

$$d_{A\eta}\left(G_1 \oplus G_2, |\cdot|_i^{\oplus m} \oplus |\cdot|_i^{\oplus k}\right) \geq d_{\max(l,m)(\varepsilon/4)^{-1}\sqrt{5\eta A \max(l,m)}}(B_1, |\cdot|^{\oplus m})$$

$$+ d_{\max(l,m)(\varepsilon/4)^{-1}\sqrt{5A\eta \max(l,m))}}(B_2, |\cdot|^{\oplus k}).$$

Setting $\eta = \frac{\varepsilon^{4/3}}{A\max(l,m)\cdot 5^{1/3}}$ we have

$$d_\eta(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}) \geq d_{\frac{\varepsilon}{A}}(B_1, |\cdot|^{\oplus m}) + d_{\frac{\varepsilon}{A}}(B_2, |\cdot|^{\oplus k})$$

$$\geq d_\varepsilon(B_1, \|\cdot\|^{\oplus k}) + d_\varepsilon(B_2, \|\cdot\|^{\oplus k}).$$

Since $B_i \supseteq \{x \in C_i : \|x\| \geq \frac{\varepsilon}{4}\}$ we have

$$d_\varepsilon(B_1, \|\cdot\|^{\oplus k}) + d_\varepsilon(B_2, \|\cdot\|^{\oplus k}) = d_\varepsilon(G_1, \|\cdot\|^{\oplus k}) + d_\varepsilon(G_2, \|\cdot\|^{\oplus k}).$$

Let $E_i \subseteq (V_i/F_i^{\perp})^{\oplus l}$ be a linear subspace of minimal dimension which ε-contains C_i with respect to $\|\cdot\|^{\oplus l}$ ($l = k$, if $i = 1$, and $l = m$ if $i = 2$.) Let $\widetilde{E_i} \subseteq V_i$ be a linear subspace such that $\dim E_i = \dim \widetilde{E_i}$ and $\pi_i^{\oplus l}(\widetilde{E_i}) = E_i$. Set $W_i = \widetilde{E_i} + F_i^{\oplus l}$. Then W_i has dimension at most $\dim E_i + lc_i$ with $\lim_{i\to\infty} \frac{c_i}{\dim V_i} = \kappa$, since $\dim V_i \to \infty$, and $K_i \subseteq_{\varepsilon,\|\cdot\|} V_i$. Thus

$$d_\varepsilon(G_i, \|\cdot\|^{\oplus l}) \geq \widehat{d}_\varepsilon(K_i, \|\cdot\|^{\oplus l}) - lc_i.$$

Since $\varepsilon \to 0$ as $\eta \to 0$ (and vice versa) we conclude that

$$\dim_\Sigma(S_1 \oplus S_2, \Gamma, \|\cdot\|_{S,T,i}) \geq -\kappa(k+m) + \dim_\Sigma(S_1, \Gamma, \|\cdot\|_{S,i}) + \underline{\dim}_\Sigma(Y_2, \Gamma, \|\cdot\|_{T,i}).$$

Since κ is arbitrary this proves the desired inequality.

□

Corollary 3.3.8. *Let $2 \leq p < \infty$.*

(a) Let Γ be a sofic group with uniformly bounded actions on separable Banach spaces X, Y and let Σ be a sofic approximation. Then

$$\dim_{\Sigma,\ell^p}(X \oplus Y, \Gamma) \geq \dim_{\Sigma,\ell^p}(X, \Gamma) + \underline{\dim}_{\Sigma,\ell^p}(Y, \Gamma)$$

$$\underline{\dim}_{\Sigma,\ell^p}(X \oplus Y, \Gamma) \geq \underline{\dim}_{\Sigma,\ell^p}(X, \Gamma) + \underline{\dim}_{\Sigma,\ell^p}(Y, \Gamma)$$

(b) Let Γ be an $\mathcal{R}^\omega$-embeddable group with uniformly bounded actions on separable Banach spaces X, Y and let Σ be an embedding sequence. Then

$$\dim_{\Sigma,S^p}(X \oplus Y, \Gamma) \geq \dim_{\Sigma,S^p}(X, \Gamma) + \underline{\dim}_{\Sigma,S^p}(Y, \Gamma)$$

$$\underline{\dim}_{\Sigma,S^p}(X \oplus Y, \Gamma) \geq \underline{\dim}_{\Sigma,S^p}(X, \Gamma) + \underline{\dim}_{\Sigma,S^p}(Y, \Gamma).$$

Proof. For $1 \leq q \leq \infty$, let B_q be the unit ball of $L^q(\{1, \ldots, n\}, \mu_n)$ where μ_n is the uniform measure.

It is known that for all q,

$$\inf_n \left(\frac{\mathrm{vol}(B_q)}{\mathrm{vol}(B_2)}\right)^{1/n} > 0,$$

$$\sup_n \left(\frac{\mathrm{vol}(B_q)}{\mathrm{vol}(B_2)}\right)^{1/n} < \infty,$$

(see the computation on page 11 of [21]). Similarly if we let C_q be the unit ball of $\{A \in M_n(\mathbb{C}) : A = A^*\}$ in the norm $\|\cdot\|_{L^p(\frac{1}{n}\mathrm{Tr})}$, it is known that for all q,

$$\inf_n \left(\frac{\mathrm{vol}(C_q)}{\mathrm{vol}(C_2)}\right)^{1/n} > 0,$$

$$\sup_n \left(\frac{\mathrm{vol}(C_q)}{\mathrm{vol}(C_2)}\right)^{1/n} < \infty,$$

(see [25]) Apply the preceding theorem. □

We note one last property of ℓ^2-dimension for representations, which will be used in a later section to show that our dimension agrees with von Neumann dimension in the ℓ^2-case.

Proposition 3.3.9. *Let H be a separable unitary representation of a $\mathcal{R}^\omega$-embeddable group Γ. Let Σ be an embedding sequence of Γ. Suppose that $H = \overline{\bigcup_{k=1} H_k}$ with H_k increasing, closed invariant subspaces, and that each H_k has a finite dynamically generating sequence. Then*

$$\dim_{\Sigma,\ell^2}(H,\Gamma) = \sup_k \dim_{\Sigma,\ell^2}(H_k,\Gamma),$$
$$\underline{\dim}_{\Sigma,\ell^2}(H,\Gamma) = \sup_k \underline{\dim}_{\Sigma,\ell^2}(H_k,\Gamma).$$

Proof. We will do the proof for dim only, the other cases are the same. By Proposition 3.3.2 we know that $\dim_{\Sigma,\ell^2}$ is monotone for unitary representations, so we only need to show

$$\dim_{\Sigma,\ell^2}(H,\Gamma) \geq \sup_k \dim_{\Sigma,\ell^2}(H_k,\Gamma).$$

Let $\{\xi_1^{(k)}, \ldots, \xi_{r_k}^{(k)}\}$ be unit vectors which dynamically generate H_k. Let S_N be the sequence

$$\xi_1^{(1)}, \ldots, \xi_{r_1}^{(1)}, \xi_1^{(2)}, \ldots, \xi_{r_2}^{(2)}, \ldots, \xi_1^{(N)}, \ldots, \xi_{r_N}^{(N)},$$

i.e. the ℓ^{th} term of S_N is

$$\xi_{q_l}^{(i)}$$

if i is the largest integer such that

$$C_i = \sum_{j \leq i} r_j < l,$$

and

$$q_l = l - \sum_{j \leq i} r_j.$$

Let S be the sequence obtained by the infinite concatenation of the S_N's. We will use S_N to compute $\dim_{\Sigma,\ell^2}(H_N,\Gamma)$ and S to compute $\dim_{\Sigma,\ell^2}(H,\Gamma)$, we also use the pseudonorms

$$\|T\|_{S,i} = \sum_{j=1}^{\infty} \frac{1}{2^j} \|T(\xi_j)\|$$

$$\|T\|_{S_N,i} = \sum_{j=1}^{\infty} \frac{1}{2^j} \|T(\xi_j)\|.$$

Fix $\varepsilon > 0$, and let M be such that $2^{-M} < \varepsilon$. Suppose $F \subseteq \Gamma$ is finite,$\delta > 0$ and $m \in \mathbb{N}$ with $m > C_M$. Let $P_M \in B(H)$ be the projection onto H_M. Suppose V is a subspace of $B(H_M, \mathbb{C}^{d_i})$ of minimal dimension such that

$$\mathrm{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i) \subseteq_{\varepsilon, \|\cdot\|_{S,i}} V,$$

let $\widetilde{V} \subseteq B(H, \mathbb{C}^{d_i})$ be the image of V under the map $T \to T \circ P_M$. If $T \in \mathrm{Hom}_{\Gamma, \ell^2(d_i)}(S, F, m, \delta, \sigma_i)$ then $\widetilde{T} = T\big|_{H_M}$ is in $\mathrm{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i)$, and there exists $\phi \in V$ such that $\|\phi - \widetilde{T}\|_{S_M,i} < \varepsilon$. Then

$$\|\phi \circ P - T\|_{S,i} \leq 2 \sum_{n=C_M+1}^{\infty} \frac{1}{2^n} + \|\phi - \widetilde{T}\|_{S_M,i} \leq 2^{-m+1} + \varepsilon \leq 3\varepsilon.$$

Thus

$$\mathrm{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i) \subseteq_{3\varepsilon, \|\cdot\|_{S,i}} \widetilde{V},$$

so

$$d_{3\varepsilon}(\mathrm{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i), \|\cdot\|_{S,i}) \leq d_{\varepsilon}(\mathrm{Hom}_{\Gamma}(S_M, F, m, \delta, \sigma_i), \|\cdot\|_{S_M,i}).$$

Thus

$$\dim_{\Sigma,\ell^2}(S, \Gamma, 3\varepsilon, \|\cdot\|_{S,i,2}) \leq \dim_{\Sigma,\ell^2}(S_M, 3\varepsilon, \|\cdot\|_{S,i,2}) \leq \sup_M \dim_{\Sigma,\ell^2}(\pi_M)$$

and similarly for $\underline{\dim}$. Taking the supremum over $\varepsilon > 0$ completes the proof. $\square$

Corollary 3.3.10. *Let Γ be a $\mathcal{R}^{\omega}$-embeddable group, and let $\Sigma = (\sigma_i \colon \Gamma \to U(d_i))$ be an embedding sequence. Let $\pi_k \colon \Gamma \to U(H_k)$ be a representations of Γ such that each π_k has a finite dynamically generating sequence. Then*

$$\dim_{\Sigma,\ell^2}\left(\bigoplus_{k=1}^{\infty} \pi_k\right) \leq \sum_{k=1}^{\infty} \dim_{\Sigma,\ell^2}(\pi_k)$$

$$\underline{\dim}_{\Sigma,\ell^2}\left(\bigoplus_{k=1}^{\infty} \pi_k\right) \geq \sum_{k=1}^{\infty} \underline{\dim}_{\Sigma,\ell^2}(\pi_k).$$

Proof. The corollary is a simple consequence of the above proposition and Theorem 3.3.7. $\square$

3.4 Computation of $\dim_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma)$, and $\dim_{\Sigma,S^p,conj}(\ell^p(\Gamma,V),\Gamma)$.

In this section we show that if Σ is a sofic approximation of Γ and $1\leq p\leq 2$, then

$$\dim_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma)=\dim V,$$

for V finite dimensional. Similarly if Σ is a embedding sequence of Γ and $1\leq p\leq 2$, we show that

$$\dim_{\Sigma,S^p,conj}(\ell^p(\Gamma,V),\Gamma)=\dim V,$$

$$\dim_{\Sigma,\ell^2}(\ell^2(\Gamma,\ell^2(n)),\Gamma)=n,$$

again for V finite dimensional.

The proof for sofic groups will be relatively simple, but the proof for $\mathcal{R}^\omega$-embeddable groups requires a few more lemmas.

Let ν be the unique $U(n)$ invariant Borel probability measure on S^{2n-1}, for the next lemma we need that if $T\colon\mathbb{C}^n\to\mathbb{C}^n$ is linear, then

$$\frac{1}{n}\operatorname{Tr}(T)=\int_{S^{2n-1}}\langle T\xi,\xi\rangle\,d\nu(\xi).$$

This follows from the fact that Tr is, up to scaling, the unique linear functional on $M_n(\mathbb{C})$ invariant under conjugation by $U(n)$.

Additionally, we will use the following concentration of measure fact (see [?] Page 295), if f is a Lipschitz function on S^{n-1}, then

$$\mathbb{P}(|f-\mathbb{E}f|>t)\leq 4e^{\frac{-nt^2}{\|f\|_{\mathrm{Lip}}^2 72\pi^2}}.$$

Lemma 3.4.1. *Let Γ be a $\mathcal{R}^\omega$-embeddable group, let $\sigma_i\colon\Gamma\to U(d_i)$ be an embedding sequence, and fix $E\subseteq\Gamma$ finite,$m\in\mathbb{N}$. For $j\in\{1,\dots,m\},\xi,\eta\in S^{2d_i-1}$ define*

$$T_{\xi,j}\colon\ell^2(\Gamma\times\{1,\dots,m\})\to\ell^2(d_i),$$

$$T_{\xi,\eta,j}\colon\ell^p(\Gamma\times\{1,\dots,m\})\to S^p(d_i)$$

by

$$T_{\xi,j}(f) = \sum_{s\in E} f(s,j)\sigma_i(s)\xi,$$

$$T_{\xi,\eta,j}(f) = \sum_{s\in E} f(s,j)\sigma_i(s)\xi \otimes \overline{\sigma_i(s)\eta}.$$

Then for any $\delta > 0$ *and* $1 \leq p < \infty$,

(a)

$$\lim_{i\to\infty} \mathbb{P}(\{\xi \in S^{2d_i-1} : \|T_{\xi,j} : \ell^2(\Gamma \times \{1,\dots,m\}) \to \ell^2(d_i)\| < 1+\delta\}) = 1,$$

(b)

$$\{(\xi,\eta) \in (S^{2d_i-1})^2 : \|T_{\xi,\eta,j} \colon \ell^p(\Gamma \times \{1,\dots,m\}) \to S^p(d_i)\| < 1+\delta\} \supseteq A_i \times A_i,$$

where $A_i \subseteq S^{2d_i-1}$ *has* $\nu(A_i) \to 1$.

Proof. Let $\kappa > 0$ which will depend upon $\delta > 0, p$ in a manner to be determined later. Let

$$A = \bigcap_{s\neq t, s,t\in E} \{\xi \in S^{2d_i-1} : |\langle \sigma_i(s)\xi, \sigma_i(t)\xi\rangle| < \kappa\},$$

since

$$\int_{S^{2d_i-1}} \langle \sigma_i(s)\xi, \sigma_i(t)\xi\rangle \, d\nu(\xi) = \frac{1}{d_i} \operatorname{Tr}(\sigma_i(t)^{-1}\sigma_i(s)) \to 0$$

for $s \neq t$, the concentration of measure estimate mentioned before the Lemma implies that

$$\nu(A) \to 1.$$

For the proof of $(a), (b)$ we prove that if $\xi, \eta \in A$ then

$$\|T_{\xi,j}\|_{\ell^2\to\ell^2} \leq 1+\delta,$$

$$\|T_{\xi,\eta,j}\|_{\ell^p\to S^p} \leq 1+\delta,$$

if $\kappa > 0$ is sufficiently small.

(a) For $f \in \ell^2(\Gamma \times \{1,\dots,m\}), \xi \in A$ we have

$$\begin{aligned}\|T_{\xi,j}(f)\|_2^2 &= \sum_{s,t\in E} f(s,j)\overline{f(t,j)}\langle\sigma_i(s)\xi,\sigma_i(t)\xi\rangle \\ &\le \|f\chi_E\|_2^2 + \sum_{s\neq t, s,t\in E}\|f\|_2^2\kappa \\ &\le \|f\|_2^2(1+\kappa|E|^2) \\ &\le (1+\delta)\|f\|_2^2\end{aligned}$$

if $\kappa < \frac{\delta}{|E|^2}$.

(b) Fix $\varepsilon > 0$ to be determined later. If κ is sufficiently small, then for any $(\xi,\eta)\in A^2$ we can find $(\xi_s)_{s\in E}(\eta_s)_{s\in E}$ such that $\langle\xi_s,\xi_t\rangle = \delta_{s=t}$, $\langle\eta_s,\eta_t\rangle = \delta_{s=t}$ and

$$\|\xi_s - \sigma_i(s)\xi\| < \varepsilon, \|\eta_s - \sigma_i(s)\eta\| < \varepsilon.$$

Then

$$\left\|T_{\xi,\eta,j}(f) - \sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right\|_p \le \|f\|_p\sum_{s\in E}(\|\xi_s-\sigma_i(s)\xi\| + \|\sigma_i(s)\eta-\eta_s\|) \le 2|E|\varepsilon\|f\|_p.$$

Note that

$$\left|\sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right|^2 = \sum_{s,t\in E}\overline{f(s)}f(t)\langle\xi_t,\xi_s\rangle\eta_s\otimes\overline{\eta_t} =$$
$$\sum_{s\in E}|f(s)|^2\eta_s\otimes\overline{\eta_s}.$$

Thus

$$\left\|\sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right\|_p^p = \|f\chi_E\|_p^p \le \|f\|_p^p.$$

So if $\varepsilon < \frac{\delta}{2|E|}$ the claim follows. □

The following Lemma will allow us to get the lower bound we need and is similar to Lemma 7.8 in [27].

Lemma 3.4.2. *Let H be a Hilbert space, and $\eta_1,\ldots,\eta_k$ an orthonormal system in H, and $V = \mathrm{Span}\{\eta_j : 1\le j\le k\}$ and P_V the projection onto V. Let K be a Hilbert space and $T\in B(H,K)$ with $\|T\|\le 1$. Then*

$$d_\varepsilon(\{T(\eta_1),\ldots,T(\eta_k)\}) \ge -k\varepsilon + \mathrm{Tr}(P_V T^*TP_V).$$

Proof. For a subspace $E \subseteq H$ we let P_E be the projection onto E. Let W be a subspace of minimal dimension which ε-contains $\{T(\eta_1), \ldots, T(\eta_k)\}$. Then

$$\mathrm{Tr}(P_W TT^*) = \mathrm{Tr}(P_W TT^* P_W) \leq \mathrm{Tr}(P_W),$$

similarly

$$\begin{aligned}\mathrm{Tr}(P_W TT^*) &\geq \mathrm{Tr}(P_V T^* P_W T P_V)\\ &= \sum_{j=1}^{k} \langle P_W T(\eta_j), T(\eta_j)\rangle\\ &\geq -\varepsilon k + \sum_{j=1}^{k} \langle T(\eta_j), T(\eta_j)\rangle\\ &= -\varepsilon k + \mathrm{Tr}(P_V T^* T P_V).\end{aligned}$$

□

For convenience, we shall identify $L(\Gamma)$ as a set of vectors in $\ell^2(\Gamma)$. That is, we shall consider $L(\Gamma)$ to be all $\xi \in \ell^2(\Gamma)$ so that

$$\|\xi\|_{L(\Gamma)} = \sup_{\substack{f \in c_c(\Gamma),\\ \|f\|_2 \leq 1}} \|\xi * f\|_2 < \infty.$$

Here $\xi * f$ is the usual convolution product. By Theorem 2.1.16, if $\xi \in L(\Gamma)$, then for all $f \in \ell^2(\Gamma)$, $\xi * f \in \ell^2(\Gamma)$ and

$$\|\xi * f\|_2 \leq \|\xi\|_{L(\Gamma)} \|f\|_2.$$

By Theorem 2.1.16, $L(\Gamma)$ is closed under convolution and

$$(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$$

for $\xi, \eta, \zeta \in L(\Gamma)$. Finally for $\xi \in L(\Gamma)$, we set

$$\xi^*(x) = \overline{\xi(x^{-1})}.$$

If $\xi \in L(\Gamma), \zeta, \eta \in \ell^2(\Gamma)$, then

$$\langle \xi * \eta, \zeta \rangle = \langle \eta, \xi^* * \zeta \rangle.$$

Finally, for $\xi \in L(\Gamma), f \in c_c(\Gamma)$,

$$\|f * \xi\|_2 = \|\xi^* * f^*\|_2 \leq \|f^*\|_2 \|\xi^*\|_{L(\Gamma)} = \|f\|_2 \|\xi\|_{L(\Gamma)}.$$

Hence every element of $L(\Gamma)$ is bounded as a right convolution operator

Lemma 3.4.3. *Let Γ be a countable sofic group, and $\Sigma = (\sigma_i \colon \Gamma \to S_{d_i})$ a sofic approximation of Γ. Extend σ_i to a embedding sequence by Lemma 2.2.6, still denoted σ_i, of $(L(\Gamma), \tau)$ with τ the group trace. For $r, s \in \mathbb{N}$ define $\sigma_i \colon M_{h,s}(L(\Gamma)) \to M_{h,s}(M_{d_i}(\mathbb{C}))$ by $\sigma_i(A) = [\sigma_i(a_{lr})]_{1 \leq l \leq h, 1 \leq r \leq s}$. Fix $n \in \mathbb{N}$. For $1 \leq j \leq d_i, 1 \leq k \leq n$ and $E \subseteq \Gamma$ finite define $T_{j,k}^{(E)} \colon \ell^p(\Gamma)^{\oplus n} \to \ell^p(d_i)$ by*

$$T_{j,k}^{(E)}(f) = \sum_{g \in E} f_k(g) \sigma_i(g) e_j.$$

Then

(a) For all E and $(1 - o(1))nd_i$ of the j, k we have $\|T_{j,k}^{(E)}\|_{\ell^p \to \ell^p} \leq 1$ as $i \to \infty$.

(b) For $1 \leq p \leq \infty$, for all $\varepsilon > 0$, for all $f \in c_c(\Gamma), g \in \ell^p(\Gamma)^{\oplus n}$, there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\|T_{j,k}^{(E')}(f * g) - \sigma_i(f) T_{j,k}^{(E)}(g)\|_p \leq \varepsilon \|g\|_p,$$

has cardinality at least $(1 - \varepsilon))nd_i$ for all large i.

(c) For all $\varepsilon > 0$, for all $\xi \in M_{1,n}(L(\Gamma))$, (identifying $M_{1,n}(L(\Gamma))$ as a subset of $\ell^2(\Gamma)^{\oplus n}$) there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \otimes e_k)\|_2 < \varepsilon,$$

(here $e_j \otimes e_k \in \ell^2(d_i)^{\oplus n}$ is e_j in the k^{th} coordinate and zero otherwise). has cardinality at least $(1 - \varepsilon)nd_i$ for all large i.

Proof. (a) We have

$$\left\| T_{j,k}^{(E)}(f) \right\|_p^p = \sum_{r=1}^{d_i} \left| \sum_{\substack{g \in E, \\ \sigma_i(g)(j)=r}} f_k(g) \right|^p .$$

Let $C_i = \{j \in \{1, \dots, d_i\} : \sigma_i(g)(j) \neq \sigma_i(h)(j)$ for $g \neq h$ in $E\}$. By soficity, we have $\frac{|C_i|}{d_i} \to 1$, and if $j \in C_i$ we have

$$\left\| T_{j,k}^{(E)}(f) \right\|_p^p \leq \|f_k\|_p^p \leq \|f\|_p^p.$$

(b) For $A \in M_{d_i}(\mathbb{C})$,

$$\|A\|_2^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \|Ae_j\|_2^2,$$

where e_j is the vector which has j^{th} coordinate equal to 1, and all other coordinates zero. Hence by Chebyshev's inequality, the fact that $\|T_{j,k}^{(E)}\|_p \leq 1$, and the definition of embedding sequences, it is enough to verify this for $f = \delta_x, g = \delta_y$ for some $x, y \in \Gamma$. But this is trivial from the definition of soficity.

(c) Let us first verify this when $\xi \in M_{1,n}(c_c(\Gamma))$. In this case, we may again reduce to $\xi = (\delta_{a_1}, \dots, \delta_{a_k})$ for some $a_1, \dots, a_k \in \Gamma$. Then if $E \supseteq \{a_1, \dots, a_k\}$ we have

$$T_{j,k}^{(E)}(\xi) = \sigma_i(a_k)e_j = \sigma_i(\xi)(e_j \otimes e_k).$$

In the general case let $\varepsilon > 0$, given $\xi \in M_{1,n}(L(\Gamma))$ choose $f \in M_{1,n}(c_c(\Gamma))$ so that $\|f - \xi\|_2 < \varepsilon$. Thus for $(1 - (\varepsilon + o(1)))kd_i$ of the (j,k) we have

$$\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \otimes e_k)\|_2 \leq 2\varepsilon + \|(\sigma_i(\xi) - \sigma_i(f))(e_j \otimes e_k)\|.$$

By the definition of embedding sequence for all large i we have

$$\frac{1}{d_i} \sum_{j=1}^{d_i} \sum_{k=1}^{n} \|(\sigma_i(\xi) - \sigma_i(f))(e_j \otimes e_k)\|_2^2 < \varepsilon^2,$$

thus for at least $(1 - \sqrt{\varepsilon})nd_i$ of the (j,k) we have

$$\|(\sigma_i(\xi) - \sigma_i(f))(e_j \otimes e_k)\|_2 < \sqrt{\varepsilon},$$

combining these estimates completes the proof.

□

We need a similar lemma for $\mathcal{R}^\omega$-embeddable groups.

Lemma 3.4.4. *Let Γ be a countable $\mathcal{R}^\omega$-embeddable group, and $\Sigma = (\sigma_i \colon \Gamma \to U(d_i))$ an embedding sequence. Define $\rho_i \colon \Gamma \to U(S^2(d_i))$ by $\rho_i(g)A = \sigma_i(g)A\sigma_i(g)^{-1}$. Extend σ_i, ρ_i to embedding sequences by Lemma 2.2.6 , still denoted σ_i, ρ_i of $(L(\Gamma), \tau)$ with τ the group trace. For $h, s \in \mathbb{N}$ define $\sigma_i \colon M_{h,s}(L(\Gamma)) \to M_{h,s}(M_{d_i}(\mathbb{C}))$ by $\sigma_i(A) = [\sigma_i(a_{lr})]_{1 \le l \le h, 1 \le r \le s}$. Fix $n \in \mathbb{N}$. For $\xi, \eta \in \ell^2(d_i), 1 \le k \le d_i$ and $E \subseteq \Gamma$ finite define $T^{(E)}_{\xi,\eta,k} \colon \ell^p(\Gamma)^{\oplus n} \to S^p(d_i)$ by*

$$T^{(E)}_{\xi,\eta,k}(f) = \sum_{g \in E} f_k(g)\sigma_i(g)\xi \otimes \overline{\sigma_i(g)\eta}.$$

Then

(a) There exists measurable $A_i \subseteq S^{2d_i - 1}$ with $\mathbb{P}(A_i) \to 1$, so that

$$\{(\xi, \eta) \in (S^{2d_i-1})^2 : \|T^{(E)}_{\xi,\eta,k}\|_{\ell^p \to S^p} \le 2\} \supseteq A_i \times A_i,$$

for $(1 - o(1))d_i$ of the k.

(b) For all $\varepsilon > 0$, for all $f \in c_c(\Gamma), g \in \ell^p(\Gamma)^{\oplus n}$, there exists measurable $B_i \subseteq S^{2d_i-1}$, with $\mathbb{P}(B_i) \ge 1 - \varepsilon$, for all large i, a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then for $(1 - \varepsilon)d_i$ of the k and for all large i,

$$\{(\xi, \eta) \in (S^{2d_i-1})^2 : \|T^{(E')}_{\xi,\eta,k}(f * g) - \rho_i(f)T^{(E)}_{\xi,\eta,k}(g)\|_p < \varepsilon\} \supseteq B_i \times B_i$$

(c) For all $\varepsilon > 0$, for all $\zeta \in M_{1,n}(L(\Gamma))$, (identifying $M_{1,n}(L(\Gamma))$ as a subset of $\ell^2(\Gamma)^{\oplus n}$) there are measurable $C_i \subseteq S^{2d_i-1}$, with $\mathbb{P}(C_i) \ge 1 - \varepsilon$ for all large i, a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, so that for at least $(1 - \varepsilon)d_i$ of the k and for all large i,

$$\{(\xi, \eta) \in (S^{2d_i-1})^2 : \|T^{(E')}_{\xi,\eta,k}(\zeta) - \rho_i(\zeta)\xi \otimes \overline{\eta})\|_2 < \varepsilon\} \supseteq C_i \times C_i,$$

has cardinality at least $(1 - \varepsilon)nd_i$ for all large i.

Proof. Same as the preceding Lemma, but using Lemma 3.4.1. □

Finally we need one last lemma, which allows us to reduce to considering subspaces of finite direct sums of $l^p(\Gamma)$.

Lemma 3.4.5. *Let Γ be a countable discrete group. Let $H \subseteq \ell^2(\mathbb{N}, \ell^2(\Gamma))$ be a closed Γ-invariant subspace.*

(a) Define $\pi_k \colon \ell^2(\mathbb{N}, \ell^2(\Gamma)) \to \ell^2(\Gamma)^{\oplus k}$ by $\pi_k f(j) = f(j)$ for $1 \le j \le k$. Then

$$\dim_{L(\Gamma)}(H) = \sup_k \dim_{L(\Gamma)}(\overline{\pi_k(H)}^{\|\cdot\|_2}).$$

(b) The representation H is isomorphic to a direct sum of representations of the form $\ell^2(\Gamma)p$ with $p \in L(\Gamma)$ an orthogonal projection.

Proof. (a) Since $\pi_k(H)$ is dense in $\overline{\pi_k(H)}$ we have

$$\dim_{L(\Gamma)}(H) \ge \sup_k \dim_{L(\Gamma)}(\overline{\pi_k(H)}^{\|\cdot\|_2}).$$

Let us first handle the case when $\dim_{L(\Gamma)}(H) < \infty$, let P be the projection onto H. Then

$$\begin{aligned}
\dim_{L(\Gamma)}(\overline{\pi_k(H)}) &= \dim_{L(\Gamma}(\ker(\pi_k P)^{\perp}) \\
&= \dim_{L(\Gamma)}(H \cap \overline{(H^{\perp} + \ell^2(\Gamma)^{\oplus k}})) \\
&= \dim_{L(\Gamma)}(H \cap (H \cap \ell^2(\mathbb{N} \setminus \{1, \dots, k\}, \Gamma))^{\perp})).
\end{aligned}$$

Let Q_k be the projection onto $H \cap \ell^2(\mathbb{N} \setminus \{1, \dots, k\}, \Gamma)$. Then

$$\begin{aligned}
\dim_{L(\Gamma)}(H \cap \ell^2(\mathbb{N} \setminus \{1, \dots, k\}, \Gamma)) &= \sum_{n=1}^{\infty} \langle Q_k(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle \\
&= \sum_{n=k}^{\infty} \langle Q_k(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle \\
&\le \sum_{n=k}^{\infty} \langle P(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle \\
&\to 0,
\end{aligned}$$

as $\dim_{L(\Gamma)}(H) < \infty$.

In the general case, it suffices to show that we may write H as a direct sum of representations with finite von Neumann dimension. Zorn's Lemma implies that every representation is a direct sum of cyclic representations which are contained in $\ell^2(\mathbb{N}, \ell^2(\Gamma))$, so it suffices to show every cyclic representation contained in $\ell^2(\mathbb{N}, \ell^2(\Gamma))$ has finite von Neumann dimension.

For this, let $\xi \in H$ be a cyclic vector, then by Theorem A.3.1, there is a $y \in L^1(L(\Gamma), \tau)$ so that

$$\langle \pi(g)\xi, \xi \rangle = \tau(xy).$$

It is easy to see that $y \geq 0$. Setting $\zeta = |y|^{1/2}$, we see that

$$\langle \pi(g)\xi, \xi \rangle = \langle g\zeta, \zeta \rangle$$

for all $g \in \Gamma$. Thus H is isomorphic to $\overline{\mathrm{Span}}^{\|\cdot\|_2}(\Gamma\xi)$ via the unitary sending $g\xi \to g\zeta$. From this it clear that H has dimension at most 1.

(b) As in part (a), we may assume that H is a cyclic representation contained in $\ell^2(\Gamma)$. We have already seen directly before Proposition 2.1.14 that

$$H = L^2(M, \tau)p.$$

□

Theorem 3.4.6. *Let Γ be a countable discrete group, let $1 \leq p \leq 2$, and Y a closed Γ-invariant subspace of $\ell^p(\mathbb{N}, \ell^p(\Gamma))$, with Γ acting by $gf(x) = f(g^{-1}x)$. Set $H = \overline{Y}^{\|\cdot\|_2}$.*

(a) Suppose Σ is a sofic approximation of Γ, then

$$\underline{\dim}_{\Sigma,\ell^p}(Y, \Gamma) \geq \dim_{L(\Gamma)}(H).$$

(b)Suppose Σ is an embedding sequence of Γ, then

$$\underline{\dim}_{\Sigma,S^p,conj}(Y, \Gamma) \geq \dim_{L(\Gamma)}(H).$$

(c) Suppose Σ is an embedding sequence of Γ, and $H \subseteq \ell^2(\mathbb{N}, \ell^2(\Gamma))$ is Γ invariant, then

$$\underline{\dim}_{\Sigma,\ell^2}(H, \Gamma) \geq \dim_{L(\Gamma)}(H).$$

Proof. We first reduce to the case that $Y \subseteq \ell^p(\Gamma)^{\oplus h}$ with h finite.

Consider the projection

$$\pi_h \colon \ell^p(\mathbb{N}, \Gamma) \to \ell^p(\{1, \ldots, h\}, \ell^p(\Gamma))$$

given by

$$\pi_h f(j) = f(j),$$

assume we know the result for $Y \subseteq \ell^p(\Gamma)^{\oplus h}$ for each h.

Then,

$$\begin{aligned}\dim_{\Sigma,\ell^p}(Y, \Gamma) &\geq \dim_{\Sigma,\ell^p}(\overline{\pi_h(Y)}^{\|\|\cdot\|_p}, \Gamma) \\ &\geq \dim_{L(\Gamma)}(\overline{\pi_h(H)}^{\|\cdot\|_2}),\end{aligned}$$

letting $h \to \infty$ and applying the preceding Lemma proves the claim. Thus, we shall assume that $Y \subseteq \ell^p(\Gamma)^{\oplus n}$ with $n \in \mathbb{N}$.

By part (b) of the preceding Lemma, we can find vectors $(\xi^{(q)})_{q=1}^{\infty} \in H$, so that

$$\langle \lambda(g)\xi^{(s)}, \xi^{(s)} \rangle = \langle \lambda(g)q_s, q_s \rangle = q_s(g^{-1}), \text{ where } q_s \text{ is a projection in } L(\Gamma),$$

$$\sum_{s=1}^{\infty} \tau(q_s) = \dim_{L(\Gamma)}(H),$$

$$\langle \lambda(g)\xi^{(j)}, \xi^{(l)} \rangle = 0 \text{ for } j \neq l, g \in \Gamma.$$

$$H = \bigoplus_{j=1}^{\infty} \overline{L(\Gamma)\xi^{(j)}}.$$

These equations can be rewritten as

$$\sum_{i=1}^{n} \xi^{(j)} * (\xi^{(j)})^* = q_j, \text{ for } 1 \leq j \leq \infty$$

$$\sum_{i=1}^{n} \xi^{(j)} * (\xi^{(l)})^* = 0 \text{ if } j \neq l,$$

Let us illuminate these equations a little. Regard a vector $\xi \in \ell^2(\Gamma)^{\oplus n}$ as a element in $M_{1,n}(\ell^2(\Gamma))$ with the product of two matrices induced from convolution of vectors. Then the product of elements of $M_{1,n}(\ell^2(\Gamma)), M_{n,1}(L(\Gamma))$ makes sense, but may not land back in $\ell^2(\Gamma)$. The above equations then read

$$\xi^{(j)}(\xi^{(j)})^* = q_j \text{for } 1 \leq j < \infty,$$

$$\xi^{(j)}(\xi^{(l)}) = 0 \text{ for } j \neq l..$$

In particular, the above equations imply that

$$\|\xi_r^{(j)}\|_{L(\Gamma)} \leq 1.$$

So that $\xi^{(j)} \in M_{1,n}(L(\Gamma))$. Extend σ_i to a embedding sequence of $M_{n,m}(L(\Gamma))$ for all n, m and such that

$$\|\sigma_i(\xi^{(j)})\| \leq 1, \text{ for all } j$$

$$\|\sigma_i(\xi_r^{(j)})\| \leq 1, \text{ for all } j, r$$

$$\sigma_i(\xi^{(j)})\sigma_i(\xi^{(l)})^* = 0 \text{ for all } j \neq l.$$

for all j, r.

(a)

Let $S = (x_j)_{j=1}^n$ be a dynamical generating sequence for Y.

Fix $\eta > 0, t \in \mathbb{N}$ and choose a finite subset $F_1 \subseteq \Gamma, m_1 \in \mathbb{N}$, and $c_{gj}^{(s)}$ for $1 \leq s \leq t, (g, j) \in F_1 \times \{1, \ldots, m_1\}$ so that for all $1 \leq s \leq t$

$$\left\| \xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j \right\|_2 < \eta.$$

Choose finitely supported functions x_j' so that $\|x_j - x_j'\|_p < \eta'$. Since $p \leq 2$, it is easy to see that if we force η' to be sufficiently small then,

$$\left\| \xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j' \right\|_2 < \eta.$$

Let $S=(x_j)_{j=1}^{\infty}$ be a dynamically generating sequence for Y. Fix $F \subseteq \Gamma$ finite $m \in \mathbb{N}, \delta > 0$. Let $E \subseteq \Gamma$ be finite, let $T_{j,k}^{(E)}$ be defined as Lemma 3.4.3.

It is easy to see that if E is sufficiently large, then $T_{j,k}^{(E)}\big|_{Y_{F,m}} \in \operatorname{Hom}_{\Gamma}(S,F,m,\delta,\sigma_i)_2$ for $(1-o(1))nd_i$ of the j,k, and in fact $\|T_{j,k}^{(E)}\|_{\ell^p \to \ell^p} \leq 2$ for $1 \leq p \leq 2$. For such (j,k), and for all small δ, for $1 \leq s \leq t+1$

$$\left\| T_{j,k}^{(E)}(\xi^{(s)}) - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j) \right\|_2 < 2\eta,$$

$$\|T_{j,k}^{(E)}(gx_j') - T_{j,k}^{(E)}(gx_j)\|_2 < \eta.$$

Thus by Lemma 3.4.3 for at least $(1-(2014)!\varepsilon)nd_i$ of the j,k we have

$$\left\| \sigma_i(\xi^{(s)})(e_j \otimes e_k) - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j) \right\|_2 < \varepsilon + \eta.$$

Now consider the linear map $A\colon \ell^{\infty}(\mathbb{N}, \ell^p(d_i)) \to \ell^2(d_i)^{\oplus t}$ given by

$$S(f) = \left(\sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) f(j) \right)_{p=1}^{t},$$

from the above it is easy to see that if $\alpha_S(\operatorname{Hom}_{\Gamma}(S,F,m,\delta,\sigma_i)) \subseteq_{\varepsilon'} V$ and ε' is sufficiently small,

$$A(V) \supseteq_{\varepsilon, \|\cdot\|_2} \{\phi_i(e_j \otimes e_k) : (j,k) \in A_i\},$$

with

$$\frac{|A_i|}{d_i} \to (1-(2014)!\varepsilon)nd_i,$$

$$\phi_i(f) = (\sigma_i(\xi^{(1)})(f), \sigma_i(\xi^{(2)})(f), \ldots, \sigma_i(\xi^{(t)})(f)).$$

Thus ϕ_i is given in matrix form by

$$\phi_i = \begin{bmatrix} \sigma_i(\xi^{(1)}) & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)}) & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)}) \end{bmatrix}.$$

As

$$\phi_i\phi_i^* = \begin{bmatrix} \sigma_i(\xi^{(1)})\sigma_i(\xi^{(1)})^* & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)})\sigma_i(\xi^{(2)})^* & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)})\sigma_i(\xi^{(t)})^* \end{bmatrix}$$

By our choice of σ_i we have

$$\|\phi_i\| \leq 1,$$

By Lemma 3.4.2, we find that

$$\dim_{\Sigma,\ell^p}(V,\Gamma) \geq (1-(2014)!\varepsilon)n + \dim_{L(\Gamma)} H_t.$$

Letting $\varepsilon \to 0, t \to \infty$ completes the proof.

(b), (c) Same proof as in (a), one instead uses Lemma 3.4.4, Lemma 3.4.1, and the formula

$$\mathbb{P}(A) = \int_{U(d_i)} \frac{|\{j : Ue_j \in A\}| d_i}{} dU,$$

for $A \subseteq S^{2d_i-1}$, to find an orthonormal system $\zeta_1, \ldots, \zeta_q$ with $q \geq (1-\varepsilon)d_i$, so that $T^{(E)}_{\zeta_j,\zeta_p,k} \in \operatorname{Hom}_\Gamma(\cdots)$ for most k and all j, p. □

Corollary 3.4.7. *Let $1 \leq p \leq 2$, V a finite-dimensional normed vector space, and Γ a countable discrete group.*

(a) If Γ is sofic and Σ is a sofic approximation of Γ, then

$$\underline{\dim}_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma) = \dim_{\Sigma,\ell^p}(\ell^p(\Gamma,V),\Gamma) = \dim V.$$

(b) If Γ is $\mathcal{R}^\omega$-embeddable and Σ is an embedding sequence of Γ, then

$$\underline{\dim}_{\Sigma,\ell^2}(\ell^2(\Gamma,\ell^2(n)),\Gamma) = \dim_{\Sigma,\ell^2}(\ell^2(\Gamma,\ell^2(n)),\Gamma) = n.$$

$$\underline{\dim}_{\Sigma,S^p,conj}(\ell^p(\Gamma,V),\Gamma) = \dim_{\Sigma,S^p,conj}(\ell^p(\Gamma,V),\Gamma) = \dim V.$$

Proof. The lower bounds are automatic from the preceding Theorem. The upper bounds are easy since $\ell^p(\Gamma,V)$ can be generated by $\dim V$ elements.

□

Corollary 3.4.8. *Let Γ be a $\mathcal{R}^\omega$-embeddable group $1 \le p \le 2$. If V,W are finite dimensional vector spaces with $\dim V < \dim W$, then there are no Γ-equivariant bounded linear maps from $\ell^p(\Gamma,V)$ to $\ell^p(\Gamma,W)$ with dense image. Consequently if $2 \le p < \infty$, then there are no Γ-equivariant bounded linear injections from $\ell^p(\Gamma,W)$ to $\ell^p(\Gamma,V)$.*

Proof. For $1 \le p \le 2$ this is immediate from the above corollary and Proposition 3.3.1. The other result follow by duality. □

Theorem 3.4.9. *Let Γ be a $\mathcal{R}^\omega$-embeddable group, and $\pi\colon \Gamma \to U(H)$ a representation, such that $\pi \le \lambda^{\oplus\infty}$. Then for every embedding sequence Σ,*

$$\dim_{\Sigma,\ell^2}(\pi) = \underline{\dim}_{\Sigma,\ell^2}(\pi) = \dim_{L(\Gamma)}(\pi).$$

Proof. Let $\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be given by $\lambda(g)f(x) = f(g^{-1}x)$. We already know from Theorem 3.4.7 that

$$\dim_{\Sigma,\ell^2}\lambda^{\oplus n} = \underline{\dim}_{\Sigma,\ell^2}\lambda^{\oplus n} = n.$$

Let us first assume that π is cyclic with cyclic vector ξ, then as in Lemma 3.4.5 we may find a $\zeta \in \ell^2(\Gamma)$ so that

$$\langle \pi(x)\xi,\xi\rangle = \langle \lambda(x)\zeta,\zeta\rangle,$$

so $\pi \leq \lambda$. Let π' be a representation such that $\lambda = \pi \oplus \pi'$, then by Theorem 3.4.6 we have

$$\begin{aligned} 1 = \dim_{\Sigma,\ell^2} \lambda &\geq \dim_{\Sigma,\ell^2} \pi + \underline{\dim}_{\Sigma,\ell^2} \pi' \\ &\geq \underline{\dim}_{\Sigma,\ell^2} \pi + \underline{\dim}_{\Sigma,\ell^2} \pi' \\ &\geq \dim_{L(\Gamma)} \pi + \dim_{L(\Gamma)} \pi' \\ &= 1. \end{aligned}$$

Thus all the above inequalities must be equalities, in particular

$$\dim_{\Sigma,\ell^2} \pi = \underline{\dim}_{\Sigma,\ell^2} \pi = \dim_{L(\Gamma)} \pi.$$

In the general case, apply Zorn's Lemma to write $\pi = \bigoplus_{n=1}^{\infty} \pi_n$ with π_n cyclic. Then by Corollary 3.3.10

$$\underline{\dim}_{\Sigma,\ell^2}(\pi) \geq \sum_{n=1}^{\infty} \underline{\dim}_{\Sigma,\ell^2}(\pi_n) = \sum_{n=1}^{\infty} \dim_{L(\Gamma)} \pi_n = \dim_{L(\Gamma)} \pi,$$

$$\dim_{\Sigma,\ell^2}(\pi) \leq \sum_{n=1}^{\infty} \dim_{\Sigma,\ell^2}(\pi_n) = \sum_{n=1}^{\infty} \dim_{L(\Gamma)} \pi_n = \dim_{L(\Gamma)} \pi.$$

This completes the proof of the theorem.

$\square$

3.5 Triviality In The Case of Finite-Dimensional Representations

In this section we prove the following.

Theorem 3.5.1. *Let Γ be a infinite sofic group, and Σ a sofic approximation of Γ. Then for every $1 \leq p \leq \infty$, and for any uniformly bounded representation of Γ on a finite-dimensional Banach space X,*

$$\dim_{\Sigma,\ell^p}(X, \Gamma) = 0.$$

Here is the outline of the proof. We will begin by studying ℓ^p-dimension for amenable groups, using the standard technique of averaging over Følner sequences. Using this averaging technique we show that for finite Γ,

$$\dim_{\Sigma,\ell^p}(X,\Gamma)=\frac{\dim_{\mathbb{C}} X}{|\Gamma|}.$$

This easily implies proves the theorem when Γ has finite subgroups of unbounded size. We then show that

$$\dim_{\Sigma,\ell^p}(X,\mathbb{Z})=0,$$

if X is finite-dimensional. Since dimension decreases when we restrict to the action of a subgroup, we may assume that Γ has no elements of infinite order, but that there is a uniform bound on the size of a finite subgroup of Γ. A compactness argument will show that Γ has an infinite subgroup which acts on X trivially, so we only have to show that

$$\dim_{\Sigma,\ell^p}(\mathbb{C},\Gamma)=0,$$

where Γ acts trivially on $\mathbb{C}$. To prove this last statement, we will pass to a sofic equivalence relation induced by the group, and use that the full group of such an equivalence relation contains $\mathbb{Z}/n\mathbb{Z}$ for every integer n.

We first show that in the case of an action of an amenable group, we may assume that the maps we use to compute dimension are only approximately equivariant after cutting down by certain subsets. We formalize this as follows.

Definition 3.5.2. Let Γ be a sofic group with a uniformly bounded action on a Banach space X. Let $\sigma_i\colon\Gamma\to S_{d_i}$ be a sofic approximation. Fix $S=(a_j)_{j=1}^{\infty}$ a bounded sequence in X. Let $A_i\subseteq\{1,\ldots,d_i\}$. For $F\subseteq\Gamma$ finite, $m\in\mathbb{N},\delta>0$, we let $\operatorname{Hom}_{\Gamma,\ell^p,(A_i)}(S,F,m,\delta,\sigma_i)$ be the set of all linear maps $T\colon X_{F,m}\to\ell^p(d_i)$ such that $\|T\|\leq 1$, and $1\leq j,k\leq m$, and $s_1,\ldots,s_k\in F$ we have

$$\|T(s_1\cdots s_k a_j)-\sigma_i(s_1)\cdots\sigma_i(s_k)T(a_j)\|_{\ell^p(A_i)}<\delta.$$

Set

$$\dim_{\Sigma,\ell^p}(S,\Gamma,(A_i),\rho)=\sup_{\varepsilon>0}\inf_{\substack{F\subseteq\Gamma \text{ finite}\\ m\in\mathbb{N}\\ \delta>0}}\limsup_{i\to\infty}\frac{1}{d_i}d_\varepsilon(\alpha_S(\operatorname{Hom}_{\Gamma,\ell^p,(A_i)}(S,F,m,\delta,\sigma_i)),\rho_{\ell^p(d_i)}),$$

where ρ is any product norm.

Proposition 3.5.3. *Fix a product norm ρ on $\ell^{\infty}(\mathbb{N})$. Let Γ be a countable amenable group, and $\Sigma = (\sigma_i \colon \Gamma \to S_{d_i})$ a sofic approximation. Let $A_i \subseteq \{1, \dots, d_i\}$ be such that*

$$\frac{|A_i|}{d_i} \to 1.$$

Then for any uniformly bounded action of Γ on a separable Banach space X, for every generating sequence S in X, for every product norm ρ, and $1 \leq p < \infty$ we have

$$\dim_{\Sigma,\ell^p}(X, \Gamma) = \dim_{\Sigma,\ell^p}(S, \Gamma, (A_i), \rho)$$

Proof. Fix $S = (x_j)_{j=1}^{\infty}$ a dynamically generating sequence for X. As

$$\mathrm{Hom}_{\Gamma,\ell^p}(S, F, m, \delta, \sigma_i) \subseteq \mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, F, m, \delta, \sigma_i)$$

for $m, i \in \mathbb{N}, \delta > 0$ and $F \subseteq \Gamma$ finite, we have

$$\dim_{\Sigma,\ell^p}(X, \Gamma) \leq \dim_{\Sigma,\ell^p}(S, \Gamma, (A_i), \rho).$$

For the reverse inequality, first fix some notation. For E, F finite subsets of Γ containing the identity and $m \in \mathbb{N}$ define

$$P_i^{(E)} \colon B(X_{EF,m}, \ell^p(d_i)) \to B(X_{F,m}, \ell^p(d_i))$$

by

$$P_i^{(E)}(T) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) \circ T \circ s^{-1}.$$

Then $\|P_i^{(E)}\| \leq 1$. Note that for $s_1, \dots, s_k \in F$ and $T \in B(X_{EF,k}, \ell^p(d_i))$ that

$$P_i^{(E)}(T)(s_1 \cdots s_k x) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) T(s^{-1} s_1 \cdots s_k x) =$$

$$\frac{1}{|E|} \sum_{s \in s_k^{-1} \cdots s_1^{-1} E} \sigma_i(s_1 \cdots s_k s) T(s^{-1} x).$$

If $B_i \subseteq \{1, \dots, d_i\}$ is the set of all $1 \leq j \leq d_i$ such that

$$\sigma_i(s_1 \cdots s_k s)^{-1}(j) = \sigma_i(s)^{-1} \sigma_i(s_1 \cdots s_k)^{-1}(j),$$

for all $s \in E, s_1, \ldots, s_k \in F, 1 \leq k \leq m$. Then the above shows that if $T \in B(X_{FE,m}, \ell^p(B_i))$ then

$$\|\sigma_i(s_1 \cdots s_k) \circ P_i^{(E)}(T)(x_j) - P_i^{(E)}(T)(s_1 \cdots s_k x_j)\| \leq 2\frac{|E\Delta s_k^{-1} \cdots s_1^{-1}E|}{|E|}\|T\|\|x_j\|, \quad (3.4)$$

for $1 \leq j \leq m$.

Let $\varepsilon > 0$, and $M = \sup_j \|x_j\| < \infty$. Since ρ is a product norm, we may choose $N \in \mathbb{N}$, and $\kappa > 0$ so that if $f, g \in \ell^\infty(\mathbb{N}, \ell^p(d_i))$ and $\|f\|, \|g\| \leq M$ and

$$\max_{1 \leq j \leq N} \|f(j) - g(j)\|_p < \kappa$$

then

$$\rho(f - g) < \varepsilon.$$

Let $\delta > 0$ depend upon κ to be determined later. Let $m \geq \max(2, N)$ be an integer, and let $e \in F$ be a symmetric finite subset of Γ. Let $E \subseteq \Gamma$ be finite, the set E will depend upon F, m, δ in a manner to be determined later. Let $T \in \mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, EF, m, \delta, \sigma_i)$ then,

$$P_i^{(E)}(\chi_{B_i} T) = \frac{1}{|E|}\sum_{s \in E} \sigma_i(s)\chi_{B_i} T \circ s^{-1} =$$

$$\frac{1}{|E|}\sum_{s \in E} \chi_{\sigma_i(s)B_i}\sigma_i(s) T \circ s^{-1}.$$

Set $C_i = A_i \cap B_i \cap \bigcap_{s \in E} \sigma_i(s)(A_i \cap B_i)$, then $\frac{|C_i|}{d_i} \to 1$, and for $1 \leq j \leq m$

$$\|P_i^{(E)}(\chi_{B_i} T)(x_j) - T(x_j)\|_{\ell^p(C_i)} \leq \frac{1}{|E|}\sum_{s \in E}\|\sigma_i(s)T(s^{-1}x_j) - T(x_j)\|_{\ell^p(A_i)} < 2\delta. \quad (3.5)$$

By amenability of Γ, we may choose E so that

$$\max_{\substack{1 \leq k \leq m, \\ s_1, \ldots, s_1 \in F}} 2\frac{|E\Delta s_k^{-1} \cdots s_1^{-1}E|}{|E|}\|x_j\| < \delta.$$

Then by (3.4), we know $P_i^{(E)}(\chi_{B_i}(T)) \in \mathrm{Hom}_{\Gamma,\ell^p}(S, F, m, \delta, \sigma_i)$. By (3.5),

$$\max_{1 \leq j \leq m} \|\chi_{C_i}(P_i^{(E)}(\chi_{B_i} T)(x_j) - T(x_j))\|_p < \delta. \quad (3.6)$$

For $A \subseteq \{1, \dots, n\}$, we use $1 \otimes \chi_A$ for the operator on $\ell^\infty(\mathbb{N}, \ell^p(n)$ given by

$$[(1 \otimes \chi_A)f](j) = \chi_A f(j), f \in \ell^\infty(\mathbb{N}, \ell^p(n)), j \in \mathbb{N}.$$

If we now force $\delta < \kappa$, then by our choice of κ, m, N and (3.6),

$$\alpha_S(\mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, EF, m, \delta, \sigma_i)) \subseteq_{2\varepsilon,\rho_{\ell^p(d_i)}} (1 \otimes \chi_{C_i})\alpha_S(\mathrm{Hom}_{\Gamma,\ell^p}(S, F, m, \delta, \sigma_i) \\ + \{f \in \ell^\infty(\mathbb{N}, \ell^p(A_i^c) : f(j) = 0, \text{ if } j > N\}.$$

Thus,

$$d_{3\varepsilon}(\alpha_S(\mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, EF, m, \delta, \sigma_i), \rho_{\ell^p}(d_i)) \leq N|A_i^c| + d_\varepsilon(\alpha_S(\mathrm{Hom}_{\Gamma,\ell^p}(S, F, m, \delta, \sigma_i)).$$

As

$$\frac{|A_i^c|}{d_i} \to 0,$$

dividing by d_i, taking the limit supremum over i, then the limit supremum over (F, m, δ) and letting $\varepsilon \to 0$ proves that

$$\dim_{\Sigma,\ell^p}(S, \Gamma, (A_i), \rho) \leq \dim_{\Sigma,\ell^p}(X, \Gamma).$$

□

Corollary 3.5.4. *Let* Γ *be an amenable group with a uniformly bounded action on a separable Banach space* X*. Let* $\Sigma = (\sigma_i \colon \Gamma \to S_{d_i})$*,* $\Sigma' = (\sigma_i' \colon \Gamma \to S_{d_i})$ *be two sofic approximations. Then for all* $1 \leq p \leq \infty$*,*

$$\dim_{\Sigma,\ell^p}(X, \Gamma) = \dim_{\Sigma',\ell^p}(X, \Gamma).$$

Proof. An ultrafilter argument using Theorem 1 of [8] shows that we can find $\tau_i \colon S_{d_i} \to S_{d_i}$ such that

$$d_{\mathrm{Hamm}}(\tau_i \sigma_i(s) \tau_i^{-1}, \sigma_i(s)') \to 0.$$

Replacing σ_i by $\tau_i \circ \sigma_i \circ \tau_i^{-1}$, we may assume that

$$d_{\mathrm{Hamm}}(\sigma_i(s), \sigma_i'(s)) \to 0$$

for all $s \in \Gamma$. In this case, we can find $A_i \subseteq \{1, \ldots, d_i\}$ such that

$$\frac{|A_i|}{d_i} \to 1$$

and for all $s_1, \ldots, s_n \in \Gamma$, we have

$$\sigma_i(s_1 \cdots s_n)(j) = \sigma_i(s_1) \cdots \sigma_i(s_n)(j) = \sigma_i'(s_1) \cdots \sigma_i'(s_n)(j) = \sigma_i'(s_1 \cdots s_n)(j)$$

for all $j \in A_i$ and all sufficiently large i. Thus if $F \subseteq \Gamma$ is finite, $m \in \mathbb{N}$, $\delta > 0$ then for all large i,

$$\mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, F, m, \delta, \sigma_i) = \mathrm{Hom}_{\Gamma,\ell^p,(A_i)}(S, F, m, \delta, \sigma_i').$$

The corollary now follows from the preceding proposition.

$\square$

Proposition 3.5.5. *Let Γ be a finite group acting on a finite-dimensional vector space X. For $n \in \mathbb{N}$, let*

$$n = q_n|\Gamma| + r_n$$

where $0 \leq r_n < |\Gamma|$ and $q_n, r_n \in \mathbb{N}$. Let A_n be a set of size r_n and define a sofic approximation $\Sigma = (\sigma_n \colon \Gamma \to \mathrm{Sym}(\Gamma \times (\{1, \ldots, q_n\} \sqcup A_n))$ by

$$\sigma_n(s)(g, j) = (sg, j) \text{ for } s \in \Gamma, 1 \leq j \leq q_n$$

$$\sigma_n(s)(a) = a \text{ for } a \in A_n.$$

Then for any $1 \leq p \leq \infty$

$$\dim_{\Sigma,\ell^p}(X, \Gamma) = \underline{\dim}_{\Sigma,\ell^p}(X, \Gamma) = \frac{\dim_{\mathbb{C}} X}{|\Gamma|}.$$

Proof. Fix a norm on X. By finite dimensionality we may use the operator norm on $B(X, \ell^p(d_i))$ as our pseudonorm, and we replace $\mathrm{Hom}_\Gamma(S, \Gamma, m, \delta, \sigma_i)$ by the space $\mathrm{Hom}_\Gamma'(\Gamma, m, \delta, \sigma_i)$ of all operators $T \colon X \to \ell^p(d_i)$ such that

$$\|T \circ s_1 \cdots s_k - \sigma_i(s_1) \cdots \sigma_i(s_k) \circ T\| < \delta$$

for all $1 \leq k \leq m, s_1, \ldots, s_k \in \Gamma$.

For $1 \leq q \leq \infty$ define an action on $\ell^q(\Gamma \times \{1, \ldots, q_n\})$ by

$$(gf)(h, j) = f(g^{-1}h, j), h \in \Gamma, 1 \leq j \leq q_n.$$

Let $V_n \subseteq B(X, \ell^p(n))$ be the linear subspace of all linear operators

$$T \colon X \to \ell^p(\Gamma \times \{1, \ldots, q_n\})$$

which are equivariant with respect to the Γ-action. Note that we have norm one projections

$$B(X, \ell^p(n)) \to B(X, \ell^p(\Gamma \times \{1, \ldots, q_n\})$$

$$B(X, \ell^p(\Gamma \times \{1, \ldots, q_n\}) \to V_n,$$

given by multiplication by $\chi_{\{1,\ldots,q_n\}}$ and by

$$T \to \frac{1}{|\Gamma|} \sum_{s \in \Gamma} \sigma_n(s)^{-1} \circ T \circ s.$$

Let P_n denote the composition of these two projections. Since we have a norm one projection form $B(X, \ell^p(n)) \to V_n$, the Riesz Lemma implies that

$$d_\varepsilon(\{T \in V_n : \|T\| \leq 1\}, \|\cdot\|) \geq \dim_{\mathbb{C}} V_n. \tag{3.7}$$

with the norm being the operator norm. Define an action of Γ on X^* by $(g\phi)(x) = \phi(g^{-1}x)$. Let W_n be the set of all Γ-equivariant operators in $B(\ell^p(\Gamma \times \{1, \ldots, q_n\}, X^*)$, then $T \mapsto T^t$ (here T^t is the Banach space adjoint of T,) defines an isomorphism $V_n \cong W_n$. For $f \in \ell^p(\Gamma), k \in \ell^p(\{1, \ldots, q_n\})$ let $f \otimes k$ be defined by $(f \otimes k)(g, j) = f(g)k(j)$. We leave it as an exercise to the reader to verify that the map

$$\Phi \colon W_n \to B(\ell^p(\{1, \ldots, q_n), X^*)$$

given by

$$\Phi(T)(f) = T(\chi_{\{e\}} \otimes f),$$

is an isomorphism.Thus,

$$\dim_{\mathbb{C}}(V_n) = \dim_{\mathbb{C}}(W_n) = q_n \dim_{\mathbb{C}}(X).$$

For $T \in \operatorname{Hom}'_{\Gamma}(\Gamma, m, \delta, \sigma_i)$ we have

$$\|P_n(T) - T\|_{B(X, \ell^p(n))} < \delta.$$

Thus

$$d_\varepsilon(\operatorname{Hom}'_{\Gamma}(\Gamma, m, \delta, \sigma_i), \|\cdot\|) \leq (\dim_{\mathbb{C}} X) q_n + r_n, \tag{3.8}$$

and (3.7), (3.8) are enough to imply the proposition.

□

Corollary 3.5.6. *Let Γ be a finite group acting on a finite-dimensional vector space X. For any finite dimensional representation X of Γ, for any sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ of Γ and $1 \leq p \leq \infty$ we have*

$$\dim_{\Sigma, \ell^p}(X, \Gamma) = \underline{\dim}_{\Sigma, \ell^p}(X, \Gamma) = \frac{\dim_{\mathbb{C}} X}{|\Gamma|}.$$

Proof. Take

$$\Sigma' = (\rho_{d_i} : \Gamma \to S_{d_i})$$

where ρ_n is defined as in the previous proposition, then use the preceding proposition and Corollary 3.5.4.

□

Proposition 3.5.7. *Let X be a finite-dimensional Banach space with a uniformly bounded action of $\mathbb{Z}$. Let $\sigma_n : \mathbb{Z} \to \operatorname{Sym}(\mathbb{Z}/n\mathbb{Z})$ be given by the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Then for all $1 \leq p \leq \infty$,*

$$\dim_{\Sigma, \ell^p}(X, \mathbb{Z}) = 0.$$

Proof. Since all norms on a finite-dimensional space are equivalent, we may assume that X is a Hilbert space. Since X is now a Hilbert space, we will call it H instead. Let $\pi : \mathbb{Z} \to B(H)$

be the representation given by the action of $\mathbb{Z}$, and let $K = \overline{\pi(\mathbb{Z})}$. By finite-dimensionality, K is a compact group. Let $\langle \cdot, \cdot \rangle_H$ be the inner product on H. Define a new inner product on H by

$$\langle \xi, \eta \rangle = \int_K \langle T\xi, T\eta \rangle_H \, dT,$$

where the integration is with respect to the Haar measure on K. We leave it as an exercise to verify that this is indeed an inner product inducing a norm equivalent to the original norm on H, and that K acts unitarily with respect to $\langle \cdot, \cdot \rangle$. Thus we may assume that $\pi(\mathbb{Z}) \subseteq U(H)$, set $U = \pi(1)$. By passing to direct sums, we may assume that π is irreducible, so if we fix any $\xi \in H$ with $\|\xi\| = 1$, then ξ is generating. We will take $S = (\xi, 0, 0, \dots)$, and as a pseudonorm we take

$$\rho(T) = \|T(\xi)\|.$$

Fix $n \in \mathbb{N}$, we then view α_S as a map into $\ell^p(n)$.

Fix $1 > \varepsilon > 0$, and let $\varepsilon > \delta > 0$. Choose k such that $\delta^p k < \varepsilon$, (if $p = \infty$ then let k be any integer.) Since $\overline{\pi(\mathbb{Z})}$ is compact, we can find an integer m such that

$$\|U^{mj} - 1\| < \delta,$$

for $1 \leq j \leq k$. We may assume that m is large enough so that $\{U^j \xi : -m \leq j \leq -1\}$ spans H. Let $F = \{j \in \mathbb{Z} : |j| \leq m(2k+1)\}$. Let $q_n \in \mathbb{N} \cup \{0\}, 0 \leq r_n < k$ be the integers defined by

$$n = q_n mk + r_n.$$

Define $Q_j, j = 0, \dots, k-1$ by

$$Q_j = \bigcup_{l=1}^{m} \{jm + l + qm_k : 0 \leq q \leq q_n - 1\}.$$

Pictorially, if we think of $\{1, \dots, q_n mk\}$ as a rectangle formed out of mk horizontal dots and q_n vertical dots, then Q_j is the rectangle from the $jm + 1^{st}$ horizontal dot to the $(j+1)m^{th}$ horizontal dot. Let $f_j \colon Q_j \to \mathbb{C}$ be given by

$$f_j(l) = T(\xi)(\sigma_n(mj)^{-1}(l)).$$

Note that for $1 \leq p < \infty$,

$$\begin{aligned}\left\| T(\xi) - \sum_{j=0}^{k-1} f_j \right\|_{\ell^p(\{1,\dots,q_n mk\})}^p &= \sum_{j=0}^{k-1} \|T(\xi) - \sigma_n(mj)T(\xi)\|_{\ell^p(Q_j)}^p \\ &< \delta^p k + \sum_{j=0}^{k-1} \|T((U^{-mj} - 1)\xi)\|_p^p \\ &< 2\delta^p k \\ &< 2\varepsilon\end{aligned}$$

similarly for $p = \infty$,

$$\left\| T(\xi) - \sum_{j=0}^{k-1} f_j \right\|_{\ell^\infty(\{1,\dots,q_n mk\})} < 2\varepsilon.$$

Finally note that $\sum_{j=0}^{k-1} f_j$ is constant on

$$\{i, i+m, \dots, i+m(k-1)\}$$

for each $i \in Q_0$. Thus

$$\begin{aligned}\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_n)) \subseteq_{\varepsilon, \|\cdot\|_p} \ell^p(\{1,\dots,n\} \setminus \{1,\dots,q_n mk\}) + \\ \{f \in \ell^p(q_n mk) : f(i+mj) = f(i), i \in Q_0, 0 \leq j \leq k-1\}.\end{aligned}$$

So

$$\frac{1}{n} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_n), \|\cdot\|_p) \leq \frac{q_n m}{n} + \frac{r_n}{n}.$$

Letting $n \to \infty$, taking the limit supremum over (F, m, δ) and then letting $\varepsilon \to 0$ we conclude that

$$d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_n), \|\cdot\|_p) \leq \frac{1}{k}.$$

Since k becomes arbitrarily large when δ becomes small (or can be made arbitrarily large when $p = \infty$), this completes the proof.

□

We will now proceed to prove that if Γ is an infinite sofic group, and Σ is a sofic approximation of Γ, then for any finite-dimensional representation V of Γ we have

$$\dim_{\Sigma, \ell^p}(V, \Gamma) = 0.$$

The method is based on passing to an action of the group on a measure space, and then using that the corresponding equivalence relations contains an action of $\mathbb{Z}$.

We shall first work with the trivial action of Γ on $\mathbb{C}$. For this, fix a sofic group Γ and a sofic approximation Σ . For $S = (1, 0, 0, \dots)$, and the trivial action of Γ on $\mathbb{C}$, the map $T \to T(\{1\})$ identifies $\mathrm{Hom}_{\Gamma,p}(S, F, m, \delta, \sigma_i)$ with all vectors $\xi \in \ell^p(d_i)$ such that

$$\|\sigma_i(g)\xi - \xi\|_p < \delta$$

for all $g \in F$.

Lemma 3.5.8. *Let Γ be a countable discrete sofic group with a sofic approximation Σ. Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, measure-preserving action on a standard probability space (X, μ) such that there is a sofic approximation (still denoted Σ) of $\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}$ extending the sofic approximation of Γ. Let $\Sigma = (\sigma_i \colon [[\mathcal{R}]] \to [[\mathcal{R}_{d_i}]])$. Fix $\phi \in [[\mathcal{R}]]$, and $\eta > 0$. Then there are $F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$ and $C_i \subseteq \{1, \dots, d_i\}$ with $|C_i| \geq (1 - \eta)d_i$ so that for the trivial representation of Γ on $\mathbb{C}$, and $T \in \mathrm{Hom}_{\Sigma,p}((1, 0, 0, \dots), F, m, \delta, \sigma_i)$ with $\xi = T(1)$ we have*

$$\|\sigma_i(\phi)\xi - \sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)})\xi\|_{\ell^p(C_i)} < \eta,$$

for all large i.

Proof. Let $\{A_g : g \in \Gamma\}$ be a partition of $\mathrm{ran}(\phi)$ so that

$$\phi = \sum_{g \in \Gamma} \mathrm{Id}_{A_g} \alpha_g,$$

with the sum converging $d_{[[\mathcal{R}]]}$. Choose $F \subseteq \Gamma$ finite so that

$$d_{[[\mathcal{R}]]}\left(\phi, \sum_{g \in F} \mathrm{Id}_{A_g} \alpha_g\right) < \eta.$$

For $\xi \in \ell^p(d_i), \phi \in [[R_{d_i}]]$, we use

$$(\phi\xi)(j) = \chi_{\mathrm{ran}(\phi)}(j)\xi(\phi^{-1}(j)),$$

for $A \subseteq \{1, \dots, d_i\}$ we also use χ_A for the operator of multiplication by A. By soficity, for all large i, we may find a $C_i \subseteq \{1, \dots, d_i\}$ with $|C_i| \geq (1-2\eta)d_i$ so that

$$\chi_{C_i}\sigma_i(\phi) = \sum_{g\in F} \chi_{C_i}\sigma_i(\mathrm{Id}_{A_g})\sigma_i(g),$$

$$\chi_{C_i}\sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)}) = \sum_{g\in F} \chi_{C_i}\sigma_i(\mathrm{Id}_{A_g}),$$

as operators on $\ell^p(d_i)$. Let $m \in \mathbb{N}$, and let $\delta > 0$ be sufficiently small in a manner to be determined later. Thus for T, ξ as in the statement of the lemma,

$$\sigma_i(\phi)\xi = \sum_{g\in F} \sigma_i(\mathrm{Id}_{A_g})\sigma_i(g)\xi,$$

$$\sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)})\xi = \sum_{g\in F} \sigma_i(\mathrm{Id}_{A_g})\xi,$$

so

$$\|\sigma_i(\phi)\xi - \sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)})\xi\|_{\ell^p(C_i)} \leq |F|\delta.$$

So if $\delta < \frac{\eta}{|F|}$, our claim is proved.

□

Lemma 3.5.9. *Let Γ be a countably infinite discrete sofic group with sofic approximation Σ. Then for the trivial representation of Γ on $\mathbb{C}$, we have*

$$\dim_{\Sigma,\ell^p}(\mathbb{C}, \Gamma) = 0.$$

Proof. Let $\mathcal{R}$ be the equivalence relation induced by the Bernoulli action of Γ on $(X, \mu) = (\{0,1\}, u)^\Gamma$, u being the uniform measure. Extend Σ to a sofic approximation of $[[\mathcal{R}]]$, (this is essentially possible by [2] Theorem 8.1, see also [7] Proposition 7.1,[6] Theorem 5.5, [20] Theorem 2.1). Let $S = (1, 0, 0, \dots)$. Since Γ is an infinite group, by ([17] Corollary 7.6) we know that for all $n \in \mathbb{N}$, there is a subequivalence relation $\mathcal{R}_n$, generated by a free, measure-preserving action of $\mathbb{Z}/n\mathbb{Z}$ on (X, μ). Let $\alpha \in [\mathcal{R}_n]$ generate the action of $\mathbb{Z}/n\mathbb{Z}$ on (X, μ). Fix $\eta > 0$. By the preceding lemma, we may choose a finite subset $F \subseteq \Gamma, \delta > 0$ and subsets $C_i \subseteq \{1, \dots, d_i\}$ with $|C_i| \geq (1 - d_i)\eta$ so that if $T \in \mathrm{Hom}_\Gamma(S, F, 1, \delta, \sigma_i)$ and $\xi = T(1)$, then

$$\|\sigma_i(\alpha)^j\xi - \xi\|_{\ell^p(C_i)} < \eta, \text{for } 1 \leq j \leq n-1$$

for all large i. We may assume that there are $A_i \subseteq \{1,\ldots,d_i\}$ with $\frac{|A_i|}{d_i} \to \frac{1}{n}$, so that

$$\{\sigma_i(\alpha)^j(A_i) : 0 \leq j \leq n-1\} \text{ are a disjoint family.}$$

$$\sigma_i(\alpha)\Big|_{\{1,\ldots,d_i\}\setminus\bigcup_{j=0}^{n-1}\sigma_i(\alpha)^j(A_i)} = \mathrm{Id}\,.$$

Let

$$\eta = \sum_{j=1}^{n} \sigma_i(\alpha)^j \chi_{A_i}\xi = \sum_{j=1}^{n} \chi_{\sigma_i(\alpha)^j(A_i)}\sigma_i(\alpha)^j\xi.$$

Set $D_i = C_i \cap \bigcup_{j=0}^{n-1} \sigma_i(\alpha)^j(A_i)$, then

$$\chi_{D_i}\eta - \chi_{D_i}\xi = \sum_{i=1}^{n} \chi_{D_i\cap\sigma_i(\alpha)^j(A_i)}(\sigma_i(\alpha)^j\xi - \xi),$$

so

$$\|\chi_{D_i}\eta - \chi_{D_i}\xi\|_p \leq \eta n.$$

We may view α_S as a map into $\ell^p(d_i)$, then

$$\alpha_S(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)) \subseteq_{\eta n,\|\cdot\|_p} \chi_{D_i}\left\{\sum_{j=1}^{n}\sigma_i(\alpha)^j f : f \in \ell^p(A_i)\right\}$$
$$+ \ell^p(\{1,\ldots,d_i\} \setminus D_i).$$

As

$$\frac{|D_i|}{d_i} \to 1,$$
$$\frac{|A_i|}{d_i} \to \frac{1}{n}$$

we find that

$$\limsup_{(F,m,\delta)} \limsup_{i\to\infty} \frac{1}{d_i} d_{\eta n}(\alpha_S(\mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i), \|\cdot\|_p) \leq \frac{1}{n}.$$

Letting $\eta \to 0$, and then $n \to \infty$ completes the proof.

□

Theorem 3.5.10. *Let* Γ *be a countably infinite sofic group with sofic approximation* Σ*. Then, for any representation of* Γ *on a finite-dimensional vector space* V*, and for all* $1 \leq p < \infty$*,*

$$\dim_{\Sigma,\ell^p}(V,\Gamma) = 0.$$

Proof. As dimension decreases under restricting the action to a subgroup, by Corollary 3.5.6 and Proposition 3.5.7 we may assume that

$$\{|\Lambda| : \Lambda \text{ is a finite subgroup of } \Gamma\},$$

is bounded, and that every element of Γ has finite order. As in Proposition 3.5.7 we may assume that V is a Hilbert space and Γ acts by unitaries. Let M be greater than $|\Lambda|$ for any finite subgroup of Γ. Choose $\varepsilon > 0$ so that if U is a unitary on a Hilbert space and

$$\|U - 1\| < \varepsilon,$$

then $U^M \neq 1$ unless $U = 1$. Let $\pi \colon \Gamma \to U(X)$ be the homomorphism induced by the action of Γ. By finite-dimensionality, $\overline{\pi(\Gamma)}$ is compact, so we may find an infinite sequence $(g_n)_{n=1}^{\infty}$ of distinct elements of Γ with

$$\|\pi(g_n) - 1\| < \varepsilon.$$

If

$$\Lambda = \langle g_n : n \in \mathbb{N} \rangle,$$

our assumptions then imply that Λ is an infinite subgroup of Γ which acts trivially. Thus by the preceding lemma and subadditivity under exact sequences,

$$\dim_{\Sigma,\ell^p}(V,\Gamma) \leq \dim_{\Sigma,\ell^p}(V,\Lambda) = 0.$$

□

3.6 A Complete Calculation in the Case of $\bigoplus_{j=1}^{n} L^p(L(\Gamma))q_j$.

In this section, we show that if Γ is $\mathcal{R}^\omega$-embeddable, Σ is an embedding sequence and $q_1, \dots, q_n \in \operatorname{Proj}(L(\Gamma))$, then

$$\dim_{\Sigma,S^p,\mathrm{mult}}\left(\bigoplus_{j=1}^{n} L^p(L(\Gamma),\tau)q_j, \Gamma\right) = \underline{\dim}_{\Sigma,S^p,\mathrm{mult}}\left(\bigoplus_{j=1}^{n} L^p(L(\Gamma),\tau)q_j, \Gamma\right) =$$

$$\sum_{j=1}^{n} \tau(q_j)$$

where τ is the group trace. See Appendix A for the appropriate background on noncommutative L^p-spaces.

Lemma 3.6.1. *(a) Let $n \in \mathbb{N}$, suppose that $A, B \in M_n(\mathbb{C})$ are such that $|A| \leq |B|$, then for all $\beta > 0$,*

$$\mathrm{tr}(|A|^\beta) \leq \mathrm{tr}(|B|^\beta).$$

(b) Suppose that $A, B \in M_n(\mathbb{C})$ and Q is a orthogonal projection in $M_n(\mathbb{C})$. Fix $1 \leq p < \infty$, suppose that $\delta, \eta > 0$ are such that

$$\|(A-1)B\|_p < \delta, \|A - Q\|_p < \eta.$$

Then

$$\|B - \chi_{(0,\sqrt{\delta})}(|A-1|)B\|_p < \sqrt{\delta},$$

and

$$\mathrm{tr}(\chi_{(0,\sqrt{\delta})}(|A-1|)) \leq \mathrm{tr}(Q) + \left(\frac{\eta}{1-\sqrt{\delta}}\right)^p.$$

Proof. We first make the following preliminary observation: if P, Q are orthogonal projections in $M_n(\mathbb{C})$ with

$$P\mathbb{C}^n \cap Q\mathbb{C}^n = \{0\},$$

then

$$\mathrm{tr}(P) \leq 1 - \mathrm{tr}(Q).$$

This follows directly from the fact that $1 - Q$ is injective on $P\mathbb{C}^n$.

(a) First note that

$$\mathrm{tr}(T^\alpha) = \alpha \int_0^\infty t^{\alpha-1}\, \mathrm{tr}(\chi_{(t,\infty)}(T))\, dt$$

if $T \geq 0$. If $0 \leq T \leq S$, and

$$\xi \in \chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n)$$

and $\xi \neq 0$, then

$$t\|\xi\|^2 < \langle T\xi, \xi\rangle \leq \langle S\xi, \xi\rangle \leq t\|\xi\|^2,$$

which is a contradiction. Hence

$$\chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n) = \{0\},$$

so the above integral formula and our preliminary observation prove (a).

(b) Note that

$$\begin{aligned} |\chi_{[\sqrt{\delta},\infty)}(|A-1|)B|^2 &= B^*\chi_{[\sqrt{\delta},\infty)}(|A-1|)B \\ &\leq \frac{1}{\delta}B^*|A-1|^2B = \left|\frac{1}{\sqrt{\delta}}(A-1)B\right|^2, \end{aligned}$$

thus by (a)

$$\|B - \chi_{(0,\sqrt{\delta})}(|A-1|)B\|_p = \|\chi_{[\sqrt{\delta},\infty)}(|A-1|)B\|_p < \sqrt{\delta}.$$

Further if

$$\xi \in \chi_{(0,\sqrt{\delta})}(|A-1|)(\mathbb{C}^n) \cap (1-Q)(\mathbb{C}^n) \cap \chi_{[0,1-\sqrt{\delta}]}(|A-Q|)(\mathbb{C}^n),$$

is nonzero, then

$$(1-\sqrt{\delta})^2\|\xi\|^2 \geq \langle |A-Q|^2\xi, \xi\rangle = \|A\xi\|^2 > (1-\sqrt{\delta})^2\|\xi\|^2,$$

which is a contradiction. Thus

$$\operatorname{tr}(\chi_{(0,\sqrt{\delta})}(|A-1|)) \leq \operatorname{tr}(Q) + \operatorname{tr}(\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|)).$$

Since

$$\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|) \leq \frac{|A-Q|^p}{(1-\sqrt{\delta})^p},$$

we have that

$$\operatorname{tr}(\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|)) < \frac{\eta^p}{(1-\sqrt{\delta})^p}.$$

□

Proposition 3.6.2. *Let Γ be an $\mathcal{R}^\omega$-embeddable group and Σ an embedding sequence. Let $M = L(\Gamma)$ and τ the canonical group trace on M. Then, for all $1 \leq p < \infty$ and for every $q_1, \ldots, q_n \in \mathrm{Proj}(M)$ we have*

$$\dim_{\Sigma, S^p, \mathrm{mult}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j, \Gamma\right) \leq \sum_{j=1}^n \tau(q_j).$$

Proof. By subadditivity of dimension, it suffices to handle the case of $L^p(M,\tau)q$. Let $0 < \varepsilon, \kappa < 1/2$. Let A be the $*$-algebra in $L(\Gamma)$ generated by q and Γ, by Lemma 2.2.6, we may extend σ_i to (potentially nonlinear, nonmultiplicative) maps $\sigma_i \colon L(\Gamma) \to M_{d_i}(\mathbb{C})$, so that

$$\sup_i \|\sigma_i(x)\|_\infty < \infty, \text{ for all } x \in L(\Gamma),$$

$$\mathrm{tr}(\sigma_i(x)) \to \tau(x), \text{ for all } x \in L(\Gamma),$$

$$\|P(\sigma_i(x_1), \ldots, \sigma_i(x_n)) - \sigma_i(P(x_1, \ldots, x_n))\|_2 \to 0,$$

for all $x_1, \ldots, x_n \in L(\Gamma)$, and all $*$-polynomials in n-noncommuting variables.

Let $p \in L(\Gamma)$ be any orthogonal projection. Then

$$\|\rho_i(p) - \rho_i(p)^*\rho_i(p)\|_2 \to 0$$

$$\|\rho_i(p)^*\rho_i(p) - (\rho_i(p)^*\rho_i(p))^2\|_2 \to 0.$$

By functional calculus, for any $\varepsilon < 1/2$,

$$\begin{aligned}\|\chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^*\rho_i(p)) - \rho_i(p)^*\rho_i(p)\|_2 &\leq \|\chi_{[0,\infty)\setminus[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^*\rho_i(p))\rho_i(p)^*\rho_i(p)\|_2 \\ &\quad + \|\chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^*\rho_i(p))(1 - \rho_i(p)^*\rho_i(p))\|_2 \\ &\leq \frac{1}{(1-\varepsilon)}\|\rho_i(p)^*\rho_i(p) - (\rho_i(p)^*\rho_i(p))^2\|_2 \\ &\quad + \frac{1}{\varepsilon}\|\rho_i(p)^*\rho_i(p) - (\rho_i(p)^*\rho_i(p))^2\|_2.\end{aligned}$$

Thus for all $\varepsilon < 1/2$,

$$\|\rho_i(p) - \chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^*\rho_i(p))\|_2 \to 0.$$

Applying the above estimates with $p = q$, we see that we may replacing $\rho_i(q)$ with $\chi_{[3/4,5/4]}(\rho_i(q)^*\rho_i(q))$. Thus, we may assume that $\rho_i(q)$ is an orthogonal projection for all i.

Choose $f \in c_c(\Gamma)$ so that

$$\left\| q - \sum_{s \in \Gamma} f(s) u_s \right\|_p < \kappa.$$

If $T\colon L^p(M,\tau)q \to L^p(M_{d_i}(\mathbb{C}), \mathrm{tr})$, define

$$\widetilde{T}(x) = T(xq).$$

Let F be the support of f, then if $m \in \mathbb{N}, \kappa, \delta > 0$ are sufficiently small we have

$$\left\| \left(\sum_{s \in \Gamma} f(s)\sigma_i(s) - 1 \right) \widetilde{T}(q) \right\|_p < \varepsilon^2,$$

for all $T \in \mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$. Thus the proceeding lemma implies that if

$$e_i = \chi_{(\varepsilon,\infty)} \left(\left| \sum_{s \in \Gamma} f(s)\sigma_i(s) - 1 \right| \right),$$

then for all large i, we have

$$\|T(q) - e_i T(q)\|_p < \varepsilon,$$

$$\mathrm{tr}(e_i) \leq \mathrm{tr}(\rho_i(q)) + 2^p \kappa^p$$

We identify α_S as a map into $L^p(M_{d_i}(\mathbb{C}), \mathrm{tr})$, then

$$\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)) \subseteq_\varepsilon \{e_i A : A \in L^p(M_{d_i}(\mathbb{C}), \mathrm{tr})\}.$$

So

$$\frac{1}{d_i} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)) \leq \frac{1}{d_i} \mathrm{Tr}(e_i) = \mathrm{tr}(e_i) \leq \mathrm{tr}(\rho_i(q)) + 2^p \kappa^p$$

and

$$\mathrm{tr}(\rho_i(q)) \to \tau(q)$$

as $i \to \infty$. Taking the limit supremum over (F, m, δ) and then letting $\varepsilon \to 0$ proves that

$$\dim_{\Sigma, S^p, \mathrm{mult}}(L^p(M,\tau), \Gamma) \leq \tau(q) + 2^p \kappa^p.$$

Since $\kappa > 0$ is arbitrary, this proves the claim.

□

Lemma 3.6.3. *Fix $1 \leq p \leq \infty$, and a sequence of positive integers $d(n) \to \infty$, and let μ_n be the Lebesgue measure on $L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr})$ normalized so that $\mu_n(\operatorname{Ball}(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr}))) = 1$. Further, let $q_n \in \operatorname{Proj}(M_{d(n)}(\mathbb{C}))$ be such that $\frac{1}{d(n)}\operatorname{Tr}(q_n)$ converges to a positive real number. Then, there is a function*

$$\kappa\colon (0,1) \times (0,\infty) \to [0,1]$$

such that

$$\lim_{\varepsilon \to 0} \kappa(\alpha, \varepsilon) = 1, \textit{for all } \alpha > 0,$$

which satisfies the following property. For all $A_n \subseteq \operatorname{Ball}(V_n)$, and $\alpha > 0$ with

$$\limsup_{n \to \infty} \mu_n(\operatorname{Ball}(V_n))^{1/2d(n)^2} \geq \alpha.$$

We have for all $\varepsilon > 0$,

$$\limsup_{n \to \infty} \frac{1}{d(n)\operatorname{Tr}(q_n)} d_\varepsilon(A_n q_n, \|\cdot\|_p) \geq \kappa(\alpha, \varepsilon).$$

Proof. Fix $1 > \varepsilon > 0$, and suppose that

$$\limsup_{n \to \infty} \frac{1}{d(n)\operatorname{tr}(q_n)} d_\varepsilon(A_n q_n, \|\cdot\|_p) < \kappa.$$

Then for all large n,

$$d_\varepsilon(A_n q_n, \|\cdot\|_{V_n}) < d(n)\kappa \operatorname{tr}(q_n).$$

Let W_n be a subspace of dimension at most $d(n)\kappa \operatorname{tr}(q_n)$ which ε-contains $A_n q_n$, thus

$$A_n q_n \subseteq (1+\varepsilon)\operatorname{Ball}(W_n) + \varepsilon \operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \operatorname{tr}) q_n\right).$$

Let $S \subseteq (1+\varepsilon)\operatorname{Ball}(W_n)$ be a maximal family of ε-separated vectors, i.e. for all $x, y \in S$ with $x \neq y$ we have $\|x - y\| \geq \varepsilon$. Then the $\varepsilon/3$ balls centered at points in S are disjoint and so by a volume computation

$$|S| \leq \left(\frac{3+3\varepsilon}{\varepsilon}\right)^{2\dim(W_n)}.$$

By maximality, S is ε-dense in $(1+\varepsilon)\operatorname{Ball}(W_n)$. Thus

$$A_n q_n \subseteq \bigcup_{x\in S} x + 2\varepsilon \operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \operatorname{tr})q_n\right),$$

so

$$\operatorname{vol}(A_n q_n) \leq 2^{2d(n)\operatorname{Tr}(q_n)}\varepsilon^{2d(n)\operatorname{Tr}(q_n)-2\dim(W_n)\operatorname{Tr}(q_n)}(3+3\varepsilon)^{2\dim(W_n)}a_p(q_n),$$

where for $q \in \operatorname{Proj}(M_{d(n)}(\mathbb{C}))$ we use

$$a_p(q) = \operatorname{vol}(\operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \operatorname{tr})q\right)).$$

Since $A_n \subseteq A_n q_n \times \operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \operatorname{tr}\right)$, we have

$$\alpha \leq \limsup_{n\to\infty} 6\cdot 2^{\frac{1}{d(n)}\operatorname{Tr}(q_n)}\varepsilon^{(1-\kappa)\frac{\operatorname{Tr}(q_n)}{d(n)}}\left(\frac{a_p(q_n)a_p(1-q_n)}{a_p(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2}$$

Hence it suffices to show that

$$\limsup_{n\to\infty}\left(\frac{a_p(q_n)a_p(1-q_n)}{a_p(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} < \infty. \tag{3.9}$$

It is well known that

$$a_2(q) = \frac{\pi^{\operatorname{Tr}(q)}}{\operatorname{Tr}(q)!}d(n)^{-d(n)}.$$

Since $\frac{1}{d(n)}\operatorname{Tr}(q_n)$ converges to a positive real number, we may apply Stirling's formula and the above equation to see that there is a $M > 1$ so that

$$M^{-1} \leq \left(\frac{a_2(q_n)a_2(1-q_n)}{a_2(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} < M.$$

We know by [25] that there is a constant $C > 0$ so that

$$\begin{aligned}\left(\frac{a_p(q_n)a_p(1-q_n)}{a_p(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} &\leq C\left(\frac{a_p(q_n)a_p(1-q_n)}{a_2(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2}\\ &\leq CM^2\left(\frac{a_p(q_n)}{a_2(q_n)}\right)^{1/2d(n)^2}\left(\frac{a_p(1-q_n)}{a_2(1-q_n)}\right)^{1/2d(n)^2}.\end{aligned}$$

Let p' be such that $\frac{1}{p}+\frac{1}{p'} = 1$. By the Santalo inequality (see [21] Corollary 7.2), and the fact that $\frac{1}{d(n)}\operatorname{Tr}(q_n)$ converges to a positive real number, we may find a $A > 0$ so that

$$\left(\frac{a_p(q_n)}{a_2(q_n)}\right)^{1/2d(n)^2}\left(\frac{a_p(1-q_n)}{a_2(1-q_n)}\right)^{1/2d(n)^2} \leq A\left(\frac{a_2(q_n)}{a_{p'}(q_n)}\right)^{1/2d(n)^2}\left(\frac{a_2(1-q_n)}{a_{p'}(1-q_n)}\right)^{1/2d(n)^2}$$

$$\leq AM^2\left(\frac{a_2(\mathrm{Id})}{a_{p'}(q_n)a_{p'}(1-q_n)}\right)^{1/2d(n)^2}.$$

Again by [25], we can find some $D>0$ so that

$$\left(\frac{a_2(\mathrm{Id})}{a_{p'}(q_n)a_{p'}(1-q_n)}\right)^{1/2d(n)^2}\leq D\left(\frac{a_{p'}(\mathrm{Id})}{a_{p'}(q_n)a_{p'}(1-q_n)}\right)^{1/2d(n)^2}.$$

As

$$\mathrm{Ball}(L^{p'}(M_{d_i}(\mathbb{C}),\mathrm{tr}))\subseteq \mathrm{Ball}(L^{p'}(M_{d_i}(\mathbb{C}),\mathrm{tr})q_n)\times \mathrm{Ball}(L^{p'}(M_{d_i}(\mathbb{C}),\mathrm{tr})(1-q_n)),$$

we find that

$$\left(\frac{a_{p'}(\mathrm{Id})}{a_{p'}(q_n)a_{p'}(1-q_n)}\right)^{1/2d(n)^2}\leq 1.$$

Putting all these inequalities together, we find that

$$\left(\frac{a_p(q_n)a_p(1-q_n)}{a_p(\mathrm{Id}_{d(n)})}\right)^{1/2d(n)^2}\leq ACM^4D,$$

and this proves (3.9).

□

To complete the calculation, it suffices to prove the following Theorem.

Theorem 3.6.4. *Let* Γ *be an* $\mathcal{R}^\omega$*-embeddable group and* Σ *an embedding sequence. Let* $M=L(\Gamma)$ *and* τ *the canonical group trace on* M*. Then, for all* $1\leq p<\infty$ *and for every* $q_1,\dots,q_n\in\mathrm{Proj}(M)$ *we have*

$$\dim_{\Sigma,S^p,\mathrm{mult}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j,\Gamma\right)=\underline{\dim}_{\Sigma,S^p,\mathrm{mult}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j,\Gamma\right)=\sum_{j=1}^n\tau(q_j).$$

Proof. We use the generating sequence $S=(q_1,\dots,q_n,0,\dots)$ to do the calculation. By Proposition 3.6.2, we have the upper bound. So it suffices to prove the lower bound. By Lemma 2.2.6, we can find maps (not assumed to be linear) $\rho_i\colon L(\Gamma)\to M_{d_i}(\mathbb{C})$ so that

$$\rho_i(\lambda(g))=\sigma_i(g),\text{for } g\in\Gamma$$

$$\sup_i\|\rho_i(x)\|_\infty<\infty,\ \text{for all } x\in L(\Gamma),$$

$$\mathrm{tr}(\rho_i(x)) \to \tau(x), \text{for all } x \in L(\Gamma)$$

$$\|P(\rho_i(x_1), \ldots, \rho_i(x_n) - \rho_i(P(x_1, \ldots, x_n))\|_2 \to 0,$$

for all $x_1, \ldots, x_n \in L(\Gamma)$, and all $*$-polynomials P in n-noncommuting variables. As in Proposition 3.6.2, we may assume that $\rho_i(q_j)$ is an orthogonal projection for all i, j.

Fix $F \subseteq \Gamma$ finite $m \geq n$ in $\mathbb{N}$, $\delta > 0$. Let $E \subseteq \Gamma$ be a finite set which is sufficiently large in a manner to be determined later. Let

$$V_E^{(j)} = \mathrm{Span}\{u_g q : g \in E\}.$$

For $A \in M_{d_i}(\mathbb{C})$ $E \subseteq \Gamma$ finite define

$$T_A^{(j)}\left(\sum_{g \in E} c_g u_g q\right) = \sum_{g \in E} c_g \sigma_i(g)\rho_i(q_j)A.$$

Note that

$$\left\|T_A^{(j)}\left(\sum_{g \in E} c_g u_g q\right)\right\|_p \leq \|A\|_\infty \left\|\sum_{g \in E} c_g \sigma_i(g)\rho_i(q_j)\right\|_p.$$

Since σ_i is an embedding sequence, we know that

$$\left\|\sum_{g \in E} c_g \sigma_i(g)\rho_i(q_j)\right\|_p \to \left\|\sum_{g \in E} c_g u_q q_j\right\|_p,$$

pointwise. As $V_E^{(j)}$ is finite-dimensional,

$$\left\|\sum_{g \in E} c_g \sigma_i(g)\rho_i(q_j)\right\|_p \to \left\|\sum_{g \in E} c_g u_q q_j\right\|_p,$$

uniformly on the $\|\cdot\|_p$ unit ball of $V_E^{(j)}$.

If E is sufficiently large, then for all $g_1, \ldots, g_k \in F$,

$$\begin{aligned}\|T_A^{(j)}(g_1 \cdots g_k q_j) - \sigma_1(g_1) \cdots \sigma_i(g_k) T_A^{(j)}(q_j)\|_p &= \|\sigma_i(g_1 \cdots g_k)\rho_i(q_j)A - \sigma_1(g_1) \cdots \sigma_i(g_k)\rho_i(q_j)A\|_p \\ &\leq \|A\|_\infty \|\sigma_i(g_1 \cdots g_k) - \sigma_i(g_1) \cdots \sigma_i(g_k)\|_p \\ &\to 0.\end{aligned}$$

Thus if E is sufficiently large, depending upon F, m, δ then for all $A_1, \dots, A_n \in M_{d_i}(\mathbb{C})$ with $\|A_j\|_\infty \leq 1$,

$$T^{(1)}_{A_1} \oplus \cdots \oplus T^{(n)}_{A_n} \in \mathrm{Hom}_\Gamma(S, F, m\delta, \sigma_i)_n.$$

So

$$\alpha_S(\mathrm{Hom}_\Gamma(S, F, m\delta, \sigma_i)_n) \supseteq \prod_{j=1}^{n} \mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_\infty)\rho_i(q_j).$$

By [25]

$$\left(\frac{\mathrm{vol}(\mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_\infty)}{\mathrm{vol}(\mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_{L^p(1/d_i \operatorname{Tr})})}\right)^{1/2d_i^2},$$

so the theorem now follows from Lemma 3.6.3.

□

We can prove an analogue for the action of Γ on its reduced C^*-algebra but first we need a Lemma.

Lemma 3.6.5. *Let Γ be a countable discrete group, and $X \subseteq L^p(L(\Gamma), \tau_\Gamma)$ a closed Γ-invariant subspace (for the action of left multiplication by elements of Γ). Then there is an orthogonal projection $q \in L(\Gamma)$ so that $X = L^p(L(\Gamma), \tau_\Gamma)$.*

Proof. We always have the inequality

$$\|xy\|_p \leq \|x\|_\infty \|y\|_p.$$

Note that if $x_n \in L(\Gamma), \sup_n \|x_n\|_\infty < \infty$, and $x_n \to x$ in the strong operator topology on $\ell^2(\Gamma)$, then $x_n y \to xy$. Indeed, this follows by the above inequality and the density of $\ell^2(\Gamma)$ in $L^p(L(\Gamma), \tau_\Gamma)$. Thus a closed Γ-invariant subspace is the same as an $L(\Gamma)$-invariant subspace.

It suffices to prove the following two claims.

Claim 1. If $x \in L^p(L(\Gamma), \tau_\Gamma)$, then $\overline{L(\Gamma)x}^{\|\cdot\|_p} = L^p(L(\Gamma), \tau_\Gamma)\chi_{(0,\infty)}(|x|)$.

Claim 2. If e, f are orthogonal projections in $L(\Gamma)$, then

$$\overline{L^p(L(\Gamma), \tau_\Gamma)e + L^p(L(\Gamma), \tau_\Gamma)f} = L^p(L(\Gamma), \tau_\Gamma)(e \vee f).$$

Indeed, if we grant the two claims, then by separability, we can find increasing subspaces X_n of Γ of the form $L^p(L(\Gamma),\tau_\Gamma)q_n$ for some orthogonal projection q_n. Setting $q=\sup q_n$ we see that

$$X = L^p(L(\Gamma),\tau_\Gamma)q.$$

For claim 2 it suffices to note that by functional calculus

$$1-(e\vee f) = 1-(1-e)\wedge(1-f) = 1-\lim_{n\to\infty}((1-e)(1-f)(1-e))^n,$$

the limit in $\|\cdot\|_p$. As

$$1-[(1-e)(1-f)(1-e)]^n \in L^p(L(\Gamma),\tau_\Gamma)e + L^p(L(\Gamma),\tau_\Gamma)f$$

for all n, this implies that

$$L^p(L(\Gamma),\tau_\Gamma)(e\vee f) \subseteq \overline{L^p(L(\Gamma),\tau_\Gamma)e + L^p(L(\Gamma),\tau_\Gamma)f}.$$

The reverse inclusion being trivial, this proves claim 2.

For claim 1, let $x=v|x|$ be the polar decomposition. Since $|x|=v^*x$,

$$\overline{L(\Gamma)x}^{\|\cdot\|_p} = \overline{L(\Gamma)|x|}^{\|\cdot\|_p}.$$

Let

$$y_n = \chi_{(\varepsilon,\infty)}(|x|)|x|^{-1},$$

then by functional calculus

$$\|y_n|x| - \chi_{(0,\infty)}(|x|)\|_p \to 0.$$

Thus

$$\overline{L(\Gamma)|x|}^{\|\cdot\|_p} \supseteq L^p(\Gamma,\tau_\Gamma)\chi_{(0,\infty)}(|x|).$$

the reverse inclusion being trivial, we are done.

□

If Γ is a countable discrete group we use $C^*_\lambda(\Gamma)$ for $\overline{\mathbb{C}[\Gamma]}^{\|\cdot\|_\infty}$, with the closure taken in the left regular representation.

Corollary 3.6.6. *Let Γ be an $\mathcal{R}^\omega$-embeddable group and $1 \leq p < \infty$. Let $I \subseteq C^*_\lambda(\Gamma)$ be a norm closed left-ideal. Let $\overline{I}^{\mathrm{wk}^*} = L(\Gamma)q$ (with the closure taken in $L(\Gamma)$). Then*

$$\dim_{\Sigma,S^p,\mathrm{mult}}(I,\Gamma) \geq \tau(q).$$

Proof. It suffices to show that the inclusion $I \subseteq L^p(L(\Gamma),\tau)q$ has dense image. By the previous Lemma, Let $q' \in \mathrm{Proj}(L(\Gamma))$ be such that

$$\overline{I}^{\|\cdot\|_p} = L^p(L(\Gamma),\tau)q'.$$

By the argument in the previous Lemma,

$$q' = \sup_{x \in I} \chi_{(0,\infty)}(|x|).$$

So it suffices to prove the following two claims.

Claim 1. If $x \in C^*_\lambda(\Gamma)$, then $\chi_{(0,\infty)}(|x|) \in \overline{I}^{\mathrm{wk}^*}$.

Claim 2. If $e, f \in \mathrm{Proj}(\overline{I}^{\mathrm{wk}^*})$, then $e \vee f \in \mathrm{Proj}(\overline{I}^{\mathrm{wk}^*})$.

For the proof of claim I, let $x = v|x|$ be the polar decomposition. By the Kaplansky Density Theorem, we can find $v_n \in C^*_\lambda(\Gamma)$ so that $\|v_n\|_\infty \leq 1$ and $\|v_n - v\|_2 \to 0$. But then $\|v_n^* x - |x|\|_2 \to 0$, so $|x| \in \overline{I}^{\mathrm{wk}^*}$. Since

$$\chi_{(\varepsilon,\infty)}(|x|) = |x|^{-1}\chi_{(\varepsilon,\infty)}(|x|)|x|,$$

we find that $\chi_{(0,\infty)}(|x|) \in \overline{I}^{\mathrm{wk}^*}$.

For the proof of claim 2, we use the formula (proved by functional calculus):

$$e \vee f = 1 - \lim_{n\to\infty}([(1-e)(1-f)(1-e)])^n$$

where the limit is in $\|\cdot\|_2$. Since $e, f \in L(\Gamma)q$, a little calculation shows that

$$1 - ([(1-e)(1-f)(1-e)])^n \in L(\Gamma)q.$$

This proves the corollary.

□

We can also handle the case $p = \infty$ if we assume a little more.

Definition 3.6.7. A $C*$-algebra A is said to be a *matricial field algebra* if there is a injective $*$-homomorphism

$$\sigma\colon A \to \frac{\{(A_n)_{n=1}^{\infty} : A_n \in M_{d(n)}(\mathbb{C}), \sup_n \|A_n\|_{\infty} < \infty\}}{\{(A_n)_{n=1}^{\infty} : A_n \in M_{d(n)}(\mathbb{C}), \sup_n \|A_n\|_{\infty} \to 0\}},$$

for some $d(n) \in \mathbb{N}$ and $d(n) \to \infty$. A sequence $\sigma_n\colon A \to M_{d_n}(\mathbb{C})$, of potentially nonmultiplicative, nonlinear maps, such that $\sigma(a)$ is the image of $(\sigma_n(a))$ is called a *norm microstates sequence.*

Theorem 3.6.8. *Let Γ be a countable discrete group. Assume that there are norm microstates $\sigma_i\colon C^*_\lambda(\Gamma) \to M_{d_i}(\mathbb{C})$ such that*

$$\operatorname{tr}(\sigma_i(x)) \to \tau(x)$$

for all $x \in \mathbb{C}[\Gamma]$. Let $I \subseteq C^*_\lambda(\Gamma)$ be a norm-closed left ideal, and let $I^{\mathrm{wk}^*} = L(\Gamma)q$, with $q \in \operatorname{Proj}(L(\Gamma))$. Then,

$$\underline{\dim}_{\Sigma,S^\infty,\mathrm{mult}}(I,\Gamma) \geq \tau(q).$$

Proof. Let

$$A = \frac{\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_{\infty} < \infty\}}{\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_{\infty} \to 0\}},$$

then our hypothesis implies that there is an isometric $*$-homomorphism

$$\sigma\colon C^*_\lambda(\Gamma) \to A,$$

such that

$$\sigma(u_g) = \pi(\sigma_1(g), \sigma_2(g), \dots)$$

where

$$\pi\colon \left\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_{\infty} < \infty\right\} \to A,$$

is the quotient map.

As before, we may extend ϕ_i to an embedding sequence

$$\psi_i\colon L(\Gamma) \to M_{d_i}(\mathbb{C}).$$

Now let $\varepsilon > 0$, and choose a finite subset $E \subseteq \Gamma, l \in \mathbb{N}$, and $c_{gj} \in \mathbb{C}$, for $(g, j) \in E \times \{1, \dots, l\}$ so that

$$\left\| q - \sum_{\substack{g \in E \\ ,1 \leq j \leq l}} c_{gj} u_g x_j \right\|_2 < \varepsilon.$$

Fix $E \subseteq F \subseteq \Gamma$ finite, $l \leq m \in \mathbb{N}, \delta > 0$. Since all injective $*$-homomorphisms defined on C^*-algebras are isometric, it is easy to see that if we define $\rho_i = \frac{\phi_i\big|_{I_{F,m}}}{\left\|\phi_i\big|_{I_{F,m}}\right\|}$, then

$$\left\| \rho_i - \phi_i\big|_{I_{F,m}} \right\| \to 0.$$

For $B \in M_{d_i}(\mathbb{C})$ define

$$T_B\colon I_{F,m} \to M_{d_i}(\mathbb{C}),$$

by

$$T_B(x) = \rho_i(x)B.$$

If $\|B\|_\infty \leq 1$, then

$$\|T_B(x)\| \leq \|B\|_\infty.$$

Further if $\|B\|_\infty \leq 1$, and $1 \leq j, k \leq m$, and $g_1, \dots, g_k \in F$, then

$$\|T_B(g_1 \cdots g_k x_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) T_B(x_j)\| \leq \|\phi_i(g_1 \cdots g_k x_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) \phi_i(x_j)\| \to 0$$

using that

$$\pi((\phi_i(g_1 \cdots g_k x_j))_{i=1}^\infty) = \pi((\sigma_i(g_1) \cdots \sigma_i(g_k) \phi_i(x_j))_{i=1}^\infty)).$$

Now suppose $V \subseteq \ell^\infty(\mathbb{N}, M_{d_i}(\mathbb{C}))$ ε-contains $\{((\rho_i(x_j)B)_{j=1}^\infty : \|B\|_\infty \leq 1\}$. Define a map $\Phi\colon \ell^\infty(\mathbb{N}, M_{d_i}(\mathbb{C})) \to L^2(M_{d_i}(\mathbb{C}), \mathrm{tr})$ by

$$\Phi(f) = \sum_{g \in E, 1 \leq j \leq l} c_{gj} \sigma_i(g) f(j),$$

then our hypotheses imply that for all large i,

$$\Phi(V) \supseteq_{3\varepsilon,\|\cdot\|_2} \{qB : B \in \mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_\infty)\}.$$

Our methods to prove Theorem 3.6.4 can be used to complete the proof.

□

3.7 Definition of ℓ^p-Dimension Using Vectors

In this section, we give a definition of the extended von Neumann dimension using vectors instead of almost equivariant operators. Thus may be conceptually simpler, as we do not have to deal with the technicalities involving changing domains inherent to the definition of $\mathrm{Hom}_\Gamma(\cdots)$. The definition is much simpler and requires fewer preliminaries as well. However, for many theoretical purposes it will still be easier to use the notion of almost equivariant operators. We will give this alternate definition after the following lemma.

Lemma 3.7.1. *Let V be a finite-dimensional Banach space, let B be a finite set, and $(v_\beta)_{\beta\in B} \in V^B$ such that $V = \mathrm{Span}\{v_\beta : \beta \in B\}$. Then for any $\eta > 0$, there is a $\delta > 0$ so that if Y is a Banach space and $(\xi_\beta)_{\beta\in B} \in Y^B$ have the property that for all $c \in \ell^1(B)$ with $\|c\|_1 \leq 1$,*

$$\left\|\sum_{\beta\in B} c(\beta)\xi_\beta\right\| \leq \delta + \left\|\sum_{\beta\in B} c(\beta)v_\beta\right\|,$$

then there is a $T\colon V \to Y$ with $\|T\| \leq 1$, such that

$$\|T(v_\beta) - \xi_\beta\| < \eta,$$

for all $j \in B$.

Proof. Let $A \subseteq B$ be such that $\{v_\beta : \beta \in A\}$ is a basis for X. For $Y, (\xi_\beta)_{\beta\in B}$, as in the statement of the Lemma let $\widetilde{T}\colon V \to Y$ be defined by

$$\widetilde{T}(v_\beta) = \xi_\beta$$

for $\beta \in A$. By finite-dimensionality, there is a $C_V > 0$ so that

$$\sum_{\beta \in A} |c_\beta| \leq C_V \left\| \sum_{j \in A} c_\beta v_\beta \right\|.$$

Thus our hypothesis implies that

$$\|\widetilde{T}\| \leq C_V \delta + 1.$$

Set $T = \frac{1}{1+C(V)\delta} \widetilde{T}$, then $\|T\| \leq 1$. For each $\alpha \in B \setminus A$ choose $a_\beta^{(\alpha)}, \beta \in A$ so that

$$v_\alpha = \sum_{\beta \in A} a_\beta^{(\alpha)} v_\beta.$$

For $\alpha \in B \setminus A$, let

$$A_\alpha = \sum_{\beta \in A} |a_\beta^{(\alpha)}|.$$

Define $c^{(\alpha)} \in \ell^1(B)$ by

$$c^{(\alpha)}(\beta) = \frac{a_\beta^{(\alpha)}}{1 + A_\alpha}, \beta \in A$$
$$c^{(\alpha)}(\alpha) = -\frac{1}{1 + A_\alpha},$$
$$c^{(\alpha)}(\beta) = 0, \beta \in B \setminus (A \cup \{k\}).$$

Then for $\alpha \in B \setminus A$, $\|c^{(\alpha)}\|_1 = 1$, and

$$\sum_{\beta \in A} c^{(\alpha)}(\beta) v_\beta = 0.$$

Thus by our hypothesis for $\alpha \in B \setminus A$,

$$\frac{1}{1 + A_\alpha} \|\xi_\alpha - \widetilde{T}(v_\alpha)\| = \left\| \sum_{\beta \in A} c^{(\alpha)}(\beta) \xi_\beta \right\| \leq \delta,$$

so

$$\|\xi_\alpha - \widetilde{T}(x_\alpha)\| \leq (1 + A_\alpha)\delta.$$

For all $\beta \in B$,

$$\|\widetilde{T}(v_\beta) - T(v_\beta)\| = \left| 1 - \frac{1}{1 + \delta C_V} \right| \|\widetilde{T}(v_\beta)\| \leq \delta C_V \|v_\beta\|.$$

Set

$$M = \max\left(\max_{\alpha\in B\setminus A} 1 + A_\alpha, \max_{\beta\in B} C_V\|v_\beta\|\right).$$

Then M does not depend upon Y, ε and for all $\beta \in B$,

$$\|T(v_\beta) - \xi_\beta\| \leq 2M\delta,$$

so if $\delta < \frac{\eta}{2M}$, we are done.

□

Definition 3.7.2. Let X be a Banach space with a uniformly bounded action of a countable discrete group Γ and $\sigma_i\colon \Gamma \to \mathrm{Isom}(X_i)$ with X_i finite-dimensional. We let $\mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i)$ be all m-tuples $(\xi_j)_{j=1}^m$ of vectors in X such that for all $(c_{g_1,\dots,g_l,j})_{1\leq l,j\leq m, g_1,\dots,g_l\in F}$ with

$$\sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} |c_{g_1,\dots,g_m,j}| \leq 1,$$

we have

$$\left\|\sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j.l\leq m}} c_{g_1,\dots,g_l,j}\sigma_i(g_1)\cdots\sigma_i(g_m)\xi_j\right\| \leq \delta + \left\|\sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} c_{g_1,\dots,g_l,j}g_1\cdots g_l x_j\right\|.$$

Set

$$\mathrm{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i\to\infty} \frac{1}{\dim X_i} d_\varepsilon(\mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{X_i}),$$

$$\mathrm{vdim}_\Sigma(S, \varepsilon, \rho) = \inf_{F,m,\delta} \mathrm{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho),$$

$$\mathrm{vdim}_\Sigma(S, \rho) = \sup_{\varepsilon>0} \mathrm{vdim}_\Sigma(S, \varepsilon, \rho).$$

Proposition 3.7.3. *Let X be a Banach space with a uniformly bounded action of a countable discrete group Γ and $\sigma_i\colon \Gamma \to \mathrm{Isom}(X_i)$ with X_i finite-dimensional. Then for any dynamically generating sequence S, and any product norm ρ,*

$$\dim_\Sigma(X, \Gamma) = \mathrm{vdim}_\Sigma(S, \rho).$$

Proof. Let $S = (x_j)_{j=1}^{\infty}$. Fix $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$. Suppose that $T \in \operatorname{Hom}_{\Gamma}(S, F, m, \delta, \sigma_i)$ and set $\xi_j = T(x_j)$. Then for all $(c_{g_1,\dots,g_l,j})_{g_1,\dots,g_l \in F, 1\leq j,l\leq m}$ with

$$\sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} |c_{g_1,\dots,g_l,j}| \leq 1,$$

we have

$$\left\| \sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} c_{g_1,\dots,g_m,j}\sigma_i(g_1)\cdots\sigma_i(g_l)\xi_j \right\| \leq \delta + \left\| T\left(\sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} c_{g_1,\dots,g_l,j} g_1\cdots g_l \xi_j \right) \right\|$$
$$\leq \delta + \left\| \sum_{\substack{g_1,\dots,g_l\in F\\ 1\leq j,l\leq m}} c_{g_1,\dots,g_l,j} g_1\cdots g_l \xi_j \right\|.$$

So $(\xi_j)_{j=1}^m \in \operatorname{Vect}_{\Gamma}(S, F, m, \delta, \sigma_i)$ and $\operatorname{vdim} \leq \dim$.

For the opposite inequality, let $\varepsilon > 0$, and let $M = \sup_j \|x_j\|$. Since ρ is a product norm, we may find an $N \in \mathbb{N}$, and a $\kappa > 0$ so that if $f \in \ell^{\infty}(\mathbb{N})$ and $\|f\|_{\infty} \leq M$, and

$$\max_{1\leq j\leq N} |f(j)| < \kappa,$$

then

$$\rho(f) < \varepsilon.$$

Fix $e \in F \subseteq \Gamma$ finite and $m \in \mathbb{N}$ with $m \geq N$. Let $\delta' > 0$ be sufficiently small depending upon κ, in a manner to be determined later. Set

$$B = \bigsqcup_{l=1}^{m} \{(g_1, \dots, g_l, j) : g_1, \dots, g_l \in F, 1 \leq j \leq m\},$$

$$V = X_{F,m},$$

$$v_{\beta} = g_1\cdots g_l x_j, \text{ if } \beta = (g_1, \dots, g_l, j) \in B,$$

$$\eta = \delta'.$$

Let $\delta > 0$ be as in the preceding lemma for this $B, V, (v_{\beta})_{\beta\in B}, \eta$. If $(\xi_j)_{j=1}^m \in \operatorname{Vect}_{\Gamma}(S, F, m, \delta, \sigma_i)$, then by the preceding lemma, we can find a $T\colon X_{F,m} \to X_i$ with $\|T\| \leq 1$ and

$$\|T(g_1\cdots g_l x_j) - \sigma_i(g_1)\cdots\sigma_i(g_l)\xi_j\| < \delta',$$

for all $g_1, \dots, g_l \in F, 1 \leq j, l \leq m$. Thus for all $1 \leq j, l \leq m, g_1, \dots, g_l \in F$,

$$\|T(g_1 \cdots g_m x_j) - \sigma_i(g_1) \cdots \sigma_i(g_m) T(x_j)\| < 2\delta'.$$

Thus $T \in \mathrm{Hom}_\Gamma(S, F, m, 2\delta', \sigma_i)$, and

$$\max_{1 \leq j \leq m} \|T(x_j) - \xi_j\| < \delta',$$

since $e \in F$. So if we choose $\delta' < \kappa$, then since $m \geq N$, our choice of κ implies

$$\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i) \subseteq_{\varepsilon, \rho_{X_i}} \mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i),$$

so

$$d_{2\varepsilon}(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i) \leq d_\varepsilon(\mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i)).$$

Taking limits in the appropriate order, we see that $\dim \leq \operatorname{vdim}$.

$\square$

3.8 ℓ^p-Betti Numbers of Free Groups

Let X be a CW complex and let $\Delta_n(X)$ be the collection of n-simplices of X. Suppose that Γ acts properly on X with compact quotient, preserving the simplicial structure. For $v_0, \dots, v_n \in X$, let

$$[v_0, v_1, \dots, v_n]$$

be the simplex spanned by $v_0, \dots, v_n$. Let

$$V_n(X) = \{(v_0, \dots, v_n) \in X : [v_0, \dots, v_n] \in \Delta_n\}.$$

We abuse notation and let $\ell^p(\Delta_n(X))$ for $1 \leq p \leq \infty$ be all functions $f \colon V_n(X) \to \mathbb{C}$ such that

$$f(v_{\sigma(0)}, \dots, v_{\sigma(n)}) = (\operatorname{sgn} \sigma) f(v_0, \dots, v_n) \text{ for } \sigma \in \mathrm{Sym}(\{0, \dots, n\})$$

$$\sum_{[v_0, \dots, v_n] \in \Delta_n(X)} |f(v_0, \dots, v_n)|^p < \infty, \text{ for } p < \infty$$

$$\sup_{[v_0,\ldots,v_n]\in\Delta_n(X)} |f(v_0,\ldots,v_n)| < \infty \ p=\infty.$$

By our antisymmetry condition the above sum is unchanged if we use a different representative for $[v_0,\ldots,v_n]$. On $\ell^p(\Delta_n(X))$ we use the norm

$$\|f\|_p^p = \sum_{v\in\Delta_n(X)} |f(v_0,\ldots,v_n)|^p, \text{ for } p<\infty$$

$$\|f\|_\infty = \sup_{[v_0,\ldots,v_n]\in\Delta_n(X)} |f(v_0,\ldots,v_n)|.$$

Define the discrete differential $\delta\colon \ell^p(\Delta_{n-1}(X)) \to \ell^p(\Delta_n(X))$ by

$$(\delta f)(v_0,\ldots,v_n) = \sum_{j=0}^{n} (-1)^j f(v_0,\ldots,\widehat{v_j},\ldots,v_n),$$

where the hat indicates a term omitted, note that δf satisfies the appropriate antisymmetry condition. Define the n^{th} ℓ^p-cohomology space of X by

$$H^n_{\ell^p}(X) = \frac{\ker(\delta)\cap \ell^p(\Delta_n(X))}{\overline{\delta(\ell^p(\Delta_{n-1}(X))}}.$$

We define the ℓ^p-Betti numbers of X with respect to Γ by

$$\beta^{(p)}_{\Sigma,n}(X,\Gamma) = \dim_{\Sigma,\ell^p}(H^n_{\ell^p}(X),\Gamma).$$

It is known that if X is contractible and $\pi_1(X/\Gamma)\cong\Gamma$, then the ℓ^p-cohomology space only depends upon Γ, (see [14] page 219). If Γ is sofic, we may use ℓ^p-dimension to define

$$H^n_{\ell^p}(\Gamma) = H^n_{\ell^p}(X,\Gamma),$$

$$\beta^{(p)}_{\Sigma,n}(\Gamma) = \beta^{(p)}_{\Sigma,n}(X,\Gamma),$$

for such X. The definition above for $p=2$ goes back to Atiyah in [1].Attaching a number to ℓ^p-cohomology (or homology), requires some dimension theory associated to ℓ^p-spaces. Since we have done this in [16], the preceding definition of ℓ^p-Betti numbers is a new definition.

We also consider ℓ^p-homology. Define $\partial\colon \ell^p(\Delta_n(X)) \to \ell^p(\Delta_{n-1}(X))$ by

$$\partial f(v_0,\ldots,v_{n-1}) = \sum_{x:[v_0,\ldots,v_{n-1},x]\in\Delta_n(X)} f(v_0,\ldots,v_{n-1},x).$$

We use T^t for the Banach space adjoint of a bounded $T\colon X \to Y$ between Banach spaces X, Y. By direct computation

$$(\partial\colon \ell^{p'}(\Delta_n(X)) \to \ell^{p'}(\Delta_{n-1}(X))) = (\delta\colon \ell^p(\Delta_{n-1}(X)) \to \ell^p(\Delta_n(X)))^t,$$

when $\frac{1}{p} + \frac{1}{p'} = 1$. Define the ℓ^p-homology of X by

$$H_n^{\ell^p}(X) = \frac{\ker(\partial) \cap \ell^p(\Delta_n(X))}{\overline{\partial(\ell^p(\Delta_{n+1}(X))}}.$$

We shall be interested in the ℓ^p-Betti numbers of free groups. Fix $n \in \mathbb{N}$ and consider the free group $\mathbb{F}_n$ on n letters $a_1, \dots, a_n$. Let G be the Cayley graph of $\mathbb{F}_n$ with respect to $a_1, \dots, a_n$, we regard the edges of G as oriented. There is a natural 1-dimensional CW complex X associated to G, whose 0-simplices are the vertices of G, and whose 1-simplices are the edges of G, and whose attaching maps are determined by incidence of edges in the natural way. Then X is contractible, since G is a tree. Also $\pi_1(X/\mathbb{F}_n) \cong \mathbb{F}_n$, so the ℓ^p-cohomology of G is the ℓ^p-cohomology of $\mathbb{F}_n$. Let $E(\mathbb{F}_n)$ denote the set of edges of $\mathbb{F}_n$. Then $\ell^p(E(\mathbb{F}_n))$ as defined above is the set of all functions $f\colon E(\mathbb{F}_n) \to \mathbb{C}$ such that

$$f(x, s) = -f(s, x) \text{ if } (s, x) \in E(\mathbb{F}_n),$$

$$\sum_{j=1}^{n} \sum_{x \in \mathbb{F}_n} |f(x, xa_j)|^p < \infty$$

with the norm

$$\|f\|_p^p = \sum_{j=1}^{n} \sum_{x \in \mathbb{F}_n} |f(x, xa_j)|^p.$$

Note that this is indeed a norm on $\ell^p(E(\mathbb{F}_n))$, and that $\mathbb{F}_n$ acts isometrically on $\ell^p(E(\mathbb{F}_n))$ by left translation. Also $\ell^p(E(\mathbb{F}_n))$ is isomorphic to $\ell^p(\mathbb{F}_n)$ with respect to this action. If $(x, s) \in E(\mathbb{F}_n)$, we let $\mathcal{E}_{(x,s)}$ be the function on $E(\mathbb{F}_n)$ such that

$$\mathcal{E}_{(x,s)}(y, t) = 0 \text{ if } \{x, s\} \neq \{y, t\}$$

$$\mathcal{E}_{(x,s)}(x, s) = 1$$

$$\mathcal{E}_{(x,s)}(s, x) = -1.$$

We think of $\mathcal{E}_{(x,s)}$ as representing the edge going from x to s.

The discrete differential $\delta\colon \ell^p(\mathbb{F}_n) \to \ell^p(E(\mathbb{F}_n))$ we defined above is given by

$$(\delta f)(x,s) = f(s) - f(x) \ (x,s) \in E(\mathbb{F}_n)).$$

The corresponding ℓ^p-cohomology space is given by

$$H^1_{\ell^p}(\mathbb{F}_n) = \ell^p(E(\mathbb{F}_n))/\overline{\delta(\ell^p(\mathbb{F}_n)}.$$

Also, $\partial\colon \ell^p(E(\mathbb{F}_n)) \to \ell^p(\mathbb{F}_n)$ is given by

$$(\partial f)(x) = \sum_{j=1}^{n} f(x, xa_j) - \sum_{j=1}^{n} f(xa_j^{-1}, x).$$

In this section, we compute the ℓ^p-Betti numbers

$$\beta^{(p)}_{\Sigma,1}(\mathbb{F}_n),$$

for $1 \leq p \leq 2$.

Let Γ be a countable discrete group, we define $\rho\colon \Gamma \to B(\ell^p(\Gamma))$ by

$$(\rho(g)f)(x) = f(xg^{-1}).$$

Lemma 3.8.1. *Let $n \in \mathbb{N}$, with $n \geq 2$. Fix $1 \leq p < \infty$. There is a $C > 0$ so that*

$$\|\delta f\|_p \geq C\|f\|_p,$$

for all $f \in \ell^p(\mathbb{F}_n)$. In particular, the image of δ is closed.

Proof. Assume the lemma is false, then we can find $f_k \in \ell^p(\mathbb{F}_n)$, with $\|f_k\|_p = 1$ and

$$\|\delta f_k\|_p \to 0.$$

By direct computation

$$\|\delta f_k\|_p^p = \sum_{j=1}^{n} \|\rho(a_j^{-1})f_k - f_k\|_p^p,$$

where $a_1, \ldots, a_n$ are the free generators of $\mathbb{F}_n$. Thus

$$\|\rho(a_j^{-1})f_k - f_k\|_p \to 0.$$

Since $\{a_1, \ldots, a_n\}$ generate $\mathbb{F}_n$, it follows that

$$\|\rho(x)f_k - f_k\|_1 \to 0$$

for all $x \in \mathbb{F}_n$. By Theorem B.1.2 (iii), this implies that $\mathbb{F}_n$ is amenable. By the argument following Theorem B.1.2 we know that $\mathbb{F}_n$ is not amenable, so we have a contradiction.

□

Lemma 3.8.2. *Fix $n \in \mathbb{N}$, $1 \leq p < \infty$. Then the set of all images of the elements $\mathcal{E}_{(e,a_1)}, \ldots,$ $\mathcal{E}_{(e,a_{n-1})}$ are dynamically generating for $H^1_{\ell^p}(\mathbb{F}_n)$.*

Proof. It suffices to show that

$$W = \delta(\ell^p(\mathbb{F}_n)) + \operatorname{Span}\{\mathcal{E}_{(s,sa_j)} : s \in \mathbb{F}_n, 1 \leq j \leq n-1\}$$

is norm dense in $\ell^p(E(\mathbb{F}_n))$. It is enough to show that

$$\mathcal{E}_{(e,a_n)} \in \overline{W}^{\|\cdot\|}.$$

By convexity it is enough to show that $\mathcal{E}_{(e,a_n)}$ is in the weak closure of W.

We shall prove by induction on k that

$$\mathcal{E}_{(e,a_n)} \equiv \mathcal{E}_{(a_n^k, a_n^{k+1})} \mod W.$$

This is enough since

$$\mathcal{E}_{(a_n^k, a_n^{k+1})} \to 0$$

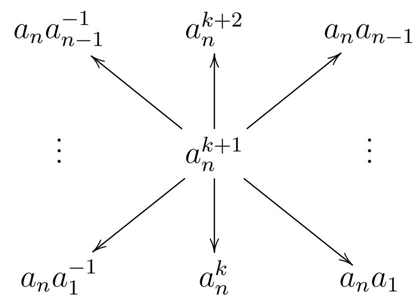
weakly.

The base case $k = 0$ is trivial, so assume the result true for some k. Then

$$\begin{aligned}\mathcal{E}_{(a_n^k,a_n^{k+1})} - \delta(\chi_{\{a_n^{k+1}\}}) &= \sum_{j=1}^{n} \mathcal{E}_{(a_n^{k+1},a_n^{k+1}a_j)} + \sum_{j=1}^{n-1} \mathcal{E}_{(a_n^{k+1},a_n^{k+1}a_j^{-1})} \\ &= \mathcal{E}_{(a_n^{k+1},a_n^{k+2})} + \sum_{j=1}^{n-1} a_n^{k+1}\mathcal{E}_{(e,a_j)} - \sum_{j=1}^{n-1} a_n^{k+1}a_j^{-1}\mathcal{E}_{(e,a_j)} \\ &= \mathcal{E}_{(a_n^{k+1},a_n^{k+2})}.\end{aligned}$$

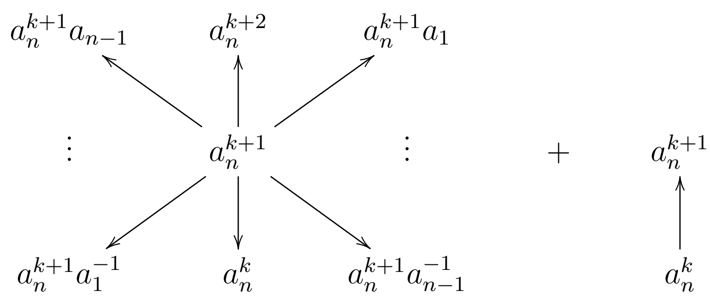
Here is a graphical explanation of the above calculation. If we think of the elements of $\ell^p(E(\mathbb{F}_n))$ as formal sums of oriented edges, then $-\delta(\chi_{a_n^{k+1}})$ is a "source" at a_n^{k+1}. It is the sum of all edges adjacent to a_n^{k+1}, directed away from a_n^{k+1}. Below is a graphical representation of $-\delta(\chi_{a_n^{k+1}})$:

$-\delta(\chi_{a_n^{k+1}})$ =

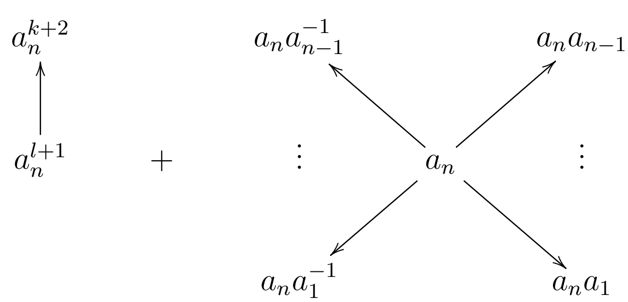


The above computation can be phrased as follows:

$$-\delta(\chi_{a_n^{k+1}}) + \mathcal{E}_{(a_n^k, a_n^{k+1})} =$$



=



and the second term on the right-hand side is easily seen to be in the span of translates of $\mathcal{E}_{(e,a_j)}, j = 1, \ldots, n-1$. This completes the induction step.

□

We shall prove the analogous claim for ℓ^p-homology of free groups, but we need a few preliminary results. These next few results must be well known, but we include proofs for completeness.

Lemma 3.8.3. *Let Γ be a non-amenable group with finite-generating set S. Let $A\colon \ell^p(\Gamma) \to \ell^p(\Gamma)$ be defined by*

$$A = \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s),$$

then for $1 < p < \infty$, there is a constant $C_p < 1$ so that $\|Af\|_p < C_p \|f\|_p$.

Proof. We use

$$\|A\|_{\ell^p \to \ell^p}$$

for the norm of A as an operator from $\ell^p(\Gamma) \to \ell^p(\Gamma)$. We know $\|A\|_{\ell^2 \to \ell^2} < 1$ from the non-amenability of Γ (see [3] Theorem 2.6.8 (8)). Since $\|A\|_{\ell^\infty \to \ell^\infty} \leq 1$, and $\|A\|_{\ell^1 \to \ell^1} \leq 1$, the lemma follows by interpolation. □

Lemma 3.8.4. *Let $n \in \mathbb{N}$ with $n \geq 2$. For $1 < p < \infty$, the operator $\partial \circ \delta\colon \ell^p(\mathbb{F}_n) \to \ell^p(\mathbb{F}_n)$, is invertible.*

Proof. Let $a_1, \dots, a_n$ be free generators for $\mathbb{F}_n$, and let $S = \{a_1, \dots, a_n\}$. We have that

$$\partial(\delta f)(x) = \sum_{s \in S \cup S^{-1}} f(x) - f(xs) = |S \cup S^{-1}| \left(f(x) - \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s) f(x) \right).$$

By the previous lemma,

$$\left\| \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s) \right\|_{\ell^p \to \ell^p} < 1,$$

for $1 < p < \infty$, so this proves that $\partial(\delta)$ is invertible for $1 < p < \infty$. □

For the next corollary we use the following notation: if X, Y, Z are Banach spaces with $Y, Z \subseteq X$, we use $X = Y \oplus Z$ to mean $Y \cap Z = \{0\}, Y + Z = X$.

Corollary 3.8.5. *Let $n \in \mathbb{N}$ with $n \geq 2$. For $1 < p < \infty$, we have the following Hodge Decomposition:*

$$\ell^p(E(\mathbb{F}_n)) = \ker(\partial\colon \ell^p(E(\mathbb{F}_n)) \to \ell^p(\Gamma)) + \delta(\ell^p(\mathbb{F}_n)).$$

Proof. By 3.8.1, $\delta(\ell^p(\mathbb{F}_n))$ is closed in $\ell^p(E(\mathbb{F}_n))$. It is clear that $\ker(\partial\colon \ell^p(E(\mathbb{F}_n)) \to \ell^p(\Gamma))$ is closed in $\ell^p(E(\mathbb{F}_n))$. If $f \in \ker(\partial\colon \ell^p(E(\mathbb{F}_n)) \to \ell^p(\mathbb{F}_n)) \cap \delta(\ell^p(\mathbb{F}_n))$ write $f = \delta(g)$, then

$$0 = \partial(f) = \partial(\delta(g)).$$

By the preceding lemma we have that $g = 0$.

If $f \in \ell^p(E(\Gamma))$, then by the preceding lemma we can find a unique g so that $\partial(f) = \partial(\delta(g))$. Then $f - \delta(g) \in \ker(\partial)$, and

$$f = f - \delta(g) + \delta(g).$$

□

Proposition 3.8.6. *Let $n \in \mathbb{N}$, and $1 < p < \infty$. Then $H_1^{\ell^p}(\mathbb{F}_n)$ can be generated by $n-1$ elements.*

Proof. The claim for $n = 1$ is clear since $H^1_{\ell^p}(\mathbb{Z}) = 0$. First, we show how to reduce to the case $n = 2$. Let $n > 2$, and let $a_1, \ldots, a_n$ be the generators of $\mathbb{F}_n$. Consider the injective homomorphisms $\phi_j\colon \mathbb{F}_2 \to \mathbb{F}_n$ for $1 \leq j \leq n-1$ given by $\phi_j(a_i) = a_{i+j}$. Let f be an element in $\ell^p(E(\mathbb{F}_2))$ so that $\mathrm{Span}(\mathbb{F}_2 f)$ is dense in $\ker(\partial) \cap \ell^p(E(\mathbb{F}_2))$. Let $f_j \in \ell^p(E(\mathbb{F}_n))$ be the element defined by

$$f_j(x,y) = \begin{cases} 0, \text{ if one of } x, y \notin \phi_j(\mathbb{F}_2) \\ f(\phi_j^{-1}(x), \phi_j^{-1}(y)), \text{ otherwise.} \end{cases}$$

Then $f_j \in \ker(\partial)$. It is easy to see from the preceding corollary and the fact that f generates $\ker(\partial) \cap \ell^p(E(\mathbb{F}_2))$, that

$$\mathcal{E}_{(e,a_j)} \in \ker(\partial) + \delta(\ell^p(\mathbb{F}_n)).$$

Again by the preceding corollary we find that $f_1, \ldots, f_{n-1}$ generate $\ker(\partial)$. Thus it suffices to handle the case $n = 2$.

We now concentrate on the case $n = 2$, and we use a, b for the generators of $\mathbb{F}_2$. Let $f\colon E(\mathbb{F}_2) \to \mathbb{R}$ defined by the following inductive procedure. Set

$$f_1 = \mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(a^{-1},e)} + \mathcal{E}_{(b^{-1},e)}.$$

Having constructed $f_1, \ldots, f_n$ so that f_j is supported on the pairs of edges which have word length at most j, define f_{n+1} as follows. For each word w of length n, let e_1, e_2, e_3 be the three oriented edges which have their terminal vertex w and the initial vertex a word of length $n+1$, and let e be the oriented edge which has its initial vertex w and its terminal vertex a word of length $n-1$. Define for $j = 1, 2, 3$

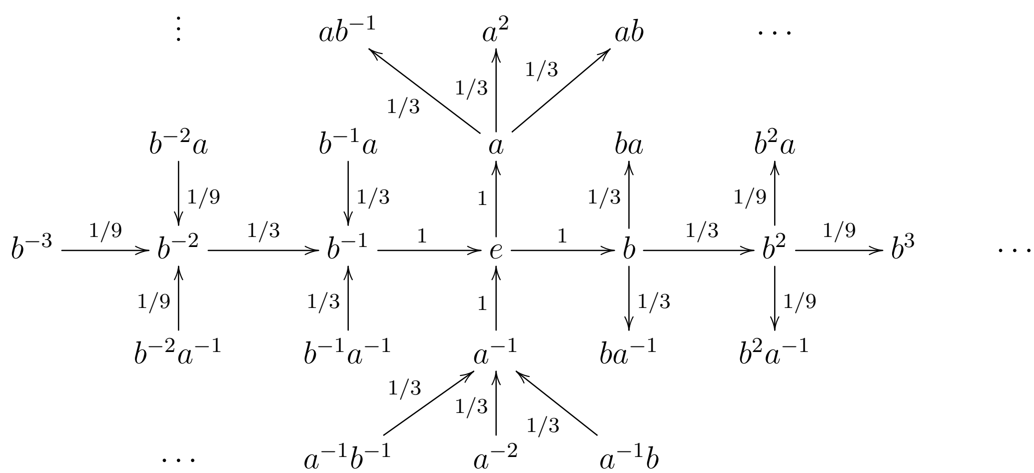
$$f_{n+1}(e_j) = \frac{1}{3} f_n(e),$$

and define

$$f_{n+1}(e) = f_n(e)$$

if both vertices of e have length at most n. It is easy to see that the f_n's as constructed above converge pointwise to a function f in $\ell^p(\mathbb{E}(F_2)) \cap \ker(\partial)$ for $1 < p \leq \infty$.

The function f is pictured below:



Set $V = \overline{\mathrm{Span}(\mathbb{F}_2 f) + \delta(\ell^p(\mathbb{F}_2))}^{\mathrm{wk}} = \overline{\mathrm{Span}(\mathbb{F}_2 f) + \delta(\ell^p(\mathbb{F}_2))}^{\|\cdot\|}$.

To show that f generates $\ker(\partial)$ it suffices, by the preceding corollary to show that

$$\mathcal{E}_{(e,a_1)}, \mathcal{E}_{(e,a_2)} \in V.$$

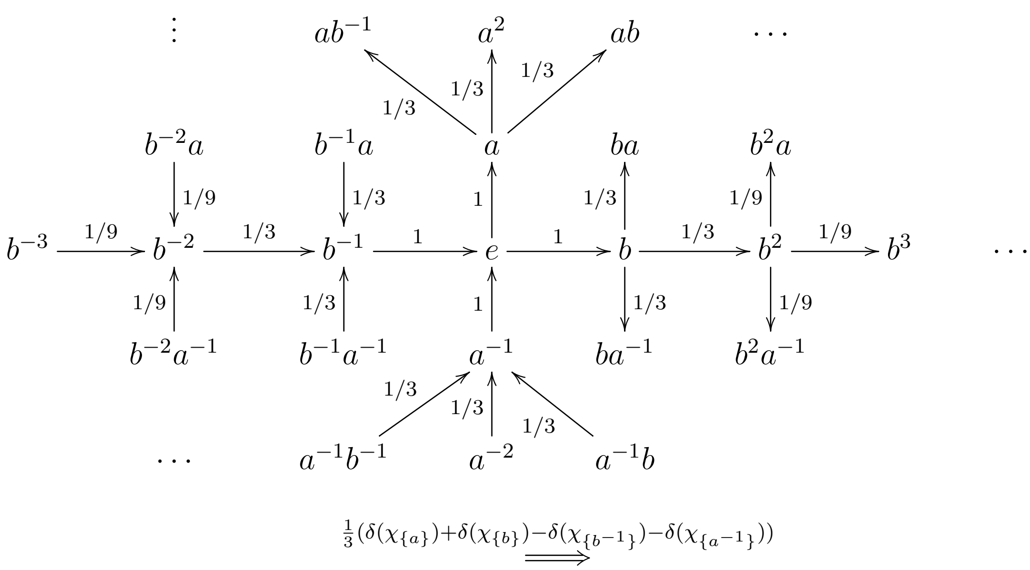
Let $B_n = \{(x, y) \in G : \|x\|, \|y\| \leq n\}$. For $n \geq 0$, let $g_n \colon E(\mathbb{F}_n) \to \mathbb{C}$, be the function defined

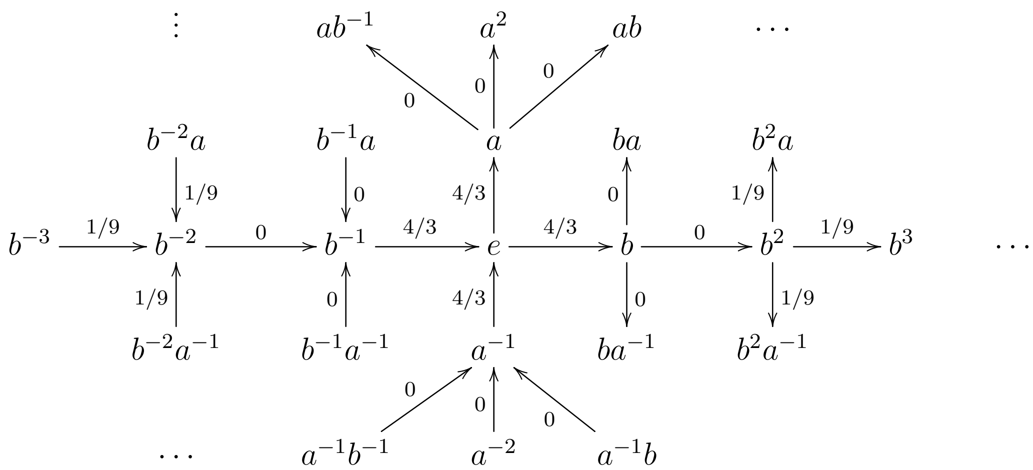
by

$$\chi_{B_n} g_n = \left(\sum_{k=0}^{n-1} (1/3)^n\right) \left(\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(a^{-1},e)} + \mathcal{E}_{(b^{-1},e)}\right),$$

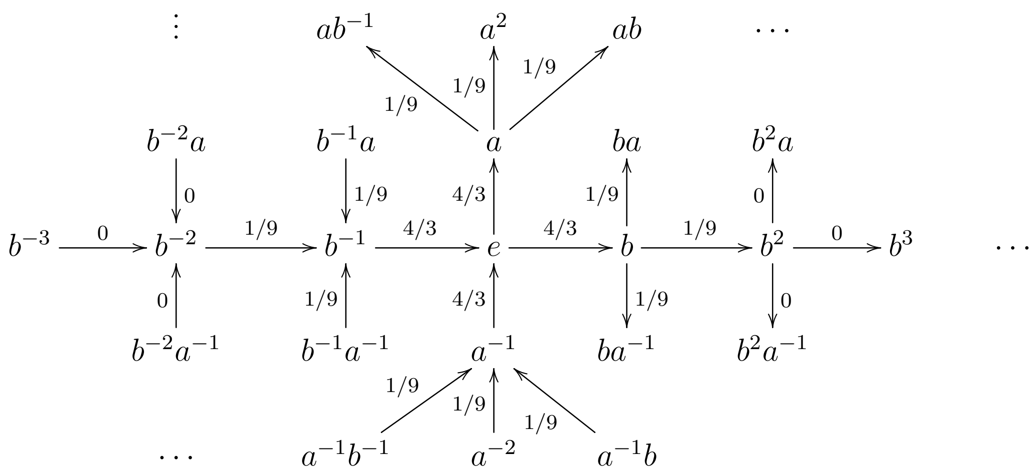
$$(1-\chi_{B_n}) g_n = (1-\chi_{B_n}) f,$$

we first show that $g_n \in \mathrm{Span}(\mathbb{F}_2 f) + \delta(\ell^p(\mathbb{F}_2))$, for all n. We prove this by induction on n, the case $n = 1$ being clear since $g_1 = f$. Suppose the claim true for some n. Then for each word w of length n, we can add either $(1/3)^n \delta(\chi_{\{w\}})$, or $-(1/3)^n \delta(\chi_{\{w\}})$, to f_n to make the value on every edge from w to a word of length $n+1$ zero. This now adds a value of $\pm(1/3)^n$ to every edge going from a word of length n to a word of length $n-1$. Now repeat for every word of length $n-1$: add on $\pm(1/3)^n \delta(\chi_{\{w\}})$ for every word w of length $n-1$ to force a value of 0 on every edge going from a word of length $n-1$ to a word of length n. Repeating this inductively until we get to words of length 1, we find by construction of f that $g_n \in \mathrm{Span}(\mathbb{F}_2 f) + \delta(\ell^p(\mathbb{F}_2))$. The first two steps of this process are pictured below:

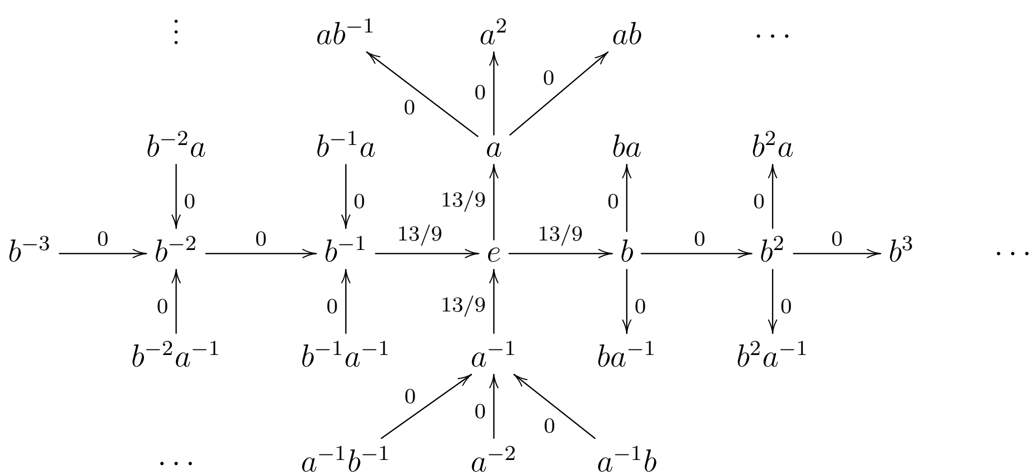




$$\xRightarrow{\frac{1}{9}(\delta(\chi_{\{b^2\}}-\delta(\chi_{\{b^{-2}\}})+\delta(\chi_{\{a^2\}})+\cdots)}$$



$$\xRightarrow{\frac{1}{9}(\delta(\chi_{\{a\}})+\delta(\chi_{\{b\}})-\delta(\chi_{\{b^{-1}\}})-\delta(\chi_{\{a^{-1}\}}))}$$



Since $\sup_n \|g_n\|_p < \infty$ we find that g_n converges weakly to

$$\frac{3}{2}(\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)}).$$

Rescaling we find that

$$\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)} \in V.$$

By adding $\pm\delta(\chi_{\{e\}})$ and scaling we find that

$$\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} \in V,$$

$$\mathcal{E}_{(e,b^{-1})} + \mathcal{E}_{(e,a^{-1})} \in V.$$

Inductively, we now see that

$$\mathcal{E}_{(e,a)} + \mathcal{E}_{((ba^{-1})^{n-1}b,(ba^{-1})^n)} \in V,$$

and taking weak limits proves that

$$\mathcal{E}_{(e,a)} \in V.$$

Subtracting $\mathcal{E}_{(e,a)}$ from $\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)}$ we find that

$$\mathcal{E}_{(e,a)}, \mathcal{E}_{(e,b)} \in V.$$

By $\mathbb{F}_2$-invariance that $V = \ell^p(E(\mathbb{F}_2))$, this completes the proof.

□

Theorem 3.8.7. *Fix $n \in \mathbb{N}$, and a sofic approximation Σ.*

(a) The dimension of the ℓ^p-cohomology groups of $\mathbb{F}_n$ satisfy

$$\dim_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n),\mathbb{F}_n) = \underline{\dim}_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n),\mathbb{F}_n) = n-1, \text{ for } 1 \leq p \leq 2,$$

$$H^m_{\ell^p}(\mathbb{F}_n) = \{0\} \text{ for } m \geq 2.$$

(b) The dimension of the ℓ^p-homology groups of $\mathbb{F}_n$ satisfy:

$$\dim_{\Sigma,\ell^p}(H_1^{\ell^p}(\mathbb{F}_n),\mathbb{F}_n) = \underline{\dim}_{\Sigma,\ell^p}(H_1^{\ell^p}(\mathbb{F}_n),\mathbb{F}_n) = n-1, \text{ for } 1 < p < 2$$

$$H_1^{\ell^1}(\mathbb{F}_n) = \ker(\partial) \cap \ell^1(E(\mathbb{F}_n)) = \{0\}.$$

$$H_m^{\ell^p}(\mathbb{F}_n) = 0 \textit{ for } m \geq 2.$$

Proof. The statements about higher-dimensional homology or cohomology are clear, since we know that the Cayley graph of $\mathbb{F}_n$ is contractible and one-dimensional.

Since the image of δ is closed, the sequence

$$0 \longrightarrow \ell^p(\mathbb{F}_n) \xrightarrow{\delta} \ell^p(E(\mathbb{F}_n)) \longrightarrow H^1_{\ell^p}(\mathbb{F}_n) \longrightarrow 0$$

is exact. Subadditivity under exact sequences, and the computation for ℓ^p-spaces implies that

$$\begin{aligned} n &= \underline{\dim}_{\Sigma,\ell^p}(\ell^p(E(\mathbb{F}_n)), \mathbb{F}_n) \\ &\leq \underline{\dim}_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n), \mathbb{F}_n) + \dim_{\Sigma,\ell^p}(\ell^p(\mathbb{F}_n) \\ &= \underline{\dim}_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n), \mathbb{F}_n) + 1. \end{aligned}$$

Thus

$$\underline{\dim}_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1.$$

On the other hand, by the Lemma 3.8.2, $H^1_{\ell^p}(\mathbb{F}_n)$ can be generated by $n-1$ elements, so

$$\dim_{\Sigma,\ell^p}(H^1_{\ell^p}(\mathbb{F}_n), \mathbb{F}_n) \leq n - 1,$$

which proves the first claim.

For the second claim, by surjectivity of ∂ for $1 < p \leq 2$, the sequence

$$0 \longrightarrow H_1^{\ell^p}(\mathbb{F}_n) \longrightarrow \ell^p(E(\mathbb{F}_n)) \xrightarrow{\partial} \ell^p(\mathbb{F}_n) \longrightarrow 0,$$

is exact. As in the first half this implies that

$$\underline{\dim}_{\Sigma,\ell^p}(H_1^{\ell^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1,$$

for $1 < p \leq 2$. The upper bound for $1 < p \leq 2$ also holds by the preceding proposition.

We turn to the last claim. If $x \in \mathbb{F}_n$, because the Cayley graph of $\mathbb{F}_n$ is a tree we can define γ_x to be the unique geodesic path from e to x. Let $|x| = d(x, e)$, and define

$$A \colon \mathbb{C}^{E(\mathbb{F}_n)} \to \mathbb{C}^{\mathbb{F}_n}$$

by

$$(Af)(x) = \sum_{j=1}^{|x|} f(\gamma_x(j-1), \gamma_x(j)),$$

note that $\delta(Af) = f$. A direct computation verifies that $A(\mathcal{E}_{(x,xa_j)}) \in \ell^\infty(\mathbb{F}_n)$, thus $\delta(\ell^\infty(\mathbb{F}_n))$ is weak* dense in $\ell^\infty(E(\mathbb{F}_n))$. By duality $\ker(\partial) \cap \ell^1(E(\mathbb{F}_n)) = \{0\}$, this completes the proof.

□

CHAPTER 4

Extended von Neumann Dimension for Equivalence Relations

Our goal in this section is to follow the methods in the group case, and introduce an extended version of von Neumann dimension for representations of a discrete, measure-preserving, sofic equivalence relation. Similar to the group case, this dimension is decreasing under equivariant maps with dense image, and in particular is an isomorphism invariant. We compute dimensions of $L^p(\mathcal{R},\overline{\mu})^{\oplus n}$ for $1 \leq p \leq 2$. We will define a upper and lower notions of ℓ^p-dimension for sofic equivalence relations, denoted $\dim_{\Sigma,\ell^p}(V,\mathcal{R}), \underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R})$ (here Σ is a sofic approximation). This extended von Neumann dimension shares some of the usual properties of von Neumann dimension, (it is an interesting problem in general to decide which properties carry over and which do not):

Property 1: $\dim_{\Sigma,\ell^p}(W,\mathcal{R}) \leq \dim_{\Sigma,\ell^p}(V,\mathcal{R})$ if there is a $\mathcal{R}$-equivariant bounded map $W \to V$ with dense image and the same for $\underline{\dim}$,

Property 2: $\mu(A)\dim_{\Sigma,\ell^p}(\operatorname{Id}_A V,\mathcal{R}_A) = \dim_{\Sigma,\ell^p}(V,\mathcal{R})$and the same for $\underline{\dim}$

Property 3: $\dim_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \dim_{\Sigma,\ell^p}(W,\mathcal{R}) + \dim_{\Sigma,\ell^p}(V/W,\mathcal{R})$, if $W \subseteq V$ is a closed $\mathcal{R}$-invariant subspace.

Property 4: $\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \underline{\dim}_{\Sigma,\ell^p}(W,\mathcal{R}) + \dim_{\Sigma,\ell^p}(V/W,\mathcal{R})$, if $W \subseteq V$ is a closed $\mathcal{R}$-invariant subspace.

Property 5: $\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \dim_{\Sigma,\ell^p}(W,\mathcal{R}) + \underline{\dim}_{\Sigma,\ell^p}(V/W,\mathcal{R})$, if $W \subseteq V$ is a closed $\mathcal{R}$-invariant subspace.

Property 6: $\underline{\dim}_{\Sigma,\ell^2}(H,\mathcal{R}) = \dim_{\Sigma,\ell^2}(H,\mathcal{R}) = \dim_{L(\Gamma)} H$ if $H \subseteq \ell^2(\mathbb{N}, L^2(\mathcal{R},\overline{\mu}))$ is a closed $\mathcal{R}$-invariant subspace.

Property 7: $\underline{\dim}_{\Sigma,\ell^p}(L^p(\mathcal{R},\overline{\mu})^{\oplus n},\mathcal{R}) = \dim_{\Sigma,\ell^p}(L^p(\mathcal{R},\overline{\mu})^{\oplus n},\mathcal{R}) = n$ for $1 \leq p \leq 2$.

In Section 4.6, if $\mathcal{R}$ is a sofic equivalence relation with sofic approximation, which satisfies a certain "finite presentation" assumption, we define a number $c^{(p)}_{1,\Sigma}(\mathcal{R})$, which is an ℓ^p-analogue of $\beta^{(2)}_1(\mathcal{R})+1$. Here $\beta^{(2)}_1(\mathcal{R})$ is the ℓ^2-Betti number as defined by Gaboriau in [12]. This number has the property that $c^{(p)}_{1,\Sigma}(\mathcal{R}) \leq c(\mathcal{R})$, where $c(\mathcal{R})$ is the cost of $\mathcal{R}$ as defined by Levitt in [19], and heavily studied by Gaboriau in [11]. Further, $\mu(A)(c^{(p)}_{1,\Sigma_A}(\mathcal{R}_A)-1) \geq c^{(p)}_{1,\Sigma}(\mathcal{R})-1$. This is if we could find an equivalence relation with vanishing ℓ^2-cohomology, but so that $c^{(p)}_{1,\Sigma}(\mathcal{R}) > 1$, for some p, then we could disprove the conjecture (due to Gaboriau in [12]) that $\beta^{(2)}_1(\mathcal{R}) = c(\mathcal{R})+1$. If in addition we could prove that $c^{(p)}_{1,\Sigma}(\mathcal{R}) > 1$ for all Σ, then $\mathcal{R}$ would necessarily have trivial fundamental group. A good reference for most of the fundamental properties of measurable equivalence relations is [17].

4.1 Definition of the Invariants

We now proceed to state the definition of our extended von Neumann dimension, again the ideas are parallel to the group case. We remark that the reader will need to recall the definition of representations of an equivalence relation in 2.1.20, and the definition of sofic equivalence relation in 2.2.3.

Definition 4.1.1. Let V be a separable Banach space with a uniformly bounded action of $\mathcal{R}$, and let $q\colon W \to V$ be a bounded linear surjective map where Y has the bounded approximation property. Let $\Phi \subseteq L(\mathcal{R})$. For $F \subseteq \Phi$ finite, we define $\mathcal{W}_k(F) = \{\phi_1 \cdots \phi_j : 1 \leq j \leq k, \phi_j \in F\}$. A *q-dynamical filtration* consists of a pair $\mathcal{F} = \big((b_{\phi,j})_{(j,\phi)\in\mathbb{N}\times\mathcal{W}(\Phi)}, (W_{F,k})_{F\subseteq\Phi \text{ finite}}\big)$ where

$$b_{\phi,j} \in W,$$

$$\sup_{(j,\phi)} \|b_{\phi,j}\| < \infty,$$

$$q(b_{\mathrm{Id},j}) \text{ is dynamically generating,}$$

$$q(b_{\phi,j}) = pv_\phi q(b_{j,\mathrm{Id}}),$$

$$W_{F,k} \subseteq W_{F',k'} \text{ if } F \subseteq F', k \leq k',$$

$$W_{F,k} = \mathrm{Span}\{b_{j,\phi} : 1 \leq j \leq k, \phi \in \mathcal{W}_k(F)\} + \ker(q) \cap W_{F,k},$$

$$\ker(q) = \overline{\bigcup_{F,k} W_{F,k} \cap \ker(q)}.$$

Definition 4.1.2. A quotient dimension tuple is a tuple $((X,\mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ where (X,μ) is a standard probability space, $\mathcal{R}$ is a discrete measure-preserving equivalence relation on (X,μ), $\Phi \subseteq L(\mathcal{R})$ is of the form $\Phi = \Phi_0 \cup \mathcal{P}$, where $\Phi_0 \subseteq [[\mathcal{R}]]$ is a graphing, and $1 \in \mathcal{P} \subseteq \mathrm{Proj}(L^\infty(X,\mu))$ has $W^*(\{\phi p \phi^{-1} : \phi \in \Phi_0, p \in \mathcal{P}) = L^\infty(X,\mu)$, $V$is a uniformly bounded representation of $\mathcal{R}$, W is a separable Banach space with the bounded approximation property, $q\colon W \to V$ is a bounded linear surjective map and $\Sigma = (\sigma_i\colon [[\mathcal{R}]] \to [[\mathcal{R}_{d_i}]])$ is a sofic approximation.

Definition 4.1.3. Let $((X,\mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ be a quotient dimension tuple. Let $\mathcal{F} = ((b_{j,\phi}, W_{F,k}))$ be a q-dynamical filtration. For $F \subseteq \Phi$ finite, $m \in \mathbb{N}, \delta > 0$ we will use $\mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)$ for all linear maps $T\colon W \to \ell^p(d_i)$ with $\|T\| \leq 1$, and such that there is an $A \subseteq \{1, \cdots, d_i\}$ with $|A| \geq (1-\delta)d_i$ so that for all $1 \leq j \leq m$, for all $\phi_1, \cdots, \phi_m \in F$ we have

$$\|T(b_{\phi_1 \cdots \phi_k, j}) - \sigma_i(\phi) \cdots \sigma_i(\phi_k) T(b_{\mathrm{Id},j})\|_{\ell^p(A)} < \delta$$

$$\|T\big|_{\ker(q) \cap W_{F,m}}\| \leq \delta.$$

The above definition is very similar to the group case. However, we caution the reader as to the necessary existence of the set A by which we cut down. This procedure will be necessary in order to pass from one graphing of $\mathcal{R}$ to another. The necessity of cutting down by A will prevent us from proving some of the analogues of the properties of extended von Neumann dimension in the group case.

Definition 4.1.4. Let $(\mathcal{R}, X, \mu)$ be a discrete measure-preserving equivalence relation with a uniformly bounded representation on a Banach space V. A *dynamically generating sequence* is a bounded sequence $S = (v_j)_{j=1}^{\infty}$ in V such that $\overline{\mathrm{Span}\{\phi v_j : j \in \mathbb{N}, \phi \in [[\mathcal{R}]]\}} = V$. If Σ is a sofic approximation of $\mathcal{R}$, and $\Phi = \Phi_0 \cup \mathcal{P} \subseteq [[\mathcal{R}]]$ with Φ_0 a graphing and $\mathcal{P}$ a set of projections so that $W^*(\{\phi^{-1} p \phi^{-1} : p \in \mathcal{P}\}) = L^\infty(X, \mu)$, then the tuple $((X, \mu), \mathcal{R}, \Phi, V, S, \Sigma)$ will be called a *dimension tuple.*

Definition 4.1.5. Let V be a Banach space and $n \in \mathbb{N}$. Let ρ be a pseudonorm on $B(V, \ell^p(n))$, if $A, B \subseteq B(V, \ell^p(n))$, for $\varepsilon, M > 0$, we say that A is (ε, M)-*contained in* B if for every $T \in A$, there is an $S \in B$, with $\|S\| \leq M$ and $C \subseteq \{1, \cdots, n\}$ with $|C| \geq (1 - \varepsilon)n$, so that $\rho(m_{\chi_C}(T - S)) < \varepsilon$. Similarly, if ρ is a pseudonorm on $\ell^\infty(\mathbb{N}, \ell^p(n))$ and $A, B \subseteq \ell^\infty(\mathbb{N}, \ell^p(n))$ we say that A is ε-contained in B if for every $f \in A$ there is a $g \in B$ and $C \subseteq \{1, \cdots, n\}$ with $|C| \geq (1 - \varepsilon)n$ so that $\rho(\chi_C(f - g)) < \varepsilon$. We shall use $d_\varepsilon(A, \rho)$, (respectively $d_{\varepsilon,M}(A, \rho)$) for the smallest dimension of a linear subspace which ε-contains (respectively (ε, M)-contains) A.

Note the difference between ε-containment as stated here and in the group case, this difference is why we have difficulty proving any sort of relation between extended von Neumann dimension for groups and for equivalence relations.

Definition 4.1.6. Let $((X, \mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ be a quotient dimension tuple, and $\mathcal{F}$ a q-dynamical filtration. For a sequence of pseudonorms $\rho = (\rho_i)$ on $B(W, \ell^p(d_i))$ we define

$$\mathrm{opdim}_{\Sigma,M,\ell^p}(\mathcal{F}, \Phi, F, m, \delta, \varepsilon, \Phi, \rho) = \limsup_{i \to \infty} \frac{1}{d_i} d_{\varepsilon,M}(\mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)),$$

$$\mathrm{opdim}_{\Sigma,\ell^p}(\mathcal{F}, \varepsilon, \Phi, \rho) = \inf_{F \subseteq \Phi \ \text{finite}, m \in \mathbb{N}, \delta > 0} \mathrm{opdim}_{\Sigma,M,\ell^p}(\mathcal{F}, F, m, \delta, \varepsilon, \Phi, \rho),$$

$$\mathrm{opdim}_{\Sigma,M,\ell^p}(\mathcal{F}, \Phi, \rho) = \sup_{\varepsilon > 0} \mathrm{opdim}_{\Sigma,M,\ell^p}(\mathcal{F}, \Phi, \varepsilon, \rho).$$

We also define $\underline{\text{opdim}}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho)$ in the same way except using a limit infimum instead of a limit supremum. For later use, we note that if ρ is a norm on $\ell^\infty(\mathbb{N})$ and $\mathcal{F}$ is as above, we use $\rho_{\mathcal{F},i}(T) = \rho(j \mapsto \|T(b_{\text{Id},j})\|)$.

Definition 4.1.7. Let $((X,\mu),\mathcal{R},\Phi,V,W,q,\Sigma)$ be a quotient dimension tuple, and $\mathcal{F}$ a q-dynamical filtration. Define $\alpha_{\mathcal{F}}\colon B(V,\ell^p(d_i)) \to \ell^\infty(\mathbb{N},\ell^p(d_i))$ by $\alpha_F(T)(n) = T(b_{\text{Id},n})$. We define

$$f.\dim_{\Sigma,\ell^p}(\mathcal{F},F,m,\delta,\varepsilon,\Phi,\rho) = \limsup_{i\to\infty}\frac{1}{d_i}d_\varepsilon(\alpha_{\mathcal{F}}(\text{Hom}_{\mathcal{R},\ell^p}(\mathcal{F},F,m,\delta,\sigma_i)),\rho_{p,d_i}),$$

$$f.\dim_{\Sigma,\ell^p}(\mathcal{F},\varepsilon,\Phi,\rho) = \inf_{F\subseteq\Phi \text{ finite},m\in\mathbb{N},\delta>0}\text{opdim}_{\Sigma,M,\ell^p}(\mathcal{F},F,m,\delta,\varepsilon,\rho),$$

$$f.\dim_{\Sigma,\ell^p}(\mathcal{F},\Phi,\rho) = \sup_{\varepsilon>0} f.\dim_{\Sigma,\ell^p}(\mathcal{F},\varepsilon,\Phi,\rho).$$

Definition 4.1.8. Let $((X,\mu),\mathcal{R},\Phi,V,S,\Sigma)$ be a dimension tuple. Let ρ be a norm on $\ell^\infty(\mathbb{N})$. Let ρ_{p,d_i} be the norm on $\ell^\infty(\mathbb{N},\ell^p(d_i))$ given by $\rho_{p,d_i}(f) = \rho(\|f\|_p)$. Let $S = (v_j)_{j=1}^\infty$, set $V_{F,m} = \text{Span}\{\phi v_j : \phi \in (F \cup \text{Id} \cup F^*)^m, 1 \le j \le m\}$. Let $\alpha_S\colon B(V_{F,m},\ell^p(d_i)) \to \ell^\infty(\mathbb{N},\ell^p(d_i))$ be given by $\alpha_S(T)(j) = \chi_{\{l\le m\}}(j)T(v_j)$. We define

$$f.\dim_{\Sigma,\ell^p}(S,F,m,\delta,\varepsilon,\Phi,\rho) = \limsup_{i\to\infty}\frac{1}{d_i}d_\varepsilon(\alpha_S(\text{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i)),\rho_{p,d_i}),$$

$$f.\dim_{\Sigma,\ell^p}(S,\varepsilon,\Phi,\rho) = \inf_{F\subseteq\Phi\text{finite},m\in\mathbb{N},\delta>0}\text{opdim}_{\Sigma,M,\ell^p}(S,F,m,\delta,\varepsilon,\rho),$$

$$f.\dim_{\Sigma,\ell^p}(S,\Phi,\rho) = \sup_{\varepsilon>0} f.\dim_{\Sigma,\ell^p}(S,\varepsilon,\Phi,\rho).$$

We shall define $\underline{f.\dim}_{\Sigma,\ell^p}(S,\Phi,\rho)$ for the same thing, except replacing all the limit suprema with limit infima.

Definition 4.1.9. A *product norm* on $\ell^\infty(\mathbb{N})$ is a norm ρ such that $\rho(f) \le \rho(g)$ if $|f| \le |g|$, and such that ρ induces the topology of pointwise convergence on $\{f : \|f\|_\infty \le 1\}$.

A typical example is

$$\rho(f) = \left(\sum_{j=1}^\infty \frac{1}{2^j}|f(j)|^p\right)^{1/p}$$

for $1 \le p < \infty$.

As in the group case, we will show that

$$f.\dim_{\Sigma,\ell^p}(S,\Phi,\rho)=f.\dim_{\Sigma,\ell^p}(S',\Phi',\rho')$$

if S,S' are two dynamically generating sequences, Φ,Φ' are two graphings and ρ,ρ' are two product norms. Thus we can define $\dim_{\Sigma,\ell^p}(V,\mathcal{R})$ to be either of these common numbers. The proof of all these facts will follow quite parallel to the proofs in the group case.

4.2 Proof of Invariance

As in the group case, Proposition 3.2.1 will be quite useful. The next Lemma will be crucially used in passing between opdim and dim.

Lemma 4.2.1. *Fix $1\leq p<\infty$. Let $((X,\mu),\mathcal{R},\Phi,V,W,q,\Sigma)$ be a quotient dimension tuple and $\mathcal{F}=(b_{j,\phi},W_{F,k})$ a (q,Φ)-dynamical filtration. Let $G\subseteq W$ be a finite-dimensional linear subspace and $\kappa>0$. Let ρ be a product norm and $\lambda>0$ so that W has the λ-bounded approximation property. Fix $M>\lambda$. Then there is a $F\subseteq\Phi$ finite, $m\in\mathbb{N}$, $\delta,\varepsilon>0$ and linear maps*

$$L_i\colon \ell^\infty(\mathbb{N},\ell^p(d_i))\to B(W,\ell^p(d_i)),$$

so that if $f\in\ell^\infty(\mathbb{N},\ell^p(d_i)),T\in\operatorname{Hom}_{\mathcal{R},\ell^p}(\mathcal{F},F,m,\delta,\sigma_i)$ and $B\subseteq\{1,\cdots,d_i\}$ has $|B|\geq(1-\varepsilon)d_i$, and $\rho_{\ell^p(d_i)}(\chi_B(\alpha_{\mathcal{F}}(T)-f))<\varepsilon$, then there is a $C\subseteq\{1,\cdots,d_i\}$ with $|C|\geq(1-\eta)d_i$ such that

$$\|L_i(f)\|_{W\to\ell^p(C)}\leq M,$$

$$\|L_i(f)\big|_G-T\big|_G\|_{G\to\ell^p(C)}\leq\kappa.$$

Proof. Note that there is a $E\subseteq\Phi$ finite, $l\in\mathbb{N}$, so that

$$\sup_{\substack{w\in G\\ \|w\|=1}}\inf_{\substack{v\in W_{E,l}\\ \|v\|=1}}\|v-w\|<\kappa.$$

Thus, we may assume that $G=W_{E,l}$ for some $E\subseteq\Phi$ finite, $l\in\mathbb{N}$.

Fix $\eta > 0$ to be determined later. By Proposition 3.2.1, we may let $\theta_{F,k}\colon W \to W_{F,k}$ be linear maps such that

$$\|\theta_{F,k}\| \leq \lambda,$$

$$\lim_{(F,k)} \|\theta_{F,k}(w) - w\| = 0 \text{ for all } w \in W.$$

Choose F, m sufficiently large so that

$$\|\theta_{F,m}\big|_{Y_{E,l}} - \mathrm{Id}\big|_{Y_{E,l}}\| < \eta.$$

Let $\mathcal{B}_{F,m} \subseteq F^m \times \{1, \cdots, m\}$ be such that $\{q(b_{\psi,j} : (\psi, j) \in \mathcal{B}_{F,m}\}$ is a basis for $V_{F,m} \colon= \mathrm{Span}\{q(b_{\psi,j}) : (\psi, j) \in F^m \times \{1, \cdots, m\}\}$. Define $\widetilde{L}_i\colon \ell^\infty(\mathbb{N}, \ell^p(d_i)) \to B(V_{F,m}, \ell^p(d_i))$ by

$$\widetilde{L}_i(q(b_{\psi,j})) = \sigma_i(\psi) f(j).$$

We claim that if $\delta, \varepsilon > 0$ are small enough, then for $f \in \ell^\infty(\mathbb{N}, \ell^p(d_i))$, $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)$, and $C \subseteq \{1, \cdots, d_i\}$ with $|C| \geq (1 - \varepsilon)d_i$ and

$$\rho(\chi_C(f - \alpha_{\mathcal{F}}(T))) < \varepsilon,$$

there is a $B \subseteq \{1, \cdots, d_i\}$ so that $|B| \geq (1 - \eta)d_i$ with

$$\|\widetilde{L}_i(f) \circ q\big|_{W_{E,l}} - T\big|_{W_{E,l}}\|_{W_{E,l} \to \ell^p(B)} \leq \eta.$$

By finite-dimensionality, there is $D(F, m) > 0$ so that if $v \in \ker(q) \cap W_{F,m}$ and $(\lambda_{\psi,r}) \in \mathbb{C}^{\mathcal{B}_{F,m}}$, then

$$\sup(\|v\|, |\lambda_{\psi,r}|) \leq D(F, m) \left\| v + \sum_{(\psi,r) \in \mathcal{B}_{F,m}} \lambda_{\psi,r} b_{\psi,r} \right\|.$$

Thus if $x \in W_{F,m}$, $\|x\| \leq 1$, and $x = v + \sum_{(\psi,r)\in\mathcal{B}_{F,m}} \lambda_{\psi,r} b_{\psi,r}$, with $v \in \ker(q) \cap W_{F,m}$, and $C \subseteq B$, then

$$\|\widetilde{L}_i(f)(q(x)) - T(x)\|_{\ell^p(C)} = \left\| T(v) + \sum_{(\psi,r)\in\mathcal{B}_{F,m}} \lambda_{\psi,r}(\sigma_i(\psi)f(j) - T(b_{\psi,r})) \right\|_{\ell^p(C)} \tag{4.1}$$
$$\leq \|T(v)\|_{\ell^p(C)} + D(F,m) \sum_{(\psi,r)\in\mathcal{B}_{F,m}} \|\sigma_i(\psi)f(j) - T(b_{\psi,r})\|_{\ell^p(C)}$$
$$\leq \|T(v)\|_{\ell^p(d_i)} + D(F,m) \sum_{(\psi,r)\in\mathcal{B}_{F,m}} \|\sigma_i(\psi)f(j) - T(b_{\psi,r})\|_{\ell^p(C)}$$
$$\leq D(F,m)\delta + D(F,m) \sum_{(\psi,r)\in\mathcal{B}_{F,m}} \|\sigma_i(\psi)f(j) - T(b_{\psi,r})\|_{\ell^p(C)},$$

where in the last line we use that $\|T\| \leq 1$.

Let $A \subseteq \{1, \cdots, d_i\}$ be such that $|A| \geq (1-\delta)d_i$, and for all $1 \leq j \leq m$, for all $\phi_1, \ldots, \phi_m \in F$,

$$\|T(b_{\phi_1\cdots\phi_m,j}) - \sigma_i(\phi_1)\cdots\sigma_i(\phi_m)T(b_{\mathrm{Id},j})\|_{\ell^p(A)} < \delta$$

and set $C = B \cap A$. Then by (4.1) we have

$$\|\widetilde{L}_i(f)(q(x)) - T(x)\|_{\ell^p(C)} \leq D(F,m)\delta + D(F,m)|F|^m m\delta + \sum_{(\psi,r)} \|f(r) - T(b_{\psi,r})\|_{\ell^p(C)},$$

so it suffices to choose $\delta, \varepsilon > 0$ sufficiently small so that

$$\delta + \varepsilon < \eta,$$

$$\delta < \frac{\eta}{2D(F,m)(1+|F|^m m)},$$

and if $g \in \ell^\infty(\mathbb{N})$ has $\rho(g) < \varepsilon$ then

$$\sum_{(\psi,r)\in\mathcal{B}_{F,m}} g(r) < \frac{\eta}{2}.$$

Now suppose that $\delta, \varepsilon > 0$ are so chosen and set $L_i(f) = \widetilde{L}_i(f) \circ q\big|_{W_{F,m}} \circ \theta_{F,m}$, then if T, f, C are as above and $w \in W_{E,l}$, then

$$\|L_i(f)(w) - T(w)\|_{\ell^p(C)} \leq (1+\eta)\|\theta_{F,m}(w) - w\| + \eta\|w\| \leq \eta(1+2\eta)\|w\|,$$

so it suffices to choose η so that

$$\eta(1+2\eta) < \kappa,$$

$$\lambda(1+\eta) < M.$$

□

Our next lemma allows us to switch between two different pseudonorms.

Lemma 4.2.2. *Fix $1 \leq p < \infty$. Let $((X,\mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ be a quotient dimension tuple and $\mathcal{F} = (b_{j,\phi}, W_{F,k})$ a (q,Φ)-dynamical filtration. Let $\mathcal{F}$ be a (q,Φ)-dynamical filtration, ρ a monotone product norm, and let $C > 0$ so that W has the C-bounded approximation property.*

(a) If $C < M < \infty$, then

$$f.\dim_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}) = \operatorname{opdim}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i})$$

$$\underline{f.\dim}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}) = \underline{\operatorname{opdim}}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}).$$

(b) If ρ' is any other product norm, then for all $M > 0$,

$$\operatorname{opdim}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}) = \operatorname{opdim}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho'_{\mathcal{F},i})$$

$$\underline{\operatorname{opdim}}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}) = \underline{\operatorname{opdim}}_{\Sigma,M,\ell^p}(\mathcal{F},\Phi,\rho'_{\mathcal{F},i}).$$

Proof. (a) Let $\mathcal{F} = ((b_{\phi,j}), (W_{F,l})_{F\subseteq \mathcal{W}(\Phi) \text{ finite}, l\in\mathbb{N}})$. Let A be such that

$$\|b_{\phi,j}\| \leq A,$$

Let $1 > \varepsilon' > 0$. Find $k \in \mathbb{N}$, so that if $\|f\|_\infty \leq 1$, and f is supported on $\{n : n \geq k\}$, then $\rho(f) < \varepsilon'$. Since ρ induces a topology weaker than the norm topology, we can find a $\varepsilon' > \kappa > 0$ so that $\rho(f) < \varepsilon'$, if $\|f\|_\infty \leq \kappa$.

Let $\mathrm{Id} \in E \subseteq \Phi$ be finite $\varepsilon' > \varepsilon > 0, m \in \mathbb{N}$, with $m \geq k$, $\delta > 0$ and $L_i \colon \ell^\infty(\mathbb{N}, \ell^p(d_i)) \to B(W, \ell^p(d_i))$ be as in the proceeding lemma for this M, κ, and the finite-dimensional subspace $W_{\{\mathrm{Id}\},k}$.

Suppose $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i), f \in \ell^\infty(\mathbb{N}, V_i)$, and $B \subseteq \{1, \cdots, d_i\}$ has $|B| \geq (1-\varepsilon)d_i$ with

$$\rho(\chi_B(f - \alpha_{\mathcal{F}}(T)) < \varepsilon.$$

By the preceding Lemma, let $C \subseteq \{1, \cdots, d_i\}$ be such that $|C| \geq (1-\kappa)d_i$,

$$\|L_i(f)\|_{W \to \ell^p(C)} \leq M,$$

$$\|L_i(f)\big|_{W_{\{\mathrm{Id}\},k}} - T\big|_{W_{\{\mathrm{Id}\},k}}\|_{W_{\{\mathrm{Id}\},k} \to \ell^p(C)} \leq \kappa.$$

Then

$$\begin{aligned}\rho_{\mathcal{F},i}(\chi_C(L_i(f) - T)) &\leq (MA+1)\varepsilon + \rho(j \to \|L_i(f)(b_{\{\mathrm{Id}\},j}) - T(b_{\{\mathrm{Id}\},j})\|_{\ell^p(C)}\chi_{\{l: l \leq m\}}(j)) \\ &\leq (MA+1)\varepsilon' + A\varepsilon'.\end{aligned}$$

Thus

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F}, F_0, m_0, \delta_0, (MA+A+1)\varepsilon', \Phi, \rho) \leq f.\dim_{\Sigma}(\mathcal{F}, F_0, m_0, \delta, \varepsilon, \Phi, \rho)$$

if $F_0 \supseteq F, m_0 \geq m, \delta_0 < \delta$. Thus

$$\mathrm{opdim}_{\Sigma,M}(\mathcal{F}, (M+A)\varepsilon', \Phi, \rho) \leq f.\dim_{\Sigma}(\mathcal{F}, \Phi, \rho),$$

and since ε' was arbitrary, we are done.

(b) This follows from compactness of $\|\cdot\|_\infty$ unit ball in the product topology.

□

We now proceed to show equality when we switching graphings, it is enough to handle the case of simply increasing the graphing.

Lemma 4.2.3. *Fix $1 \leq p < \infty$. Let $((X,\mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ be a quotient dimension tuple and $\mathcal{F} = (b_{j,\phi}, W_{F,k})$ a (q,Φ)-dynamical filtration. Let $\Phi \subseteq \Phi' \subseteq [[\mathcal{R}]]$ with Φ' countable. Let $\mathcal{F}' = ((b'_{j,\phi}), W'_{F,k})$ be a (q,Φ') dynamical filtration extending $\mathcal{F}$. Suppose that Σ' is any sofic approximation then*

$$\mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}, \Phi, \rho) = \mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}', \Phi', \rho),$$

$$\underline{\text{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho)=\underline{\text{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F}',\Phi',\rho).$$

Proof. Let $M>0$ be such that for every $v\in V$, there is a $w\in W$ so that $q(w)=v$, and

$$\|v\|\leq M\|w\|.$$

It is clear that

$$\text{opdim}_{\Sigma',\ell^p}(\mathcal{F},\Phi',\rho)\leq\text{opdim}_{\Sigma,\ell^p}(\mathcal{F},\Phi,\rho).$$

For the opposite inequality, first note that for any subset $E\subseteq *-\text{Alg}(\Phi)\cap[[\mathcal{R}]]$ (here we view $[[\mathcal{R}]]\subseteq L(\mathcal{R})$, and $*-\text{Alg}(\Phi)$ denotes the smallest $*$-subalgebra of $L(\mathcal{R})$ containing Φ) we have

$$\text{opdim}_{\Sigma}(\mathcal{F},\Phi,\rho)\leq\text{opdim}_{\Sigma}(\mathcal{F},E,\rho).$$

Our assumptions imply that for any $\eta>0$, for any $\psi\in[[\mathcal{R}]]$, there is a $\psi'\in[[\mathcal{R}]]\cap*-\text{Alg}(\Phi)$

$$\|\psi-\psi'\|_2<\eta.$$

Fix $1\in F'\subseteq\Phi'$ finite, $\delta'>0$ and $m'\in\mathbb{N}$. Let $\eta>0$ to be determined later. By our above observation, we can find a finite subset $E\subseteq *-\text{Alg}(\Phi)\cap[[\mathcal{R}]]$ such that for every $\phi'\in F'$, there is a $\phi\in E$ so that

$$\|\phi_1\cdots\phi_m a_j-\phi_1'\cdots\phi_m'a_j\|<\frac{\delta'}{M}\text{ for all }1\leq j\leq m,\text{ and }\phi_1',\cdots,\phi_m'\in F',$$

$$\|\phi_1\cdots\phi_m-\phi_1'\cdots\phi_m'\|_2<\eta\text{ for all }\phi_1',\cdots,\phi_m'\in F'.$$

Thus we can find a finite subset $E\subseteq F\subseteq\mathcal{W}(\Phi)$, and an $m\in\mathbb{N}$ and $w_{\phi_1'\cdots\phi_m',j}\in\ker(q)\cap W_{F,m}$ so that

$$\|b_{\phi_1'\cdots\phi_m',j}-b_{\phi_1\cdots\phi_m,j}-w_{\phi_1'\cdots\phi_m',j}\|<\delta'.$$

We use $\mathcal{W}(\Phi)$ for all finite products of elements in $\Phi\cup\Phi^*\cup\text{Id}$, and we use $\mathcal{W}_m(\Phi)$ for $[\Phi\cup\Phi^*\cup\text{Id}]^m$. We may assume that F,m are sufficiently large so that

$$\sup_{w\in\text{Ball}(W_{F',m'}\cap\ker(q))}\inf_{v\in W_{F,m}\cap\ker(q)}\|w-v\|<\delta',$$

$$E \subseteq \mathcal{W}_m(\Phi).$$

Let $\delta > 0$ which will depend upon δ', F', m' in a manner to be determined later. Fix $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)$ and suppose A is such that

$$\|T(b_{j,\phi_1\cdots,\phi_m}) - \sigma_i(\phi_1)\cdots\sigma_i(\phi_m)T(b_{j,\mathrm{Id}})\|_{\ell^p(A)} < \delta$$

for all $\phi_1', \cdots, \phi_m' \in F'$. Let C be the set of j in $\{1, \cdots, d_i\}$ so that whenever $\phi_1, \cdots, \phi_m \in F$, then

$$j \notin \mathrm{dom}(\sigma_i(\phi_1)\cdot\sigma_i(\phi_m))\Delta\,\mathrm{dom}(\sigma_i(\phi_1')\cdot\sigma_i(\phi_m'))$$
$$\sigma_i(\phi_m)^{-1}\cdot\sigma_i(\phi_1)^{-1}(j) = \sigma_i(\phi_m')^{-1}\cdot\sigma_i(\phi_1)^{-1}(j), \text{ if either side is defined.}$$

If η is sufficiently small, then soficity implies that for all large i, $|C| \geq (1-\delta')d_i$.

Thus for all $1 \leq j \leq m$ and $\phi_1, \cdots, \phi_m \in F$ we have

$$\begin{aligned}
\|T(b_{\phi_1'\cdots,\phi_m',j}) - \sigma_i(\phi_1')\cdots\sigma_i(\phi_m')T(b_{\mathrm{Id},j})\|_{\ell^p(A\cap C)} &= \|T(b_{\phi_1'\cdots,\phi_m',j}) - \sigma_i(\phi_1)\cdots\sigma_i(\phi_m)T(b_{\mathrm{Id},j})\|_{\ell^p(A\cap C)} \\
&\leq \delta' + \|T(w_{\phi_1'\cdots\phi_m',j})\| \\
&+ \|T(b_{\phi_1\cdots\phi_m,j}) - \sigma_i(\phi_1)\cdots\sigma_i(\phi_m)T(b_{\mathrm{Id},j})\|_{\ell^p(A\cap C)} \\
&\leq \delta' + \delta\|w_{\phi_1'\cdots\phi_m,j}\| + \delta.
\end{aligned}$$

Our assumptions on F', m' ensure that

$$\|T\big|_{\ker(q)\cap W_{F',m'}}\| \leq \delta(1+\delta') + \delta'.$$

Thus if δ is sufficiently small, we may ensure that $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F', m', 2\delta', \sigma_i)$. So for any $\varepsilon > 0$, we have

$$\mathrm{opdim}_{\Sigma,\ell^p}(\mathcal{F}, \varepsilon, \Phi, \rho) \leq \mathrm{opdim}_{\Sigma',\ell^p}(\mathcal{F}', F', m', \delta', \varepsilon, \Phi', \rho).$$

Since $F', m', \delta', \varepsilon'$ were arbitrary, we see that

$$\mathrm{opdim}_{\Sigma,\ell^p}(F, \Phi, \rho) \leq \mathrm{opdim}_{\Sigma',\ell^p}(\mathcal{F}', \Phi', \rho).$$

□

We now show that $\operatorname{opdim}_{\Sigma,\infty}(\mathcal{F},\Phi,\rho_{\mathcal{F},i})$ only depends upon Φ and the quotient map q. Because of Lemmas 4.2.1,4.2.2,4.2.3 for any other (q,Φ)-dynamical filtration $\mathcal{F}'$

$$\operatorname{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho_{\mathcal{F},i}) = \operatorname{opdim}_{\Sigma,\ell^p}(\mathcal{F}',\Phi,\rho_{\mathcal{F},i}),$$

so the only difficulty is in switching $\rho_{\mathcal{F},i}$ to $\rho_{\mathcal{F}',i}$. To do this, we will have to investigate how much our definition of dimension depends on the choice of pseudonorm.

Definition 4.2.4. Let $((X,\mu),\mathcal{R},\Phi,V,W,q,\Sigma)$ be a quotient dimension tuple and $\mathcal{F} = (b_{j,\phi},W_{F,k})$ a (q,Φ)-dynamical filtration. Let ρ_i, q_i be two sequence of pseudonorms on $B(W,\ell^p(d_i))$, we say that ρ_i is $(\mathcal{F},\Sigma)$ *weaker than* q_i and write $\rho_i \preceq_{\mathcal{F},\Sigma} q_i$, if for every $\varepsilon'>0$, there are $\varepsilon,\delta>0, m,i_0\in\mathbb{N}$, $F\subseteq\Phi$ finite, and linear maps $L_i\colon B(W,\ell^p(d_i))\to B(W,\ell^p(d_i))$ for $i\geq i_0$, so that if G is a linear subspace of $B(W,\ell^p(d_i))$ and $\operatorname{Hom}_{\mathcal{R},\ell^p}(\mathcal{F},F,m,\delta,\sigma_i)\subseteq_{\varepsilon,q_i} G$, then $\operatorname{Hom}_{\mathcal{R},\ell^p}(\mathcal{F},F,m,\delta,\sigma_i)\subseteq_{\varepsilon',\rho_i} L_i(G)$.

Lemma 4.2.5. *Let $((X,\mu),\mathcal{R},\Phi,V,W,q,\Sigma)$ be a quotient dimension tuple and $\mathcal{F} = (b_{j,\phi},W_{F,k})$ a (q,Φ)-dynamical filtration.*

(a) If ρ_i,q_i are two sequence of pseudonorms on $B(W,\ell^p(d_i))$ and $\rho_i\preceq_{\mathcal{F},\Sigma} q_i$, then

$$\operatorname{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho_i) \leq \operatorname{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,q_i),$$

$$\underline{\operatorname{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho_i) \leq \underline{\operatorname{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,q_i),$$

(b) Let $\mathcal{F}'$ be another q-dynamical filtration, then $\rho_{\mathcal{F}',i}\preceq_{\mathcal{F},\Sigma}\rho_{\mathcal{F},i}$.

Proof. (a) Follows directly from the definitions.

(b) Let $\mathcal{F} = ((b_{\phi,j}),(W_{E,l})), \mathcal{F}' = ((b'_{\phi,j}),(W'_{E,l}))$. Let $D>0$ be such that W has the C-bounded approximation property, and

$$\|b_{\phi,j}\| \leq D,$$

$$\|b'_{\phi,j}\| \leq D.$$

Fix $1 > \varepsilon' > 0$. Choose $k \in \mathbb{N}$, so that if f is supported on $\{n : n \geq k\}$ and $\|f\|_\infty \leq 1$, then $\rho(f) < \varepsilon'$, and let $\varepsilon' > \kappa > 0$ be such that $\rho(f) < \varepsilon'$ if $\|f\|_\infty \leq \kappa$. Let $F, m, \delta, \varepsilon$, and $L_i \colon \ell^\infty(\mathbb{N}, \ell^p(d_i)) \to B(W, \ell^p(d_i))$ be as Lemma 4.2.1 for this κ, $M = 2D$ and the finite-dimensional subspace $W'_{\{\mathrm{Id}\},k}$. Define $\alpha_{\mathcal{F}} \colon B(W, \ell^p(d_i)) \to \ell^\infty(\mathbb{N}, \ell^p(d_i))$ by $\alpha_{\mathcal{F}}(T)(n) = T(b_{\mathrm{Id},n})$. Set $\widetilde{L}_i(T) = L_i(\alpha_{\mathcal{F}}(T))$. We may assume that $m \geq k$.

Suppose that $\mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i) \subseteq_{\varepsilon, \rho_{\mathcal{F},i}} G$, then by Lemma 4.2.1, for every $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)$ we can find an $S \in B(W, \ell^p(d_i))$ and a $C \subseteq \{1, \cdots, d_i\}$ with $|C| \geq (1-\kappa)d_i$ so that

$$\|\widetilde{L}_i(S)\|_{W \to \ell^p(C)} \leq 2D,$$

$$\|\widetilde{L}_i(S)\big|_{W'_{\{\mathrm{Id}\},k}} - T\big|_{W'_{\{\mathrm{Id}\},k}}\| < \kappa.$$

Thus

$$\begin{aligned}\rho_{\mathcal{F},i}(m_{\chi_C}(\widetilde{L}_i(S) - T)) &\leq (2D+1)D\varepsilon' + \rho(\chi_{l \leq k}(j)\|\widetilde{L}_i(S)(b_{\mathrm{Id},j}) - T(b_{\mathrm{Id},j})\|_{\ell^p(C)}) \\ &\leq (2D+1)D\varepsilon' + D\varepsilon'\end{aligned}$$

This proves (b).

□

Corollary 4.2.6. *Fix $1 \leq p < \infty$. Let $((X,\mu), \mathcal{R}, \Phi, V, W, q, \Sigma)$ be a quotient dimension tuple, and $\mathcal{F}$ a (q,Φ)-dynamical filtration. If $\mathcal{F}'$ is another (q,Φ)-dynamical filtration, and ρ, ρ' are two product norms, then*

$$\mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}, \Phi, \rho) = \mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}', \Phi, \rho')$$

$$\underline{\mathrm{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F}, \Phi, \rho) = \underline{\mathrm{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F}', \Phi, \rho').$$

Proof. If we combine Lemmas 4.2.2-4.2.5 we obtain

$$\mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}, \Phi, \rho) \leq \mathrm{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F}', \Phi, \rho')$$

the result follows by symmetry.

□

Because of the above corollary, and Lemma 4.2.3 we may define

$$\dim_{\Sigma,\ell^p}(q) = \operatorname{opdim}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho),$$

$$\underline{\dim}_{\Sigma,\ell^p}(q) = \underline{\operatorname{opdim}}_{\Sigma,\infty,\ell^p}(\mathcal{F},\Phi,\rho),$$

where $\mathcal{F},\rho$ are as in the statement of the corollary. Then $\dim_{\Sigma,\ell^p}(q)$ only depends upon q and the action of $\mathcal{R}$ on V.

Lemma 4.2.7. *Let $((X,\mu),\mathcal{R},\Phi,V,\Sigma)$ be a dimension tuple, and let ρ be a product norm. Let S be a dynamically generating sequence in V. Then*

$$f.\dim_{\Sigma,\ell^p}(S,\Phi,\rho) = \sup_{\varepsilon>0}\liminf_{(F,m,\delta)}\limsup_{i\to\infty} f.\dim_{\Sigma,\ell^p}(S,F,m,\delta,\varepsilon,\Phi,\rho),$$

$$\underline{f.\dim}_{\Sigma,\ell^p}(S,\Phi,\rho) = \sup_{\varepsilon>0}\limsup_{(F,m,\delta)}\liminf_{i\to\infty} f.\dim_{\Sigma,\ell^p}(S,F,m,\delta,\varepsilon,\Phi,\rho).$$

Proof. Let $S=(a_j)_{j=1}^{\infty}$, and $C>0$ so that $\|a_j\|\leq C$ for all j.

Fix $\varepsilon>0$, and choose $k\in\mathbb{N}$ so that $\rho(f)<\varepsilon$ if $\|f\|_\infty\leq 2C$ and f is supported on $\{n: n\geq k\}$. Fix $F\subseteq\Phi$ finite, a natural number $m\geq k$ and $\delta>0$. Then if $F'\supseteq F$ is a finite subset of Φ, $m'\geq m$ is a natural number and $0<\delta'<\delta$, then $\operatorname{Hom}_{\mathcal{R},\ell^p}(S,F',m',\delta',\sigma_i)\subseteq \operatorname{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i)$. Further, for $f\in\ell^\infty(\mathbb{N},\ell^p(d_i))$ with $\|f\|_{\ell^\infty(\mathbb{N},\ell^p(d_i))}\leq C$ we have

$$\rho(\chi_{\{l:l\leq m\}}(j)f(j)-\chi_{\{l:l\leq m'\}}(j)f(j))<\varepsilon.$$

Thus

$$d_{2\varepsilon}(\alpha_S(\operatorname{Hom}_{\mathcal{R},\ell^p}(S,F',m',\delta',\sigma_i)),\rho)\leq d_\varepsilon(\alpha_S(\operatorname{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i),\rho).$$

This implies that

$$f.\dim_{\Sigma,\ell^p}(S,2\varepsilon,\Phi,\rho)\leq f.\dim_{\Sigma,\ell^p}(S,F,m,\delta,\varepsilon,\Phi,\rho).$$

Since $F,m,$ were arbitrarily large, $\delta>0$ was arbitrarily small we see that

$$f.\dim_{\Sigma,\ell^p}(S,2\varepsilon,\Phi,\rho)\leq \liminf_{(F,m,\delta)} f.\dim_{\Sigma,\ell^p}(S,F,m,\delta,\varepsilon,\Phi,\rho)$$

and taking the supremum over $\varepsilon>0$ completes the proof. □

Lemma 4.2.8. *Let $((X,\mu),\mathcal{R},\Phi,V,W,q,\Sigma)$ be a quotient dimension tuple. Let S be a dynamically generating sequence in V. Then for any product norm ρ we have*

$$\dim_{\Sigma,\ell^p}(q,\Phi,\rho)=f.\dim_{\Sigma,\ell^p}(S,\Phi,\rho),$$

$$\underline{\dim}_{\Sigma,\ell^p}(q,\Phi,\rho)=\underline{f.\dim}_{\Sigma,\ell^p}(S,\Phi,\rho).$$

Proof. Let $S=(a_j)_{j=1}^{\infty}$, and let $\mathcal{F}=((b_{\phi,j}),(W_{E,l}))$ be a (q,Φ)-dynamical filtration such that $q(b_{\mathrm{Id},j})=a_j$. Let $C>0$ be such that

$$\|a_j\|\leq C,$$

$$\|b_{\phi,j}\|\leq C,$$

$$\|q\|\leq C,$$

for every $v\in V$, there is a $w\in W$ such that $q(w)=v$, and $\|w\|\leq C\|v\|$,

W has the C-bounded approximation property.

Let $\theta_{F,k}\colon W\to W_{F,k}$ be such that $\|\theta_{F,k}\|\leq C$ and

$$\lim_{(F,k)}\|\theta_{F,k}(w)-w\|=0 \text{ for all } w\in W.$$

We first show that

$$f.\dim_{\Sigma,\ell^p}(\mathcal{F},\Phi,\rho)\geq f.\dim_{\Sigma}(S,\Phi,\rho).$$

Fix $\varepsilon>0$, and choose $k\in\mathbb{N}$, so that $\rho(f)<\varepsilon$ if $\|f\|_\infty\leq 1$ and f is supported on $\{n:n\geq k\}$. Choose $\kappa>0$ so that $\rho(f)<\varepsilon$ if $\|f\|_\infty\leq\kappa$. Let $\mathrm{Id}\in E\subseteq\Phi$ finite and $l\in\mathbb{N}$, so that if $F\supseteq E, m\geq l$ then

$$\|\theta_{F,m}(b_{\mathrm{Id},j})-b_{\mathrm{Id},j}\|\leq\kappa$$

for all $1\leq j\leq k$.

Fix $E\subseteq F\subseteq\Phi$ finite $l\leq m\in\mathbb{N}$, and $\delta>0$. We claim that we can find a $F'\subseteq\Phi$ finite, and $m'\in\mathbb{N}$ and $\delta'>0$ so that

$$\mathrm{Hom}_{\mathcal{R},\ell^p}(S,F',m',\delta',\sigma_i)\circ q\big|_{W_{F',m'}}\circ\theta_{F',m'}\subseteq\mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F},F,m,\delta,\sigma_i)_{C^2}.$$

If $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i)$, $B \subseteq \{1, \cdots, d_i\}$ is as in the definition of $\mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i)$, if $1 \leq j \leq m$ and $\phi_1, \cdots, \phi_m \in F$ then

$$\begin{aligned}
&\|T \circ q \circ \theta_{F',m'}(b_{\phi_1 \cdots \phi_m, j}) - \sigma_i(\phi_1) \cdots \sigma_i(\phi_m) T(q(\theta_{F',m'}(b_{\mathrm{Id},j}))\|_{\ell^p(B)} \\
&\leq C\|\theta_{F',m'}(b_{\phi_1 \cdots \phi_m, j}) - b_{\phi_1 \cdots \phi_m, j}\|_{\ell^p(B)} + C\|\theta_{F',m'}(b_{\mathrm{Id},j}) - b_{\mathrm{Id},j}\|_{\ell^p(B)} + \delta'.
\end{aligned}$$

For $w \in \ker(q) \cap W_{F,m}$ we have

$$\|T \circ q \circ \theta_{F',m'}(w)\| \leq C\|\theta_{F',m'}(w) - w\|,$$

so it suffices to choose $\delta' < \delta$ and then F', m' large so that for all $1 \leq j \leq m, \psi \in F^m$,

$$C\|\theta_{F',m'}(b_{\psi,j}) - b_{\psi,j}\| + C\|\theta_{F',m'}(b_{\mathrm{Id},j}) - b_{\mathrm{Id},j}\| < \delta - \delta',$$
$$\|\theta_{F',m'}\big|_{W_{F,m}} - W_{F,m}\| < \frac{\delta}{C}.$$

Suppose that F', m', δ' are so chosen, and that $m' \geq k$. If $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta)$, then

$$\begin{aligned}
\rho(\alpha_{\mathcal{F}}(T \circ q \circ \theta_{F',m'}) - \alpha_S(T)) &\leq (C^2 + 1)\varepsilon + \rho(\chi_{\{l: l \leq k\}}(j)\|T \circ q \circ \theta_{F',m'}(b_{\mathrm{Id},j}) - T \circ q(b_{\mathrm{Id},j})\|) \\
&\leq (C^2 + C + 1)\varepsilon.
\end{aligned}$$

Thus

$$f.\dim_{\Sigma,\ell^p}(S, F, m, \delta, (C^2 + C + 2)\varepsilon, \Phi, \rho) \leq \mathrm{opdim}_{\Sigma}, \ell^p(\mathcal{F}, F', m', \delta', \varepsilon, \rho)_C,$$

since F', m' were arbitrarily large and δ' arbitrarily small we have

$$f.\dim_{\Sigma,\ell^p}(S, F, m, \delta, \varepsilon, \Phi, \rho) \leq \mathrm{opdim}_{\Sigma}(\mathcal{F}, \varepsilon, \rho)_C,$$

taking the limit supremum over (F, m, δ) and then the supremum over $\varepsilon > 0$ we find that

$$f.\dim_{\Sigma,\ell^p}(S, \Phi, \rho) \leq \dim_{\Sigma,\ell^p}(q, \Phi, \rho).$$

For the opposite inequality, let $1 > \varepsilon > 0$, and let k, E, l, κ be as in the first half of the proof. Fix $E \subseteq F \subseteq \Phi$ finite, $m \geq \max(k, l)$ and $0 < \delta < \kappa$. By Lemma 3.2.8, choose a $0 < \delta' < \delta$ a $F \subseteq F' \subseteq \Phi$ finite, a $m \leq m' \in \mathbb{N}$ so that if E is Banach space and

$$T \colon W_{F',m'} \to E$$

is a contraction with

$$\|T\big|_{\ker(q)\cap W_{F',m'}}\| \leq \delta''$$

then there is a linear map $A\colon V_{F,m} \to E$ with $\|A\| \leq 2C$ and

$$\|T(b_{\psi,j}) - A(\psi a_j)\| < \delta \text{ for all } 1 \leq j \leq m, \text{ and } \psi \in F^m$$

Let F', m' be as above and $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F', m', \delta', \sigma_i)$ and chose S as in the preceding paragraph. Let $B \subseteq \{1, \cdots, d_i\}$ be as in the definition for $\mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F', m', \delta', \sigma_i)$. Then for all $1 \leq j \leq m$ and $\phi_1, \cdots, \phi_m \in F$ we have

$$\begin{aligned}
\|A(\phi_1 \cdots \phi_m a_j) - \sigma_i(\phi) \cdots \sigma_i(\phi_m) A(a_j)\|_{\ell^p(B)} &\leq 2\delta + \|T(b_{\phi_1 \cdots \phi_m, j}) - \sigma_i(\phi) \cdots \sigma_i(\phi_m) T(b_{\mathrm{Id},j})\|_{\ell^p(B)} \\
&\leq 2\delta + \delta' \\
&< 3\delta.
\end{aligned}$$

Further

$$\rho(\alpha_S(A) - \alpha_{\mathcal{F}}(T)) \leq (2C^2 + C)\varepsilon + \rho(\chi_{\{l : l \leq k\}}(j)\|A(a_j) - T(b_{\mathrm{Id},j})\|) \leq (2C^2 + C + 1)\varepsilon.$$

Thus

$$f.\dim_{\Sigma,\ell^p}(\mathcal{F}, (2C^2 + C + 2)\varepsilon, \Phi, \rho) \leq f.\dim_{\Sigma,\ell^p}(S, F, m, 3\delta, \varepsilon, \Phi, \rho)$$

so taking a limit infimum over (F, m, δ) and then a supremum over ε completes the proof.

□

We now prove the necessary invariance to show that ℓ^p-dimension is well-defined.

Theorem 4.2.9. *Let $(\mathcal{R}, X, \mu)$ be a sofic, discrete, measure-preserving equivalence relation, and V a separable Banach space with a uniformly bounded action of $\mathcal{R}$. Let $\Sigma = (\sigma_i\colon [[\mathcal{R}]] \to [[\mathcal{R}_{d_i}]])$ be a sofic approximation.*

(i): If $q\colon Y \to V, q'\colon Y' \to V'$ are two bounded linear surjections where Y, Y' have the bounded approximation property,

$$\dim_{\Sigma,\ell^p}(q) = \dim_{\Sigma,\ell^p}(q'),$$

in the sense of the definition given after Corollary 4.2.6.

(ii): Let S be a dynamically generating sequence in V. There is a separable Banach space Y with the bounded approximation property, and a bounded linear surjection $q\colon Y \to V$, so that

$$f.\dim_{\Sigma,\ell^p}(S,\phi,\rho) = \dim_{\Sigma,\ell^p}(q)$$

for any graphing Φ and any product norm ρ on $\ell^\infty(\mathbb{N})$.

Proof. (i): Let $S = (a_j)_{j=1}^\infty$ be a dynamically generating sequence in V. We may choose dynamically filtrations $\mathcal{F} = (W_{F,m}, (b_{\psi,j})), \mathcal{F}' = (W'_{F,m}, (b'_{\psi,j}))$ for q, q' so that

$$q(b_{\mathrm{Id},j}) = a_j = q'(b_{\mathrm{Id},j}).$$

Now (i) follows from the preceding Lemma.

(ii): It is a standard exercise that there is a bounded linear surjection $q\colon \ell^1(\mathbb{N}) \to V$. Let $S = (a_j)_{j=1}^\infty$, there is a dynamically filtration $\mathcal{F} = (W_{F,m}, (b_{\psi,j}))$ for q so that

$$q(b_{\mathrm{Id},j}) = a_j.$$

Now apply the preceding Lemma.

□

By the above Theorem, we can set

$$\dim_{\Sigma,\ell^p}(V,\mathcal{R}) = \dim_{\Sigma,\ell^p}(S,\Phi,\rho),$$

$$\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R}) = \underline{\dim}_{\Sigma,\ell^p}(S,\Phi,\rho),$$

and this is independent of our choice of S, Φ, ρ.

4.3 Properties of Extended von Neumann Dimension

Definition 4.3.1. Let $(\mathcal{R}, X, \mu)$ be as above and V a Banach space representation of $\mathcal{R}$. If $v \in V$, then since (X, μ) is standard there is a unique (up to measure zero) set A such that $\operatorname{Id}_A v = v$ and $\operatorname{Id}_{A^c} v = 0$. We call A the support of v, and denote it by $\operatorname{supp} v$.

The following inequality is frequently useful, and will be used to great extent in Section 4.6.

Proposition 4.3.2. *Let $((X, \mu), \mathcal{R}, V, \Phi, \Sigma)$ be a dimension tuple. Let $S = (a_j)_{j=1}^{\infty}$ be a dynamically generating sequence in V, then for any sofic approximation Σ, and $1 \leq p < \infty$,*

$$\dim_{\Sigma,\ell^p}(V, \mathcal{R}) \leq \sum_{j=1}^{\infty} \mu(\operatorname{supp} a_j).$$

Proof. Let $A_j = \operatorname{supp} a_j$. Fix $\varepsilon > 0$, let $F \subseteq \Phi$ be finite, $m \in \mathbb{N}, \delta > 0$, if F is sufficiently large, then there is a $B_j \subseteq X$ measurable with $\operatorname{Id}_{B_j} \in F^m$ so that

$$\| \operatorname{Id}_{B_j} a_j - a_j \| < \varepsilon,$$

$$\mu(B_j \Delta A_j) < \delta.$$

Thus for all large i, and for all $T \in \operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, m, \delta, \sigma_i)$ we can find a set $C \subseteq \{1, \cdots, d_i\}$ with $|C| \geq (1 - 2\delta(1 + m))d_i$ so that

$$\|T(a_j) - \sigma_i(\operatorname{Id}_{A_j})T(a_j)\|_{\ell^p(C)} < \delta,$$

for all $1 \leq j \leq m$. So if δ is sufficiently small (depending only upon ε, m) we have shown that

$$(T(a_1), \cdots, T(a_m)) \subseteq_{\varepsilon} \bigoplus_{j=1}^{n} \sigma_i(\operatorname{Id}_{A_j})(\ell^p(d_i)),$$

so for all large i,

$$\frac{1}{d_i} d_{\varepsilon}(\operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, m, \delta, \sigma_i)) \leq \frac{1}{d_i} \sum_{j=1}^{n} \operatorname{Tr}(\sigma_i(\operatorname{Id}_{A_j})) \to \sum_{j=1}^{m} \mu(A_j).$$

Thus

$$f.\dim_{\Sigma}(S,F,m,\varepsilon,\delta,\sigma_i) \le \sum_{j=1}^{\infty}\mu(A_j),$$

since the above is true for all F, m, sufficiently large and δ sufficiently small (depending only on ε) the proof is complete.

□

Proposition 4.3.3. *Let $((X,\mu),\mathcal{R},\Phi,V,\Sigma)$ be a dimensional tuple, and let W be another representation of $\mathcal{R}$. If $T\colon W \to V$ is a bounded equivariant map with dense image, then*

$$\dim_{\Sigma,\ell^p}(V,\mathcal{R}) \le \dim_{\Sigma,\ell^p}(W,\mathcal{R}).$$

Proof. If if S is a dynamically generating sequence in W, then $T(S)$ is a dynamically generating for W. If $\phi \in \mathrm{Hom}_{\mathcal{R},\ell^p}(T(S),F,m,\delta,\sigma_i)$, then $\phi\circ T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i)$ and

$$\alpha_S(\phi\circ T) = \alpha_{T(S)}(\phi)$$

we are done.

□

We also handle how dimension behaves under compressions. This implies in particular that dimension is in fact invariant under *weak isomorphism* (we recall that two representations V, W of an discrete, measure-preserving equivalence relation $(\mathcal{R}, X, \mu)$ are weakly isomorphic if for any $\varepsilon > 0$ there is a measurable $A \subseteq X$ with $\mu(A) \ge 1-\varepsilon$ and $\mathrm{Id}_A V \cong \mathrm{Id}_A W$ are isomorphic as representations of $\mathcal{R}_A$).

Proposition 4.3.4. *Fix $1 \le p < \infty$. Let $((X,\mu),\mathcal{R},\Phi,V,\Sigma)$ be a dimensional tuple with $\mathcal{R}$ ergodic and (X,μ) diffuse. For a measurable $A \subseteq X$, let Σ_A be defined by $\sigma_{A,i}(\phi) = \sigma_i(\mathrm{Id}_A)\sigma_i(\phi)\sigma_i(\mathrm{Id}_A)$. Then*

$$\mu(A)\dim_{\Sigma_A,\ell^p}(\mathrm{Id}_A V,\mathcal{R}_A) = \dim_{\Sigma,\ell^p}(V,\mathcal{R})$$

$$\mu(A)\underline{\dim}_{\Sigma_A,\ell^p}(\mathrm{Id}_A V,\mathcal{R}_A) = \underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R})$$

Proof. We will handle the case of dim only. Let $S_A = (a_j)_{j=1}^{\infty}$ be a dynamically generating sequence for V_A. By ergodicity, we may find $\psi_1, \cdots, \psi_k \in [[\mathcal{R}]]$ with $\psi_1 = \mathrm{Id}_A, \mathrm{dom}(\psi_j) = A$ for $1 \leq j \leq k, \mathrm{dom}(\psi_k) \subseteq A$, and up to sets of measure zero,

$$X = \bigsqcup_{j=1}^{k} \mathrm{ran}(\psi_j).$$

Set $A_j = \psi_j(A)$. Let S be an enumeration of $(\psi_k a_j)_{j,k}$.

We will first prove that $\dim_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \mu(A)\dim_{\Sigma_A,\ell^p}(V_A,\mathcal{R}_A)$ when $\mu(A) = 1/n$.

It is easy to see that

$$\dim_{\Sigma_{A_j},\ell^p}(\mathrm{Id}_{A_j} V, \mathcal{R}_{A_j})$$

is independent of j. For $T\colon V \to \ell^p(d_i)$, let $T_{A_j}\colon V_{A_j} \to \ell^p(\sigma(\mathrm{Id}_{A_j})(\{1,\cdots,d_i\}))$ be given by

$$T_{A_j}(x) = \sigma_i(\mathrm{Id}_{A_j})T(x).$$

Fix $\varepsilon' > 0$, and let $\varepsilon > 0$ depend upon ε' in a manner to be determined later.

Given $F \subseteq \Phi_A, m \in \mathbb{N}, \delta > 0$, there is a $F' \subseteq \Phi, m' \in \mathbb{N}, \delta' > 0$ so that $T \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i)$ implies $T_A \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S_A, F, m, \delta, \sigma_{i,A})$. If we choose F', m', δ' appropriately and

$$\alpha_{S_A}(\mathrm{Hom}_{\mathcal{R}_A,\ell^p}(S_A, F, m, \delta, \sigma_{i,A})) \subseteq_{\varepsilon,\|\cdot\|_p} W,$$

then

$$\alpha_S(\mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i)) \subseteq_{\varepsilon',\|\cdot\|_p} \left\{\sum_{k=1}^{n} \sigma_i(\psi_k)\xi : \xi \in W\right\}.$$

Since

$$\frac{\mathrm{tr}(\sigma_i(\mathrm{Id}_A))}{d_i} \to \frac{1}{n},$$

we find that

$$\dim_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \frac{1}{n}\dim_{\Sigma,\ell^p}(V_A,\mathcal{R}_A).$$

$$\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \frac{1}{n}\underline{\dim}_{\Sigma,\ell^p}(V_A,\mathcal{R}_A).$$

We now show that $\dim_{\Sigma,\ell^p}(V,\mathcal{R}) \leq \mu(A)\dim_{\Sigma_A,\ell^p}(V_A,\mathcal{R}_A)$ for general A (not necessarily with $\mu(A) = 1/n$). Fix $F \subseteq [[\mathcal{R}]]$ finite $r \in \mathbb{N}, \delta > 0$. Fix $\kappa > 0$ which will depend upon δ in a manner to be determined. Let

$$F' \supseteq \{\psi_i^{-1}\phi\psi_q : 1 \leq i, q \leq k, \phi \in F\}$$

Fix $r' \in \mathbb{N}, \delta' > 0$ which will depend upon r, δ in a manner to be determined shortly. Suppose that $T \in \mathrm{Hom}_{\mathcal{R}_A,\ell^p}(S_A, F', r', \delta', \sigma_i)$, define

$$\widetilde{T}(x) = \sum_{j=1}^{k} \sigma_i(\psi_i)T(\psi_i^{-1}x).$$

Then

$$\|\widetilde{T}\| \leq M,$$

where $M > 0$ is some constant.

Choose $C \subseteq \{1, \cdots, d_i\}$ of cardinality at least $(1-\delta')d_i$ for T as in the definition of $\mathrm{Hom}_{\mathcal{R}_A,\ell^p}(S_A, F', r', \delta', \sigma_i)$. It is easy to see that if F', r' are sufficiently large and δ' is sufficiently small, then

$$\|T(\psi_i^{-1}\phi\psi_q\psi_q^{-1}a_l) - \sigma_i(\psi_i^{-1}\phi\psi_q^{-1})T(\psi_q^{-1}a_l)\|_{\ell^p(C)} < \kappa.$$

We have

$$\psi_j^{-1}\phi = \sum_{q=1}^{k} \psi_j^{-1}\phi\psi_q\psi_q^{-1},$$

hence

$$T(\psi_j^{-1}\phi a_l) = \sum_{q=1}^{k} T(\psi_j^{-1}\phi\psi_q\psi_q^{-1}a_l),$$

so

$$\left\| T(\psi_j^{-1}\phi a_l) - \sum_{q=1}^{k} \sigma_i(\psi_j^{-1}\phi\psi_q)T(\psi_q^{-1}a_l) \right\|_{\ell^p(C)} < \frac{\delta}{k},$$

if $\kappa > 0$ is sufficiently small. Since

$$\sum_{i=1}^{k} \psi_j\psi_j^{-1}\phi\psi_q = \phi\psi_q,$$

for all sufficiently large i we can find a set $C' \subseteq \{1, \cdots, d_i\}$ of size at least $(1-\delta')d_i$ so that

$$\chi_{C'} \sum_{i=1}^{k} \sigma_i(\psi)\sigma_i(\psi_j^{-1}\phi\psi_q) = \chi_{C'}\sigma_i(\phi)\sigma_i(\psi_q),$$

as elements of $M_{d_i}(\mathbb{C})$. Then

$$\begin{aligned}
&\left\|\widetilde{T}(\phi a_l) - \sigma_i(\phi)T(a_l)\right\|_{\ell^p(C\cap C')} \\
&= \left\|\sum_{i=1}^{k} \sigma_i(\psi_j)T(\psi_j^{-1}\phi a_l) - \sum_{i=1}^{k} \sigma_i(\phi)\sigma_i(\psi_j)T(\psi_j^{-1}a_l)\right\|_{\ell^p(C\cap C')} \\
&\leq \delta + \left\|\sum_{1\leq j,q\leq k} \sigma_i(\psi_j)\sigma_i(\psi_j^{-1}\phi\psi_q)T(\psi_q^{-1}a_l) - \sum_{q=1}^{k} \sigma_i(\psi_q)\sigma_i(\psi_q)T(\psi_j^{-1}a_l)\right\|_{\ell^p(C\cap C')} \\
&= \delta.
\end{aligned}$$

Thus $\widetilde{T} \in \operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, r, \delta, \sigma_i)_M$. Further, since $\psi_1 = \operatorname{Id}_A$,

$$\sum_{j=1}^{k} \operatorname{Id}_A \psi_j\psi_j^{-1} = \operatorname{Id}_A,$$

so

$$\sigma_i(\operatorname{Id}_A)\widetilde{T}(a_j) = \sum_{j=1}^{k} \sigma_i(\operatorname{Id}_A)\sigma_i(\psi_j)T(\psi_j^{-1}a_j),$$

hence $\sigma_i(\operatorname{Id}_A)\widetilde{T}(a_j)$ agrees with $T(a_j)$ on a set of size at least $(1-\varepsilon)d_i$ if i is sufficiently large.

So, if W is a subspace of $\ell^\infty(\mathbb{N}, \ell^p(d_i))$ which ε-contains $\alpha_S(\operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, r, \delta, \sigma_i))$, then $\sigma_i(\operatorname{Id}_A)(W)$ 2ε-contains $\alpha_{S_A}(\operatorname{Hom}_{\mathcal{R}_A,\ell^p}(S_A, F', r', \delta', \sigma_i))$. This shows that

$$\dim_{\Sigma,\ell^p}(\operatorname{Id}_A V, R_A) \leq \frac{1}{\mu(A)} \dim_{\Sigma,\ell^p}(V, R).$$

Note that this implies $\mu(A)\dim_{\Sigma_A,\ell^p}(V_A, \mathcal{R}_A) = \dim_{\Sigma,\ell^p}(V, \mathcal{R})$ when $\mu(A)$ is rational. If $\mu(A)$ is not rational, then since (X, μ) is diffuse we may find measurable $A_n \subseteq A \subseteq B_n$ with A_n increasing, B_n decreasing $\mu(A_n), \mu(B_n)$ are rational and $\mu(A_n), \mu(B_n) \to \mu(A)$. Then by considering compressions

$$\frac{1}{\mu(A_n)} \dim_{\Sigma,\ell^p}(V, \mathcal{R}) = \dim_{\Sigma_{A_n},\ell^p}(V_{A_n}, \mathcal{R}_{A_n}) \leq \frac{\mu(A)}{\mu(A_n)} \dim_{\Sigma_A,\ell^p}(V_A, \mathcal{R}_A)$$

$$\frac{1}{\mu(B_n)}\dim_{\Sigma,\ell^p}(V,\mathcal{R})=\dim_{\Sigma_{B_n},\ell^p}(V_{B_n},\mathcal{R}_{B_n})\geq\frac{\mu(B_n)}{\mu(A)}\dim_{\Sigma_A,\ell^p}(V_A,\mathcal{R}_A),$$

let $n\to\infty$ to complete the proof.

□

We now show that dimension is subadditive under exact sequences. Unfortunately, we cannot handle superadditivity even in the case of direct sums, not even in the case of Hilbert spaces. Unfortunately, the proof for superadditivity given in Theorem 3.3.7 does not carry over to our setting. The difficulty is in getting a bound analogous to Lemma 3.3.3 for our different version of approximate dimension.

Theorem 4.3.5. *Let $((X,\mu),\mathcal{R},\Phi,V,\Sigma)$ be a dimensional tuple, and let $W\subseteq V$ be a closed $\mathcal{R}$-invariant subspace. Then for every $1\leq p<\infty$, we have the following inequalities:*

$$\dim_{\Sigma,\ell^p}(V,\mathcal{R})\leq\dim_{\Sigma,\ell^p}(V/W,\mathcal{R})+\dim_{\Sigma,\ell^p}(W,\mathcal{R}),$$

$$\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R})\leq\dim_{\Sigma,\ell^p}(V/W,\mathcal{R})+\underline{\dim}_{\Sigma,\ell^p}(W,\mathcal{R}),$$

$$\underline{\dim}_{\Sigma,\ell^p}(V,\mathcal{R})\leq\underline{\dim}_{\Sigma,\ell^p}(V/W,\mathcal{R})+\dim_{\Sigma,\ell^p}(W,\mathcal{R}).$$

Proof. Let $S_2=(w_j)_{j=1}^\infty$ be a dynamically generating sequence for W, and $(a_j)_{j=1}^\infty$ a dynamically generating sequence for V/W. Let $v_j\in V$ be such that $v_j+W=a_j$, and $\|v_j\|\leq 2\|a_j\|$. Let S be the sequence

$$v_1,w_1,v_2,w_2,\cdots$$

we shall use S and the pseudonorms

$$\|T\|_{S_1,i}=\sum_{j=1}^\infty\frac{1}{2^j}\|T(a_j)\|,$$

$$\|T\|_{S_2,i}=\sum_{j=1}^\infty\frac{1}{2^j}\|T(w_j)\|,$$

$$\|T\|_{S,i}=\sum_{j=1}^\infty\frac{1}{2^j}\|T(w_j)\|+\sum_{j=1}^\infty\frac{1}{2^j}\|T(v_j)\|$$

to do our calculation.

Let $\varepsilon > 0$, and choose $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$. Let $e \in F_1 \subseteq \Phi$ be finite, $m \leq m_1 \in \mathbb{N}$ and $\delta_1 > 0$ to be determined later. By 3.2.8 choose $0 < \delta < \delta_1$, and $F_1 \subseteq E \subseteq \Phi$ finite and $m_1 \leq k \in \mathbb{N}$ so that if G is a Banach space and

$$T\colon V_{E,2k} \to G$$

has $\|T\| \leq 2$, and

$$\|T\big|_{W \cap V_{E,2k}}\| < \delta,$$

then there is a $A\colon (V/W)_{F_1,m_1} \to G$ with $\|A\| \leq 3$, and

$$\|A(\psi a_j) - T(\psi x_j)\| < \delta_1,$$

for all $1 \leq j, k \leq m_1$ and $\psi \in (F_1 \cup F_1^* \cup \{e\})^{m_1}$.

By finite-dimensionality, we can find a $F' \supseteq E$, $m' \geq 2k$, and $0 < \delta' < \delta_1$ so that if G is a Banach space and $T\colon V_{F',m'} \to G$ has

$$\|T(\psi x_j)\| \leq \delta' \|\psi x_j\|$$

for all $1 \leq j \leq m', \psi \in (F' \cup F'^* \cup \{\mathrm{Id}\})^{m'}$ then

$$\|T\big|_{W \cap V_{E,2k}}\| < \delta.$$

Define $\Xi\colon\ \mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i) \to \mathrm{Hom}_{\mathcal{R},\ell^p}(S_2, F', m', \delta', \sigma_i)$ by

$$\Xi(T) = T\big|_{W_{F',m'}}.$$

Find

$$\Theta\colon\ \mathrm{im}(\Xi) \to \mathrm{Hom}_{\mathcal{R},\ell^p}(S, F', m', \delta', \sigma_i)$$

so that

$$\Xi \circ \Theta = \mathrm{Id}\,.$$

Then

$$(T - \Theta(\Xi(T))(\psi v_j) = 0$$

for all $1 \le j \le m'$ and $\psi \in (F_1 \cup F_1' \cup \{id\})^{m'}$. Thus our assumption implies that we can find a $A\colon (V/W)_{F_1,m_1} \to \ell^p(d_i)$ so that

$$\|T(\psi x_j) - A(\psi a_j)\| < \delta_1$$

for all $1 \le j \le m_1, \psi \in (F_1 \cup F_1^* \cup \{\mathrm{Id}\})^{m_1}$, with $\|A\| \le 3$.

Thus whenever $\psi \in (F_1 \cup F_1^* \cup \{\mathrm{Id}\})^{m_1}$, and $C \subseteq \{1, \cdots, d_i\}$ we have

$$\|A(\psi a_j) - \sigma_i(\psi)A(a_j)\|_{\ell^p(C)} \le 2\delta_1 + \|T(\psi x_j) - \sigma_i(\psi)A(a_j)\|_{\ell^p(C)},$$

so $A \in \mathrm{Hom}_{\mathcal{R},\ell^p}(S_1, F_1, m_1, 3\delta_1, \sigma_i)_3$. The rest follows as in Proposition 3.3.2.

□

4.4 Preliminary Results on Direct Integrals

Definition 4.4.1. Let (X, μ) be a standard measure space, and $V = (V_x)_{x\in X}$ a family of Banach spaces. We say that V is *measurable* if there are sequences $(v_x^{(j)})_{x\in X, j\in\mathbb{N}}, (\phi_x^{(j)})_{x\in X, j\in\mathbb{N}}$ with $v_x^{(j)} \in V_x, \phi_x^{(j)} \in V_x^*$ satisfying the following properties

Property 1: $x \mapsto \langle v_x^{(j)}, \phi_x^{(k)} \rangle_{x\in X}$ is measurable for all j, k

Property 2: $\overline{\mathrm{Span}}^{\|\cdot\|}\{v_x^{(j)} : j \in \mathbb{N}\} = V_x$ for almost every x

Property 3: $\overline{\mathrm{Span}}^{\mathrm{wk}^*}\{\phi_x^{(j)} : j \in \mathbb{N}\} = V_{x^*}$ for almost every x

Property 4: $x \mapsto \|\sum_j f(j)v_x^{(j)}\|$ is measurable for all $f \in c_c(\mathbb{N})$

Property 5: $x \mapsto \|\sum_j f(j)\phi_x^{(j)}\|$ is measurable for all $f \in c_c(\mathbb{N})$

It is a fact that if we are given properties $1 - 3$, then property 4 is actually equivalent to property 5.

We define the set of measurable vector fields, $\mathrm{Meas}(X, V)$, to be all fields $(v_x)_{x\in X}$ of vectors in X such that $v_x \in V_x$ for all x and $x \mapsto \langle v_x, \phi_x^{(j)} \rangle$ is measurable for all $j \in \mathbb{N}$. Note that our axioms imply that

$$\|v_x\| = \sup_{\substack{f \in c_c(\mathbb{N},\mathbb{Q}[i]),\\ \left\|\sum_j f(j)\phi_x^{(j)}\right\| < 1}} \left| \sum_j f(j)\langle v_x, \phi_x^{(j)}\rangle \right|.$$

so that the norm of a measurable vector fields is a measurable function. We also define $\mathrm{Meas}(X, V^*)$ to be all fields of vectors $(\phi_x)_x$ such that $\phi_x \in V_x^*$ for all $x \in X$ and $x \mapsto \langle v_x^{(j)}, \phi_x\rangle$ is measurable for all $j \in \mathbb{N}$. As above $\|\phi_x\|$ is measurable. We leave it as an exercise to verify that if $v \in \mathrm{Meas}(X, V), \phi \in \mathrm{Meas}(X, V^*)$ then $x \mapsto \langle v_x, \phi_x\rangle$ is measurable.

For $1 \leq p < \infty$, we define the L^p-direct integral of V denoted

$$\int_X^{\oplus_p} V_x \, d\mu(x)$$

to be all $v \in \mathrm{Meas}(X, V)$ so that

$$\|v\|_p^p = \int_X \|v_x\|^p \, d\mu(x) < \infty.$$

Hölder's inequality shows that $\int_X^{\oplus_p} V_x \, d\mu(x)$ is a vector space.

Proposition 4.4.2. *Let (X, μ) be a standard measure space and V a measurable field of Banach spaces over X. Then for $1 \leq p < \infty$,*

$$\int_X^{\oplus_p} V_x \, d\mu(x)$$

is a separable Banach space. Further a sequence $(w^{(j)})_{j=1}^\infty$ in $\int_X^{\oplus_p} V_x \, d\mu(x)$ has

$$\mathrm{Span}\{\chi_A w^{(j)} : A \textit{ measurable}, j \in \mathbb{N}\}$$

dense in $\int_X^{\oplus_p} V_x \, d\mu(x)$ if and only if for almost every x, $(w_x^{(j)})_{j=1}^\infty$ spans a dense subspace.

Proof. Let $v_x^{(j)}, \phi_x^{(j)}$ be as in the definition of measurable vector field. We first prove completeness.

Suppose that $w^{(n)}$ in $\int_X^{\oplus_p} V_x \, d\mu(x)$ has

$$\sum_{n=1}^\infty \|w^{(n)}\|_p < \infty.$$

Then,

$$\begin{aligned}\int_X \sum_{n=1}^{\infty} \|w_x^{(n)}\|^p \, d\mu(x) &\leq \liminf_{N\to\infty} \int_X \sum_{n=1}^{N} \|w_x^{(n)}\|^p \, d\mu(x) \\ &\leq \left(\sum_{n=1}^{N} \|w^{(n)}\|_p \right)^p \\ &\leq \left(\sum_{n=1}^{\infty} \|w^{(n)}\|_p \right)^p \\ &< \infty.\end{aligned}$$

So for almost every x, $w_x = \sum_{n=1}^{\infty} w_x^{(n)}$ is norm convergent in V_x. It is easy to see by taking limits that $w \in \mathrm{Meas}(X, V)$. By the same inequalities as above we also see that

$$\left\| w - \sum_{n=1}^{N} w^{(n)} \right\|_p \leq \sum_{n=N+1}^{\infty} \|w^{(n)}\|_p \to 0,$$

as $N \to \infty$, and this proves completeness.

For the second fact, first suppose that $w^{(j)}$ in $\int_X^{\oplus_p} V_x \, d\mu(x)$ is such that $\mathrm{Span}\{w_x^{(j)} : j \in \mathbb{N}\}$ is dense in V_x for almost every $x \in X$. Let $v \in \int_X^{\oplus_p} V_x \, d\mu(x)$ and $\varepsilon > 0$. then up to sets of measure zero,

$$X = \bigcup_{f \in c_c(\mathbb{N}, \mathbb{Q}[i])} \left\{ x \in X : \left\| \sum_{j=1}^{\infty} f(j) w_x^{(j)} - v_x \right\| < \varepsilon \right\}.$$

Thus by the usual arguments we can find a measurable $f \colon X \to c_c(\mathbb{N}, \mathbb{Q}[i])$ such that for almost every $x \in X$, we have

$$\left\| \sum_{j=1}^{\infty} f(x)(j) w_x^{(j)} - v_x \right\| < \varepsilon.$$

Let F_n be finite subsets of $\mathbb{Q}[i]$ which increase to $\mathbb{Q}[i]$, and so that $0 \in F_n$ for all n. For $n \in \mathbb{N}$, set

$$X_n = \{x \in X : f(x)(j) = 0 \text{ for } j \geq n, f(x)(j) \in F_n \text{ for all } j\}.$$

If n is sufficiently large then,

$$\int_{X_n^c} \|v_x\|^p \, d\mu(x) < \varepsilon.$$

Thus

$$\int_X \left\| \sum_{j=1}^{\infty} \chi_{X_n} f(x)(j) w_x^{(j)} - v_x \right\|^p \, d\mu(x) < \varepsilon^p + \varepsilon,$$

and it is easy to see that

$$\sum_{j=1}^{\infty} \chi_{X_n} f(x)(j) w_x^{(j)}$$

is a finite linear combination of elements of the form $\chi_A w_x^{(j)}$. This proves one implication.

Conversely, suppose that $\chi_A w_x^{(j)}$ densely span $\int_X^{\oplus_p} V_x \, d\mu(x)$, but that

$$A = \{x \in X : w_x^{(j)} \text{ does not densely span } V_x\}$$

has positive measure. Then there is a $v \in \operatorname{Meas}(A, V)$ so that

$$d\left(v_x, \operatorname{Span}\{w_x^{(j)}\}\right) \geq 1$$

for all $x \in A$. But we can find $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$, $j_1, \cdots, j_k \in \mathbb{N}$ and sets $A_1, \cdots, A_k$ so that

$$\left\| v - \sum_{j=1}^{k} \lambda_j \chi_{A_j} w^{(j)} \right\|_p < 1/2.$$

Replacing A_j with $A \cap A_j$ we may assume $A_j \subseteq A$. This clear implies that there is some $x \in A$ so that

$$\left\| v - \sum_{j=1}^{k} \lambda_j \chi_{A_j}(x) w_x^{(j)} \right\|_p < 1/2,$$

and this is a contradiction.

$\square$

If V_x, W_x are measurable fields of Banach spaces over (X, μ), and $T_x \colon V_x \to W_x$ are bounded linear operators with

$$x \mapsto (T(v))_x \in \operatorname{Meas}(X, W) \text{ for all } w \in \operatorname{Meas}(W, X)$$

$$x \mapsto \|T_x\|_{V_x \to W_x} \text{ is in } L^{\infty}$$

then we let

$$T = \int_X^{\oplus_p} T_x \, d\mu(x)$$

denote the operator

$$\int_X^{\oplus_p} V_x \, d\mu(x) \to \int_X^{\oplus_p} W_x \, d\mu(x)$$

defined by

$$(T(v))_x = T_x(v_x) \text{ for all } v \in \operatorname{Meas}(X, V).$$

Direct integrals arise naturally in the context of representations of equivalence relations.

Definition 4.4.3. Let $(\mathcal{R}, X, \mu)$ be a discrete measure preserving equivalence relation, and let $x \to V_x$ be measurable field of Banach spaces over X. *A representation* π *of* $\mathcal{R}$ *on* V consists of bounded linear maps $\pi(x, y) \colon V_y \to V_x$ so that $\pi(z, x)\pi(x, y) = \pi(z, y)$ for $x \sim y \sim z$, $\pi(x, x) = \mathrm{Id}$, and for each $v \in \operatorname{Meas}(X, V), \phi \in \operatorname{Meas}(X, V^*)$ we have that $(x, y) \to \langle \pi(x, y)v_y, \phi_x \rangle$ is a measurable map $\mathcal{R} \to \mathbb{C}$. We say that π is *uniformly bounded* if there is a $C > 0$ so that $\|\pi(x, y)v\| \leq C$ for all $(x, y) \in \mathcal{R}$, $v \in V_y$.

Note that if π is uniformly bounded, then for every $1 \leq p < \infty$, we get a uniformly bounded action of $\mathcal{R}$ on $\int_X^{\oplus_p} V_x \, d\mu(x)$ by

$$(\phi \cdot v)_x = \chi_{\operatorname{ran}(\phi)}(x)\pi(x, \phi^{-1}(x))v_{\phi^{-1}(x)}.$$

Our work in this section has the following corollary which will allow us to work fiberwise in the case of representations on measurable fields. This will be used quite heavily in Section 4.6.

Corollary 4.4.4. *Let* $(\mathcal{R}, X, \mu)$ *be a discrete measure-preserving equivalence relation, with a representation* π *on a measurable field of Banach spaces* $x \to V_x$*. If* $w^{(j)} \in \int_X^{\oplus_p} V_x \, d\mu(x)$ *is bounded, then* $w^{(j)}$ *is dynamically generating if and only if for almost every* x*,*

$$\overline{\operatorname{Span}\{\pi(x, y)w_y^{(j)} : y \sim x\}}^{\|\cdot\|_{V_x}} = V_x.$$

4.5 Computations for $L^p(\mathcal{R}, \overline{\mu})$.

Here we prove that

$$\dim_{\Sigma, \ell^p}(L^p(\mathcal{R}, \overline{\mu})^{\oplus n}, \mathcal{R}) = \underline{\dim}_{\Sigma, \ell^p}(L^p(\mathcal{R}, \overline{\mu})^{\oplus n}, \mathcal{R}) = n.$$

We must take a different approach than the group case, as the operators defined there will not fill up enough space if we use our different version of ε-dimension. Instead, we shall take a more probabilistic approach.

Proposition 4.5.1. *Fix $1 \leq p < \infty$. Let ν_n be the uniform probability measure on $\{1, \dots, n\}$. Let $A_n \subseteq B(\ell^p(n, \nu_n))$, be measurable, where ν_n is the uniform measure, and suppose that*

$$\liminf_{n\to\infty} \left(\frac{\operatorname{vol}(A_n)}{\operatorname{vol}(B(\ell^p(n,\nu_n))}\right)^{1/2n} \geq \alpha.$$

Then there is a $\kappa(\alpha, \varepsilon, p) \geq 0$ with

$$\lim_{\varepsilon\to 0} \kappa(\alpha, \varepsilon, p) = 1,$$

so that

$$\liminf_{n\to\infty} \frac{1}{n} d_\varepsilon(A_n, \|\cdot\|_p) \geq \kappa(\alpha, \varepsilon, p).$$

Proof. If the claim is false, then there is a $\kappa < 1$, so that for every $\varepsilon > 0$,

$$\kappa > \liminf_{i\to\infty} \frac{1}{n} d_\varepsilon(A_n, \|\cdot\|_p),$$

Then for all large n, we can find a subspace $W \subseteq \ell^p(n)$ with $\dim(W) \leq \kappa n$, and $A \subseteq_\varepsilon W$. This implies that

$$A \subseteq \bigcup_{\substack{B\subseteq\{1,\cdots,n\},\\ |B|\leq\varepsilon n}} ((1+\varepsilon)\operatorname{Ball}(\chi_{B^c}(W)) + \varepsilon \operatorname{Ball}(\ell^p(B^c, \nu_{B^c})) \times \operatorname{Ball}(\ell^p(B, \nu_B)).$$

Since $\chi_{B^c}(W)$ has dimension at most κn, we can find a ε-dense subset of $(1+\varepsilon)\operatorname{Ball}(\chi_{B^c}(W))$ of cardinality at most $\left(\frac{2+4\varepsilon}{\varepsilon}\right)^{2\kappa n}$. Thus

$$\operatorname{vol}((1+\varepsilon)\operatorname{Ball}(\chi_{B^c}(W)) + \varepsilon \operatorname{Ball}(\ell^p(B^c, \nu_{B^c})) \leq$$
$$\left(\frac{2+4\varepsilon}{\varepsilon}\right)^{2\kappa n} \operatorname{vol}(\operatorname{Ball}(\ell^p(B^c, \nu_{B^c}))(2\varepsilon)^{2|B^c|}.$$

So $\frac{\operatorname{vol}(A)}{\operatorname{vol}(\operatorname{Ball}(\ell^p(n,\nu_n))}$ is at most

$$\sum_{\substack{B\subseteq\{1,\cdots,d_i\}\\ |B|\leq\varepsilon n}} 2^{|B^c|}(\varepsilon)^{2(|B^c|-\kappa n)}(2+4\varepsilon)^{2\kappa n} \frac{\operatorname{vol}(\operatorname{Ball}(\ell^p(B^c, \nu_{B^c})) \operatorname{vol}(\operatorname{Ball}(\ell^p(B, \nu_B))}{\operatorname{vol}(\operatorname{Ball}(\ell^p(n, \nu_n))}.$$

We have that the above sum is

$$\sum_{r=0}^{\lfloor \varepsilon n \rfloor} 2^{n-r}(\varepsilon)^{2(n(1-\kappa)-r)}(2+4\varepsilon)^{2\kappa n}\binom{n}{r}V(r,n,p)$$

where

$$V(r,n,p) = \frac{r^{2r/p}(n-r)^{2(n-r)/p}\Gamma(1+\frac{2n}{p})}{\Gamma(1+\frac{2r}{p})\Gamma(1+\frac{2n-2r}{p})n^{2n/p}}.$$

By Stirling's Formula we see that

$$V(r,n,p) \leq C(p),$$

where $C(p)$ is a constant which depends only on p.

Further if n is sufficiently large and $\varepsilon < 1/2$, then by Stirling's Formula

$$\binom{n}{r} \leq \binom{n}{\lfloor \varepsilon n \rfloor} \leq A\left(\frac{n}{\lfloor \varepsilon n \rfloor}\right)^{\lfloor \varepsilon n \rfloor}\left(\frac{n}{n-\lfloor \varepsilon n \rfloor}\right)^{n-\lfloor \varepsilon n \rfloor},$$

for some constant $A > 0$.

Putting this altogether, we have that

$$\alpha \leq \sqrt{2}\varepsilon^{(1-\kappa)-\varepsilon}(2+4\varepsilon)^{\kappa}\left(\frac{1}{\varepsilon}\right)^{\varepsilon}\left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon}.$$

Since $\kappa < 1$, the right-hand side tends to zero as $\varepsilon \to 0$ so we have a contradiction.

□

Theorem 4.5.2. *Let $\mathcal{R}$ be a sofic discrete measure-preserving equivalence relation on a standard probability space (X, μ). For all $1 \leq p \leq 2$, we have*

$$\dim_{\Sigma,\ell^p}(L^p(\mathcal{R},\overline{\mu})^{\oplus n},\mathcal{R}) = \underline{\dim}_{\Sigma,\ell^p}(L^p(\mathcal{R},\overline{\mu})^{\oplus n},\mathcal{R}) = n.$$

Proof. We shall present the proof when $n = 1$. Since our approach is probabilistic, it is not hard to generalize the proof for general n.

Let Σ be a sofic approximation of $\mathcal{R}$, and let $\mathrm{Id} \in \Phi = \Phi_0 \cup \mathcal{P}$, where Φ_0 is a graphing of $\mathcal{R}$, and $\mathcal{P}$ is generating family of projections in $L^\infty(X,\mu)$. Let $\mathrm{Id} \in F \subseteq \Phi$ be finite, $m \in \mathbb{N}, \delta > 0$. We use $S = (\chi_\Delta)$ to do our computation. It is clear that

$$\dim_{\Sigma,\ell^p}(L^p(R,\overline{\mu}),\mathcal{R}) \leq 1,$$

so it suffices to show that

$$\underline{\dim}_{\Sigma,\ell^p}(L^p(R,\overline{\mu}),\mathcal{R}) \geq 1.$$

Let

$$C = W^*(\{v_\psi p v_\psi^* : \psi \in F^m, p \in \mathcal{P} \cap F^m\}),$$

and let $\chi_{B_1}, \cdots, \chi_{B_r}$ be the minimal projections in C. Let $\{A_1, \cdots, A_q\}$ be a partition refining $\{B_1, \cdots, B_r\}$, which we will assume to be sufficiently fine in a manner to be determined later. We may assume that Σ is eventually a homomorphism on $W^*(\{A_j\}_{j=1}^q)$, there are $E_j \subseteq [[\mathcal{R}]]$,

$$\mathcal{O}_{A_j} := \{(x,y) \in R : x \in A_j\} = \bigsqcup_{\psi \in E_j} \operatorname{graph}(\psi),$$

and that

$$F^m \subseteq E_1^{-1} + E_2^{-1} + \cdots + E_q^{-1}.$$

We may also assume that for every $\psi \in E_j$ and for all large i, we have that $\operatorname{dom}(\sigma_i(\psi)) \subseteq \sigma_i(A_j)$.

Note that if $f \in L^p(\mathcal{R},\overline{\mu})$, then we can uniquely write

$$\operatorname{Id}_{A_j} f = \sum_{\psi \in E_j} f_\psi \chi_{\operatorname{graph}(\psi)},$$

where $f_\psi \in L^p(\operatorname{dom}(\psi),\mu)$ and the sum converges in $\|\cdot\|_p$. Fix $\eta > 0$, and let $F_j \subseteq E_j$ be finite and so that for all $\psi \in F^m$,

$$\operatorname{dist}_{\|\cdot\|_2}(\psi, F^m) < \eta.$$

Let ν_i be the uniform probability measure on $\{1,\ldots,d_i\}$. For $\xi \in \ell^p(d_i,\nu_i)$ define

$$T_\xi^{(j)}(f) = \sum_{\psi \in F_j} \mathbb{E}_{\operatorname{dom}(\psi)}(f_\psi)\sigma_i(\psi^{-1})\xi,$$

where for a measurable $A \subseteq X$, and $f \in L^1(A,\mu)$ we use

$$\mathbb{E}_A(f) = \frac{1}{\mu(A)}\int_A f\,d\mu.$$

Finally set

$$T_\xi = \sum_{j=1}^{q} T_\xi^{(j)}(f).$$

We claim that if $\{A_1, \cdots, A_q\}$ is sufficiently fine, and i is sufficiently large, then

$$\frac{\mu(\{\xi \in \mathrm{Ball}(\ell^2(d_i,\nu_i)) : \|T_\xi\|_{L^p\to\ell^p} \leq 2, \text{ for all } 1 \leq p \leq 2\})}{\mu(\mathrm{Ball}(\ell^2(d_i,\nu_i)} \to 1. \tag{4.2}$$

By interpolation it suffices to show that

$$\frac{\mu(\{\xi \in \mathrm{Ball}(\ell^2(d_i,\nu_i)) : \|T_\xi\|_{L^1\to\ell^1} \leq 2,)}{\mu(\mathrm{Ball}(\ell^2(d_i,\nu_i)} \to 1,$$

$$\frac{\mu(\{\xi \in \mathrm{Ball}(\ell^2(d_i,\nu_i)) : \|T_\xi\|_{L^2\to\ell^2} \leq 2,)}{\mu(\mathrm{Ball}(\ell^2(d_i,\nu_i)} \to 1,$$

Let us first do the ℓ^2 case. We have that

$$\|T_\xi^{(j)}(f)\|_2^2 \leq \sum_{\psi\in F_j} |\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)|^2 \|\sigma_i(\psi)^{-1}\xi\|_2^2 +$$

$$\sum_{\phi\neq\psi\in F_j} |\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)\mathbb{E}_{(\mathrm{dom}(\phi)}(f_\phi)||\langle\sigma_i(\psi)^{-1}\xi, \sigma_i(\phi)^{-1}\xi\rangle| \leq$$

$$\sum_{\psi\in F_j} \frac{\|f_\psi\|_2^2}{\mu(\mathrm{dom}(\psi)} \|\sigma_i(\psi)^{-1}\xi\|_2^2 +$$

$$\sum_{\phi\neq\psi\in F_j} \frac{\|f_\psi\|_2 \|f_\phi\|_2}{\mu(\mathrm{dom}(\psi)^{1/2}\mu(\mathrm{dom}(\phi)^{1/2}} |\langle\sigma_i(\psi)^{-1}\xi, \sigma_i(\phi)^{-1}\xi\rangle|.$$

Since

$$\frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i)} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} \|\sigma_i(\psi)^{-1}\xi\|_2^2 \, d\xi \leq \frac{|\operatorname{dom}(\sigma_i(\psi)^{-1}|}{d_i} \to \mu(\mathrm{dom}(\psi)),$$

$$\frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i)} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} \langle\sigma_i(\psi)^{-1}\xi, \sigma_i(\phi)^{-1}\xi\rangle \, d\xi = \frac{2n}{2n+2} \operatorname{tr}(\sigma_i(\phi)\sigma_i(\psi)) \to 0,$$

it follows by concentration of measure that $\mathbb{P}(\|T_\xi^{(j)}\| \leq 2 \text{ for all } j) \to 1$. If $\|T^{(j)}(\xi)\|_2 \leq 2$ for all j, then

$$\|T(f)\|_2^2 = \sum_{j=1}^{q} \|T_\xi^{(j)}(\mathrm{Id}_{A_j} f)\|_2^2 \leq 4 \sum_{j=1}^{q} \|\mathrm{Id}_{A_j} f\|_2^2 \leq 4\|f\|_2^2.$$

For the ℓ^1-case, simply note that

$$\|T_\xi^{(j)}(f)\|_1 \leq \sum_{\psi \in F_j} \frac{\|f_\psi\|_1}{\mu(\mathrm{dom}(\psi))} \|\sigma_i(\psi)^{-1}\xi\|.$$

Since

$$\frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} \|\sigma_i(\psi)^{-1}\xi\|_1 \, d\xi =$$
$$\frac{|\operatorname{dom}(\sigma_i(\psi)^{-1})|}{d_i} \frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} |\xi_1| d\xi \leq$$
$$\frac{|\operatorname{dom}(\sigma_i(\psi)^{-1})|}{d_i} \left(\frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} |\xi_1|^2 \, d\xi \right)^{1/2} =$$
$$\frac{|\operatorname{dom}(\sigma_i(\psi)^{-1})|}{d_i} \left(\frac{1}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))} \int_{\mathrm{Ball}(\ell^2(d_i,\nu_i)} \|\xi\|_2^2 \, d\xi \right)^{1/2} \leq$$
$$\frac{|\operatorname{dom}(\sigma_i(\psi)^{-1})|}{d_i} \to \mu(\mathrm{dom}(\psi)).$$

So, again by concentration of measure

$$\mathbb{P}(\{\xi : \|T_\xi^{(j)}\|_{L^1 \to \ell^1} \leq 2 \text{ for all } j\}) \to 1.$$

If $\|T_\xi^{(j)}\|_{L^1 \to \ell^1} \leq 2$ for all j, it is again easy to see that $\|T_\xi\|_{L^1 \to \ell^1} \leq 1$. Thus (4.2) holds. Suppose $\phi \in F^m$, by our choice of $E_1, \cdots, E_q$, we may write

$$\phi = \sum_{j=1}^{q} \sum_{\psi \in E_j} c_{j,\psi} \psi^{-1},$$

where $c_{j,\psi} \in \{0,1\}$, further

$$\left\| \phi - \sum_{j=1}^{q} \sum_{\psi \in F_j} c_{j,\psi} \psi^{-1} \right\|_2^2 < \eta^2.$$

So

$$\|T(\chi_{\mathrm{graph}(\phi)}) - \sigma_i(\phi) T(\chi_\Delta)\|_2 = \left\| \left(\sum_{j=1}^{q} \sum_{\psi \in F_j} c_{j,\psi} \sigma_i(\psi)^{-1} - \sigma_i(\phi) \right) \xi \right\|_2.$$

As in Lemma 3.4.1,

$$\int_{S^{2d_i-1}} \left\| \left(\sum_{j=1}^{q} \sum_{\psi \in F_j} c_{j,\psi} \sigma_i(\psi)^{-1} - \sigma_i(\phi) \right) \xi \right\|_2^2 d\xi = \left\| \sum_{j=1}^{q} \sum_{\psi \in F_j} c_{j,\psi} \sigma_i(\psi)^{-1} - \sigma_i(\phi) \right\|_2^2$$

and so for most ξ,

$$\left\|\left(\sum_{j=1}^{q}\sum_{\psi\in F_j} c_{j,\psi}\sigma_i(\psi)^{-1}-\sigma_i(\phi)\right)\xi\right\|_2<2\eta,$$

by concentration of measure. Thus we have shown that

$$\frac{\mathrm{vol}(\alpha_S(\mathrm{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i)_2))}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))}\to 1,$$

and since

$$\inf_i\left(\frac{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i))}{\mathrm{vol}(\mathrm{Ball}(\ell^p(d_i,\nu_i))}\right)^{1/2d_i}>0,$$

we are done by the proceeding Lemma.

□

We can prove a nice fact in the case of the action of $\mathcal{R}$ on $L^2(\mathcal{R},\overline{\mu})$ but we will need a generalization of our previous volume packing Lemma.

Proposition 4.5.3. *There is a function $\kappa(\alpha,\varepsilon)$ with*

$$\lim_{\varepsilon\to 0}\kappa(\alpha,\varepsilon)=1$$

for all α which has the following property. Let d_i be a sequence of integers going to infinity, and let $A_i\subseteq \mathrm{Ball}(\ell^2(d_i))$, and let p_i be a projection on $\ell^2(d_i)$, so that $\mathrm{tr}(p_i)$ converges. If

$$\liminf_{i\to\infty}\left(\frac{\mathrm{vol}(A_i)}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i)))}\right)^{1/d_i}\geq\alpha,$$

then

$$\liminf_{i\to\infty}\frac{1}{d_i\,\mathrm{tr}(p_i)}d_\varepsilon(p_iA_i,\|\cdot\|_2)\geq\kappa(\alpha,\varepsilon).$$

Proof. If the claim is false, then we can find $\kappa<1$, so that for every $\varepsilon>0$ there are sets A_i as in the proposition, and subspaces $V_i\subseteq\ell^2(d_i)$ with $\dim(V_i)\leq\kappa\,\mathrm{tr}(p_i)d_i$, so that $p_iA_i\subseteq_\varepsilon V_i$. This implies that

$$p_iA_i\subseteq\bigcup_{\substack{B\subseteq\{1,\cdots,d_i\},\\|B|\leq\varepsilon d_i}}[(1+\varepsilon)\,\mathrm{Ball}_{\|\cdot\|_2}(\chi_{B^c}(V_i)+\varepsilon\,\mathrm{Ball}(\ell^2(B^c))]\times\mathrm{Ball}(p_i\ell^2(B)).$$

Let $q_i = \operatorname{tr}(p_i), q = \lim q_i$, also let $V(k)$ be the volume of the k-dimensional ball in $\ell^2(d_i)$. Then we have

$$\operatorname{vol}(p_i A_i) \leq \operatorname{vol}[(1+\varepsilon)\operatorname{Ball}(p_i\chi_{B^c}(V_i) + \varepsilon \operatorname{Ball}(\ell^2(B^c))]V(\dim(p_i\ell^2(B))).$$

Let S_B be a maximal ε-separated subset of $(1+\varepsilon)\operatorname{Ball}(p_i\chi_{B^c}(V_i))$, then

$$|S_B| \leq \left(\frac{2+2\varepsilon}{\varepsilon}\right)^{\dim(p_i\chi_{B^c}(V_i))} \leq \left(\frac{2+2\varepsilon}{\varepsilon}\right)^{\kappa q_i d_i}.$$

Thus

$$\begin{aligned}\operatorname{vol}(p_i A_i) &\leq \sum_{\substack{B\subseteq\{1,\cdots,d_i\},\\ |B|\leq \varepsilon d_i}} \left(\frac{2+2\varepsilon}{\varepsilon}\right)^{\kappa q_i d_i} (2\varepsilon)^{\dim(p_i\ell^2(B^c))} V(\dim(p_i\ell^2(B)))V(\dim(p_i\ell^2(B^c)) \\ &\leq \sum_{\substack{B\subseteq\{1,\cdots,d_i\},\\ |B|\leq \varepsilon d_i}} 4^{\kappa q_i d_i} 2^{d_i q_i} \varepsilon^{d_i(1-\kappa)q_i} V(\dim(p_i\ell^2(B))V(\dim(p_i\ell^2(B^c)).\end{aligned}$$

Thus

$$\frac{\operatorname{vol}(p_i A_i)}{V(q_i d_i)} \leq \sum_{\substack{B\subseteq\{1,\cdots,d_i\},\\ |B|\leq \varepsilon d_i}} 4^{\kappa q_i d_i} 2^{d_i q_i} \varepsilon^{d_i(1-\kappa)q_i} \frac{V(\dim(p_i\ell^2(B))V(\dim(p_i\ell^2(B^c))}{V(q_i d_i)}.$$

Now

$$V(k) = \frac{\pi^k}{k!},$$

so by Stirling's Formula there is a constant $C > 0$ so that

$$\frac{V(\dim(p_i\ell^2(B)))V(\dim(p_i\ell^2(B^c))}{V(q_i d_i)} \leq C\pi^{\varepsilon d_i} \frac{(q_i d_i)^{q_i d_i} e^{\varepsilon d_i}\sqrt{2\pi q_i d_i}}{(q_i-\varepsilon)^{(q_i-\varepsilon)d_i}\sqrt{2\pi(q_i-\varepsilon)d_i}}.$$

Thus

$$\liminf_{i\to\infty} \left(\frac{\operatorname{vol}(p_i A_i)}{V(q_i d_i)}\right)^{1/d_i} \leq \frac{q^q}{(q-\varepsilon)^{q-\varepsilon}\varepsilon^{\varepsilon}(1-\varepsilon)^{(1-\varepsilon)}} 4^{\kappa q} 2^q \varepsilon^{(1-\kappa)q}.$$

Since $\text{vol}(A_i) \leq \text{vol}(p_i A)V((1-q_i)d_i)$ we have

$$\alpha \leq \frac{q^q}{(q-\varepsilon)^{q-\varepsilon}\varepsilon^\varepsilon(1-\varepsilon)^{(1-\varepsilon)}} 4^{\kappa q} 2^q \varepsilon^{(1-\kappa)q} \times$$
$$\liminf_{i\to\infty} \left(\frac{\text{vol}(q_i d_i)V((1-q_i)d_i)}{V(d_i)} \right)^{1/d_i}$$
$$= \frac{(1-q)^{1-q}}{(q-\varepsilon)^{q-\varepsilon}\varepsilon^\varepsilon(1-\varepsilon)^{(1-\varepsilon)}} 4^{\kappa q} 2^q \varepsilon^{(1-\kappa)q}.$$

Letting $\varepsilon \to 0$, we obtain a contradiction.

□

Theorem 4.5.4. *Let $\mathcal{R}$ be a discrete-measure preserving sofic equivalence relation on (X, μ). Let $\mathcal{H}$ be a separable unitary representation of $\mathcal{R}$ such that the action of $\mathcal{R}$ on $\mathcal{H}$ extends to the von Neumann algebra $L(\mathcal{R})$. For any sofic approximation Σ of $\mathcal{R}$, we have*

$$\dim_{\Sigma,\ell^2}(\mathcal{H}, \mathcal{R}) = \underline{\dim}_{\Sigma,\ell^2}(\mathcal{H}, \mathcal{R}) = \dim_{L(\mathcal{R})}(\mathcal{H}).$$

Proof. We first show that

$$\underline{\dim}_{\Sigma,\ell^2}(\mathcal{H}, \mathcal{R}) \geq \dim_{L(\mathcal{R})}(\mathcal{H}).$$

Our hypothesis implies that as a representation of $\mathcal{R}$,

$$\mathcal{H} \cong \bigoplus_{j=1}^{\infty} L^2(\mathcal{R}, \overline{\mu})q_j,$$

with $q_j \in \text{Proj}(L(\mathcal{R}))$.

As in Theorem 4.5.2 we shall deal with the case that $\mathcal{H} = L^2(\mathcal{R}, \mu)q$ for some $q \in \text{Proj}(L(\mathcal{R}))$, it is easy to see that our proof generalizes.

Thus $\mathcal{H}$ is unitarily equivalent to a subrepresentation of $L^2(R, \overline{\mu})$ so we may as well assume that it is a subrepresentation of $L^2(R, \overline{\mu})$. Let p be the projection onto $\mathcal{H}$, we use $\widehat{p} = p\chi_\Delta$ to do our calculation. Fix a graphing Φ of $\mathcal{R}$, and

$$\sigma_i \colon [[\mathcal{R}]] \to [[\mathcal{R}_{d_i}]],$$

a sofic approximation. By Lemma 2.2.6 we may extend σ_i to an embedding sequence

$$\sigma_i\colon L(\mathcal{R}) \to M_{d_i}(\mathbb{C}).$$

By perturbing elements slightly, we may assume that $p_i = \widetilde{\sigma}_i(p)$ is a projection for all i. Let T_ξ be the operator constructed in the proof of Theorem 4.5.2. Fix $F \subseteq \Phi$ finite, $m \in \mathbb{N}, \delta > 0$.

We know that for every $F' \subseteq \Phi$ finite, $m' \in \mathbb{N}$, $\delta' > 0$ that

$$\frac{\operatorname{vol}(\{\xi \in \operatorname{Ball}(\ell^2(d_i)) : T_\xi \in \operatorname{Hom}_{\mathcal{R},\ell^2}(\{\chi_\Delta\}, F, m, \delta, \sigma_i)\})}{\operatorname{vol}(\operatorname{Ball}(\ell^2(d_i))} \to 1,$$

and that $T_\xi(\chi_\Delta)$ is close to ξ.

It is easy to see that if F', m', δ' are chosen wisely then

$$\operatorname{Hom}_{\mathcal{R},\ell^2}(\{\chi_\Delta\}, F', m', \delta', \sigma_i))\big|_{pL^2(R,\overline{\mu})} \subseteq \operatorname{Hom}_{\mathcal{R},\ell^2}(\{\widehat{p}\}, F, m, \delta, \sigma_i),$$

and that $T_\xi(\widehat{p})$ is close to p_i. Thus the preceding proposition proves the lower bound.

For the upper bound, let $S = (\chi_\Delta q_j)_{j=1}^\infty$. Fix $\varepsilon > 0$, $m \in \mathbb{N}$, it is easy to see that if F is large, and $\delta > 0$ is small enough then

$$\{(T(\chi_\Delta q_1), \cdots, T(\chi_\Delta q_m)) : T \in \operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, m, \delta, \sigma_i)\} \subseteq_\varepsilon \bigoplus_{j=1}^m \sigma_i(q_j) \operatorname{Ball}(\ell^2(d_i)),$$

as

$$\operatorname{tr}(\sigma_i(q_j))) \to \tau(q_j),$$

the desired upperbound is proved.

□

We close this section with a complete computation in the case of direct integrals of finite-dimensional representations.

Proposition 4.5.5. *Let (R, X, μ) be a discrete, measure-preserving equivalence relation. Suppose that for some $n \in \mathbb{N}$, $|\mathcal{O}_x| = n$ for almost for every $x \in X$. Let V_x be a measure-field of finite dimensional vector spaces and π a representation of $\mathcal{R}$ on V_x. Then for all $1 \leq p < \infty$, and for every sofic approximation Σ of $\mathcal{R}$,*

$$\dim_{\Sigma,\ell^p}\left(\int_X^{\oplus_p} V_x\, d\mu(x), \mathcal{R}\right) = \underline{\dim}_{\Sigma,\ell^p}\left(\int_X^{\oplus_p} V_x\, d\mu(x), \mathcal{R}\right) = \frac{1}{n}\int_X \dim(V_x)\, d\mu(x).$$

Proof. We shall only handle the case when $\dim(V_x)$ is almost surely constant, say equal to k. The general case will follow by more or less the same proof. Without loss of generality $V_x = \mathbb{C}^k$ with the Euclidean norm and $\pi(x,y)$ is a unitary for almost every $(x,y) \in \mathcal{R}$. Let $\alpha \in [\mathcal{R}]$ be n-periodic and so that up to sets of measure zero, $\mathcal{R} = \{(x, \alpha^j(x)) : 0 \leq j \leq n-1\}$. Let

$$b(\alpha^j x) = \pi(x, \alpha^j x), x \in A, 0 \leq j \leq n-1.$$

Then

$$b(\alpha^j x) b(\alpha^k x)^{-1} = \pi(\alpha^j x, \alpha^k x),$$

that is

$$b(y) b(x)^{-1} = \pi(y, x)$$

for $x, y \in \mathcal{R}$.

Define $T \colon L^p(X, \mu, \mathbb{C}^k) \to L^p(X, \mu, \mathbb{C}^k)$ by

$$(Tf)(x) = b(x) f(x).$$

For $\phi \in [[\mathcal{R}]]$, we have

$$\phi \cdot (Tf)(x) = \chi_{\mathrm{ran}(\phi)}(x) \pi(x, \phi^{-1}x) b(\phi^{-1}x) f(\phi^{-1}x) = \chi_{\mathrm{ran}(\phi)}(x) b(x) f(\phi^{-1}x) =$$

$$T(f \circ \phi^{-1})(x).$$

Thus we may assume that $\pi(x,y) = \mathrm{Id}$ for all $(x,y) \in \mathcal{R}$. Find $A \subseteq X$ so that up to sets of measure zero,

$$X = \bigsqcup_{j=0}^{n-1} \alpha^j(A).$$

Let $S = (e_j \otimes \chi_A)_{j=1}^n$, where $v \otimes f(x) = f(x)v$ for $f \colon X \to \mathbb{C}$ measurable and $v \in \mathbb{C}^k$. Set

$$\rho_i(f) = \sum_{j=1}^k \|f_j\|,$$

for $f \in \ell^\infty(k, \ell^p(d_i))$. Fix $\Phi \subseteq [[\mathcal{R}]]$ containing $\{\mathrm{Id}, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ and a set $\mathcal{P}$ of projections in $L^\infty(A, \mu)$ so that there is a sequence $\mathcal{P}_n$ of partitions of A in $\mathcal{P}$ so that $\mathcal{P}_n \to \mathrm{Id}$. Without loss of generality, we may assume that for each n, σ_i is eventually a $*$-homomorphism on $W^*(\mathcal{P}_n, \alpha)$ with $\operatorname{tr}(\sigma_i(\operatorname{Id})) \to 1$. Let $\mathcal{P}_n = \{B_{1,n}, \cdots, B_{m_n,n}\}$.

Fix $\varepsilon > 0$, and $N \in \mathbb{N}$. Suppose $F \subseteq \Phi$ is finite, and contains $\{\operatorname{Id}, \alpha, \alpha^2, \cdots, \alpha^{n-1}, \operatorname{Id}_A\}$, $m \in \mathbb{N}, \delta > 0$. It is easy to see that $\alpha_S(\operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, m, \delta, \sigma_i))$ is almost contained in $\ell^p(\sigma_i(\operatorname{Id}_A)\{1, \cdots, d_i\})^{\oplus k}$. Thus

$$\dim_{\Sigma,\ell^p}(F, m, \delta, \varepsilon, \rho_i) \leq \lim_{i\to\infty} \frac{k \operatorname{tr}(\sigma_i(\operatorname{Id}_A)}{d_i} = \frac{k}{n}.$$

Define

$$T_{\xi,N} \colon L^p(X, \mu, \mathbb{C}^k) \to \ell^p(d_i)$$

by

$$T_{\xi,N}(f) = \sum_{j=0}^{n-1} \sum_{k=1}^{m_N} \left(\frac{1}{\mu(B_{k,N})} \int_{B_{k,N}} f \circ \alpha^j \, d\mu \right) \sigma_i(\alpha)^j \sigma_i(\operatorname{Id}_{B_{k,N}})\xi.$$

Simple estimates prove that

$$\|T_{\xi,N}(f)\|_p^p \leq \sum_{j=0}^{n-1} \sum_{k=1}^{m} \left(\int_{\alpha^j(B_{k,N})} |f|^p \, d\mu \right) \frac{\|\sigma_i(\operatorname{Id}_{B_{k,N}})\xi\|_p^p}{\mu(B_k)^p}.$$

As

$$\frac{1}{\operatorname{vol}(\operatorname{Ball}(\ell^p(d_i, \nu_i))} \int_{\operatorname{Ball}(\ell^p(d_i,\nu_i))} \|\sigma_i(\operatorname{Id}_{B_{k,N}})\xi\|_p \, d\mu = \operatorname{tr}(\sigma_i(\operatorname{Id}_{B_{k,N}})) \to \mu(B_{k,N}),$$

there are $C_i \subseteq \operatorname{Ball}(\ell^p(d_i, \nu_i)$ with $\liminf_i \frac{\operatorname{vol}(C_i)}{\operatorname{vol}(\operatorname{Ball}(\ell^p(d_i,\nu_i))} \geq 1/3$, so that $\|T_{\xi,N}\| \leq 2$ if $\xi \in C_i$.

For all large i,

$$T_{\xi,N}(\alpha f) = \sigma_i(\alpha) T_\xi(f).$$

$$T_{\xi,N}(\operatorname{Id}_{B_{k,N}} f) = \sigma_i(\operatorname{Id}_{B_{k,N}}) T_{\xi,N}(f)$$

thus if N is large enough $T_{\xi,N} \in \operatorname{Hom}_{\mathcal{R},\ell^p}(S, F, m, \delta, \sigma_i)$ if $\|\xi\|_p \leq 1$.

As

$$T_{\xi,N}(\chi_A) = \sigma_{\operatorname{Id}}(A)\xi,$$

we have

$$\alpha_S(\mathrm{Hom}_{\mathcal{R},\ell^p}(S,F,m,\delta,\sigma_i)) \supseteq \{\sigma_{\mathrm{Id}}(A)\xi : \xi \in C_i\}.$$

so

$$\dim_{\Sigma,\ell^p}(L^p(X,\mu,\mathbb{C}^k),\mathcal{R}) \geq \frac{k}{n}.$$

□

Corollary 4.5.6. *Let $(X,\mu,\mathcal{R})$ be a discrete measure-preserving equivalence relation. Suppose that $\mathcal{O}_x$ is infinite for almost every x. Let V be a measurable field of finite-dimensional vector spaces with an action of $\mathcal{R}$. Then for $1 \leq p < \infty$, we have*

$$\dim_{\Sigma,\ell^p}\left(\int_X^{\oplus_p} V_x\, d\mu(x),\mathcal{R}\right) = 0.$$

Proof. This is simple from the preceding proposition, since for every $n \in \mathbb{N}$, there is a subequivalence relation $\mathcal{R}_n \subseteq \mathcal{R}$, where $\mathcal{R}_n$ has orbits of size n for almost every $x \in X$ (see Proposition 2.1.24)

□

4.6 ℓ^p-Homology of Equivalence Relations

Let G be a locally finite graph. We let $\mathcal{E}(G)$ be the set of oriented edges of G, and $E(G)$ the set of unoriented edges of G, also we let $V(G)$ be the set of vertices of G. If $x,y \in V(G)$, we let (x,y) be the oriented edge from x to y, and $[x,y]$ be the unoriented edge between x and y. We shall abuse notation and use $\mathbb{C}^{E(G)}$ for all functions $f\colon \mathcal{E}(G) \to \mathbb{C}$ such that $f(x,y) = -f(y,x)$ for all $(x,y) \in \mathcal{E}(G)$. We let $\ell^p(E(G))$ be the functions in $\mathbb{C}^{E(G))}$ so that

$$\|f\|_p^p = \sum_{[x,y]\in E(G)} |f(x,y)|^p < \infty$$

(note $|f(x,y)|$ does not depend on the orientation of $[x,y]$). Similar remarks apply for $c_c(E(G))$ and other function spaces.

If (x, y) is an oriented edge in G, define $\mathcal{E}_{x,y}(u, v) = 0$ if one of u, v is not x or y, 1 if $(u, v) = (x, y)$ and -1 if $(u, v) = (y, x)$. If $\gamma\colon \{0, \cdots, k\} \to V(G)$ is a path (i.e. for all $1 \leq j \leq k$, $(\gamma(j), \gamma(j)) \in \mathcal{E}(G)$) we think of γ as a an element of $\ell^p(E(G))$ by having γ correspond to

$$\sum_{j=1}^{k} \mathcal{E}_{(\gamma(j-1),\gamma(j))}.$$

For $f\colon \mathcal{E}(G) \to \mathbb{C}$ and γ a path as above, we define

$$\int_\gamma f = \sum_{j=1}^{k} f(\gamma(j-1), \gamma(j)).$$

For a general graph G, define $\delta\colon \mathbb{C}^{V(G)} \to \mathbb{C}^{E(G))}$, $\partial\colon \mathbb{C}^{E(G)} \to \mathbb{C}^{V(G)}$ by

$$\delta f(v, w) = f(w) - f(v)$$

$$(\partial f)(v) = \sum_{w \in V(G):(w,v)\in\mathcal{E}(G)} f(w, v).$$

Then δ and ∂ are dual in the following sense: if $f \in c_c(E(G))$, and $g \in \mathbb{C}^{V(G)}$ then

$$\langle \partial f, g\rangle = -\langle f, \delta g\rangle$$

where

$$\langle h, k\rangle = \sum_{[x,y]\in E(G)} h(x, y)k(x, y)$$

for $h \in c_c(E(G)), k \in \mathbb{C}^{E(G)}$, (again this is independent of orientation). Similarly if $f \in \mathbb{C}^{E(G)}, g \in c_c(V(G))$, then

$$\langle g, \partial f\rangle = -\langle \delta g, f\rangle.$$

Let

$$B_1(G) = \mathrm{Span}\{\gamma \in c_c(E(G)) : \gamma \text{ is a loop}\}.$$

$$Z_1(G) = \{f \in c_c(E(G)) : \partial f = 0\}$$

$$Z^1(G) = \left\{f \in \mathbb{C}^{E(G)} : \int_\gamma f = 0 \text{ for all loops } \gamma\right\}$$

If $f \in Z^1(G)$ and v, w are vertices in G, and $\gamma\colon \{0, \cdots, k\} \to V(G)$ is a path from v to w, then

$$\int_\gamma f$$

depends only on v and w since f integrates to zero along all loops. We will use

$$\int_{v \to_G w} f,$$

for this number. Note that $Z^1(G) = \{\delta_G h : h \in \mathbb{C}^{V(G)}\}$. In fact, if $f \in Z^1(G)$, and $(G_j)_{j \in J}$ are the connected components, then for fixed $x_j \in V(G_j)$

$$h(v) = \int_{x_j \to_G v} f,$$

for $v \in V(G_j)$ has $\delta_G h = f$.

Define the space of ℓ^p-cocycles by

$$Z^1_{(p)}(G) = Z^1(G) \cap \ell^p(E(G)).$$

Define the space of ℓ^p-boundaries by

$$B_1^{(p)}(G) = \overline{B_1(G)}^{\|\cdot\|_p}.$$

If $G' \subseteq G$ is a subgraph we identify $\mathbb{C}^{E(G')} \subseteq \mathbb{C}^{E(G)}$ by extending by zero. This allows us to make sense of all the function spaces above for G' as subsets of $\mathbb{C}^{E(G)}$.

Definition 4.6.1. Let $(\mathcal{R}, X, \mu)$ be a discrete measure-preserving equivalence relation. A *measurable field of graphs fibered over* $\mathcal{R}$ is a field $\{\Phi_x\}_{x \in X}$ of graphs having vertex set $\mathcal{O}_x$, such that $\Phi_x = \Phi_y$ for almost every $(x, y) \in \mathcal{R}$, and $\bigcup_{x \in X} \mathcal{E}(\Phi_x)$ is a measurable subset of $\mathcal{R}$ which intersects the diagonal in a set of measure zero.

We set $\mathcal{E}(\Phi) = \bigcup_{x \in X} \mathcal{E}(\Phi_x)$.

If Φ is a measurable field of graphs fibered over $\mathcal{R}$, the cost of Φ by (originally defined by Levitt in [19]) is defined by

$$c(\Phi) = \frac{1}{2} \int_X \deg(x)\, d\mu(x)$$

where $\deg(x)$ is the degree of the vertex x. This is also

$$\frac{1}{2}\overline{\mu}(\mathcal{E}(\Phi)).$$

We recall that the cost of $\mathcal{R}$, as defined by Levitt is given by

$$c(\mathcal{R}) = \inf c(\Phi)$$

where the infimum is over all measurable fields of graphs fibered over $\mathcal{R}$ which are connected for almost every x. Many important properties of cost are discussed in [11]. In particular, Gaboriau proves a formula for how cost behaves under compression.

For any $\Phi = (\phi_j)_{j \in J}$, with J countable and $\phi_j \in [[\mathcal{R}]]$, and for each $x \in X$, we define a graph whose vertices are $\mathcal{O}_x$ and whose oriented edges are $\{(u, v) : u \sim x, v = \phi^{\pm 1}(u), \text{ for some } \phi \in \Phi\}$. If Φ_x denotes the corresponding graph note that

$$c(\Phi) = \sum_{j \in J} \mu(\mathrm{dom}(\phi_j)).$$

Note that Φ_x is connected almost everywhere if and only if Φ is a graphing, and in this case $c(\Phi)$ Is simply the cost of Φ as previously defined. We leave it as an exercise to use the measurable selection theorem (Theorem 2.1.18) to show that any measurable field of graphs over X comes from a graphing of a subequivalence relation.

If $x \to \Phi_x$ is a measurable field of graphs over X, let $L^p(E(\Phi))/B_1^{(p)}(E(\Phi))$ be the L^p-direct integral of the space $\ell^p(E(\Phi_x))/B_1^{(p)}(\Phi_x)$. Note that $\mathcal{R}$ has a representation π on $\ell^p(E(\Phi_x))/B_1^{(p)}(\Phi_x)$, given by $\pi(x, y) = \mathrm{Id}$ for all $(x, y) \in \mathcal{R}$.

We will show that if $\mathcal{R}$ has finite cost and satisfies a "finite presentation" assumption, then $\dim_\Sigma(L^p(E(\Phi))/B_1^{(p)}(E(\Phi)), \mathcal{R})$ does not depend on the choice of finite cost graph Φ.

Definition 4.6.2. Let $(X, \mu, \mathcal{R})$ be a discrete measure-preserving equivalence relation. Let $\Phi = (\phi_j)_{j \in J}$ be a graphing of $\mathcal{R}$. We say that Φ is *finitely presented* if there are measurable fields of loops $(L^{(j)})_{j=1}^{\infty}$ such that for almost every $x \in X$,

$$\mathrm{Span}\{L_y^{(j)} : y \sim x, j \in \mathbb{N}\} = B_1(\Phi_x),$$

and

$$\sum_{j=1}^{\infty} \mu(\operatorname{supp} L^{(j)}) < \infty.$$

We say that $\mathcal{R}$ is *finitely presented* if it has a finitely presented graphing. For example, if $\mathcal{R}$ is a induced by a free action of a finitely presented group, then $\mathcal{R}$ is finitely presented.We will proceed to show that if $\mathcal{R}$ is finitely presented, then in fact every graphing is finitely presented. It may be useful to consider the group analogue first.

Suppose $\Gamma = \langle s_1, \cdots, s_n | r_1, r_2, \cdots, r_m \rangle$ is a finitely presented group. And suppose that $t_1, \cdots, t_k$ also generate Γ. Choose words w_i in $t_1, \cdots, t_k$ so that

$$w_i(t_1, \cdots, t_k) = s_i$$

and choose words v_i in $s_1, \cdots, s_n$ so that

$$t_i = v_i(s_1, \cdots, s_n).$$

Set

$$\sigma_i = r_i(w_1, \cdots, w_n),$$

$$\eta_i = v_i(w_1, \cdots, w_n),$$

then one can show that

$$\Gamma = \langle t_1, t_2, \cdots, t_k | \sigma_1, \cdots, \sigma_m, \eta_1 t_1^{-1}, \eta_2 t_2^{-1}, \cdots, t_k a_k^{-1} \rangle.$$

Graphically, choosing words w_i, v_i as above corresponds to finding a path in $\operatorname{Cay}(\Gamma, \{t_1, \cdots, t_k\})$ from e to s_i and vice versa. So we will simply express the above proof in the language of graphs and this will allow us to generalize to the case of equivalence relations.

Lemma 4.6.3. *Let G, G' be two connected locally finite graphs with the same vertex set. Choose paths $\{\sigma_{y,z}\}_{(y,z) \in \mathcal{E}(G)}$ in G' from y to z such that $\sigma_{yz} = -\sigma_{zy}$. Similarly, choose paths $\{\gamma_{v,w}\}_{(v,w) \in \mathcal{E}(G')}$ in G from v to w such that $\gamma_{vw} = -\gamma_{wv}$. Suppose that $\{L_j : j \in J\}$ is a family of loops in G so that*

$$B_1(G) = \operatorname{Span}\{L_j : j \in J\}.$$

Define $T\colon c_c(E(G)) \to c_c(E(G'))$, *by*

$$Tf = \sum_{[y,z]\in E(G)} f(y,z)\sigma_{yz},$$

Then

$$B_1(G') = \mathrm{Span}\{T(L_j) : j \in J\} + \mathrm{Span}\{T(\gamma_{v,w}) - \mathcal{E}_{(v,w)} : (v,w) \in \mathcal{E}(G')\}.$$

Proof. Note that

$$c_c(E(G)) = \bigcup_{F\subseteq E(G) \text{ finite}} c_c(F),$$

give $c_c(E(G))$ the direct limit topology with respect to this filtration. That is, if $f_n \in c_c(E(G))$ then a sequence $f_n \in c_c(E(G))$ converges to $f \in c_c(E(G))$ if and only if there is a finite subset $F \subseteq E(G)$ so that $\mathrm{supp}\{f_n\} \subseteq F$ for all n and $f_n \to f$ pointwise. It is easy to see that every subspace of $c_c(E(G))$ is closed in this topology, and that $c_c(E(G))^* = \mathbb{C}^{E(G)}$ with respect to the pairing

$$\langle f, g\rangle = \sum_{[y,z]\in E(G)} f(y,z)g(y,z)$$

(the above sum being independent of the orientation of edges).

Let $g \in \mathbb{C}^{E(G')}$ be such

$$\int_{T(L_j)} g = 0,\ g(v,w) = \int_{(T(\gamma_{v,w}))} g.$$

Note that the topological vector space adjoint

$$T^t\colon \mathbb{C}^{E(G')} \to \mathbb{C}^{E(G)},$$

is given by

$$T^t f(y,z) = \int_{\sigma_{yz}} f,$$

Thus

$$\int_{L_j} T^t g = \int_{T(L_j)} g = 0,$$

for all j. As the L_j span $B_1(G)$, This implies that there is a $h\colon V(G) \to \mathbb{C}$ such that $\delta_G h = T^t g$. Note that for all $(v,w) \in \mathcal{E}(G')$,

$$h(w) - h(v) = \int_{\gamma_{vw}} T^t g = \int_{T(\gamma_{v,w})} g = g(v,w).$$

Therefore,

$$\delta_{G'} h = g.$$

This implies that $g \in Z^1(G')$. As $Z^1(G')$ is the annihilator of $B_1(G)$, and $c_c(E(G))$ is a locally convex space, the Hahn-Banach Theorem now completes the proof.

□

Lemma 4.6.4. *Let $(\mathcal{R}, X, \mu)$ be a discrete measure-preserving equivalence relation, if $\mathcal{R}$ is finitely presented then every finite cost graphing of $\mathcal{R}$ is finitely presented.*

Proof. Let Φ be finitely presented and let $L^{(j)}$ be as in the definition of finitely presented. By measurable selection, (Theorem 2.1.18) we may let $(\mathcal{E}_k)_{k\in K}$ be a countable family of partially defined measurable functions from X to X with the following properties:

1: $\mathcal{E}(\Phi) = \bigcup_{k\in K}\{(x, \mathcal{E}_k(x)) : x \in \mathrm{dom}(\mathcal{E}_k)\} \cup \{(\mathcal{E}_k(x), x) : x \in \mathrm{dom}(\mathcal{E}_k)\}$

2: for all j, k $\{(x, \mathcal{E}_j(x)) : x \in \mathrm{dom}(\mathcal{E}_j)\} \cap \{(\mathcal{E}_k(x), x) : x \in \mathrm{dom}(\mathcal{E}_k)\} = \varnothing$

3: for all $j \neq k$,$\{(x, \mathcal{E}_j(x)) : x \in \mathrm{dom}(\mathcal{E}_j)\} \cap \{(x, \mathcal{E}_k(x)) : x \in \mathrm{dom}(\mathcal{E}_k)\} = \varnothing$

By measurable selection, we may choose each $k \in K$, a measurable path $\sigma_x^{(k)}$ in Ψ so that for almost every x, $\sigma_x^{(k)}$ is a path from x to $\mathcal{E}_k(x)$. Define

$$T_x\colon c_c(E(\Phi_x)) \to c_c(E(\Psi_x))$$

by

$$T_x f = \sum_{y\sim x} \sum_{k\in K : y\in\mathrm{dom}(\mathcal{E}_k)} f(y, \mathcal{E}_k(y))\sigma_y^{(k)}.$$

Then $T_x = T_y$ if $y \sim x$. Let $(\mathcal{D}_\alpha)_{\alpha\in A}$ be a countable family of partially defined measurable functions from X to X in Ψ following properties:

1: $\mathcal{E}(\Psi) = \bigcup_{k \in A}\{(x, \mathcal{D}_k(x)) : x \in \operatorname{dom}(\mathcal{D}_k)\} \cup \{(\mathcal{D}_k(x), x) : x \in \operatorname{dom}(\mathcal{D}_k)\}$

2: for all j, k $\{(x, \mathcal{D}_j(x)) : x \in \operatorname{dom}(\mathcal{D}_j)\} \cap \{(\mathcal{D}_k(x), x) : x \in \operatorname{dom}(\mathcal{E}_k)\} = \varnothing$

3: for all $j \neq k$,$\{(x, \mathcal{D}_j(x)) : x \in \operatorname{dom}(\mathcal{D}_j)\} \cap \{(x, \mathcal{D}_k(x)) : x \in \operatorname{dom}(\mathcal{D}_k)\} = \varnothing$

Let $\gamma_x^{(\alpha)}$ be a measurable family of paths in Φ so that for almost x, $\gamma_x^{(\alpha)}$ is a path from x to $\mathcal{D}_\alpha(x)$. From the preceding lemma, it then follows that $(T_x L_x^{(j)})_{j=1}^\infty, (T_x(\gamma_x^{(\alpha)}) - \mathcal{E}_{(x, \mathcal{D}_x^{(\alpha)})})$ is a measurable family of loops in Ψ_x and

$$\operatorname{Span}\{\{T_y L_y^{(j)} : j \in \mathbb{N}, y \sim x\} \cup \{(T_y(\gamma_y^{(\alpha)}) - \mathcal{E}_{(y, \mathcal{D}_y^{(\alpha)})}) : y \sim x, \alpha \in A\} = B_1(\Psi_x),$$

(note that to apply the preceding Lemma we need to use that $T_x L_y^{(j)} = T_y L_x^{(j)}$, and similarly for $\gamma_y^{(\alpha)}$).Further,

$$\sum_j \mu(\operatorname{supp} T(L^{(j)})) \leq \sum_{j=1}^\infty \mu(\operatorname{supp} L^{(j)}) < \infty,$$
$$\sum_\alpha \mu(\operatorname{supp}(T(\gamma^{(\alpha)}) - \mathcal{E}_{(\cdot, \mathcal{D}_\alpha(\cdot))})) \leq c(\Psi) < \infty.$$

□

We now proceed to prove that $\dim_{\Sigma, \ell^p}(L^p(E(\Phi))/B_1^{(p)}(\Phi), \mathcal{R})$ does not depend upon the choice of finite cost graphing when $\mathcal{R}$ is finitely presented. Our methods are similar to Gaboriau's in [12]. We must be more careful, however, since we do not have monotonicity of our dimension. We will need the following "Continuity Lemma."

Lemma 4.6.5. *Fix $1 \leq p, q < \infty$. Let $(\mathcal{R}, X, \mu)$ be a sofic, discrete, measure-preserving equivalence relation, with $\mathcal{R}$ finitely presented. If Φ is a finite cost graphing of $\mathcal{R}$, and $\Phi^{(n)}$ is an increasing sequence of subgraphs of $\mathcal{R}$ so that*

$$\Phi_x = \bigcup_{n=1}^\infty \Phi_x^{(n)}$$

for almost every x, then

$$\dim_{\Sigma, \ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}), \mathcal{R}) \to \dim_\Sigma(L^p(E(\Phi))/B_1^{(p)}(\Phi), \mathcal{R}),$$

$$\dim_{\Sigma,\ell^q}(B_1^{(p)}(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \to 0,$$

$$\underline{\dim}_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \to \underline{\dim}_{\Sigma}(L^p(E(\Phi))/B_1^{(p)}(\Phi),\mathcal{R}),$$

$$\underline{\dim}_{\Sigma,\ell^q}(B_1^{(p)}(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \to 0.$$

Proof. Let $E\colon L^p(E(\Phi^{(n)})) \to L^p(E(\Phi))$ be defined by extension by zero. It is easy to see that E descends to a well-defined map, still denoted E

$$L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}) \to L^p(E(\Phi))/B_1^{(p)}(\Phi).$$

By subadditivity under exact sequences, and the fact that E is surjective,

$$\begin{aligned}
\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi),\mathcal{R}) &\leq \dim_{\Sigma,\ell^q}(\overline{\operatorname{im} E}),\mathcal{R}) + \dim_{\Sigma,\ell^q}([L^p(E(\Phi))/B_1^{(p)}(\Phi)]/\overline{\operatorname{im} E},\mathcal{R}),\\
&\leq \dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R})\\
&\quad + \dim_{\Sigma,\ell^q}([L^p(E(\Phi))/B_1^{(p)}(\Phi)]/\overline{\operatorname{im} E},\mathcal{R}),
\end{aligned}$$

where in the last line we use that dim is decreasing under equivariant maps with dense image. It is easy to see that there is a $\mathcal{R}$-equivariant map

$$L^p(E(\Phi \setminus \Phi^{(n)})) \to [L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})]/\overline{\operatorname{im} E}$$

with dense image. Thus

$$\begin{aligned}
\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi),\mathcal{R}) \leq \dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R})\\
+ c(\Phi \setminus \Phi^{(n)}).
\end{aligned}$$

Since $\Phi^{(n)}$ increasing to Φ, and $c(\Phi) < \infty$, we know that

$$c(\Phi \setminus \Phi^{(n)}) = \frac{1}{2}\overline{\mu}(\mathcal{E}(\Phi) \setminus \mathcal{E}(\Phi^{(n)})) \to 0.$$

Thus,

$$\dim_{\Sigma,\ell^q}([L^p(E(\Phi))/B_1^{(p)}(\Phi)]/\overline{\operatorname{im} E},\mathcal{R}) \leq \liminf_{n\to\infty} \dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}).$$

For the opposite inequality, consider the restriction map

$$R\colon L^p(E(\Phi)) \to L^p(E(\Phi^{(n)})),$$

then R descends to an surjective $\mathcal{R}$-equivariant map (still denoted R)

$$L^p(E(\Phi))/B_1^{(p)}(\Phi^{(n)}) \to L^p(E(\Phi))/B_1^{(p)}(\Phi^{(n)}).$$

Thus

$$\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \leq \dim_{\Sigma}(L^p(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}).$$

Considering the exact sequence

$$0 \longrightarrow \frac{B_1^{(p)}(\Phi)}{B_1^{(p)}(\Phi^{(n)})} \longrightarrow \frac{L^p(\Phi)}{B_1^{(p)}(\Phi^{(n)})} \longrightarrow \frac{L^p(\Phi)}{B_1^{(p)}(\Phi)} \longrightarrow 0,$$

we find that

$$\begin{aligned}\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \leq \dim_{\Sigma,\ell^q}&\left(\frac{B_1^{(p)}(\Phi)}{B_1^{(p)}(\Phi^{(n)})},\mathcal{R}\right)\\ &+\dim_{\Sigma,\ell^q}\left(\frac{L^p(E(\Phi))}{B_1^{(p)}(\Phi)},\mathcal{R}\right).\end{aligned}$$

So it suffices to prove the second limiting statement. Since $\mathcal{R}$ is finitely presented, by the preceding lemma we can find measurable fields of loops $(L^{(j)})_{j=1}^{\infty}$ which generate $B_1^{(p)}(\Phi)$ and so that

$$\sum_{j=1}^{\infty}\mu(\operatorname{supp} L^{(j)}) < \infty.$$

Since

$$\dim_{\Sigma,\ell^q}(B_1^{(p)}(E(\Phi))/B_1^{(p)}(\Phi^{(n)}),\mathcal{R}) \leq \sum_{j=1}^{\infty}\mu(\{x : L_x^{(j)} \text{ is not supported in } \Phi_x^{(n)}\})$$

and

$$\mu(\{x : L_x^{(j)} \text{ is not supported in } \Phi_x^{(n)}\}) \to 0,$$

$$\mu(\{x : L_x^{(j)} \text{ is not supported in } \Phi_x^{(n)}\}) \leq \mu(\operatorname{supp} L^{(j)}),$$

the Dominated Convergence Theorem implies that

$$\dim_{\Sigma,\ell^q}(B_1^{(p)}(\Phi)/B_1^{(p)}(\Phi^{(n)}),\mathcal{R})\to 0,$$

as desired.

□

Theorem 4.6.6. *Fix $1\leq p,q<\infty$. Let $(\mathcal{R},X,\mu)$ be a sofic, discrete, measure-preserving equivalence relation with $\mathcal{R}$ finitely presented and of finite cost. Let Φ,Ψ be two finite cost graphings of $\mathcal{R}$. Then*

$$\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi),\mathcal{R})=\dim_{\Sigma,\ell^q}(L^p(E(\Psi))/B_1^{(p)}(\Psi),\mathcal{R}),$$

$$\underline{\dim}_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi),\mathcal{R})=\underline{\dim}_{\Sigma,\ell^q}(L^p(E(\Psi))/B_1^{(p)}(\Psi),\mathcal{R}),$$

Proof. Let $\Phi=(\phi_j)_{j=1}^{\infty}$. Let $\Phi_x^{(n)},\Psi_x^{(nm)}$ be the subgraphs defined by

$$\mathcal{E}(\Phi_x^{(n)})=\{(y,\phi_j^{\pm 1}(y)):1\leq j\leq n, y\in\operatorname{dom}(\phi_j^{\pm 1}), y\sim x\}$$

$$\mathcal{E}(\Psi_x^{(n,m)})=\{(y,z)\in\mathcal{E}(\Psi_x):d_{\Phi_x^{(n)}}(y,z)\leq m\},$$

here $d_{\Phi_x^{(n)}}$ is the graph distance defined as the infimum over all k so that there exists $x_0,x_1,\dots,x_k\in V(\Phi_x^{(n)}),x_0=y,x_k=z,(x_{j-1},x_j)\in\mathcal{E}(\Phi_x^{(n)})$ for all $1\leq j\leq n$.

Note that if $\gamma_{yz},\gamma'_{y,z}$ are two paths from y to z in $\Phi^{(n)}$, then their difference is a loop in $\Phi^{(n)}$. Thus for $(y,z)\in\mathcal{E}(\Psi_x^{(n,m)})$ we have a well-defined element σ_{yz} of $\ell^p(E(\Phi^{(n)})_x)/B_1^{(p)}(\Phi_x^{(n)})$ given as the equivalence class of any path from y to z in $\Phi^{(n)}$.

Then for each n,m we have a well-defined bounded linear map with T_x (whose norm is bounded uniformly in x)

$$T_x\colon \ell^p(E(\Psi_x^{(n,m)}))/B_1^{(p)}(\Psi^{(n,m)})\to\ell^p(E(\Phi_x^{(n)}))/B_1^{(p)}(\Phi^{(n)})$$

by

$$T_xf=\sum_{[y,z]\in E(\Phi^{(n)})}f(y,z)\sigma_{yz},$$

Let

$$T = \int_X^{\oplus} T_x \, d\mu(x).$$

It is straightforward to check that $T_x = T_y$ for almost every $(x, y) \in \mathcal{R}$ so that T is an $\mathcal{R}$-equivariant map

$$L^p(E(\Psi^{(n,m)}))/B_1^{(p)}(\Psi^{(n,m)}) \to L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}).$$

By subadditivity of dim under exact sequences, and the fact that dim decreases under bounded, linear, equivariant maps with dense image,

$$\begin{aligned}
\dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}), \mathcal{R}) &\le \dim_{\Sigma,\ell^q}(\overline{\operatorname{im} T}, \mathcal{R}) \\
&\quad + \dim(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/\overline{\operatorname{im} T}, \mathcal{R}) \\
&\le \dim_{\Sigma,\ell^q}(L^p((E\Psi^{(n,m)}))/B_1^{(p)}(\Psi^{(n,m)}), \mathcal{R}) \\
&\quad + \dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/\overline{\operatorname{im} T}, \mathcal{R}).
\end{aligned}$$

Now suppose that $x \sim y \sim z$ in X, and y, z are in the same connected component in $\Psi_x^{(n,m)}$. Then we can find $x_1, \cdots, x_n$ with $y = x_1, x_n = z$ which are adjacent and

$$\sigma_{yz} = \sum_{i=1}^{n-1} \sigma_x^{x_i x_{i+1}},$$

is a path from y to z in B_x. Further, if σ_{yz} is any other such path, then again there difference is a loop, so σ_{yz} represents a well-defined element in $\operatorname{im}(T)$. Let

$$Y_x^{(n)} = \overline{\operatorname{Span}}^{\|\cdot\|_p}\{\sigma_{yz} : y, z \text{ are connected in } \Psi_x^{(n,m)}\},$$

$$Y^{(n)} = \int_X^{\oplus_p} Y_x^{(n)} \, d\mu(x).$$

Then

$$\dim_\Sigma(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/\overline{\operatorname{im} T}, \mathcal{R}) \le \dim_\Sigma(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/Y^{(n)}, \mathcal{R}).$$

Now let $V_x^{(n)} \subset \ell^p(G_x^B)$ be defined by

$$V_x^{(n)} = \overline{\operatorname{Span}}^{\|\cdot\|_p}\{\gamma^{yz} : \gamma^{yz} \text{ is a path from } y \text{ to } z \text{ in } \Phi_x^{(n)},\ y, z \text{ connected in } \Psi_x^{(n,m)}\},$$

$$V^{(n)} = \int_X^{\oplus_p} V_x^{(n)}\, d\mu(x)$$

Then we have a surjective equivariant map

$$L^p(E(\Phi^{(n)}))/V^{(n)} \to (L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/Y^{(n)},$$

so

$$\dim_\Sigma(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)})/Y_n, \mathcal{R}) \leq \dim_\Sigma(L^p(E(\Phi^{(n)}))/V^{(n)}, \mathcal{R}).$$

Let $(E_j)_{j=1}^\infty$ be disjoint edges generating $L^p(E(\Phi^{(n)}))$ such that

$$\sum_{j=1}^\infty \mu(\mathrm{supp}(E_j)) = c(\Phi^{(n)}).$$

Writing $E_x^{(j)} = (f(x), g(x))$. Then

$$\begin{aligned}\dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/V^{(n)}, \mathcal{R}) &\leq \sum_{j=1}^\infty \mu(\{x \in \mathrm{supp}(E_j) : (f(x), g(x)) \notin \mathcal{C}(\Psi_x^{(m,n)})\}) \\ &= c(\Phi^{(n)} \setminus \mathcal{C}(\Psi^{(n,m)}))\end{aligned}$$

where

$$\mathcal{C}(\Psi^{(n,m)}) = \{(y,z) \in \mathcal{R} : y \text{ is connected to } z \text{ in } \Phi_x^{(n)}\}.$$

Putting this altogether we have

$$\begin{aligned}\dim_{\Sigma,\ell^q}(L^p(E(\Phi^{(n)}))/B_1^{(p)}(\Phi^{(n)}), \mathcal{R}) \leq{}& \dim_{\Sigma,\ell^q}(L^p(\Psi^{(n,m)})/B_1^{(p)}(\Psi^{(n,m)}), \mathcal{R}) \\ &+ c(\Phi^{(n)} \setminus \mathcal{C}(\Psi^{(n,m)})),\end{aligned}$$

choose an increasing sequence of integers m_n so that

$$c(\Psi_x \cap \mathcal{C}(\Phi^{(n)})) \setminus \Psi^{(n,m_n)}) \to 0$$

$$c([\Phi^{(n)} \setminus \mathcal{C}(\Psi_x \cap \mathcal{C}(\Phi_x^{(n)}))] \setminus [\Phi^{(n)} \setminus \mathcal{C}(\Psi^{(n,m_n)})]) \to 0.$$

Then $\Psi^{(n,m_n)}$ increases to Ψ, and it is easy to see that

$$c(\Phi^{(n)} \setminus \mathcal{C}(\Psi^{(n,m_n)})) \to 0.$$

Thus letting $n \to \infty$ and applying the preceding lemma we find that

$$\dim_{\Sigma,\ell^q}(L^p(E(\Phi))/B_1^{(p)}(\Phi), \mathcal{R}) \leq \dim_{\Sigma,\ell^q}(L^p(E(\Psi))/B_1^{(p)}(\Psi), \mathcal{R})$$

the proposition now follows by symmetry.

$\square$

Definition 4.6.7. Let $(X, \mu, \mathcal{R})$ be a sofic, discrete, measure-preserving equivalence relation, with $\mathcal{R}$ finitely presented and of finite cost, and let Σ be a sofic approximation of $\mathcal{R}$. By the above Theorem, the number $c_{1,\Sigma}^{(p)}(\mathcal{R}) = \dim_{\Sigma,\ell^p}(L^p(E(\Phi))/B_1^{(p)}(\Phi), \mathcal{R})$ is independent on the choice of a finite cost graphing Φ. Similar remarks apply to $\underline{c}_{1,\Sigma}^{(p)}(\mathcal{R}) = \underline{\dim}_{\Sigma,\ell^p}(L^p(E(\Phi))/B_1^{(p)}(\Phi), \mathcal{R})$.

It is easy to see that $c_1^{(p)}(\mathcal{R}) \leq c(\mathcal{R})$. By Theorem 4.5.4, if $\mathcal{R}$ has infinite orbits we then have

$$\beta_1^{(2)}(\mathcal{R}) + 1 = \underline{c}_{1,\Sigma}^{(2)}(\mathcal{R}) = c_{1,\Sigma}^{(2)}(\mathcal{R}) \leq c(\mathcal{R}).$$

In [12], Gaboriau asked the following question "is $c(\mathcal{R}) = \beta_1^{(2)}(\mathcal{R}) + 1$?" (see [12] page 129). If we can find an example where

$$c_{1,\Sigma}^{(p)}(\mathcal{R}) > c_{1,\Sigma}^{(2)}(\mathcal{R})$$

for some $1 < p < \infty$, this would automatically produce a counterexample to this conjecture.

Theorem 4.6.8. *Let $(X, \mu, \mathcal{R})$ be a ergodic, finitely presented, sofic, discrete, measure-preserving equivalence relation, and let Σ be a sofic approximation of $\mathcal{R}$. Let $A \subseteq \mathcal{R}$, and define $\sigma_{i,A} \colon L(\mathcal{R}_A) \to M_{d_i}(\mathbb{C})$ by $\sigma_{i,A}(x) = \sigma_i(\mathrm{Id}_A)\sigma_i(x)\sigma_i(\mathrm{Id}_A)$. Then*

$$\mu(A)(c_{1,\Sigma_A}^{(p)}(\mathcal{R}_A) - 1) \geq c_{1,\Sigma}^{(p)}(\mathcal{R}) - 1.$$

Proof. Let Ψ be a graphing of $\mathcal{R}_A$. Let $n \in \mathbb{N} \cup \{0\}$ be such that $n\mu(A) \leq 1 < (n+1)\mu(A)$. By ergodicity, we may find $A = A_1, A_2, \cdots, A_n$ essentially disjoint measurable sets such that there exists $\phi_i \in [[\mathcal{R}]], 1 \leq i \leq n$ with $\mathrm{dom}(\phi_i) = A, \mathrm{ran}(\phi_i) = A_i, 1 \leq i \leq n$, and a $A' \subseteq A$ such that there is $\phi_{n+1} \in [[\mathcal{R}]]$ with $\mathrm{dom}(\phi_{n+1}) = A'$, and

$$\mathrm{ran}(\phi_{n+1}) = X \setminus \bigcup_{j=1}^{n} A_j.$$

Let $\Phi = \Psi \cup \{\phi_j\}_{j=1}^{n+1}$. We use $L^p(E(\Psi\big|_A)), B_1^{(p)}(\Psi\big|_A)$, for

$$\int_A^{\oplus_p} \ell^p(E(\Psi)_x)\, d\mu(x),$$

$$\int_A^{\oplus_p} B_1^{(p)}(\Psi_x)\, d\mu(x),$$

and $L^p(E(\Psi)), B_1^{(p)}(\Psi)$, for

$$\int_X^{\oplus_p} \ell^p(E(\Psi)_x)\, d\mu(x),$$

$$\int_X^{\oplus_p} B_1^{(p)}(\Psi_x)\, d\mu(x).$$

Then

$$\chi_A L^p(E(\Psi)) = L^p(E(\Psi|_A)),$$

$$\chi_A B_1^{(p)}(\Psi) = B_1^{(p)}(\Psi\big|_A),$$

and

$$B_1^{(p)}(\Phi) = B_1^{(p)}(\Psi).$$

Considering the exact sequence

$$0 \longrightarrow L^p(E(\Phi \setminus \Psi)) \longrightarrow \frac{L^p(E(\Phi))}{B_1^{(p)}(\Psi)} \longrightarrow \frac{L^p(E(\Psi))}{B_1^{(p)}(\Phi)} \longrightarrow 0,$$

we have

$$\begin{aligned} c_{1,\Sigma}^{(p)}(\mathcal{R}) &\leq c(\Phi \setminus \Psi) + \dim_{\Sigma,\ell^p}(L^p(E(\Psi))/B_1^{(p)}(\Phi), \mathcal{R}) \\ &= 1 - \mu(A) + \dim_{\Sigma,\ell^p}(L^p(E(\Psi))/B_1^{(p)}(\Phi), \mathcal{R}). \end{aligned}$$

By Proposition 4.3.4, we thus have

$$c^{(p)}_{1,\Sigma}(\mathcal{R}) \leq 1 - \mu(A) + \mu(A)\dim_{\Sigma_A,\ell^p}(L^p(E(\Psi\big|_A))/B^{(p)}_1(\Psi\big|_A), \mathcal{R}_A)$$
$$= 1 - \mu(A) + \mu(A)c^{(p)}_{1,\Sigma_A}(\mathcal{R}_A).$$

Rearranging proves the inequality.

□

Let $(\mathcal{R}, X, \mu)$ be an ergodic, discrete, measure-preserving equivalence relation. We let $\mathcal{F}(\mathcal{R})$ be the set of all $t \in \mathbb{R}$ so that there exists measurable subsets A and B of X with

$$t = \frac{\mu(A)}{\mu(B)}$$

and

$$\mathcal{R}_A \cong R_B.$$

Then $\mathcal{F}(\mathcal{R})$ is a subgroup of the positive reals. We call $\mathcal{F}(\mathcal{R})$ the fundamental group of $\mathcal{R}$.

Corollary 4.6.9. *Let $(X, \mu, \mathcal{R})$ be a sofic, ergodic, finitely presented, discrete, measure-preserving equivalence relation. If for some p we have*

$$\inf_{\Sigma} c^{(p)}_{1,\Sigma}(\mathcal{R}) > 1,$$

where the infimum is over all sofic approximations, then the fundamental group of $\mathcal{R}$ is trivial.

We will deduce more about $c^{(p)}_{1,\Sigma}(\mathcal{R})$ in the non-amenable case, but we will first need to discuss the discrete Hodge decomposition for amenable graphs.

Let G be a countably infinite connected graph of uniformly bounded degree. Since G is infinite, δ is always injective. We say that G is *amenable* if for some $1 \leq p < \infty$, we have $\delta(\ell^p(V(G))$ is a closed subspace of $\ell^p(E(G))$. Equivalently, there is some $C > 0$ so that

$$\|\delta f\|_p \geq C\|f\|_p.$$

Note that if p is as above, then for all $1 < q < \infty$, we have $\delta(\ell^q(G))$ is closed in $\ell^q(E(G))$. If $\delta(\ell^q(G))$ were not closed, then we could find $f_n \in \ell^q(G)$ of norm one so that $\|\delta f_n\|_q \to 0$. We can argue as in Lemma 3.8.1.

By duality G is amenable if and only if ∂ is surjective as an operator from $\ell^p(E(G)) \to \ell^p(V(G))$ for some $1 < p < \infty$, and this is also equivalent to saying that ∂ is surjective as an operator from $\ell^p(E(G)) \to \ell^p(V(G))$ for all $1 < p < \infty$. For notation we let $\Delta = \partial \circ \delta$.

Proposition 4.6.10. *Let G be an infinite amenable graph of uniformly bounded degree, then Δ is invertible as an operator from $\ell^p(V(G)) \to \ell^p(V(G))$ for all $1 < p < \infty$.*

Proof. Let $d(x)$ be the degree of x, and let M_d be the operator on $\ell^p(V(G))$ given by multiplication by d. Define

$$Af(x) = \frac{1}{d(x)} \sum_{y:[x,y]\in E(G)} f(y),$$

and note that $\Delta = M_d(A - \mathrm{Id})$.

Regard d as a measure on $V(G)$, then since G has uniformly bounded degree we know that

$$\ell^p(V(G)) = \ell^p(V(G), d)$$

with equivalent norms. Regard δ as an operator from $\ell^p(V(G), d) \to \ell^p(E(G))$ and let $-\partial_d$ be its adjoint, also let $\Delta_d = \partial_d \circ \delta$. Since

$$\langle \delta f, g\rangle_{\ell^p(E(G))} = -\langle f, \partial g\rangle_{\ell^p(V(G))} = -\langle f, M_d M_{d^{-1}} \partial g\rangle_{\ell^p(V(G))} =$$

$$-\langle f, M_{d^{-1}} \partial g\rangle_{\ell^p(V(G),d)},$$

we find that $\partial_d = M_{d^{-1}}\partial$, so

$$\Delta_d = M_{d^{-1}}\Delta = A - \mathrm{Id},$$

hence it suffices to show that Δ_d is invertible for all $1 < p < \infty$ as an operator from $\ell^p(V(G), d) \to \ell^p(V(G), d)$.

Let $\varepsilon > 0$ be such that $\|\delta f\|_{\ell^2(E(G))} \geq \varepsilon \|f\|_{\ell^2(V(G),d)}$, then

$$\varepsilon^2 \leq -\Delta_d,$$

as an operator on $\ell^2(V(G),d)$. Since $-\Delta_d = 1 - A$, this implies that $A \leq 1 - \varepsilon^2$ as an operator on $\ell^2(V(G),d)$.

Thus

$$|\langle Af, f\rangle_{\ell^2(V(G),d)}| \leq \langle A|f|, |f|\rangle_{\ell^2(V(G),d)} \leq (1-\varepsilon^2)\|f\|_2^2.$$

Since A is a self-adjoint operator, this implies that $\|A\|_{\ell^2(V(G),d)\to\ell^2(V(G),d)} < 1$. Since $\|A\|_{\ell^1(V(G),d)\to\ell^1(V(G),d)} \leq 1$, $\|A\|_{\ell^\infty(V(G),d)\to\ell^\infty(V(G),d)} \leq 1$, by interpolation we find that there is a $C_p < 1$ so that

$$\|A\|_{\ell^p(V(G),d)\to\ell^p(V(G),d)} \leq C_p.$$

Thus Δ_d is invertible on ℓ^p for $1 < p < \infty$ as desired.

□

Corollary 4.6.11 (Discrete Hodge Decomposition)**.** *Let G be an infinite non-amenable graph of uniformly bounded degree, then for every $1 < p < \infty$ we have the direct sum decomposition*

$$\ell^p(E(G)) = Z_1^{(p)}(G) \oplus B^1_{(p)}(G).$$

A projection onto $B^1_{(p)}(G)$ relative to this decomposition may be given by $\delta \circ \Delta^{-1} \circ \partial$.

To apply this to the case of equivalence relations, we prove the following Lemma.

Lemma 4.6.12. *Let $(X,\mu,\mathcal{R})$ be a finite cost discrete measure-preserving equivalence relation with $\mathcal{O}_x$ infinite for almost every x. The following are equivalent*

(i) There is a finite subset $\Phi \subseteq [[\mathcal{R}]]$, such that for almost every x, every connected component of the graph Φ_x is not amenable,

(ii) for every $\mathcal{R}$-invariant measurable $A \subseteq X$ with $\mu(A) > 0$ we have $\mathcal{R}_A$ is not amenable,

(iii) for every $A \subseteq X$ with $\mu(A) > 0$ we have that $\mathcal{R}_A$ is not amenable.

Proof. It is clear that (iii) implies (ii).

The fact that (ii) implies (i) is the content of Lemma 9.5 in [17].

Suppose (iii) fails and (i) holds. Let A with $\mu(A) > 0$ be such that $\mathcal{R}_A$ is amenable, let B be the $\mathcal{R}$-saturation of A. Since (i) holds, we know that

$$C_x = \inf_{\substack{f \in \ell^1(\mathcal{O}_x), \\ \|f\|_1 = 1}} \|\delta_{\Phi_x} f\|_1 > 0,$$

for almost every $x \in X$ and is constant of equivalence classes. Thus replacing A with a subset, we may assume that there is a $C > 0$ so that

$$\|\delta_{\Phi_x} f\|_1 \geq C\|f\|_1,$$

for all $x \in B$.

Since $\mathcal{R}_A$ is amenable, we may find measurable fields of vectors $\xi_x^{(n)} \in \ell^1(\mathcal{O}_x \cap A)$ so that $\|\xi_x^{(n)}\|_1 = 1$, and $\|\xi_x^{(n)} - \xi_y^{(n)}\|_1 \to 0$ for $(x, y) \in \mathcal{R}_A$. Let $\{\phi_j\}_{j \in J} \subseteq [[\mathcal{R}]]$ with J countable be such that $\{\mathrm{ran}(\phi_j)\}_{j \in J}$ is a disjoint family, $\mathrm{dom}(\phi_j) \subseteq A$, and

$$B = \bigcup_{j \in J} \mathrm{ran}(\phi_j).$$

Define $\lambda_x^{(n)} \in \mathrm{Meas}(\ell^1(\mathcal{O}_x)$ for $x \in X$ by $\lambda_x^{(n)} = \xi_{\phi_j^{-1}(x)}^{(n)}$ if $x \in B$ and j is such that $x \in \mathrm{ran}(\phi_j)$, and $\lambda_x^{(n)} = 0$ for $x \notin B$.

Define $\zeta^{(n)} \in \mathrm{Meas}(\ell^1(\mathcal{O}_x))$, by $\zeta_x^{(n)}(y) = \lambda_y^{(n)}(x)$. Then

$$\int_X \|\delta_{\Phi_x} \zeta_x^{(n)}\|_1 \, d\mu(x) \leq \int_B \sum_{\phi \in \Phi} \|\lambda_y^{(n)} - \lambda_{\phi(y)}^{(n)}\|_1 \chi_{\mathrm{dom}(\phi)}(y), d\mu(y),$$

and since Φ has finite cost, this goes to zero by the Dominated Convergence Theorem. But on the other hand,

$$\int_X \|\delta_{\Phi_x} \zeta_x^{(n)}\|_1 \, d\mu(x) \geq C \int_X \|\zeta_x^{(n)}\|_1 \, d\mu(x) = C \int_{\mathcal{R}} |\lambda_x^{(n)}(y)| \, d\overline{\mu}(x, y) = C\mu(B),$$

which is a contradiction.

□

If Φ is a graphing of $\mathcal{R}$, we may define the ℓ^p-cohomology space of $\mathcal{R}$ as the direct integral of $Z_1^{(p)}(\Phi_x)/B_1^{(p)}(\Phi_x)$ and we denote it by $H_1^{(p)}(\Phi)$. We set $\beta_{1,\Sigma}^{(p)}(\phi) = \dim_{\Sigma,\ell^p}(H_1^{(p)}(\Phi), \mathcal{R})$, $\underline{\beta}_{1,\Sigma}^{(p)}(\phi) = \underline{\dim}_{\Sigma,\ell^p}(H_1^{(p)}(\Phi), \mathcal{R})$.

Corollary 4.6.13. *Let $(X,\mu,\mathcal{R})$ be a discrete, sofic, measure-preserving equivalence relation such that $\mathcal{R}_A$ is not amenable for any $A \subseteq X$ with $\mu(A) > 0$. Suppose $\mathcal{R}$ has finite cost and is finitely presented, and fix a sofic approximation Σ of $\mathcal{R}$. Then for any graphing Φ of $\mathcal{R}$, we have*

$$c^{(p)}_{1,\Sigma}(\mathcal{R}) \leq \beta^{(p)}_{1,\Sigma}(\Phi) + 1,$$

$$\underline{c}^{(p)}_{1,\Sigma}(\mathcal{R}) \leq \underline{\beta}^{(p)}_{1,\Sigma}(\Phi) + 1,$$

Proof. First, express $\Phi = \bigcup_n \Phi^{(n)}$ by the above Lemma, we find that up to sets of measure zero,

$$X = \bigcup_{n=1}^{\infty} \{x : \Phi^{(n)}_x \text{ is not amenable }\},$$

and each of the above sets is $\mathcal{R}$-invariant. From this, it is not hard to see that we may choose $\Phi^{(n)}$ so that for every n, either $\Phi^{(n)}_x$ is non-amenable or zero.

By the Discrete Hodge Decomposition, we have the following exact sequence

$$0 \longrightarrow B^1_{(p)}(\Phi^{(n)}) \longrightarrow \frac{L^p(E(\Phi^{(n)}))}{B^{(p)}_1(\Phi^{(n)})} \longrightarrow \frac{Z^{(p)}_1(\Phi)}{B^{(p)}_1(\Phi^{(n)})} \longrightarrow 0,$$

now apply subadditivity under exact sequences, and Lemma 4.6.5 to complete the proof.

□

Corollary 4.6.14. *Fix $n \in \mathbb{N}$, suppose $\mathcal{R}$ is the equivalence relation induced by a free action of $\mathbb{F}_n$ on a standard probability space (X,μ). Then for any sofic approximation Σ of $\mathcal{R}$, we have that*

$$\underline{c}^{(p)}_{1,\Sigma}(\mathcal{R}) = c^{(p)}_{1,\Sigma}(\mathcal{R}) = n,$$

in particular for $n \geq 1$,

$$\underline{\beta}^{(p)}_{1,\Sigma}(\Phi) \geq n - 1$$

for any graphing Φ. If Φ is a treeing of $\mathcal{R}$, then

$$\underline{\beta}^{(p)}_{1,\Sigma}(\Phi) = \beta^{(p)}_{1,\Sigma}(\Phi) = n - 1.$$

Thus, if $\mathcal{R}$ has infinite orbits and is amenable then

$$\underline{c}^{(p)}_{1,\Sigma}(\mathcal{R}) = c^{(p)}_{1,\Sigma}(\mathcal{R}) = 1.$$

Proof. If Φ is the graphing provided by the canonical generating set of $\mathbb{F}_n$, then

$$B_1^{(p)}(\Phi) = \{0\},$$

$$L^p(E(\Phi)) \cong L^p(\mathcal{R}, \overline{\mu})^{\oplus n},$$

and the proof of the first statement is thus complete.

By Lemma 3.8.2, we know that $H_1^{(p)}(\Phi)$ can be generated by $n-1$ elements, and this proves the upperbound.

The last statement follows from the standard fact that a amenable equivalence relation with infinite orbits is induced by a free action of $\mathbb{Z}$ (see [17] Theorem 6.6).

□

Proposition 4.6.15. *Let $(\mathcal{R}, X, \mu)$ be a discrete measure-preserving equivalence relation such that $\mathcal{O}_x$ is infinite for almost every $x \in X$. Then $\underline{c}_{1,\Sigma}^{(p)}(\mathcal{R}) \geq 1$.*

Proof. By the ergodic decomposition (Theorem 2.1.23) , we can find $\mathcal{R}$-invariant measurable subsets A, B of X so that $\mu(A \cap B) = 0$, with $\mathcal{R}_A$ amenable, and $\mathcal{R}_B$ has no amenable compression. Let $\alpha \in [\mathcal{R}_A]$ generate $\mathcal{R}_A$. Let Φ_0 be any countable graphing of $\mathcal{R}_B$, and set $\Phi = \{\alpha\} \cup \Phi_0$. Then as representations of $\mathcal{R}$:

$$\frac{L^p(E(\Phi))}{B_1^{(p)}(\Phi)} = L^p(\mathcal{R}_A, \overline{\mu}) \oplus \frac{L^p(E(\Phi_0))}{B_1^{(p)}(\Phi_0)},$$

and by the Discrete Hodge decomposition we have a surjective $\mathcal{R}$-equivariant map

$$\frac{L^p(E(\Phi_0))}{B_1^{(p)}(\Phi_0)} \to \frac{L^p(E(\Phi_0))}{Z_1^{(p)}(\Phi_0)} \cong L^p(\mathcal{R}_B, \overline{\mu}).$$

Thus $\frac{L^p(E(\Phi))}{B_1^{(p)}(\Phi)}$ has an $\mathcal{R}$-equivariant surjection onto $L^p(\mathcal{R}, \overline{\mu})$ and this completes the proof.

□

We would like to prove one last property of our ℓ^p-Betti numbers. Namely, that

$$c_{1,\Sigma}^{(p)}(\mathcal{R}) \geq \beta_1^{(2)}(\mathcal{R}) + 1.$$

As we already know that

$$c_{1,\Sigma}^{(p)}(\mathcal{R}) \leq c(\mathcal{R}),$$

this gives one more relation between the problem of evaluation $c_{1,\Sigma}^{(p)}(\mathcal{R})$ and the cost versus ℓ^2-Betti number problem.

We need the following Lemma, which is a technical refinement of Proposition 4.5.1.

Lemma 4.6.16. *Let ν_n be the uniform measure on $\{1,\dots,n\}$. Let q_n be a sequence of orthogonal projections in $M_n(\mathbb{C})$ such that $\operatorname{tr}(q_n)$ converges to some $q \in [0,1]$. Then there is a function $\kappa\colon (0,1]\times(0,\infty)\to[0,1]$ such that*

$$\lim_{\varepsilon\to 0}\kappa(\alpha,\varepsilon) = 1 \textit{ for all } \alpha > 0$$

and which satisfies the following. For all $A_n \subseteq \operatorname{Ball}(\ell^2(n,\nu_n))$, with A_n measurable and

$$\liminf_{n\to\infty}\left(\frac{\operatorname{vol}(A_n)}{\operatorname{vol}(\operatorname{Ball}(\ell^2(n,\nu_n))}\right)^{1/2n} \geq \alpha$$

we have

$$\lim_{n\to\infty}\frac{1}{n}d_\varepsilon(q_nA_n,\|\cdot\|_p) \geq \kappa(\alpha,\varepsilon)q.$$

Proof. Suppose $\kappa > 0$ so that for all $\varepsilon > 0$ it is true that for all large n we can find measurable $A_n \subseteq \operatorname{Ball}(\ell^2(n,\nu_n))$, with $q_nA_n \subseteq_\varepsilon W$ and $\dim(W) \leq \kappa n$. Fix such $A_n, \kappa, W, \varepsilon$. We wish to get a lower bound on κ. We have

$$q_nA_n \subseteq \bigcup_{\substack{B\subseteq\{1,\dots,n\},\\ |B|\leq\varepsilon n}} (1+\varepsilon)\operatorname{Ball}(\chi_{B^c}(W)) + \varepsilon\operatorname{Ball}(\ell^p(B^c,\nu_n) + \varepsilon\operatorname{Ball}(\ell^p(B,\nu_n)).$$

By a volume-packing argument, we may select ε-dense subsets

$$S_B \subseteq \operatorname{Ball}(\chi_{B^c}(W),$$

$$T_B \subseteq \operatorname{Ball}(\ell^p(B,\nu_n),$$

so that

$$|S_B| \leq \left(\frac{2+4\varepsilon}{\varepsilon}\right)^{2\kappa n}$$

$$|T_B| \leq \left(\frac{2+4\varepsilon}{\varepsilon}\right)^{2\varepsilon n}.$$

Thus

$$q_nA_n \subseteq \bigcup_{\substack{B\subseteq\{1,\dots,n\},\\ |B|\leq \varepsilon n}} \bigcup_{\xi\in S_B,\zeta\in T_B} \xi+\zeta+2\varepsilon\,\mathrm{Ball}(\ell^p(n,\nu_n)).$$

Similarly, we may select a ε-dense (in the ℓ^2-norm) subset $\mathcal{F}$ of $\mathrm{Ball}((1-q_n)\ell^2(n,\nu_n),\|\cdot\|_2)$ with

$$|\mathcal{F}| \leq \left(\frac{2+2\varepsilon}{\varepsilon}\right)^{2(n-\mathrm{Tr}(q_n))}.$$

Since $A_n \subseteq \mathrm{Ball}(\ell^2(n,\nu_n))$ we have $(1-q_n)A_n \subseteq_{\varepsilon,\|\cdot\|_2} \mathcal{F}$ so

$$\begin{aligned}
A_n &\subseteq q_nA_n + (1-q_n)A_n\\
&\subseteq \bigcup_{\substack{B\subseteq\{1,\dots,n\},\\ |B|\leq\varepsilon n}} \bigcup_{\substack{\xi\in S_B,\\ \zeta\in T_B,\\ \eta\in\mathcal{F}}} \xi+\zeta+\eta+2\varepsilon\,\mathrm{Ball}(\ell^p(n,\nu_n))+\varepsilon\,\mathrm{Ball}(\ell^2(n,\nu_n)),\\
&\subseteq \bigcup_{\substack{\xi\in S_B,\\ \zeta\in T_B,\\ \eta\in\mathcal{F}}} \xi+\zeta+\eta+3\varepsilon\,\mathrm{Ball}(\ell^p(n,\nu_n)),
\end{aligned}$$

where in the last line we use that $p \leq 2$. Thus if $\varepsilon < 1/2$, we have

$$\mathrm{vol}(A_n)^{1/2n} \leq \lfloor n\varepsilon\rfloor^{1/2n}\binom{n}{\lfloor n\varepsilon\rfloor}^{1/2n}|\mathcal{F}|^{1/2n}|S_B|^{1/2n}|T_B|^{1/2n}3\varepsilon\,\mathrm{vol}(\mathrm{Ball}(\ell^p(n,\mu_n))^{1/2n}.$$

By the calculation on [21] page 11, there is a $C>0$ so that

$$\mathrm{vol}(\mathrm{Ball}(\ell^p(n,\mu_n))^{1/2n} \leq C\,\mathrm{vol}(\mathrm{Ball}(\ell^2(n,\mu_n))^{1/2n}.$$

Thus,

$$\begin{aligned}
\left(\frac{\mathrm{vol}(A_n)}{\mathrm{vol}(\mathrm{Ball}(\ell^2(n,\mu_n))}\right)^{1/2n} &\leq C\lfloor n\varepsilon\rfloor^{1/2n}\binom{n}{\lfloor n\varepsilon\rfloor}^{1/2n}|\mathcal{F}|^{1/2n}|S_B|^{1/2n}|T_B|^{1/2n}3\varepsilon\\
&\leq 3C\varepsilon^{\kappa+1-\mathrm{Tr}(q_n)}\lfloor n\varepsilon\rfloor^{1/2n}\binom{n}{\lfloor n\varepsilon\rfloor}^{1/2n}(2+2\varepsilon)^{1-\mathrm{tr}(q_n)}(2+4\varepsilon)^{\kappa+\varepsilon}.
\end{aligned}$$

By Stirling's Formula,

$$\lim_{n\to\infty}\binom{n}{\lfloor n\varepsilon\rfloor}^{1/2n} = (1-\varepsilon)^{-(1-\varepsilon)/2}\varepsilon^{-\varepsilon/2}.$$

Thus,

$$\alpha \leq 3C\varepsilon^{-\kappa+1-\varepsilon/2-q}(2+2\varepsilon)^{1-q}(2+4\varepsilon)^{\kappa+\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)/2},$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ implies that

$$-\kappa + 1 - q \leq 0,$$

so

$$\kappa \geq 1 - q.$$

□

Theorem 4.6.17. *Let $(\mathcal{R}, X, \mu)$ be a sofic, discrete, measure-preserving equivalence relation with sofic approximation Σ. Let Φ be a countable subset of $[[\mathcal{R}]]$, with $c(\Phi) < \infty$ and $1 \leq p \leq 2$. Let V_x be a measurable field of closed subspaces of $\ell^p(\mathcal{E}(\Phi_x))$, such that $V_x = V_y$ for almost every $(x, y) \in \mathcal{R}$. Set $\mathcal{H}_x = \overline{V_x}^{\|\cdot\|_2}$, and*

$$W = \int_X^{\oplus_p} \ell^p(\mathcal{E}(\Phi_x))/V_x \, d\mu(x),$$

$$\mathcal{K} = \int_X^{\oplus_2} \ell^2(\mathcal{E}(\Phi_x) \cap \mathcal{H}_x^{\perp} \, d\mu(x).$$

Then,

$$\dim_{\Sigma,\ell^p}(W, \mathcal{R}) \geq \dim_{L(\mathcal{R})}(\mathcal{K}).$$

Proof. Since $c(\Phi) < \infty$, we may argue as in Lemma 3.4.5 to reduce to the case that Φ is finite. Let $\Phi = \{\phi_1, \ldots, \phi_n\}$. We may view $L^p(E(\Phi))$ as a subset of $L^p(\mathcal{R}, \overline{\mu})^{\oplus n}$ in such a way that the measurable vector field $x \mapsto \mathcal{E}_{(x,\phi_j(x))}$ is identified with $\mathrm{Id}_{\mathrm{ran}(\phi_j)}\, \chi_\Delta \otimes e_j$ (recall that $f \otimes e_j$ is the vector on $L^p(\mathcal{R}, \overline{\mu})^{\oplus n}$ which is zero in every coordinate except the j^{th} where it is f). Let

$$q_\Phi = \bigoplus_{j=1}^{n} \mathrm{Id}_{\mathrm{ran}(\phi_j)} \in M_n(L(\mathcal{R}))$$

and

$$Q \colon L^p(E(\Phi)) \to W$$

the canonical quotient map. Fix a graphing Ψ of $\mathcal{R}$ and $\mathcal{P}$ a set of projections in $L^\infty(X,\mu)$ so that

$$W^*(\{\phi p \phi^{-1} : \phi \in \Psi, p \in \mathcal{P}\}) = L^\infty(X,\mu).$$

We will use Q-dynamical filtrations to do our calculation. So let $V = \ker(Q)$, and $\mathcal{F} = (V_{F,m}, S)$ be a Q-dynamical filtration where

$$S = (\phi\,\mathrm{Id}_{\mathrm{ran}(\phi_1)}\,\chi_\Delta \otimes e_1, \dots, \phi\,\mathrm{Id}_{(\mathrm{ran}_n)}\,\chi_\Delta \otimes e_n)_{\phi \in \Psi, 1 \le j \le n}.$$

By Lemma 2.2.6, extend σ_i to maps

$$\sigma_i \colon L(\mathcal{R}) \to M_{d_i}(\mathbb{C})$$

such that

$$\sup_i \|\sigma_i(x)\|_\infty < \infty, \text{ for all } x \in L(\mathcal{R})$$

$$\mathrm{tr}(\sigma_x) \to \tau(x) \text{ for all } x \in L(\mathcal{R})$$

$$\|P(\sigma_i(x_1), \dots, \sigma_i(x_n) - \sigma_i(P(x_1, \dots, x_n))\|_2 \to 0,$$

for all $x_1, \dots, x_n \in L(\mathcal{R})$ and all $*$-polynomials P in n non-commuting variables.

Define $\sigma_i \colon M_n(L(\mathcal{R})) \to M_n(M_{d_i}(\mathbb{C}))$ by

$$\sigma_i(A)_{jj} = \sigma_i(A_{jj}) \text{ for } 1 \le j \le n.$$

Let q be the orthogonal projection onto $\mathcal{H}$, since we view $L^p(E(\Phi)) \subseteq L^p(\mathcal{R}, \overline{\mu})^{\oplus n}$, we have $q \in M_n(L(\mathcal{R}))$. As in Proposition 3.6.2, we may find q_i orthogonal projections in $M_n(M_{d_i}(\mathbb{C}))$ so that

$$\|\sigma_i(q) - q_i\|_2 \to 0.$$

Set

$$q_{\phi,i} = \bigoplus_{l=1}^n \sigma_i(\mathrm{Id}_{\mathrm{dom}(\phi_l)}).$$

Let $F \subseteq \Psi$ be given, $m \in \mathbb{N}$, and $\delta > 0$. Set

$$C = W^*(\{\phi p \phi^{-1} : p \in L^\infty(X,\mu) \cap F, \phi \in F\}),$$

and let $\chi_{B_1}, \ldots, \chi_{B_r}$ be the minimal projections in C. Let $\{A_1, \ldots, A_s\}$ be a sufficiently fine partition refining $\{B_1, \ldots, B_r\}$ in a manner to be determined later. We may assume that Σ is eventually a homomorphism on $W^*(\{\chi_{A_j}, \chi_{\mathrm{dom}(\phi_l)}\}_{1 \leq j \leq q, 1 \leq l \leq n})$, there are $E_j \subseteq [[\mathcal{R}]]$,

$$\mathcal{O}_{A_j} \colon = \{(x,y) \in R : x \in A_j\} = \bigsqcup_{\psi \in E_j} \mathrm{graph}(\psi),$$

and that

$$F^m \subseteq E_1^{-1} + E_2^{-1} + \cdots + E_q^{-1}.$$

We may also assume that for every $\psi \in E_j$ and for all large i, we have that $\mathrm{dom}(\sigma_i(\psi)) \subseteq \sigma_i(A_j)$, and that $q_i \leq q_{\phi,i}$.

Note that if $f \in L^p(\mathcal{R}, \overline{\mu})$, then we can uniquely write

$$\mathrm{Id}_{A_j} f = \sum_{\psi \in E_j} f_\psi \chi_{\mathrm{graph}(\psi)},$$

where $f_\psi \in L^p(\mathrm{dom}(\psi), \mu)$ and the sum converges in $\| \cdot \|_p$. Fix $\eta > 0$, and let $F_j \subseteq E_j$ be finite and so that for all $\psi \in F^m$,

$$\mathrm{dist}_{\|\cdot\|_2}(\psi, F^m) < \eta.$$

Let ν_i be the uniform probability measure on $\{1, \ldots, d_i\}$. On $\ell^p(d_i, \nu_i)^{\oplus n}$ we use the norm

$$\|f\|_p^p = \frac{1}{n} \sum_{j=1}^n \|f(j)\|_p^p.$$

For $\xi \in \ell^2(d_i, \nu_i)^{\oplus n}, 1 \leq j \leq q, 1 \leq k \leq n$ and $f \in L^p(\mathcal{R}, \overline{\mu})$,

$$S_\xi^{(j)}(f) = \sum_{\psi \in F_j} \mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)(q_{\Phi,i} - q_i)\sigma_i(1_{M_n(\mathbb{C})} \otimes \psi^{-1})\xi,$$

where for a measurable $A \subseteq X$, and $f \in L^1(A, \mu)$ we use

$$\mathbb{E}_A(f) = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

For $1 \leq k \leq n$, let $\pi_k \colon \ell^p(d_i, \nu_i)^{\oplus n} \to \ell^p(d_i, \nu_i)$ by

$$\pi_k(f) = f(k).$$

Finally define $S_\xi\colon L^p(\mathcal{R},\overline{\mu})\to \ell^p(d_i,\nu_i)^{\oplus n}$, $T_\xi\colon L^p(\mathcal{E}(\Phi))\to \ell^p(d_i,\nu_i)$ by

$$S_\xi=\sum_{j=1}^{s}T_\xi^{(j)}(f).$$

$$T_\xi=\sum_{k=1}^{n}\pi_k(S_\xi(f(k))),$$

in the last line we are using the identification

$$L^p(E(\Phi))\cong\bigoplus_{j=1}^{n}L^p(\mathrm{graph}(\phi_j),\overline{\mu}).$$

We claim that if $\{A_1,\dots,A_s\}$ is sufficiently fine then,

$$\frac{\mathrm{vol}(\{\xi\in\mathrm{Ball}(\ell^2(d_i,\nu_i)^{\oplus n}):\|S_\xi\|_{L^p\to\ell^p}\leq 2\text{ for }1\leq p\leq 2\})}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i,\nu_i)^{\oplus n})}\to 1.$$

As in Theorem 4.5.2, it suffices to do this for $p=1,2$. We have

$$\|S_\xi^{(j)}(f)\|_1\leq\sum_{\psi\in F_j}|\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)|\|\xi\|_2\leq\sum_{\psi\in F_j}|\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)|\leq 1.$$

So we need to do the case $p=2$. From the definition of S_ξ it follows that we may choose $\kappa>0$ so that

$$|\langle\sigma_i(\phi^{-1})(q_{\Phi,i}-q_i)\xi,\sigma_i(\psi^{-1})(q_{\Phi,i}-q_i)\xi\rangle-\tau(\psi(q_\phi-q)\phi^{-1})|<\kappa \tag{4.3}$$

for all $\phi,\psi\in F_j, 1\leq j\leq n$ implies

$$\|S_\xi(f)\|_2\leq\alpha(\kappa)\left\|\sum_{j=1}^{s}\sum_{\psi\in F_j}\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)\chi_{\mathrm{graph}(\psi)}\right\|_2+\left\|(q_\Phi-q)\sum_{j=1}^{q}\sum_{\psi\in F_j}\mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi)\chi_{\mathrm{graph}(\psi)}\right\|_2, \tag{4.4}$$

for all $f\in L^2(\mathcal{R},\overline{\mu})$ with

$$\lim_{\kappa\to 0}\alpha(\kappa)=0.$$

By the integral equation,

$$\int_{S^{2d_i-1}}\langle T\xi,\xi\rangle,d\xi=\mathrm{tr}(T),$$

and concentration of measure, for any $\kappa > 0$ the set of ξ so that (4.3) holds has probability tending to 1. Thus (4.4) holds with high probability. Thus the set of ξ so that

$$\|S_\xi(f)\|_{L^2 \to \ell^2} \leq 2$$

has probability tending to 1. Thus by interpolation,

$$\frac{\mathrm{vol}(\{\xi \in \mathrm{Ball}(\ell^2(d_i, \nu_i)^{\oplus n}) : \|S_\xi\|_{L^p \to \ell^p} \leq 2n \text{ for } 1 \leq p \leq 2\})}{\mathrm{vol}(\mathrm{Ball}(\ell^2(d_i, \nu_i)^{\oplus n}))} \to 1.$$

As in Theorem 4.5.2, if $\{A_1, \ldots, A_s\}$ is sufficiently fine, then the set of $\xi \in \mathrm{Ball}(\ell^2(d_i, \nu_i)^{\oplus n})$ with

$$\|S_\xi(\psi \operatorname{Id}_{\mathrm{ran}(\phi_j)} \chi_\Delta) - \psi S_\xi(\operatorname{Id}_{\mathrm{ran}(\phi_j)} \chi_\Delta)\|_2 < \delta$$

has probability tending to 1 for all $1 \leq j \leq n$.

We now show that if $\{A_1, \ldots, A_s\}$ is sufficiently fine then

$$\|T_\xi\big|_{W_{F,m}}\|_{L^p \to \ell^p} \leq \delta,$$

with high probability in ξ. For this, let $F \subseteq \mathrm{Ball}(W_{F,m})$ be a finite $\frac{\delta}{2n(2014)!}$-dense set. It is then enough to show that

$$\|T_\xi(f)\|_p \leq \frac{\delta}{(2014)!} \text{ for all } f \in F,$$

for a high probability set of ξ. Using that $p \leq 2$, it is enough to show that

$$\|T_\xi(f)\|_2 \leq \frac{\delta}{(2014)!} \text{ for all } f \in F,$$

for a high probability set of ξ. Fix $f \in F$ by (4.4), we have for any $\alpha > 0$ that the set of ξ with

$$\|T_\xi(f)\|_2 \leq \alpha \|f\|_2 + \left\|(q_\Phi - q) \sum_{j=1}^{q} \sum_{\psi \in F_j} \mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi) \chi_{\mathrm{graph}(\psi)}\right\|_2$$

has probability tending to 1, here $\mathbb{E}_A(f)$ is defined as before but f is view as a map $\mathcal{R} \to \mathbb{C}^n$, and the integral is vector-valued. Since $qf = f$, we have

$$\left\|(q_\Phi - q) \sum_{j=1}^{n} \sum_{\psi \in F_j} \mathbb{E}_{\mathrm{dom}(\psi)}(f_\psi) \chi_{\mathrm{graph}(\psi)}\right\|_2$$

can be made arbitrarily small by making $\{A_1, \ldots, A_q\}$ sufficiently fine and $\eta > 0$ sufficiently small.

It now follows that if $\{A_1, \ldots, A_s\}$ sufficiently fine, then with high probability we have $T_\xi \in \mathrm{Hom}_{\mathcal{R},\ell^p}(\mathcal{F}, F, m, \delta, \sigma_i)$. Moreover, the above estimates show that

$$(T_\xi(\mathrm{Id}_{\mathrm{ran}(\phi_1)}\, \chi_\Delta \otimes e_j)$$

is close to

$$\pi_j((q_{\Phi,i} - q_i)\xi)$$

with high probability if $\{A_1, \ldots, A_s\}$ are sufficiently fine and $\eta > 0$ is sufficiently small. Thus

$$(T_\xi(\mathrm{Id}_{\mathrm{ran}(\phi_1)}\, \chi_\Delta \otimes e_1), \ldots, T_\xi(\mathrm{Id}_{\mathrm{ran}(\phi_n)}\, \chi_\Delta \otimes e_n))$$

is close to

$$(q_{\phi,i} - q_i)\xi$$

with high probability if $\{A_1, \ldots, A_s\}$ are sufficiently fine and $\eta > 0$ is sufficiently small, so the desired lower bound now follows from the preceding Lemma.

$\square$

The following corollary is automatic from the preceding theorem and the definition of $c^{(p)}_{1,\Sigma}(\mathcal{R})$.

Corollary 4.6.18. *Let $(\mathcal{R}, X, \mu)$ be a sofic, discrete,measure-preserving equivalence relation. Then,*

$$\underline{c}^{(p)}_{1,\Sigma}(\mathcal{R}) \geq \beta^{(1)}_2(\mathcal{R}) + 1.$$

As we mentioned before, it is easy from the definition that

$$c^{(p)}_{1,\Sigma}(\mathcal{R}) \leq c(\mathcal{R}),$$

so one may hope that this inequality and the above corollary shed some light on the cost versus ℓ^2-Betti number problem.

APPENDIX A

Noncommutative L^p Spaces

A.1 Definition of Noncommutative L^p

In this section, we will define the noncommutative L^p-spaces associated to a von Neumann algebra. We will not do this in the full generality. For the informed reader we mention that it is possible to generalize our methods and define noncommutative L^p-spaces in the case of a semifinite trace, but we will not pursue this. Instead we will only define and prove the basic properties of $L^p(M,\tau)$ for (M,τ) a tracial von Neumann algebra.

Definition A.1.1. Let (M,τ) be a tracial von Neumann algebra, for $x \in M$, and $1 \leq p < \infty$, we set

$$\|x\|_p^p = \tau(|x|^p),$$

if $p = \infty$, we let $\|x\|_\infty$ be the operator norm of x.

Let us give some intuition. If M is abelian, then we know that

$$(M,\tau) \cong L^\infty(X,\mu)$$

in this case

$$\|f\|_p^p = \int |f|^p \, d\mu$$

is the usual L^p-norm.

We wish to prove that $\|x\|_p$ is a norm on (M,τ), and that the usual Holder inequalities hold true in the usual noncommutative case. We first prove the special case of L^1.

Proposition A.1.2. *Let (M, τ) be a tracial von Neumann algebra. Then*

(i): $|\tau(x)| \leq \|x\|_1$ for all $x \in M$,

(ii): $\|x\|_1 = \|x^*\|_1$for all $x \in M$,

(iii): $\|xy\|_1 \leq \|x\|_\infty \|y\|_1$ for all $x, y \in M$,

(iv): $\|xy\|_1 \leq \|x\|_1 \|y\|_\infty$ for all $x, y \in M$.

Proof. (i): We first note that following inequality if $\mathcal{H}$ is a Hilbert space, and $a, b \in B(\mathcal{H})$:

$$\mathrm{Re}(a^*b) \leq \frac{|a|^2 + |b|^2}{2}.$$

Indeed this follows from

$$|a - b|^2 = |a|^2 - 2\,\mathrm{Re}(a^*b) + |b|^2.$$

For $x \in M$, let $x = u|x|$ be the polar decomposition of x. Then by the above inequality

$$\begin{aligned}\mathrm{Re}(\tau(x)) &= \tau(\mathrm{Re}(u|x|^{1/2}|x|^{1/2}) \\ &\leq \frac{1}{2}\|x\|_1 + \frac{1}{2}\tau(u|x|u^*) \\ &= \frac{1}{2}\|x\|_1 + \frac{1}{2}\tau(|x|u^*u).\end{aligned}$$

As $u^*u = P_{\ker(x)^\perp} = P_{\ker(|x|)^\perp}$, so

$$|x| = |x|u^*u.$$

Thus

$$\mathrm{Re}(\tau(x)) \leq \|x\|_1.$$

Now choose $\lambda \in \mathbb{C}, |\lambda| = 1$, so that $\tau(\lambda x) = |\tau(x)|$, then

$$|\tau(x)| = \mathrm{Re}(\tau(\lambda x)) \leq \|\lambda x\|_1 = \|x\|_1.$$

(ii): Let $x = u|x|$ be the Polar decomposition. Then $x^* = |x|u^*$, so

$$|x^*|^2 = xx^* = u|x|^2u^*.$$

Additionally,

$$(u|x|u^*)^2 = u|x|u^*u|x|u,$$

as in part (i), $|x|u^*u = |x|$, so

$$(u|x|u^*)^2 = |x^*|^2$$

$$|x^*| = u|x|u^*.$$

Thus

$$\|x^*\|_1 = \tau(u|x|u^*) = \tau(|x|u^*u) = \tau(|x|) = \|x\|_1.$$

(iii): We have

$$|xy|^2 = y^*x^*xy \leq \|x\|_\infty^2 |y|^2,$$

by operator monotonicity of the square root ([4] Exercise VIII.3.12) we know that

$$|xy| \leq \|x\|_\infty |y|.$$

Thus

$$\|xy\|_1 \leq \|x\|_\infty \|y\|_1.$$

(iv): Combine (ii) and (iii).

□

Our approach to proving that the L^p-norms are norms will be through noncommutative decreasing rearrangements. The advantage of this approach is that the proofs are very short and totally general, and reduce to the commutative case. One can use the same techniques to prove that other analogues of L^p-spaces (e.g. Lorenz spaces) have noncommutative analogues, and moreover this reduces to the commutative case.

Definition A.1.3. Let (M, τ) be a tracial von Neumann algebra. For $x \in M$, and $t \in [0, 1]$ define the *noncommutative decreasing rearrangement of* x, $s_x \colon [0, 1] \to [0, \infty)\}$ by

$$s_x(t) = \inf\{\lambda \in [0, \infty) : \tau(\chi_{(\lambda,\infty)}(|x|)) \leq t\}.$$

Note that normality of τ implies that

$$\tau(\chi_{(s_x(t),\infty)}(|x|)) \leq t.$$

We will prove some basic properties of non-commutative decreasing rearrangements, for which we need the following Lemma. For projections $p, q \in M \subseteq B(\mathcal{H})$ we use $p \vee q$ for the projection onto $\overline{p\mathcal{H} + q\mathcal{H}}$, and $p \wedge q$ for the projection onto $p\mathcal{H} \cap q\mathcal{H}$.

Lemma A.1.4. *Let (M, τ) be a tracial von Neumann algebra, and $p, q \in M$. If $p \wedge (1-q) = 0$, then*

$$\tau(p) \leq \tau(q).$$

Proof. The lemma is equivalent to the statement that

$$\dim_M(\rho(p)L^2(M,\tau)) \leq \dim_M(\rho(q)L^2(M,\tau)),$$

which follows from the fact that $\rho(q)$ restricted to $\rho(p)L^2(M,\tau)$ has kernel

$$\rho(p \wedge q)L^2(M,\tau).$$

□

Proposition A.1.5. *Let (M, τ) be a tracial von Neumann algebra, and use m for the Lebesgue measure on $[0,1]$. Then*

(i): For $x \in M$, and $\lambda \in [0,1]$

$$m(\{t : s_x(t) > \lambda\}) = \tau(\chi_{(\lambda,\infty)}(|x|))$$

and for $1 \leq p \leq \infty$,

$$\|x\|_p = \|s_x\|_{L^p([0,1])}.$$

(ii): For $x, y \in M$

$$s_{xy} \leq \|x\|_\infty s_y$$

$$s_{x^*} = s_x$$

$$s_{yx} \leq \|x\|_\infty s_y.$$

(iii): For a projection $p \in M$, and any $x \in M$,

$$\|xp\|_1 \leq \int_0^{\tau(p)} s_x(\lambda) d\lambda.$$

Proof. (i): We have

$$s_x(t) > \lambda$$

if and only if

$$\tau(\chi_{(\lambda,\infty)}(|x|)) > t.$$

Thus

$$\{t : s_x(t) > \lambda\} = [0, \tau(\chi_{(\lambda,\infty)})].$$

The above equality implies that

$$\|s_x\|_\infty = \|x\|_\infty.$$

For $1 \leq p < \infty$, we have that

$$|x|^p = p \int_0^1 \lambda^{p-1} \chi_{(\lambda,\infty)}(|x|) \, d\lambda$$

by functional calculus.Thus

$$\tau(|x|^p) = p \int_0^1 \lambda^{p-1} \tau(\chi_{(\lambda,\infty)}(|x|)) \, d\lambda.$$

By Fubini,

$$\|s_x\|_p^p = p \int_0^1 \lambda^{p-1} m(\{t : s_x(t) > \lambda\}) \, d\lambda,$$

so the second part follows.

(ii):

First note that

$$\chi_{(s_y(t)\|x\|_\infty,\infty)}(|xy|) \wedge \chi_{[0,s_y(t)]}(|y|) = 0.$$

Indeed, suppose

$$\xi \in \lambda(\chi_{(s_y(t)\|x\|_\infty,\infty)}(|xy|) \wedge \chi_{[0,s_y(t)]}(|y|))L^2(M,\tau)$$

and

$$\|\xi\| = 1.$$

Let

$$xy = u|xy|,$$

$$y = v|y|$$

be the Polar decompositions. Then

$$\begin{aligned} s_y(t)\|x\|_\infty &< \langle |xy|\xi, \xi\rangle \\ &= \langle u^*xv|y|\xi, \xi\rangle \\ &= \langle |y|\xi, v^*x^*u\xi\rangle \\ &\leq \||y|\xi\|\|x\|_\infty \\ &= \langle |y|^2\xi, \xi\rangle^{1/2}\|x\|_\infty \\ &\leq s_y(t)\|x\|_\infty, \end{aligned}$$

a contradiction.

By the preceding Lemma, we know that

$$\tau(\chi_{(s_y(t)\|x\|_\infty, \infty)}(|xy|)) \leq t,$$

and thus

$$s_{xy}(t) \leq s_y(t)\|x\|_\infty.$$

If we prove that

$$s_x = s_{x^*},$$

then it will follow that

$$s_{yx}(t) \leq s_y(t)\|x\|_\infty.$$

So it remains to show $s_x = s_{x^*}$, for this it is enough to show that

$$\tau(\chi_{(\lambda,\infty)}(|x|)) = \tau(\chi_{(\lambda,\infty)}(|x^*|)).$$

Let $x = u|x|$ be the polar decomposition. First, we claim that

$$\chi_{(t,\infty)}(|x^*|) \wedge [u\chi_{(t,\infty)}(|x|)u^*]^{\perp} = 0. \tag{A.1}$$

We have that

$$[u\chi_{(t,\infty)}(|x|)u^*]^{\perp} = u\chi_{[0,t]}(|x|)u^* + P_{\overline{\mathrm{im}(x)}} = u\chi_{[0,t]}(|x|)u^* + P_{\ker(x^*)} = u\chi_{[0,t]}(|x|)u^* + P_{\overline{\mathrm{im}(x)}}.$$

Suppose

$$\xi \in \lambda(\chi_{(t,\infty)}(|x^*|) \wedge [u\chi_{(t,\infty)}(|x|)u^*]^{\perp})L^2(M,\tau)$$

and $\|\xi\| = 1$. Let

$$\xi = \xi_0 + \xi_1$$

with $\xi_1 \in \ker(x^*), \xi_0 \perp_{\ker(x^*)}$. We saw in Proposition A.1.2 that $|x^*| = u|x|u^*$, so

$$\begin{aligned} t < \langle |x^*|\xi, \xi\rangle &= \langle |x^*|\xi_0, \xi_0\rangle \\ &= \langle u|x|u^*\xi_0, \xi_0\rangle. \end{aligned}$$

Write $\xi_0 = u\eta$ with $\eta \in \lambda(\chi_{(t,\infty)}(|x|))L^2(M,\tau)$ and $\|\eta\| = \|\xi_0\| \leq 1$. Then,

$$\langle u|x|u^*\xi_0, \xi_0\rangle = \langle u|x|\eta, \eta\rangle \leq \|\eta\|\langle |x|^2\eta, \eta\rangle^{1/2} \leq t,$$

we thus get a contradiction, and this proves $(A.1)$.

By the Lemma,

$$\tau(\chi_{(t,\infty)}(|x^*|)) \leq \tau(\chi_{(t,\infty)}(|x|)u^*u) = \tau(\chi_{(t,\infty)}(|x|))$$

as

$$\chi_{(t,\infty)}(|x|) \leq u^*u.$$

The claim now follows by symmetry.

(iii): We have

$$s_{xp} \leq s_x.$$

Also for any $t > 0$,

$$\chi_{(t,\infty)}(|xp|) \le p_{\ker(xp)^\perp} \le p.$$

So $s_{xp}(t) = 0$ for all $t > \tau(p)$. Thus

$$\|xp\|_1 = \int_0^{\tau(p)} s_{xp}(t)\,dt \le \int_0^{\tau(p)} s_x(t)\,dt.$$

□

We now have several corollaries which prove many of the analogues of inequalities which are known in the commutative case.

Corollary A.1.6. *Let (M, τ) be a tracial von Neumann algebra. For $x, y \in M$ and $1 \le p \le \infty$, we have*

$$\|xy\|_p \le \|x\|_\infty \|y\|_p$$

$$\|x\|_p = \|x^*\|_p,$$

$$\|yx\|_p \le \|x\|_\infty \|y\|_p.$$

Proof. The second inequality follows from part (ii) of the preceding proposition. The third inequality is a consequence of the first. For the first inequality, by part (ii) of the preceding proposition,

$$\|xy\|_p^p = \|s_{xy}\|_p^p = \int_0^1 s_{xy}(t)^p\,dt \le \|x\|_\infty^p \int_0^1 s_y(t)^p\,dt.$$

□

Corollary A.1.7 (Noncommutative Decreasing Rearrangement Inequality)**.** *Let (M, τ) be a tracial von Neumann algebra. For $x, y \in (M, \tau)$ we have*

$$\|xy\|_1 \le \int_0^1 s_x(t) s_y(t)\,dt.$$

Proof. Let $y = u|y|$, $|xy| = v^* xy$ be the polar decomposition. By Borel functional calculus,

$$|xy| = v^* x u |y| = \int_0^\infty v^* x u \chi_{(\lambda,\infty)}(|y|)\,d\lambda.$$

By the preceding proposition,

$$\begin{aligned}\|xy\|_1 &= \int_0^\infty \tau(v^* x u \chi_{(\lambda,\infty)}(|y|))\,d\lambda \\ &\leq \int_0^\infty \int_0^{\tau(\chi_{(\lambda,\infty)}(|y|))} s_{v^*xu}(t)\,dt\,d\lambda \\ &\leq \int_0^\infty \int_0^{\tau(\chi_{(\lambda,\infty)}(|y|))} s_x(t)\,dt\,d\lambda.\end{aligned}$$

By definition, we have that $t < \tau(\chi_{(\lambda,\infty)}(|y|))$ if and only if $\lambda < s_y(t)$. Thus by Fubini,

$$\|xy\|_1 \leq \int_0^\infty \int_0^{s_y(t)} s_x(t)\,d\lambda, dt = \int_0^\infty s_y(t)s_x(t)\,dt.$$

□

Corollary A.1.8. *Let* (M,τ) *be a tracial von Neumann algebra, and* $1 \leq p \leq \infty$ *and let* $1 \leq p' \leq \infty$ *be such that* $\frac{1}{p} + \frac{1}{p'} = 1$*, then*

$$\|xy\|_1 \leq \|x\|_p \|y\|_{p'},$$

further

$$\|x\|_p = \sup\{|\tau(xa)| : a \in L^{p'}(M,\tau), \|a\|_{p'} \leq 1\}.$$

Proof. We have

$$\|xy\|_1 \leq \int_0^\infty s_x(t)s_y(t)\,dt \leq \|s_x\|_p \|s_y\|_{p'} = \|x\|_p \|y\|_{p'},$$

by the usual Hölder's inequality, and Proposition A.1.5.

For the second statement, the preceding and Proposition A.1.2 proves that

$$\|x\|_p \geq \sup\{|\tau(xa)| : a \in L^{p'}(M,\tau), \|a\|_{p'} \leq 1\}.$$

For the reverse, let $x = u|x|$ be the polar decomposition of x, let

$$a = \frac{|x|^{p-1}u^*}{\|x\|_p^{p-1}}.$$

Then

$$\tau(xa) = \frac{1}{\|x\|_p^{p-1}}\tau(u|x|^p u^*) = \frac{1}{\|x\|_p^{p-1}}\tau(|x|^p u^* u) = \|x\|_p,$$

as

$$|x|^p u^* u = |x|^p.$$

By Corollary A.1.6, we have

$$\||x|^{p-1}u^*\|_{p'}^{p'} \leq \||x|^{p-1}\|_{p'}^{p'} = \tau(|x|^p) = \|x\|_p^p.$$

Thus,

$$\|a\|_{p'} \leq \|x\|_p^{\frac{p}{p'}-p-1} = 1.$$

□

Corollary A.1.9. *Let (M,τ) be a tracial von Neumann algebra. Then, $\|\cdot\|_p$ is a norm on M.*

Proof. Let us first prove that $\|\cdot\|_p$ is a norm. The only nontrivial fact is the triangle inequality. For this, let $x, y \in M$, let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $z \in M$,

$$|\tau((x+y)z)| \leq |\tau(xz)| + |\tau(yz)| \leq \|xz\| + \|yz\|_1 \leq \|x\|_p + \|y\|_p,$$

by the preceding corollary. Taking the supremum over all such z and applying the preceding corollary again proves the triangle inequality.

□

Definition A.1.10. By the preceding Corollary, we may define $L^p(M,\tau)$ to be the completion of M with respect to the norm $\|\cdot\|_p$, note that this agrees with the previous definition of $L^2(M,\tau)$. Also by the preceding corollary, we have a bilinear map

$$M\colon L^p(M,\tau) \times L^{p'}(M,\tau) \to L^1(M,\tau)$$

uniquely defined by requiring that $M(x,y) = xy$ for $x, y \in M$, and $\|M(x,y)\|_1 \leq \|x\|_p\|y\|_{p'}$. For $\xi \in L^p(M,\tau), \eta \in L^{p'}(M,\tau)$ we denote $M(\xi,\eta)$ by $\xi\eta$.

Note that for $\xi \in L^2(M,\tau), x \in M$, that by the conventions in the preceding definition

$$x\xi = \lambda(x)\xi, \xi x = \rho(x)\xi,$$

so we will typically drop the λ, ρ from here on out. We will similarly denote ξ^* for $\xi \in L^p(M,\tau)$ the unique isometric extension of the map $x \to x^*$ on M. We denote $\tau\colon L^1(M,\tau) \to \mathbb{C}$ the unique continuous extension of τ to $L^1(M,\tau)$.

By density, we have

$$\tau(xy) = \tau(yx) \ x \in L^p(M,\tau), y \in L^{p'}(M,\tau)$$

$$\langle x\xi, \eta\rangle = \tau(\eta^* x\xi),\ \xi, \eta \in L^2(M,\tau).$$

A.2 Noncommutative L^p-Spaces as Unbounded Operators

We would like to view $L^p(M,\tau)$ as a space of *operators* instead of a completion of a space of operators. This will allow us to apply functional calculus arguments to $L^p(M,\tau)$. This will makes certain arguments easier, in particular computing the dual of $L^p(M,\tau)$, and invariant subspaces under the action of M. The price that we have to pay to do this, is to pass to *unbounded* operators. This is to be expected as in the commutative case, functions in $L^p(X,\mu)$ are in general unbounded. For basics of unbounded operators, we refer to [4] Chapter X.

Definition A.2.1. Let (M,τ) be a tracial von Neumann algebra, and view $M \subseteq B(L^2(M,\tau))$. We let $\mathrm{Meas}(M)$ be the set of all closed, densely-defined, unbounded operators T on $L^2(M,\tau)$ with $xT \subseteq Tx$ for all $x \in M'$.

Equivalently, the graph of T is invariant under the diagonal action of M' on $L^2(M,\tau)^{\oplus 2}$.

We will see later that if $(M,\tau) = L^\infty(X,\mu)$, then $\mathrm{Meas}(M)$ is equal to all M_f for measurable $f\colon X \to \mathbb{C}$, where M_f is defined by

$$\mathrm{dom}(M_f) = \{\xi \in L^2(X,\mu) : f\xi \in L^2(M,\mu)\},$$

$$M_f\xi = f\xi \text{ for all } \xi \in \operatorname{dom}(M_f) \ .$$

We proceed to collect some basic properties about $\operatorname{Meas}(M)$. For this we need the following definition.

Definition A.2.2. Let (M, τ) be a tracial von Neumann algebra. A linear, M'-invariant subspace $V \subseteq L^2(M, \tau)$ is said to be *essentially dense* if for every $\varepsilon > 0$, there is a projection $p \in M$ so that $pL^2(M, \tau) \subseteq V$ and $\tau(p) \geq 1 - \varepsilon$.

Proposition A.2.3. *Let (M, τ) be a tracial von Neumann algebra.*

(i): Let T be a closeable, densely-defined unbounded operator on $L^2(M, \tau)$. Suppose that $A \subseteq M$ has strong operator topology dense linear span in M and $\rho(a)T \subseteq T\rho(a)$ for all $a \in A$. Then the closure of T is measurable.

(ii): Let $T, S \in \operatorname{Meas}(M)$. Suppose that $V \subseteq \operatorname{dom}(T) \cap \operatorname{dom}(S)$ is essentially dense, and $T\xi = S\xi$ for all $\xi \in V$. Then $T = S$.

(iii): Let T be an closed operator on $L^2(M, \tau)$, and let $T = U|T|$ be its polar decomposition. Then $T \in \operatorname{Meas}(M)$ if and only if $U \in M$, and $\chi_B(|T|) \in M$ for all $B \subseteq \mathbb{C}$ Borel.

(iv): Any essentially dense subset of $L^2(M, \tau)$ is norm dense.

(v): Let $T \in \operatorname{Meas}(M)$, and $V \subseteq L^2(M, \tau)$ essentially dense. Then $T^{-1}(V)$ is essentially dense. In particular, $\operatorname{dom}(T) = T^{-1}(L^2(M, \tau))$ is essentially dense.

(vi): A countable intersection of essentially dense subspaces of $L^2(M, \tau)$ is essentially dense.

Proof. (i): The graph of T is a $\rho(A)$-invariant subspace of $L^2(M, \tau)^{\oplus 2}$. Hence its closure is M-invariant.

(ii): Let G_T, G_S be the graphs of T, S. Since T is densely-defined, we have a M'-equivariant injection with dense image

$$G_T \to L^2(M, \tau)$$

by

$$(\xi, T\xi) \mapsto \xi,$$

so

$$\dim_{M'}(G_T) = 1.$$

Similarly,

$$\dim_{M'}(G_S) = 1.$$

By symmetry, it suffices to show that $G_T \cap G_S = G_T$. By the above, it suffices to show that

$$\dim_{M'}(G_T \cap G_S) \geq 1.$$

For this, let V be an essentially dense subspace of $L^2(M, \tau)$ on which S and T agree. Given $\varepsilon > 0$, we can find a $p \in M$ so that

$$pL^2(M, \tau) \subseteq V.$$

Then

$$\{(p\xi, Tp\xi) : \xi \in L^2(M, \tau)\} = \{p\xi, Sp\xi) : \xi \in L^2(M, \tau)\},$$

and so

$$\{(p\xi, Tp\xi) : \xi \in L^2(M, \tau)\} \subseteq G_T \cap G_S.$$

Hence, there is a surjective map

$$G_T \cap G_S \to pL^2(M, \tau)$$

given by $(\xi, \eta) \mapsto p\xi$, so

$$\mathrm{dom}_{M'}(G_T \cap G_S) \geq \tau(p) \geq 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof.

(iii): First suppose $U \in M$, and that $\chi_B(|T|) \in M$ for all $B \subseteq \mathbb{C}$ Borel. Let $x \in M$, and $\xi \in \mathrm{dom}(T) = \mathrm{dom}(|T|)$. Let E be the spectral measure on $[0, \infty)$ so that

$$|T| = \int_{[0,\infty)} t\, dE(t).$$

For $B \subseteq \mathbb{C}$ Borel, we have

$$\langle E(B)x\xi, x\xi\rangle = \langle \rho(x)E(B)\xi, \rho(x)E(B)\xi\rangle \leq \|x\|_\infty \langle E(B)\xi, \xi\rangle.$$

Thus we have the following inequality of measures

$$d\langle E(t)\rho(x)\xi, \rho(x)\xi\rangle \leq \|x\|_\infty d\langle E(t)\xi, \xi\rangle,$$

so

$$\int t^2 d\langle E(t)\rho(x)\xi, \rho(x)\xi\rangle \leq \|x\|_\infty \int t^2 d\langle E(t)\xi, \xi\rangle < \infty$$

so $x\xi \in \mathrm{dom}(T)$. Further, from the equality of spectral measure

$$d\langle E(T)\rho(x)\xi, \eta\rangle = d\langle E(T)\xi, \rho(x)^*\eta\rangle,$$

it is straightforward to see that $|T|(\rho(x)\xi) = \rho(x)|T|(\xi)$. Hence

$$T(\rho(x)\xi) = U\rho(x)|T|(\xi) = \rho(x)U|T|(\xi) = \rho(x)T(\xi),$$

so $T \in \mathrm{Meas}(M)$.

Conversely, suppose that $T \in \mathrm{Meas}(M)$. Let $u \in \mathcal{U}(M)$, then since $T \in \mathrm{Meas}(M)$ we have

$$\rho(u)T = T\rho(u),$$

$$\rho(u^*)T = T\rho(u^*).$$

Thus,

$$|T|^2 = \rho(u)|T|^2\rho(u^*),$$

hence for all $B \subseteq \mathbb{C}$ Borel we have

$$\chi_B(|T|^2) = \rho(u)\chi_B(|T|^2)\rho(u^*).$$

This easily implies that

$$\rho(u)\chi_B(|T|) = \chi_B(|T|)\rho(u)$$

for all $B \subseteq \mathbb{C}$ Borel. Since M is the linear span of its unitaries, we find that

$$\chi_B(|T|) \in M$$

for all $B \subseteq \mathbb{C}$ Borel. From this, it is not hard to argue as in the first half of the proof that $|T| \in \text{Meas}(M)$. Thus, for all $u \in \mathcal{U}(M)$, we have

$$T = \rho(u)T\rho(u)^* = \rho(u)U\rho(u)^*|T|,$$

and uniqueness of the polar decomposition implies that

$$\rho(u)U = U\rho(u).$$

As before, this implies that $U \in M$.

(iv): Let V be an essentially dense subspace of $L^2(M,\tau)$. For all $n \in \mathbb{N}$, choose $p_n \in M$ with $\tau(p_n) \geq 1 - 2^{-n}$, and

$$p_n L^2(M,\tau) \subseteq V.$$

Set

$$q_n = \bigwedge_{m \geq n} p_m,$$

then

$$\tau(1 - q_n) \leq 2^{-n+1}.$$

As q_n are increasing, we have

$$\dim_{M'}\left(\overline{\bigcup_{n=1}^{\infty} q_n L^2(M,\tau)}\right) = \lim_{n\to\infty} \tau(q_n) = 1,$$

and this implies that

$$\overline{\bigcup_{n=1}^{\infty} q_n L^2(M,\tau)} = L^2(M,\tau).$$

Since

$$q_n L^2(M,\tau) \subseteq V,$$

it follows that V is norm dense.

(v): Let $T = U|T|$ be the polar decomposition of T. For $n \in \mathbb{N}$, set

$$q_n = \chi_{(1/n,n)}(|T|).$$

Note that

$$\|T\xi\| \geq \frac{1}{n}\|\xi\|$$

on $q_n L^2(M,\tau)$, thus $T\big|_{q_n L^2(M,\tau)}$ is an injection with closed image. Set

$$\mathcal{H}_n = T(q_n L^2(M,\tau)),$$

and let $\phi_n \colon \mathcal{H}_n \to q_n L^2(M,\tau)$ be the inverse to T.

Let $\varepsilon > 0$, and choose $p \in M$ so that

$$pL^2(M,\tau) \subseteq V,$$

and

$$\tau(p) \geq 1 - \varepsilon.$$

Choose n so that

$$\tau(q_n) \geq \dim_{M'}((\ker(T))^{\perp}) - \varepsilon.$$

Since

$$q_n L^2(M,\tau) \cap (1-p)L^2(M,\tau) \subseteq (1-p)L^2(M,\tau),$$

we have

$$\begin{aligned}\dim_{M'}(q_n L^2(M,\tau) \cap pL^2(M,\tau)) &\geq \dim_{M'}(q_n L^2(M,\tau)) - \varepsilon \\ &\geq \dim_{M'}(\ker(T)^{\perp}) - 2\varepsilon.\end{aligned}$$

Let

$$K_n = \phi_n(q_n L^2(M,\tau) \cap pL^2(M,\tau)),$$

then

$$\ker(T) + K_n \subseteq T^{-1}(V),$$

and

$$\dim_{M'}(\ker(T)+K_n) = \dim_{M'}(\ker(T)) + \dim_{M'}(K_n) \geq \dim_{M'}(\ker(T)^{\perp}) - 2\varepsilon + \dim_{M'}(\ker(T)) = 1 - 2\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we see that $T^{-1}(V)$ is essentially dense.

(vi): Let $(V_n)_{n=1}^{\infty}$ be essentially dense subspaces of $L^2(M,\tau)$, and let $\varepsilon > 0$. For each $n \in \mathbb{N}$, choose $p_n \in M$ so that

$$\tau(p_n) \geq 1 - \frac{\varepsilon}{2^{-n}},$$

and

$$p_n L^2(M,\tau) \subseteq V_n.$$

Set

$$p = \bigwedge_{n=1}^{\infty} p_n,$$

then

$$\tau(p) \geq 1 - \varepsilon,$$

and

$$pL^2(M,\tau) \subseteq \bigcap_{n=1}^{\infty} V_n.$$

□

Let (M,τ) be a tracial von Neumann algebra. Let $x, y \in \mathrm{Meas}(M)$, consider the two operators

$$P_{xy} \colon y^{-1}(\mathrm{dom}(x)) \to L^2(M,\tau)$$

$$S_{xy} \colon \mathrm{dom}(x) \cap \mathrm{dom}(y) \to L^2(M,\tau)$$

by

$$P_{xy}(\xi) = xy\xi,$$

$$S_{xy}(\xi) = x\xi + y\xi.$$

By (v),(vi) of the above proposition, we know that P_{xy}, S_{xy} are densely-defined. As

$$P_{xy}^* \supseteq P_{y^*x^*}$$

$$S_{xy}^* \supseteq S_{x^*y^*},$$

we know that P_{xy}, S_{xy} are densely-defined. Thus we may define

$$xy, x+y$$

to be the closures of P_{xy}, S_{xy}. By (i) of the above proposition, we know that $xy, x+y \in \mathrm{Meas}(M)$. One can use the above proposition to argue that these operations turn $\mathrm{Meas}(M)$ into an algebra. For example

$$(xy)z, x(yz)$$

agree on $z^{-1}(y^{-1}(\mathrm{dom}(x))$, and so by (v) and (ii) of the above proposition, we know

$$(xy)z = x(yz).$$

We leave the similar proofs of the other axioms to the reader.

We can turn $\mathrm{Meas}(M)$ into a topological $*$-algebra with a basis of opens neighborhoods of the identity given by

$$U_{\varepsilon,t} = \{T \in \mathrm{Meas}(M) : \tau(\chi_{(t,\infty)}(|T|) < \varepsilon\}.$$

We leave the proofs of the axioms of a topological vector space to the reader (see [24] IX.2 for detailed proofs). We call the result topology the *measure topology.*

For intuition, let us discuss what the above proposition implies in the abelian case.

Proposition A.2.4. *Let X be a compact metrizable space, and let μ a Borel probability measure on X. View $L^\infty(X,\mu)$ on operators on $L^2(X,\mu)$. For a μ-measurable $f\colon X \to \mathbb{C}$, define a densely-defined operator by*

$$\mathrm{dom}(M_f) = \{\xi \in L^2(X,\mu) : f\xi \in L^2(X,\mu)\},$$

and

$$M_f\xi = f\xi$$

for $\xi \in \mathrm{dom}(M_f)$. Thus $\mathrm{Meas}(M)$ is indeed a generalization of the algebra of measurable functions associated to a measure space. Then a closed operator T on $L^2(X,\mu)$ is in $\mathrm{Meas}(L^\infty(X,\mu))$ if and only if $T = M_f$ for some measurable $f\colon X \to \mathbb{C}$.

Proof. First we note that M_f is a closed operator. Suppose $\xi_n \in \mathrm{dom}(M_f)$, and $\xi_n \to \xi$ in $L^2(X,\mu)$, and $f\xi_n \to g \in L^2(X,\mu)$. Passing to a subsequence, we may assume that $\xi_n \to \xi, f\xi_n \to f\xi$ pointwise almost everywhere. By Fatou's Lemma,

$$\|f\xi\|_2 \leq \liminf_{n\to\infty} \|f\xi_n\| < \infty$$

as $f\xi_n \to g$. Thus for almost every $x \in X$,

$$g(x) = \lim_{n\to\infty} f\xi_n(x) = f\xi(x).$$

Hence M_f is closed. It is easy to see that the polar decomposition of f is given by

$$M_f = M_\alpha M_{|f|}$$

where $\alpha(x) = \chi_{\{x : f(x)\neq 0\}} \frac{f(x)}{|f(x)|}$. Additionally, for all $B \subseteq [0,\infty)$ Borel,

$$\chi_E(M_{|f|}) = M_{\chi_E(|f|)}.$$

Thus, it follows that $M_f \in \mathrm{Meas}(L^\infty(X,\mu))$.

Suppose that $T \in \mathrm{Meas}(L^\infty(X,\mu))$, and let

$$T = U|T|$$

be its polar decomposition. Then $U = M_\alpha$ for some α with $|\alpha(x)| \in \{0,1\}$ for almost every $x \in X$. Let

$$|T| = \int_{[0,\infty)} t\, dE(t)$$

be the polar decomposition of T. Set

$$S_n = \int_{[1/n,n]} t\, dE(t).$$

Since $E(B)$ commutes with $L^\infty(X,\mu)$ for all $B \subseteq [0,\infty)$ Borel, it is not hard to argue that S_n commutes with $L^\infty(X,\mu)$ and thus not $S_n = M_{f_n}$ for a unique (up to measure zero) $f_n \colon X \to [1/n,n]$. Further $f_n \leq f_{n+1}$ almost everywhere. Removing a countable collection of null sets, we may assume that for all n, for all $x \in X$, we have

$$f_n(x) \leq f_{n+1}(x).$$

Set

$$f(x) = \sup_n f_n(x).$$

Note that

$$\mu(\{x \in X : f(x) \geq M\}) = \lim_{n\to\infty} \mu(\{x \in X : f_n(x) \geq M\}).$$

For each n, we have $E((M,n)) = \chi_{\{x: f_n(x) \geq M\}}$ for some decreasing sequence of sets A_n. Since $E((M,n)) \to E((M,\infty))$ in the strong operator topology as $n \to \infty$, if we choose $A_M \subseteq X$ measurable so that $E((M,\infty)) = \chi_{A_m}$ we find that

$$\mu(\{x \in X : f(x) \geq M\}) = \mu(A_M).$$

Since $E((M,\infty)) \to 0$ in the strong operator topology we find that

$$\mu(\{x \in X : f(x) \geq M\}) = \mu(A_M) \to 0$$

as $M \to \infty$. Hence, we find that $f(x) < \infty$ for almost every x. By construction

$$M_{\chi_{(t,\infty)}(|f|)} = E(\chi_{(t,\infty)}).$$

From this, it is not hard to argue that

$$M_f = |T|.$$

Setting $g = \alpha|f|$, we find that

$$T = M_g.$$

□

We now turn to our alternate definition of $L^p(M,\tau)$. For $1 \leq p < \infty$, let

$$\mathcal{L}^p = \left\{ T \in \text{Meas}(M) : \int_{[0,\infty)} t^p \, d\langle E_{|L|}(t)1, 1\rangle < \infty \right\},$$

for $T \in \mathcal{L}^p$, set

$$\|T\|_{\mathcal{L}^p}^p = \int_{[0,\infty)} t^p \, d\langle E_{|L|}(t)1, 1\rangle.$$

By the above Proposition, if (M,τ) is an abelian tracial von Neumann algebra, then $L^p(M.,\tau)$ can be canonically and isometrically identified with $\mathcal{L}^p$. It is thus reasonable to wonder if this is true in the nonabelian case we now proceed to show that this is true.

Proposition A.2.5. *Let (M, τ) be a tracial von Neumann algebra.*

(i): We have that $\mathcal{L}^p$ is a vector space, and

$$\|T + S\|_{\mathcal{L}^p} \leq 2^p(\|T\|_{\mathcal{L}^p} + \|S\|_{\mathcal{L}^p}).$$

(ii): If $T_n \in \mathcal{L}^p$, and $\|T - T_n\|_{\mathcal{L}^p} \to 0$, then

$$\|T\|_{\mathcal{L}^p} \leq \liminf_{n \to \infty} \|T_n\|_{\mathcal{L}^p}.$$

Proof. (i): As in proposition A.1.5 (ii), we have

$$\tau(\chi_{(t,\infty)}(|T + S|)) \leq \tau(\chi_{(t/2,\infty)}(|T|)) + \tau(\chi_{(t/2,\infty)}(|S|)).$$

Thus,

$$\begin{aligned}
\|T + S\|_{\mathcal{L}^p}^p &= p \int_0^\infty \tau(\chi_{(t,\infty)}(|T + S|))\, dt \\
&\leq p \int_0^\infty t^{p-1} \tau(\chi_{(t/2,\infty)}(|T|))\, dt + p \int_0^\infty t^{p-1} \tau(\chi_{(t/2,\infty)}(|S|))\, dt \\
&= 2^p p \int_0^\infty t^{p-1} \tau(\chi_{(t,\infty)}(|T|))\, dt + 2^p p \int_0^\infty t^{p-1} \tau(\chi_{(t,\infty)}(|S|))\, dt \\
&= 2^p(\|T\|_{\mathcal{L}^p}^p + \|S\|_{\mathcal{L}^p}^p).
\end{aligned}$$

(ii): As in proposition A.1.5 (ii)

$$\begin{aligned}
\tau(\chi_{(t+\varepsilon,\infty)}(|T|)) &\leq \tau(\chi_{(t,\infty)}(|T_n|)) + \tau(\chi_{(\varepsilon,\infty)}(|T - T_n|)) \\
&\leq \tau(\chi_{(t,\infty)}(|T_n|)) + \frac{1}{\varepsilon^p} \|T - T_n\|_{\mathcal{L}^p}^p.
\end{aligned}$$

Thus for all $t > 0, \varepsilon > 0$,

$$\tau(\chi_{(t+\varepsilon,\infty)}(|T|)) \leq \liminf_{n \to \infty} \tau(\chi_{(t,\infty)}(|T_n|)).$$

By normality of τ, we know that

$$\tau(\chi_{[t,\infty)}(|T|)) = \tau(\chi_{(t,\infty)}(|T|))$$

for all but countably many t. Letting $\varepsilon \to 0$, we find that

$$\tau(\chi_{(t\infty)}(|T|)) \le \liminf_{n\to\infty} \tau(\chi_{(t,\infty)}(|T_n|)),$$

for all but countably many t. Thus (ii) follows from Fatou's Lemma and the equality

$$\|S\|_{\mathcal{L}^p} = p\int_0^\infty t^{p-1}\tau(\chi_{(t,\infty)}(|S|))\, dt.$$

□

Theorem A.2.6. *Let (M,τ) be a tracial von Neumann algebra.*

Then $\|\cdot\|_{\mathcal{L}^p}$ is a norm which turns $\mathcal{L}^p$ into a Banach space, and there is a unique isometry

$$L^p(M,\tau) \to \mathcal{L}^p$$

which is the identity on M.

Proof. For $T, S \in \mathcal{L}^p$, let $T = U|T|, S = V|S|$ be the polar decomposition, let $T_n = U\chi_{[0,n]}(|T|)|T|, S_n = V\chi_{[0,n]}(|S|)|S|$. Then,

$$\|T_n\|_{\mathcal{L}^p} \to \|T\|_{\mathcal{L}^p}, \|T - T_n\|_{\mathcal{L}^p} \to 0,$$

$$\|S_n\|_{\mathcal{L}^p} \to \|S\|_{\mathcal{L}^p}, \|S - T_n\|_{\mathcal{L}^p} \to 0,$$

and the preceding proposition implies that

$$\|T + S\|_{\mathcal{L}^p} \le \liminf_{n\to\infty} \|T_n + S_n\|_{\mathcal{L}^p}.$$

Since $T_n, S_n \in M$, we know that

$$\|T_n\|_{\mathcal{L}^p} = \|T_n\|_p,$$

$$\|S_n\|_{\mathcal{L}^p} = \|S_n\|_p,$$

and so the triangle inequality now follows from the fact that $\|\cdot\|_p$ is a norm. The existence and the uniqueness of the isometry follows from the fact that

$$\|x\|_p = \|x\|_{\mathcal{L}^p}$$

for $x \in M$, and the density of M in $\mathcal{L}^p$, once we prove that $\mathcal{L}^p$ is a Banach space.

To prove that $\mathcal{L}^p$, let $x_n \in \mathcal{L}^p$ with $\|x_n\|_p < 3^{-n}$, it is enough to show that

$$\sum_{n=1}^{\infty} x_n$$

converges in $\mathcal{L}^p$. Let

$$\mathcal{K} = \left\{ \xi \in \bigcap_{n=1}^{\infty} \mathrm{dom}(x_n) : \sum_{n=1}^{\infty} \|x_n \xi\|_2 < \infty \right\},$$

and define T on $\mathcal{K}$ by

$$T(\xi) = \sum_{n=1}^{\infty} x_n \xi.$$

We claim that T is densely defined and closeable, and that its closure is measurable with respect to M.

For this, set

$$p_{t,n} = \bigvee_{m \geq n} \chi_{(t2^{-n},\infty)}(|x_n|),$$

then

$$\tau(p_{t,n}) \leq \sum_{m=n}^{\infty} \tau(\chi_{(t2^{-m},\infty)}(|x_n|)) \leq \frac{1}{t^p} \left(\sum_{m=n}^{\infty} \left(\frac{2}{3}\right)^{mp} \right),$$

and $\mathcal{K} \supseteq (1 - p_{t,1}) L^2(M, \tau)$, this proves that $\mathcal{K}$ is dense. As

$$\mathrm{dom}(T^*) \supseteq \left\{ \xi \in \bigcap_{n=1}^{\infty} \mathrm{dom}(x_n) : \sum_{n=1}^{\infty} \|x_n^* \xi\|_2 < \infty \right\},$$

the same logic implies that T is closeable. It is also straightforward to check that the domain of $\mathcal{K}$ is $\rho(M)$ invariant, and that $T(\xi x) = T(\xi)x$ for $\xi \in \mathcal{K}, x \in M$. Thus the closure of T is a measurable operator affiliated to M, we let x be the closure of T.

As in Proposition A.1.2

$$\chi_{(t(1+2^{-N}),\infty)}(|x|) \wedge \chi_{(t,\infty)}\left(\left| \sum_{n=1}^{N} x_n \right| \right) \wedge p_{t,N+1} = 0,$$

so

$$\tau(\chi_{(t(1+2^{-N}),\infty)}(|x|)) \leq \tau\left(\chi_{(t,\infty)}\left(\left| \sum_{n=1}^{N} x_n \right| \right) \right) + \frac{1}{t^p} \sum_{n=N+1}^{\infty} \frac{2^{np}}{3^{np}},$$

as in the preceding proposition this implies that

$$\tau(\chi_{(t,\infty)}(|x|)) \leq \liminf_{N\to\infty} \tau\left(\chi_{(t,\infty)}\left(\left|\sum_{n=1}^{N} x_n\right|\right)\right)$$

for all but countable many t. Thus by Fatou's Lemma

$$\begin{aligned}
\|x\|_{\mathcal{L}^p} &= p\int_0^\infty t^{p-1}\tau(\chi_{(t,\infty)}(|x|)) \\
&\leq \liminf_{N\to\infty} p\int_0^\infty t^{p-1}\tau\left(\chi_{(t,\infty)}\left(\left|\sum_{n=1}^{N} x_n\right|\right)\right) dt \\
&= \liminf_{N\to\infty}\left\|\sum_{n=1}^{N} x_n\right\|_{\mathcal{L}^p} \\
&\leq \sum_{n=1}^{\infty}\|x_n\|_{\mathcal{L}^p},
\end{aligned}$$

so $x \in \mathcal{L}^p$.

By the same logic,

$$\tau\left(\chi_{(t,\infty)}\left(\left|x - \sum_{n=1}^{N} x_n\right|\right)\right) \leq \liminf_{M\to\infty} \tau\left(\chi_{(t,\infty)}\left(\left|\sum_{n=N+1}^{M} x_n\right|\right)\right),$$

and thus

$$\left\|x - \sum_{n=1}^{N} x_n\right\|_{\mathcal{L}^p} \leq \sum_{n=N+1}^{\infty}\|x_n\|_{\mathcal{L}^p} \to 0$$

as $N \to \infty$. This completes the proof.

□

A.3 Duality

We now extend the usual duality between $L^p(M,\tau)$ and $L^{p'}(M,\tau)$ from the commutative case to the noncommutative case. Let us first start with the dual of L^1.

Theorem A.3.1. *Let (M,τ) be a tracial von Neumann algebra. Under the duality*

$$(x,y) \to \tau(xy)$$

we have an isometric identification

$$L^1(M,\tau)^* \cong M.$$

Proof. For $x \in M$, we denote $\tau(x\cdot)$ the element of $L^1(M,\tau)^*$ defined by

$$\tau(x\cdot)(y) = \tau(xy).$$

We have already seen that for $x \in M, y \in L^1(M,\tau)$ we have

$$|\tau(xy)| \leq \|x\|_\infty \|y\|_1,$$

thus $\|\tau(x\cdot)\| \leq \|x\|_\infty$.

Let $\phi \in L^1(M,\tau)^*$. For $\xi, \eta \in L^2(M,\tau)$ we have

$$|\phi(\xi\eta^*)| \leq \|\phi\| \|\xi\eta^*\|_1 \leq \|\xi\|_2 \|\eta\|_2 \|\phi\|.$$

Thus there is a unique $T \in B(L^2(M,\tau))$ with $\|T\| \leq \|\phi\|$ and

$$\langle T(\xi), \eta \rangle = \phi(\xi\eta^*).$$

For $y \in M$, we have

$$\langle T(\xi y), \eta \rangle = \phi(\xi y \eta^*) = \langle T(\xi), \eta y^* \rangle = \langle T(\xi) y, \eta \rangle,$$

thus $T \in (M')' = M$. For $y \in L^1(M,\tau)$, we have

$$\phi(y) = \phi(u|y|^{1/2}|y|^{1/2}) = \langle Tu|y|^{1/2}, |y|^{1/2} \rangle = \tau(|y|^{1/2}Tu|y|^{1/2}) = \tau(Ty).$$

Further

$$\|T\| \leq \|\phi\| = \|\tau(T\cdot)\|.$$

This proves the theorem.

□

We have a similar result for M.

Theorem A.3.2. *Let (M, τ) be a tracial von Neumann algebra and $\phi\colon M \to \mathbb{C}$ a normal linear functional. Then, there is a unique $y \in L^1(M, \tau)$ so that $\phi(x) = \tau(xy)$ for all $x \in M$.*

Proof. It is well known that

$$(X^*, weak^*)^* = X,$$

so it is enough to show that ϕ is continuous in the weak* topology coming from M as the dual of $L^1(M, \tau)$. For this, it is enough to show that

$$\ker(\phi) \cap M$$

is weak* closed. By the Krein-Smulian theorem (see [4] V.12.1), it is enough to show that

$$\ker(\phi) \cap \{x \in M : \|x\|_\infty \leq 1\}$$

is weak* closed. For this, it is enough to show that $\phi\big|_{\{x \in M : \|x\|_\infty \leq 1\}}$ is weak* continuous. Let $x_i \in M$, with $\|x_i\|_\infty \leq 1$, and suppose that $x_i \to x$ weak*. Given $\xi, \eta \in L^2(M, \tau)$ we have

$$\langle x_i \xi, \eta \rangle = \tau(x_i \xi \eta^*) \to \tau(x \xi \eta^*),$$

as $\xi\eta^* \in L^1(M, \tau)$. Thus $x_i \to x$ in the weak operator topology, and the normality of ϕ implies that

$$\phi(x_i) \to \phi(x).$$

This proves the theorem.

□

Theorem A.3.3. *Let (M, τ) be a tracial von Neumann algebra, and $1 \leq p < \infty$, let $1 < p' \leq \infty$ be given by*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The duality

$$(x, y) \mapsto \tau(xy)$$

gives an isometric identification $L^p(M, \tau)^* \cong L^{p'}(M, \tau)$.

Proof. By Theorem A.3.1, we may assume $p > 1$. For $x \in L^{p'}(M, \tau)$, let

$$\tau_x \colon L^p(M, \tau) \to \mathbb{C}$$

by

$$\tau_x(y) = \tau(xy).$$

By Corollary A.1.8, we know

$$\|\tau_x\| = \|x\|_{p'}.$$

Let $\phi\colon L^p(M, \tau)^*$, note that $\phi\big|_M$ is normal. For this, suppose that $x_i \in M$, $\|x_i\|_\infty < 1$, and $x_i \to x$ in the strong operator topology. If $p \le 2$, then

$$\|x_i - x\|_p \le \|x_i - x\|_2 \to 0,$$

if $p > 2$, then

$$\|x_i - x\|_p^p \tau(|x_i - x|^2 |x_i - x|^{p-2}) \le 2^{p-2} \|x_i - x\|_2 \to 0.$$

So

$$\phi(x_i) \to \phi(x),$$

and thus ϕ is normal. So by Proposition 2.1.8 (iv) and the preceding theorem, we have $\phi(x) = \tau(xy)$ for some $y \in L^1(M, \tau)$ and all $x \in M$. We may regard y as a densely-defined unbounded operator. Let $y = u|y|$ be the polar decomposition, and let

$$y_n = \chi_{[0,n]}(|y|)|y|^{p'-1}u^*.$$

Then,

$$\|y_n\|_p \|\phi\| \ge \tau(y y_n) = \int_{[0,n]} t^{p'} \langle dE_{|y|}(t)1, 1\rangle.$$

As

$$\|y_n\|_p \le \|\chi_{[0,n]}(|y|)|y|^{p-1}\|_p = \left(\int_{[0,n]} t^{p'} \langle dE_{|y|}(t)1, 1\rangle\right)^{1/p},$$

we have

$$\|\phi\| \ge \left(\int_{[0,n]} t^{p'} \langle dE_{|y|}(t)1, 1\rangle\right)^{1/p'},$$

letting $n \to \infty$ implies that $y \in L^p(M, \tau)$. By density $\phi = \tau_y$.

□

A.4 Interpolation

We wish to generalize the usual Riesz-Thorin interpolation theorem for L^p-spaces to the more general noncommutative L^p-spaces. We wish actually prove something more general, using the abstract version of interpolation theory for Banach spaces.

Definition A.4.1. A *compatible pair of Banach spaces* is a pair of Banach spaces (X, Y) together with continuous inclusions into a Hausdorff locally convex topological vector space Z. We will usually identify X, Y with their images in Z. For $a \in X + Y$, we define the norm

$$\|a\| = \inf\{\|x\| + \|y\| : x \in X, y \in Y, a = x + y\}.$$

Note that we have a natural isometry

$$\frac{X \oplus Y}{\{(x, y) : x = -y\}} \to X + Y,$$

where $X \oplus Y$ is given the norm

$$\|(x, y)\| = \|x\| + \|y\|.$$

Since the inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous, we know that

$$\{(x, y) : x = -y\}$$

is closed. Thus $X + Y$ is a Banach space.

Let $\Omega = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, we let $\mathcal{A}(X, Y)$ be set of all continuous functions $f : \overline{\Omega} \to X + Y$ such that $f\big|_{\Omega}$ is holomorphic, and $f(it) \in X$ for all $t \in \mathbb{R}$, $f(1 + it) \in Y$ for all $t \in \mathbb{R}$, and

$$\|f\| = \sup_t \max(\|f(it)\|_X, \|f(1 + it)\|_Y).$$

For $0 < \theta < 1$, we set

$$(X, Y)_\theta = \{f(\theta) : f \in \mathcal{A}(X, Y)\},$$

with the norm

$$\|p\|_\theta = \inf_f \|f\|,$$

where the infimum is over all $f \in \mathcal{A}(X,Y)$ so that $f(\theta) = p$. We call $(X,Y)_\theta$ the θ-interpolation between X and Y.

Proposition A.4.2. *Let (X,Y) be a compatible pair of Banach spaces. Then $\mathcal{A}(X,Y)$ and $(X,Y)_\theta, 0 < \theta < 1$ are all Banach spaces.*

Proof. We first prove that $\mathcal{A}(X,Y)$ is a Banach space, the only nontrivial issue being completeness. We need a preliminary observation. Let $\phi \in (X+Y)^*$, $f \in \mathcal{A}(X,Y)$ and consider the function $g\colon \overline{\Omega} \to \mathbb{C}$ given by

$$g(z) = \phi(f(z)).$$

Then g is continuous on $\overline{\Omega}$ and holomorphic on Ω. Thus by the Three-Lines Lemma we have

$$\sup_z |g(z)| \leq \sup_t \max(|g(it)|, |g(1+it)|) \leq \|f\|\|\phi\|.$$

If we fix z and taking the supremum over all ϕ we find that

$$\|f(z)\|_{X+Y} \leq \|f\|$$

for all $z \in \overline{\Omega}$. Thus

$$\sup_{z\in\overline{\Omega}} \|f(z)\|_{X+Y} \leq \|f\|.$$

Now suppose that $f_n \in \mathcal{A}(X,Y)$ are Cauchy. From the above estimates, we see that f_n converges uniformly to a continuous function $f\colon \overline{\Omega} \to X+Y$, clearly $f(it) \in X$, $f(1+it) \in Y$ for all $t \in \mathbb{R}$. Further, if $\phi \in (X+Y)^*$, then

$$\phi \circ f_n \to \phi \circ f$$

uniformly, and thus $\phi \circ f$ is holomorphic on Ω. Since this is true for all ϕ, we know that f is holomorphic (see [4] Exercise VII.3.4) on Ω. Since $f_n \to f$ uniformly, we have that

$$\|f_n - f\|_{\mathcal{A}(X,Y)} \to 0.$$

Thus $\mathcal{A}(X,Y)$ is a Banach space.

Fix $0 < \theta < 1$, then $(X, Y)_\theta$ can be isometrically identified with

$$\mathcal{A}(X, Y)/\{f \in \mathcal{A}(X, Y) : f(\theta) = 0\},$$

as $\{f \in \mathcal{A}(X, Y) : f(\theta) = 0\}$ is closed, we find that $(X, Y)_\theta$ is a Banach space.

□

We present the main theorem on interpolation spaces. For a linear operator $T\colon X \to Y$ between Banach spaces, we use $\|T\|_{X\to Y}$ for the operator norm.

Theorem A.4.3. *Let $(X_1, Y_1), (X_2, Y_2)$ be a compatible pairs of Banach spaces. Let $T\colon X_1 + Y_1 \to X_2 + Y_2$ be a linear operator such that $T(X_1) \subseteq X_2, T(Y_1) \subseteq Y_2$ and $T\big|_{X_1}\colon X_1 \to X_2, T\big|_{Y_1}\colon Y_1 \to Y_2$ are bounded operators. Then for all $0 < \theta < 1$, we have that $T((X_1, Y_1)_\theta) \subseteq (X_2, Y_2)_\theta$ and*

$$\|T\|_{(X_1,Y_1)_\theta \to (X_2,Y_2)_\theta} \leq \|T\|^{1-\theta}_{X_1\to X_2} \|T\|^{\theta}_{Y_1\to Y_2}.$$

Proof. First note that T is a bounded linear operator $X_1 + Y_1 \to X_2 + Y_2$ and

$$\|T\|_{X_1+Y_1\to X_2+Y_2} \leq \|T\|.$$

Thus for all $f \in \mathcal{A}(X_1, Y_1)$ we have $T \circ f \in \mathcal{A}(X_2, Y_2)$. Fix M_1, M_2 real numbers so that

$$M_1 > \|T\|_{X_1\to X_2}, M_2 > \|T\|_{Y_1\to Y_2}.$$

Fix $0 < \theta < 1$. Let $p \in (X_1, Y_1)_\theta$ and $f \in \mathcal{A}(X_1, Y_1)$ such that $f(\theta) = p$. Set

$$g(z) = T(f(z))(M_1 M_2^{-1})^{z-\theta},$$

then $g \in \mathcal{A}(X_2, Y_2)$ and $g(\theta) = T(f(\theta))$. For $t \in \mathbb{R}$,

$$\|g(it)\| \leq \|f\|\|T\|_{X_1\to X_2} M_1^{-\theta} M_2^{\theta} \leq M_1^{1-\theta} M_2^{\theta} \|f\|,$$

$$\|g(1 + it)\| \leq \|f\|\|T\|_{Y_1\to Y_2} M_1^{1-\theta} M_2^{\theta-1} \leq M_1^{1-\theta} M_2^{\theta} \|f\|.$$

Thus

$$\|T(p)\|_\theta \leq \|f\| M_1^{1-\theta} M_2^{\theta},$$

and taking the infimum over all f proves

$$\|T(p)\|_\theta \leq \|p\|_\theta M_1^{1-\theta} M_2^\theta.$$

Thus

$$\|T\|_{(X_1,Y_1)_\theta \to (X_2,Y_2)_\theta} \leq M_1^{1-\theta} M_2^\theta$$

and letting $M_1 \to \|T\|_{X_1 \to X_2}, M_2 \to \|T\|_{Y_1 \to Y_2}$ completes the proof.

□

We wish to apply the above interpolation theory to noncommutative L^p-spaces. For this, if we view L^p as unbounded operators, we then have continuous inclusions

$$L^p(M,\tau) \subseteq \text{Meas}(M)$$

where $\text{Meas}(M)$ is given the measure topology. It then suffices to prove the following theorem.

Theorem A.4.4. *Let (M,τ) be a tracial von Neumann algebra, and $1 \leq p_0, p_1 \leq \infty$. Define p_θ for $0 < \theta < 1$ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$(L^{p_0}(M,\tau), L^{p_1}(M,\tau))_\theta = L^{p_\theta}(M,\tau),$$

with equality of norms.

Proof. We may assume that $p_0 \neq p_1$, hence $1 < p_\theta < \infty$.

For $1 \leq p \leq \infty$ we let p' be defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Note that

$$\frac{1-\theta}{p_0'} + \frac{\theta}{p_1'} = \frac{1}{p_\theta'}$$

Let $f \in \mathcal{A}(L^{p_0}(M,\tau), L^{p_1}(M,\tau))$. Suppose that $x \in M$, and that

$$|x| \geq \varepsilon \chi_{(0,\infty)}(|x|)$$

for some $\varepsilon > 0$. Let $x = u|x|$ be the polar decomposition, and set

$$g(z) = u|x|^{\frac{p_{\theta'}}{p_0'}+z\left(\frac{p_\theta'}{p_1'}-\frac{p_\theta'}{p_0'}\right)}\|x\|_{p_{\theta'}}^{1-\frac{p_\theta'}{p_0'}+z\left(\frac{p_\theta'}{p_1'}-\frac{p_\theta'}{p_0'}\right)}.$$

Since

$$|x| \geq \varepsilon\chi_{(0,\infty)}(|x|),$$

we see by functional calculus that

$$z \mapsto |x|^{\frac{p_{\theta'}}{p_0'}+z\left(\frac{p_\theta'}{p_1'}-\frac{p_\theta'}{p_0'}\right)}$$

is holomorphic in $\|\cdot\|_\infty$. Thus $g(z)$ is a holomorphic as a $L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$ valued function.

Thus $\psi(z) = \tau(f(z)g(z))$ is holomorphic, and

$$\sup_{t\in\mathbb{R}}|\psi(it)| \leq \|f\| \sup_t \|g(it)\|_{p_0'},$$

for $t \in \mathbb{R}$, using $\|ab\|_{p_1} \leq \|a\|_\infty\|b\|_{p_1}$ for $a \in M, b \in L^{p_1}(M,\tau)$ and the fact that $|x|^{is}$ is unitary for all $s \in \mathbb{R}$,

$$\|g(it)\|_{p_1} \leq \|x\|_{p_{\theta'}}^{1-\frac{p_{\theta'}}{p_0}}\||x|^{\frac{p_\theta'}{p_1'}}\|_{p_1'} = \|x\|_{p_\theta'}.$$

Thus

$$\sup_{t\in\mathbb{R}}|\psi(it)| \leq \|f\|\|x\|_{p_{\theta'}}.$$

Similarly

$$\sup_{t\in\mathbb{R}}|\psi(1+it)| \leq \|f\|\|x\|_{p_{\theta'}}.$$

By the Three-Lines Lemma, we find that

$$|\tau(f(\theta)x)| \leq \|f\|\|x\|_{p_{\theta'}}.$$

Thus if $y \in (L^{p_0}(M,\tau), L^{p_1}(M,\tau))_\theta$ then

$$|\tau(yx)| \leq \|y\|_\theta\|x\|_{p_{\theta'}}.$$

Now let $y = v|y|$ be the polar decomposition of y, and let

$$|y| = \int_{[0,\infty)} t\, dE(t)$$

be the spectral decomposition of y. For $\varepsilon > 0$, set $x_\varepsilon = \chi_{(\varepsilon,1/\varepsilon)}|y|^{p_\theta - 1}v^*$. Then,

$$x_\varepsilon^* x_\varepsilon \geq \varepsilon^{p_\theta - 1} v \chi_{(\varepsilon,1/\varepsilon)}(|x|)v^* = \varepsilon^{p_\theta - 1}\chi_{(0,\infty)}(|x_\varepsilon|).$$

Hence, by what we just saw

$$\begin{aligned}\int_{(\varepsilon,1/\varepsilon)} t^p d\tau \circ E(t) &= \tau(x_\varepsilon y) \\ &\leq \|y\|_\theta \|x_\varepsilon\|_{p_{\theta'}} \\ &\leq \|y\|_\theta \|\chi_{(\varepsilon,1/\varepsilon)}(|y|)|y|^{p_\theta - 1}\|_{p_{\theta'}} \\ &= \|y\|_\theta \left(\int_{(\varepsilon,1/\varepsilon)} t^p d\tau \circ E(t)\right)^{1/p_{\theta'}}\end{aligned}$$

Thus

$$\left(\int_{(\varepsilon,1/\varepsilon)} t^p d\tau \circ E(t)\right)^{1/p} \leq \|y\|_\theta,$$

letting $\varepsilon \to 0$ and applying the Monotone Convergence Theorem we see that

$$\|y\|_{p_\theta} \leq \|y\|_{p_\theta}.$$

For the reverse inequality, let $y \in L^{p_\theta}(M, \tau)$ and let $y = u|y|$ be the polar decomposition of y. Define $f\colon \overline{\Omega} \to \mathrm{Meas}(M)$.

$$f(z) = u|y|^{\frac{p_\theta}{p_0} + z\left(\frac{p_\theta}{p_1} - \frac{p_\theta}{p_0}\right)} \|y\|_{p_\theta}^{1 - \frac{p_\theta}{p_0} + z\left(\frac{p_\theta}{p_1} - \frac{p_\theta}{p_0}\right)}.$$

Note that $f(\theta) = y$. Using again $\|ab\|_{p_\theta} \leq \|a\|_\infty \|b\|_{p_\theta}$ for $a \in M, b \in L^{p_\theta}(M, \tau)$ and that $|y|^{is}$ is unitary for $s \in \mathbb{R}$,

$$\sup_t \|f(it)\|_{p_0} \leq \|y\|_{p_\theta}^{1 - \frac{p_\theta}{p_0}} \||y|^{\frac{p_\theta}{p_0}}\|_{p_0} = \|y\|_{p_\theta}.$$

Similarly

$$\sup_t \|f(1 + it)\|_{p_1} \leq \|y\|_{p_\theta}.$$

We claim that f has image inside $L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$, that f is a continuous as a map $\overline{\Omega} \to L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$ and that f is holomorphic as a map $\Omega \to L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$. The preceding inequalities then show that

$$\|f\|_{\mathcal{A}(L^{p_0}(M,\tau),L^{p_1}(M,\tau))} \leq \|y\|_{p_\theta},$$

and this will complete the proof.

As $\|u\|_\infty \leq 1$, the claim that f maps into $L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$ reduces to the statement that

$$|y|^{\frac{p_\theta}{p_0}+z\left(\frac{p_\theta}{p_1}-\frac{p_\theta}{p_0}\right)}$$

is in $L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$ which is true by functional calculus. The continuity claim reduces to the fact that

$$z \mapsto |y|^{\frac{p_\theta}{p_0}+z\left(\frac{p_\theta}{p_1}-\frac{p_\theta}{p_0}\right)}$$

is continuous for the $L^{p_0}(M,\tau)+L^{p_1}(M,\tau)$ norm which is true by functional calculus and the commutative case.

Similarly, the holomorphicity claim reduces to the statement that

$$z \mapsto |y|^{\frac{p_\theta}{p_0}+z\left(\frac{p_\theta}{p_1}-\frac{p_\theta}{p_0}\right)}$$

is holomorphic, and for this we may assume that M is abelian and represented on a separable Hilbert space. Since $p_0 \neq p_1$, we know that $(L^{p_0}(X,\mu)+L^{p_1}(X,\mu))^*$ is identified with $L^{p_0'}(X,\mu) \cap L^{p_1'}(X,\mu)$, from this observation it is not hard to argue weak holomorphicity of the above map. This completes the proof.

$\square$

APPENDIX B

Amenable Groups and Equivalence Relations

B.1 Amenable Groups

The concept of an amenable group is probably the most ubiquitous notion in the study of harmonic analysis on discrete groups. An amenable groups is roughly one over which you can average, as we will see shortly there are many equivalent ways of phrasing this (there are even more than the ones we will list below). Each of these different ways lead to many generalizations: weaker approximation properties for groups, amenability properties for graphs, equivalence relations, Banach algebras, subfactors, each of which have proved to be useful in their respective fields. Moreover, amenability of groups has seen tremendous applications in the study of operator algebras, ergodic theory, L^2-invariants, and other related fields. It turns out that amenable groups are related to the Banach-Tarski paradox, as a crucial step in the proof of the Banach-Tarski paradox is that $\mathbb{F}_2$ is not amenable (here $\mathbb{F}_n$ is the free group on n letters).

Definition B.1.1. Let Γ be a countable discrete group. A *Følner sequence for* Γ is a sequence F_n of finite nonempty subsets of Γ. So that

$$\frac{|gF_n \Delta F_n|}{|F_n|} \to 0,$$

for all $g \in \Gamma$. We say that Γ is *amenable* if it has a Følner sequence.

A Følner sequence gives a way of averaging functions over a group: given a $f \in \ell^\infty(\Gamma)$, we can consider the sequence of averages

$$\frac{1}{|F_n|} \sum_{g \in F_n} f(g)$$

and these will be approximately invariant. This is related to (v) of the next theorem. It also evokes the usual properties of the intervals $\{-n, \ldots, n\}$ inside $\mathbb{Z}$, and averages over such intervals are already useful in classical harmonic analysis. Often, properties of the integers have generalizations to amenable groups. In the next section, a precise and deep relation between arbitrary amenable groups and $\mathbb{Z}$ will be discussed.

We collect many equivalent definitions of amenable in the next theorem.

Theorem B.1.2. *Let Γ be a countable discrete group, then the following are equivalent.*

(i): Γ is amenable.

(ii): For all $1 \leq p < \infty$, there is a sequence $f_n \in \ell^p(\Gamma)$ so that

$$\|\lambda(g)f_n - f_n\|_p \to 0.$$

(iii): For some $1 \leq p < \infty$, there is a sequence $f_n \in \ell^p(\Gamma)$ so that $\|f_n\|_p = 1$, and

$$\|\lambda(g)f_n - f_n\|_p \to 0.$$

(iv): There is a $\phi \in \ell^\infty(\Gamma)^*$ so that $\phi(\lambda(g)f) = \phi(f)$ for all $f \in \ell^\infty(\Gamma)$.

(v): Every affine action of Γ on a nonempty compact convex set in a locally convex space by homeomorphisms has a fixed point.

(vi): Every action of Γ on a compact metrizable space has an invariant measure.

Proof. (vi) implies (v): Let X be a locally convex space, and $K \subseteq X$ a compact convex set.

We first prove a two preliminary claims.

Claim 1: Every action of Γ on a compact *Hausdorff* space has an invariant measure.

For this, let X be a compact Hausdorff space. For every finite subset F of $C(X)$ containing the identity, let A_F be the C^*-subalgebra of $C(X)$ generated by $\{gf : g \in \Gamma, f \in F\}$. By Gelfand Theory, $A_F \cong C(Y_F)$, and the action of Γ by automorphisms on A_F gives rise to an action by homeomorphisms on Y_F. As A_F is separable, we know that Y_F is metrizable. By hypothesis, we can find a positive linear functional

$$\phi_F \colon A_F \to \mathbb{C}$$

of norm 1 with $\phi(1) = 1$, and which is invariant under the action of Γ.

Extend ϕ_F to a linear functional $\psi_F \colon C(X) \to \mathbb{C}$ by Hahn-Banach with $\|\psi_F\| = 1$. By compactness, we can find a weak* cluster point ψ of ψ_F. Then ψ is a positive linear functional, invariant under the action of Γ. Again, by Gelfand duality we know that ψ corresponds to a Γ-invariant probability measure on X.

Claim 2: For every $\mu \in \operatorname{Prob}(K)$, there is a unique $x \in K$, so that

$$\int_K \phi(y)\, d\mu(y) = \phi(x)$$

for all $\phi \in X^*$. We shall abbreviate the above statement as

$$\int_K y\, d\mu(y) = x.$$

Since X is a locally convex space, uniqueness of x is obvious from the Hahn-Banach theorem. To prove existence, first suppose that

$$\mu = \sum_{j=1}^{n} \lambda_j \delta_{y_j},$$

with $\lambda_j \geq 0, \sum \lambda_j = 1$, and $y_1, \ldots, y_n \in K$. In this case, we have

$$x = \sum_{j=1}^{n} \lambda_j y_j.$$

In general, let μ_α be a net of atomic probability measures on K with $\mu_\alpha \to \mu$ weak*. By what we just saw, there is a $x_\alpha \in K$ so that

$$\int_K y\, d\mu_\alpha(y) = x_\alpha.$$

By compactness, we may assume that there is an $x \in K$ so that $x_\alpha \to x$. Then for all $\phi \in X^*$

$$\phi(x) = \lim_\alpha \phi(x_\alpha) = \lim_\alpha \int_K \phi(y)\, d\mu_\alpha(y) = \int_K \phi(y)\, d\mu(y),$$

this proves Claim 2.

By Claim 1, we can find a Γ-invariant measure μ on K. By uniqueness, and the fact that the action is affine, it is hard to see that

$$\int_K y\, d\mu(y)$$

is a fixed point in K.

(v) implies (iv): Given $l^\infty(\Gamma)^*$ the weak* topology. Let Γ act on $l^\infty(\Gamma)^*$ by

$$(g\phi)(f) = \phi(\lambda(g)^{-1}f).$$

Then

$$K = \{\phi \in l^\infty(\Gamma)^* : \phi(f) \geq 0 \text{ for all } f \in l^\infty(\Gamma), \phi(1) = 1\}$$

is invariant under the action of Γ. Any fixed point under this action gives an element as in (*iv*).

(iv) implies (iii): We take $p = 1$. We identify Prob(Γ) as a subset of $\ell^1(\Gamma)$. Let ϕ be as in (*iv*). We first prove the following claim.

Claim: View $\ell^1(\Gamma) \subseteq \ell^\infty(\Gamma)^*$, we have

$$\phi \in \overline{\mathrm{Prob}(\Gamma)}^{\mathrm{weak}^*}.$$

If the claim is false, then by geometric Hahn-Banach we can find a weak* continuous linear functional

$$F : \ell^\infty(\Gamma)^* \to \mathbb{C}$$

and real numbers $\alpha < \beta$ so that

$$\mathrm{Re}(F(\phi)) < \alpha < \beta < \mathrm{Re}(F(\mu))$$

for all $\mu \in \mathrm{Prob}(\Gamma)$. It is a standard functional analysis exercise that $F(\phi) = \phi(f)$ for a unique $f \in l^\infty(\Gamma)$. Replacing f with $\mathrm{Re}(f)$, we may assume that f is real. Write $f = f^+ - f^-$, where $f^+f^- = 0$, and $f^+, f^- \geq 0$. Taking the infimum over all $\mu \in \mathrm{Prob}(\Gamma)$, we find that

$$\alpha < \beta \leq -\|f^-\|_\infty.$$

However, as $f \geq -\|f^-\|_\infty$, we have

$$\alpha > \mathrm{Re}(\phi(f)) \geq -\|f^-\|_\infty,$$

which is a contradiction.

Let F be a finite subset of Γ, and $\varepsilon > 0$. It suffices to show that there is a $f \in \mathrm{Prob}(\Gamma)$ so that

$$\max_{g \in F} \|\lambda(g)f - f\|_1 < \varepsilon.$$

Let

$$K = \bigoplus_{g \in F} \{\lambda(g)f - f : g \in F, f \in \mathrm{Prob}(\Gamma)\}.$$

It suffices to show that $0 \in \overline{K}^{\|\cdot\|_1}$. By convexity, it suffices to show that $0 \in \overline{K}^{\mathrm{weak}}$. By the claim, we find a net $f_\alpha \in \mathrm{Prob}(\Gamma)$ so that

$$f_\alpha \to \phi$$

weak*. Thus, for all $k \in \ell^\infty(\Gamma)$, for all $g \in F$,

$$\langle \lambda(g)f_\alpha - f_\alpha, k\rangle = \langle f_\alpha, \lambda(g)^{-1}k - k\rangle \to \phi(\lambda(g)^{-1}k - k) = 0.$$

Thus $0 \in \overline{K}^{\mathrm{weak}}$, and we are done.

(iii) implies (ii): Let f_n be as in (*iii*). By the triangle inequality,

$$\|\lambda(g)|f_n| - |f_n|\|_p \leq \|\lambda(g)f_n - f_n\|_p,$$

for all $g \in \Gamma$. So we may assume that $f_n \geq 0$. Let $1 \leq q < \infty$, and set $k_n = f_n^{p/q}$, we will show that

$$\|\lambda(g)k_n - k_n\|_q \to 0$$

for all $g \in \Gamma$. Clearly we may assume $p \neq q$.

Let us first handle the case that $p > q$. For $a, b \in [0, \infty)$ we have by elementary calculus:

$$|a^{p/q} - b^{p/q}| \leq \frac{p}{q}\max(|a|^{\frac{p}{q}-1}, |b|^{\frac{p}{q}-1})|a-b| \leq \frac{p}{q}|a|^{\frac{p}{q}-1}|a-b| + \frac{p}{q}|b|^{\frac{p}{q}-1}|a-b|.$$

Thus,

$$\|\lambda(g)k_n - k_n\|_q^q \leq \frac{p}{q}\sum_{g\in\Gamma}\left(|f_n|^{\frac{p}{q}-1}|f_n(x) - f_n(g^{-1}x)| + |f_n(g^{-1}x)|^{\frac{p}{q}-1}|f_n(x) - f_n(g^{-1}x)|\right)^q$$
$$\leq 2^q\left(\frac{p}{q}\right)^q\sum_{g\in\Gamma}\left(|f_n|^{p-q}|f_n(x) - f_n(g^{-1}x)|^q + |f_n(g^{-1}x)|^{p-q}|f_n(x) - f_n(g^{-1}x)|^q\right).$$

Where in the last line we use the inequality $(a+b)^q \leq 2^q(a^q + b^q)$, for $a, b \in [0,\infty)$.

Since $p > q$, we may apply Hölder's inequality to see that

$$\|\lambda(g)k_n - k_n\|_q^q \leq 2^{q+1}\left(\frac{p}{q}\right)\|f_n - \lambda(g)f_n\|_p^q,$$

as

$$\|f_n\|_p = 1.$$

Thus

$$\|\lambda(g)k_n - k_n\|_q \to 0$$

for all $g \in \Gamma$.

Now we handle the case $p < q$. We leave it as an exercise to the reader to verify that

$$|a^{p/q} - b^{p/q}| \leq |a-b|^{p/q}$$

for $a, b \in [0,\infty)$. Thus for all $g \in \Gamma$,

$$\|\lambda(g)k_n - k_n\|_q^q \leq \|\lambda(g)f_n - f_n\|_p^p \to 0.$$

(ii) implies (i): We take $p = 1$. Let f_n be as the statement of (ii). As in the proof of (iii) implies (ii), we may assume that $f_n \geq 0$. Let $\varepsilon > 0$, and $K \subseteq \Gamma$ be finite. It is enough to find a $F \subseteq \Gamma$ finite so that

$$\max_{g\in K}\frac{|gF\Delta F|}{|F|} < \varepsilon.$$

Take n sufficiently large so that

$$\sum_{g\in K}\|\lambda(g)f_n - f_n\|_1 < \varepsilon.$$

By Tonelli, for each n we have

$$\|\lambda(g)f_n - f_n\|_1 = \int_0^1 |\{f_n \geq t\}\Delta g\{f_n \geq t\}|\,dt.$$

Thus

$$\int_0^1 \sum_{g\in K} |\{f_n \geq t\}\Delta g\{f_n \geq t\}|\,dt < \varepsilon = \varepsilon \int_0^1 |\{f_n \geq t\}|\,dt.$$

Hence we can find a $t > 0$ with

$$\sum_{g\in K} |\{f_n \geq t\}\Delta g\{f_n \geq t\}| < \varepsilon|\{f_n \geq t\}|,$$

and so we may take $F = \{f_n \geq t\}$.

(i) implies (vi): Let X be a compact metrizable space and $\Gamma \curvearrowright X$ by homeomorphisms. Let $x_0 \in X$, and let F_n be a Følner sequence for Γ. Set

$$\mu_n = \frac{1}{|F_n|}\sum_{g\in F_n} \delta_{gx_0}.$$

Then,

$$\|g_*\mu_n - \mu_n\| \leq \frac{|gF_n\Delta F_n|}{|F_n|}$$

hence any weak* limit point of μ_n is a Γ-invariant measure on X. □

A ϕ as in (iv), is called an *invariant mean* for Γ. Let us use the above theorem to prove that $\mathbb{F}_2$ is not amenable. Suppose $\mathbb{F}_2$ is amenable, and let ϕ be an invariant mean for $\mathbb{F}_2$. Let a, b be free generators for $\mathbb{F}_2$. Let A^+, (respectively A^-) be the set of all words in $\mathbb{F}_2$ beginning with a (respectively a^{-1}), and similarly define B^+, B^- in terms of b, b^{-1}. Let $S = \{1, b, b^2, \dots\}$. Then,

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus S) \sqcup (B^- \cup S) = A^+ \sqcup aA^- = b^{-1}(B^+ \setminus S) \sqcup (B^- \cup S).$$

Thus,

$$\begin{aligned}
1 = \phi(1) &= \phi(\chi_{A^+}) + \phi(\chi_{A^-}) + \phi(\chi_{B^+\setminus S}) + \phi(\chi_{B^-\sqcup S}) \\
&= \phi(\chi_{A^+}) + \phi(\lambda(a)\chi_{A^-}) + \phi(\lambda(b)^{-1}\chi_{B^+\setminus S}) + \phi(\chi_{B^-\sqcup S}) \\
&= \phi(\chi_{A^+} + (\lambda(a)\chi_{A^-} + \lambda(b)^{-1}\chi_{B^+\setminus S} + \chi_{B^-\sqcup S}) \\
&= \phi(2) \\
&= 2,
\end{aligned}$$

a contradiction. By (i) of the next proposition, and the fact that $\mathbb{F}_n$ embeds into $\mathbb{F}_2$ for all $n \geq 2$, we see that $\mathbb{F}_n$ is non-amenable for all n.

We now prove various permanence properties of amenable groups.

Proposition B.1.3. *(i): Every subgroup of amenable group is amenable.*

(ii): Let Γ be a countable discrete group, and let Γ_n are an increasing sequence of subgroups with

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n.$$

Then Γ is amenable if and only if Γ_n is amenable for all n.

(iii): Let

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 1,$$

be a short exact sequence of countable discrete groups. Then Γ is amenable if and only if Λ and Δ are amenable.

Proof. (i): Let $\varepsilon > 0$, and $K \subseteq \Lambda$ finite. As a representation of Λ we have

$$\ell^2(\Gamma) \cong \bigoplus_{\Gamma/\Lambda}^{(2)} \ell^2(\Lambda),$$

where the superscript indicates we are taking the ℓ^2-direct sum. By (ii) in the preceding theorem, we may find Let $f \in \ell^2(\Gamma)$ with $\|f\|_2 = 1$ such that

$$\sum_{g \in K} \|\lambda(g)f - f\|_2^2 < \varepsilon.$$

Let $f_c, c \in \Gamma/\Lambda$ be such that

$$f = \bigoplus_{c \in \Gamma/\Lambda} f_c,$$

under the decomposition

$$\ell^2(\Gamma) \cong \bigoplus_{\Gamma/\Lambda}^{(2)} \ell^2(\Lambda).$$

Then

$$\sum_{g \in K} \sum_{c \in \Gamma/\Lambda} \|\lambda(g)f_c - f_c\|_2^2 = \sum_{g \in K} \|\lambda(g)f - f\|_2^2 < \varepsilon = \varepsilon \sum_{c \in \Gamma/\Lambda} \|f_c\|_2^2.$$

Thus there is some $c \in \Gamma/\Lambda$ with$\|f_c\|_2 \neq 0$, and

$$\sum_{g \in K} \|\lambda(g)f_c - f_c\|_2^2 < \varepsilon \|f_c\|_2^2.$$

Hence if we set $k = \frac{f_c}{\|f_c\|_2}$, then $\|k\|_2 = 1$, and

$$\max_{g \in K} \|\lambda(g)k - k\|_2 < \sqrt{\varepsilon}.$$

This proves (ii) for Λ.

(ii): By (i), we have that if Γ is amenable, then so is each Γ_n. Suppose Γ_n is amenable for all n. Let $K \subseteq \Gamma$ be finite and $\varepsilon > 0$. For n large enough, we have $K \subseteq \Gamma_n$. Thus there is a $F \subseteq \Gamma_n$ finite so that

$$\max_{g \in K} \frac{|gF \Delta F|}{|F|} < \varepsilon.$$

Now apply a diagonal argument to argue that Γ has a Følner sequence.

(iii): Without loss of generality, $\Lambda \triangleleft \Gamma$ and $\Delta = \Gamma/\Lambda$. By (i), we know that Λ is amenable. To see that Δ is amenable, let K be a compact convex set in a locally convex space, and $\Delta \curvearrowright K$ by affine homeomorphisms. The we have a Γ action on K by

$$gx = (g\Lambda)x, \ g \in \Gamma, x \in K.$$

By (v) in the preceding theorem, we know that Γ has a fixed point under this action. Any fixed point for Γ is one for Δ, so Δ is amenable.

Now suppose that Λ and Δ are amenable. Let K be a compact convex set in a locally convex space, and let $\Gamma \curvearrowright K$ by affine homeomorphisms. By amenability of Λ,

$$K' = \{x \in K : \lambda x = x \text{ for all } \lambda \in \Lambda\},$$

is nonempty. Since Γ acts by affine homeomorphisms we know that K' is a compact convex set. Define an action of Δ on K' by

$$(g\Lambda)x = gx$$

for $g \in \Gamma, x \in K'$. By normality of Λ, and the definition of K' we see that this is a well-defined action of Δ by affine homeomorphisms. By amenability of Δ, there is a fixed point $x \in K'$ for the action of Δ. It is easy to see that x is a fixed point under the action of Γ, hence Γ is amenable.

□

Let us now give some examples of amenable groups. First, every finite group is amenable. This follows because

$$\phi(f) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(g),$$

is easily seen to be an invariant mean for a finite group Γ. The integers are amenable. Indeed, $\{-n, \ldots, n\}$ is easily seen to be a Følner sequence for $\mathbb{Z}$. By repeated applications of *(iii)*, and the fundamental theorem of finitely generated abelian groups it follows that all finitely generated abelian groups are amenable. By *(ii)*, it follows that every abelian group is amenable. If we let $\mathcal{C}$ be the smallest class of countable discrete groups containing all abelian and finite groups, and which is closed under taking subgroups, extensions and direct unions, it follows that every group in $\mathcal{C}$ is amenable. The class $\mathcal{C}$ is called the class of *elementary amenable* groups. In particular, every locally solvable group (i.e. every finitely generated subgroup is solvable) is amenable.

B.2 Amenable Equivalence Relations

Following our philosophy that properties of a group should translate into properties of their induced equivalence relations, we will discuss the concept of an amenable equivalence relation. We list below as a theorem the following equivalent definitions of an amenable equivalence relation. As it would take us too far afield, we will not prove these equivalences, instead referring the reader to [23] Theorem 4.10, and [17] Chapter II where the below conditions are taken from.

Theorem B.2.1. *Let X be a standard probability space, and μ a Borel probability measure on X, and $\mathcal{R}$ a discrete measure-preserving equivalence relation over (X,μ). The following are equivalent.*

(i): For every Borel $B \subseteq X$ such that

$$x \mapsto |\{(y \in X : (x,y) \in B\}|,$$

$$y \mapsto |\{x \in X : (x,y) \in B\}|,$$

are in $L^{\infty}(X,\mu)$, and for every $\varepsilon > 0$, there is a Borel $A \subseteq X$ so that

$$\frac{|\{x \in \mathcal{O}_z \cap A : (x,y) \in B, y \text{ for some } y \in X \setminus A\}|}{|\mathcal{O}_z \cap A}| < \varepsilon,$$

for almost every $z \in X$.

(ii): For all $1 \leq p < \infty$, there is a sequence $f_n \in L^p(R,\overline{\mu})$ so that $\|f_{n,x}\|_p = 1$, where $f_{n,x}\colon \mathcal{O}_x \to \mathbb{C}$ is defined by $f_{n,x}(y) = f_n(x,y)$, and

$$\|\phi f_n - \mathrm{Id}_{\mathrm{ran}(\phi)} f_n\|_p \to 0,$$

for all $\phi \in [[\mathcal{R}]]$.

(iii): For some $1 \leq p < \infty$, there is a sequence $f_n \in L^p(R,\overline{\mu})$ so that $\|f_{n,x}\|_p = 1$, where $f_{n,x}\colon \mathcal{O}_x \to \mathbb{C}$ is defined by $f_{n,x}(y) = f_n(x,y)$, and

$$\|\phi f_n - \mathrm{Id}_{\mathrm{ran}(\phi)} f_n\|_p \to 0,$$

for all $\phi \in [[\mathcal{R}]]$.

(iv): There is a positive linear map $P\colon L^\infty(R,\overline{\mu}) \to L^\infty(X,\mu)$ *so that*

$$P(gf) = gP(f), \text{ for all } f \in L^\infty(R,\overline{\mu}), g \in L^\infty(X,\mu)$$

$$P(\phi f) = \phi P(f), \text{ for all } \phi \in [[\mathcal{R}]].$$

(v): There is an increasing sequence $\mathcal{R}_n$ *of subequivalence relations of* $\mathcal{R}$, *so that* $\{y : (x,y) \in \mathcal{R}_n\}$ *is finite for almost every* $x \in X$, *and all* $n \in \mathbb{N}$, *and*

$$\overline{\mu}\left(\mathcal{R} \setminus \bigcup_{n=1}^{\infty} \mathcal{R}_n\right) = 0.$$

In the theorem, the conditions (i) through (iv) are the analogues of (i) through (iv) in Theorem B.1.2. Conditions (v) is not, and in fact is rather surprising. Condition (v) is analogous to saying that Γ is a union of finite groups. We will discuss later, that when $\mathcal{R}$ has infinite orbits, then there is a free measure-preserving $\mathbb{Z} \curvearrowright (X,\mu)$ so that $\mathcal{R} = \mathcal{R}_{\mathbb{Z}\curvearrowright(X,\mu)}$. This is analogous to saying that $\Gamma \cong \mathbb{Z}$! This is another instance where many properties of groups become simpler when we pass to equivalence relations.

We now turn to permanence properties of amenable equivalence relations.

Proposition B.2.2. *Let* X *be a standard Borel space, let* μ *be a Borel probability measure on* X, *and* $\mathcal{R}$ *a discrete measure-preserving equivalence relation on* (X,μ).

(i): If $\mathcal{R}$ *is amenable, then so is any Borel subequivalence relation.*

(ii): If $\mathcal{R}$ *is amenable, and* $A \subseteq X$ *is Borel, then* $\mathcal{R}_A$ *is amenable.*

(iii): Suppose that $(\phi_n)_{n=1}^\infty$ *is a sequence in* $[[\mathcal{R}]]$, *and* $1 \leq p < \infty$. *Suppose that* $f_n \in L^p(\mathcal{R},\overline{\mu})$, *and* $f_{n,x}\colon \mathcal{O}_x \to \mathbb{C}$ *is defined as in the preceding theorem, and*

$$\|\phi_j f_n - \mathrm{Id}_{\mathrm{ran}(\phi_j)} f_n\|_1 \to 0.$$

If

$$\overline{\mu}\left(\mathcal{R} \setminus \bigcup_{j=1}^{\infty}\{(x,\phi_j) : x \in \mathrm{dom}(\phi_j)\}\right) = 0,$$

then $\mathcal{R}$ is amenable.

(iv): If Γ is amenable group, and $\Gamma \curvearrowright (X,\mu)$ is a measure-preserving action so that $\mathcal{R} = \{(x,gx) : g \in \Gamma\}$, then $\mathcal{R}$ is amenable.

(v): If $\mathcal{R}_n$ are an increasing sequence of amenable subequivalence relations, then

$$\bigcup_{n=1}^{\infty} \mathcal{R}_n$$

is amenable.

(vi): If A_n are an increasing sequence of measurable subsets of X so that $\mathcal{R}_{A_n}$ is amenable and

$$\mu\left(X \setminus \bigcup_{n=1}^{\infty} A_n\right) = 0,$$

then $\mathcal{R}$ is amenable.

(vii): If $A \subseteq X$ is Borel, and $\mathcal{R}\big|_A$ is amenable, then so is $\mathcal{R}_{\mathcal{R}A}$.

Proof. (i): Let $\mathcal{S} \subseteq \mathcal{R}$ be a Borel subequivalence relation. Let $\mathcal{R}_n$ be as in (v). Set

$$\mathcal{S}_n = \mathcal{R}_n \cap \mathcal{S}.$$

Then $\mathcal{S}_n$ are subequivalence relations of $\mathcal{S}$, and

$$\{y \in X : (x,y) \in \mathcal{S}_n\}$$

is finite for almost every X. Choose a conull $X_0 \subseteq X$ so that $\{y \in X : (x,y) \in \mathcal{R}_n\}$ if finite for all $x \in X_0$, and

$$\{y : (x,y) \in \mathcal{R}\} = \bigcup_{n=1}^{\infty} \{y : (x,y) \in \mathcal{R}_n\}$$

for all $x \in X_0$. Then

$$\{y : (x,y) \in S\} = \bigcup_{n=1}^{\infty} \{y : (x,y) \in \mathcal{S}_n\}$$

for all $x \in X_0$. Thus $\mathcal{S}$ verifies (v) of the above theorem, and hence is amenable.

(ii): Let P be as in (iv) for the above theorem. For $f \in L^\infty(\mathcal{R}_A, \overline{\mu})$, and define $\widetilde{f} \in L^\infty(\mathcal{R}, \overline{\mu})$ by $\widetilde{f}(x,y) = \chi_A(x)\chi_A(y)f(x,y)$. Define $\widetilde{P} \colon L^\infty(\mathcal{R}_A, \overline{\mu}) \to L^\infty(A,\mu)$ by

$$\widetilde{P}(f) = P(\widetilde{f}),$$

then it is straightforward to verify (iv) of the above theorem for $\widetilde{P}$.

(iii): Passing to a subsequence, we may assume that

$$\|\phi_j f_{n,x} - \mathrm{Id}_{\mathrm{ran}(\phi)} f_{n,x}\|_p \to 0$$

for almost every $x \in X$ and all $j \in \mathbb{N}$. Since $(x,y) \mapsto (y,x)$ preserves $\overline{\mathcal{R}}$, it follows that

$$\overline{\mu}\left(\mathcal{R} \setminus \bigcup_{j=1}^{\infty}\{(x, \phi_j^{-1}(x)) : x \in \mathrm{ran}(\phi_j)\}\right) = 0.$$

Given $\phi \in [[\mathcal{R}]]$, let $m_\phi \colon \mathrm{ran}(\phi) \to \mathbb{N} \cup \{\infty\}$ be defined by

$$m_\phi(x) = \inf\{k : \phi^{-1}(x) = \phi_j^{-1}(x)\}.$$

Then,

$$\begin{aligned}\|\phi f_n - \mathrm{Id}_{\mathrm{ran}(\phi)} f_n\|_p^p &= \int_{\mathrm{ran}(\phi)} \sum_{y\sim x} |f_n(\phi^{-1}x, y) - f_n(x,y)|^p \, d\mu(x) \\ &= \int_{\mathrm{ran}(\phi)} \sum_{m=1}^{\infty} \chi_{\{x : m_\phi(x) = m\}} \sum_{y \sim x} |f_n(\phi_j^{-1}(x), y) - f_n(x,y)|^p \, d\mu(x).\end{aligned}$$

Now

$$\sum_{y\sim x} |f_n(\phi_j^{-1}(x), y) - f_n(x,y)|^p \to 0,$$

almost everywhere by assumption. And since $\|f_{n,x}\|_p = 1$, we find that

$$\sum_{y\sim x} |f_n(\phi_j^{-1}(x), y) - f_n(x,y)|^p \leq 2^p.$$

Thus, the dominated convergence theorem implies that

$$\|\phi f_n - \mathrm{Id}_{\mathrm{ran}(\phi)} f_n\|_p \to 0.$$

(iv): Let $f_n \in \ell^1(\Gamma)$ be such that

$$\|\lambda(g) f_n - f_n\|_1 \to 0$$

for all $g \in \Gamma$, as in the proof of Theorem B.1.2, we may assume $f_n \geq 0$. Define $\widetilde{f_n} \colon \mathcal{R} \to [0,\infty)$ by

$$\widetilde{f_n}(x,y) = \sum_{g \in \Gamma : gx = y} f_n(g^{-1}),$$

then $\|f_{n,x}\|_1 = 1$ for almost every $x \in X$. Let $\alpha_g \in [\mathcal{R}]$ be defined by $\alpha_g(x) = gx$. Then

$$\begin{aligned}\|\alpha_g \widetilde{f}_n - \widetilde{f}_n\|_1 &= \int_X \sum_{y \sim x} \left| \sum_{h \in \Gamma : hg^{-1}x = y} f_n(h^{-1}) - \sum_{h \in \Gamma : hx = y} f_n(h^{-1}) \right| d\mu(x) \\ &= \int_X \sum_{y \sim x} \left| \sum_{h \in \Gamma : hx = y} |f_n(g^{-1}h^{-1}) - f_n(h^{-1}) \right| d\mu(x) \\ &\leq \|\lambda(g) f_n - f_n\|_1 \to 0.\end{aligned}$$

Hence, by (iv) we know that Γ is amenable.

(v): For $\phi \in [[\mathcal{R}]]$, define $n_\phi \colon X \to \mathbb{N} \cup \{\infty\}$ by

$$n_\phi(x) = \inf\{k : (x, \phi(x)) \in \mathcal{R}_n\},$$

by assumption n_ϕ is finite almost everywhere. Let $\phi_1, \dots, \phi_k \in [[\mathcal{R}]]$, and $\varepsilon > 0$. Choose $N \in \mathbb{N}$, and $A \subseteq X$ so that $\mu(A) \geq 1 - \varepsilon$, and

$$n_{\phi_j}(x) \leq N$$

for al $x \in A$. For $1 \leq j \leq k$, define $\psi_j \in [[\mathcal{R}_N]]$, by $\psi_j = \phi_j \operatorname{Id}_A$. Let $f \in L^1(\mathcal{R}_N, \overline{\mu})$ be such that $\|f_x\|_1 = 1$ for almost every $x \in X$, and

$$\max_{1 \leq j \leq k} \|\psi_j f - f\|_1 < \varepsilon.$$

Define $\widetilde{f} \in L^1(\mathcal{R}, \overline{\mu})$ by declaring $\widetilde{f}(x, y) = 0$ for $(x, y) \in \mathcal{R} \setminus \mathcal{R}_N$. For any $1 \leq j \leq k$,

$$\begin{aligned}\|\phi_j \widetilde{f} - \operatorname{Id}_{\operatorname{ran}(\phi_j)} \widetilde{f}\| = \int_X |\phi_j \widetilde{f}_x - \operatorname{Id}_{\operatorname{ran}(\phi_j)} \widetilde{f}_x\|_1 \, d\mu(x) &\leq 2\varepsilon + \int_A \|\phi_j f_x - \operatorname{Id}_{\operatorname{ran}(\phi_j)} f_x\|_1 \\ &= 2\varepsilon + \|\psi_j f - \operatorname{Id}_{\operatorname{ran}(\psi_j)} f\|_1 \\ &< 3\varepsilon.\end{aligned}$$

By a diagonal argument and (iv), we see that $\mathcal{R}$ is amenable.

(vi): This is proved in a similar manner to (v) as above.

(vii): By (v), it is enough to show that if $\phi \in [[\mathcal{R}]]$, $\operatorname{dom}(\phi) \subseteq A$, and $\operatorname{ran}(\phi) \cap A = \varnothing$, then $\mathcal{R}_{A \cup \phi(A)}$ is amenable. Let us first setup notation. For $f \in L^\infty(\mathcal{R}, \overline{\mu})$, define $(f_{ij})_{1 \leq i,j \leq 2}$

in $L^\infty(\mathcal{R}_A, \overline{\mu})$ by

$$f_{11} = \mathrm{Id}_A\, f\, \mathrm{Id}_A,$$

$$f_{12} = \mathrm{Id}_A\, f \phi^{-1},$$

$$f_{21} = \phi f\, \mathrm{Id}_A,$$

$$f_{22} = \phi f \phi^{-1}.$$

For $\psi \in [[\mathcal{R}_{A\cup\phi(A)}]]$ define $(\psi_{ij})_{1\leq i,j\leq 2}$ in $[[\mathcal{R}_A]]$ by

$$\psi_{11} = \mathrm{Id}_A\, \psi\, \mathrm{Id}_A,$$

$$\psi_{12} = \mathrm{Id}_A\, \psi \phi^{-1},$$

$$\psi_{21} = \phi \psi\, \mathrm{Id}_A,$$

$$\psi_{22} = \phi \psi \phi^{-1}.$$

Then for $1 \leq i, j \leq 2$

$$(\psi f)_{ij} = \sum_{l=1}^{2} \psi_{il} f_{lj}.$$

For $k \in L^\infty(X, \mu)$, set

$$k_1 = \mathrm{Id}_A\, k,$$

$$k_2 = \phi k.$$

Then, for $i = 1, 2$, and ψ, k as above:

$$(\psi k)_i = \sum_{l=1}^{2} \psi_{il} k_l.$$

Let P be as in (iv) of the preceding theorem for $L^\infty(\mathcal{R}_A, \overline{\mu})$. Define $\widetilde{P}\colon L^\infty(\mathcal{R}_{A\cup\phi(A)}, \overline{\mu}) \to L^\infty(X, \mu)$ by

$$\widetilde{P}(f)_i = \sum_{l=1}^{2} P(f_{il}),$$

for $i = 1, 2$. For $\psi \in [[\mathcal{R}_{A\cup\phi(A)}]]$, and $f \in L^\infty(\mathcal{R}_{A\cup\phi(A)}, \overline{\mu}), i = 1, 2$ we have

$$\begin{aligned} \widetilde{P}(\psi f)_i = \sum_{l=1}^{2}\sum_{j=1}^{2} \psi_{ij} P(f_{jl}) &= \sum_{j=1}^{2} \psi_{ij} \left(\sum_{l=1}^{2} P(f_{jl})\right) \\ &= \sum_{j=1}^{2} \psi_{ij} \widetilde{P}(f)_j \\ &= (\psi \widetilde{P}(f))_i. \end{aligned}$$

It is even easier to show that $\widetilde{P}(gf) = g\widetilde{P}(f)$, for $g \in L^\infty(A\cup\phi(A), \mu), f \in L^\infty(\mathcal{R}_{A\cup\phi(A)}, \overline{\mu})$. This proves (vii).

□

Let us note what amenability means in the context of free actions of groups.

Proposition B.2.3. *Let (X, μ) be a standard probability space, and $\Gamma \curvearrowright (X, \mu)$ a free measure-preserving action. Then, $R_{\Gamma\curvearrowright(X,\mu)}$ is amenable if and only if Γ is amenable.*

Proof. By the preceding proposition, we know that if Γ is amenable, then $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$ is amenable.

Now suppose that $R_{\Gamma\curvearrowright(X,\mu)}$ is amenable. Let f_n be as in (ii) In the above theorem for $p = 1$. As

$$\|\phi f_n - \mathrm{Id}_{\mathrm{ran}(\phi)} f_n\|_1 \leq \|\phi|f_n| - \mathrm{Id}_{\mathrm{ran}(\phi)} |f_n|\|_1,$$

we may assume that $f_n \geq 0$. Let $\widetilde{f_n} \colon \Gamma \to \mathbb{C}$ be defined by

$$\widetilde{f_n}(g) = \int_X f_n(gx, x)\, d\mu(x).$$

As $f_n \geq 0$,

$$\sum_{g\in\Gamma} \widetilde{f_n}(g) = \sum_{g\in\Gamma} \int_X f_n(gx, x)\, d\mu(x) = \sum_{g\in\Gamma} \int_X f_n(x, g^{-1}x)\, d\mu(x),$$

where in the last equality we use that the action is measure-preserving. As the action is free,

$$\sum_{g\in\Gamma} \int_X f_n(x, g^{-1}x)\, d\mu(x) = \int_X \sum_{g\in\Gamma} f_n(x, g^{-1}x)\, d\mu(x) = \int_{\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}} f_n(x, y)\, d\overline{\mu}(x, y) = 1,$$

since $f_n \geq 0$ and $\|f_n\|_1 = 1$.

For all $g \in \Gamma$,

$$\begin{aligned}\|\lambda(g)\widetilde{f}_n - \widetilde{f}_n\|_1 &= \sum_{h\in\Gamma} \left| \int_X f_n(g^{-1}hx, x)\, d\mu(x) - \int_X f_n(hx, x)\, d\mu(x) \right| \\ &= \sum_{h\in\Gamma} \left| \int_X f_n(g^{-1}x, h^{-1}x) - \int_X f_n(x, h^{-1}x)\, d\mu(x) \right|,\end{aligned}$$

where in the last equality we use that the action is measure-preserving. Thus,

$$\|\lambda(g)\widetilde{f}_n - \widetilde{f}_n\|_1 \leq \int_X \sum_{h\in\Gamma} |f_n(g^{-1}x, h^{-1}x) - f_n(x, h^{-1}x)|\, d\mu(x) = \|gf_n - f_n\|_1,$$

where in the last line we use the action is free. Thus,

$$\|\lambda(g)\widetilde{f}_n - \widetilde{f}_n\|_1 \to 0,$$

and so Γ verifies Theorem B.1.2 (iii) for $p = 1$.

□

The following is a fundamental and rather surprising theorem about amenable equivalence relations see [17] for the proof.

Theorem B.2.4 (Dye, Connes-Feldman-Weiss)**.** *Let X and Y be standard Borel spaces, and let μ, ν be Borel probability measures on X, Y respectively. Let $\mathcal{R}, \mathcal{S}$, be discrete, measure-preserving, ergodic equivalence relations over $(X, \mu), (Y, \nu)$ respectively. Suppose that $\mathcal{O}_x, \mathcal{O}_y$ are infinite for almost every $x \in X, y \in Y$. Then, $\mathcal{R}$ and $\mathcal{S}$ are isomorphic.*

Thus, from the point of view of equivalence relations there is only one amenable equivalence relation with infinite orbits. This implies that, from the point of view of equivalence relations, there is only one infinite amenable group. We present this formally below.

Corollary B.2.5. *Let X be a standard probability space and μ a Borel probability measure on X. Let Γ be any infinite amenable group. If $\mathcal{R}$ is a ergodic, discrete, measure-preserving equivalence relation over (X, μ) and $\mathcal{O}_x$ is infinite for almost every x, then there is a free* *measure-preserving action $\Gamma \curvearrowright (X,\mu)$ so that $\mathcal{R} = \mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$. In particular, if Λ is any other infinite amenable group, and $\Lambda \curvearrowright (X,\mu)$ is a free probability measure-preserving ergodic action, then there is an action $\Gamma \curvearrowright (X,\mu)$ so that $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)} = \mathcal{R}_{\Lambda\curvearrowright(X,\mu)}$.*

Proof. The in particular part follows from the preceding proposition. Fix some standard probability space (Y,ν) and a ergodic action $\Gamma \curvearrowright (Y,\nu)$ (for example we can consider a nontrivial Bernoulli action). By the preceding Theorem, there is a bimeasurable bijection

$$\Phi\colon Y \to X$$

so that $\Phi_*\nu = \mu$, and

$$\{\Phi(gy) : g \in \Gamma\} = \{x \in X : (x, \Phi(y)) \in \mathcal{R}\},$$

for almost every $y \in Y$. We may define an action of Γ on X by

$$gx = \Phi(g\Phi^{-1}(x)).$$

For this action $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)} = \mathcal{R}$.

$\square$

We remark that we may actually remove the ergodicity assumption in the above corollary by applying the ergodic decomposition and Borel selection. We leave this as an exercise to the reader. We close this section with a theorem showing that a hyperfinite equivalence relation is roughly the "smallest" equivalence relation.

Theorem B.2.6. *[Jackson-Kechris-Louveau,Lemma 23.2 in [17]] Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving equivalence relation such that $\mathcal{O}_x$ is infinite for almost every $x \in X$. Then, there is a amenable subequivalence relation $\mathcal{S}$ of $\mathcal{R}$ with infinite orbits almost everywhere.*

References

[1] M.F. Atiyah. Elliptic operators, discrete groups and von neumann algebras,. In *Colloque "Analyse et Topologie" en l'Honner de Henri Cartan*, pages 43–72, Paris, 1976. Soc. Math. France.

[2] L. Bowen. Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc*, 23:217–245, 2010.

[3] N. Brown and N. Ozawa. *C^*-Algebras and Finite-Dimensional Approximations*. Cambridge University Press, 1994.

[4] J. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer, New York, NY, 2 edition, 2000.

[5] R. Lyons D. Gaboriau. A measurable-group theoretic solution to von neumann's problem. *Invent. Math*, 177(3):533–540, 2009.

[6] K. Dykema, D. Kerr, and M. Pichot. Sofic dimension for discrete measurable groupoids. *To appear in Trans. Amer. Math. Soc.*

[7] G. Elek and G. Lippner. Sofic equivalence relations. *Journal of Functional Analysis*, 258:1692–1708, 2010.

[8] G. Elek and E. Szabo. Sofic representations of amenable groups. *Proceedings of the AMS*, 139(2011):4285–4291.

[9] G. Elek and E. Szabo. On sofic groups. *Journal of Group Theory*, 9(2):161–171, 2006.

[10] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology, and von neumann algebras. *Bull. Amer. Math. Soc.*, 81(5), September 1975.

[11] D. Gaboriau. Coû t des relations d'equivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.

[12] D. Gaboriau. Invariants ℓ^2 de relations d'équivalence et de groupes. *Publ. math. Inst. Hautes Etudes Sci.*, 95(1):93–150, 2002.

[13] A. Gournay. A dynamical approach to von neumann dimension. *Discrete and Continuous Dynamical Systems*, 26(3), 2010.

[14] M. Gromov. l_P-cohomology. In *Geometric Group Theory*, volume 2 of *Lecture Note Series*, pages 219–263, Cambridge, 1991. London Mathematical Society, Cambridge Univeristy Press.

[15] M. Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. i. *Math. Phys. Anal. Geom*, 2:323–415, 1999.

[16] B. Hayes. An l^p-version of von neumann dimension for banach space representations of sofic groups. *Journal of Functional Analysis*, 266(2):989–1040, 2014.

[17] A.S. Kechris and B.D. Miller. *Topics in Orbit Equivalence*. Lecture Notes in Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, 2000.

[18] D. Kerr and H. Li. Topological entropy and the variational principle for actions of sofic groups. *Invent. Math*, 186:501–558, 2011.

[19] G. Levitt. On the cost of generating an equivalence relation. *Ergodic Theory and Dynamical Systems*, 15(6):1173–1181, 1995.

[20] L. Paunescu. On sofic actions and equivalence relations. *Journal of Functional Analysis*, 9:2461–2485, 261.

[21] G. Pisier. *The Volume of Convex Bodies and Banach Space Geometry*. Cambridge University Press, 1989.

[22] S.M. Srivastava. *A Course on Borel Sets*. Graduate Texts In Mathematics. Springer, New York, 1991.

[23] M. Takesaki. *Theory of Operator Algebras III*, volume 127 of *Encyclopaedia of Mathematical Sciences*. Springer, 2001.

[24] M. Takesaki. *Theory of Operator Algebras II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer, New York, NY, 2003.

[25] N. Tomczak-Jaegermann. Banach-mazur distance and finite-dimensional operator ideals. Pitman Monographs, 38, 1989.

[26] V.S. Varadarajan. Groups of automorphisms of borel spaces. *Trans. Amer. Math. Soc*, (109):191–220, 1963.

[27] D. Voiculescu. Dynamical approximation entropies and topological entorpy in operator algebras. *Commun. Math. Phys.*, 170:249–281, 1996.