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SPACE CHARGE EXPANSION OF ION BUNCHES DRIFTING DOWN A CONDUCTING PIPE

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SPACE CHARGE EXPANSION OF ION BUNCHES DRIFTING DOWN A CONDUCTING PIPE

A. Garren

July 13, 1951

Berkeley, California

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SPACE CHARGE EXPANSION OF ION BUNCHES DRIFTING DOWN A CONDUCTING PIPE

A. Garren

Radiation Laboratory, Department of Physics
University of California, Berkeley, California

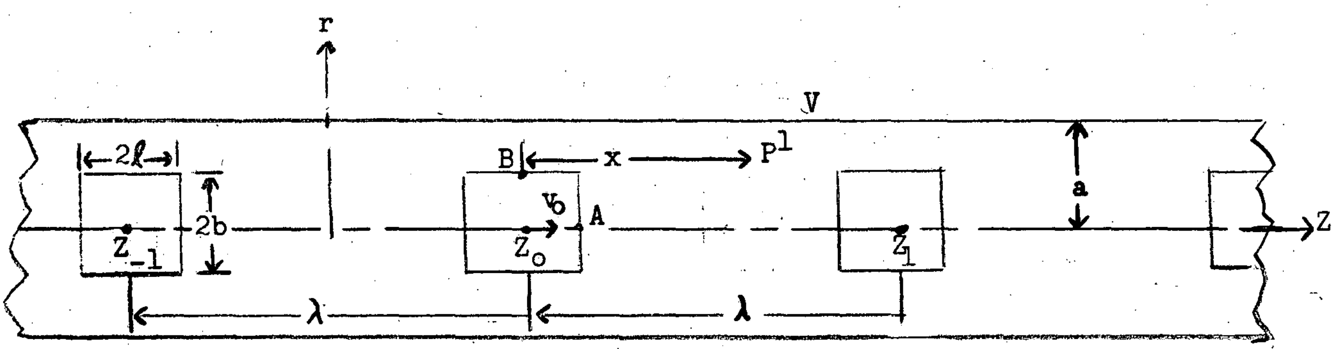
July 13, 1951

Introduction

In a previous report "Longitudinal Diffusion due to Space Charge of Ion Bunches during Acceleration", the writer made a rather crude calculation for this effect in a particular case when bunches were to be accelerated down a pipe. The following calculation pertains to the expansion of originally cylindrical bunches under the influence of space charge forces as they drift down a pipe at constant velocity. Subject only to the non-critical assumption that the bunches retain a cylindrical shape and uniform density, and that the velocity of motion down the pipe is non-relativistic, exact equations of motion are derived. Tables have been prepared to facilitate the numerical evaluation of functions entering these equations for a considerable range in the arguments. For one particular case of interest this numerical evaluation is carried out completely and the equations of motion integrated by the Differential Analyzer.

Schematic Representation of Problem

As said previously it is assumed that the bunches remain cylindrical in shape, and of uniform density. Thus it suffices to consider only the variation in length and radius of the bunches, or equivalently the motion of the representative points A and B in the sketch below.



We assume that the bunches are identical and extend infinitely in both directions. The separation of their centers is λ, the radius is b and the length 2ℓ. The common velocity is v_o the charge density ρ. If

V = accelerating potential

m = mass of ions

ν = r.f. frequency of bunch formation

I = average current

then

$$v_o = C\sqrt{\frac{2\epsilon V}{mc^2}}$$

$$\lambda = \frac{C}{\nu}\sqrt{\frac{2eV}{mc^2}}$$

$$\rho = \frac{I}{2\pi\nu\ell b^2} \quad \text{inside bunches}$$

$$= 0 \quad \text{outside}$$

Calculation of the Field

Apart from terms of order $(v/c)^2$ where v is the velocity of the charges, the electric field due to an arbitrary charge distribution is given by the negative gradient of a scalar potential Ψ :

$$\Psi(\vec{r}',t) = \iiint \rho(\vec{r}, t - r/c)\, G(\vec{r}, \vec{r}')\, d\vec{r} + \Psi_o(\vec{r}',t) \tag{1}$$

where Ψ_o is the potential in the absence of any charge or current distribution, and $G(\vec{r}, \vec{r}')$ is Green's function. Since we are considering the non-relativistic case we can use $\rho(\vec{r},t)$ instead of $\rho(\vec{r},t - r/c)$ without appreciable error.

The Green's function for a cylinder is given by Smythe. When azimuthal symmetry obtains it is

$$G(\vec{r},\vec{r}') = \frac{1}{2\pi a\epsilon} \sum_j \frac{J_0(j\frac{r}{a})\, J_0(j\frac{r'}{a})}{j\, J_1^2(j)} e^{-j\frac{|z - z_o|}{a}} \tag{2}$$

where the summation is over all the zeroes j of $J_o(x)$. If (2) is inserted in (1) and the charge distribution ρ described in the previous section is used then one obtains

$$\Psi(\vec{r}',t) - \Psi_o = \frac{I\,a}{\pi\epsilon\nu\ell b} \sum_j \frac{J_1(j\frac{b}{a})\, J_0(j\frac{r'}{a})}{j^3\, J_1^2(j)} \times \begin{cases} \sinh j\frac{\ell}{a}\, \dfrac{e^{-j\frac{|x|}{a}} + e^{-j\frac{\lambda - |x|}{a}}}{1 - e^{-j\frac{\lambda}{a}}} \\[2ex] 1 - \cosh j\frac{x}{a}\, \dfrac{e^{-j\frac{\ell}{a}} - e^{-j\frac{\lambda-\ell}{a}}}{1 - e^{-j\frac{\lambda}{a}}} \end{cases}$$

for $|x| \gtrless \ell$ and $x = z' - z_o$. (3)

Equations of Motion

For the non-relativistic velocities here considered

$$\frac{m\,d^2\vec{r}'}{dt^2} = -e\vec{\nabla}\Psi \tag{4}$$

We take the origin of time so that $z_o = v_o t$, $\frac{d}{dt} = v_o \frac{d}{d z_o}$. Since $\frac{d^2 z'}{d z_o^2} = \frac{d^2 x}{d z_o^2}$, from (3) and (4) we can write down the equations of motion from points A and B. If from now on all lengths are expressed in units of a the result is

$$\frac{d^2 \ell}{dS^2} = P(\ell, b)$$

$$\frac{d^2 b}{dS^2} = Q(\ell, b) \tag{5}$$

where

$$S = \sqrt{k}\ z_o$$

$$k = \frac{I}{4\pi\epsilon \nu V a} \tag{6}$$

$$P(\ell,b) = \frac{1}{\ell b} \sum_j \frac{J_1(jb)}{[j\ J_1\ (j)]^2}\ \frac{1 - e^{-2j\ell} - e^{-j(\lambda - 2\ell)} + e^{-j\lambda}}{1 - e^{-j\lambda}}$$

$$Q(\ell,b) = \frac{1}{\ell b} \sum_j \frac{J_1^2(jb)}{[j\ J_1\ (j)]^2}\ \frac{1 - e^{-j\ell} + e^{-j(\lambda - \ell)} - e^{-j\lambda}}{1 - e^{-j\lambda}} \tag{7}$$

The boundary conditions are $S = 0$, $\ell = \ell_o$, $b = b_o$, $\frac{d\ell}{dS} = \frac{db}{dS} = 0$.

The equations can probably only be solved numerically. To do this tables of P and Q for different ℓ,b must be calculated. This has been done for $\lambda = 1.5$. Tables of $\frac{J_1(jb)}{[j\ J_1(j)]^2}$, $\frac{J_1^2(jb)}{[j\ J_1(j)]^2}$, and e^{-jx} follow, and can be used to calculate tables for P and Q for different values of λ. For $\lambda = 1.5$ the calculated tables of P and Q were used to integrate (5) on the differential analyzer for one set of initial conditions.

Application to MTA

To estimate whether space charge debunching would render bunching of the beam impractical a specific calculation was carried out for the following case: (all lengths are in units of the pipe radius a)

$V = 80$ kv

$\nu = 12 \times 10^6$ cycles/sec

$a = 6"$

$I = 100$ ma

$2\ell_o = \lambda/4$

$b_o = 1/2 = 0.5$

particle: deuteron

whence

$\lambda = 1.515$

$\ell_o = 0.1894$

$k = 0.0061516$

$\sqrt{k} = 0.07843$

for simplicity we took instead

$\lambda = 1.5$

$\ell_o = 0.2$

For the special case of $\lambda = 1.5$ tables of P and Q were computed and equations (5) were integrated on the differential analyzer for $\ell_o = 0.2$, $b_o = 0.5$*. We see that the bunches double in length i.e., $\ell = 0.4$ when $S = 0.434$ or

$$z_o = \frac{S}{\sqrt{k}} = \frac{.434}{.07843} = 5.534 \text{ in units of a}, \quad = 33.2"$$

At $z_o = 20"$ $\ell = 0.277$ which corresponds to 125°.

Thus by the time a bunch has gone 33" or about four times as far as the space between the bunches, its length will have doubled and its diameter increased by about 20 percent. It will be recalled that we assumed the bunches to be identical and to extend infinitely in both directions. Actually they are formed at one point and then drift to the right, expanding as they go. In other words at a given time the bunches increase in size to the right. However we

* See curves at end of paper

see that in the particular case treated above they expand slowly enough so that the assumption of identical size should not be critical. In fact in this case the influence of neighboring bunches is not very strong anyway. We have of course neglected the perturbing field of the bunching mechanism and other complications, so that the results here obtained have only an order of magnitude significance.

$$\frac{J_1(j_n b)}{[j_n J_1(j\ )]^2} \quad \text{for} \quad J_0(j_n) = 0, \quad 0 < j_n b < 20$$

n \ b	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
1	.07656	.14349	.21671	.27429	.32017	.35251	.37009	.37247	.35989	.33313
2	.07528	.13381	.16292	.15724	.11980	.06114	-.00366	-.05882	-.09178	-.07423
3	.07128	.10499	.08551	.02781	-.03269	-.06213	-.04802	-.00506	.03607	.04920
4	.06554	.07033	.01617	-.03757	-.03934	.00235	.03491	.02328	-.01380	-.03091
5	.05858	.03620	-.02371	-.02967	.01331	.02630	-.00711	-.02386	.00261	.02172
6	.05054	.00778	-.02996	.00542	.02094	-.01304	-.01216	.01651	.00353	-.01631
7	.04188	-.01132	-.01417	.02019	-.00758	-.00929	.01532	-.00685	-.00677	
8	.03150	-.02001	.00569	.00690	-.01345	.01293	-.00678	-.00155		
9	.02432	-.01950	.01496	-.01007	.00504	-.00198	-.00376			
10	.01613	-.01279	.01121	-.01023	.00954	-.00900				
11	.00953	-.00366	.00047	.00187	-.00365					
12	.00244	.00448	-.00800	.00878	-.00724					
13	-.00345	.00926	-.00871	.00338						
14	-.00649	.00989	-.00287	-.00506						
15	-.00897	.00698	.00396	-.00554						
16	-.01022	.00220	.00658	.00100						
17	-.01029	-.00252	.00378							
18	-.00948	-.00559	-.00141							
19	-.00789	-.00619	-.00473							
20	-.00586	-.00454	-.00394							

$$\frac{J_1^2 (j_n b)}{[j_n J_1(j_n)]^2} \quad \text{for} \quad \begin{matrix} J_o(j_n) = 0 \\ 0 < j_n b < 20 \end{matrix}$$

n \ b	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
1	.00913	.03502	.07318	.11723	.15973	.19363	.21343	.21618	.20183	.17292
2	.02000	.06318	.09365	.08724	.05065	.01319	.00005	.01221	.02971	.03230
3	.02804	.06083	.04035	.00427	.00590	.02130	.01272	.00014	.00718	.01336
4	.03230	.03719	.00197	.01061	.01164	.00004	.00916	.00407	.00143	.00718
5	.03263	.01246	.00534	.00837	.00168	.00066	.00048	.00541	.00007	.00448
6	.02939	.00070	.01033	.00034	.00505	.00196	.00170	.00314	.00014	.00306
7	.02371	.00173	.00271	.00551	.00078	.00117	.00317	.00063	.00062	
8	.01537	.00621	.00050	.00073	.00280	.00259	.00071	.00004		
9	.01036	.00666	.00392	.00178	.00044	.00007	.00025			
10	.00508	.00320	.00245	.00205	.00178	.00158				
11	.00195	.00029	.00000	.00007	.00029					
12	.00014	.00047	.00150	.00181	.00123					
13	.00030	.00101	.00194	.00029						
14	.00116	.00269	.00023	.00070						
15	.00237	.00144	.00046	.00091						
16	.00328	.00015	.00136	.00003						
17	.00355	.00021	.00048							
18	.00319	.00111	.00007							
19	.00234	.00144	.00084							
20	.00136	.00081	.00061							

$e^{-j_n x}$ where $J_o(j_n) = 0$

n \ x	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
1	.7862	.6182	.4861	.3822	.3005	.2362	.1857	.1460	.1148	.0903
2	.5758	.3294	.1909	.1099	.0633	.0364	.0210	.0118	.0070	.0040
3	.4209	.1772	.0746	.0314	.0132	.0056	.0023	.0010	.0004	.0002
4	.3075	.0946	.0291	.0089	.0028	.0008	.0003	.0001	.0000	.0000
5	.2247	.0505	.0013	.0025	.0006	.0001	.0000	.0000		
6	.1641	.2069	.0044	.0007	.0001	.0000				
7	.1199	.0144	.0017	.0002	.0000					
8	.0792	.0077	.0007	.0001	.0000					
9	.0640	.0041	.0003	.0000						
10	.0467	.0022	.0001	.0000						
11	.0355	.0012	.0000							
12	.0249	.0006								
13	.0173	.0003								
14	.0133	.0002								
15	.0097	.0001								
16	.0071	.0001								
17	.0052	.0000								
18	.0038									
19	.0028									
20	.0020									

n \ x	1.1	1.2	1.3	1.4	1.5
1	.0710	.0558	.0439	.0345	.0271
2	.0023	.0013	.0008	.0004	.0002
3	.0001	.0000	.0000	.0000	.0000
4	.0000				

P(ℓ,b) for λ = 1.5

b \ ℓ	.1	.2	.3	.4	.5	.6	.7
.1	39.0	22.0	15.5	11.6	9.19	7.19	4.42
.2	14.6	9.34	6.69	5.02	3.91	2.86	1.36
.3	7.72	5.43	4.02	3.02	2.32	1.61	0.68
.4	4.73	3.53	2.69	2.01	1.54	1.02	0.392
.5	3.181	2.468	1.917	1.430	1.088	0.700	0.257
.6	2.243	1.796	1.418	1.053	0.800	0.503	0.177
.7	1.709	1.384	1.102	0.815	0.619	0.386	0.135
.8	1.330	1.087	0.868	0.642	0.488	0.301	0.105
.9	1.039	0.860	0.705	0.518	0.394	0.237	0.081
1.0	1.003	0.805	0.639	0.471	0.359	0.225	0.080

ℓ b P(ℓ,b) for λ = 1.5

b \ ℓ	.1	.2	.3	.4	.5	.6	.7
.1	.390	.450	.465	.466	.459	.431	.310
.2	.291	.374	.401	.402	.391	.344	.191
.3	.232	.326	.362	.362	.348	.289	.142
.4	.189	.282	.322	.322	.307	.245	.110
.5	.159	.247	.287	.286	.272	.210	.090
.6	.135	.216	.255	.253	.240	.181	.074
.7	.120	.194	.231	.228	.217	.162	.066
.8	.106	.174	.208	.205	.195	.145	.059
.9	.094	.155	.190	.186	.177	.128	.051
1.0	.100	.161	.191	.188	.180	.135	.056

Q(ℓ,b) for λ = 1.5

b \ ℓ	.1	.2	.3	.4	.5	.6	.7
.1	18.8	11.15	7.80	5.98	4.84	4.07	3.50
.2	6.812	4.607	3.459	2.756	2.286	2.129	1.705
.3	3.418	2.705	2.128	1.745	1.482	1.290	1.144
.4	2.378	1.796	1.540	1.230	1.066	0.944	0.851
.5	1.693	1.316	1.084	0.934	0.824	0.740	0.676
.6	1.151	0.947	0.832	0.731	0.656	0.597	0.551
.7	1.052	0.845	0.723	0.638	0.575	0.526	0.487
.8	0.882	0.721	0.623	0.554	0.501	0.459	0.426
.8	0.748	0.631	0.550	0.489	0.408	0.403	0.373
1.0	0.779	0.618	0.519	0.451	0.400	0.362	0.327

ℓb Q(ℓ,b) for λ = 1.5

b \ ℓ	.1	.2	.3	.4	.5	.6	.7
.1	.188	.223	.234	.239	.242	.244	.245
.2	.136	.184	.208	.221	.229	.234	.239
.3	.103	.162	.192	.209	.222	.232	.240
.4	.095	.144	.185	.197	.213	.227	.238
.5	.085	.132	.163	.187	.206	.222	.237
.6	.069	.114	.150	.176	.197	.215	.231
.7	.074	.118	.152	.179	.201	.221	.239
.8	.071	.115	.150	.177	.200	.220	.239
.9	.067	.114	.148	.176	.198	.218	.235
1.0	.078	.124	.156	.180	.200	.217	.229

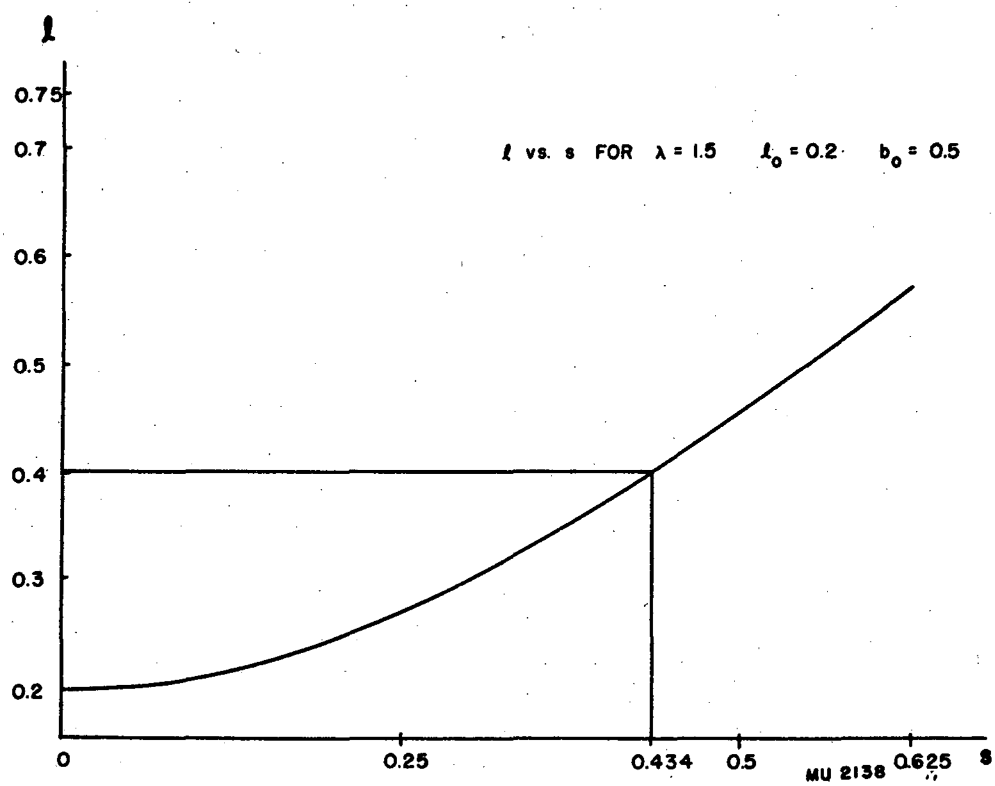

ℓ
0.75
0.7
0.6
0.5
0.4
0.3
0.2
ℓ vs. s FOR λ = 1.5 ℓ₀ = 0.2 b₀ = 0.5
0
0.25
0.434
0.5
0.625
s
MU 2138

Fig. 1

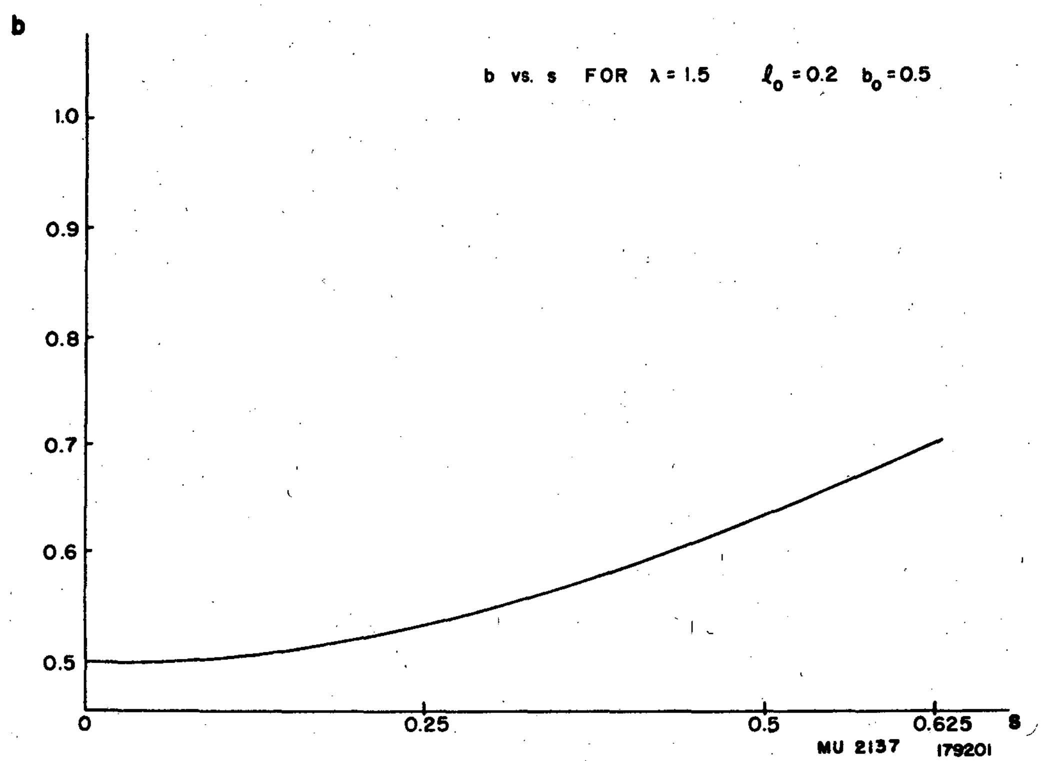

b
b vs. s FOR λ = 1.5 ℓ0 = 0.2 b0 = 0.5
1.0
0.9
0.8
0.7
0.6
0.5
0
0.25
0.5
0.625
s
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Fig. 2