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ABSTRACT

A new method of calculating phase shifts for a Bethe-Salpeter equation is presented. The scattering amplitude is calculated below elastic threshold using the differential equation and variational methods, and then continued to the elastic scattering region to find phase shifts.

Recently Schlessinger and Schwartz presented a method of finding phase shifts in potential theory by solving the Schrödinger differential equation for the scattering amplitude for energies below threshold and continuing it to the scattering region.[1] In this paper we report a variation on their method involving an on-mass-shell continuation that has proven successful in solving a Bethe-Salpeter equation.[2] The on-shell amplitude satisfies a simple unitarity relation, and this can be used advantageously in performing the continuation. We calculate below threshold in order to avoid the problems of solving a singular integral equation for the phase shifts.[3]

The differential Bethe-Salpeter equation in the ladder approximation for spinless particles of equal mass, m, is of the form

$$\mathcal{O}\,\psi_k(x) = V(x)\,\psi_k(x), \qquad (1)$$

where $k = (0,\underset{\sim}{k})$, and $|\underset{\sim}{k}|^2 = |E^2/4| - m^2$. We are interested in this equation below elastic threshold $(E^2 < 4m^2)$, where the Wick rotation can be performed.[4] In the four-dimensional euclidean metric, $\mathcal{O}$ takes the form[5]

$$\mathcal{D} = \left(-\Box - \frac{E^2}{4} + m^2\right)^2 - E^2\left(\frac{\partial}{\partial x_4}\right)^2 , \qquad (2)$$

where

$$\Box = \sum_{\nu=1}^{4} \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial x_\nu}$$

For mass μ exchange, the potential is

$$V(x) = \frac{4\mu\lambda}{|x|} K_1(\mu|x|). \qquad (3)$$

The T matrix is defined as

$$T(k',k) = \int d^4x \, e^{-k' \cdot x} \, V(x) \, \psi_k(x). \qquad (4)$$

Let us define the scattered part of the wave function $\chi_k(x)$ by

$$\psi_k(x) = \phi_k(x) + \chi_k(x), \qquad (5)$$

where $\phi_k(x)$ is the free wave term, $e^{ik\cdot x}$. The differential equation for $\chi_k(x)$ is

$$\mathcal{D} \chi_k(x) = V(x) \, \chi_k(x) + V(x) \, \phi_k(x) . \qquad (6)$$

We can write a Kohn type variational principle[6] for the T matrix based on Eq. (6)

$$[T(k',k)] = \int d^4x\, \chi^*_{k'}(x)\, \{\mathcal{D} - V(x)\}\, \chi_k(x) + \int d^4x\, \chi^*_{k'}(x)\, V(x)\, \phi_k(x)$$

$$+ \int d^4x\, \phi^*_{k'}(x)\, V(x)\, \chi_k(x) + \int d^4x\, \phi^*_{k'}(x)\, V(x)\, \phi_k(x) \; . \tag{7}$$

This variational principle can be applied when the integrals are well defined. For an energy above threshold, the asymptotic behavior of the wave function $\chi_k(x)$ for large x_4 is a <u>growing</u> exponential,[7] and thus the derivative term in Eq. (7) is not well defined. However, below threshold, the wave function is exponentially damped,[7] and there exists an energy region where all the integrals are convergent. In practice, the application of this variational principle is considerably simpler than the Schwinger variational principle based on the integral equation used by Schwartz and Zemach.[5] Our method amounts to solving the bound-state equations using the method of Schwartz[8] but with an inhomogeneous term.

If we do a partial-wave analysis of these equations, we can calculate $T_\ell(E^2)$ in the region $4(m^2 - \mu^2) < E^2 < 4m^2$, i.e., between threshold and the second Born contribution to the left-hand cut. The integrals diverge below this point, because $V(x)\, \phi_k(x)$ grows exponentially there.

The analytic continuation is performed using the K matrix defined by

$$K_\ell(E^2) = \frac{T_\ell(E^2)}{2 + 2i\rho\, T_\ell(E^2)}, \qquad (8)$$

where

$$T_\ell(E^2) = \frac{e^{i\delta_\ell} \sin\delta_\ell}{\rho}, \quad \rho = \frac{\left(\frac{E^2}{4} - m^2\right)^{\frac{1}{2}}}{8\pi E}, \qquad (9)$$

The analytically continued unitarity equation implies that $K_\ell(E^2)$ is analytic in E^2 at threshold, and thus by employing the K matrix, we have removed the threshold branch point.[9] Figure 1 shows the cut structure of $K_\ell(E^2)$ in the E^2 plane in the region of interest.

Before doing the continuation to the scattering region, we first remove the cut contribution $K_{cut}(E^2)$ between $4m^2 - 4\mu^2$ and $4m^2 - \mu^2$, thus enlarging the region of analyticity. The continuation is done using a Padé form as in Ref. 1:

$$K_\ell(E^2) - K_{cut}(E^2) = \frac{\sum_{i=0}^{n} a_i\, (E^2)^i}{1 + \sum_{i=1}^{n} b_i\, (E^2)^i}. \qquad (10)$$

We extrapolate with these functions and then add $K_{cut}(E^2)$ back in.

For a strong attractive potential with a deeply bound state, i.e., $\lambda = 3$, $\mu = m$, the S-wave phase shift was obtained to at least 2% in the entire elastic-scattering region. Close to elastic threshold and for weaker potentials, the accuracy was considerably better. The input numbers for the extrapolation were good to four or five places. Table I gives a sample of the stability of the extrapolation.

In conclusion, we find that because of the high accuracy of the variational method below threshold, it is possible to get phase shifts in the elastic region by extrapolation using the simpler differential-equation methods and modest computer time. Roughly two significant figures are lost in the extrapolation.

I wish to thank Professor Charles Schwartz for suggesting this problem and for many enlightening discussions. Also I would like to thank Leonard Schlessinger for helpful discussions on the numerical continuation.

FOOTNOTES AND REFERENCES

* Work done under the auspices of the U. S. Atomic Energy Commission.

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5. C. Schwartz and C. Zemach 141, 1454 (1966). We use the notation and conventions from this article.

6. See Sec. IV-C of Ref. 5 for a discussion and further references.

7. See Sec. II-D of Ref. 5.

8. C. Schwartz, Phys. Rev. 137, B717, (1965).

9. See for example R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorn, The Analytic S Matrix, (Cambridge University Press, New York, 1966), p. 231.

Table I. A sample of the convergence of the extrapolation for two attractive potentials upon increasing the order of fitting. The S.Z. values were taken from the Schwartz and Zemach calculation described in Ref. 5 (private communication). Cancellations in the fitting generally limit the meaningful size of fitting functions to $n = 5$ or 6 for the accuracy of our input numbers for the extrapolation.

n	$(\delta_0/\pi) - 1$ [a]	δ_0/π [b]
2	- 0.2420	0.2703
3	- 0.3147	0.2674
4	- 0.3277	0.2672
5	- 0.3048	0.2666
6	- 0.3083	0.2670
7	- 0.3093	0.2663
8	- 0.3119	0.2731
S.Z. values	- 0.3097	0.2684

a. $E^2 = 5.6$, $\lambda = 3$, $\mu = m = 1$.

b. $E^2 = 5.2$, $\lambda = 0.7$, $\mu = m = 1$.

FIGURE CAPTION

Fig. 1. Cut structure of $K_\ell(E^2)$ showing the first inelastic threshold and the first two contributions to the left-hand cut.

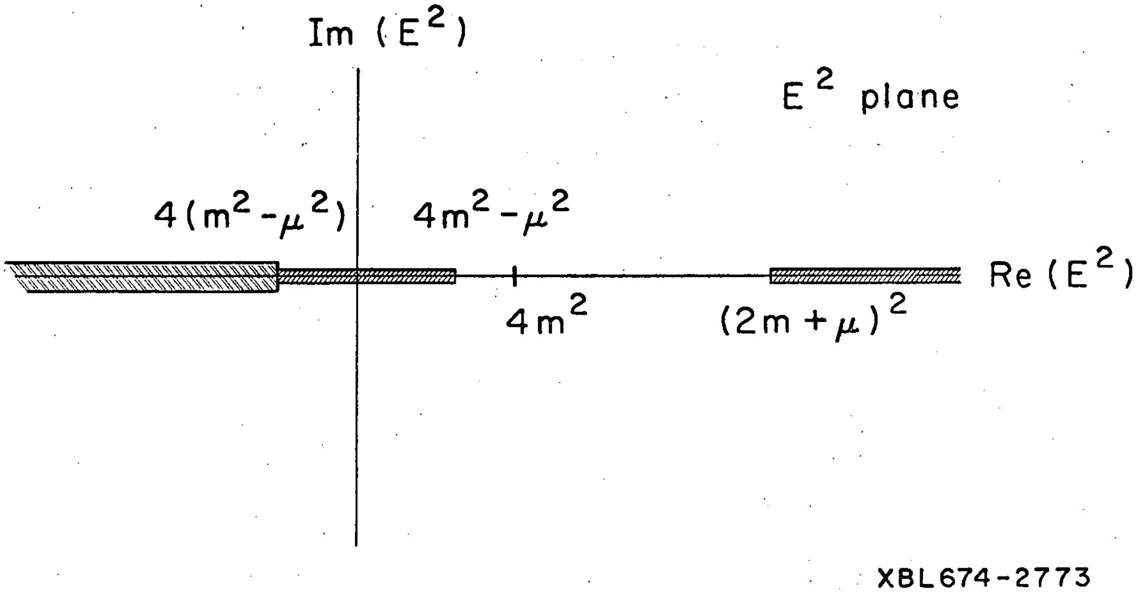

Fig. 1