Rigid structures in traffic probability: with a view toward random matrices

by

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Abstract

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Traffic probability is an operadic non-commutative probability theory recently introduced by Male that generalizes the standard non-commutative probabilistic framework. This additional operad structure admits a corresponding notion of independence, the so-called *traffic independence*. At the same time, traffic probability captures certain aspects of both classical and free probability. An as yet incomplete understanding of this relationship yields insightful feedback between the different theories. In this dissertation, we study this problem through two complementary angles: first, in the context of the universal enveloping traffic space; and second, in the context of large random matrices. For a tracial non-commutative probability space $(\mathcal{A}, \varphi)$, Cébron, Dahlqvist, and Male constructed an enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ that extends the trace. The CDM construction provides a universal object that allows one to appeal to the traffic probability framework in generic situations, prioritizing an understanding of its structure. In Chapter 3, we study the structure of the universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ as a general non-commutative probability space $(\mathcal{B}, \psi)$, particularly in relation to non-commutative notions of independence. We show that $(\mathcal{B}, \psi)$ admits a canonical free product decomposition $\mathcal{B} = \mathcal{A} * \mathcal{A}^\intercal * \Theta(\mathcal{B})$, regardless of the choice of $(\mathcal{A}, \varphi)$. If $(\mathcal{A}, \varphi)$ is itself a free product, then we show how this additional structure lifts into $(\mathcal{B}, \psi)$. Here, we find a duality between classical independence and free independence. Our proof relies on the existence of a natural homomorphic conditional expectation in $(\mathcal{B}, \psi)$ that takes $\Theta(\mathcal{B})$ to a commutative subalgebra $\Delta(\mathcal{B})$. Up to degeneracy, we further show that $\Delta(\mathcal{B})$ is spanned by tree-like graph operations. In Chapter 4, we utilize the traffic framework to study the asymptotics of large random multi-matrix models. As a starting point, we compute the limiting traffic distribution of the classical ensembles of Wigner, Ginibre, and Wishart-Laguerre. This allows us to apply our free product decomposition from Chapter 3 to prove the asymptotic freeness of a large class of dependent random matrices, generalizing and providing a unifying framework for results of Bryc, Dembo, and Jiang and of Mingo and Popa. We further prove general Markov-type concentration inequalities for the joint traffic distribution of our matrices. We then extend our analysis to random band matrices and investigate the extent to which the joint traffic distribution of these matrices deviates from the

classical case. We also pursue an orthogonal computation, namely, that of a Haar distributed orthogonal random matrix. Altogether, our formulas suggest a convenient cactus-cumulant correspondence, the details of which we commit in the last section. Our results related to the universal enveloping traffic space form part of a joint work in progress with Camille Male.

To my family, with admiration.

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Chapter 1

Introduction

Non-commutative (NC) probability is a generalization of classical probability theory that extends the probabilistic perspective to non-commuting random variables. To accommodate this non-commutativity, one must first revisit the notion of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the classical framework, a random variable simply corresponds to a measurable function $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{C}$. Here, one often restricts attention to a special class of random variables: for example, $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $1 \leq p \leq \infty$ or

$$L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) = \bigcap_{p=1}^{\infty} L^p(\Omega, \mathcal{F}, \mathbb{P}),$$

the space of random variables with finite moments of all orders. For such random variables, one can define the expectation $\mathbb{E}[X]$, which encodes the probability measure $\mathbb{P}(E) = \mathbb{E}[\mathbb{1}_E]$. For $p \in \{\infty-, \infty\}$, $L^p(\Omega, \mathcal{F}, \mathbb{P})$ further possesses the additional structure of a $\mathbb{C}$-algebra, one that is both unital and commutative. One arrives at the NC framework by forgoing this second property.

Definition 1.1 (NC probability space)**.** A *NC probability space* is a pair $(\mathcal{A}, \varphi)$ consisting of a unital NC $\mathbb{C}$-algebra $\mathcal{A}$ together with a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$. We refer to elements $a \in \mathcal{A}$ as *NC random variables* (or simply *random variables*) with φ playing the role of the expectation.

One often works with additional structure in a NC probability space: for example, the algebra $\mathcal{A}$ could be a $*$-algebra, a C^*-algebra, or a W^*-algebra, while the functional φ could be positive, tracial, or faithful. Surprisingly, even at the most basic level of this definition, one sees the emergence of interesting and uniquely NC phenomena.

Naturally, the classical framework provides the first example. In particular, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ begets a NC probability space $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$. The fundamental notion of independence, defined for sub-σ-fields $(\mathcal{F}_i)_{i \in I}$, amounts to an elementary factorization property for the subalgebras $(L^{\infty-}(\Omega, \mathcal{F}_i, \mathbb{P}))_{i \in I}$ over the expectation $\mathbb{E}$. By adapting this concrete example, one can formulate the notion of *classical independence* for subalgebras

$(\mathcal{A}_i)_{i\in I}$ of an abstract NC probability space $(\mathcal{A}, \varphi)$. Of course, in order to conform to the classical notion, one requires that the subalgebras $(\mathcal{A}_i)_{i\in I}$ commute as part of the definition.

At the same time, genuinely NC examples abound. Historically, operator algebras provide the first avenue for such investigations. In this context, Voiculescu discovered a remarkable analogue of classical independence for non-commuting random variables [Voi85]. In the group von Neumann algebra $(L(G), \tau_G)$, a free product structure $G = *_{i\in I} G_i$ defines a universal factorization property for the subalgebras $(L(G_i))_{i\in I}$ over the canonical trace τ_G, the so-called *free independence*. By taking free independence as the fundamental concept, Voiculescu developed what is now known as *free probability*. Free independence appears as a ubiquitous phenomenon in many guises [Spe17], with myriad applications both pure [VDN92,HP00,NS06,MS17] and applied [TV04,CD11]. One also finds free analogues of many classical notions: for example, the free central limit theorem [Voi85], free cumulants [Spe94], free entropy [Voi93,Voi94], a free stochastic calculus [BS98], and free extreme values [BAV06].

Yet, despite the parallels, free independence exists purely as a NC phenomenon, distinct from the classical notion. One can justifiably ask if free independence deserves to occupy such a distinguished role in the NC probabilistic framework. Is free independence truly *the* NC analogue of classical independence? Voiculescu's landmark paper [Voi91] answers this question in the affirmative, establishing a connection between the two notions via random matrices. Random matrices provide a fertile intermediate ground where one can consider classical notions in a genuinely NC setting. In particular, one can lift the classical independence of random variables

$$(L^{\infty-}(\Omega, \mathcal{F}_i, \mathbb{P}))_{i\in I} \quad \text{in} \quad (L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$$

to independent random matrices

$$(L^{\infty-}(\Omega, \mathcal{F}_i, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C}))_{i\in I} \quad \text{in} \quad \left(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C}), \mathbb{E}\frac{1}{N}\operatorname{tr}\right).$$

Naturally, one can ask what becomes of the independence relation (as NC random variables) in the process. Does the $\mathbb{E}$-factorization property give rise to a rigid $\mathbb{E}\frac{1}{N}\operatorname{tr}$ behavior? Of course, the answer depends on how one constructs the random matrices in question; however, remarkably, free independence describes the asymptotic behavior of such matrices in many generic situations. Wigner matrices provide an illustrative example of this phenomenon, a fact first established in [Voi91] for the highly structured Gaussian Unitary Ensemble (GUE) and later extended to general Wigner matrices in [Dyk93]. In the Wigner model, the entries of the matrix form an independent family of random variables up to the symmetry constraint on the matrix. The results of [Voi91,Dyk93] show that independent Wigner matrices become freely independent in the large N limit. Thus, free independence emerges precisely from this process of "non-commutification": we formulate this relationship in the heuristic equation

$$\text{classical independence} \quad \overset{\otimes}{\underset{}{\xrightarrow[]{N\to\infty}}} \quad \text{free independence.} \tag{1.1}$$

On the other hand, one increasingly finds interest in models that lie beyond the scope of the standard free probabilistic machinery. Nevertheless, one might reasonably expect to still be able to apply the NC probabilistic perspective. At the combinatorial level, classical independence and free independence simply amount to rules for calculating mixed moments in independent random variables from the pure moments. The free probability heuristic (1.1), while affirming the suitability of free independence, says nothing about the existence of other NC notions of independence (i.e., other such rules). Of course, such a rule should satisfy certain natural properties to warrant consideration as a probabilistic notion. In the setting of Definition 1.1, Speicher showed that if one requires the rule to be suitably universal in an algebraic sense, then in fact classical independence and free independence comprise the full set of the possibilities [Spe97] (see also [BGS02] for a categorical axiomatization). One can maneuver past this dichotomy by relaxing the requirements: in particular, eschewing the unital framework allows for an additional possibility, the so-called *Boolean independence* [Spe97, BGS02]. If one further allows for asymmetric notions of independence (i.e., X is independent of $Y \not\Longleftrightarrow Y$ is independent of X), then *monotone independence* and *anti-monotone independence* come into the picture [Mur02, Mur03]. Despite their seemingly exotic nature, such notions of independence possess interesting structure in their own right, particularly in relation to spectral graph theory [HO07, Oba17].

Up till now, the approach to Speicher's dichotomy proceeds by opting for less (even though often desirable) structure. In a different direction, one can consider enriching the structure. Optimistically, the resulting framework would be sufficiently robust to accommodate a new notion of independence in addition to recovering features of the existing notions. Traffic probability provides precisely such a framework. Motivated by the study of permutation invariant random matrices, Male introduced an operadic NC probability theory based on graph operations that extends the usual NC probabilistic framework [Mal]. This additional operad structure admits a new notion of independence, the so-called *traffic independence*. Notably, independent permutation invariant random matrices provide a canonical model of traffic independence in the large N limit.

Traffic independence circumvents Speicher's dichotomy while retaining the desirable properties of a probabilistic notion. To accomplish this, one works in the setting of a *traffic space*, which can be thought of as a NC probability space with additional structure. Informally, an *algebraic traffic space* $(\mathcal{A}, \tau)$ consists of a complex vector space $\mathcal{A}$ over the operad $\mathcal{G}$ of graph operations together with a linear functional $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ defined on a particular family of graphs with edge labels in $\mathcal{A}$ satisfying certain compatibility conditions. This operad structure further defines a unital $\mathbb{C}$-algebra structure on $\mathcal{A}$, while the functional τ defines an expectation $\varphi_\tau : \mathcal{A} \to \mathbb{C}$, recovering the framework of Definition 1.1. The novelty of traffic independence owes to its formulation in terms of the functional τ: whereas classical independence and free independence correspond to polynomial relationships over the expectation φ (relying on the algebra structure), traffic independence corresponds to a graphical decomposition over the functional τ (relying on the operad structure).

Of course, one can still define the usual notions of independence in an algebraic traffic space $(\mathcal{A}, \tau)$ by virtue of the induced expectation φ_τ. In particular, this subsumption allows for an interplay between the different notions of independence in the traffic framework. Indeed, one finds many striking relationships between them: for example, general criteria for when traffic independence implies free independence or classical independence [Mal, CDM]. The reader will no doubt anticipate such a relationship based on the random matrix heuristic. For example, unitarily invariant random matrices fall into the domain of free probability [Voi91]. Of course, unitarily invariant random matrices are also permutation invariant, so one sees an overlap for many classical random matrix ensembles. Similarly, diagonal matrices with i.i.d. entries are also permutation invariant, bridging to the classical probabilistic framework. Still, crucially, the notions far from align. For example, traffic independence alone governs the behavior of heavy Wigner matrices [Mal17] and sparse random graphs [MP]. The traffic CLT further interpolates between the classical, free, and Boolean CLTs [Mal]. Even in the absence of any sort of strong distributional invariance, the traffic framework still proves advantageous, particularly in the case of random band matrices [Au].

At this point, the reader should pause to ask a natural question: when can one actually appeal to the traffic framework? For reference, the basic setting of Definition 1.1 requires very little in the way of assumptions. Despite its merits, the operad structure of a traffic space could in principle prove to be prohibitively specific, limiting the scope of the traffic machinery. Fortunately, this is not the case. In practice, one often specializes Definition 1.1 to the case of a tracial $*$-probability space $(\mathcal{A}, \varphi)$. In this setting, Cébron, Dahlqvist, and Male constructed a universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ that extends the trace $\varphi_{\tau_\varphi}|_{\mathcal{A}} = \varphi$ [CDM]. Thus, effectively, one can always appeal to the traffic framework. In finite dimensions, this extension corresponds to randomly rotating a matrix to bring it into generic position. This construction even provides a concrete limit object for large random matrices: if a family of unitarily invariant random matrices $(\mathbf{M}_N^{(i)})_{i\in I}$ converges in $*$-distribution to a family of random variables $(a_i)_{i\in I}$ in $(\mathcal{A}, \varphi)$ and satisfies a mild factorization condition, then $(\mathbf{M}_N^{(i)})_{i\in I}$ further converges in traffic distribution to $(a_i)_{i\in I}$ in $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ [CDM].

In Chapter 3, we study the probabilistic structure of the universal enveloping traffic space, but with minimal reference to traffic independence. Instead, we show that $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ can be viewed quite profitably as a general NC probability space, particularly in relation to the usual notions of independence. Nevertheless, we make considerable use of the traffic framework. Having hopefully convinced the reader of the merits of traffic probability, we delay the precise statement of the main results until Section 3.1, after the necessary prerequisites. To this end, we devote Chapter 2 to a crash course in traffic probability.

In Chapter 4, we study the asymptotics of large random multi-matrix models through the lens of traffic probability. For concreteness, we revisit a number of classical random matrix ensembles, only to find a departure from the usual free probabilistic universality. As an application, we show how our structural results for the universal enveloping traffic space can be realized in the large N limit of our matrices. We also consider band matrix variants of our ensembles, which lack the homogeneity of their classical counterparts. Our analysis suggests a correspondence between the free cumulants and the injective traffic distribution in the case of cactus-type random variables. We explore this connection in the last section.

Chapter 2

A crash course on traffic probability

For the convenience of the reader, we include a condensed exposition of traffic probability. We refer the reader to [Mal, CDM] for the definitive references. Section 2.1 reviews the basic framework of *non-commutative probability*, largely following [NS06]. Section 2.2 introduces the *operad of graph operations*, which formalizes the additional algebraic structure in the traffic probability framework. Section 2.3 then sets up the traffic probability machinery. Finally, Section 2.4 revisits the CDM construction of the universal enveloping traffic space.

2.1 Non-commutative probability

We start by specializing Definition 1.1 to the setting of a tracial $*$-probability space, the primary setting of the remainder of the article.

Definition 2.1.1 ($*$-probability space)**.** A $*$-*probability space* is a pair $(\mathcal{A}, \varphi)$ consisting of a unital $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ together with a state $\varphi : \mathcal{A} \to \mathbb{C}$. Equivalently, $(\mathcal{A}, \varphi)$ is a NC probability space equipped with a conjugate linear anti-isomorphic involution $* : \mathcal{A} \to \mathcal{A}$ such that $\varphi(a^*a) \geq 0$ for every $a \in \mathcal{A}$. The state φ is said to be *tracial* if it vanishes on the commutators of $\mathcal{A}$ (i.e., $\varphi(ab) = \varphi(ba)$ for every $a, b \in \mathcal{A}$).

Example 2.1.2. In keeping with the introduction, a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ begets a $*$-probability space $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ under the complex conjugate $X^* = \overline{X}$. We abstract another feature from the classical framework: in the setting of a $*$-probability space $(\mathcal{A}, \varphi)$, we say that the state φ is *faithful* if $\varphi(a^*a) = 0$ implies $a = 0$. ♢

Example 2.1.3. Let $\mathrm{Mat}_N(\mathbb{C})$ denote the $*$-algebra of $N \times N$ complex matrices under the usual matrix adjoint. The normalized trace $\frac{1}{N}\,\mathrm{tr}$ is clearly positive (indeed, faithful), giving rise to the $*$-probability space $(\mathrm{Mat}_N(\mathbb{C}), \frac{1}{N}\,\mathrm{tr})$. The trace of course vanishes on the commutators. ♢

Example 2.1.4. Combining the two previous examples, we obtain the $*$-probability space $(\mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})), \mathbb{E}\frac{1}{N}\operatorname{tr})$ of random $N \times N$ matrices whose entries have finite moments of all orders. We leave it to the reader to verify that $\mathbb{E}\frac{1}{N}\operatorname{tr}$ is indeed a faithful trace. We alternate between the notation $\mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})) = L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C})$ as convenient. $\lozenge$

In the NC framework, the distribution of a random variable simply corresponds to the information of its moments. One records this data in a generic setting to facilitate the comparison of such distributions. More precisely, for an index set I, we write $\mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle$ for the free unital $*$-algebra on the indeterminates $\mathbf{x} = (x_i)_{i \in I}$. For a family of random variables $\mathbf{a} = (a_i)_{i \in I}$ in a $*$-probability space $(\mathcal{A}, \varphi)$, one can then define the natural evaluation map

$$\mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle \ni P \mapsto P(\mathbf{a}) \in \mathcal{A}.$$

Definition 2.1.5 (Joint $*$-distribution)**.** Let $\mathbf{a} = (a_i)_{i \in I}$ be a family of random variables in a $*$-probability space $(\mathcal{A}, \varphi)$. The *joint $*$-distribution* of $\mathbf{a}$ is the linear functional

$$\mu_{\mathbf{a}} : \mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle \to \mathbb{C}, \qquad P \mapsto \varphi(P(\mathbf{a})).$$

A sequence of families $\mathbf{a}_n = (a_n^{(i)})_{i \in I}$, each living in a $*$-probability space $(\mathcal{A}_n, \varphi_n)$, is said to *converge in $*$-distribution* to $\mathbf{a}$ if the corresponding joint $*$-distributions $\mu_{\mathbf{a}_n}$ converge pointwise to $\mu_{\mathbf{a}}$, i.e.,

$$\lim_{n \to \infty} \mu_{\mathbf{a}_n}(P) = \mu_{\mathbf{a}}(P), \qquad \forall P \in \mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle.$$

For a family of self-adjoint random variables $a_i^* = a_i$, we simply refer to the *joint distribution* $\mu_{\mathbf{a}} : \mathbb{C}\langle \mathbf{x} \rangle \to \mathbb{C}$, which is defined in the obvious way.

With distributions in mind, we segue into NC notions of independence. But first, we introduce some notation to facilitate the definitions. For a collection of random variables $\mathcal{S} \subset \mathcal{A}$ in a NC probability space $(\mathcal{A}, \varphi)$, we write $\mathring{\mathcal{S}} = \{a \in \mathcal{S} : \varphi(a) = 0\}$ for the subcollection (possibly empty) of *centered* random variables.

Definition 2.1.6 (Classical independence)**.** Let $(\mathcal{A}, \varphi)$ be a NC probability space. We say that unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ are *classically independent* if the $(\mathcal{A}_i)_{i \in I}$ commute (i.e., $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$) and φ is multiplicative across the $(\mathcal{A}_i)_{i \in I}$ in the following sense: for any $k \geq 1$ and distinct indices $i(1), \ldots, i(k) \in I$,

$$\varphi\biggl(\prod_{j=1}^{k} a_{i(j)}\biggr) = \prod_{j=1}^{k} \varphi(a_{i(j)}), \qquad \forall a_{i(j)} \in \mathcal{A}_{i(j)}. \tag{2.1}$$

We note that the multiplicative property (2.1) is equivalent to

$$\varphi\biggl(\prod_{j=1}^{k} a_{i(j)}\biggr) = 0, \qquad \forall a_{i(j)} \in \mathring{\mathcal{A}}_{i(j)}. \tag{2.1$'$}$$

The reader should carefully compare this definition with

Definition 2.1.7 (Free independence)**.** Let $(\mathcal{A}, \varphi)$ be a NC probability space. We say that unital subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ are *freely independent* (or simply *free*) if for any $k \geq 1$ and consecutively distinct indices $i(1) \neq i(2) \neq \cdots \neq i(k) \in I$,

$$\varphi\bigg(\prod_{j=1}^{k} a_{i(j)}\bigg) = 0, \qquad \forall a_{i(j)} \in \mathring{\mathcal{A}}_{i(j)}. \tag{2.2}$$

Independence (classical or free) for a collection of subsets $(\mathcal{S}_i)_{i\in I}$ of $\mathcal{A}$ is defined as the independence of the generated unital subalgebras $(\mathrm{alg}(1_{\mathcal{A}}, \mathcal{S}_i))_{i\in I}$. To emphasize the $*$-structure, we often use the terms $*$-*classically independent* and $*$-*free* as appropriate, which refer to the independence of the generated unital $*$-subalgebras $(*\text{-alg}(1_{\mathcal{A}}, \mathcal{S}_i))_{i\in I}$.

The reader will no doubt notice that equations (2.1′) and (2.2) are identical; however, the admissible indices $i(j)$ to which they apply crucially differ. The corresponding CLTs, recorded below, illustrate the considerable extent to which these two notions diverge.

Theorem 2.1.8 (CLTs, classical and free)**.** *Let (a_n) be a sequence of identically distributed self-adjoint random variables in a $*$-probability space $(\mathcal{A}, \varphi)$. Assume that the a_n are centered with unit variance, i.e., $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, and write $s_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} a_i$ for the normalized sum. We consider two cases:*

(i) If the a_n are classically independent, then (s_n) converges in distribution to a standard normal random variable, i.e.,

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{\mathbb{R}} t^m \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt, \qquad \forall m \in \mathbb{N}.$$

(ii) If the a_n are freely independent, then (s_n) converges in distribution to a standard semi-circular random variable, i.e.,

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{-2}^{2} t^m \cdot \frac{1}{2\pi}\sqrt{4 - t^2}\, dt, \qquad \forall m \in \mathbb{N}.$$

Proof. See Theorems 8.5 and 8.10 in [NS06]. ■

An independence relation between collections of random variables, say $\mathcal{S}_1$ and $\mathcal{S}_2$, allows us to determine the joint distribution from the marginals. This is perhaps less transparent in the case of free independence, where one must iterate the defining property (2.2) to centered shifts

$$\prod_{j=1}^{k} a_{i(j)} = \prod_{j=1}^{k} \Big((a_{i(j)} - \varphi(a_{i(j)})) + \varphi(a_{i(j)}) \Big)$$

to reduce any mixed moment in $\mathcal{S}_1 \cup \mathcal{S}_2$ into a polynomial of pure moments in $\mathcal{S}_1$ and $\mathcal{S}_2$. Of course, freeness can also be characterized by precisely such a universal polynomial relationship between the pure moments, but the explicit determination of these polynomials becomes highly intractable for large k. Instead, one can parameterize the moments using a combinatorial gadget known as *cumulants*, which repackage the independence relation in both an elegant and functional form. To this end, let $(\mathcal{NC}(n), \leq)$ denote the poset of non-crossing partitions of $[n]$ with the reversed refinement order and μ the corresponding Möbius function. We write 0_n for the minimal element consisting of singletons and 1_n for the maximal element consisting of a single block.

Definition 2.1.9 (Free cumulants)**.** Let $(\mathcal{A}, \varphi)$ be a NC probability space. For a partition $\pi \in \mathcal{NC}(n)$, we define the multilinear functional $\varphi_\pi : \mathcal{A}^n \to \mathbb{C}$ by

$$\varphi_\pi[a_1, \ldots, a_n] = \prod_{B \in \pi} \varphi(B)[a_1, \ldots, a_n],$$

where a block $B = (i_1 < \cdots < i_k)$ defines a partial product

$$\varphi(B)[a_1, \ldots, a_n] = \varphi\bigg(\prod_{j=1}^{k} a_{i(j)}\bigg).$$

The *free cumulant* κ_π is the multilinear functional $\kappa_\pi : \mathcal{A}^n \to \mathbb{C}$ given by the Möbius convolution

$$\kappa_\pi[a_1, \ldots, a_n] = \sum_{\substack{\sigma \in \mathcal{NC}(n) \\ \text{s.t. } \sigma \leq \pi}} \varphi_\sigma[a_1, \ldots, a_n]\mu(\sigma, \pi).$$

One recovers the expectation from the free cumulants via the Möbius inversion

$$\varphi_\pi[a_1, \ldots, a_n] = \sum_{\substack{\sigma \in \mathcal{NC}(n) \\ \text{s.t. } \sigma \leq \pi}} \kappa_\sigma[a_1, \ldots, a_n].$$

The free cumulants satisfy many other desirable properties [NS06], far too many to list here. Instead, we will make do with just a few. To begin, we write $\kappa_n(a_1, \ldots, a_n) := \kappa_{1_n}[a_1, \ldots, a_n]$. Notably, the free cumulants are *multiplicative*, i.e.,

$$\kappa_\pi[a_1, \ldots, a_n] = \prod_{B \in \pi} \kappa(B)[a_1, \ldots, a_n],$$

where $B = (i_1 < \cdots < i_k)$ is a block as before and $\kappa(B)[a_1, \ldots, a_n] = \kappa_k(a_{i(1)}, \ldots, a_{i(k)})$. Thus, the full set of free cumulants $(\kappa_\pi)_{\pi \in \mathcal{NC}(n), n \in \mathbb{N}}$ can be recovered from $(\kappa_n)_{n \in \mathbb{N}}$. Furthermore, the vanishing of mixed cumulants characterizes free independence!

Proposition 2.1.10. *Let $(\mathcal{A}, \varphi)$ be a NC probability space and $(\kappa_n)_{n \in \mathbb{N}}$ its free cumulant sequence. For unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$, the following two conditions are equivalent:*

(i) The $(\mathcal{A}_i)_{i \in I}$ are freely independent;

(ii) For any $n \geq 2$ and $a_1, \ldots, a_n$ such that $a_j \in \mathcal{A}_{i(j)}$,

$$\exists i(j) \neq i(k) \implies \kappa_n(a_1, \ldots, a_n) = 0.$$

Proof. See Theorem 11.16 in [NS06]. ∎

Naturally, the motivation for the free cumulant construction comes from the theory of classical cumulants. More precisely, in the classical construction, one works with general partitions $\mathcal{P}(n)$ instead of non-crossing partitions $\mathcal{NC}(n)$. Indeed, this changeover from general partitions to non-crossing partitions when passing from the classical framework to the free framework becomes a recurring theme [NS06]. In the traffic framework, one instead considers partitions of vertices $\mathcal{P}(V)$ of a graph $G = (V, E)$, which allows for an intertwining of the two notions. In anticipation of this discussion in Section 3.3, we recall the notion of a NC conditional expectation.

Definition 2.1.11 (Conditional expectation)**.** Let $\mathcal{B} \subset \mathcal{A}$ be a unital $*$-subalgebra of a $*$-probability space $(\mathcal{A}, \varphi)$. A linear map $\mathscr{E} : \mathcal{A} \to \mathcal{B}$ is said to be a *conditional expectation (onto $\mathcal{B}$)* if it satisfies:

(i) $\varphi(\mathscr{E}(a)) = \varphi(a)$ for every $a \in \mathcal{A}$;
(ii) $\mathscr{E}(b) = b$ for every $b \in \mathcal{B}$;
(iii) $\mathscr{E}(a^*) = \mathscr{E}(a)^*$ for every $a \in \mathcal{A}$.
(iv) $\mathscr{E}(b_1 a b_2) = b_1 \mathscr{E}(a) b_2$ for every $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

Example 2.1.12. Consider the $*$-probability space $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C}), \mathbb{E}\frac{1}{N}\operatorname{tr})$. The projection onto the diagonal $\Delta : L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C}) \to L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Diag}_N(\mathbb{C})$,

$$X_{ij} \otimes e_{ij} \mapsto \delta_{ij}(X_{ij} \otimes e_{ij}),$$

defines a conditional expectation onto the (commutative) unital $*$-subalgebra of diagonal matrices. We come back to this example shortly. ◊

We assume hereafter that the state φ is tracial; however, in general, we do not assume that φ is faithful. The traciality of φ implies that the subspace of degenerate elements

$$\mathcal{D} = \{a \in \mathcal{A} : \varphi(ab) = 0 \text{ for every } b \in \mathcal{A}\}$$

further has the structure of a two-sided $*$-ideal. We say that two random variables $a, b \in \mathcal{A}$ are *equal up to degeneracy* if $a - b \in \mathcal{D}$, for which we use the notation $a \equiv b \pmod{\varphi}$. This equivalence conforms with our intuition from the classical setting: for example, if $a \equiv b \pmod{\varphi}$, then one can interchange a and b in a joint $*$-distribution or free cumulant without consequence.

Finally, we recall one of the most basic (and frequently appearing) families of random variables in the NC framework.

Definition 2.1.13 (Semicircular family)**.** Let $(\beta_{i,j})_{i,j\in I}$ be a positive definite matrix. A family of self-adjoint random variables $(s_i)_{i\in I}$ in a $*$-probability space $(\mathcal{A},\varphi)$ is said to be a *semicircular family of covariance* $(\beta_{i,j})_{i,j\in I}$ if for any $n \geq 1$ and indices $i(1),\dots,i(n) \in I$,

$$\varphi(s_{i(1)}\cdots s_{i(n)}) = \sum_{\pi\in\mathcal{NC}_2(n)} \kappa_\pi[s_{i(1)},\dots,s_{i(n)}],$$

where $\mathcal{NC}_2(n)$ denotes the set of non-crossing pair partitions of $[n]$ and

$$\kappa_\pi[s_{i(1)},\dots,s_{i(n)}] = \prod_{(j,k)\in B} \beta_{i(j),i(k)}.$$

In particular, if $(\beta_{i,j})_{i,j\in I}$ is the identity matrix, then the $(s_i)_{i\in I}$ are freely independent standard semicircular random variables (a so-called *semicircular system*).

2.2 The operad of graph operations

An *operad* is an algebraic structure that formalizes the interaction of many natural mathematical operations in a unified setting. May first introduced the notion of an operad in a purely topological context [May72], but this abstract framework is now employed in a number of different fields to great success [MSS02]. We recall the primary example of interest for our purposes, the operad of graph operations [Mal].

Definition 2.2.1 (Graph operation)**.** A *multidigraph* $G = (V, E, \mathrm{src}, \mathrm{tar})$ consists of a nonempty set of vertices V, a set of edges E, and a pair of maps $\mathrm{src}, \mathrm{tar} : E \to V$ specifying the *source* $\mathrm{src}(e)$ and *target* $\mathrm{tar}(e)$ of each edge $e \in E$. Such a graph G is said to be *bi-rooted* if it has a pair of distinguished vertices $(v_{\mathrm{in}}, v_{\mathrm{out}}) \in V^2$, the coordinates of which we term the *input* and the *output* respectively. A *graph operation* is a finite, connected, bi-rooted multidigraph $g = (V, E, \mathrm{src}, \mathrm{tar}, v_{\mathrm{in}}, v_{\mathrm{out}}, o)$ together with an ordering of its edges $o : E \overset{\sim}{\to} [\#(E)]$. We interpret $g = g(\cdot_1,\dots,\cdot_K)$ as a function of $K = \#(E)$ arguments, one for each edge $e \in E$, with coordinates specified by the ordering o. In particular, we call such a graph g a *K-graph operation*. We write $\mathcal{G}_K$ for the set of all K-graph operations and $\mathcal{G} = \bigcup_{K\geq 0} \mathcal{G}_K$ for the graded set of all graph operations.

Example 2.2.2. We introduce some conventions for depicting graph operations that the reader will hopefully discern from the examples below. In particular, we enumerate

$$\mathcal{G}_0 = \left\{ \underset{\text{in/out}}{\cdot} \right\}$$

and

$$\mathcal{G}_1 = \left\{ \underset{\text{out}}{\cdot} \overset{1}{\leftarrow} \underset{\text{in}}{\cdot} \quad , \quad \underset{\text{out}}{\cdot} \overset{1}{\rightarrow} \underset{\text{in}}{\cdot} \quad , \quad \begin{matrix} \cdot \\ \uparrow 1 \\ \underset{\text{in/out}}{\cdot} \end{matrix} \quad , \quad \begin{matrix} \cdot \\ \downarrow 1 \\ \underset{\text{in/out}}{\cdot} \end{matrix} \quad , \quad \begin{matrix} 1 \\ \circlearrowleft \\ \underset{\text{in/out}}{\cdot} \end{matrix} \right\}.$$

$\diamondsuit$

When there is little ambiguity, we omit the ordering of the edges in the figure. For instance, this can done in the examples above. For a slightly less trivial example, consider the graph operation $g(\cdot_1, \cdot_2) = \underset{\text{out}}{\cdot} \leftleftarrows \underset{\text{in}}{\cdot}$. Nevertheless, we emphasize the importance of the ordering in distinguishing distinct graph operations: for example,

$$\underset{\text{out}}{\cdot} \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \underset{\text{in}}{\cdot} \quad \neq \quad \underset{\text{out}}{\cdot} \overset{2}{\leftarrow} \cdot \overset{1}{\leftarrow} \underset{\text{in}}{\cdot}$$

One can define an action of the symmetric group on the graph operations by permuting the ordering of the edges. Formally, for a permutation $\sigma \in \mathfrak{S}_K$ and a K-graph operation g as before, the permuted graph operation follows as $g_\sigma = (V, E, \mathrm{src}, \mathrm{tar}, v_{\mathrm{in}}, v_{\mathrm{out}}, \sigma \circ o)$. Under this action, the set of graph operations $\mathcal{G}$ carries the structure of a symmetric operad, namely,

Definition 2.2.3 (Operad of graph operations)**.** Let $g = (V, E, \mathrm{src}, \mathrm{tar}, v_{\mathrm{in}}, v_{\mathrm{out}}, o)$ be a K-graph operation. For a K-tuple of graph operations $(g_1, \ldots, g_K)$ with

$$g_i = (V_i, E_i, \mathrm{src}_i, \mathrm{tar}_i, v_{\mathrm{in}}^{(i)}, v_{\mathrm{out}}^{(i)}, o_i) \in \mathcal{G}_{L_i},$$

we define the *composite graph operation*

$$g(g_1, \ldots, g_K) \in \mathcal{G}_{\sum_{i=1}^K L_i}$$

by substitution. Formally, one removes each edge $e \in E$ and installs a copy of $g_{o(e)}$ in its place by identifying the vertices $\mathrm{src}(e) \sim v_{\mathrm{in}}^{(o(e))}$ and $\mathrm{tar}(e) \sim v_{\mathrm{out}}^{(o(e))}$. The composite graph operation $g(g_1, \ldots, g_K)$ then inherits the obvious ordering and direction of its edges.

The reader can easily verify that this composition is *associative*, i.e.,

$$g(g_1(g_{1,1}, \ldots, g_{1,L_1}), \ldots, g_K(g_{K,1}, \ldots, g_{K,L_K})) = (g(g_1, \ldots, g_K))(g_{1,1}, \ldots, g_{1,L_1}, \ldots, g_{K,1}, \ldots, g_{K,L_K}),$$

and *equivariant*, i.e.,

$$g_\sigma(g_1, \ldots, g_K) = (g(g_{\sigma(1)}, \ldots, g_{\sigma(K)}))_{\pi(\sigma)}, \qquad \forall \sigma \in \mathfrak{S}_K,$$

and

$$g((g_1)_{\sigma_1}, \ldots, (g_K)_{\sigma_K}) = g(g_1, \ldots, g_K)_{\sigma_1 \oplus \cdots \oplus \sigma_K}, \qquad \forall \sigma_i \in \mathfrak{S}_{L_i},$$

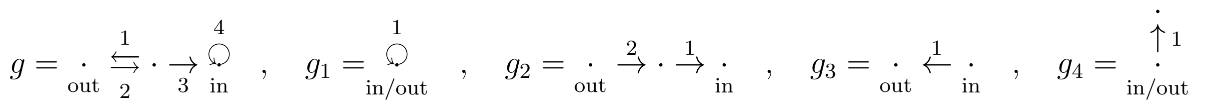
where $\oplus$ denotes the direct sum of permutations and

$$\pi(\sigma) = \prod_{i=1}^K \begin{pmatrix} \sum_{j=1}^{\sigma^{-1}(i)-1} L_{\sigma(j)} + 1 & \sum_{j=1}^{\sigma^{-1}(i)-1} L_{\sigma(j)} + 2 & \cdots & \sum_{j=1}^{\sigma^{-1}(i)} L_{\sigma(j)} \\ \sum_{j=1}^{i-1} L_j + 1 & \sum_{j=1}^{i-1} L_j + 2 & \cdots & \sum_{j=1}^{i} L_j \end{pmatrix} \in \mathfrak{S}_{\sum_{i=1}^K L_i}.$$

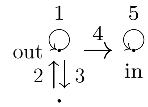
The graph operation $\mathrm{id}_{\mathcal{G}} = \underset{\text{out}}{\cdot} \leftarrow \underset{\text{in}}{\cdot} \in \mathcal{G}_1$ is the *unit* for this composition, namely,

$$\mathrm{id}_{\mathcal{G}}(g) = g(\mathrm{id}_{\mathcal{G}}, \ldots, \mathrm{id}_{\mathcal{G}}) = g, \qquad \forall g \in \mathcal{G}.$$

Example 2.2.4. If



then

$$g(g_1, g_2, g_3, g_4) =$$ 

$\Diamond$

Definition 2.2.5 ($\mathcal{G}$-algebra)**.** A *$\mathcal{G}$-algebra* is a complex vector space $\mathcal{A}$ together with an action $(Z_g)_{g\in\mathcal{G}}$ of the operad of graph operations. By this, we mean that each graph operation $g \in \mathcal{G}_K \subset \mathcal{G}$ defines a linear map

$$Z_g : \mathcal{A}^{\otimes K} \to \mathcal{A}$$

satisfying the following properties:

(i) (Associativity) The action of the composite graph operation $Z_{g(g_1,\dots,g_K)}$ factors through the action of the graph operation Z_g via

$$Z_{g(g_1,\dots,g_K)} = Z_g \circ (Z_{g_1} \otimes \cdots \otimes Z_{g_K});$$

(ii) (Equivariance) The ordering of the edges only plays a formal role in defining the action of a graph operation by assigning the location of each argument to a specific edge. Any equivalent assignment of edge locations produces the same action, i.e.,

$$Z_{g_\sigma} = Z_g \circ P_\sigma, \qquad \forall \sigma \in \mathfrak{S}_K,$$

where

$$P_\sigma : \mathcal{A}^{\otimes K} \to \mathcal{A}^{\otimes K}, \qquad a_1 \otimes \cdots \otimes a_K \mapsto a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(K)};$$

(iii) (Identity) The operad unit $\mathrm{id}_{\mathcal{G}} = \underset{\text{out}}{\cdot} \leftarrow \underset{\text{in}}{\cdot}$ defines the stable action

$$Z_{\mathrm{id}_{\mathcal{G}}} : \mathcal{A} \to \mathcal{A}, \qquad a \mapsto a.$$

A *sub-$\mathcal{G}$-algebra* $\mathcal{B}$ is a subspace $\mathcal{B} \subset \mathcal{A}$ that is closed under the action of the graph operations. Similarly, the *$\mathcal{G}$-algebra generated by a subset* $\mathcal{S} \subset \mathcal{A}$ is the smallest $\mathcal{G}$-algebra containing $\mathcal{S}$, which can be characterized as the span of $\bigcup_{K\geq 0} \bigcup_{g\in\mathcal{G}_K} Z_g(\mathcal{S}^{\otimes K})$.

A *morphism of $\mathcal{G}$-algebras* (or *$\mathcal{G}$-morphism*) is a linear map $f : \mathcal{A} \to \mathcal{B}$ between $\mathcal{G}$-algebras that respects the action of the graph operations, namely,

$$f \circ Z_g = Z_g(f \otimes \cdots \otimes f), \qquad \forall g \in \mathcal{G}.$$

Symbolically, we represent the action Z_g on a K-tuple $a_1 \otimes \cdots \otimes a_K$ by placing each argument a_i in the location prescribed by the ordering. For example, we can formulate the identity axiom (iii) as

$$\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} = a, \qquad \forall a \in \mathcal{A}.$$

A $\mathcal{G}$-algebra structure on $\mathcal{A}$ defines a unital $\mathbb{C}$-algebra structure on $\mathcal{A}$ via the product

$$a \cdot_{\mathcal{G}} b := \underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \cdot \overset{b}{\leftarrow} \underset{\text{in}}{\cdot}\,. \tag{2.3}$$

In particular, the unit $1_{\mathcal{A}} = Z_{\underset{\text{in/out}}{\cdot}}(1)$ comes from the action $Z_{\underset{\text{in/out}}{\cdot}} : \mathbb{C} \to \mathcal{A}$ of the trivial graph operation $\underset{\text{in/out}}{\cdot} \in \mathcal{G}_0$. The reader should verify that the axioms of a $\mathcal{G}$-algebra ensure the well-definedness of this unital $\mathbb{C}$-algebra structure. When referring to a $\mathcal{G}$-algebra, we implicitly assume the additional $\mathbb{C}$-algebra structure defined above. As such, a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ also defines a morphism of unital $\mathbb{C}$-algebras.

One can further define a pair of natural involutions on the operad of graph operations through role reversals. In particular, for a graph operation $g = (V, E, \text{src}, \text{tar}, v_{\text{in}}, v_{\text{out}}, o)$, one obtains the *transpose* $g^\intercal = (V, E, \text{src}, \text{tar}, v_{\text{out}}, v_{\text{in}}, o)$ by interchanging the input and output. Similarly, one obtains the *flip* $g_\to = (V, E, \text{tar}, \text{src}, v_{\text{in}}, v_{\text{out}}, o)$ by interchanging the maps src and tar, reversing the direction of each edge $e \in E$. If our $\mathcal{G}$-algebra also comes equipped with an involution, then we can further ask that these operations obey a natural adjoint relation. This leads us to

Definition 2.2.6 ($\mathcal{G}^*$-algebra)**.** Let $\mathcal{A}$ be a $\mathcal{G}$-algebra with a conjugate linear involution $* : \mathcal{A} \to \mathcal{A}$. We say that $\mathcal{A}$ further has the structure of a *$\mathcal{G}^*$-algebra* if

$$Z_{g^\intercal_\to} \circ (* \otimes \cdots \otimes *) = * \circ Z_g, \qquad \forall g \in \mathcal{G}.$$

A *sub-$\mathcal{G}^*$-algebra* $\mathcal{B}$ is a $*$-subspace $\mathcal{B} \subset \mathcal{A}$ that is closed under the action of the graph operations. The *$\mathcal{G}^*$-algebra generated by a subset* $\mathcal{S} \subset \mathcal{A}$ is the smallest $\mathcal{G}^*$-algebra containing $\mathcal{S}$, which can characterized as the span of $\bigcup_{K \geq 0} \bigcup_{g \in \mathcal{G}_K} Z_g((\mathcal{S} \cup \mathcal{S}^*)^{\otimes K})$.

A *morphism of $\mathcal{G}^*$-algebras* (or *$\mathcal{G}^*$-morphism*) is a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ between $\mathcal{G}^*$-algebras that further respects the involution operations, namely,

$$*_{\mathcal{B}} \circ f = f \circ *_{\mathcal{A}}.$$

A $\mathcal{G}^*$-algebra structure extends the involution $* : \mathcal{A} \to \mathcal{A}$ to a conjugate linear anti-isomorphism for the product defined by (2.3). Thus, as before, when referring to a $\mathcal{G}^*$-algebra, we implicitly assume the additional $*$-algebra structure so defined. In particular, a $\mathcal{G}^*$-morphism $f : \mathcal{A} \to \mathcal{B}$ also defines a morphism of unital $*$-algebras.

Our first example provides a useful backdrop for distributional considerations.

Example 2.2.7 ($*$-graph polynomial)**.** Let $\mathbf{x} = (x_i)_{i \in I}$ be a set of indeterminates. A *$*$-graph monomial $t = (G, \gamma, \varepsilon)$ in* $\mathbf{x}$ is a bi-rooted multidigraph $G = (V, E, \text{src}, \text{tar}, v_{\text{in}}, v_{\text{out}})$

with edge labels $\gamma : E \to I$ and $\varepsilon : E \to \{1, *\}$ in $\langle \mathbf{x}, \mathbf{x}^* \rangle$. We define the *transpose* $t^\intercal$ as before, interchanging the input and the output. We also define the *conjugate* $\bar{t}$ as the $*$-flip $\bar{t} = (V, E, \mathrm{tar}, \mathrm{src}, v_{\mathrm{in}}, v_{\mathrm{out}}, \gamma, \varepsilon^*)$, which flips both the direction and the $*$-label of each edge. Finally, we define the *adjoint* t^* as the conjugate transpose $t^* = \bar{t}^\intercal$. We write $\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ for the set of all $*$-graph monomials and $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ for the complex vector space spanned by $\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$, the so-called $*$-*graph polynomials*. We extend the adjoint operation to a conjugate linear involution $* : \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle \to \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$.

The reader should verify that the $*$-graph polynomials $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ form a $\mathcal{G}^*$-algebra under the action of composition: for $*$-graph monomials $t_1, \ldots, t_K$, we define $Z_g(t_1 \otimes \cdots \otimes t_K)$ as the $*$-graph monomial obtained by concatenating the t_i according to g as in the composite graph construction $g(g_1, \ldots, g_K)$. The $*$-graph polynomials generalize the usual $*$-polynomials. In particular, one obtains an embedding of unital $*$-algebras $\eta : \mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle \hookrightarrow \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ via

$$x_i \mapsto \underset{\text{out}}{\cdot} \xleftarrow{x_i} \underset{\text{in}}{\cdot} \qquad \text{and} \qquad 1 \mapsto \underset{\text{in/out}}{\cdot} . \tag{2.4}$$

$\lozenge$

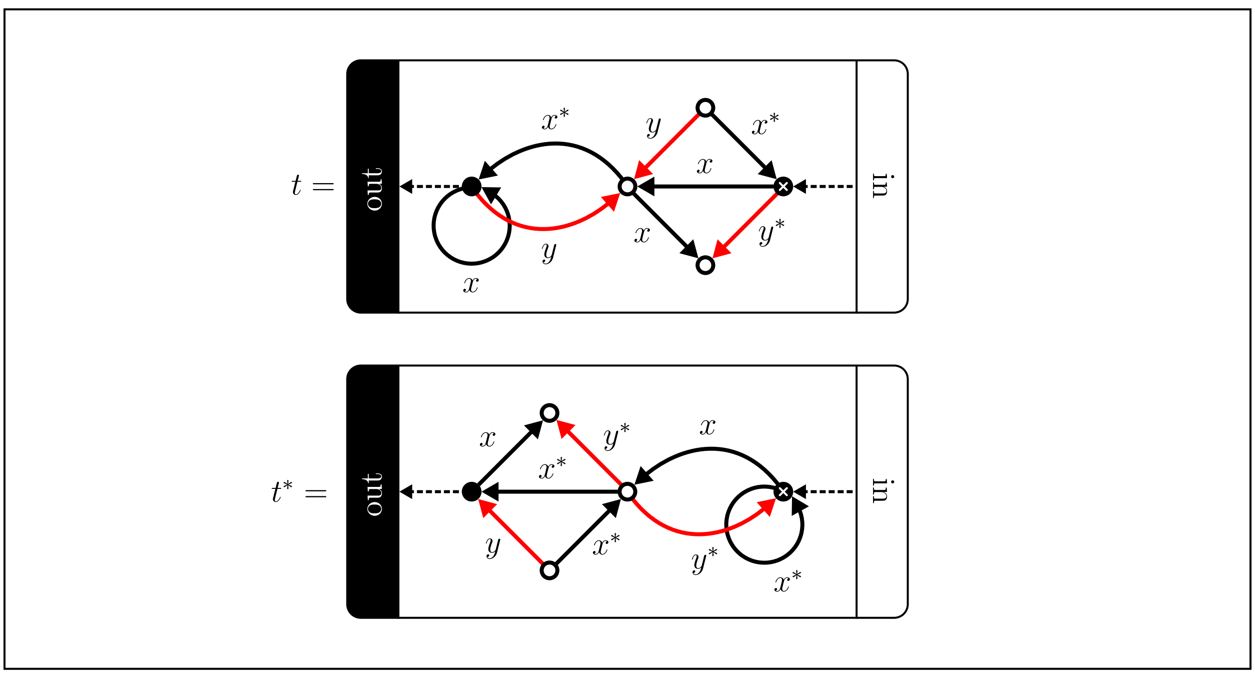


Figure 2.1: An example of a $*$-graph monomial t (in the indeterminates $\{x, y\}$) and its adjoint t^*. We adopt the convention in [Mal], plotting the graph G inside of a box for which we then specify two sides to orient the distinguished vertices, right to left (hence the dashed lines). We plot the distinguished vertices in solid black (the input with a cross) and use different colors for edges labeled by different indeterminates.

Earlier notions of our next example appear in the work [Jon] of Jones on planar algebras and [MS12] of Mingo and Speicher on sums of products of matrix entries.

Example 2.2.8 (Graph of matrices)**.** Let $\mathrm{Mat}_N(\mathbb{C})$ denote the $*$-algebra of $N \times N$ matrices over $\mathbb{C}$. For a K-graph operation g, we define the *graph of matrices*

$$Z_g : \mathrm{Mat}_N(\mathbb{C})^{\otimes K} \to \mathrm{Mat}_N(\mathbb{C})$$

by the coordinate formula

$$Z_g(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})(i,j) = \sum_{\substack{\phi : V \to [N] \text{ s.t.} \\ \phi(v_{\mathrm{out}})=i, \ \phi(v_{\mathrm{in}})=j}} \prod_{e \in E} \mathbf{A}_N^{(o(e))}(\phi(\mathrm{tar}(e)), \phi(\mathrm{src}(e))). \tag{2.5}$$

For notational convenience, we often write $\phi(e) := (\phi(\mathrm{tar}(e)), \phi(\mathrm{src}(e)))$. The action (2.5) defines a $\mathcal{G}^*$-algebra structure on $\mathrm{Mat}_N(\mathbb{C})$ that recovers the usual matrix multiplication:

$$\mathbf{A}_N \cdot_{\mathcal{G}} \mathbf{B}_N = \underset{\mathrm{out}}{\cdot} \xleftarrow{\mathbf{A}_N} \cdot \xleftarrow{\mathbf{B}_N} \underset{\mathrm{in}}{\cdot} = \mathbf{A}_N\mathbf{B}_N.$$

The action of the graph operations also produces matrices of additional linear algebraic structure: for example,

1. (Transpose) For any $g \in \mathcal{G}$,
$$Z_{g^\intercal} = Z_g^\intercal,$$
where on the right-hand side of the equality we have used the same notation $\intercal$ for the usual matrix transpose. In particular,
$$\underset{\mathrm{out}}{\cdot} \xrightarrow{\mathbf{A}_N} \underset{\mathrm{in}}{\cdot} = \mathbf{A}_N^\intercal;$$

2. (Hadamard-Schur product) Parallel edges correspond to entrywise products. In particular,
$$\underset{\mathrm{out}}{\cdot} \overset{\mathbf{A}_N}{\underset{\mathbf{B}_N}{\leftleftarrows}} \underset{\mathrm{in}}{\cdot} = \mathbf{A}_N \circ \mathbf{B}_N = \big(\mathbf{A}_N(i,j)\mathbf{B}_N(i,j)\big)_{1 \le i,j \le N};$$

3. (Diagonal) The action of a graph operation with $v_{\mathrm{in}} = v_{\mathrm{out}}$ produces a diagonal matrix. In particular,
$$\underset{\mathrm{in/out}}{\overset{\mathbf{A}_N}{\circlearrowleft}} = \Delta(\mathbf{A}_N) = \big(\mathbf{A}_N(i,i)\big)_{1 \le i \le N};$$

4. (Degree) Similarly, one can obtain the diagonal matrix of row sums (resp., column sums) as
$$\begin{matrix} \cdot \\ \downarrow \mathbf{A}_N \\ \underset{\mathrm{in/out}}{\cdot} \end{matrix} = \mathrm{rDeg}(\mathbf{A}_N) = \left(\sum_{j=1}^N \mathbf{A}_N(i,j)\right)_{1 \le i \le N}$$

$$\left(\text{resp.,} \quad \underset{\text{in/out}}{\overset{\cdot}{\uparrow}} \mathbf{A}_N = \mathrm{cDeg}(\mathbf{A}_N) = \left(\sum_{j=1}^{N} \mathbf{A}_N(j,i)\right)_{1 \leq i \leq N}\right).$$

◊

Remark 2.2.9. The trace tr of a graph of matrices $Z_g(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})$ only depends on the graph operation $g = (V, E, \mathrm{src}, \mathrm{tar}, v_{\mathrm{in}}, v_{\mathrm{out}}, o)$ up to the *unrooted* graph

$$T = \widetilde{\Delta}(g) := (\widetilde{V}, E, \mathrm{src}, \mathrm{tar}, o)$$

obtained from g by identifying the input and the output $v_{\mathrm{in}} \sim v_{\mathrm{out}}$ and forgetting their distinguished roles. Indeed,

$$\begin{aligned}
\mathrm{tr}\left[Z_g(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})\right] &= \sum_{i=1}^{N} Z_g(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})(i,i) \\
&= \sum_{i=1}^{N} \sum_{\substack{\phi: V \to [N] \text{ s.t.} \\ \phi(v_{\mathrm{out}}) = \phi(v_{\mathrm{in}}) = i}} \prod_{e \in E} \mathbf{A}_N^{(o(e))}(\phi(e)) \\
&= \sum_{\phi: \widetilde{V} \to [N]} \prod_{e \in E} \mathbf{A}_N^{(o(e))}(\phi(e)) \\
&=: \mathrm{tr}\left[T(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})\right],
\end{aligned}$$

where in the last equality we define $\mathrm{tr}\left[T(\mathbf{A}_N^{(1)} \otimes \cdots \otimes \mathbf{A}_N^{(K)})\right]$ as the appropriate sum. This observation will motivate the traffic space construction in the next section.

Of course, the example above applies equally well to *random matrices* with the appropriate modifications, particularly $L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathrm{Mat}_N(\mathbb{C})$ with the expected trace $\mathbb{E}\frac{1}{N}\,\mathrm{tr}$. We adapt the notation from the matricial setting to general $\mathcal{G}$-algebras: for example,

$$a^{\intercal} = \underset{\text{out}}{\cdot} \overset{a}{\rightarrow} \underset{\text{in}}{\cdot} \quad , \quad a \circ b = \underset{\text{out}}{\cdot} \underset{b}{\overset{a}{\leftleftarrows}} \underset{\text{in}}{\cdot} \quad , \quad \Delta(a) = \underset{\text{in/out}}{\overset{a}{\circlearrowleft}} \quad ,$$

and so forth. In particular, the action Δ defines a projection $\Delta = \Delta^2$ on a $\mathcal{G}$-algebra (resp., $\mathcal{G}^*$-algebra) $\mathcal{A}$ whose image $\Delta(\mathcal{A})$ is a commutative sub-$\mathcal{G}$-algebra (resp., sub-$\mathcal{G}^*$-algebra), the so-called *diagonal (sub)algebra of* $\mathcal{A}$.

2.3 Traffic probability

We are almost ready to define a traffic space. But first, we will need a few more definitions. To capture the intuition behind Remark 2.2.9, we formalize

Definition 2.3.1 (n-graph monomial)**.** An *n-graph monomial* $t = (G, \gamma, \mathbf{v})$ *in* $\mathcal{S}$ is a finite, connected multidigraph $G = (V, E, \mathrm{src}, \mathrm{tar})$ with edge labels $\gamma : E \to \mathcal{S}$ and an n-tuple of distinguished (not necessarily distinct) vertices $\mathbf{v} = (v_1, \dots, v_n) \in V^n$. We write $\mathcal{G}^{(n)}\langle\mathcal{S}\rangle$ for the set of all n-graph monomials in $\mathcal{S}$ and $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{S}\rangle$ for the complex vector space spanned by $\mathcal{G}^{(n)}\langle\mathcal{S}\rangle$, the so-called *$n$-graph polynomials*.

Similarly, an *n^*-graph monomial* $t = (G, \gamma, \varepsilon, \mathbf{v})$ *in* $\mathcal{S}$ is an n-graph monomial t in $\mathcal{S}$ with the additional information of $*$-labels $\varepsilon : E \to \{1, *\}$. We write $\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle$ for the set of all n^*-graph monomials in $\mathcal{S}$ and $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle$ for the complex vector space spanned by $\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle$, the so-called *$n^*$-graph polynomials*. We define the *$*$-flip* of an n^*-graph monomial as before, namely, $\bar{t} = (V, E, \mathrm{tar}, \mathrm{src}, \gamma, \varepsilon^*, \mathbf{v})$ and extend this operation to a conjugate linear involution on $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle$.

We highlight the special case of $n = 0$, in which there are no distinguished vertices. We refer to 0-graph monomials (resp., 0^*-graph monomials) as *test graphs* (resp., *$*$-test graphs*) and use the notation

$$T = (G, \gamma) \in \mathcal{T}\langle\mathcal{S}\rangle = \mathcal{G}^{(0)}\langle\mathcal{S}\rangle \qquad (\text{resp., } T = (G, \gamma, \varepsilon) \in \mathcal{T}\langle\mathcal{S}, \mathcal{S}^*\rangle = \mathcal{G}^{(0)}\langle\mathcal{S}, \mathcal{S}^*\rangle)$$

as opposed to the lower case t.

For $n \geq 1$, we define the bilinear *gluing map*

$$\bowtie_n : \mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle \times \mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{S}, \mathcal{S}^*\rangle \to \mathbb{C}\mathcal{T}\langle\mathcal{S}, \mathcal{S}^*\rangle$$

on pairs of n^*-graph monomials $(t_1, t_2) = ((G_1, \gamma_1, \varepsilon_1, \mathbf{v}), (G_2, \gamma_2, \varepsilon_2, \mathbf{w}))$ by identifying the distinguished vertices $\mathbf{v}$ and $\mathbf{w}$ coordinatewise $v_i \sim w_i$ and then forgetting their distinguished roles.

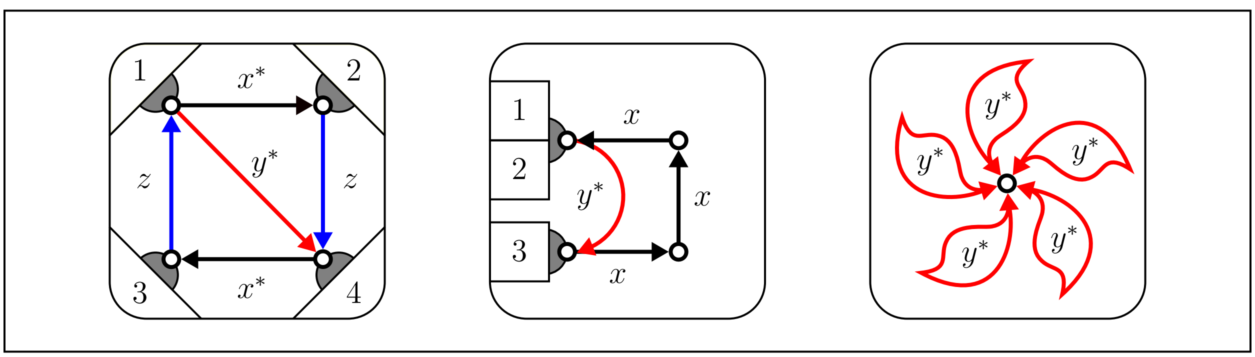


Figure 2.2: Examples of n^*-graph monomials. In the last example, we have a $*$-test graph.

The reader will note that a $*$-graph polynomial in $\mathbf{x}$ (Example 2.2.7) is just an n^*-graph polynomial for $n = 2$. We often consider n^*-graph monomials in a set of indeterminates $\mathbf{x} = (x_i)_{i \in I}$, in which case it is important to separate the labels γ and ε. At the same time, we will also consider n-graph monomials in sets that already carry a defined involution $*$ (for example, a $\mathcal{G}^*$-algebra). In this case, one can still define the $*$-flip of an n-graph monomial by conjugating the labels $\bar{t} = (V, E, \mathrm{tar}, \mathrm{src}, \gamma^*, \mathbf{v})$. We can now finally set about

Definition 2.3.2 (Traffic space)**.** An *algebraic traffic space* is a pair $(\mathcal{A}, \tau)$ consisting of a $\mathcal{G}$-algebra $\mathcal{A}$ together with a $\mathcal{G}$-compatible linear functional $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ as follows:

(i) (Unity) The trivial test graph consisting of a single isolated vertex evaluates to 1,

$$\tau\big[\,\cdot\,\big] = 1;$$

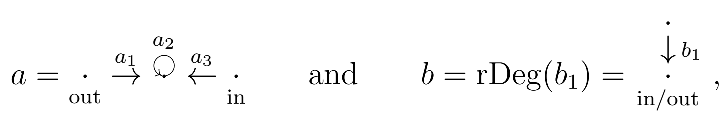
(ii) (Substitution) The functional τ respects the $\mathcal{G}$-action: in particular, if a test graph $T = (V, E, \mathrm{src}, \mathrm{tar}, \gamma) \in \mathcal{T}\langle\mathcal{A}\rangle$ has an edge $e \in E$ with label

$$\gamma(e) = a = Z_g(a_1 \otimes \cdots \otimes a_K),$$

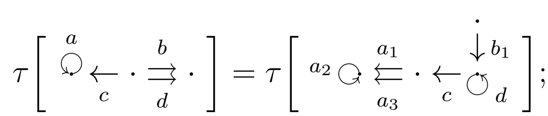
then τ returns the same value on the test graph $T_{e,Z_g(a_1\otimes\cdots\otimes a_K)}$ obtained from T by substituting the graph represented by the action $Z_g(a_1 \otimes \cdots \otimes a_K)$ in for the edge e. Formally, one removes the edge $e \in E$ and installs a copy of the graph $Z_g(a_1 \otimes \cdots \otimes a_K)$ in its place by identifying the vertices $\mathrm{src}(e) \sim v_{\mathrm{in}}$ and $\mathrm{tar}(e) \sim v_{\mathrm{out}}$, in which case

$$\tau\big[T\big] = \tau\big[T_{e,Z_g(a_1\otimes\cdots\otimes a_K)}\big].$$

For example, if



then



(iii) (Multilinearity) For a fixed test graph, the functional τ is multilinear with respect to the edge labels. Formally, fixing the underlying graph $G = (V, E, \mathrm{src}, \mathrm{tar})$ of a test graph $T = (G, \gamma)$ defines a $\#(E)$-linear function

$$\tau\big[T(\bigtimes_{e\in E} \cdot_e)\big] : \mathcal{A}^E \to \mathbb{C}$$

of the edges E via the labels $\gamma : E \to \mathcal{A}$.

A *traffic space* is pair $(\mathcal{A}, \tau)$ consisting of a $\mathcal{G}^*$-algebra $\mathcal{A}$ together with a $\mathcal{G}^*$-compatible linear functional $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$, namely, in addition to (i)-(iii) above:

(iv) (Positivity) Recall that we can define the $*$-flip for n-graph polynomials in $\mathcal{A}$. The functional τ is positive with respect to the $*$-flip operation via the gluing map $\bowtie_n$:

$$\tau\big[\bowtie_n(\overline{p}, p)\big] \geq 0, \qquad \forall p \in \mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle.$$

Accordingly, we refer to τ as the *traffic state*. For convenience, we use the same term even in the case of an algebraic traffic space, where we do not assume the positivity axiom.

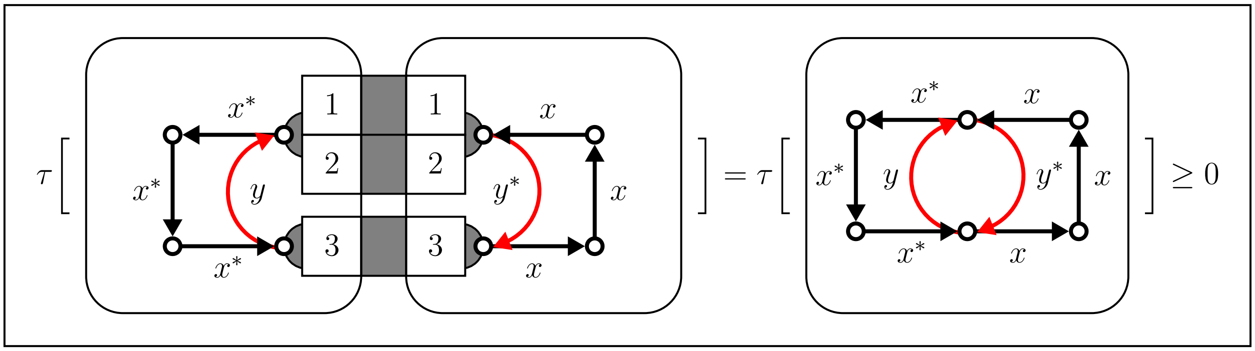


Figure 2.3: The positivity condition for the 3^*-graph monomial from Figure 2.2.

The traffic state τ of an algebraic traffic space $(\mathcal{A}, \tau)$ defines a consistent unital linear functional $\varphi_\tau : \mathcal{A} \to \mathbb{C}$ in the form of

$$\varphi_\tau(a) = \varphi_\tau\Bigg(\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} \Bigg) = \tau\Bigg[\overset{a}{\circlearrowleft} \Bigg].$$

The reader should verify that the axioms of a traffic space ensure the well-definedness of this functional φ_τ. In particular, the substitution axiom implies that the expectation φ_τ of a random variable $a = Z_g(a_1 \otimes \cdots \otimes a_K)$ only depends on the test graph obtained from a by identifying the input of g with the output of g and forgetting their distinguished roles. In the notation of Remark 2.2.9, this amounts to

$$\varphi_\tau(a) = \varphi_\tau(\Delta(a)) = \tau\big[\widetilde{\Delta}(a)\big].$$

For example, this implies that the expectation φ_τ is necessarily tracial. Indeed,

$$\varphi_\tau(ab) = \varphi_\tau\Bigg(\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \cdot \overset{b}{\leftarrow} \underset{\text{in}}{\cdot} \Bigg) = \tau\Bigg[\cdot \underset{b}{\overset{a}{\rightleftarrows}} \cdot \Bigg] = \tau\Bigg[\cdot \underset{a}{\overset{b}{\rightleftarrows}} \cdot \Bigg] = \varphi_\tau\Bigg(\underset{\text{out}}{\cdot} \overset{b}{\leftarrow} \cdot \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} \Bigg) = \varphi_\tau(ba).$$

Thus, when referring to an algebraic traffic space $(\mathcal{A}, \tau)$, we implicitly assume the additional tracial NC probability space structure $(\mathcal{A}, \varphi_\tau)$ defined above. When there is little ambiguity, we omit the subscript τ and simply write φ. We refer to elements $a \in \mathcal{A}$ as *traffic random variables* (or simply *traffics*) to emphasize the (algebraic) traffic space structure.

In the case of a traffic space $(\mathcal{A}, \tau)$, the positivity axiom (iv) ensures that the induced trace φ_τ is in fact a state:

$$\varphi_\tau(a^*a) = \varphi_\tau\Bigg(\underset{\text{out}}{\cdot} \overset{a^*}{\leftarrow} \cdot \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} \Bigg) = \tau\Bigg[\cdot \underset{a}{\overset{a^*}{\rightleftarrows}} \cdot \Bigg] = \tau\Bigg[\bowtie_2 \Bigg(\underset{2}{\cdot} \overset{a^*}{\rightarrow} \underset{1}{\cdot}, \underset{2}{\cdot} \overset{a}{\leftarrow} \underset{1}{\cdot} \Bigg) \Bigg] \geq 0.$$

As before, when referring to a traffic space, we implicitly assume the additional tracial $*$-probability space structure $(\mathcal{A}, \varphi_\tau)$ so defined.

Note that the information of the trace φ_τ recovers the traffic state τ. We introduce an additional parameterization of the traffic state, the so-called *injective traffic state*

$$\tau^0 : \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle \to \mathbb{C}, \qquad T \mapsto \sum_{\pi \in \mathcal{P}(V)} \tau\big[T^\pi\big]\mu(0_V, \pi), \tag{2.6}$$

where $(\mathcal{P}(V), \leq)$ is the poset of partitions of V with the reversed refinement order, μ is the corresponding Möbius function, and T^π is the test graph obtained from T by identifying the vertices within each block $B \in \pi$. One recovers the traffic state via the inversion

$$\tau\big[T\big] = \sum_{\pi \in \mathcal{P}(V)} \tau^0\big[T^\pi\big]. \tag{2.7}$$

For example,

$$\varphi_\tau(ab) = \varphi_\tau\left(\underset{\text{out}}{\cdot} \xleftarrow{a} \cdot \xleftarrow{b} \underset{\text{in}}{\cdot} \right) = \tau\left[\cdot \overset{a}{\underset{b}{\rightleftarrows}} \cdot \right] = \tau^0\left[\cdot \overset{a}{\underset{b}{\rightleftarrows}} \cdot \right] + \tau^0\left[\overset{a}{\underset{b}{\circlearrowleft\!\!\circlearrowleft}} \right].$$

The injective traffic state is unital (i) and edge-multilinear (iii), but in general it fails the substitution axiom (ii) (but do see [Mal, Lemma 4.17])

Henceforth, we work in the context of a traffic space $(\mathcal{A}, \tau)$. The notion of a distribution in the traffic setting can pass through any of the three equivalent routes described above. To make this precise, we will need a natural generalization of the usual polynomial evaluation map $\mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle \ni P \mapsto P(\mathbf{a}) \in \mathcal{A}$. In particular, for a family of traffics $\mathbf{a} = (a_i)_{i \in I}$, one can define an evaluation map on the $*$-graph polynomials

$$\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle \ni t \mapsto t(\mathbf{a}) \in \mathcal{A},$$

where $t(\mathbf{a})$ represents the action of the graph operation $Z_g(\mathbf{a})$ with edge labels prescribed by the substitution $x_i \mapsto a_i$. In fact, the image of this evaluation map is precisely the $\mathcal{G}^*$-algebra generated by $\mathbf{a}$. Similarly, one defines an evaluation on the $*$-test graphs

$$\mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle \ni T \mapsto T(\mathbf{a}) \in \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle.$$

As a sanity check, note that we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle & \xrightarrow{\;t(\cdot)\;} & \mathcal{A} & & \\ & & & \searrow^{\varphi_\tau} & \\ \Big\downarrow \widetilde{\Delta} & & & & \mathbb{C} \\ & & & \nearrow_{\tau} & \\ \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle & \xrightarrow[T(\cdot)]{} & \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle & & \end{array} \tag{2.8}$$

This motivates the definition of the traffic distribution (cf. Definition 2.1.5).

Definition 2.3.3 (Traffic distribution)**.** Let $\mathbf{a} = (a_i)_{i\in I}$ be a family of random variables in a traffic space $(\mathcal{A}, \tau)$. The *traffic distribution* of $\mathbf{a}$ is the linear functional

$$\nu_{\mathbf{a}} : \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle \to \mathbb{C}, \qquad T \mapsto \tau\big[T(\mathbf{a})\big].$$

A sequence of families $\mathbf{a}_n = (a_n^{(i)})_{i\in I}$, each living in a traffic space $(\mathcal{A}_n, \tau_n)$, is said to *converge in traffic distribution* to $\mathbf{a}$ if the corresponding traffic distributions $\nu_{\mathbf{a}_n}$ converge pointwise to $\nu_{\mathbf{a}}$, i.e.,

$$\lim_{n\to\infty} \nu_{\mathbf{a}_n}(T) = \nu_{\mathbf{a}}(T), \qquad \forall T \in \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle.$$

At times, it will be more convenient to consider the *injective traffic distribution*

$$\nu^0_{\mathbf{a}} : \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle \to \mathbb{C}, \qquad T \mapsto \tau^0\big[T(\mathbf{a})\big].$$

Of course, the relationships (2.6) and (2.7) imply that convergence in traffic distribution is equivalent to convergence in injective traffic distribution, where the latter notion is defined in the obvious way.

Remark 2.3.4. The diagram (2.8) shows that the information of the traffic distribution $\tau_{\mathbf{a}}$ is equivalent to that of the functional

$$\mu^{\mathcal{G}}_{\mathbf{a}} : \mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^*\rangle \to \mathbb{C}, \qquad t \mapsto \varphi_\tau(t(\mathbf{a})),$$

which itself is a natural generalization of the $*$-distribution $\mu_{\mathbf{a}} : \mathbb{C}\langle \mathbf{x}, \mathbf{x}^*\rangle \to \mathbb{C}$; however, contrary to the functionals $\mu_{\mathbf{a}}$ and $\mu^{\mathcal{G}}_{\mathbf{a}}$, the traffic distribution does not pass through the traffic space $t(\mathbf{a}) \in \mathcal{A}$. Instead, it simply reads off the values of the traffic state τ on test graphs $T(\mathbf{a}) \in \mathcal{T}(\mathcal{A})$. For this reason, we sometimes refer to $\nu_{\mathbf{a}}$ as the *combinatorial distribution* of $\mathbf{a}$.

Remark 2.3.5. One can always find a realization of a traffic convergent sequence $(\mathbf{a}_n)_{n\in\mathbb{N}}$ via the family $\mathbf{x} = (x_i)_{i\in I}$ in the induced traffic space $(\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^*\rangle, \lim_{n\to\infty} \nu_n)$. In particular, the properties (i)-(iv) of a traffic state are preserved in the limit.

Example 2.3.6 (The traffic space of random matrices)**.** As suggested by Remark 2.2.9, the $\mathcal{G}^*$-algebra $\mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}))$ admits a traffic state

$$\tau\big[T\big] = \mathbb{E}\left[\frac{1}{N}\,\mathrm{tr}\,\big[T\big]\right] := \mathbb{E}\left[\frac{1}{N}\sum_{\phi: V\to[N]}\prod_{e\in E}\gamma(e)(\phi(e))\right], \qquad \forall T \in \mathcal{T}\langle \mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}))\rangle,$$

that recovers the trace

$$\varphi_\tau(\mathbf{A}_N) = \mathbb{E}\left[\frac{1}{N}\,\mathrm{tr}(\mathbf{A}_N)\right], \qquad \forall \mathbf{A}_N \in \mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})).$$

The injective traffic state τ^0 admits an explicit formula without reference to the Möbius function in the matricial setting, namely,

$$\tau^0[T] = \mathbb{E}\left[\frac{1}{N}\operatorname{tr}^0[T]\right] := \mathbb{E}\left[\frac{1}{N}\sum_{\substack{\phi:V\to[N] \text{ s.t.} \\ \phi \text{ is injective}}}\prod_{e\in E}\gamma(e)(\phi(e))\right] = \mathbb{E}\left[\frac{1}{N}\sum_{\phi:V\hookrightarrow[N]}\prod_{e\in E}\gamma(e)(\phi(e))\right],$$

whence the name *injective* traffic state. $\diamondsuit$

The traffic framework provides a novel perspective on the spectral analysis of large random matrices, including ensembles traditionally outside of the domain of free probability (e.g., heavy Wigner matrices [Mal17], sparse random graphs [MP], and random band matrices [Au]). In fact, the notion of traffic independence was first discovered in the context of permutation invariant random matrices [Mal]. We review the definition.

Definition 2.3.7 (Graph of colored components)**.** Let $\mathcal{S} = \bigsqcup_{i\in I} S_i$ be a disjoint union of labeling sets S_i, each thought of as a "color" i. For a test graph $T \in \mathcal{T}\langle\mathcal{S}\rangle$, we define the *graph of colored components* $\chi(T)$ as the simple bipartite graph obtained from T as follows. For each $i \in I$, let $(T_{i,\ell})_{\ell=1}^{k(i)}$ denote the connected components of the subgraph of T spanned by the labels S_i so that

$$T_{i,\ell} \in \mathcal{T}\langle S_i\rangle, \qquad \forall \ell \in [k(i)].$$

In other words, the $(T_{i,\ell})_{\ell=1}^{k(i)}$ are the connected components of the test graph T restricted to the color i. Note that $\sum_{i\in I} k(i) < \infty$ since T is a finite graph. We write $(v_m)_{m=1}^n$ for the vertices of T that belong to more than one of the components $(T_{i,\ell})_{i\in I,\ell\in[k(i)]}$. The components $(T_{i,\ell})_{i\in I,\ell\in[k(i)]}$ and the vertices $(v_m)_{m=1}^n$ form the vertices of $\chi(T)$ with edges determined by inclusion, i.e.,

$$v_m \sim_{\chi(T)} T_{i,\ell} \iff v_m \in T_{i,\ell}.$$

In particular, we say that T is a *free product* in $(S_i)_{i\in I}$ if the graph of colored components $\chi(T)$ is a tree.

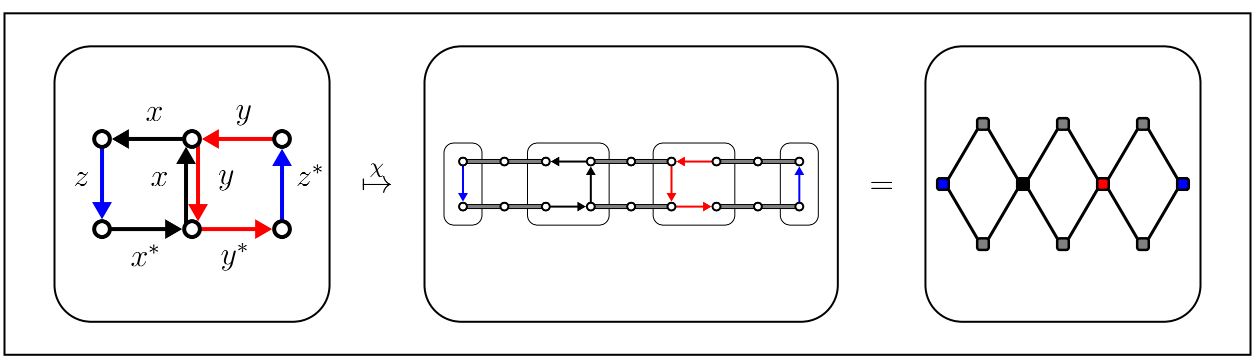


Figure 2.4: An example of the construction of a graph of colored components $\chi(T)$ from a test graph T. Here, we color the vertices of $\chi(T)$ to clarify the construction. Note that T is not a free product in this example.

Definition 2.3.8 (Traffic independence)**.** Let $(\mathcal{A}, \tau)$ be a traffic space. We say that families of random variables $(\mathbf{a}_i)_{i \in I}$ of $\mathcal{A}$ with union $\mathbf{a} = \bigcup_{i \in I} \mathbf{a}_i = (a_{i,j})_{i \in I, j \in J_i}$ are *traffic independent* if for any $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle = \mathcal{T}\langle \bigsqcup_{i \in I} \langle \mathbf{x}_i, \mathbf{x}_i^* \rangle \rangle$,

$$\tau^0\big[T(\mathbf{a})\big] = \begin{cases} \prod_{i \in I} \prod_{\ell=1}^{k(i)} \tau^0\big[T_{i,\ell}(\mathbf{a}_i)\big] & \text{if } T \text{ is a free product in } \langle \mathbf{x}_i, \mathbf{x}_i^* \rangle_{i \in I}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Similarly, suppose that for each $n \in \mathbb{N}$ we have families of random variables $(\mathbf{a}_n^{(i)})_{i \in I}$ of a traffic space $(\mathcal{A}_n, \tau_n)$ with union $\mathbf{a}_n = \bigcup_{i \in I} \mathbf{a}_n^{(i)} = (a_n^{(i,j)})_{i \in I, j \in J_i}$. We say that the $(\mathbf{a}_n^{(i)})_{i \in I}$ are *asymptotically traffic independent* if the joint traffic distributions $\nu_{\mathbf{a}_n} : \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle \to \mathbb{C}$ converge pointwise to a limit ν such that for any $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle = \mathcal{T}\langle \bigsqcup_{i \in I} \langle \mathbf{x}_i, \mathbf{x}_i^* \rangle \rangle$,

$$\nu^0\big[T\big] = \begin{cases} \prod_{i \in I} \prod_{\ell=1}^{k(i)} \nu^0\big[T_{i,\ell}\big] & \text{if } T \text{ is a free product in } \langle \mathbf{x}_i, \mathbf{x}_i^* \rangle_{i \in I}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Remark 2.3.9. The relationships (2.6) and (2.7) characterize the joint traffic distribution of traffic independent random variables in terms of the corresponding marginal traffic distributions. Moreover, note that the traffic distribution of a family of random variables $\mathbf{a}_i$ specifies the traffic distribution of the generated traffic space $\mathcal{A}_i$. In particular, the traffic independence of the families $(\mathbf{a}_i)_{i \in I}$ extends to the generated traffic spaces $(\mathcal{A}_i)_{i \in I}$ [Mal, Proposition 2.14]. In the context of Remark 2.3.5, this implies that we actually have the traffic independence of the traffic spaces $(\mathbb{C}\mathcal{G}\langle \mathbf{x}_i, \mathbf{x}_i^* \rangle)_{i \in I}$ in $(\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle, \nu)$ in the asymptotic case.

For a family of traffic spaces $(\mathcal{A}_i, \tau_i)_{i \in I}$, we can find a traffic independent realization of the $(\mathcal{A}_i, \tau_i)_{i \in I}$ inside of a larger traffic space $(\mathcal{A}, \tau)$. Intuitively, we imagine $\mathcal{A}$ as a suitable set of graphs with edge labels in $\bigsqcup_{i \in i} \mathcal{A}_i$, while equation (2.9) completely determines our choice of $\tau = *_{i \in I} \tau_i$. The formal construction involves a number of technical details: most notably, in establishing the positivity of τ. We refer the reader to [CDM] for the existence of such a (traffic) free product; however, one need not appeal to the free product construction in order to find instances of traffic independence. More concretely, Theorem 1.8 in [Mal], recorded below, shows that traffic independence describes the asymptotic behavior of permutation invariant random matrices.

Theorem 2.3.10 (Criteria for asymptotic traffic independence)**.** *Let I be an index set. For each $i \in I$ and $N \in \mathbb{N}$, let $\mathcal{M}_N^{(i)} = (\mathbf{M}_N^{(i,j)})_{j \in J_i} \subset \mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}))$ be a family of random matrices satisfying the following properties:*

(i) (Independence) The families $(\mathcal{M}_N^{(i)})_{i \in I}$ are independent;

(ii) (Permutation invariance) The distribution of all but at most one of the families $\mathcal{M}_N^{(i)}$ is invariant under conjugation by the permutation matrices, i.e.,

$$\mathbf{P}_\sigma \mathcal{M}_N^{(i)} \mathbf{P}_\sigma^* \stackrel{d}{=} \mathcal{M}_N^{(i)}, \qquad \forall \sigma \in \mathfrak{S}_N;$$

(iii) (Convergence in traffic distribution) For each $i \in I$, the sequence $(\mathcal{M}_N^{(i)})_{N\in\mathbb{N}}$ converges in traffic distribution;

(iv) (Factorization) For any finite collection of $*$-test graphs $T_1, \ldots, T_\ell \in \mathcal{T}\langle \mathbf{x}_i, \mathbf{x}_i^* \rangle$,

$$\lim_{N\to\infty} \mathbb{E}\Big[\prod_{m=1}^{\ell} \frac{1}{N}\mathrm{tr}\big[T_m(\mathcal{M}_N^{(i)})\big]\Big] = \prod_{m=1}^{\ell}\Big(\lim_{N\to\infty} \mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\big[T_m(\mathcal{M}_N^{(i)})\big]\Big]\Big), \tag{2.11}$$

where the limits on the right-hand side exist by (iii).

Then the families $(\mathcal{M}_N^{(i)})_{i\in I}$ are asymptotically traffic independent and satisfy the joint factorization property

$$\lim_{N\to\infty} \mathbb{E}\Big[\prod_{m=1}^{\ell} \frac{1}{N}\mathrm{tr}\big[T_m(\mathcal{M}_N)\big]\Big] = \prod_{m=1}^{\ell}\Big(\lim_{N\to\infty} \mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\big[T_m(\mathcal{M}_N)\big]\Big]\Big)$$

for any finite collection of $*$-test graphs $T_1, \ldots, T_\ell \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle$, where $\mathcal{M}_N = \bigcup_{i\in I} \mathcal{M}_N^{(i)}$.

The assumptions of Theorem 2.3.10 turn out to be surprisingly mild in practice and hold for many classical random matrix ensembles: for example, Wigner matrices, Haar distributed unitary random matrices, and uniformly distributed random permutation matrices [Mal]. Of course, these ensembles are already well-studied within the context of free probability, but Theorem 2.3.10 also applies to random matrix ensembles outside of the scope of free probability: for example, heavy Wigner matrices [Mal17].

We conclude with a central limit theorem for traffic independence. The version stated below is contained in the more general Theorem 6.4 of [Mal] and interpolates between the classical CLT and the free CLT (cf. Theorem 2.1.8).

Theorem 2.3.11 (Traffic CLT)**.** *Let (a_n) be a sequence of identically distributed self-adjoint random variables in a traffic space $(\mathcal{A}, \tau)$. Assume that the a_n are centered with unit variance, i.e., $\varphi_\tau(a_n) = 0$ and $\varphi_\tau(a_n^2) = 1$, and write $s_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n a_i$ for the normalized sum. We split the variance of a_n as*

$$1 = \varphi_\tau(a_n^2) = \varphi_\tau\Big(\underset{\text{out}}{\cdot} \xleftarrow{a_n} \cdot \xleftarrow{a_n} \underset{\text{in}}{\cdot} \Big) = \tau\Big[\cdot \overset{a_n}{\underset{a_n}{\rightleftarrows}} \cdot \Big] = \tau^0\Big[\cdot \overset{a_n}{\underset{a_n}{\rightleftarrows}} \cdot \Big] + \tau^0\Big[\overset{a_n}{\underset{a_n}{\circlearrowleft\!\!\circlearrowright}} \Big] = \alpha + (1-\alpha).$$

If the (a_n) are traffic independent, then (s_n) converges in distribution to the free convolution $\mu_\alpha = \mathcal{SC}(0,\alpha) \boxplus \mathcal{N}(0, 1-\alpha)$ of the semicircle distribution of mean 0 and variance α with the normal distribution of mean 0 and variance $1-\alpha$, i.e.,

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{\mathbb{R}} t^m\, \mu_\alpha(dt), \qquad \forall m \in \mathbb{N}.$$

Remark 2.3.12. We note that (2.6) and the positivity of the traffic state imply that both

$$\begin{aligned}\alpha = \tau^0\left[\,\cdot \overset{a_n}{\underset{a_n}{\rightleftarrows}} \cdot\,\right] &= \tau\left[\,\cdot \overset{a_n}{\underset{a_n}{\rightleftarrows}} \cdot\,\right] - \tau\left[\,\overset{a_n}{\underset{a_n}{\circlearrowleft\!\circlearrowright}}\,\right] \\ &= \varphi_\tau((a_n - \Delta(a_n))^2) \\ &= \varphi_\tau((a_n - \Delta(a_n))^*(a_n - \Delta(a_n))) \geq 0\end{aligned}$$

and

$$1 - \alpha = \tau^0\left[\,\overset{a_n}{\underset{a_n}{\circlearrowleft\!\circlearrowright}}\,\right] = \tau\left[\,\overset{a_n}{\underset{a_n}{\circlearrowleft\!\circlearrowright}}\,\right] \geq 0.$$

Chapter 6 in [Mal] shows how one can realize the traffic CLT for the values $\alpha \in \{0, 1\}$, recovering the usual CLTs. We show how one can realize the traffic CLT for the intermediate values $\alpha \in (0, 1)$ in Section 4.2.

Remarkably, in the context of a tracial $*$-probability space, one can *always* appeal to the traffic framework, bringing this seemingly specialized machinery to bear in generic situations, with further implications for random matrices [CDM]. We revisit the CDM construction in the next section.

2.4 The universal enveloping traffic space

The construction of the universal enveloping traffic space involves a number of technical details, particularly in proving the positivity of the associated traffic state, but the basic idea is quite intuitive. For a tracial $*$-probability space $(\mathcal{A}, \varphi)$, we would like to introduce a consistent traffic space structure that extends the original $*$-probability space. For starters, we must define a $\mathcal{G}^*$-algebra structure on $\mathcal{A}$. Of course, in general, we cannot expect to do this without enlarging our space; however, to qualify as universal, this construction should be kept to a minimum. Assuming an action of the graph operations $(Z_g)_{g\in\mathcal{G}}$, the $\mathcal{G}^*$-algebra generated by $\mathcal{A} = \mathcal{A}^*$ (Definition 2.2.6) can be characterized as

$$\operatorname{span}\left(\bigcup_{K\geq 0}\bigcup_{g\in\mathcal{G}_K} Z_g(\mathcal{A}^{\otimes K})\right),$$

and so it remains to define said action $(Z_g)_{g\in\mathcal{G}}$.

Recall our earlier convention of depicting the action of a graph operation on a K-tuple $Z_g(a_1 \otimes \cdots \otimes a_K)$ by assigning each argument a_i to the edge prescribed by the ordering. This suggests the naïve approach of simply declaring the enlarged space to be the span of such edge-labeled graphs *ab initio*. In particular, consider the set of 2-graph polynomials $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ (Definition 2.3.1), where we adopt the convention from before and refer to the coordinates of the pair of distinguished vertices $(v_1, v_2) = (v_{\text{in}}, v_{\text{out}})$ as the input and the output respectively. For simplicity, we drop the prefix 2- and simply refer to graph monomials/polynomials in $\mathcal{A}$.

The graph polynomials $\mathbb{C}\mathcal{G}^{(2)}(\mathcal{A})$ carry a natural $\mathcal{G}^*$-algebra structure under the action of composition, virtually identical to the $*$-graph polynomials (Definition 2.2.7). In fact, recall that the $*$-graph polynomials generalize the usual $*$-polynomials via the embedding (2.4). Similarly, we have a formal embedding

$$F : \mathcal{A} \hookrightarrow \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle, \qquad a \mapsto \underset{\text{out}}{\cdot} \xleftarrow{a} \underset{\text{in}}{\cdot},$$

which is neither linear nor homomorphic. For example,

$$z_1ab + z_2c + z_3 \mapsto \underset{\text{out}}{\cdot} \xleftarrow{z_1ab+z_2c+z_3} \underset{\text{in}}{\cdot} \neq z_1\left(\underset{\text{out}}{\cdot} \xleftarrow{a} \cdot \xleftarrow{b} \underset{\text{in}}{\cdot}\right) + z_2\left(\underset{\text{out}}{\cdot} \xleftarrow{c} \underset{\text{in}}{\cdot}\right) + z_3\left(\underset{\text{in/out}}{\cdot}\right).$$

The obstruction comes from the basic definition of the 2-graph polynomials, which are ultimately just linear combinations of formal objects (graphs) that do not remember much of the structure of $\mathcal{A}$ beyond its involution $*$. To account for this defect, we must transfer the algebraic structure of $\mathcal{A}$ to $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ on the level of graphs. To this end, for any graph operation $g \in \mathcal{G}$ and polynomial $a_1 = P(b_1, \ldots, b_n) \in \mathcal{A}$, we make the identification

$$Z_g\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1} \underset{\text{in}}{\cdot} \otimes \cdots \otimes \underset{\text{out}}{\cdot} \xleftarrow{a_K} \underset{\text{in}}{\cdot}\right) \cong Z_g\left(P\left(\underset{\text{out}}{\cdot} \xleftarrow{b_1} \underset{\text{in}}{\cdot}, \ldots, \underset{\text{out}}{\cdot} \xleftarrow{b_n} \underset{\text{in}}{\cdot}\right) \otimes \cdots \otimes \underset{\text{out}}{\cdot} \xleftarrow{a_K} \underset{\text{in}}{\cdot}\right),$$

where

$$P\left(\underset{\text{out}}{\cdot} \xleftarrow{b_1} \underset{\text{in}}{\cdot}, \ldots, \underset{\text{out}}{\cdot} \xleftarrow{b_n} \underset{\text{in}}{\cdot}\right) \in \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle.$$

Recall that multiplication in $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ is defined simply as concatenation. So, for example, if $P(x, y) = cyx^2y$, then

$$P\left(\underset{\text{out}}{\cdot} \xleftarrow{b_1} \underset{\text{in}}{\cdot}, \underset{\text{out}}{\cdot} \xleftarrow{b_2} \underset{\text{in}}{\cdot}\right) = c\left(\underset{\text{out}}{\cdot} \xleftarrow{b_2} \cdot \xleftarrow{b_1} \cdot \xleftarrow{b_1} \cdot \xleftarrow{b_2} \underset{\text{in}}{\cdot}\right).$$

In particular, if $P \equiv z \in \mathbb{C}$, then as a graph polynomial,

$$P = z\left(\underset{\text{in/out}}{\cdot}\right) \in \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle.$$

We note that the equivariance property of a $\mathcal{G}^*$-algebra allows us to formulate the desired identifications entirely within the first argument of a graph operation. As a simple example,

$$\underset{\text{out}}{\cdot} \xleftarrow{z_1ab+z_2} \overset{\overset{\cdot}{\uparrow}\, z_3d+z_4}{\underset{\text{in}}{\cdot}} \cong z_1z_3\left(\underset{\text{out}}{\cdot} \xleftarrow{a} \cdot \xleftarrow{b} \overset{\overset{\cdot}{\uparrow}\, d}{\underset{\text{in}}{\cdot}}\right) + z_2z_3\left(\overset{\overset{\cdot}{\uparrow}\, d}{\underset{\text{in/out}}{\cdot}}\right) + z_1z_4\left(\underset{\text{out}}{\cdot} \xleftarrow{a} \cdot \xleftarrow{b} \underset{\text{in}}{\cdot}\right) + z_2z_4\left(\underset{\text{in/out}}{\cdot}\right),$$

which conforms to our expectations for a suitable lifting of the algebraic structure of $\mathcal{A}$.

We write $\mathcal{G}(\mathcal{A})$ for the quotient of $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ by these relations, the so-called *free $\mathcal{G}^*$-algebra generated by $\mathcal{A}$*. Equivalently, $\mathcal{G}(\mathcal{A})$ is the quotient of $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ by the two-sided $*$-ideal $\mathcal{I}$ spanned by elements of the form

$$Z_g\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1} \underset{\text{in}}{\cdot} \otimes \cdots \otimes \underset{\text{out}}{\cdot} \xleftarrow{a_K} \underset{\text{in}}{\cdot}\right) - Z_g\left(P\left(\underset{\text{out}}{\cdot} \xleftarrow{b_1} \underset{\text{in}}{\cdot}, \ldots, \underset{\text{out}}{\cdot} \xleftarrow{b_n} \underset{\text{in}}{\cdot}\right) \otimes \cdots \otimes \underset{\text{out}}{\cdot} \xleftarrow{a_K} \underset{\text{in}}{\cdot}\right),$$

from which the quotient $\mathcal{G}^*$-algebra structure on $\mathcal{G}(\mathcal{A})$ becomes clear. In practice, we still consider the elements of $\mathcal{G}(\mathcal{A})$ as (linear combinations of) graphs as opposed to equivalence classes of such graphs, and we adopt the same conventions for their depictions. In fact, for much of our analysis, the (multi)linearity of the object under consideration allows us to restrict our attention to graph monomials $\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$, viewed of course as elements in $\mathcal{G}(\mathcal{A})$. When necessary, we address any issues of well-definedness, though this will often be self-evident.

The formal embedding from before now becomes a genuine embedding of unital $*$-algebras

$$F^{\mathcal{G}} : \mathcal{A} \hookrightarrow \mathcal{G}(\mathcal{A}) = \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle/\mathcal{I}, \qquad a \mapsto \underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot}\ ,$$

grace of the identifications $\mathcal{I}$. This allows us to consider our original $*$-algebra $\mathcal{A}$ as a canonical subalgebra of the enlarged space $\mathcal{G}(\mathcal{A})$. Indeed,

$$\mathcal{G}(\mathcal{A}) = \operatorname{span}\left(\bigcup_{K\geq 0}\bigcup_{g\in\mathcal{G}_K} Z_g(F^{\mathcal{G}}(\mathcal{A})\otimes\cdots\otimes F^{\mathcal{G}}(\mathcal{A}))\right);$$

or, in words, $\mathcal{G}(\mathcal{A})$ is the $\mathcal{G}^*$-algebra generated by the image $F^{\mathcal{G}}(\mathcal{A})$. By a slight abuse of notation, we identify $\mathcal{A} \cong F^{\mathcal{G}}(\mathcal{A})$. Thus, moving forward,

$$\mathcal{A} = \left(\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} \;\middle|\; a \in \mathcal{A} \right) \subset \mathcal{G}(\mathcal{A}).$$

Again, we note that in this notation, $\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot}$ stands for an equivalence class of graphs.

The free $\mathcal{G}^*$-algebra satisfies the following universal property, the proof of which we leave as a simple but instructive exercise for the reader.

Proposition 2.4.1. *Let $\mathcal{A}$ be a unital $*$-algebra, $\mathcal{B}$ a $\mathcal{G}^*$-algebra, and $f : \mathcal{A} \to \mathcal{B}$ a morphism of unital $*$-algebras. Then there exists a unique $\mathcal{G}^*$-morphism $\mathcal{G}(f) : \mathcal{G}(\mathcal{A}) \to \mathcal{B}$ such that*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad f \quad} & \mathcal{B} \\ & {\scriptstyle F^{\mathcal{G}}}\searrow \quad \nearrow{\scriptstyle \mathcal{G}(f)} & \\ & \mathcal{G}(\mathcal{A}) & \end{array}$$

Remark 2.4.2. Starting with a $\mathbb{C}$-algebra $\mathcal{A}$, one can also define the free $\mathcal{G}$-algebra generated by $\mathcal{A}$ in a similar manner. In fact, the same construction produces the correct object $\mathcal{G}(\mathcal{A})$ except that we do not have an involution on $\mathcal{G}(\mathcal{A})$ coming from the $*$-flip/transpose combination (since the $*$-flip cannot be defined without the involution on $\mathcal{A}$).

Example 2.4.3. The free $\mathcal{G}^*$-algebra generated by the $*$-polynomials $\mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle$ is isomorphic to the $*$-graph polynomials $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$. The reader should take care to understand why we have phrased this as an isomorphism as opposed to an outright equality. Indeed, one can think of $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ as a canonical set of representatives in $\mathcal{G}(\mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle)$. This example highlights our perspective on elements of the free $\mathcal{G}^*$-algebra as graphs (as opposed to equivalence classes of graphs). $\Diamond$

Having defined the $\mathcal{G}^*$-algebra extension of $\mathcal{A}$, we now turn our attention to the matter of the traffic state τ_φ. As it turns out, the appropriate construction passes through the injective traffic state τ^0_φ in the form of cactus graphs.

Definition 2.4.4 (Cactus graph)**.** A finite, connected multigraph $G = (V, E)$ is said to be a *cactus* if every edge $e \in E$ belongs to a unique simple cycle. A finite, connected, multidigraph $G = (V, E, \mathrm{src}, \mathrm{tar})$ is said to be a *cactus* if the underlying undirected multigraph (V, E) is a cactus and further an *oriented cactus* if each simple cycle is further a directed cycle. We refer to the cycles of a cactus G as *pads* and denote the set of such cycles by $\mathrm{Pads}(G)$.

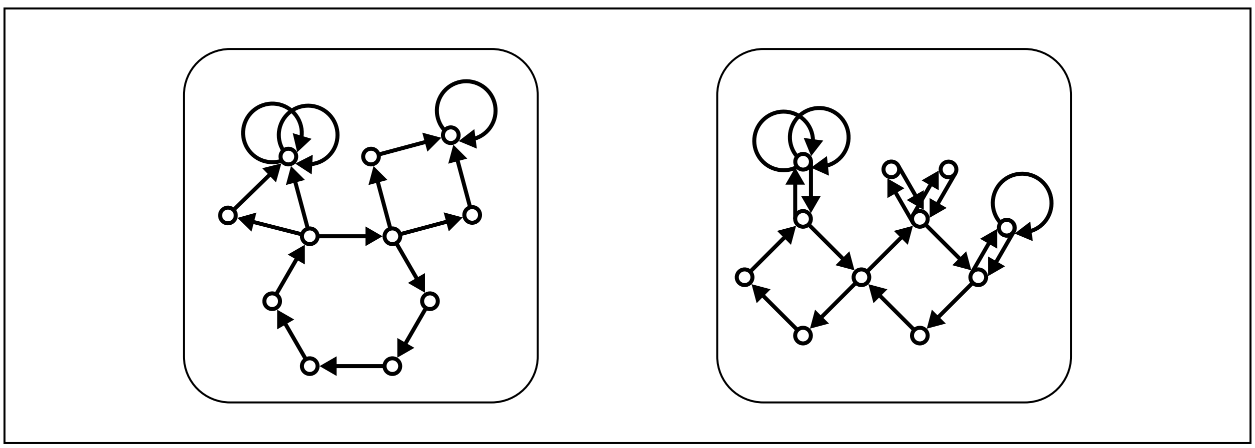


Figure 2.5: An example of a cactus and an oriented cactus respectively.

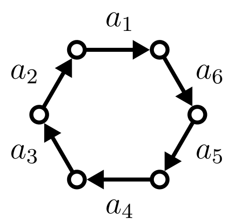
For reasons that will soon become apparent, we define a functional $\tau^0_\varphi : \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle \to \mathbb{C}$ that is supported on oriented cacti and is further multiplicative with respect to the pads, namely,

(i) If T is a directed cycle of length n in the clockwise orientation with edges labeled by $a_1, \ldots, a_n$ *counterclockwise*, then

$$\tau^0_\varphi[T] = \kappa_n(a_1, \ldots, a_n),$$

where κ_n is the nth free cumulant functional of $(\mathcal{A}, \varphi)$. For example,

$$\tau^0_\varphi\Big[$$



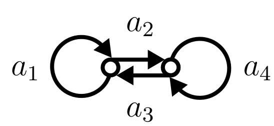
$$\Big] = \kappa_6(a_1, a_2, a_3, a_4, a_5, a_6).$$

(ii) If T is an oriented cactus, then

$$\tau^0_\varphi[T] = \prod_{C \in \mathrm{Pads}(T)} \tau^0[C].$$

For example,

$$\tau^0_\varphi\Big[$$



$$\Big] = \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4).$$

(iii) Otherwise,

$$\tau^0_\varphi[T] = 0.$$

Since φ is a trace, the free cumulant functionals $(\kappa_n)_{n\in\mathbb{N}}$ are cyclically invariant, i.e.,

$$\kappa_n(a_1, \ldots, a_n) = \kappa_n(a_2, \ldots, a_n, a_1) = \cdots = \kappa_n(a_n, a_1, \ldots, a_{n-1}).$$

This implies that the definition in (i) does not depend on where we choose to start reading off the elements a_i in the cycle as long as we proceed in a counterclockwise fashion, ensuring that the construction is well-defined.

As advertised, we use the inversion (2.7) to define a functional $\tau_\varphi : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$. This gives rise to a linear functional $\varphi_{\tau_\varphi} : \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle \to \mathbb{C}$ through the now familiar route of identifying the roots and forgetting their distinguished roles: formally,

$$\varphi_{\tau_\varphi} = \tau_\varphi \circ \widetilde{\Delta}.$$

Proposition 4.4 in [CDM] shows that the functional φ_{τ_φ} respects the identifications defining $\mathcal{G}(\mathcal{A}) = \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle/\mathcal{I}$, namely,

$$\mathcal{I} \subset \ker(\varphi_{\tau_\varphi}).$$

Thus, by a slight abuse of notation, we also write $\varphi_{\tau_\varphi} : \mathcal{G}(\mathcal{A}) \to \mathbb{C}$ for the quotient functional. In fact, the proof of this statement proceeds by showing that τ_φ drops to a well-defined (algebraic) traffic state on $\mathcal{G}(\mathcal{A})$, which we again denote by τ_φ. Theorem 4.13 in [CDM] further establishes the positivity of τ_φ, which leads us to

Proposition 2.4.5 (Universal enveloping traffic space [CDM]). *Let $(\mathcal{A},\varphi)$ be a tracial $*$-probability space. We call $(\mathcal{G}(\mathcal{A}),\tau_\varphi)$ the universal enveloping traffic space of $(\mathcal{A},\varphi)$. In particular, the embedding $F^{\mathcal{G}}:\mathcal{A}\hookrightarrow\mathcal{G}(\mathcal{A})$ further preserves the trace:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad\varphi\quad} & \mathbb{C} \\ & {\scriptstyle F^{\mathcal{G}}}\searrow \quad \nearrow{\scriptstyle \varphi_{\tau_\varphi}} & \\ & \mathcal{G}(\mathcal{A}) & \end{array}$$

Thus, effectively, we can realize $(\mathcal{A},\varphi)$ as a sub-$*$-probability space of $(\mathcal{G}(\mathcal{A}),\tau_\varphi)$. Since $\varphi_{\tau_\varphi}|_{\mathcal{A}}=\varphi$, we unencumber the notation and simply write $\varphi=\varphi_{\tau_\varphi}:\mathcal{G}(\mathcal{A})\to\mathbb{C}$ when there is no ambiguity.

Proof. It only remains to prove that $F^{\mathcal{G}}$ preserves the trace, but this follows immediately from the definitions:

$$\varphi_{\tau_\varphi}\circ F^{\mathcal{G}}(a)=\varphi_{\tau_\varphi}\left(\underset{\text{out}}{\cdot}\overset{a}{\leftarrow}\underset{\text{in}}{\cdot}\right)=\tau_\varphi\left[\overset{a}{\circlearrowleft}\right]=\kappa_1(a)=\varphi(a).$$

■

For the purposes of this article, we will not need the positivity of the traffic state in the universal enveloping traffic space. Instead, we will do just fine with algebraic traffic space structure. In the applications to random matrices, we inherit the positivity from the matricial setting in the large N limit, and so we do not need to rely on the abstract construction.

The motivation behind the cactus structure of τ_φ^0 comes from the one-to-one correspondence between non-crossing partitions of $[n]$ and cactus graphs obtained as quotients of cycles of length n. In particular, consider a cycle graph $C=(V,E)$ of length n with vertices $V=(v_1,\ldots,v_n)$ in counterclockwise order and edges $E=(e_1,\ldots,e_n)$ connecting $v_i\overset{e_i}{\sim}v_{i+1}$. The mapping $[n]\ni i\mapsto v_i\in V$ induces a one-to-one correspondence

$$f:\mathcal{NC}(n)\to\mathcal{NC}(V)$$

between non-crossing partitions of $[n]$ and non-crossing partitions of V, where the latter notion comes from drawing $C=(V,E)$ as a circle. Similarly, the mapping $[\overline{n}]\ni\overline{i}\mapsto e_i\in E$ induces a one-to-one correspondence

$$g:\mathcal{NC}(\overline{n})\to\mathcal{NC}(E)$$

between non-crossing partitions of $[\overline{n}]=\{\overline{1}<\cdots<\overline{n}\}$ and non-crossing partitions of E. Furthermore, the Kreweras complement $K:\mathcal{NC}(n)\to\mathcal{NC}(\overline{n})$, defined as a non-crossing partition of $[\overline{n}]=\{\overline{1},\ldots,\overline{n}\}$ via the interlacing

$$1<\overline{1}<\cdots<n<\overline{n},$$

corresponds to the Kreweras complement $K : \mathcal{NC}(V) \to \mathcal{NC}(E)$ defined as a non-crossing partition of the edges E of C via the interlacing

$$v_1 \overset{e_1}{\sim} v_2 \overset{e_2}{\sim} \cdots \overset{e_{n-1}}{\sim} v_n \overset{e_n}{\sim} v_1.$$

Formally, this amounts to the equality

$$g \circ K = K \circ f.$$

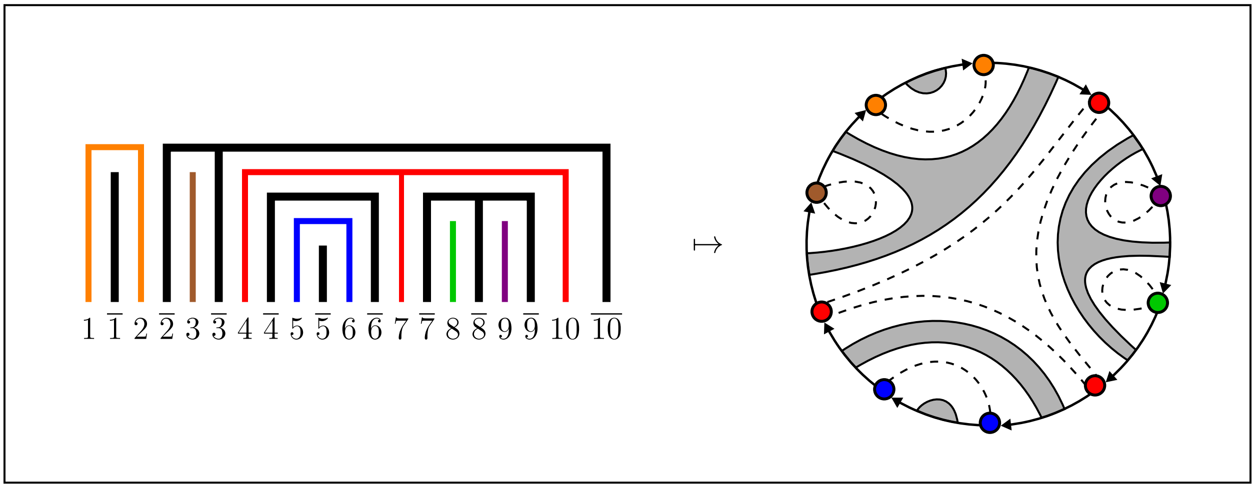


Figure 2.6: An example of the correspondence between non-crossing partitions on the line and non-crossing partitions on the circle.

The following crucial observation is contained in Remark 4.5 of [CDM].

Proposition 2.4.6. *Let $C = (V, E)$ be a cycle graph, say of length n and with vertices $V = (v_1, \ldots, v_n)$ in counterclockwise order and edges $E = (e_1, \ldots, e_n)$ connecting $v_i \overset{e_i}{\sim} v_{i+1}$. For a partition $\pi \in \mathcal{P}(V)$, the following two conditions are equivalent:*

(i) The quotient graph C^π is a cactus;

(ii) The partition π is non-crossing.

Furthermore, if $\pi \in \mathcal{NC}(V)$, then the pads of the cactus C^π correspond to the blocks of the Kreweras complement $K(\pi) \in \mathcal{NC}(E)$ via the map

$$K(\pi) \ni B = (e_{i_1}, \ldots, e_{i_k}) \mapsto (v_{i_1}, v_{i_1+1} \overset{\pi}{\sim} v_{i_2}, \ldots, v_{i_{k-1}+1} \overset{\pi}{\sim} v_{i_k}, v_{i_k+1} \overset{\pi}{\sim} v_{i_1}) \in \mathrm{Pads}(C^\pi),$$

where $(i_1 < \cdots < i_k)$. In particular, assume that $C = (V, E, \mathrm{src}, \mathrm{tar})$ is also directed (though not necessarily a directed cycle). We say that an edge e_i is oriented counterclockwise if $\mathrm{src}(e_i) = v_i$*; otherwise,* $\mathrm{src}(e_i) = v_{i+1}$ *and we say that e_i is oriented clockwise. For a partition $\pi \in \mathcal{P}(V)$, the following two conditions are equivalent:*

(I) The quotient graph C^π is an oriented cactus;

(II) The partition π is non-crossing and each block of $K(\pi) \in \mathcal{NC}(E)$ only contains edges of a uniform orientation.

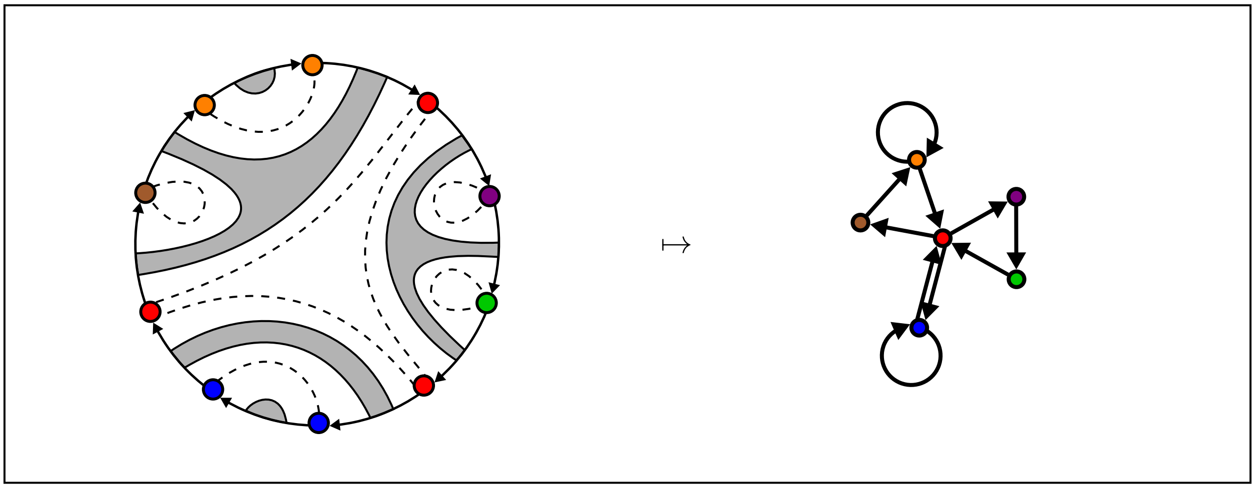


Figure 2.7: An example of Proposition 2.4.6 in action. Note that we can track the edges of each pad in the resulting cactus via the Kreweras complement.

We prove Proposition 2.4.6 in Section 3.2 after developing the combinatorics of cactus graphs. For now, we return to our discussion on the cactus structure of τ^0_φ. In particular, note that

$$\varphi_{\tau_\varphi}\left(\underset{\text{out}}{\cdot} \overset{a_1}{\longleftarrow} \cdot \overset{a_2}{\longleftarrow} \cdots \cdot \overset{a_n}{\longleftarrow} \underset{\text{in}}{\cdot} \right) = \tau_\varphi\big[C(a_1, \ldots, a_n)\big], \tag{2.12}$$

where $C = (V, E, \mathrm{src}, \mathrm{tar}, \gamma) \in \mathcal{T}\langle\mathcal{A}\rangle$ is a directed cycle of length n drawn with vertices $V = (v_1, \ldots, v_n)$ ordered counterclockwise and edges $E = (e_1, \ldots, e_n)$ satisfying

$$\mathrm{src}(e_i) = v_{i+1}, \quad \mathrm{tar}(e_i) = v_i, \quad \text{and} \quad \gamma(e_i) = a_i.$$

We think of C as the underlying graph with the arguments $a_1, \ldots, a_n$ indicating the edge labels $\gamma(e_i)$, hence the notation $C(a_1, \ldots, a_n)$. We continue the calculation in terms of τ^0_φ:

$$\begin{aligned}
\tau_\varphi\big[C(a_1, \ldots, a_n)\big] &= \sum_{\pi \in \mathcal{P}(V)} \tau^0_\varphi\big[C^\pi(a_1, \ldots, a_n)\big] \\
&= \sum_{\pi \in \mathcal{NC}(V)} \tau^0_\varphi\big[C^\pi(a_1, \ldots, a_n)\big] \\
&= \sum_{\pi \in \mathcal{NC}(V)} \prod_{B = (e_{i_1}, \ldots, e_{i_k}) \in K(\pi)} \kappa_{\#(B)}(a_{i_1}, \ldots, a_{i_k})
\end{aligned}$$

$$\begin{aligned} &= \sum_{\pi \in \mathcal{NC}(n)} \kappa_{K(\pi)}[a_1, \ldots, a_n] \\ &= \sum_{\pi \in \mathcal{NC}(n)} \kappa_{\pi}[a_1, \ldots, a_n] = \varphi(a_1 \cdots a_n). \end{aligned}$$

The reader should take care to note that we have applied both Proposition 2.4.6 and the discussion preceding it in this chain of equalities, thereby justifying the choice of τ_φ^0.

Of course, since $(\mathcal{G}(\mathcal{A}), \varphi_{\tau_\varphi})$ is also a $*$-probability space, we can likewise consider its free cumulant sequence $(\kappa_n^{\mathcal{G}})_{n\in\mathbb{N}}$. At a glance, we note that

$$\varphi_{\tau_\varphi}|_{\mathcal{A}} = \varphi \qquad \Longrightarrow \qquad \kappa_n^{\mathcal{G}}|_{\mathcal{A}^n} = \kappa_n, \qquad \forall n \in \mathbb{N}.$$

We can use this simple observation to compute the free cumulants of the transposed algebra

$$\mathcal{A}^{\intercal} = \left(a^{\intercal} = \underset{\text{out}}{\cdot} \overset{a}{\to} \underset{\text{in}}{\cdot} \;\middle|\; a \in \mathcal{A} \right) \subset \mathcal{G}(\mathcal{A}).$$

In particular,

$$\begin{aligned} \varphi(a_1^{\intercal} \cdots a_n^{\intercal}) &= \varphi\left(\underset{\text{out}}{\cdot} \overset{a_1}{\longrightarrow} \cdots\cdots \overset{a_n}{\longrightarrow} \underset{\text{in}}{\cdot} \right) \\ &= \tau_\varphi\big[C_{\to}(a_1, \ldots, a_n)\big] \\ &= \tau_\varphi\big[C(a_n, \ldots, a_1)\big] = \varphi(a_n \cdots a_1), \end{aligned}$$

where $C_{\to} = (V, E, \mathrm{tar}, \mathrm{src}, \gamma)$ is the flip of the cycle $C = (V, E, \mathrm{src}, \mathrm{tar}, \gamma)$ in (2.12). It follows that $\intercal : (\mathcal{A}, \varphi) \to (\mathcal{A}^{\intercal}, \varphi)$ defines an involutive anti-isomorphism of $*$-probability spaces such that

$$\kappa_n^{\mathcal{G}}(a_1^{\intercal}, \ldots, a_n^{\intercal}) = \kappa_n(a_n, \ldots, a_1).$$

In fact, the same argument shows that $\intercal : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ defines an involutive anti-automorphism of $*$-probability spaces such that

$$\kappa_n^{\mathcal{G}}(t_1^{\intercal}, \ldots, t_n^{\intercal}) = \kappa_n^{\mathcal{G}}(t_n, \ldots, t_1), \qquad \forall t_i \in \mathcal{G}(\mathcal{A}).$$

Based on the matrix heuristic, the transposed algebra $\mathcal{A}^{\intercal}$ emerges as a natural $*$-subalgebra of $\mathcal{G}(\mathcal{A})$. Similarly, one can also consider the degree algebras

$$\mathrm{rDeg}(\mathcal{A}) = \mathbb{C}\left\langle \left(\begin{matrix} \cdot \\ \downarrow a \\ \underset{\text{in/out}}{\cdot} \end{matrix} \;\middle|\; a \in \mathcal{A} \right) \right\rangle, \qquad \mathrm{cDeg}(\mathcal{A}) = \mathbb{C}\left\langle \left(\begin{matrix} \cdot \\ \uparrow a \\ \underset{\text{in/out}}{\cdot} \end{matrix} \;\middle|\; a \in \mathcal{A} \right) \right\rangle,$$

and

$$\mathrm{Deg}(\mathcal{A}) = \mathbb{C}\langle \mathrm{rDeg}(\mathcal{A}), \mathrm{cDeg}(\mathcal{A}) \rangle,$$

the $*$-subalgebra generated by $\mathrm{rDeg}(\mathcal{A})$ and $\mathrm{cDeg}(\mathcal{A})$. Here, we find the first indication of rigidity in the universal enveloping traffic space.

Proposition 2.4.7. *The unital $*$-subalgebras $\mathcal{A}, \mathcal{A}^\intercal$, and $\operatorname{Deg}(\mathcal{A})$ are freely independent in the universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \varphi)$.*

Proof. See Corollary 4.7 in [CDM]. ■

The proof of this result in [CDM] proceeds by directly establishing the alternating moments characterization (2.2) of freeness. In particular, this free independence structure arises regardless of the choice of tracial $*$-probability space $(\mathcal{A}, \varphi)$. We extend this result to general graph operations in Section 3.3 to give a complete free product decomposition of the universal enveloping traffic space. As a warm-up, we give a second proof of this result for $\mathcal{A}$ and $\mathcal{A}^\intercal$, relying on the cactus characterization in Proposition 2.4.6 to prove the vanishing of mixed cumulants.

Proof. Let $a_1, \ldots, a_n \in \mathcal{A}$ with transpose labels $\hat{\intercal} : [n] \to \{1, \intercal\}$. We compute the expectation of the product $a_1^{\hat{\intercal}(1)} \cdots a_n^{\hat{\intercal}(n)}$ as before:

$$\varphi(a_1^{\hat{\intercal}(1)} \cdots a_n^{\hat{\intercal}(n)}) = \varphi\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1^{\hat{\intercal}(1)}} \cdots\cdot \xleftarrow{a_n^{\hat{\intercal}(n)}} \underset{\text{in}}{\cdot} \right) = \tau_\varphi\big[C(a_1^{\hat{\intercal}(1)}, \ldots, a_n^{\hat{\intercal}(n)})\big].$$

Consider the following modification of the usual directed cycle $C = (V, E, \operatorname{src}, \operatorname{tar}, \gamma)$:

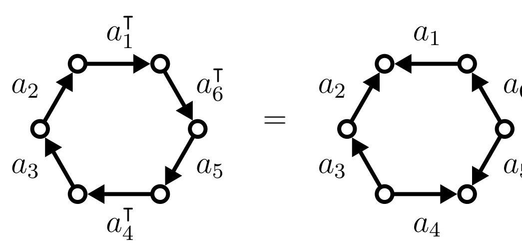
(i) If $\hat{\intercal}(i) = 1$, then $\widetilde{\operatorname{src}}(e_i) = \operatorname{src}(e_i)$ and $\widetilde{\operatorname{tar}}(e_i) = \operatorname{tar}(e_i)$;

(ii) If $\hat{\intercal}(i) = \intercal$, then $\widetilde{\operatorname{src}}(e_i) = \operatorname{tar}(e_i)$ and $\widetilde{\operatorname{tar}}(e_i) = \operatorname{src}(e_i)$.

In words, $\widetilde{C} = (V, E, \widetilde{\operatorname{src}}, \widetilde{\operatorname{tar}}, \gamma)$ flips the original direction of an edge e_i according to the transpose label $\hat{\intercal}(i)$. By construction,

$$\tau_\varphi\big[C(a_1^{\hat{\intercal}(1)}, \ldots, a_n^{\hat{\intercal}(n)})\big] = \tau_\varphi\big[\widetilde{C}(a_1, \ldots, a_n)\big].$$

For example,

$C(a_1^\intercal, a_2, a_3, a_4^\intercal, a_5, a_6^\intercal) =$  $= \widetilde{C}(a_1, a_2, a_3, a_4, a_5, a_6).$

We are now in a position to apply Proposition 2.4.6. Note that an edge $e_i \in \widetilde{C}$ is oriented clockwise if $\hat{\intercal}(i) = 1$ and counterclockwise if $\hat{\intercal}(i) = \intercal$. To separate the two cases, let E_1 denote the subset of clockwise edges and E_2 the subset of counterclockwise edges. This allows us to continue the calculation along the lines

$$\tau_\varphi\big[\widetilde{C}(a_1, \ldots, a_n)\big] = \sum_{\pi \in \mathcal{P}(V)} \tau_\varphi^0\big[\widetilde{C}^\pi(a_1, \ldots, a_n)\big] = \sum_{\substack{\pi \in \mathcal{NC}(V) \text{ s.t.} \\ K(\pi) = \sigma \cup \rho \in \mathcal{NC}(E) \\ \text{for some } \sigma \in \mathcal{NC}(E_1) \\ \text{and } \rho \in \mathcal{NC}(E_2)}} \tau_\varphi^0\big[\widetilde{C}^\pi(a_1, \ldots, a_n)\big].$$

Putting this into the multiplicative cactus formula for τ^0_φ, this becomes

$$\sum_{\substack{\pi\in\mathcal{NC}(n) \text{ s.t.}\\ \pi=\sigma\cup\rho \text{ for some}\\ \sigma\in\mathcal{NC}(\hat{\intercal}^{-1}(1))\\ \text{and } \rho\in\mathcal{NC}(\hat{\intercal}^{-1}(\intercal))}} \prod_{B_1=(i_1<\cdots<i_k)\in\sigma} \kappa_{\#(B_1)}(a_{i_1},\ldots,a_{i_k}) \prod_{B_2=(j_1<\cdots<j_\ell)\in\rho} \kappa_{\#(B_2)}(a_{j_\ell},\ldots,a_{j_1}).$$

We can rewrite this in terms of the free cumulants of $\mathcal{G}(\mathcal{A})$ to obtain the equivalent expression

$$\sum_{\substack{\pi\in\mathcal{NC}(n) \text{ s.t.}\\ \pi=\sigma\cup\rho \text{ for some}\\ \sigma\in\mathcal{NC}(\hat{\intercal}^{-1}(1))\\ \text{and } \rho\in\mathcal{NC}(\hat{\intercal}^{-1}(\intercal))}} \prod_{B_1=(i_1<\cdots<i_k)\in\sigma} \kappa^{\mathcal{G}}_{\#(B_1)}(a_{i_1},\ldots,a_{i_k}) \prod_{B_2=(j_1<\cdots<j_\ell)\in\rho} \kappa^{\mathcal{G}}_{\#(B_2)}(a^{\intercal}_{j_1},\ldots,a^{\intercal}_{j_\ell}),$$

and so

$$\varphi(a_1^{\hat{\intercal}(1)}\cdots a_n^{\hat{\intercal}(n)}) = \sum_{\substack{\pi\in\mathcal{NC}(n) \text{ s.t.}\\ \pi=\sigma\cup\rho \text{ for some}\\ \sigma\in\mathcal{NC}(\hat{\intercal}^{-1}(1))\\ \text{and } \rho\in\mathcal{NC}(\hat{\intercal}^{-1}(\intercal))}} \kappa^{\mathcal{G}}_\pi[a_1^{\hat{\intercal}(1)},\ldots,a_n^{\hat{\intercal}(n)}].$$

In particular, we see that there are no contributions from mixed cumulants in $\mathcal{A}$ and $\mathcal{A}^\intercal$. It follows that mixed cumulants in $\mathcal{A}$ and $\mathcal{A}^\intercal$ vanish (see, e.g., Remarks 11.19 (2) in [NS06]), as was to be shown. ∎

The choice of the traffic state τ_φ is further justified by the following properties of the universal enveloping traffic space (Proposition 4.8 and Theorem 4.9 in [CDM] respectively).

Proposition 2.4.8. *Let $(\mathcal{A},\varphi)$ be a tracial $*$-probability space. For unital $*$-subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$, the following two conditions are equivalent:*
(i) The $(\mathcal{A}_i)_{i\in I}$ are freely independent in $(\mathcal{A},\varphi)$;
(ii) The $(\mathcal{A}_i)_{i\in I}$ are traffic independent in $(\mathcal{G}(\mathcal{A}),\tau_\varphi)$.

Of course, since $\varphi_{\tau_\varphi}|_{\mathcal{A}} = \varphi$, we could also state the first condition in the larger space $(\mathcal{G}(\mathcal{A}),\varphi_{\tau_\varphi})$; however, we emphasize the crucial assumption that $\mathcal{A}_i \subset \mathcal{A}$. Furthermore, because traffic independence is a property of $\mathcal{G}$-algebras, the second condition is actually equivalent to the traffic independence of the generated traffic spaces $(\mathcal{G}(\mathcal{A}_i))_{i\in I}$. In Section 3.4, we show that one can also describe the independence structure between the sub-traffic spaces $(\mathcal{G}(\mathcal{A}_i))_{i\in I}$ entirely in terms of the familiar notions of free independence and classical independence.

The results above show that the cactus structure of τ^0_φ lifts the free independence structure of $(\mathcal{A},\varphi)$ to a canonical independence structure in $(\mathcal{G}(\mathcal{A}),\varphi_{\tau_\varphi})$. Even more, under some further assumptions, convergence in $*$-distribution to $\mathbf{a}\subset(\mathcal{A},\varphi)$ can be lifted to convergence in traffic distribution to $\mathbf{a}\subset(\mathcal{G}(\mathcal{A}),\tau_\varphi)$.

Proposition 2.4.9. *Let $\mathcal{M}_N = (\mathbf{M}_N^{(i)})_{i\in I} \subset L^{\infty-}(\Omega,\mathcal{F},\mathbb{P})\otimes \mathrm{Mat}_N(\mathbb{C})$ be a family of random matrices satisfying the following properties:*

(i) (Unitary invariance) For any unitary matrix $\mathbf{U}_N \in \mathcal{U}(N)$,

$$\mathbf{U}_N \mathcal{M}_N \mathbf{U}_N^* = (\mathbf{U}_N \mathbf{M}_N^{(i)} \mathbf{U}_N^*)_{i\in I} \overset{d}{=} (\mathbf{M}_N^{(i)})_{i\in I} = \mathcal{M}_N;$$

(ii) (Convergence in ∗-distribution) For any ∗-polynomial $P \in \mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle$,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(P(\mathcal{M}_N))\right] \in \mathbb{C};$$

(iii) (Factorization) For any ∗-polynomials $P_1, \ldots, P_\ell \in \mathbb{C}\langle \mathbf{x}, \mathbf{x}^* \rangle$, *we have the equality*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{k=1}^{\ell} \frac{1}{N}\operatorname{tr}(P_k(\mathcal{M}_N))\right] = \prod_{k=1}^{\ell} \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(P_k(\mathcal{M}_N))\right],$$

where the product on the right-hand side exists by (ii).

In particular, (ii) implies that $\mathcal{M}_N$ *converges in ∗-distribution to a family of random variables* $\mathbf{a} = (a_i)_{i\in I}$ *in a tracial ∗-probability space* $(\mathcal{A}, \varphi)$. *Under these assumptions,* $\mathcal{M}_N$ *further converges in traffic distribution to* $\mathbf{a}$ *in the universal enveloping traffic space* $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$.

Let us discuss some immediate consequences of this convergence. A surprising result of Mingo and Popa states that a unitarily invariant random matrix is asymptotically free from its transpose [MP16]: this despite the complete lack of independence between the entries of the two matrices. The [MP16] result operates under a bounded cumulants assumption (which is absent in the traffic formulation) and further extends to the level of second-order freeness (which is not addressed in the traffic formulation). At the level of (first-order) freeness, the transpose phenomenon can be seen as a corollary of Propositions 2.4.7 and 2.4.9. Indeed, Proposition 2.4.9 shows that the universal enveloping traffic space provides an extended limit object for unitarily invariant random matrices. In particular, this extended limit object captures the interaction with the transpose: this is encoded in the transpose graph operation $\underset{\text{out}}{\cdot} \overset{1}{\rightarrow} \underset{\text{in}}{\cdot}$. The distinguishing feature of the transpose graph is that its direction is reversed, and this reversal does not play well with the usual direction $\mathrm{id}_{\mathcal{G}} = \underset{\text{out}}{\cdot} \overset{1}{\leftarrow} \underset{\text{in}}{\cdot}$, resulting in a graphical analogue of the vanishing of mixed cumulants (Proposition 2.4.6), and hence the observed freeness in Proposition 2.4.7. In this way, the traffic approach further provides a top-level explanation of the transpose phenomenon in [MP16].

Moreover, by framing the result as a consequence of the structure in the limit object $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$, we can easily obtain generalizations for random matrix ensembles that lack the restrictive unitary invariance property. For example, the limiting traffic distribution of the Wigner matrices $\mathbf{W}_N$ subdivides according to the common pseudo-variance $\beta = N \cdot \mathbb{E}[\mathbf{W}_N(i,j)^2]$ of the strictly upper triangular entries. Since the GUE satisfies the assumptions of Proposition 2.4.9 [Mal], we can conclude that $\mathbf{W}_N$ and $\mathbf{W}_N^\intercal$ are asymptotically free if the parameter $\beta = 0$. This line of reasoning suggests the following definition.

Definition 2.4.10 (Unitarily invariant traffics)**.** Let $\mathbf{a} = (a_i)_{i \in I}$ be a family of traffics in a traffic space $(\mathcal{A}, \tau)$. By forgetting the traffic space structure and simply considering the underlying $*$-probability space $(\mathcal{A}, \varphi_\tau)$, we can construct the universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_{\varphi_\tau})$. Let us write $\mathbf{b} = (b_i)_{i \in I}$ for the image $\digamma^{\mathcal{G}}(\mathbf{a}) \subset (\mathcal{G}(\mathcal{A}), \tau_{\varphi_\tau})$. We say that the family $\mathbf{a}$ is *unitarily invariant in the traffic sense (or UIT for short)* if $\mathbf{a}$ and $\mathbf{b}$ have the same traffic distribution $\nu_{\mathbf{a}} = \nu_{\mathbf{b}}$.

Unitarily invariant traffics frequently appear as limit objects for a wide variety of random matrix ensembles, including those quite far away from any such symmetry in the traditional sense: for example, random band matrices of slow growth [Au].

Note that our analysis seemingly precludes random matrices over $\mathbb{R}$. Indeed, one quickly sees that the same results cannot possibly hold in this generality. For example, the transpose result fails for real symmetric Wigner matrices $\mathbf{W}_N^\intercal = \mathbf{W}_N$. Still, with some maneuvering, one can interpret the structural results for $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$ in the real case: for example, to extend the results of Bryc, Dembo, and Jiang on random Markov matrices [BDJ06]. We come back to these ideas and give a precise statement for all of these results in Chapter 4 once we have analyzed the structure of the universal enveloping traffic space. At this point, we are sufficiently prepared to move onto the main part of the article.

Chapter 3

Rigid structures in the universal enveloping traffic space

The construction of the universal enveloping traffic space entails an unexpectedly rigid probabilistic structure, the investigation of which we carry out in the remainder of this chapter. We begin with a precise statement of the main results in Section 3.1. From there, we move onto the combinatorics of cactus graphs in Section 3.2: the simple but surprisingly effective tools developed here will be frequently employed in the sequel. Section 3.3 realizes the universal enveloping traffic space as a free product of three natural sub-$*$-probability spaces via a canonical conditional expectation onto a space spanned by elementary graph operations. This motivates the direction of Section 3.4, which establishes a duality between classical independence and free independence by way of the diagonal algebra. At the same time, our conditional expectation suggests a great deal redundancy in the action of the graph operations. In fact, up to degeneracy, the universal enveloping traffic space is spanned by tree-like graph operations: this follows from the cycle pruning algorithm in Section 3.5. The results in this chapter form part of a joint work in progress with Camille Male [AM].

3.1 Introduction and main results

Our first result extends the inherent free independence structure in the universal enveloping traffic space to general graph operations (cf. Proposition 2.4.7). For notational convenience, we write $(\mathcal{B}, \psi) = (\mathcal{G}(\mathcal{A}), \varphi_{\tau_\varphi})$ to emphasize the underlying $*$-probability space structure of the universal enveloping traffic space. We distinguish two special classes of graph operations:

$$\Delta(\mathcal{G}_K) = \{g \in \mathcal{G}_K : \text{input}(g) = \text{output}(g)\}$$

and

$$\Theta(\mathcal{G}_K) = \{g \in \mathcal{G}_K : \text{there exists an undirected cycle in } g \\ \text{that visits both the input and the output}\}.$$

In particular, we allow for a repetition of vertices in the cycle, but not edges. For example, $\underset{\text{out}}{\cdot} \overset{1}{\underset{2}{\leftleftarrows}} \cdot \overset{3}{\underset{4}{\leftleftarrows}} \underset{\text{in}}{\cdot} \in \Theta(\mathcal{G}_4)$. In this notation, the diagonal algebra becomes

$$\Delta(\mathcal{B}) = \Delta(\mathcal{G}(\mathcal{A})) = \operatorname{span}\left(\bigcup_{K\geq 0} \bigcup_{g\in\Delta(\mathcal{G}_K)} Z_g(\mathcal{A}^{\otimes K}) \right).$$

Similarly, we define the unital $*$-subalgebra

$$\Theta(\mathcal{B}) = \Theta(\mathcal{G}(\mathcal{A})) = \operatorname{span}\left(\bigcup_{K\geq 0} \bigcup_{g\in\Theta(\mathcal{G}_K)} Z_g(\mathcal{A}^{\otimes K}) \right).$$

Note that

$$\mathrm{rDeg}, \mathrm{cDeg} \in \Delta(\mathcal{G}_K) \subset \Theta(\mathcal{G}_K),$$

and so

$$\mathrm{Deg}(\mathcal{A}) \subset \Delta(\mathcal{B}) \subset \Theta(\mathcal{B}).$$

Theorem 3.1.1. *Let $(\mathcal{A}, \varphi)$ be a tracial $*$-probability space. Then the unital $*$-subalgebras $\mathcal{A}$, $\mathcal{A}^{\intercal}$, and $\Delta(\mathcal{B})$ are freely independent in the universal enveloping traffic space $(\mathcal{B}, \psi)$. Accordingly, we identify their free product as a sub-$*$-probability space*

$$(\mathcal{A}, \psi|_{\mathcal{A}}) * (\mathcal{A}^{\intercal}, \psi|_{\mathcal{A}^{\intercal}}) * (\Delta(\mathcal{B}), \psi|_{\Delta(\mathcal{B})}) \subset (\mathcal{B}, \psi).$$

Moreover, there exists a homomorphic conditional expectation $\mathscr{E} : \mathcal{B} \to \mathcal{A} * \mathcal{A}^{\intercal} * \Delta(\mathcal{B})$ such that $\mathscr{E}^{-1}(\Delta(\mathcal{B})) = \Theta(\mathcal{B})$. In fact, the map $\mathscr{E}$ is the unique homomorphic conditional expectation that satisfies the commutation

$$\begin{CD} \mathcal{B}\otimes\mathcal{B} @>{\Delta\otimes\Delta}>> \mathcal{B}\otimes\mathcal{B} \\ @V{\circ}VV @VV{\cdot_{\mathcal{G}}}V \\ \mathcal{B} @>{\mathscr{E}}>> \mathcal{B} \end{CD} \tag{3.1}$$

where $\circ(\cdot_1, \cdot_2) = \underset{\text{out}}{\cdot} \overset{\cdot_1}{\underset{\cdot_2}{\leftleftarrows}} \underset{\text{in}}{\cdot}$ and $\cdot_{\mathcal{G}}$ is the standard multiplication induced by the graph operations. Altogether, this further implies the free product decomposition

$$(\mathcal{B}, \psi) = (\mathcal{A}, \psi|_{\mathcal{A}}) * (\mathcal{A}^{\intercal}, \psi|_{\mathcal{A}^{\intercal}}) * (\Theta(\mathcal{B}), \psi|_{\Theta(\mathcal{B})}). \tag{3.2}$$

Notably, Theorem 3.1.1 holds regardless of the choice of tracial $*$-probability space $(\mathcal{A}, \varphi)$. Naturally, one can then ask how an existing free product structure

$$(\mathcal{A}, \varphi) = *_{i \in I} (\mathcal{A}_i, \varphi_i) \tag{3.3}$$

behaves in this construction. Proposition 4.8 in [CDM] shows that the free independence of the $(\mathcal{A}_i)_{i \in I}$ is equivalent to the traffic independence of the sub-traffic spaces

$$(\mathcal{G}(\mathcal{A}_i))_{i \in I} \subset (\mathcal{G}(\mathcal{A}), \tau_\varphi).$$

From a different perspective, we can study these sub-traffic spaces as sub-$*$-probability spaces

$$(\mathcal{B}_i, \psi|_{\mathcal{B}_i})_{i \in I} = (\mathcal{G}(\mathcal{A}_i), \varphi_{\tau_\varphi}|_{\mathcal{G}(\mathcal{A}_i)})_{i \in I}.$$

Of course, the free product decomposition (3.2) still applies, and so we know the behavior of the cross-terms in the decomposition

$$(\mathcal{B}_i)_{i \in I} = (\mathcal{A}_i * \mathcal{A}_i^\intercal * \Theta(\mathcal{B}_i))_{i \in I}$$

grace of the inclusions

$$\mathcal{A}_i \subset \mathcal{A}, \quad \mathcal{A}_i^\intercal \subset \mathcal{A}^\intercal, \quad \text{and} \quad \Theta(\mathcal{B}_i) \subset \Theta(\mathcal{B}).$$

Thus, it remains to understand the relationship within each of the three "columns"

$$(\mathcal{A}_i)_{i \in I}, \quad (\mathcal{A}_i^\intercal)_{i \in I}, \quad \text{and} \quad (\Theta(\mathcal{B}_i))_{i \in I}.$$

By assumption, the $(\mathcal{A}_i)_{i \in I}$ are freely independent, which is preserved in the universal enveloping traffic space since $\psi|_{\mathcal{A}} = \varphi$. Moreover, recall that the transpose map $\intercal : \mathcal{A} \to \mathcal{A}^\intercal$ defines an involutive anti-isomorphism of $*$-probability spaces. Thus, the free product structure (3.3) is directly transported to the transposed algebras

$$(\mathcal{A}^\intercal, \psi|_{\mathcal{A}^\intercal}) = *_{i \in I} (\mathcal{A}_i^\intercal, \psi|_{\mathcal{A}_i^\intercal}). \tag{3.3$^\intercal$}$$

Finally, we come to the last column $(\Theta(\mathcal{B}_i))_{i \in I}$. Applying our conditional expectation $\mathscr{E}$, we can further reduce this problem to understanding the relationship between the diagonal algebras

$$(\mathscr{E}(\Theta(\mathcal{B}_i)))_{i \in I} = (\Delta(\mathcal{B}_i))_{i \in I}.$$

Since the diagonal algebra $\Delta(\mathcal{B}_i) \subset \Delta(\mathcal{B})$ is commutative, the trend of free independence in the first two columns cannot possibly continue. Instead, we find an interesting connection to the classical framework.

Theorem 3.1.2. *Let $(\mathcal{A}, \varphi)$ be a tracial $*$-probability space with freely independent unital $*$-subalgebras $(\mathcal{A}_i)_{i \in I}$. Then the commutative sub-traffic spaces $(\Delta(\mathcal{B}_i))_{i \in I}$ are classically independent in $(\mathcal{B}, \psi)$. We formulate this in the heuristic equation*

$$\text{classical independence} \quad \overset{\Delta}{\Longleftarrow} \quad \text{free independence.} \tag{3.4}$$

Let us discuss the content of equation (3.4), particularly in relation to equation (1.1). Before, starting with classical independence, we obtain free independence through a natural process of "non-commutification," namely, passing to a matrix algebra and taking a limit. In the opposite direction, Theorem 3.1.2 starts with free independence; however, the route back to the commutative world becomes unclear. We could hope to make use of the traffic framework, where the diagonal algebra emerges as a natural "commutification" of our space. In particular, by pushing $(\mathcal{A}, \varphi)$ up to the universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$, we can project down to the diagonal algebra $\Delta(\mathcal{G}(\mathcal{A}))$. As it turns out, the shadow cast by free independence in this projection is precisely classical independence. In this way, equations (1.1) and (3.4) establish a duality between classical independence and free independence.

Traditionally, we think of classical independence as the first concept with freeness arising as a second concept when abstracting to the NC framework. Here, we take freeness as the first concept and recover classical independence by projecting to the commutative world. We can find a simple manifestation of this principle at the level of random matrices: for example, independent GOE matrices $(\mathbf{W}_N^{(i)})_{i\in I}$ are asymptotically free semicircular random variables, whereas their degree matrices $(\mathrm{rDeg}(\mathbf{W}_N^{(i)}))_{i\in I} = (\mathrm{cDeg}(\mathbf{W}_N^{(i)}))_{i\in I}$ are asymptotically classically independent Gaussian random variables.

Theorem 3.1.2 completes the characterization of the joint $*$-distribution of the $(\mathcal{B}_i)_{i\in I}$. In particular, we note that the $(\Theta(\mathcal{B}_i))_{i\in I}$ are *not* classically independent as they do not commute. At the same time, the conditional expectation $\mathscr{E}$ allows us to compute the trace on $*$-alg$(\bigcup_{i\in I}\Theta(\mathcal{B}_i))$ as if they did: namely, for any $t_1, \ldots, t_n \in *$-alg$(\bigcup_{i\in I}\Theta(\mathcal{B}_i)) \subset \Theta(\mathcal{B})$,

$$\begin{aligned}\psi(t_1\cdots t_n) &= \psi(\mathscr{E}(t_1\cdots t_n))\\ &= \psi(\mathscr{E}(t_1)\cdots\mathscr{E}(t_n))\\ &= \psi(\mathscr{E}(t_{\pi(1)})\cdots\mathscr{E}(t_{\pi(n)}))\\ &= \psi(\mathscr{E}(t_{\pi(1)}\cdots t_{\pi(n)})) = \psi(t_{\pi(1)}\cdots t_{\pi(n)}), \qquad \forall \pi \in \mathfrak{S}_n.\end{aligned}$$

In fact, the conditional expectation $\mathscr{E}$ satisfies the stronger property

$$\mathscr{E}(t) \equiv t \pmod{\psi}, \qquad \forall t \in \mathcal{B}.$$

Thus, from the distributional point of view, the reduction from $\Theta(\mathcal{B})$ to $\mathscr{E}(\Theta(\mathcal{B})) = \Delta(\mathcal{B})$ comes without loss of generality. This redundancy in the action of the graph operations extends a great deal further. To make this precise, we introduce a subclass of the diagonal graph operations

$$\Delta_{\mathrm{tree}}(\mathcal{G}_K) = \{g \in \Delta(\mathcal{G}_K) : \ g \text{ is a tree}\},$$

with the obvious definition for the subalgebra $\Delta_{\mathrm{tree}}(\mathcal{B})$.

Theorem 3.1.3. *For any $t \in \Delta(\mathcal{B})$, there exists a $\mathbf{T}(t) \in \Delta_{\mathrm{tree}}(\mathcal{B})$ such that*

$$\mathbf{T}(t) \equiv t \pmod{\psi}.$$

We can apply Theorem 3.1.3 to each of the diagonal components in the free product decomposition $\mathcal{B} \equiv \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B}) \pmod{\psi}$ to show that a similar statement holds for general graph polynomials $t \in \mathcal{B}$. In this case, the graph polynomial of trees $\mathbf{T}(t) \equiv t \pmod{\psi}$ is obtained as a linear combination of graph operations Z_g such that $g \in \mathcal{G}$ is a tree. As such, we think of the universal enveloping traffic space as being spanned by tree-like graph operations.

To prove all of these results, we will need a better understanding of the structure of cactus graphs, which we undertake in the next section.

3.2 The combinatorics of cactus graphs

To begin, we recall some basic notions from graph theory, largely following [Bol98, GR01].

Definition 3.2.1 (Connectivity)**.** Let $G = (V, E)$ be a finite multigraph. An *edge cutset* in G is a subset of edges $E' \subset E$ whose deletion increases the number of connected components. In particular, a *cut-edge* is an edge $e \in E$ such that the singleton $\{e\}$ is an edge cutset. Similarly, a *vertex cutset* in G is a subset of vertices $V' \subset V$ whose deletion (along with all edges adjacent to V') increases the number of connected components. A *cut-vertex* is defined in the obvious way. A *block* of G is a maximal cut-vertex-free connected subgraph $H \subset G$. Note that any two distinct blocks of G can only have at most one vertex in common (necessarily a cut-vertex of G). Conversely, every cut-vertex of G belongs to at least two distinct blocks.

Suppose now that G is further connected. The *edge connectivity* of G is the size of the smallest edge cutset in G, which we denote by $\lambda(G) \geq 1$. We say that G is k*-edge-connected* if $k \leq \lambda(G)$: in words, deleting any $\ell < k$ edges of G does not affect its connectivity. In particular, we say that G is *two-edge-connected* (or *t.e.c.* for short) if it has no cut-edges.

We can define a similar notion for vertices even if G is not connected. The *edge connectivity* of two distinct vertices $v, w \in V$ is the size of the smallest subset of edges whose deletion disconnects v and w, which we denote by $\lambda(v, w) \geq 0$. We say that v and w are k*-edge-connected* if $k \leq \lambda(v, w)$.

We will need the following version of Menger's theorem for edge connectivity [Men27].

Theorem 3.2.2. *Let v and w be distinct vertices of a finite multigraph G. Then the edge connectivity of v and w is equal to the maximum number of edge-disjoint paths from v to w.*

In view of Menger's theorem, we also say that k-edge-connected vertices v and w form a k*-connection*, particularly when we want to emphasize the number of edge-disjoint paths connecting v and w. We apply this to obtain a simple characterization of cactus graphs.

Proposition 3.2.3. *A finite multigraph $G = (V, E)$ is a cactus iff*

$$\lambda(v, w) = 2, \qquad \forall v \neq w \in V.$$

Proof. First, suppose that G is a cactus. Note that a cactus can be reconstructed from its pads by "growing" the cactus as follows: start at level 0 by choosing an arbitrary pad $C^{(0)} \in \text{Pad}(G)$ to be the base. At level 1, attach all of the remaining pads $C_1^{(1)}, \ldots, C_{\ell_1}^{(1)}$ that share a vertex with $C^{(0)}$. Of course, each pad can only share at most one vertex with another pad, so we imagine each $C_i^{(1)}$ as growing from a vertex in $C^{(0)}$. Furthermore, note that if any two pads at this level share a vertex, then it has to be the same vertex that they each share with $C^{(0)}$. Otherwise, the cactus has grown in on itself and one can easily find an edge that belongs to more than one simple cycle. At level n, we attach all of the remaining pads that share a vertex with a pad at level $n-1$. In particular, we can think of each pad at level $n-1$ as a new base and growing the remaining pads on each base. In this case, we further note that a pad at level n can only be attached to a single pad at level $n-1$; otherwise, one can again find an edge that belongs to more than one simple cycle.

Now, since every edge belongs to a unique simple cycle, G is necessarily t.e.c. This implies that

$$\lambda(v, w) \geq \lambda(G) \geq 2, \qquad \forall v \neq w \in V.$$

If v and w belong to a common pad, then we can assume that this pad is the base $C^{(0)}$. In this case, deleting the two edges adjacent to v in $C^{(0)}$ clearly disconnects v and w in G. If v and w do not belong to a common pad, then we can again take a pad containing v to be the base $C^{(0)}$. In this case, w belongs to a pad that was successively grown from an ancestor $C_1^{(1)}$ on level 1, say attached to a vertex $u \in C^{(0)}$. Deleting the two edges adjacent to u in $C_1^{(1)}$ then clearly disconnects v and w in G. It follows that

$$\lambda(v, w) = \lambda(G) = 2, \qquad \forall v \neq w \in V.$$

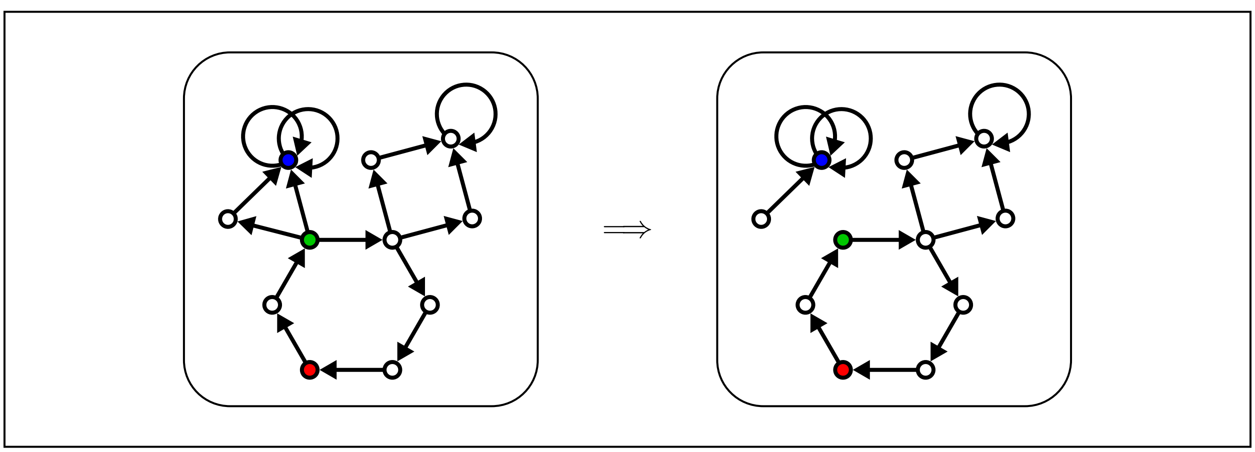


Figure 3.1: An example of the edge connectivity of two distinct vertices in the cactus from Figure 2.5. Here, we disconnect the red vertex $v \in C^{(0)}$ and the blue vertex $w \in C_1^{(1)}$. The green vertex denotes the ancestor vertex $u \in C^{(0)}$.

In the opposite direction, assume that the edge connectivity of every pair of vertices in G is equal to two. This implies that G is connected with $\lambda(G) = 2$, and so every edge belongs to a simple cycle. For a contradiction, assume that there exists an edge $e \in E$ that belongs to two distinct simple cycles C_1 and C_2. Here, we allow for common edges between the two cycles; however, there must be at least one edge $e_1 \in C_1 \setminus C_2$ (and one edge $e_2 \in C_2 \setminus C_1$). Let E' be a separating pair of edges for the distinct vertices v and w adjacent to e (a loop belongs to a unique simple cycle). Of course, it must be that $e \in E'$. In fact, the second edge in E' must be another edge in $C_1 \cap C_2$ since otherwise v and w are not separated. Thus, deleting e and e_1 from G does not separate v and w. Let P be a simple path from v to w in the (e, e_1)-deleted graph $\widetilde{G}$. By construction, P cannot stay in C_1. Let e_P be the first edge along the path P that leaves C_1, and define v_P to be the vertex in C_1 adjacent to e_P. Similarly, let f_P be the first edge along this path that returns to C_1, and define w_P to be the vertex in C_1 adjacent to f_P. By construction, $v_P \neq w_P$ form a 3-connection in G: two edge-disjoint paths come from inside the cycle C_1, while a third comes from the truncation of P outside of C_1. Menger's theorem then implies that $\lambda(v_P, w_P) \geq 3$, a contradiction. ■

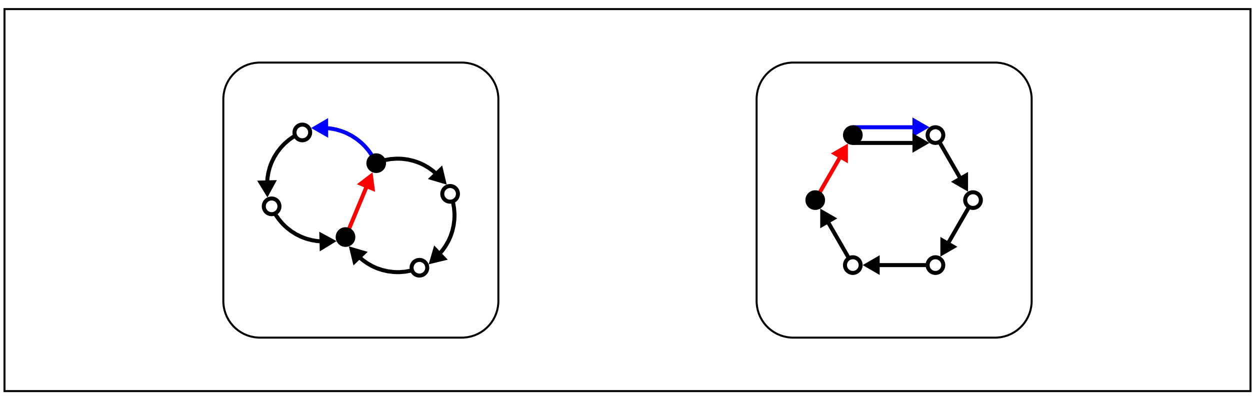


Figure 3.2: Two representative cases of the contradiction argument in the above proof. Here, we have colored the edge e belonging to two distinct simple cycles red, its adjacent vertices v and w black, and the edge $e_1 \in C_1 \setminus C_2$ blue. The reader should identify the vertices v_P and w_P as well as the associated 3-connection.

As a first application, we use Proposition 3.2.3 to give a proof of Proposition 2.4.6.

Proof of Proposition 2.4.6. We will only prove the equivalence of (i) and (ii) as the rest of the proposition follows almost immediately. First, suppose that C^π is a cactus. For a contradiction, assume that $\pi \notin \mathcal{NC}(V)$. After a suitable rotation of the cycle, this implies that there exist $i_1 < i_2 < i_3 < i_4 \in [n]$ such that $v_{i_1} \overset{\pi}{\sim} v_{i_3}$ and $v_{i_2} \overset{\pi}{\sim} v_{i_4}$ belong to different blocks of π. In this case, v_{i_2} and v_{i_3} form a 3-connection in C^π, which is absurd.

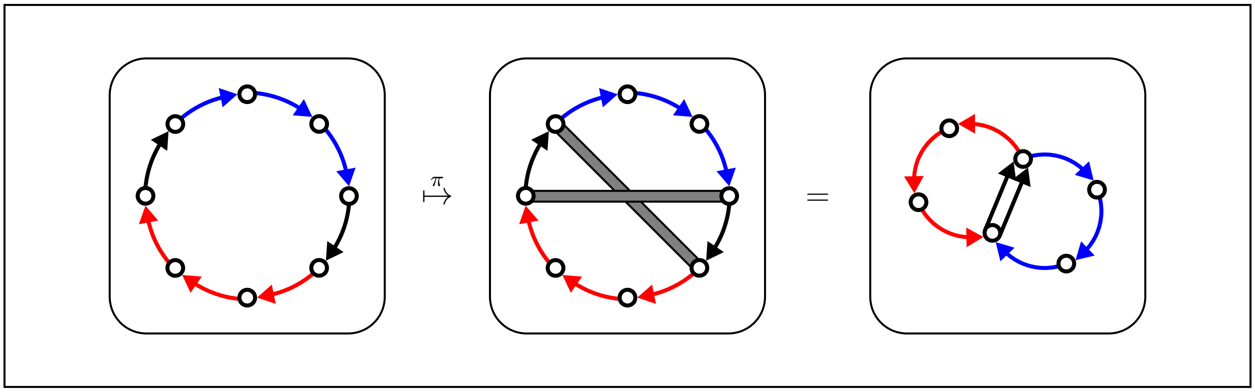


Figure 3.3: An example of a crossing leading to a 3-connection.

In the opposite direction, suppose that $\pi \in \mathcal{NC}(V)$. Naturally, we can identify the blocks $B_1, \dots, B_{\#(\pi)} \in \pi$ with the vertices of C^π. Since the original graph C is a cactus, we know that

$$\lambda(v_{i_1}, v_{i_2}) = 2, \qquad \forall i_1 \neq i_2 \in [n].$$

At the same time, edge-disjoint paths in C induce edge-disjoint paths in the quotient C^π, which implies that

$$\lambda(B_{j_1}, B_{j_2}) \geq 2, \qquad \forall j_1 \neq j_2 \in [\#(\pi)].$$

Now, because π is non-crossing, two distinct blocks $B_{j_1}, B_{j_2} \in \pi$ can only take one of two relative positions up to symmetry: in particular, after a suitable rotation of the cycle, either

$$v_1 \in B_{j_1} \quad \text{and} \quad \max\{i \in [n] : v_i \in B_{j_1}\} < \min\{i \in [n] : v_i \in B_{j_2}\}$$

or

$$\exists v_{i_1}, v_{i_2} \in B_{j_2} : \forall v_i \in B_{j_1} \qquad i_1 < i < i_2.$$

In words, the first case corresponds to when B_{j_1} and B_{j_2} lie on two non-intersecting arcs of the cycle C drawn as a circle, whereas the second case corresponds to when B_{j_1} lies on an arc trapped between two vertices of B_{j_2}. In either case, deleting the edges

$$e_{\min\{i \in [n] : v_i \in B_{j_1}\}-1} \quad \text{and} \quad e_{\max\{i \in [n] : v_i \in B_{j_1}\}}$$

disconnects B_{j_1} and B_{j_2} in C^π. We think of the identification

$$v_{\min\{i \in [n] : v_i \in B_{j_1}\}} \overset{\pi}{\sim} v_{\max\{i \in [n] : v_i \in B_{j_1}\}}$$

as pinching off the arc supporting B_{j_1}. By removing the edges at the boundary of this arc, we have separated the vertices lying on this arc from the rest of the vertices, even in the quotient C^π (any identification of vertices across the two arcs would be crossing). It follows that

$$\lambda(B_{j_1}, B_{j_2}) = 2, \qquad \forall j_1 \neq j_2 \in [\#(\pi)],$$

and so C^π is a cactus. ■

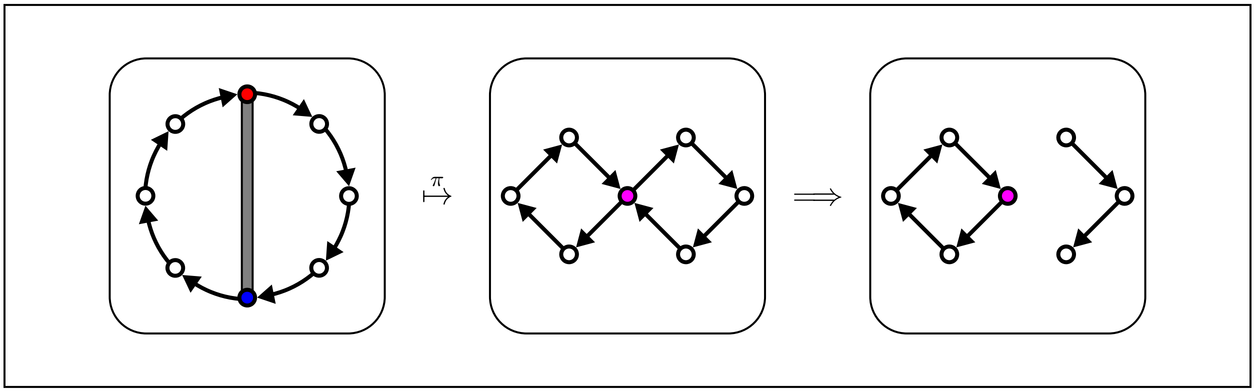


Figure 3.4: A depiction of the pinching argument and the corresponding edge removals that disconnect the arc. Here, we color the vertices $v_{\min\{i\in[n]:v_i\in B_{j_1}\}}$ and $v_{\max\{i\in[n]:v_i\in B_{j_1}\}}$ red and blue respectively.

We emphasize an important point in the proof above, namely, that edge-disjoint paths in a graph G induce edge-disjoint paths in a quotient G^π. Thus, if two vertices $v \neq w$ in G are not identified $v \overset{\pi}{\nsim} w$ in G^π, then their edge connectivity (weakly) increases $\lambda_{G^\pi}(v,w) \geq \lambda_G(v,w)$. This immediately implies the following useful lemma for verifying cactus quotients.

Lemma 3.2.4 (A 3-connection criteria for cacti)**.** *Let $G = (V,E)$ be a finite multigraph, and suppose that the vertices $v \neq w \in V$ form a 3-connection in G. If a partition $\pi \in \mathcal{P}(V)$ induces a cactus G^π, then $v \overset{\pi}{\sim} w$.*

We give a number of applications of the 3-connection lemma to computations in the universal enveloping traffic space $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$.

Proposition 3.2.5. *Let $T = (V,E,\gamma)$ be a test graph in $\mathcal{A}$, and suppose that the vertices $v \neq w \in V$ form a 3-connection in T. We write $T_{v\sim w}$ for the test graph obtained from T by identifying the vertices v and w, in which case*

$$\tau_\varphi[T] = \tau_\varphi[T_{v\sim w}].$$

Proof. Since τ_φ^0 is supported on (oriented) cacti, the 3-connection lemma implies that

$$\tau_\varphi[T] = \sum_{\pi\in\mathcal{P}(V)} \tau_\varphi^0[T^\pi] = \sum_{\substack{\pi\in\mathcal{P}(V)\\ \text{s.t. } v\overset{\pi}{\sim}w}} \tau_\varphi^0[T^\pi] = \tau_\varphi[T_{v\sim w}].$$

■

Corollary 3.2.6. *Let $t \in \mathcal{G}(\mathcal{A})$ be a graph monomial with vertices $v \neq w$ that form a 3-connection. We write $t_{v\sim w}$ for the graph monomial obtained from t by identifying the vertices v and w, in which case*

$$t_{v\sim w} \equiv t \pmod{\psi}.$$

Proof. This amounts to proving that

$$\psi(tt') = \psi(t_{v\sim w}t'), \qquad \forall t' \in \mathcal{G}(\mathcal{A}).$$

In particular, without loss of generality, we may assume that t' is also a graph monomial. For a graph monomial s, recall that $\widetilde{\Delta}(s)$ denotes the test graph obtained from s by identifying the input and the output and forgetting their distinguished roles. Then there are only two possibilities: either v and w are identified in $\widetilde{\Delta}(tt')$ and $\widetilde{\Delta}(tt') = \widetilde{\Delta}(t_{v\sim w}t')$ outright, or v and w are not identified in $\widetilde{\Delta}(tt')$ and $(\widetilde{\Delta}(tt'))_{v\sim w} = \widetilde{\Delta}(t_{v\sim w}t')$. In the second case, $v \neq w$ still form a 3-connection in $\widetilde{\Delta}(tt')$. Thus, in any event,

$$\psi(tt') = \tau_\varphi\big[\widetilde{\Delta}(tt')\big] = \tau_\varphi\big[\widetilde{\Delta}(t_{v\sim w}t')\big] = \psi(t_{v\sim w}t').$$

■

Iterating the 3-connection lemma, we arrive at the following definition.

Definition 3.2.7 (Quasi-cactus)**.** A finite, connected multigraph $G = (V, E)$ is said to be a *quasi-cactus* if

$$\lambda(v, w) \leq 2, \qquad \forall v \neq w \in V.$$

Equivalently, every edge $e \in E$ belongs to at most one simple cycle.

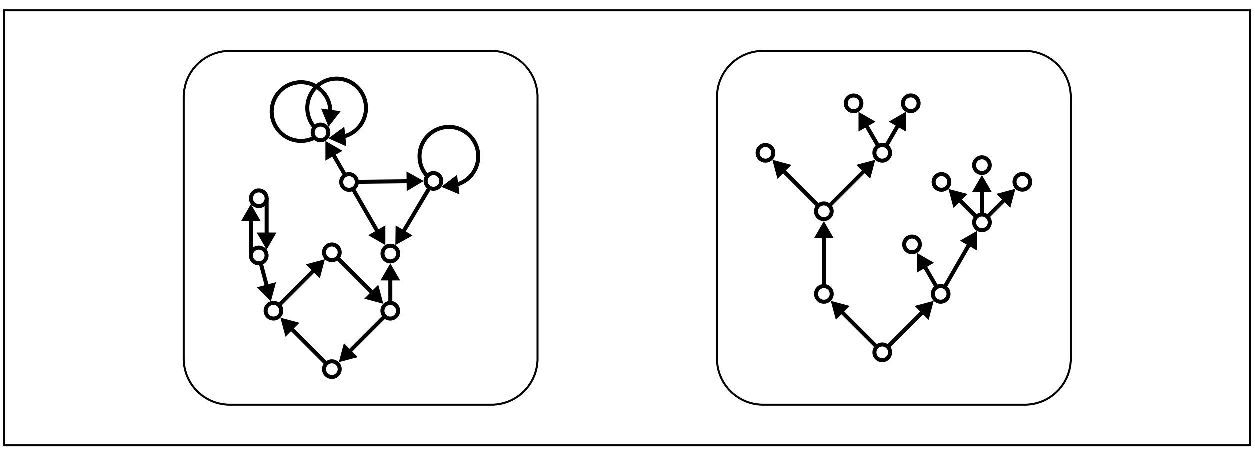


Figure 3.5: Examples of quasi-cacti. We can think of a quasi-cactus as a wiring of cacti, as in the first example above. Of course, a quasi-cactus could consist entirely of the wires, as in the second example above.

Corollary 3.2.8. *For every graph monomial $t \in \mathcal{G}(\mathcal{A})$, there exists a quasi-cactus quotient t^π such that*

$$t^\pi \equiv t \ (\mathrm{mod}\ \psi).$$

Graph monomials with a cycle that visits both the input and the output play a special role in the free product decomposition of $\mathcal{G}(\mathcal{A})$. In particular, the construction of the conditional expectation in Theorem 3.1.1 crucially relies on the following equivalence.

Corollary 3.2.9. *Let $t \in \Theta(\mathcal{B})$ be a graph monomial. Then*

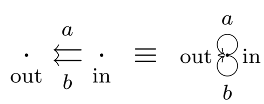
$$t \equiv \Delta(t) \pmod{\psi}.$$

Proof. Of course, if $t \in \Delta(\mathcal{B})$, then $\Delta(t) = t$ and we are done. Otherwise, the cycle condition ensures a 2-connection between $v := v_{\text{in}} \neq v_{\text{out}} =: w$ in t. Let $t' \in \mathcal{G}(\mathcal{A})$ be a graph monomial. If $t' \in \Delta(\mathcal{B})$, then $\Delta(tt') = \Delta(t)t'$. Otherwise, v and w form a 3-connection in $\widetilde{\Delta}(tt')$ with a third edge-disjoint path coming from the edges of t' ("going out the back door"). In particular, note that $\widetilde{\Delta}(tt')_{v\sim w} = \widetilde{\Delta}(\Delta(t)t')$. Thus, in any event,

$$\psi(tt') = \psi(\Delta(tt')) = \tau_\varphi\big[\widetilde{\Delta}(tt')\big] = \tau_\varphi\big[\widetilde{\Delta}(\Delta(t)t')\big] = \psi(\Delta(t)t').$$

■

Example 3.2.10. Let $a, b \in \mathcal{A}$. Then $a \circ b \equiv \Delta(a)\Delta(b) \pmod{\psi}$. Pictorially,



$(\text{mod } \psi)$.

◊

In fact, we can even use the 3-connection lemma to outright prune t.e.c. subgraphs that are attached at a single vertex. We generalize this idea to obtain a generic cycle pruning algorithm in Section 3.5.

Lemma 3.2.11. *Let $T_1 = (V_1, E_2, \gamma_1)$ and $T_2 = (V_2, E_2, \gamma_2)$ be test graphs in $\mathcal{A}$, and suppose that T_2 is t.e.c. We write $T_1 \sharp T_2 = (V_1 \sharp V_2, E_1 \sharp E_2, \gamma_1 \sharp \gamma_2)$ for the test graph obtained from T_1 and T_2 by identifying an arbitrary vertex v_1 of T_1 with an arbitrary vertex v_2 of T_2, in which case*

$$\tau_\varphi\big[T_1 \sharp T_2\big] = \tau_\varphi\big[T_1\big]\tau_\varphi\big[T_2\big].$$

In particular, this factorization is independent of the choice of vertices v_1 and v_2.

Proof. To begin, note that $V_1 \sharp V_2 = (V_1 \sqcup V_2)/(v_1 \sim v_2)$, $E_1 \sharp E_2 = E_1 \sqcup E_2$, and

$$\gamma_1 \sharp \gamma_2 : E_1 \sharp E_2 \to \mathcal{A}, \qquad (\gamma_1 \sharp \gamma_2)|_{E_i} = \gamma_i.$$

In particular, we denote the amalgamated vertex $v_1 \sim v_2 \in V_1 \sharp V_2$ by ρ.

For any pair of partitions $\pi_1 \in \mathcal{P}(V_1)$ and $\pi_2 \in \mathcal{P}(V_2)$, we define the subset of partitions $\mathcal{P}_\rho(\pi_1, \pi_2) \subset \mathcal{P}(V_1 \sharp V_2)$ by

$$\mathcal{P}_\rho(\pi_1, \pi_2) = \{\pi \in \mathcal{P}(V_1 \sharp V_2) : \pi|_{V_i} = \pi_i\}.$$

In words, $\mathcal{P}_\rho(\pi_1,\pi_2)$ consists of the partitions $\pi \in \mathcal{P}(V_1\sharp V_2)$ obtained from (π_1,π_2) by either keeping a block $B_i \in \pi_i$ (so $B_i \in \pi$) or merging it with at most a single block $B_j \in \pi_j$ where $j \neq i$ (so $B_i \cup B_j \in \pi$). Of course, the block in π_1 containing v_1 and the block in π_2 containing v_2 are necessarily merged. Indeed, the minimal element $\pi_\rho(\pi_1,\pi_2) \in \mathcal{P}_\rho(\pi_1,\pi_2)$ for the usual reversed refinement order keeps every block of π_1 and π_2 separate otherwise.

By construction,

$$\bigsqcup_{(\pi_1,\pi_2)\in\mathcal{P}(V_1)\times\mathcal{P}(V_2)} \mathcal{P}_\rho(\pi_1,\pi_2) = \mathcal{P}(V_1\sharp V_2),$$

and so we can compute

$$\tau_\varphi\big[T_1\sharp T_2\big] = \sum_{\pi\in\mathcal{P}(V)} \tau_\varphi^0\big[(T_1\sharp T_2)^\pi\big] = \sum_{(\pi_1,\pi_2)\in\mathcal{P}(V_1)\times\mathcal{P}(V_2)} \sum_{\pi\in\mathcal{P}_\rho(\pi_1,\pi_2)} \tau_\varphi^0\big[(T_1\sharp T_2)^\pi\big].$$

Assume that $\pi \in \mathcal{P}_\rho(\pi_1,\pi_2)\setminus\{\pi_\rho(\pi_1,\pi_2)\}$. Then there exists a vertex v in T_1 and a vertex w in T_2 such that

$$v \overset{\pi}{\sim} w \overset{\pi}{\not\sim} \rho.$$

Since T_2 is t.e.c., w and ρ form a 2-connection in T_2. But then $v \overset{\pi}{\sim} w$ and ρ form a 3-connection in $(T_1\sharp T_2)^\pi$ with a third edge-disjoint path coming from the edges of T_1. In this case, the 3-connection lemma implies that $\tau_\varphi^0\big[(T_1\sharp T_2)^\pi\big] = 0$, and so

$$\sum_{\pi\in\mathcal{P}_\rho(\pi_1,\pi_2)} \tau_\varphi^0\big[(T_1\sharp T_2)^\pi\big] = \tau_\varphi^0\big[(T_1\sharp T_2)^{\pi_\rho(\pi_1,\pi_2)}\big].$$

The cactus structure of τ_φ^0 further implies that if S is a test graph composed of otherwise disjoint test graphs $S_1,\ldots,S_n$ all attached at a single vertex, then

$$\tau_\varphi^0\big[S\big] = \prod_{i=1}^n \tau_\varphi^0\big[S_i\big]. \tag{3.5}$$

In particular, $\tau_\varphi^0\big[(T_1\sharp T_2)^{\pi_\rho(\pi_1,\pi_2)}\big] = \tau_\varphi^0\big[T_1^{\pi_1}\big]\tau_\varphi^0\big[T_2^{\pi_2}\big]$, and so

$$\begin{aligned}\tau_\varphi\big[T_1\sharp T_2\big] &= \sum_{(\pi_1,\pi_2)\in\mathcal{P}(V_1)\times\mathcal{P}(V_2)} \tau_\varphi^0\big[T_1^{\pi_1}\big]\tau_\varphi^0\big[T_2^{\pi_2}\big] \\ &= \Bigg(\sum_{\pi_1\in\mathcal{P}(V_1)} \tau_\varphi^0\big[T_1^{\pi_1}\big]\Bigg)\Bigg(\sum_{\pi_2\in\mathcal{P}(V_2)} \tau_\varphi^0\big[T_2^{\pi_2}\big]\Bigg) = \tau_\varphi\big[T_1\big]\tau_\varphi\big[T_2\big].\end{aligned}$$

■

Corollary 3.2.12. *Let $t \in \mathcal{G}(\mathcal{A})$ be a graph monomial, and suppose that T is a t.e.c. test graph in $\mathcal{A}$. We write $t\sharp T$ for the graph monomial obtained from t and T by identifying an arbitrary vertex v of t with an arbitrary vertex w of T, in which case*

$$t\sharp T \equiv \tau\big[T\big]t \pmod{\psi}.$$

In particular, this equivalence is independent of the choice of vertices v and w.

Proof. As usual, let $t' \in \mathcal{G}(\mathcal{A})$ be a graph monomial. We abbreviate $T_1 = \widetilde{\Delta}(tt')$ and $T_2 = \widetilde{\Delta}((t\sharp T)t')$. In the notation of Lemma 3.2.11, we can write

$$T_2 = T_1 \sharp T,$$

where we attach T to T_1 by identifying the vertex w of T with the (image of the) vertex $v \in t$ in T_1. Since T is t.e.c., this implies that

$$\psi((t\sharp T)t') = \tau_\varphi[T_2] = \tau_\varphi[T_1 \sharp T] = \tau_\varphi[T_1]\tau_\varphi[T] = \psi(tt')\tau_\varphi[T] = \psi((\tau_\varphi[T]t)t').$$

■

Example 3.2.13. Suppose that $t \in \Delta(\mathcal{B})$ is a graph monomial such that the underlying graph of t is a cactus. Then, up to degeneracy, t is a constant, namely,

$$t \equiv \prod_{C \in \text{Pads}(t)} \tau_\varphi[C] \pmod{\psi}.$$

◊

Of course, all of these results follow more or less from the same basic idea as captured in the 3-connection lemma, namely, that certain identifications must be made in order to contribute to the calculation of the injective traffic state τ_φ^0, and so it makes no difference if we make these identifications ahead of time. In a slightly different direction, we can also use the edge connectivity characterization of cactus graphs to track the image of t.e.c. subgraphs in a cactus quotient. In particular, we obtain the following simple but useful consequence.

Corollary 3.2.14. *Let G be a finite multigraph (V, E) with a t.e.c. subgraph $H = (W, F)$. If a partition $\pi \in \mathcal{P}(V)$ induces a cactus G^π, then the sub-quotient $H^{\pi|_W}$ is also a cactus. In particular, if G is further a multidigraph and G^π an oriented cactus, then the sub-quotient $H^{\pi|_W}$ is also an oriented cactus.*

Proof. We will only prove the first statement as the second statement follows almost immediately. Since H is t.e.c., so too is the quotient $H^{\pi|_w} = (\widetilde{W}, F)$. At the same time, $H^{\pi|_W}$ is also a subgraph of the cactus $G^\pi = (\widetilde{V}, E)$. Altogether, this implies that

$$2 \leq \lambda_{H^{\pi|_W}}(v, w) \leq \lambda_{G^\pi}(v, w) = 2, \qquad \forall v \neq w \in \widetilde{W} \subset \widetilde{V}.$$

■

3.3 Free products and conditional expectations

We will prove Theorem 3.1.1 in steps.

Lemma 3.3.1. *Let $(\mathcal{A}, \varphi)$ be a tracial $*$-probability space. Then the unital $*$-subalgebras $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$ are freely independent in the universal enveloping traffic space $(\mathcal{B}, \psi)$.*

Proof. We will show that any mixed cumulant in $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$ vanishes. To begin, let $t_1 \cdots t_{2n}$ be an alternating product in the subalgebras $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$, namely,

$$t_{2i+1} \in \Delta(\mathcal{B}) \quad \text{and} \quad t_{2i} \in \mathcal{A} \cup \mathcal{A}^\intercal.$$

In fact, we can always write a mixed product in $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$ in this form by strategically inserting the identity element

$$\underset{\text{out}}{\cdot} \xleftarrow{1_{\mathcal{A}}} \underset{\text{in}}{\cdot} = \underset{\text{out}}{\cdot} \xrightarrow{1_{\mathcal{A}}} \underset{\text{in}}{\cdot} = \underset{\text{in/out}}{\overset{1_{\mathcal{A}}}{\circlearrowleft}} = \underset{\text{in/out}}{\cdot} = 1_{\mathcal{B}} \in \mathcal{A} \cap \mathcal{A}^\intercal \cap \Delta(\mathcal{B}).$$

The multilinearity of the free cumulant functionals $(\kappa_n^{\mathcal{G}})_{n\in\mathbb{N}}$ allows us to further assume that each t_i is a graph monomial, and so we write

$$d_i = t_{2i+1} \in \Delta(\mathcal{B}) \quad \text{and} \quad a_{\bar{i}}^{\hat{\intercal}(\bar{i})} = t_{2i} \in \mathcal{A}^{\hat{\intercal}(\bar{i})},$$

where $\hat{\intercal} : [\overline{n}] \to \{1, \intercal\}$ indicates the transpose label and the indices $i \in [n]$ and $\bar{i} \in [\overline{n}]$ interlace as before. In particular, we write $[n + \overline{n}] = \{1 < \overline{1} < \cdots < n < \overline{n}\}$. The Kreweras complement then defines a function

$$K : \mathcal{NC}(n) \cup \mathcal{NC}(\overline{n}) \to \mathcal{NC}(n) \cup \mathcal{NC}(\overline{n})$$

such that

$$K(\mathcal{NC}(n)) = \mathcal{NC}(\overline{n}), \quad K(\mathcal{NC}(\overline{n})) = \mathcal{NC}(n), \quad \text{and} \quad K^2 = \text{id}.$$

Let us compute the trace of our alternating product. Our first step takes the form

$$\psi(d_1 a_{\overline{1}}^{\hat{\intercal}(\overline{1})} \cdots d_n a_{\overline{n}}^{\hat{\intercal}(\overline{n})}) = \tau_\varphi\big[T_n\big] = \sum_{\pi \in \mathcal{P}(V)} \tau_\varphi^0\big[T_n^\pi\big], \tag{3.6}$$

where $T_n = \widetilde{\Delta}(d_1 a_{\overline{1}}^{\hat{\intercal}(\overline{1})} \cdots d_n a_{\overline{n}}^{\hat{\intercal}(\overline{n})}) \in \mathcal{T}(\mathcal{A})$ with the vertex set V. In words, T_n consists of a cycle C_n of length n with vertices $v_1, \ldots, v_n$ in counterclockwise order; edges $v_i \overset{e_i}{\sim} v_{i+1}$ in the clockwise orientation if $\hat{\intercal}(\bar{i}) = 1$ (resp., in the counterclockwise orientation if $\hat{\intercal}(\bar{i}) = \intercal$); labels $\gamma(e_i) = a_{\bar{i}}$; and each monomial d_i attached at the vertex v_i via the identification

$$\text{input}(d_i) = \text{output}(d_i) \sim v_i.$$

One can think of the notation T_n as a shortening of $T_n(d_1, a_{\overline{1}}, \ldots, d_n, a_{\overline{n}})$. In particular, we visualize the test graph T_n as a cycle with a single loop (or "petal") attached at each vertex and the arguments $d_i, a_{\bar{i}}$ as indicating the edge labels: the loops are labeled by the d_i and the edges of the cycle are labeled by the $a_{\bar{i}}$. In the case of a loop, the edge label stands in for the graph monomial d_i that is to be rooted at that location (for example, by substitution). We draw the loops as undirected since the orientation plays no role in the substitution; the edges of the cycle are oriented according the transpose label $\hat{\intercal}$ (recall the proof of Proposition 2.4.7).

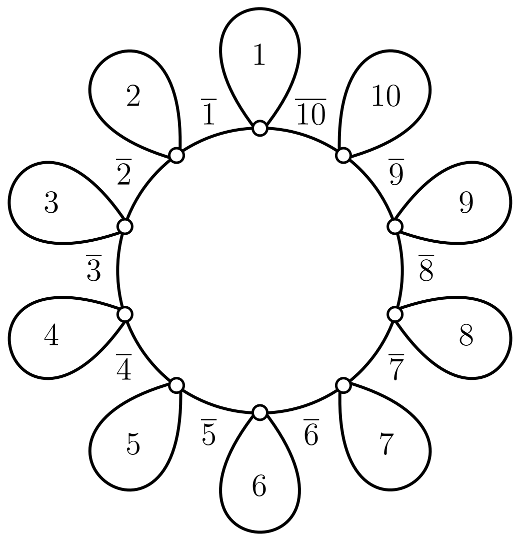


Figure 3.6: A visualization of the test graph T_n for $n = 10$. To highlight the relevant features, we omit the ordering of the edges and their labels. We will see how a non-crossing partition of the cycle C_n acts on the petals of T_n shortly.

Of course, not every partition contributes in the calculation of (3.6). We can narrow down the summands by finding necessary conditions for $\pi \in \mathcal{P}(V)$ to have $\tau^0_\varphi[T^\pi] \neq 0$. For starters, T^π_n must be an oriented cactus. If so, Corollary 3.2.14 implies that the sub-quotient $C_n^{\pi|_{C_n}}$ is also an oriented cactus, in which case Proposition 2.4.6 already tells us that π restricts to a non-crossing partition of C_n (that further satisfies condition (II)). The enumeration of the vertices $v_1, \ldots, v_n$ (resp., edges $v_i \overset{e_i}{\sim} v_{i+1}$) of the cycle C_n allows us to consider $\pi|_{C_n} \in \mathcal{NC}([n])$ (resp., $K(\pi|_{C_n}) \in \mathcal{NC}(\overline{n})$) as convenient. The blocks

$$B = (i_1 < \cdots < i_{\#(B)}) \in \pi|_{C_n}$$

then group the petals d_i into bunches (or "flowers")

$$\left(\prod_{i \in B} d_i\right)_{B \in \pi|_{C_n}},$$

each attached at a single vertex B in $C_n^{\pi|_{C_n}}$. Suppose that π makes an identification across different flowers ("cross-pollinates"), i.e., there exist vertices $u \in d_i$ and $w \in d_j$ such that $u \overset{\pi}{\sim} w$ and $i \overset{\pi|_{C_n}}{\not\sim} j$. Then the vertices v_i and v_j of the cycle form a 3-connection in T_n^π: two edge-disjoint paths come from the cactus $C_n^{\pi|_{C_n}}$, and a third comes from the edges of d_i and d_j. In this case, T_n^π cannot possibly be a cactus, and so $\tau_\varphi^0\big[T_n^\pi\big] = 0$.

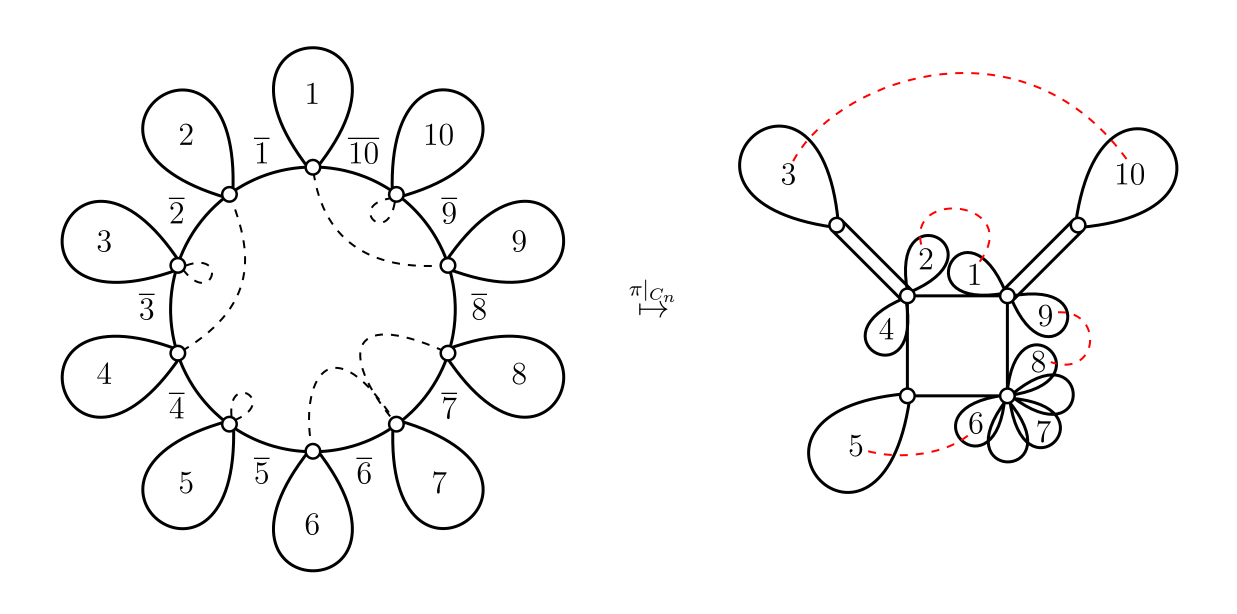


Figure 3.7: A visualization of the flowering process. The red dotted lines on the right indicate identifications across different flowers. Such cross-pollination will not produce a cactus.

Thus, we are left to consider partitions $\pi \in \mathcal{P}(V)$ such that $\pi|_{C_n} \in \mathcal{NC}(n)$ and π does not cross-pollinate. Iterating the vertex factorization property (3.5) of the injective traffic state at each vertex $B \in \pi|_{C_n}$ in T_n^π, we can rewrite the sum in (3.6) as

$$\sum_{\pi\in\mathcal{P}(V)} \tau_\varphi^0\big[T_n^\pi\big] = \sum_{\pi\in\mathcal{NC}(n)} \left(\prod_{B\in\pi} \sum_{\eta_B\in\mathcal{P}(V_B)} \tau_\varphi^0\left[\widetilde{\Delta}\left(\prod_{i\in B} d_i\right)^{\eta_B}\right]\right)\left(\tau_\varphi^0\big[C_n(a_{\overline{1}},\dots,a_{\overline{n}})^\pi\big]\right),$$

where V_B is the vertex set of the test graph $\widetilde{\Delta}\big(\prod_{i\in B} d_i\big)$. Moreover, by definition,

$$\sum_{\eta_B\in\mathcal{P}(V_B)} \tau_\varphi^0\left[\widetilde{\Delta}\left(\prod_{i\in B} d_i\right)^{\eta_B}\right] = \tau_\varphi\left[\widetilde{\Delta}\left(\prod_{i\in B} d_i\right)\right] = \psi\left(\Delta\left(\prod_{i\in B} d_i\right)\right) = \psi\left(\prod_{i\in B} d_i\right),$$

and so

$$\prod_{B\in\pi} \sum_{\eta_B\in\mathcal{P}(V_B)} \tau_\varphi^0\left[\widetilde{\Delta}\left(\prod_{i\in B} d_i\right)^{\eta_B}\right] = \prod_{B\in\pi} \psi\left(\prod_{i\in B} d_i\right) = \psi_\pi[d_1,\dots,d_n] = \sum_{\substack{\omega\in\mathcal{NC}(n)\\ \text{s.t. } \omega\le\pi}} \kappa_\omega^{\mathcal{G}}[d_1,\dots,d_n].$$

At the same time, condition (II) of Proposition 2.4.6 imposes an additional constraint on π. Altogether, our expression for the trace becomes

$$\begin{aligned}\psi(d_1 a_{\overline{1}}^{\intercal(\overline{1})} \cdots d_n a_{\overline{n}}^{\intercal(\overline{n})}) &= \sum_{\substack{\pi\in\mathcal{NC}(n) \text{ s.t.} \\ K(\pi)=\sigma\cup\rho\in\mathcal{NC}(\overline{n}) \\ \text{for some } \sigma\in\mathcal{NC}(\hat{\intercal}^{-1}(1)) \\ \text{and } \rho\in\mathcal{NC}(\hat{\intercal}^{-1}(\intercal))}} \Bigg(\sum_{\substack{\omega\in\mathcal{NC}(n) \\ \text{s.t. } \omega\leq\pi}} \kappa_\omega^{\mathcal{G}}[d_1,\ldots,d_n]\Bigg)\Big(\kappa_{K(\pi)}^{\mathcal{G}}[a_{\overline{1}}^{\intercal(\overline{1})},\ldots,a_{\overline{n}}^{\intercal(\overline{n})}]\Big) \\ &= \sum_{\substack{\pi\in\mathcal{NC}(n+\overline{n}) \text{ s.t.} \\ \pi=\pi_1\cup\pi_2, \\ \text{where } \pi_2=\sigma\cup\rho\in\mathcal{NC}(\overline{n}) \\ \text{for some } \sigma\in\mathcal{NC}(\hat{\intercal}^{-1}(1)) \\ \text{and } \rho\in\mathcal{NC}(\hat{\intercal}^{-1}(\intercal)) \\ \text{and } \pi_1\leq K(\pi_2)\in\mathcal{NC}(n)}} \kappa_\pi^{\mathcal{G}}[d_1, a_{\overline{1}}^{\hat{\intercal}(\overline{1})},\ldots,d_n,a_{\overline{n}}^{\hat{\intercal}(\overline{n})}],\end{aligned}$$

where we have applied the Kreweras complement K to reindex the sum in the second equality. In particular, we see that there are no contributions from mixed cumulants in $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$. It follows that mixed cumulants in $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$ vanish, as was to be shown. ∎

We move on to the construction of the conditional expectation $\mathscr{E} : \mathcal{B} \to \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B})$. For a finite, connected simple graph $G = (V, E)$, we recall the construction of the *block-cut tree* $bc(G)$ of G. The vertices of $bc(G)$ consist of both the cut-vertices of G and the blocks of G with edges determined by inclusion: we connect a cut-vertex v to a block H if $v \in H$. As the name suggests, the block-cut tree is indeed a tree. It will be convenient to distinguish between the two different classes of vertices in $bc(G)$. In particular, we use circular vertices for the cut-vertices and square vertices for the blocks.

We will need a simple modification of the block-cut tree construction in the case of a bi-rooted multidigraph $G = (V, E, \text{src}, \text{tar}, v_{\text{in}}, v_{\text{out}})$. Allowing for multiple (directed) edges does not materially affect the construction; however, allowing for loops creates an issue when determining the blocks. Specifically, a loop based at a cut-vertex will belong to more than one block of G. To account for this, we temporarily remove any loop based at a cut-vertex of G, resulting in a graph $\widetilde{G}$. We reintroduce the loops in the block-cut tree $bc(\widetilde{G})$ by adding a single block for each set of loops based at a given cut-vertex with an edge between the two to indicate the inclusion. This process ensures that we have a faithful reconstruction of the original graph G from our modified block-cut tree. Furthermore, if either distinguished vertex v_{in} or v_{out} is not a cut-vertex, then we add it to our tree as a circular vertex, colored black, and attach it to its corresponding (unique) block (if $v_{\text{in}} = v_{\text{out}}$ then we only add a single vertex). A moment's thought shows that the resulting graph, which we denote $bcd(G)$, is of course still a tree.

We apply our modified block-cut tree construction to graph monomials $t \in \mathcal{B}$ to prove

Lemma 3.3.2. *There exists a homomorphic conditional expectation $\mathscr{E} : \mathcal{B} \to \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B})$ such that*

$$\mathscr{E}^{-1}(\mathcal{A}) = \mathcal{A}, \quad \mathscr{E}^{-1}(\mathcal{A}^\intercal) = \mathcal{A}^\intercal, \quad \mathscr{E}^{-1}(\Delta(\mathcal{B})) = \Theta(\mathcal{B}),$$

and

$$\mathscr{E}(t) \equiv t \pmod{\psi}, \qquad \forall t \in \mathcal{B}. \tag{3.7}$$

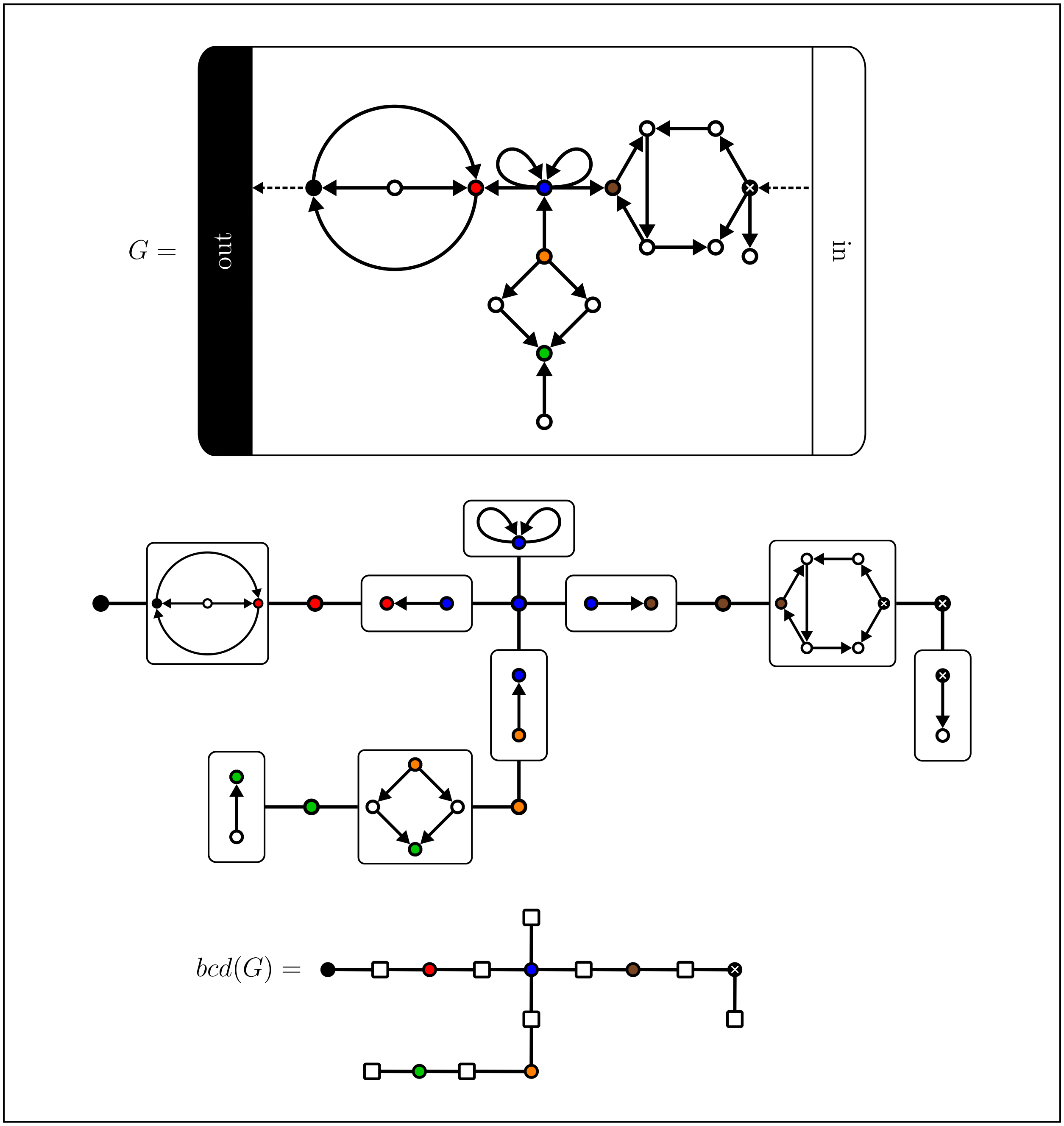


Figure 3.8: An example of the modified block-cut tree construction. Starting with a bi-rooted multidigraph G, we color each of the non-root cut vertices. Note that in this case the output v_{out} is not a cut-vertex, and so we append it to the standard block-cut tree.

Proof. Let $t = (V, E, \mathrm{src}, \mathrm{tar}, v_{\mathrm{in}}, v_{\mathrm{out}}, \gamma) \in \mathcal{B}$ be a graph monomial with modified block-cut tree $bcd(t)$. By construction, there exists a unique path $P = (v_1, B_1, \ldots, v_{n-1}, B_{n-1}, v_n)$ from v_{in} to v_{out} in $bcd(t)$ (in particular, $v_1 = v_{\mathrm{in}}$ and $v_n = v_{\mathrm{out}}$). For each non-block vertex v_i on this path, we consider the connected component of v_i in $bcd(t)$ off of the path P, namely, let $\widetilde{C}(v_i)$ denote the connected component containing v_i after removing the edges of the path P. Similarly, for each block B_i on the path P, let $\widetilde{C}(B_i)$ denote the connected component containing B_i in $bcd(t)$ off of the path P. Note that each connected component $\widetilde{C}(v_i)$ (resp., $\widetilde{C}(B_i)$) corresponds to a connected edge-labeled subgraph $C(v_i)$ (resp., $C(B_i)$) of the original graph monomial t. Each of the subgraphs $C(v_i)$ (resp., $C(B_i)$) further defines a graph monomial after a natural choice of distinguished vertices

$$d_i = (C(v_i), v_i, v_i) \in \Delta(\mathcal{B}) \qquad (\text{resp., } m_i = (C(B_i), v_i, v_{i+1}) \in \mathcal{B}).$$

For example, if $v_{\mathrm{in}} = v_{\mathrm{out}} \in t$, then

$$P = (v_1), \quad \widetilde{C}(v_1) = bcd(t), \quad \text{and} \quad t = d_1 \in \Delta(\mathcal{B}).$$

In fact, the last equality is simply a special case of the general factorization

$$t = d_n m_{n-1} d_{n-1} \cdots m_1 d_1. \tag{3.8}$$

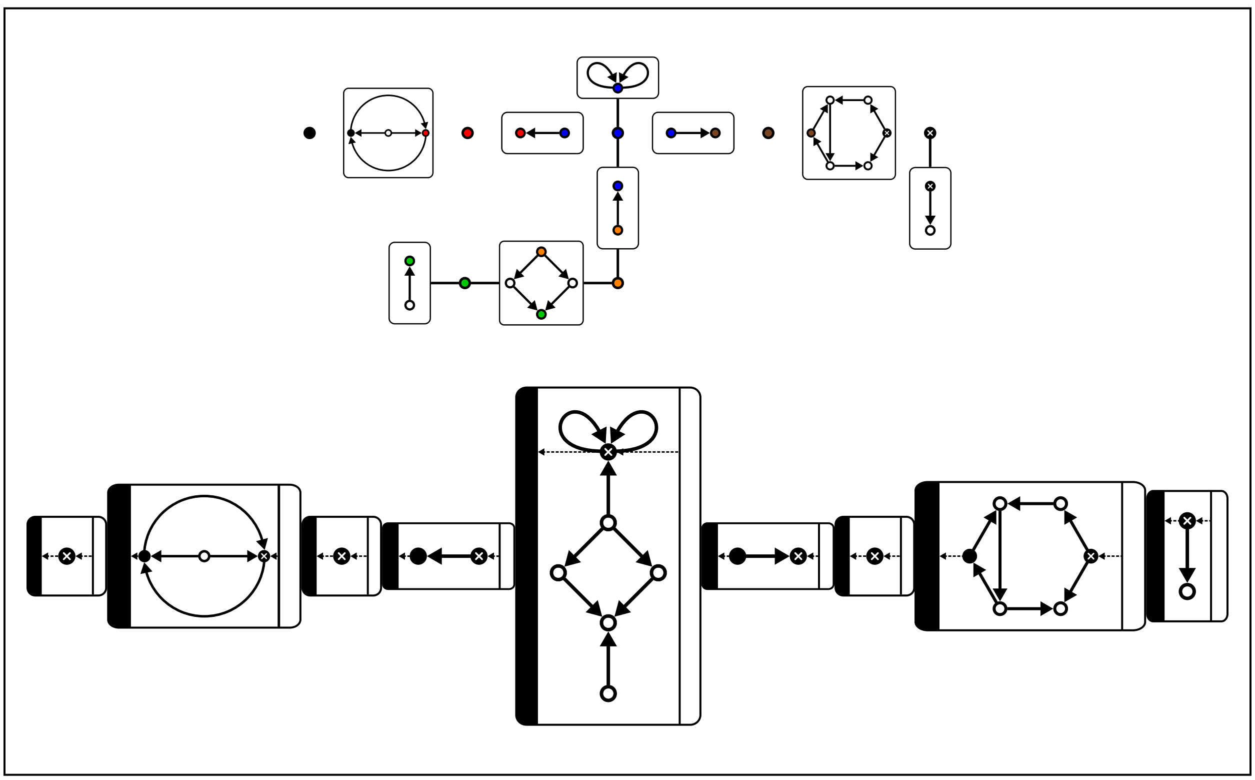


Figure 3.9: Removing the edges of the path P. We apply this procedure to the block-cut tree from Figure 3.8 to identify the components and produce the factorization (3.8).

Suppose that a factor $m_i \notin \mathcal{A} \cup \mathcal{A}^\intercal$. Then there must be a simple cycle in m_i that visits both the input v_i and the output v_{i+1}. Indeed, this follows from the vertex version of Menger's theorem: if $v_i \neq v_{i+1}$ are the only vertices in m_i, then $m_i \notin \mathcal{A} \cup \mathcal{A}^\intercal$ implies that there are multiple edges between v_i and v_{i+1}; if $v_i \neq v_{i+1}$ are not the only vertices in m_i, then the lack of cut-vertices in m_i implies the existence of such a simple cycle. Corollary 3.2.9 then implies that

$$m_i \equiv \Delta(m_i) \pmod{\psi}.$$

To apply this to our factorization (3.8), we define a linear operator $\nabla : \mathcal{B} \to \mathcal{B}$ on graph monomials m by the formula

$$\nabla(m) = \begin{cases} m & \text{if } m \in \mathcal{A} \cup \mathcal{A}^\intercal, \\ \Delta(m) & \text{otherwise.} \end{cases}$$

In particular, $\mathscr{E}(t) := d_n \nabla(m_{n-1}) d_{n-1} \cdots \nabla(m_1) d_1 \in \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B})$ satisfies

$$\mathscr{E}(t) \equiv t \pmod{\psi}.$$

Extending $\mathscr{E}$ to $\mathcal{B}$ by linearity, properties (i)-(iii) of a conditional expectation (Definition 2.1.11) follow almost immediately. Moreover, property (iv) will follow if we can prove that $\mathscr{E}$ is a homomorphism $\mathscr{E}(t_1 t_2) = \mathscr{E}(t_1)\mathscr{E}(t_2)$. As before, we can assume that t_1 and t_2 are graph monomials, say with block-cut tree factorizations

$$t_1 = d_{n_1}^{(1)} m_{n_1-1}^{(1)} d_{n_1-1}^{(1)} \cdots m_1^{(1)} d_1^{(1)} \quad \text{and} \quad t_2 = d_{n_2}^{(2)} m_{n_2-1}^{(2)} d_{n_2-1}^{(2)} \cdots m_1^{(2)} d_1^{(2)}$$

coming from the paths

$$P_1 = (v_1^{(1)}, B_1^{(1)}, \ldots, v_{n_1-1}^{(1)}, B_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \quad \text{and} \quad P_2 = (v_1^{(2)}, B_1^{(2)}, \ldots, v_{n_2-1}^{(2)}, B_{n_2-1}^{(2)}, v_{n_2}^{(2)})$$

in $bcd(t_1)$ and $bcd(t_2)$ respectively. The reader will now see why we have insisted on including the distinguished vertices in the modified block-cut tree. In particular, the amalgamated vertex $v_{\text{out}}^{(2)} = v_{\text{in}}^{(1)} \in t_1 t_2$ is either a cut-vertex of $t_1 t_2$ or once again a distinguished vertex v_{in} or $v_{\text{out}} \in t_1 t_2$. The path from v_{in} to v_{out} in $bcd(t_1 t_2)$ can then be written as

$$P = (v_1^{(2)}, B_1^{(2)}, \ldots, v_{n_2-1}^{(2)}, B_{n_2-1}^{(2)}, v_{n_2}^{(2)} = v_1^{(1)}, B_1^{(1)}, \ldots, v_{n_1-1}^{(1)}, B_{n_1-1}^{(1)}, v_{n_1}^{(1)}),$$

which gives rise to the factorization

$$t_1 t_2 = d_{n_1}^{(1)} m_{n_1-1}^{(1)} d_{n_1-1}^{(1)} \cdots m_1^{(1)} (d_1^{(1)} d_{n_2}^{(2)}) m_{n_2-1}^{(2)} d_{n_2-1}^{(2)} \cdots m_1^{(2)} d_1^{(2)}.$$

We conclude that

$$\begin{aligned} \mathscr{E}(t_1 t_2) &= d_{n_1}^{(1)} \nabla(m_{n_1-1}^{(1)}) d_{n_1-1}^{(1)} \cdots \nabla(m_1^{(1)}) (d_1^{(1)} d_{n_2}^{(2)}) \nabla(m_{n_2-1}^{(2)}) d_{n_2-1}^{(2)} \cdots \nabla(m_1^{(2)}) d_1^{(2)} \\ &= (d_{n_1}^{(1)} \nabla(m_{n_1-1}^{(1)}) d_{n_1-1}^{(1)} \cdots \nabla(m_1^{(1)}) d_1^{(1)}) (d_{n_2}^{(2)} \nabla(m_{n_2-1}^{(2)}) d_{n_2-1}^{(2)} \cdots \nabla(m_1^{(2)}) d_1^{(2)}) = \mathscr{E}(t_1)\mathscr{E}(t_2). \end{aligned}$$

Finally, the equalities

$$\mathscr{E}^{-1}(\mathcal{A}) = \mathcal{A}, \quad \mathscr{E}^{-1}(\mathcal{A}^\intercal) = \mathcal{A}^\intercal, \quad \text{and} \quad \mathscr{E}^{-1}(\Delta(\mathcal{B})) = \Theta(\mathcal{B})$$

follow virtually by definition. ■

Remark 3.3.3. The construction of our conditional expectation $\mathscr{E}$ relies on taking a particular (graphical) realization of a graph monomial $t \in \mathcal{B}$. Of course, one should then verify that this construction is well-defined. We can restrict our attention to the monomials m_i coming from the blocks B_i since $\mathscr{E}$ only acts on these factors. Moreover, since the action of $\mathscr{E}$ is defined on each factor individually, we can further restrict to a single factor m_i. Here, there are two cases to consider. First, suppose that a factor $m_i = P(a_1, \ldots, a_k) \in \mathcal{A}$. Then

$$m_i = \underset{\text{out}}{\cdot} \xleftarrow{P(a_1,\ldots,a_k)} \underset{\text{in}}{\cdot} = P\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1} \underset{\text{in}}{\cdot}, \cdots, \underset{\text{out}}{\cdot} \xleftarrow{a_k} \underset{\text{in}}{\cdot} \right).$$

Running through the algorithm for $\mathscr{E}$ on m_i, we have the equality

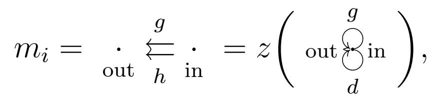
$$\begin{aligned} \mathscr{E}\left(P\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1} \underset{\text{in}}{\cdot}, \cdots, \underset{\text{out}}{\cdot} \xleftarrow{a_k} \underset{\text{in}}{\cdot} \right)\right) &= P\left(\underset{\text{out}}{\cdot} \xleftarrow{a_1} \underset{\text{in}}{\cdot}, \cdots, \underset{\text{out}}{\cdot} \xleftarrow{a_k} \underset{\text{in}}{\cdot} \right) \\ &= \underset{\text{out}}{\cdot} \xleftarrow{P(a_1,\ldots,a_k)} \underset{\text{in}}{\cdot} \\ &= \mathscr{E}\left(\underset{\text{out}}{\cdot} \xleftarrow{P(a_1,\ldots,a_k)} \underset{\text{in}}{\cdot} \right), \end{aligned}$$

and similarly for $m_i = P(a_1, \ldots, a_k)^\intercal$.

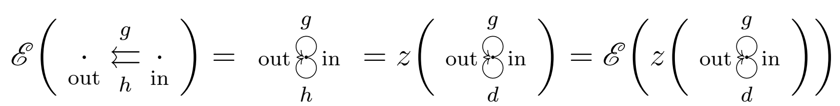
Next, suppose that $h \in \mathcal{B}$ is a graph monomial such that $h = zd \in \Delta(\mathcal{B})$ for some $z \in \mathbb{C}$ and graph monomial $d \in \Delta(\mathcal{B})$. For example, it could be that $z \in \mathbb{R}_+$ and

$$h = \underset{\text{out}}{\cdot} \xleftarrow{\sqrt{z}} \cdot \xrightarrow{\sqrt{z}} \underset{\text{in}}{\cdot} = z\left(\underset{\text{in/out}}{\cdot} \right).$$

A factor m_i could then take the form



where g stands in for an arbitrary graph monomial. Again, running through the algorithm for $\mathscr{E}$ on m_i,



Note that these are the only cases where the identifications defining $\mathcal{B} = \mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle/\mathcal{I}$ can affect the path P in $bcd(t)$: the former by expanding the block b_i by introducing cut vertices in m_i; the latter by compressing the block b_i by identifying the vertices $v_i \neq v_{i+1} \in m_i$. In any case, we see that the action of $\mathscr{E}$ is well-defined.

In general, a conditional expectation is only unique up to degeneracy. In particular, if $\mathscr{F} : \mathcal{B} \to \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B})$ is also a conditional expectation, then

$$\mathscr{F}(t) \equiv \mathscr{E}(t) \pmod{\psi}, \qquad \forall t \in \mathcal{B}.$$

Indeed, even with the additional properties stated in Lemma 3.3.2, one can still find such maps $\mathscr{F} \neq \mathscr{E}$. To see this, note that our algorithm for $\mathscr{E}$ only operates on the cut-vertices of t along the path P. The map $\mathscr{E}$ satisfies the equivalence (3.7) precisely because it only identifies redundant vertices (i.e., vertices that would need to be identified anyway in order to contribute to the calculation of the injective traffic state). Yet, there can be many such redundant vertices, whereas $\mathscr{E}$ only considers a "minimal" subset of them. One can modify the map $\mathscr{E}$ while preserving all of the desired properties by specifying a more vigilant approach to dealing with such redundancies within each component $C(v_i)$ and $C(B_i)$ defined by the path P. Doing so clearly defines other maps $\mathscr{F} \neq \mathscr{E}$. We encourage the reader to consider the example in Figure 3.8. At the same time, we can formalize this notion of minimality to characterize our map $\mathscr{E}$.

Corollary 3.3.4. *The map $\mathscr{E}$ is the unique homomorphic conditional expectation that satisfies the commutation in (3.1):*

$$\mathscr{E}\left(\underset{\text{out}}{\cdot} \overset{t}{\underset{t'}{\leftleftarrows}} \underset{\text{in}}{\cdot} \right) = \text{out} \overset{t}{\underset{t'}{\infty}} \text{in}, \qquad \forall t, t' \in \mathcal{B}.$$

Proof. Suppose that $\mathscr{F}$ is such a map. Then for a monomial $t \in \mathcal{B}$ with block-cut tree factorization $t = d_n m_{n-1} d_{n-1} \cdots m_1 d_1$,

$$\mathscr{F}(t) = \mathscr{F}(d_n)\mathscr{F}(m_{n-1})\mathscr{F}(d_{n-1}) \cdots \mathscr{F}(m_1)\mathscr{F}(d_1) = d_n \mathscr{F}(m_{n-1}) d_{n-1} \cdots \mathscr{F}(m_1) d_1,$$

and so it suffices to prove that $\mathscr{F}(m_i) = \nabla(m_i)$. If $m_i \in \mathcal{A} \cup \mathcal{A}^\intercal$, then $\mathscr{F}(m_i) = m_i = \nabla(m_i)$ and we are done. Otherwise, there is a simple in cycle in m_i that visits both the input and the output of m_i (recall the proof of Lemma 3.3.2), which means that we can write

$$m_i = \underset{\text{out}}{\cdot} \overset{t}{\underset{t'}{\leftleftarrows}} \underset{\text{in}}{\cdot}$$

for some $t, t' \in \mathcal{B}$. But then, by assumption,

$$\mathscr{F}(m_i) = \mathscr{F}\left(\underset{\text{out}}{\cdot} \overset{t}{\underset{t'}{\leftleftarrows}} \underset{\text{in}}{\cdot} \right) = \text{out} \overset{t}{\underset{t'}{\infty}} \text{in} = \nabla(m_i).$$

■

Finally, the last piece of Theorem 3.1.1 now follows almost immediately.

Corollary 3.3.5. *The universal enveloping traffic space $(\mathcal{B}, \psi) = (\mathcal{G}(\mathcal{A}), \varphi_{\tau_\varphi})$ admits the free product decomposition*

$$(\mathcal{B}, \psi) = (\mathcal{A}, \psi|_{\mathcal{A}}) * (\mathcal{A}^\intercal, \psi|_{\mathcal{A}^\intercal}) * (\Theta(\mathcal{B}), \psi|_{\Delta(\mathcal{B})}).$$

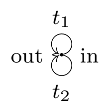
Proof. Our modified block-cut tree algorithm already proves the *algebraic* free product decomposition $\mathcal{B} = \mathcal{A} * \mathcal{A}^\intercal * \Theta(\mathcal{B})$. We can further use our conditional expectation $\mathscr{E}$ to pull back the free independence of $\mathcal{A}$, $\mathcal{A}^\intercal$, and $\Delta(\mathcal{B})$ to $\mathscr{E}^{-1}(\mathcal{A}) = \mathcal{A}$, $\mathscr{E}^{-1}(\mathcal{A}^\intercal) = \mathcal{A}^\intercal$, and $\mathscr{E}^{-1}(\Delta(\mathcal{B})) = \Theta(\mathcal{B})$. ■

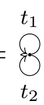
3.4 A duality between classical and free

It suffices to prove Theorem 3.1.2 for a pair of unital $*$-subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$.

Lemma 3.4.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be freely independent unital $*$-subalgebras of a tracial $*$-probability space $(\mathcal{A}, \varphi)$. Then the commutative sub-traffic-spaces $\Delta(\mathcal{B}_1) = \Delta(\mathcal{G}(\mathcal{A}_1))$ and $\Delta(\mathcal{B}_2) = \Delta(\mathcal{G}(\mathcal{A}_2))$ are classically independent in $(\mathcal{B}, \psi)$.*

Proof. Let $t_i \in \Delta(\mathcal{B}_i)$ be a graph monomial. Then

$t_1 t_2 =$ 

and $\widetilde{\Delta}(t_1 t_2) =$ 

,

and we can compute the trace as

$$\psi(t_1 t_2) = \tau_\varphi\big[\widetilde{\Delta}(t_1 t_2)\big] = \sum_{\pi \in \mathcal{P}(V)} \tau_\varphi^0\big[\widetilde{\Delta}(t_1 t_2)^\pi\big].$$

We think of the edges of t_1 (resp., t_2) as being colored black (resp., red) to indicate the edge labels in $\mathcal{A}_1$ (resp., $\mathcal{A}_2$). Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent, $\tau_\varphi^0\big[\widetilde{\Delta}(t_1 t_2)^\pi\big] = 0$ unless $\widetilde{\Delta}(t_1 t_2)^\pi$ is an oriented cactus whose pads are each of a uniform color (a *colored oriented cactus*). But this implies that $\widetilde{\Delta}(t_i)^{\pi|_{\widetilde{\Delta}(t_i)}}$ is a sub-cactus of $\widetilde{\Delta}(t_1 t_2)^\pi$. Moreover, note that the sub-cacti $\widetilde{\Delta}(t_1)^{\pi|_{\widetilde{\Delta}(t_1)}}$ and $\widetilde{\Delta}(t_2)^{\pi|_{\widetilde{\Delta}(t_2)}}$ can only have one vertex in common as two such vertices would form a 4-connection, with two edge-disjoint paths coming from the black edges and two edge-disjoint paths coming from the red edges. Of course, this common vertex must be

$$\rho := \text{input}(t_1) = \text{output}(t_1) = \text{input}(t_2) = \text{output}(t_2) \in \widetilde{\Delta}(t_1 t_2),$$

in which case

$$\tau_\varphi^0\big[\widetilde{\Delta}(t_1 t_2)^\pi\big] = \tau_\varphi^0\big[\widetilde{\Delta}(t_1)^{\pi|_{\widetilde{\Delta}(t_1)}}\big]\tau_\varphi^0\big[\widetilde{\Delta}(t_2)^{\pi|_{\widetilde{\Delta}(t_2)}}\big].$$

In particular, if $\widetilde{\Delta}(t_1 t_2)^\pi$ is a colored oriented cactus with $v_i \in \widetilde{\Delta}(t_i)$ such that $v_1 \overset{\pi}{\sim} v_2$, then it is necessarily the case that $v_1 \overset{\pi}{\sim} \rho \overset{\pi}{\sim} v_2$. This allows us to factor the trace

$$\begin{aligned}\psi(t_1 t_2) = \sum_{\pi \in \mathcal{P}(V)} \tau_\varphi^0\big[\widetilde{\Delta}(t_1 t_2)^\pi\big] &= \Bigg(\sum_{\pi_1 \in \mathcal{P}(V_1)} \tau_\varphi^0\big[\widetilde{\Delta}(t_1)^{\pi_1}\big]\Bigg)\Bigg(\sum_{\pi_2 \in \mathcal{P}(V_2)} \tau_\varphi^0\big[\widetilde{\Delta}(t_2)^{\pi_2}\big]\Bigg) \\ &= \tau_\varphi\big[\widetilde{\Delta}(t_1)\big]\tau_\varphi\big[\widetilde{\Delta}(t_2)\big] = \psi(t_1)\psi(t_2),\end{aligned}$$

as was to be shown. ∎

The general case of Theorem 3.1.2 now follows from the associativity of free independence. Our proof relies on an explicit calculation made possible by the cactus structure of the injective traffic state τ_φ^0. At the same time, one can also realize this duality by appealing to the relationship between traffic independence and classical/free independence. More precisely, Proposition 4.8 in [CDM] states that the free independence of the $(\mathcal{A}_i)_{i\in I}$ in $(\mathcal{A}, \varphi)$ amounts to the traffic independence of the $(\mathcal{G}(\mathcal{A}_i))_{i\in I}$ in $(\mathcal{G}(\mathcal{A}), \tau_\varphi)$. We can specialize this to the traffic independence of the sub-traffic-spaces $(\Delta(\mathcal{G}(\mathcal{A}_i)))_{i\in I}$, where $\Delta(\mathcal{G}(\mathcal{A}_i)) \subset \mathcal{G}(\mathcal{A}_i)$. Theorem 5.5 of [Mal] proves that traffic independence and classical independence are equivalent for diagonal traffic random variables $\Delta(t) = t$, and so the result follows.

3.5 A cycle pruning algorithm

In this section, we generalize the ideas of Section 3.2 and 3.3 to formulate a generic cycle pruning algorithm. However, this generality comes at a cost: in contrast to our earlier results, our equivalence now takes the form of a graph polynomial.

Theorem 3.5.1. *Let $t = \Delta(d_0 a_0^{\hat{\intercal}(0)} \cdots d_n a_n^{\hat{\intercal}(n)})$ be a graph monomial in $\mathcal{B}$, where $d_i \in \Delta(\mathcal{B})$ and $a_i \in \mathcal{A}$ are graph monomials with the transpose labels $\hat{\intercal}(i) \in \{1, \intercal\}$. For any subset $A = \{i_1 < \cdots < i_{\#(A)}\} \subset [n]$, we define the A-segmented factors*

$$m_{A,k} = a_{i_k}^{\hat{\intercal}(i_k)} d_{i_k+1} \cdots a_{i_{k+1}-2}^{\hat{\intercal}(i_{k+1}-2)} d_{i_{k+1}-1} a_{i_{k+1}-1}^{\hat{\intercal}(i_{k+1}-1)}$$

for $0 \leq k \leq \#(A)$, where $i_0 = 0$ and $i_{\#(A)+1} = n+1$. In particular, note that

$$d_0 a_0^{\hat{\intercal}(0)} \cdots d_n a_n^{\hat{\intercal}(n)} = d_0 m_{A,0} d_{i_1} m_{A,1} \cdots d_{i_{\#(A)}} m_{A,\#(A)}.$$

Then

$$t \equiv \sum_{A \subset [n]} \Big(\psi(m_A)(d_0 \prod_{i\in A} d_i) \Big) \pmod{\psi},$$

where

$$m_A = \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} \Big((-1)^{\#(B)-\#(A)} \prod_{i \in B\setminus A} d_i \prod_{k=0}^{\#(B)} \Delta(m_{B,k}) \Big).$$

Proof. If $n = 0$, then $t = \Delta(d_0 a_0^{\hat{\intercal}(0)}) = d_0 \Delta(a_0^{\hat{\intercal}(0)})$. In this case, t has a loop e with edge label $\gamma(e) = a_0$. Corollary 3.2.12 then implies that

$$t \equiv \psi(a_0) d_0 \pmod{\psi},$$

and so we are done.

Otherwise, assume that $n \geq 1$. As before, we think of $t = \Delta(d_0 a_0^{\hat{\intercal}(0)} \cdots d_n a_n^{\hat{\intercal}(n)})$ as a flower: in this case, a cycle of length $n+1$ with the petals d_i based at each vertex v_i. For any subset $A \subset [n]$, we define the graph monomial t_A by the identifying the vertices $v_0 \sim v_i$ for $i \in A$. As a sanity check, note that

$$t_A = d_0 \prod_{i\in A} d_i \prod_{k=0}^{\#(A)} \Delta(m_{A,k}).$$

We further define the graph polynomials

$$p_A(t) = \sum_{\substack{B\subset[n] \\ \text{s.t. } A\subset B}} (-1)^{\#(B)-\#(A)} t_B,$$

which satisfy the identity

$$\sum_{A\subset[n]} p_A(t) = \sum_{A\subset[n]} \sum_{\substack{B\subset[n] \\ \text{s.t. } A\subset B}} (-1)^{\#(B)-\#(A)} t_B = \sum_{B\subset[n]} \left(\sum_{A\subset B} (-1)^{\#(B)-\#(A)} \right) t_B = t_\emptyset = t.$$

So, the result will follow if we can show that

$$p_A(t) \equiv \psi(m_A) d_0 \prod_{i\in A} d_i \pmod{\psi}.$$

To this end, let $t' \in \mathcal{B}$ be a graph monomial. We will prove that

$$\psi(p_A(t)t') = \sum_{\substack{B\subset[n] \\ \text{s.t. } A\subset B}} (-1)^{\#(B)-\#(A)} \psi(t_B t') = \psi(m_A)\psi((d_0 \prod_{i\in A} d_i)t').$$

For starters, note that we can factor

$$\Delta(t_B t') = \left(d_0 \prod_{i\in A} d_i \Delta(t') \right) \left(\prod_{i\in B\setminus A} d_i \prod_{k=0}^{\#(B)} \Delta(m_{B,k}) \right) =: l_A(B) r_A(B)$$

into a left side $l_A = l_A(B)$ and a right side $r_A(B)$ joined at the single vertex

$$\rho := \text{input}(l_A) = \text{output}(l_A) = \text{input}(r_A(B)) = \text{output}(r_A(B)) \in \Delta(t_B t').$$

Similarly, we define the test graphs

$$L_A = \widetilde{\Delta}(l_A) = (V_{L_A}, E_{L_A}, \gamma_{L_A}) \quad \text{and} \quad R_A(B) = \widetilde{\Delta}(r_A(B)) = (V_{R_A(B)}, E_{R_A(B)}, \gamma_{R_A(B)}),$$

in which case

$$\widetilde{\Delta}(t_B t') = L_A \sharp R_A(B) = (V_{L_A\sharp R_A(B)}, E_{L_A\sharp R_A(B)}, \gamma_{L_A\sharp R_A(B)}).$$

Here, we use $L_A \sharp R_A(B)$ to denote the test graph obtained from L_A and $R_A(B)$ by identifying the vertices $\text{input}(l_A) = \text{output}(l_A) \in L_A$ and $\text{input}(r_A(B)) = \text{output}(r_A(B)) \in R_A(B)$. For convenience, we write $r_A = r_A(A)$ and $R_A = R_A(A)$.

Now, by definition,

$$\psi(t_B t') = \psi(l_A r_A(B)) = \tau_\varphi\big[L_A \sharp R_A(B)\big] = \sum_{\pi_B \in \mathcal{P}(V_{L_A\sharp R_A(B)})} \tau_\varphi^0\big[(L_A \sharp R_A(B))^{\pi_B}\big]. \tag{3.9}$$

It will be convenient to reindex the sum in terms of partitions $\pi \in \mathcal{P}(V_{L_A} \sharp V_{R_A})$. For any pair of partitions $\pi_L \in \mathcal{P}(V_{L_A})$ and $\pi_R \in \mathcal{P}(V_{R_A})$, we define the class of partitions

$$\mathcal{P}_\rho(\pi_L, \pi_R) = \{\pi \in \mathcal{P}(V_{L_A \sharp R_A}) : \pi|_{V_{L_A}} = \pi_L \text{ and } \pi|_{V_{R_A}} = \pi_R\}.$$

We recall the interpretation for $\mathcal{P}_\rho(\pi_L, \pi_R)$ from Lemma 3.2.11: $\mathcal{P}_\rho(\pi_L, \pi_R)$ consists of the partitions $\pi \in \mathcal{P}(V_{L_A \sharp R_A})$ obtained from (π_L, π_R) by either keeping a block $V \in \pi_L \cup \pi_R$ (so $V \in \pi$) or merging it with at most a single block from V' from the other side (so $V \cup V' \in \pi$). Of course, the block in π_L containing $\text{input}(l_A) = \text{output}(l_A)$ and the block in π_R containing $\text{input}(r_A) = \text{output}(r_A)$ are necessarily merged. As before, we write $\pi_\rho(\pi_L, \pi_R)$ for the minimal element in $\mathcal{P}_\rho(\pi_L, \pi_R)$ for the reversed refinement order. By construction,

$$\bigsqcup_{(\pi_L, \pi_R) \in \mathcal{P}(V_{L_A}) \times \mathcal{P}(V_{R_A})} \mathcal{P}_\rho(\pi_L, \pi_R) = \mathcal{P}(V_{L_A} \sharp V_{R_A}).$$

Moreover, note that for every partition $\pi_B \in \mathcal{P}(V_{L_A} \sharp V_{R_A(B)})$, there exists a unique partition $\pi \in \mathcal{P}(V_{L_A} \sharp V_{R_A})$ such that

$$(L_A \sharp R_A(B))^{\pi_B} = (L_A \sharp R_A)^{\pi}.$$

Indeed, one can construct π from π_B by simply expanding the amalgamated vertex $v_i \sim v_j$ for $i, j \in B$ in π_B into the vertices $v_i \sim v_j$ for $i, j \in A \subset B$ and v_k for $k \in B \setminus A$. For a partition $\pi_R \in \mathcal{P}(V_{R_A})$, we then define

$$B_{\pi_R} = \{i : v_i \overset{\pi_R}{\sim} v_0\} \cup A \subset [n].$$

This allows us to rewrite (3.9) as

$$\psi(t_B t') = \sum_{\pi_B \in \mathcal{P}(V_{L_A \sharp R_A(B)})} \tau_\varphi^0\big[(L_A \sharp R_A(B))^{\pi_B}\big] = \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B \subset B_{\pi_R}}} \sum_{\pi \in \mathcal{P}_\rho(\pi_L, \pi_R)} \tau_\varphi^0\big[(L_A \sharp R_A)^{\pi}\big],$$

in which case

$$\begin{aligned}
\psi(p_A(t)t') &= \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} (-1)^{\#(B) - \#(A)} \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B \subset B_{\pi_R}}} \sum_{\pi \in \mathcal{P}_\rho(\pi_L, \pi_R)} \tau_\varphi^0\big[(L_A \sharp R_A)^{\pi}\big] \\
&= \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\pi_R \in \mathcal{P}(V_{R_A})} \sum_{\pi \in \mathcal{P}_\rho(\pi_L, \pi_R)} \Bigg(\sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B \subset B_{\pi_R}}} (-1)^{\#(B) - \#(A)} \Bigg) \tau_\varphi^0\big[(L_A \sharp R_A)^{\pi}\big] \\
&= \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B_{\pi_R} = A}} \sum_{\pi \in \mathcal{P}_\rho(\pi_L, \pi_R)} \tau_\varphi^0\big[(L_A \sharp R_A)^{\pi}\big]
\end{aligned}$$

since

$$\sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B \subset B_{\pi_R}}} (-1)^{\#(B) - \#(A)} = 0$$

unless $A = B_{\pi_R}$.

Continuing the calculation, suppose that $\pi \neq \pi_\rho(\pi_L, \pi_R) \in \mathcal{P}_\rho(\pi_L, \pi_R)$, where $B_{\pi_R} = A$. Then $(L_A \sharp R_A)^\pi$ cannot possibly be a cactus. Indeed, since $B_{\pi_R} = A$, the vertices $v_i \neq \rho$ form a 2-connection in $(L_A \sharp R_A)^\pi$ for $i \in B \setminus A$ via the edges of

$$\widetilde{\Delta}(a_{i_k}^{\hat{\dagger}(i_k)} \cdots a_{i_{k+1}-1}^{\hat{\dagger}(i_{k+1}-1)}) \subset \widetilde{\Delta}(m_{A,k}) \subset R_A,$$

where $A = \{i_1 < \cdots < i_{\#(A)}\}$ and

$$i_k < i < i_{k+1}.$$

Now since $\pi \neq \pi_\rho(\pi_L, \pi_R)$, it must be that π identifies a vertex v in

$$\widetilde{\Delta}(d_i) \subset \widetilde{\Delta}(m_{A,k}) = \widetilde{\Delta}(a_{i_k}^{\hat{\dagger}(i_k)} d_{i_k+1} \cdots a_{i_{k+1}-2} d_{i_{k+1}-1} a_{i_{k+1}-1}^{\hat{\dagger}(i_{k+1}-1)})$$

with a vertex w in $L_A \subset L_A \sharp R_A$ for some $i \in B \setminus A$. Of course, w is already connected to ρ in L_A, so this identification creates a path from v_i to ρ using only the edges of L_A and $\widetilde{\Delta}(d_i)$, which implies that v_i and ρ form a 3-connection in $(L_A \sharp R_A)^\pi$. The lone contribution in our sum over $\mathcal{P}_\rho(\pi_L, \pi_R)$ then comes from the minimum element $\pi_\rho(\pi_L, \pi_R)$, and so

$$\begin{aligned}
\sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B_{\pi_R} = A}} \sum_{\pi \in \mathcal{P}_\rho(\pi_L, \pi_R)} \tau_\varphi^0\big[(L_A \sharp R_A)^\pi\big] &= \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B_{\pi_R} = A}} \tau_\varphi^0\big[(L_A \sharp R_A)^{\pi_\rho(\pi_L, \pi_R)}\big] \\
&= \sum_{\pi_L \in \mathcal{P}(V_{L_A})} \tau_\varphi^0\big[L_A^{\pi_L}\big] \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B_{\pi_R} = A}} \tau_\varphi^0\big[R_A^{\pi_R}\big].
\end{aligned}$$

Now, by definition,

$$\sum_{\pi_L \in \mathcal{P}(V_{L_A})} \tau_\varphi^0\big[L_A^{\pi_L}\big] = \psi(l_A) = \psi((d_0 \prod_{i \in A} d_i) t'),$$

whereas

$$\begin{aligned}
\sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B_{\pi_R} = A}} \tau_\varphi^0\big[R_A^{\pi_R}\big] &= \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} \left((-1)^{\#(B) - \#(A)} \sum_{\substack{\pi_R \in \mathcal{P}(V_{R_A}) \\ \text{s.t. } B \subset B_{\pi_R}}} \tau_\varphi^0\big[R_A^{\pi_R}\big] \right) \\
&= \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} (-1)^{\#(B) - \#(A)} \psi(r_A(B)) \\
&= \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} (-1)^{\#(B) - \#(A)} \psi\left(\prod_{i \in B \setminus A} d_i \prod_{k=0}^{\#(B)} \Delta(m_{B,k}) \right) = \psi(m_A).
\end{aligned}$$

We conclude that

$$\psi(p_A(t) t') = \sum_{\substack{B \subset [n] \\ \text{s.t. } A \subset B}} (-1)^{\#(B) - \#(A)} \psi(t_B t') = \psi(m_A) \psi((d_0 \prod_{i \in A} d_i) t'),$$

as was to be shown. ∎

In words, Theorem 3.5.1 says that the flower $t = \Delta(d_0 a_0^{\hat{\dagger}(0)} \cdots d_n a_n^{\hat{\dagger}(n)})$ is equivalent to a polynomial in the petals d_i. In fact, the result still holds even if $\text{input}(t) = \text{output}(t)$ is not located in the cycle $\Delta(a_0^{\hat{\dagger}(0)} \cdots a_n^{\hat{\dagger}(n)}) \subset t$. Indeed, suppose that $t = \Delta(t)$ has a simple cycle $C \subset t$. Without loss of generality, we may assume that t is a quasi-cactus (Corollary 3.2.8). For any vertex $v \in C$, we define t_v to be the graph monomial obtained from t by changing both the input and the output to v. Since t is a quasi-cactus, we can write $t_v = \Delta(d_0 a_0^{\hat{\dagger}(0)} \cdots d_n a_n^{\hat{\dagger}(n)})$ as a flower. Moreover, by construction,

$$\widetilde{\Delta}(t) = \widetilde{\Delta}(t_v) \in \mathcal{T}\langle \mathcal{A} \rangle, \qquad \forall v \in C.$$

Specifically, we choose the unique vertex $v \in C$ such that the petal $d_0 \subset t_v$ rooted at v contains the original input/output of t. In the notation of Theorem 3.5.1, the vertex v now becomes v_0. We can then apply our cycle pruning algorithm to obtain a graph polynomial

$$p(t_v) := \sum_{A \subset [n]} \left(\psi(m_A)(d_0 \prod_{i \in A} d_i) \right) \equiv t_v \ (\text{mod}\ \psi).$$

For any subset $A \subset [n]$, let $\hat{d}_A$ be the unique graph monomial such that

$$\widetilde{\Delta}(\hat{d}_A) = \widetilde{\Delta}\left(d_0 \prod_{i \in A} d_i \right)$$

with $\text{input}(\hat{d}_A) = \text{output}(\hat{d}_A) = \text{input}(t) = \text{output}(t) \in d_0$. Then

$$p(t) := \sum_{A \subset [n]} \left(\psi(m_A) \hat{d}_A \right) \equiv t \ (\text{mod}\ \psi). \tag{3.10}$$

To see this, let $t' \in \mathcal{B}$ be a graph monomial. Note that the construction of the polynomial $p(t_v)$ leaves the initial petal d_0 intact throughout. In particular, any modification to this petal does not affect the coefficients $\psi(m_A)$, and so we only need to account for the change in the terms $d_0 \prod_{i \in A} d_i$ for $A \subset [n]$. This implies that

$$\begin{aligned}
\psi(tt') = \tau_\varphi\big[\widetilde{\Delta}(tt')\big] = \tau_\varphi\big[\widetilde{\Delta}(t_v) \sharp \widetilde{\Delta}(t')\big] &= \psi(t_v \sharp \widetilde{\Delta}(t')) \\
&= \sum_{A \subset [n]} \psi(m_A) \psi\left((d_0 \sharp \widetilde{\Delta}(t')) \prod_{i \in A} d_i \right) \\
&= \sum_{A \subset [n]} \psi(m_A) \tau_\varphi\left[\widetilde{\Delta}(d_0 \prod_{i \in A} d_i) \sharp \widetilde{\Delta}(t') \right] \\
&= \sum_{A \subset [n]} \psi(m_A) \tau_\varphi\left[\widetilde{\Delta}(\hat{d}_A) \sharp \widetilde{\Delta}(t') \right] \\
&= \sum_{A \subset [n]} \psi(m_A) \psi(\hat{d}_A t') = \psi(p(t) t'),
\end{aligned}$$

where, in every case, $\sharp$ denotes the appropriate object (test graph or graph monomial) obtained by identifying the vertices $\mathrm{input}(t) = \mathrm{output}(t)$ (seen as vertices of t_v, d_0, or $\hat{d}_A$) and $\mathrm{input}(t') = \mathrm{output}(t') \in \widetilde{\Delta}(t')$. The equivalence (3.10) now follows.

Our work above commits the formal details of the proof, but it also suggests a simple interpretation of the result. For simplicity, we think of every diagonal element d_i as a petal of the flower $\Delta(d_0 a_0^{\dagger(0)} \cdots d_n a_n^{\dagger(n)})$; however, this neglects the fact that d_0 plays a special role in the construction. Instead, we should think of d_0 as the *stem* of the flower. Theorem 3.5.1 then tells us how to prune the flower before reattaching it to the stem. If $\mathrm{input}(t) = \mathrm{output}(t)$ is not located in the cycle, then we simply need to orient ourselves properly to apply the algorithm. So, we designate the stem according to the location of the distinguished vertex, in which case everything goes through as before. Iterating the algorithm, we can gradually remove every cycle of $t = \Delta(t)$. Of course, the diagonality assumption greatly simplifies the analysis, but we can always reduce to this case. Indeed, recall that $\mathcal{B} \equiv \mathcal{A} * \mathcal{A}^\intercal * \Delta(\mathcal{B}) \pmod{\psi}$ via the conditional expectation $\mathscr{E}$. In the notation of Lemma 3.3.2, if t is a graph monomial with block-cut tree factorization $t = d_n m_{n-1} d_{n-1} \cdots m_1 d_1$, then

$$\mathscr{E}(t) = d_n \nabla(m_{n-1}) d_{n-1} \cdots \nabla(m_1) d_1;$$

however, note that if $\nabla(m_i) \in \mathcal{A} \cup \mathcal{A}^\intercal$, then $\nabla(m_i)$ is a cut-edge. So, it must be that every cycle in $\mathscr{E}(t)$ belongs to a factor $\nabla(m_i) \in \Delta(\mathcal{B})$ or $d_i \in \Delta(\mathcal{B})$. Passing to a quasi-cactus equivalent, we can then apply our cycle pruning algorithm. Moreover, the quasi-cactus property is preserved by our algorithm (remove the flower, attach petals); so, even though passing to a quasi-cactus equivalent may create more cycles, this is a one-time cost that then allows us to iterate our algorithm to eventually prune *every* cycle (recall that Corollary 3.2.12 already takes care of loops). Theorem 3.1.3 now follows, and so too does its extension to $\mathcal{B}$, namely,

Theorem 3.5.2. *For any $t \in \mathcal{B}$, there exists a graph polynomial $\mathbf{T}(t)$ of trees such that*

$$\mathbf{T}(t) \equiv t \pmod{\psi}.$$

Chapter 4

Applications to random multi-matrix models

In this chapter, we apply the traffic framework to study the asymptotics of random multi-matrix models. Random matrices provide the most salient source of applications for the traffic framework. Here, the action of the graph operations provides a unified setting that captures the interaction of a number of linear algebraic structures. For example, as opposed to thinking of the transpose as an operation on the entries of a matrix, we can think of the transpose as a particular graph operation, which allows us to consider its relation with other graph operations. Naturally, the results in the previous chapter will come into play. In particular, the results in this chapter combine work in both [AM] and [Au].

4.1 Introduction and main results

For a matrix $\mathbf{A}_N \in \mathrm{Mat}_N(\mathbb{C})$, let $(\lambda_k(\mathbf{A}_N))_{1\leq k\leq N}$ denote the eigenvalues of $\mathbf{A}_N$, counting multiplicity, arranged in a radially non-increasing order. In particular, we write

$$\mu(\mathbf{A}_N) = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k(\mathbf{A}_N)}$$

for the empirical spectral distribution (or ESD for short) of $\mathbf{A}_N$, where

$$|\lambda_1(\mathbf{A}_N)| \geq \cdots \geq |\lambda_N(\mathbf{A}_N)|$$

and

$$0 \leq \arg(\lambda_k(\mathbf{A}_N)) \leq \arg(\lambda_{k+1}(\mathbf{A}_N)) < 2\pi \quad \text{if} \quad |\lambda_k(\mathbf{A}_N)| = |\lambda_{k+1}(\mathbf{A}_N)|.$$

For a random matrix $\mathbf{A}_N$, the ESD $\mu(\mathbf{A}_N)$ then becomes a random probability measure on the complex plane. For the most part, we restrict to the case of a random real symmetric or complex Hermitian matrix, in which case the ESD $\mu(\mathbf{A}_N)$ becomes a random probability measure on the real line.

Wigner initiated the modern study of random matrices by proving the weak convergence of the ESD in the large N limit for a general class of random real symmetric matrices [Wig55, Wig58]. We recall the so-called Wigner matrices, formulated deliberately in such a way below in order to suit our purposes later. In particular, we consider a family of independent Wigner matrices with a strong uniform control on the moments of the entries.

Definition 4.1.1 (Wigner matrix)**.** Let I be an index set. For each $i \in I$ and $N \in \mathbb{N}$, let $(\mathbf{X}_N^{(i)}(j,k))_{1\leq j<k\leq N}$ and $(\mathbf{X}_N^{(i)}(j,j))_{1\leq j\leq N}$ be independent families of random variables: the former, real-valued (resp., complex-valued), centered, and of unit variance; the latter, real-valued and of finite variance. We further assume that

$$\sup_{N\in\mathbb{N}} \sup_{i\in I_0} \sup_{1\leq j\leq k\leq N} \mathbb{E}[|\mathbf{X}_N^{(i)}(j,k)|^\ell] \leq m_\ell^{(I_0)} < \infty, \qquad \forall I_0 \subset I : \#(I_0) < \infty, \tag{4.1}$$

where the random variables $(\mathbf{X}_N^{(i)}(j,k))_{1\leq j\leq k\leq N,\, i\in I}$ are independent with *parameter*

$$\mathbb{E}[\mathbf{X}_N^{(i)}(j,k)^2] = \beta_i, \qquad \forall j < k.$$

Taken together, the two families $(\mathbf{X}_N^{(i)}(j,k))_{1\leq j<k\leq N}$ and $(\mathbf{X}_N^{(i)}(j,j))_{1\leq j\leq N}$ define a random real symmetric (resp., complex Hermitian) matrix $\mathbf{X}_N^{(i)} \in \mathrm{Mat}_N(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P}))$. We call such a matrix $\mathbf{X}_N^{(i)}$ an *unnormalized real (resp., complex) Wigner matrix*.

We introduce the standard normalization via a Hadamard-Schur product: let $\mathbf{J}_N$ denote the $N \times N$ all-ones matrix, and define $\mathbf{N}_N = N^{-1/2}\mathbf{J}_N$. We call the random real symmetric (resp., complex Hermitian) matrix $\mathbf{W}_N^{(i)}$ defined by

$$\mathbf{W}_N^{(i)} = \mathbf{N}_N \circ \mathbf{X}_N^{(i)} = \frac{1}{\sqrt{N}}\mathbf{X}_N^{(i)}$$

a *normalized real (resp., complex) Wigner matrix*. We simply refer to *Wigner matrices* when the context is clear, or when considering the definition altogether.

Remark 4.1.2. Note that a Wigner matrix is a real Wigner matrix iff its parameter $\beta = 1$ (i.e., the common pseudo-variance of its unnormalized strictly upper triangular entries), and so we can specify a Wigner matrix by its parameter. We further note that the distribution of a Wigner matrix is invariant under conjugation by the permutation matrices only if its parameter $\beta \in \mathbb{R}$ (in general, $\beta \in \mathbb{D} \subset \mathbb{C}$.). This in turn is equivalent to the real and imaginary parts of $\mathbf{X}_N^{(i)}(j,k)$ being uncorrelated.

Wigner identified the standard semicircle distribution μ_{SC} as the universal limiting spectral distribution (or LSD for short) of the Wigner matrices, where

$$\mu_{SC}(dx) = \frac{1}{2\pi}(4 - x^2)_+^{1/2}\, dx. \tag{4.2}$$

Considerable work has since been done on the Wigner matrices and other classical random matrix ensembles, e.g., on questions related to maximal eigenvalues, central limit theorems, concentration inequalities, joint eigenvalue distribution, large deviations, eigenvalue spacing, and free probability. The recent monograph [AGZ10] by Anderson, Guionnet, and Zeitouni provides an excellent introduction to this end.

Our first result identifies the limiting traffic distribution (LTD) of the Wigner ensemble in the large N limit in a slight generalization of Proposition 3.1 in [Mal]. In particular, we allow for a general parameter $\beta \in \mathbb{C}$; we do not assume that the entries are identically distributed; and we allow the distribution of the entries to depend on N. As opposed to the usual universality (4.2), the LTD of the Wigner ensemble subdivides according to the parameter β. The precise form of this LTD requires a good deal of preparation, and we do not state it here in the introduction (see Proposition 4.2.2). Instead, we use the support of this LTD to apply our results from the previous chapter. To this end, let us introduce some notation. For a matrix $\mathbf{A}_N$, we write

$$\Theta(\mathbf{A}_N) = \operatorname{span}\Bigg(\bigcup_{K \geq 0} \bigcup_{g \in \Theta(\mathcal{G}_K)} Z_g(\{\mathbf{A}_N, \mathbf{A}_N^*\})\Bigg).$$

Theorem 4.1.3. *Let $\mathbf{W}_N$ be a Wigner matrix. If $\beta \in \mathbb{R}$, then $(\mathbf{W}_N, \mathbf{W}_N^\intercal)$ and $\Theta(\mathbf{W}_N)$ are asymptotically free. Furthermore, $\mathbf{W}_N$ and $\mathbf{W}_N^\intercal$ are asymptotically free iff $\beta = 0$.*

Theorem 4.1.3 generalizes the results of [BDJ06] on random Markov matrices and of [MP16] on freeness from the transpose. In particular, for a real Wigner matrix $\mathbf{W}_N$, Bryc, Dembo, and Jiang showed that the ESD $\mu(\mathbf{M}_N)$ of the random Markov matrix

$$\mathbf{M}_N = \mathbf{W}_N - \operatorname{rDeg}(\mathbf{W}_N) = \underset{\text{out}}{\cdot} \overset{\mathbf{W}_N}{\longleftarrow} \underset{\text{in}}{\cdot} \quad - \quad \underset{\text{in/out}}{\overset{\cdot \,\downarrow \mathbf{W}_N}{\cdot}}$$

converges weakly almost surely to the free convolution $\mathcal{SC}(0,1) \boxplus \mathcal{N}(0,1)$. Individually, we know that

$$\mu(\mathbf{W}_N) \xrightarrow{w} \mathcal{SC}(0,1) \quad \text{and} \quad \mu(\operatorname{rDeg}(\mathbf{W}_N)) \xrightarrow{w} \mathcal{N}(0,1) \quad \text{as} \quad N \to \infty,$$

but the two matrices are certainly not independent, so we would not expect that the free convolution should appear. Even more surprising, Theorem 4.1.3 extends this result to the asymptotic freeness of $\mathbf{W}_N$ and $\operatorname{rDeg}(\mathbf{W}_N) \in \Theta(\mathbf{W}_N)$. For example, as a consequence, this implies that $\mathbf{W}_N - \operatorname{rDeg}(\mathbf{W}_N)$ and $\mathbf{W}_N + \operatorname{rDeg}(\mathbf{W}_N)$ have the same LSD, $\mathcal{SC}(0,1) \boxplus \mathcal{N}(0,1)$, if $\beta = 1$.

Similarly, for a GUE matrix $\mathbf{W}_N$, Mingo and Popa showed that $\mathbf{W}_N$ and $\mathbf{W}_N^\intercal$ are asymptotically free: this despite the complete lack of independence between the two matrices. Of course, a GUE matrix is a special case of a Wigner matrix with parameter $\beta = 0$. Theorem 4.1.3 extends this result to general Wigner matrices of parameter $\beta = 0$ and shows that this is also a necessary condition for freeness from the transpose.

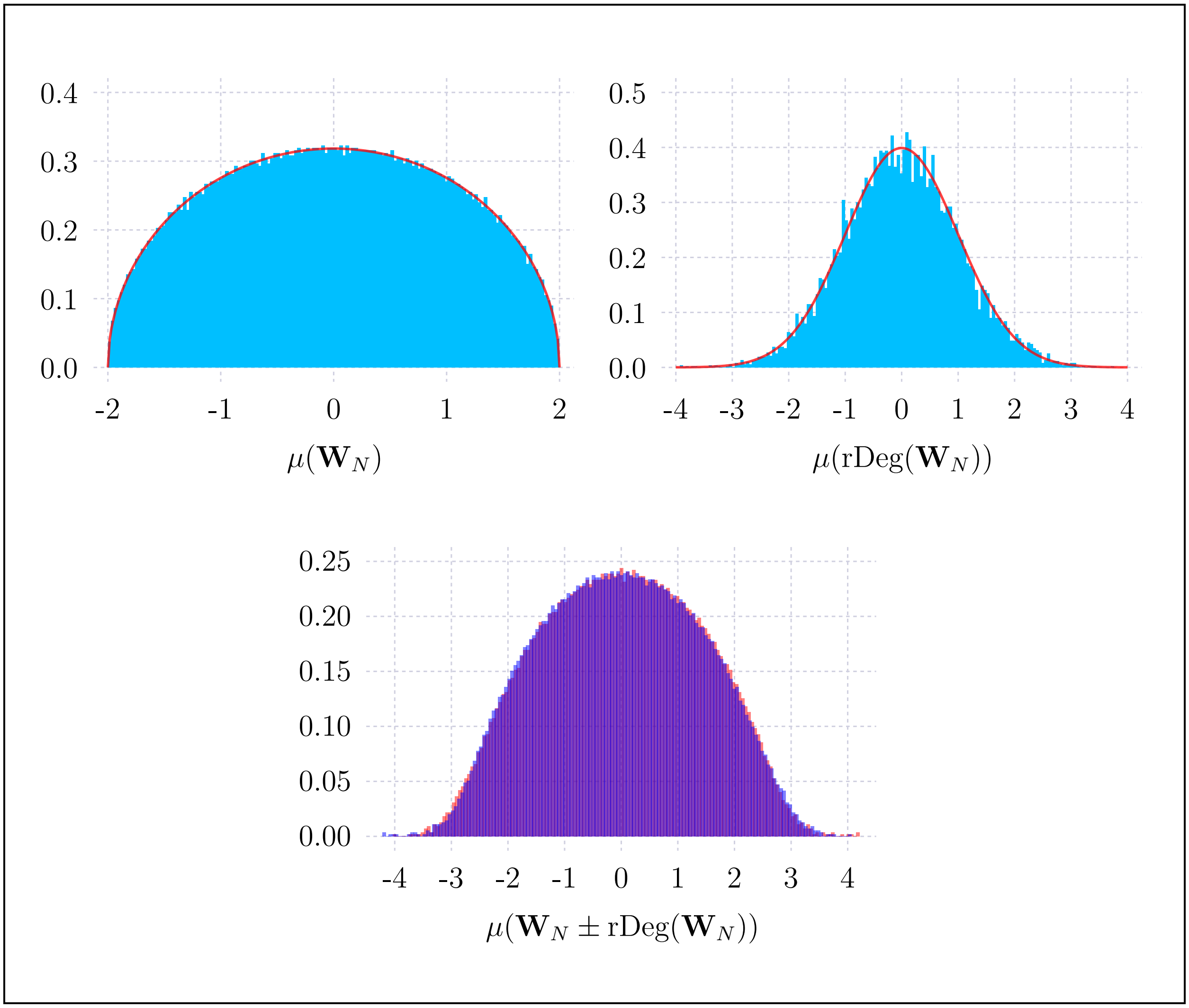


Figure 4.1: Histograms of the ESDs of random matrices constructed from a single realization of a normalized 10,000 by 10,000 GOE matrix. In the first two histograms, we overlay the density of the LSD in red for comparison. In the last histogram, we overlay the ESDs of the sum $\mathbf{W}_N + \mathrm{rDeg}(\mathbf{W}_N)$ and the difference $\mathbf{W}_N - \mathrm{rDeg}(\mathbf{W}_N)$, plotted in blue and red respectively. The overlapping region is colored blue + red = purple and dominates the graph, as predicted by the asymptotic freeness of $\mathbf{W}_N$ and $\mathrm{rDeg}(\mathbf{W}_N)$.

Naturally, we extend our analysis to other well-studied ensembles. In particular, we recall

Definition 4.1.4 (Ginibre matrix)**.** Let I be an index set. For each $i \in I$ and $N \in \mathbb{N}$, let $(\mathbf{Y}_N^{(i)}(j,k))_{1\leq j,k\leq N}$ be an independent family of random variables (real or complex) where the off-diagonal entries are centered and of unit variance. We further assume that

$$\sup_{N\in\mathbb{N}} \sup_{i\in I_0} \sup_{1\leq j,k\leq N} \mathbb{E}[|\mathbf{Y}_N^{(i)}(j,k)|^\ell] \leq m_\ell^{(I_0)} < \infty, \qquad \forall I_0 \subset I : \#(I_0) < \infty, \tag{4.3}$$

where the random variables $(\mathbf{Y}_N^{(i)}(j,k))_{1\leq j\leq k\leq N,\, i\in I}$ are independent with *parameter*

$$\mathbb{E}[\mathbf{Y}_N^{(i)}(j,k)^2] = \zeta_i, \qquad \forall j \neq k.$$

The family of random variables $(\mathbf{Y}_N^{(i)}(j,k))_{1\leq j,k\leq N}$ defines a non-normal random matrix $\mathbf{Y}_N^{(i)} \in \mathrm{Mat}_N(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P}))$. We call such a matrix $\mathbf{Y}_N^{(i)}$ an *unnormalized Ginibre matrix.* The same normalization as in Definition 4.1.1 defines a *normalized Ginibre matrix*

$$\mathbf{G}_N^{(i)} = \mathbf{N}_N \circ \mathbf{Y}_N^{(i)} = \frac{1}{\sqrt{N}}\mathbf{Y}_N^{(i)}.$$

We simply refer to *Ginibre matrices* when the context is clear, or when considering the definition altogether.

Remark 4.1.5. As before, a Ginibre matrix is a real Ginibre matrix iff its parameter $\zeta = 1$. In contrast, a strictly complex ζ no longer precludes the permutation invariance of the matrix.

The spectral theory of non-normal matrices require a great deal more care. In particular, the method of moments fails in this setting, and more sophisticated tools must be used to carry out the analysis. In this case, the analogue of Wigner's semicircle law for the Ginibre ensemble, the so-called *circular law*, was only established recently in [TV10], which itself builds on a long line of work. We refer the reader to the survey [BC12] for the history of this problem and future directions.

Much like the method of moments, the traffic distribution fails to completely capture the spectral behavior of non-normal matrices. Nevertheless, we can still apply the traffic framework to study the asymptotic behavior of the Ginibre ensemble. We identify the LTD of the Ginibre ensemble in the large N limit: as before, we save the precise statement of this result for later (see Proposition 4.3.1). Instead, we use the support of this LTD to once again apply the results of the previous chapter.

Theorem 4.1.6. *Let $\mathbf{G}_N$ be a normalized Ginibre matrix. Then $(\mathbf{G}_N, \mathbf{G}_N^\intercal)$ and $\Theta(\mathbf{G}_N)$ are asymptotically $*$-free. Furthermore, $\mathbf{G}_N$ and $\mathbf{G}_N^\intercal$ are asymptotically $*$-free iff $\zeta = 0$.*

The graph operations allow us to easily extend the results above to the self-adjoint Wishart-Laguerre ensemble $\mathbf{L}_N = \mathbf{G}_N\mathbf{G}_N^*$ [Wis28]. For the convenience of the reader, we state the conclusion separately.

Theorem 4.1.7. *Let $\mathbf{L}_N$ be a normalized Wishart-Laguerre matrix. Then $(\mathbf{L}_N, \mathbf{L}_N^\intercal)$ and $\Theta(\mathbf{L_N})$ are asymptotically free. Furthermore, $\mathbf{L}_N$ and $\mathbf{L}_N^\intercal$ are asymptotically free iff $\zeta = 0$.*

Theorem 4.1.7 generalizes another result in [MP16] on freeness from the transpose. In particular, if the unnormalized entries of the matrix $\mathbf{G}_N$ are i.i.d. standard complex normal (a complex Gaussian Ginibre matrix), then Mingo and Popa showed that the matrices $\mathbf{L}_N$ and $\mathbf{L}_N^\intercal$ are asymptotically free. As in the Wigner case, we extend this result to general $\zeta = 0$, which is again a necessary condition for freeness from the transpose.

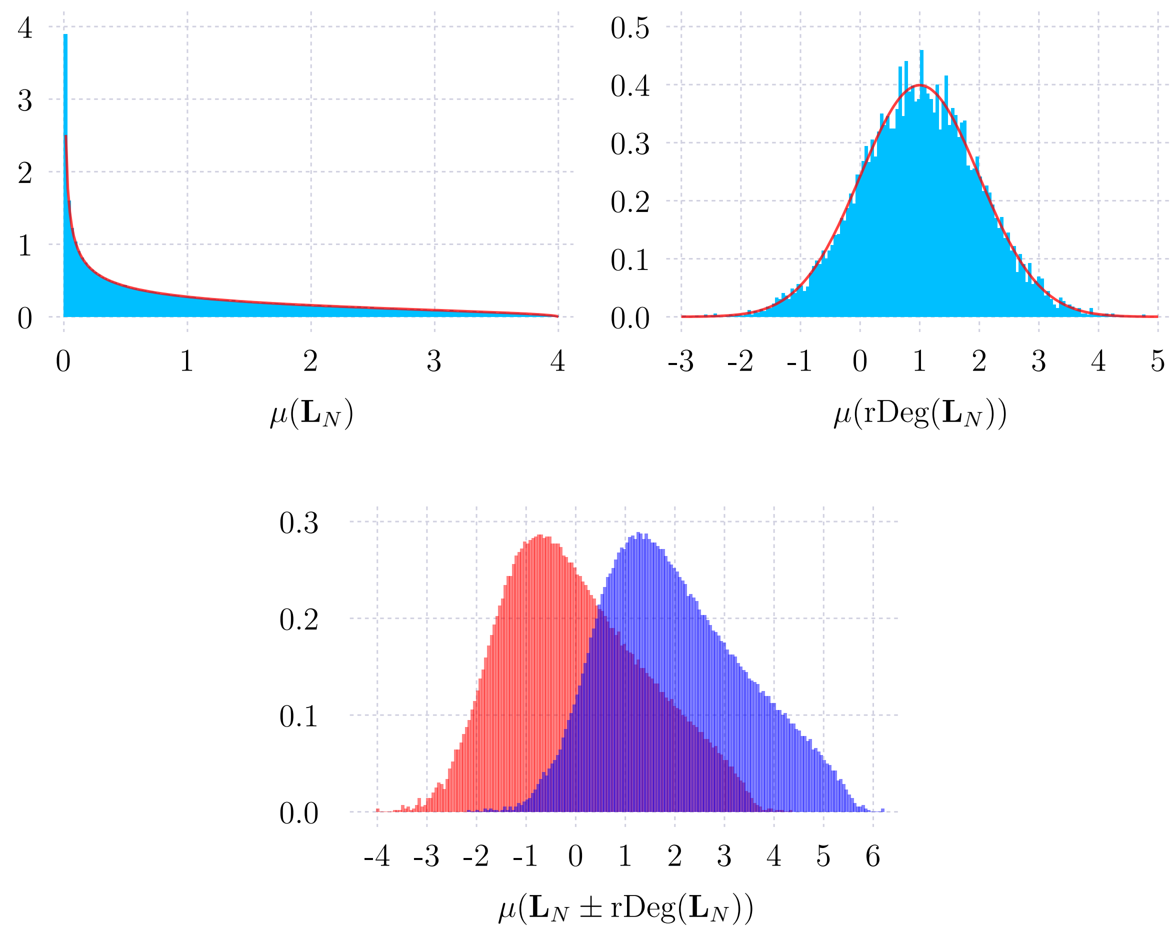


Figure 4.2: Histograms of the ESDs of random matrices constructed from a single realization of a normalized 10,000 by 10,000 real Gaussian Ginibre matrix. As before, in the first two histograms, we overlay the density of the LSD in red for comparison. In the last histogram, we overlay the ESDs of the sum $\mathbf{L}_N + \mathrm{rDeg}(\mathbf{L}_N)$ and the difference $\mathbf{L}_N - \mathrm{rDeg}(\mathbf{L}_N)$, plotted in blue and red respectively. The overlapping region is colored blue + red = purple, and we see that the two ESDs appear to be translations of one another, as predicted by the asymptotic freeness of $\mathbf{L}_N$ and $\mathrm{rDeg}(\mathbf{L}_N)$.

We can also use Theorem 4.1.7 to prove the Wishart-Laguerre analogue of the Markov matrix construction in [BDJ06]. More precisely, let $\mathbf{L}_N$ be a real Wishart-Laguerre matrix and $\mathrm{rDeg}(\mathbf{L}_N) = \mathrm{cDeg}(\mathbf{L}_N)$ its degree matrix. Individually, we know that

$$\mu(\mathbf{L}_N) \xrightarrow{w} \mathcal{MP}(1,1) \quad \text{and} \quad \mu(\mathrm{rDeg}(\mathbf{L}_N)) \xrightarrow{w} \mathcal{N}(1,1) \quad \text{as} \quad N \to \infty,$$

where $\mathcal{MP}(1,1)$ denotes the Marčenko-Pastur of mean 1 and variance 1 [MP67]. The asymptotic freeness of $\mathbf{L}_N$ and $\mathrm{rDeg}(\mathbf{L}_N) \in \Theta(\mathbf{L}_N)$ then implies that

$$\begin{aligned} \mu(\mathbf{L}_N - \mathrm{rDeg}(\mathbf{L}_N)) &\xrightarrow{w} \mathcal{MP}(1,1) \boxplus \mathcal{N}(-1,1) \\ \mu(\mathbf{L}_N + \mathrm{rDeg}(\mathbf{L}_N)) &\xrightarrow{w} \mathcal{MP}(1,1) \boxplus \mathcal{N}(1,1) \end{aligned} \qquad \text{as} \quad N \to \infty.$$

In particular, the two LSDs are simply translations of one another.

In a different direction, the universality of *non-invariant* ensembles constitutes a major ongoing program of research. We recall one prominent model of interest: the random band matrices.

Definition 4.1.8 (Band matrix)**.** Let (b_N) be a sequence of nonnegative integers. We write $\mathbf{B}_N$ for the corresponding $N \times N$ band matrix of ones with band width b_N, i.e.,

$$\mathbf{B}_N(i,j) = \mathbb{1}\{|i-j| \leq b_N\}.$$

Let $\mathbf{X}_N$ be an unnormalized Wigner matrix. We call the random matrix $\mathbf{\Xi}_N$ defined by

$$\mathbf{\Xi}_N = \mathbf{B}_N \circ \mathbf{X}_N$$

an *unnormalized random band matrix*. We introduce a normalization based on the growth rate of the band width b_N. We say that (b_N) is of *slow growth* (resp., *proportional growth*) if

$$\lim_{N\to\infty} b_N = \infty \quad \text{and} \quad b_N = o(N) \qquad \left(\text{resp., } \lim_{N\to\infty} \frac{b_N}{N} = c \in (0,1]\right),$$

in which case we use the normalization

$$\mathbf{\Upsilon}_N = (2b_N)^{-1/2}\mathbf{J}_N \qquad (\text{resp., } \mathbf{\Upsilon}_N = (2c-c^2)^{-1/2}N^{-1/2}\mathbf{J}_N).$$

We call c the *proportionality constant*: we say that (b_N) is of *full proportion* if $c = 1$ and *proper* otherwise. For a fixed band width $b_N \equiv b$, we use the normalization $\mathbf{\Upsilon}_N = (2b+1)^{-1/2}\mathbf{J}_N$. In any case, we call the random matrix $\mathbf{\Theta}_N$ defined by

$$\mathbf{\Theta}_N = \mathbf{\Upsilon}_N \circ \mathbf{\Xi}_N$$

a *normalized random band matrix*. We simply refer to random band matrices (or RBMs for short) when the context is clear, or when considering the definition altogether.

Following Wigner, one expects universality to hold for any large quantum system of sufficient complexity (see [Meh04] for more on this perspective; see [BEYY17,EY17] and the references therein for progress in this direction). In particular, a fundamental conjecture of Fyodorov and Mirlin proposes a dichotomy for the local spectral statistics of RBMs [FM91]: random matrix theory statistics (weak disorder) for large band widths; Poisson statistics (strong disorder) for small band widths; and a sharp transition around the critical value $b_N = \sqrt{N}$ (again, we refer the reader to [BEYY17, EY17] for progress in this direction).

At the macroscopic level, Bogachev, Molchanov, and Pastur proved that the classes of band widths in Definition 4.1.8 determine the global universality classes of the RBMs [BMP91]: for slow growth RBMs, $\mu(\mathbf{\Theta}_N)$ converges to the semicircle distribution μ_{SC}; for proportional growth RBMs of proper proportion, $\mu(\mathbf{\Theta}_N)$ converges to a non-semicircular distribution μ_c of bounded support; and for fixed band width RBMs having a symmetric distribution for the entries, $\mu(\mathbf{\Theta}_N)$ converges to a non-universal symmetric distribution μ_b. The authors further proved a continuity result for these distributions, namely,

$$\lim_{c\to 0^+} \mu_c = \lim_{c\to 1^-} \mu_c = \mu_{SC} \quad \text{and} \quad \lim_{b\to\infty} \mu_b = \mu_{SC}. \tag{4.4}$$

The work [BMP91] considered the distribution of a single RBM: naturally, this invites the question of the joint distribution of such matrices. Shlyakhtenko showed that freeness *with amalgamation* in the context of *operator-valued* free probability governs what he called Gaussian RBMs [Shl96]; otherwise, to our knowledge, RBMs have not received much attention from the non-commutative probabilistic perspective. Nevertheless, we show that the framework of traffic probability allows for effective, tractable computations in multiple RBMs. In particular, we identify the LTD of independent RBMs of possibly mixed band width types.

Theorem 4.1.9. *Let $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$ be a family of independent unnormalized Wigner matrices. We assume that the parameters $\beta_i \in \mathbb{R}$ and write $\mathcal{W}_N = (\mathbf{W}_N^{(i)})_{i\in I}$ for the corresponding family of normalized Wigner matrices. Consider a family of band widths*

$$(b_N^{(i)})_{i\in I} = (b_N^{(i)})_{i\in I_1} \cup (b_N^{(i)})_{i\in I_2} \cup (b_N^{(i)})_{i\in I_3} \cup (b_N^{(i)})_{i\in I_4}$$

of slow growth, proper proportion, full proportion, and fixed band width respectively, and form the corresponding family of normalized RBMs $\mathcal{O}_N = (\mathbf{\Theta}_N^{(i)})_{i\in I}$. Then the family $\mathcal{O}_N$ converges in traffic distribution. In fact, the LTDs of the families $(\mathbf{\Theta}_N^{(i)})_{i\in I_1\cup I_3}$ and $(\mathbf{W}_N^{(i)})_{i\in I_1\cup I_3}$ are identical.

Knowledge of the traffic distribution, which is defined in terms of test graphs, can often be difficult to interpret. At the same time, the equality of the LTD for $(\mathbf{\Theta}_N^{(i)})_{i\in I_1\cup I_3}$ and $(\mathbf{W}_N^{(i)})_{i\in I_1\cup I_3}$ allows us to transfer all of our results for $(\mathbf{W}_N^{(i)})_{i\in I_1\cup I_3}$ to $(\mathbf{\Theta}_N^{(i)})_{i\in I_1\cup I_3}$ at no additional cost. For example, this implies that the analogue of Theorem 4.1.3 holds for RBMs of slow growth. We can even apply this to consider the joint distribution of independent RBMs. We highlight one particular consequence.

Corollary 4.1.10. *The mixed family of RBMs $(\mathbf{\Theta}_N^{(i)})_{i\in I_1\cup I_3}$ converges in distribution to a semicircular system.*

Remark 4.1.11. We do not make any assumptions on the relative rates of growth for the band widths $(b_N^{(i)})_{i\in I_1}$; thus, for example, it could be that $(b_N^{(i_1)}, b_N^{(i_2)}, b_N^{(i_3)}, b_N^{(i_4)})$ are each of slow growth with $b_N^{(i_1)}, b_N^{(i_2)} \ll \sqrt{N} \ll b_N^{(i_3)}, b_N^{(i_4)}$. In particular, perhaps not surprisingly, we fail to observe any sort of transition around the conjectured critical value for the local spectral statistics at the level of (first-order) freeness.

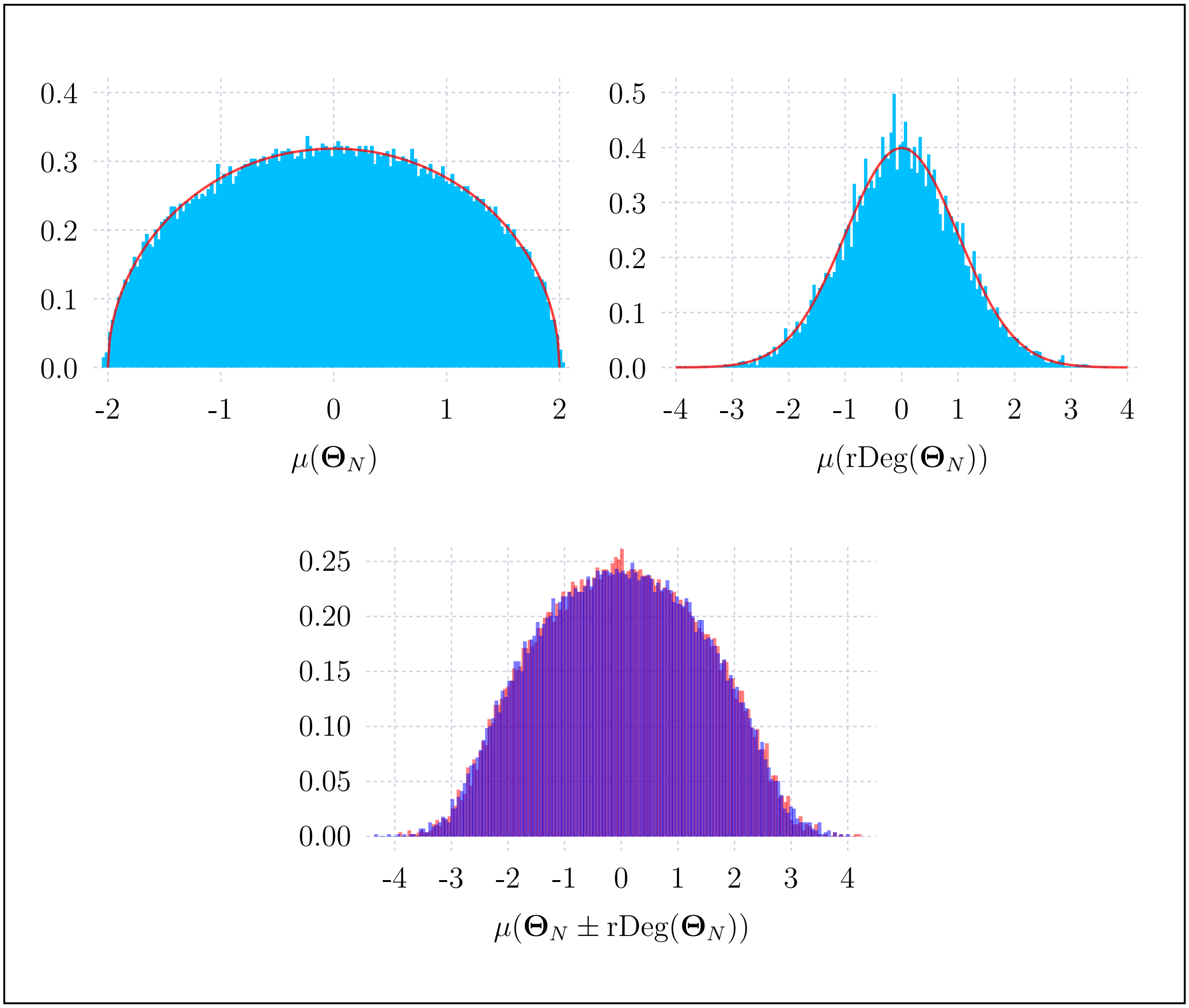


Figure 4.3: Histograms of the ESDs of random matrices constructed from a single realization of a 10,000 by 10,000 GOE matrix $\mathbf{X}_N$. We construct a RBM $\boldsymbol{\Theta}_N = \boldsymbol{\Upsilon}_N \circ \mathbf{B}_N \circ \mathbf{X}_N$ of slow growth $b_N = \sqrt{N}$. The ESDs closely resemble those of the standard Wigner ensemble in Figure 4.1, as predicted by LTD.

We note that the same analysis applies *mutatis mutandis* to band matrix versions of the Ginibre ensemble.

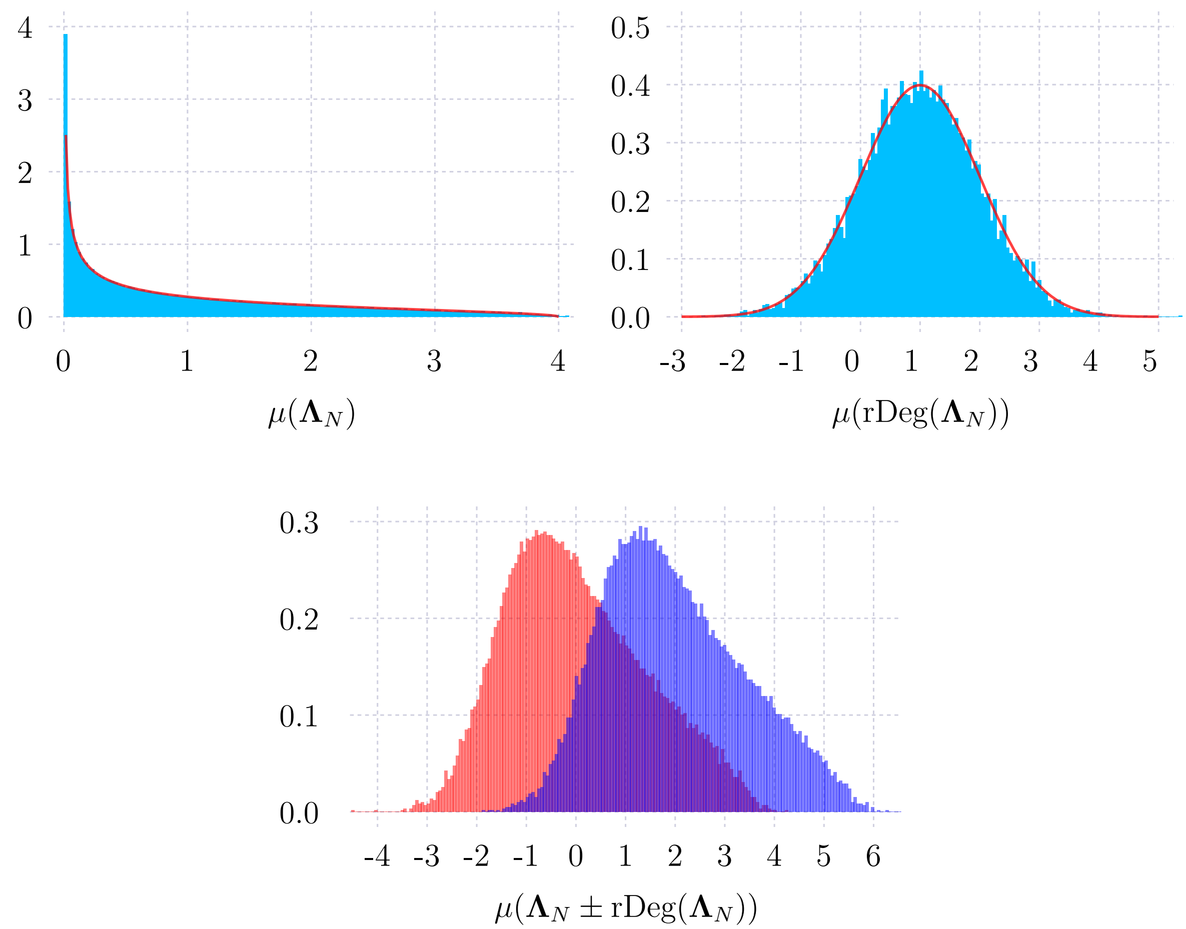


Figure 4.4: Histograms of the ESDs of random matrices constructed from a single realization of a 10,000 by 10,000 real Gaussian Ginibre matrix $\mathbf{Y}_N$. We construct a banded Ginibre matrix $\mathbf{\Gamma}_N = \mathbf{\Upsilon}_N \circ \mathbf{B}_N \circ \mathbf{Y}_N$ of slow growth $b_N = \sqrt{N}$ with the same normalization $\mathbf{\Upsilon}_N$ as in Definition 4.1.8. We then consider the banded Wishart-Laguerre matrix $\mathbf{\Lambda}_N = \mathbf{\Gamma}_N\mathbf{\Gamma}_N^*$, which itself is a band matrix of band width $2b_N$. The ESDs closely resemble those of the standard Wishart-Laguerre ensemble in Figure 4.2, as predicted by LTD.

Our results suggest a further investigation into the differences between the real and complex versions of a random matrix ensemble. For the classical compact groups, this comparison becomes that of Haar distributed orthogonal random matrices and Haar distributed unitary random matrices. An analysis of the unitary case in the traffic framework can be found in [Mal]. We consider the orthogonal case in Section 4.5.

Our formulas for the limiting traffic distribution evoke many of the formulas for free cumulants. This can be seen as a consequence of the correspondence between cactus graphs and non-crossing partitions as spelled out in Proposition 2.4.6. In Section 4.6, we give a simple procedure for determining the free cumulants from the injective traffic distribution in the case of cactus-type random variables. For example, as a simple application of this correspondence, we obtain the following corollary.

Corollary 4.1.12. *Let $\mathbf{W}_N$ be a Wigner matrix of parameter $\beta \in \mathbb{R}$. Then $(\mathbf{W}_N, \mathbf{W}_N^\intercal)$ converges in distribution to a semicircular family (s_1, s_2) of covariance $\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$.*

4.2 The Wigner ensembles

Let $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$ be a family of Wigner matrices as before. In particular, recall that

$$\mathbb{E}[\mathbf{X}_N^{(i)}(j,k)^2] = \beta_i, \qquad \forall j < k.$$

For technical reasons, we first assume that the real and imaginary parts of an off-diagonal entry $\mathbf{X}_N^{(i)}(j,k)$ are uncorrelated so that

$$\mathbb{E}[\mathbf{X}_N^{(i)}(j,k)^2] = \beta_i = \overline{\beta_i} = \mathbb{E}[\mathbf{X}_n^{(i)}(k,j)^2]. \tag{4.5}$$

For example, this includes the class of all real Wigner matrices ($\beta_i = 1$), but also circularly-symmetric ensembles such as the GUE ($\beta_i = 0$). We comment on the general case of $\beta_i \in \mathbb{C}$ when possible, though the situation becomes much different and often intractable (especially for RBMs). Thus, unless stated otherwise, we assume that $\beta_i = \overline{\beta_i} \in \mathbb{R}$.

Under this assumption, we prove the traffic convergence of the normalized Wigner matrices $\mathcal{W}_N = (\mathbf{W}_N^{(i)})_{i\in I}$. For simplicity, we restrict our attention to test graphs. The general case of a $*$-test graph follows from the self-adjointness of our ensembles. To describe the LTD, we will need some definitions.

Definition 4.2.1 (Colored double tree)**.** Let $T = (V, E, \gamma)$ be a test graph in $\mathbf{x} = (x_i)_{i\in I}$. We say that T is a *fat tree* if when disregarding the orientation and multiplicity of the edges, T becomes a tree. We further specify that T is a *double tree* if there are exactly two edges between adjacent vertices. We call the pair of edges connecting adjacent vertices in a double tree *twin edges*: *congruent* if they have the same orientation, *opposing* otherwise. Finally, we say that T is a *colored double tree* if T is a double tree such that each pair of twin edges $\{e, e'\}$ shares a common label $\gamma(e) = \gamma(e') \in I$. We record the number $c_i(T)$ of pairs of congruent twin edges with the common label i in a colored double tree T.

We introduce some notation to emphasize the relevant features of our test graphs. This notation will greatly simplify our analysis and features prominently in the remainder of the article. We start with a finite (not necessarily connected) multidigraph $G = (V, E)$.

We partition the set of edges $E = L \cup \mathcal{N}$ to distinguish between the loops L and the non-loop edges $\mathcal{N} = L^c$. As suggested by Definition 4.2.1, we define $\widetilde{G} := (V, \widetilde{E})$ as the undirected graph obtained from G by disregarding the orientation and multiplicity of the edges. Formally, $\widetilde{E} = E/\sim$ consists of equivalence classes in E, where

$$e \sim e' \quad \Longleftrightarrow \quad \{\mathrm{src}(e), \mathrm{tar}(e)\} = \{\mathrm{src}(e'), \mathrm{tar}(e')\}.$$

In this case, our partition $E = L \cup \mathcal{N}$ projects down to a partition $\widetilde{E} = \widetilde{L} \cup \widetilde{\mathcal{N}}$ between equivalence classes of loops and equivalence classes of non-loops respectively. We may then write the underlying simple graph $\underline{G}$ of $G = (V, E)$ as $\underline{G} = (V, \widetilde{\mathcal{N}})$.

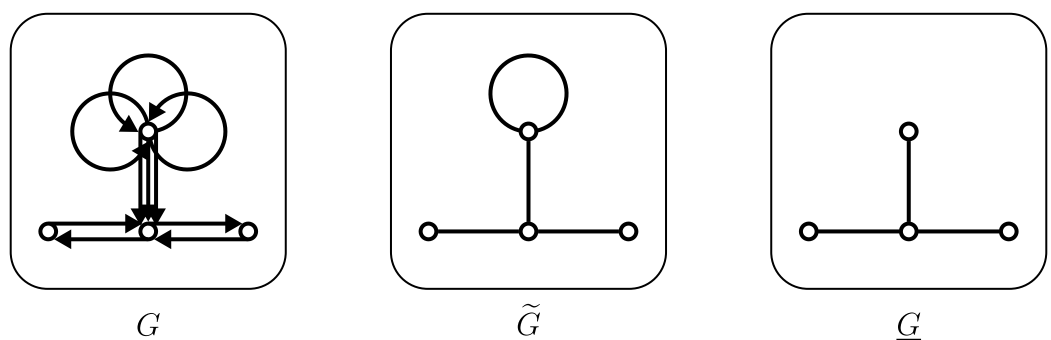


Figure 4.5: Examples of the projections $\widetilde{G}$ and $\underline{G}$ starting from a multidigraph G.

Now suppose that our graph G comes with edge labels $\gamma : E \to I$. We count the (undirected) multiplicity of a label i in a class of edges $[e] = \{e' \in E : e \sim e'\} \in \widetilde{E}$ with

$$m_{i,[e]} = \#(\gamma^{-1}(\{i\}) \cap [e]) \geq 0.$$

Summing this over the labels in I, we of course obtain the multiplicity of the class $[e]$,

$$m_{[e]} = \sum_{i \in I} m_{i,[e]} = \#([e]).$$

If $T = (G, \gamma)$ is a colored double tree, then

$$m_{i,[e]} \in \{0, 2\} \quad \text{and} \quad m_{[e]} = 2, \qquad \forall (i, [e]) \in I \times \widetilde{E}. \tag{4.6}$$

In this case, we write

$$\gamma([e]) = \gamma(e) \tag{4.7}$$

for the common label $\gamma(e) = \gamma(e')$ of twin edges $[e] = \{e, e'\}$. Conversely, if (4.6) and (4.7) hold for a test graph T whose projection $\widetilde{T}$ is a tree, then T is a colored double tree.

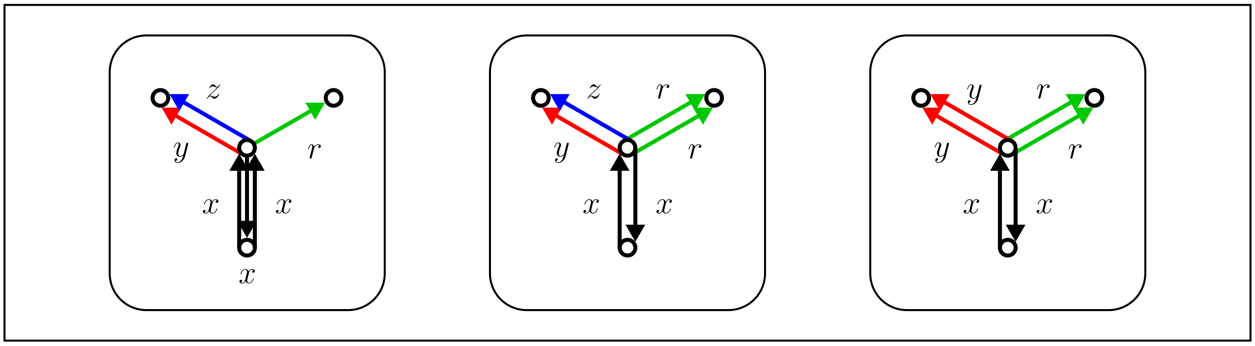


Figure 4.6: Examples of a fat tree, a double tree, and a colored double tree respectively.

Proposition 4.2.2 (β-semicircular traffics). *For any test graph $T \in \mathcal{T}\langle \mathbf{x} \rangle$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{W}_N)\big] = \begin{cases} \prod_{i\in I} \beta_i^{c_i(T)} & \textit{if } T \textit{ is a colored double tree,} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.8}$$

Proof. Suppose that $T = (V, E, \gamma)$. By definition, we have that

$$\begin{aligned} \tau^0\big[T(\mathcal{W}_N)\big] &= \mathbb{E}\Bigg[\frac{1}{N} \sum_{\phi: V \hookrightarrow [N]} \prod_{e\in E} \mathbf{W}_N^{(\gamma(e))}(\phi(e))\Bigg] \\ &= \frac{1}{N^{1+\frac{\#(E)}{2}}} \sum_{\phi: V \hookrightarrow [N]} \mathbb{E}\Bigg[\prod_{e\in E} \mathbf{X}_N^{(\gamma(e))}(\phi(e))\Bigg]. \end{aligned} \tag{4.9}$$

We analyze the asymptotics of (4.9) by working piecemeal in order to count the number of contributing maps ϕ (i.e., maps such that the summand is nonzero). First, we note that the independence of the random variables $\mathbf{X}_N^{(i)}(j,k)$ and the injectivity of the maps ϕ allow us to factor the product over the expectation, provided that we take into account multi-edges. The relevant information is precisely contained in the projected graph $\widetilde{T} = (V, \widetilde{E})$, which allows us to recast (4.9) as

$$\frac{1}{N^{1+\frac{\#(E)}{2}}} \sum_{\phi: V \hookrightarrow [N]} \Bigg(\prod_{[\ell]\in\widetilde{L}} \mathbb{E}\Bigg[\prod_{\ell'\in[\ell]} \mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell'))\Bigg]\Bigg)\Bigg(\prod_{[e]\in\widetilde{\mathcal{N}}} \mathbb{E}\Bigg[\prod_{e'\in[e]} \mathbf{X}_N^{(\gamma(e'))}(\phi(e'))\Bigg]\Bigg). \tag{4.10}$$

For non-loop edges $e' \in \mathcal{N}$, the independence of the centered random variables $\mathbf{X}_N^{(i)}(\phi(e'))$ implies that the second expectation in (4.10) vanishes if there exists a lone edge $e_0 \in [e]$ with the label $\gamma(e_0) = i_0$. Thus, in order for a summand to be non-zero, each label i present in a class $[e]$ must occur with multiplicity

$$m_{i,[e]} \geq 2. \tag{4.11}$$

This in turn implies that

$$\#(\mathcal{N}) \geq 2\#(\widetilde{\mathcal{N}}). \tag{4.12}$$

The underlying simple graph $\underline{T} = (V, \widetilde{\mathcal{N}})$ is of course still connected, whence the inequality

$$\#(\widetilde{\mathcal{N}}) \geq \#(V) - 1. \tag{4.13}$$

Finally, we make use of our strong moment assumption (4.1) to bound the summands in (4.10) uniformly in ϕ and N. In particular, our bound only depends on T, i.e.,

$$\Big(\prod_{[\ell] \in \widetilde{L}} \mathbb{E}\Big[\prod_{\ell' \in [\ell]} \mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell')) \Big] \Big) \Big(\prod_{[e] \in \widetilde{\mathcal{N}}} \mathbb{E}\Big[\prod_{e' \in [e]} \mathbf{X}_N^{(\gamma(e'))}(\phi(e')) \Big] \Big) \leq C_T < \infty. \tag{4.14}$$

Putting everything together, we arrive at the asymptotic

$$\tau^0\big[T(\mathcal{W}_N)\big] = O_T(N^{-1-\frac{\#(E)}{2}} N^{\#(V)}) = O_T(N^{-(\frac{\#(\mathcal{N})}{2} - (\#(V)-1))} N^{-\frac{\#(L)}{2}}). \tag{4.15}$$

The inequalities (4.11)-(4.13) then imply that $\tau^0\big[T(\mathcal{W}_N)\big]$ vanishes in the limit unless T is a colored double tree. For such a test graph T, (4.10) becomes

$$\frac{N^{\underline{\#(V)}}}{N^{\#(V)}} \prod_{[e] \in \widetilde{E}} \Big(\mathbb{1}\{[e] \text{ are opposing}\} + \beta_{\gamma([e])} \mathbb{1}\{[e] \text{ are congruent}\} \Big), \tag{4.16}$$

where $N^{\underline{\#(V)}}$ denotes the falling factorial $N(N-1)\cdots(N-(\#(V)-1))$. The limit (4.8) now follows. ■

Equation (4.16) explains the apparent asymmetry in the LTD of the Wigner matrices. In particular, if we record the number $o_i(T)$ of pairs of opposing twin edges with the common label i in a colored double tree T, then we can rewrite the nontrivial part of (4.8) as

$$\prod_{i \in I} \beta_i^{c_i(T)} = \prod_{i \in I} 1^{o_i(T)} \beta_i^{c_i(T)}. \tag{4.8$'$}$$

Working directly with this LTD, one can prove the asymptotic traffic independence of the Wigner matrices $\mathcal{W}_N$. To the same end, we can instead appeal to Theorem 2.3.10 by choosing a permutation invariant realization of our ensemble and concluding the general result by universality.

The careful reader will notice that we have made use of (4.5) in formulating (4.16): by assuming that $\beta_i = \overline{\beta_i}$, we were able to disregard the ordering on the vertices induced by the maps ϕ and conclude that congruent twin edges $[e]$ always give a contribution of $\beta_{\gamma([e])}$. In general, for a colored double tree T, a summand $S_\phi(T)$ of (4.10) will depend on ϕ, namely,

$$S_\phi(T) = \prod_{[e] \in \widetilde{E}} \Big(\mathbb{1}\{[e] \text{ are opposing}\} + \beta_{\gamma([e])} \mathbb{1}\{[e] \text{ are congruent and } \phi(\text{tar}([e])) < \phi(\text{src}([e]))\} + \overline{\beta_{\gamma([e])}} \mathbb{1}\{[e] \text{ are congruent and } \phi(\text{tar}([e])) > \phi(\text{src}([e]))\} \Big).$$

To compute the limit, we must then keep track of the ordering ψ_ϕ on the vertices, where

$$\psi_\phi : [\#(V)] \xrightarrow{\sim} V, \qquad \phi(\psi_\phi(1)) > \cdots > \phi(\psi_\phi(\#(V))).$$

Note that if $\phi_1 : V \hookrightarrow [N_1]$ and $\phi_2 : V \hookrightarrow [N_2]$ induce the same ordering $\psi_{\phi_1} = \psi_{\phi_2}$, then the corresponding summands are equal, i.e.,

$$S_{\phi_1}(T) = \mathbb{E}\bigg[\prod_{e\in E} \mathbf{X}_{N_1}^{(\gamma(e))}(\phi_1(e))\bigg] = \mathbb{E}\bigg[\prod_{e\in E} \mathbf{X}_{N_2}^{(\gamma(e))}(\phi_2(e))\bigg] = S_{\phi_2}(T).$$

Thus, for an ordering $\psi : [\#(V)] \xrightarrow{\sim} V$, we write $S_\psi(T)$ for the common value of

$$\{S_\phi(T) : \psi_\phi = \psi\}.$$

In this case, (4.16) becomes

$$\sum_{\psi:[\#(V)]\xrightarrow{\sim}V} \frac{\sum_{\phi:V\hookrightarrow[N]} \mathbb{1}\{\psi_\phi = \psi\}}{N^{\#(V)}} S_\psi(T). \tag{4.17}$$

One can intuitively verify that

$$\lim_{N\to\infty} \frac{\sum_{\phi:V\hookrightarrow[N]} \mathbb{1}\{\psi_\phi = \psi\}}{N^{\#(V)}} = \frac{1}{\#(V)!}, \qquad \forall \psi : [\#(V)] \xrightarrow{\sim} V;$$

however, in anticipation of Section 4.4, we give a natural integral representation of this limit instead. To this end, we introduce a set of indeterminates $\mathbf{x}_V = (x_v)_{v\in V}$ indexed by the vertices of our graph. A straightforward weak convergence argument then shows that

$$\lim_{N\to\infty} \frac{\sum_{\phi:V\hookrightarrow[N]} \mathbb{1}\{\psi_\phi = \psi\}}{N^{\#(V)}} = \int_{[0,1]^V} \mathbb{1}\{x_{\psi(1)} \geq \cdots \geq x_{\psi(\#(V))}\}\, d\mathbf{x}_V = \frac{1}{\#(V)!}. \tag{4.18}$$

Indeed, for each $N \in \mathbb{N}$, we can scale a labeling $\phi : V \hookrightarrow [N]$ by N to associate the image $\phi(V) = (\phi(v))_{v\in V}$ with a point p_ϕ of the latticed hypercube $[0,1]^V$, namely,

$$p_\phi = \left(\frac{\phi(v)}{N}\right)_{v\in V}.$$

We imagine integrating the indicator $\mathbb{1}\{x_{\psi(1)} \geq \cdots \geq x_{\psi(\#(V))}\}$ against the atomic measure

$$\mu_N = \frac{1}{N^{\#(V)}} \sum_{\phi:V\hookrightarrow[N]} \delta_{p_\phi}$$

to obtain the left-hand side of (4.18) (up to an asymptotically negligible correction factor). The limit $N \to \infty$ then converts this discretization into the uniform measure on $[0,1]^V$.

Finally, we arrive at the analogue of (4.8) for general $\beta_i \in \mathbb{C}$,

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{W}_N)\big] = \begin{cases} \displaystyle\sum_{\psi:[\#(V)]\xrightarrow{\sim} V} \frac{1}{\#(V)!} S_\psi(T) & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

In contrast to Proposition 4.2.2, the LTD (4.19) does not necessarily describe asymptotically traffic independent matrices $\mathcal{W}_N$. In fact, if we divide our index set I into two camps $I = I_\mathbb{R} \cup I_\mathbb{C} = \{i \in I : \beta_i \in \mathbb{R}\} \cup \{i \in I : \beta_i \in \mathbb{C} \setminus \mathbb{R}\}$, then the two families $\mathcal{W}_N^\mathbb{R} = (\mathbf{W}_N^{(i)})_{i\in I_\mathbb{R}}$ and $\mathcal{W}_N^\mathbb{C} = (\mathbf{W}_N^{(i)})_{i\in I_\mathbb{C}}$ are asymptotically traffic independent, but the matrices $\mathcal{W}_N^\mathbb{C}$ are not.

For the first statement, we need only to note that the representative value $S_\psi(T)$ does not depend on the ordering of the vertices that are only adjacent to edges with labels $i \in I_\mathbb{R}$, for which $\beta_i = \overline{\beta_i}$. We can formalize this by considering the subgraphs $T_\mathbb{R} = (V_\mathbb{R}, E_\mathbb{R})$ and $T_\mathbb{C} = (V_\mathbb{C}, E_\mathbb{C})$ of T with edge labels in $I_\mathbb{R}$ and $I_\mathbb{C}$ respectively. We write $T_\mathbb{C} = C_1^\mathbb{C} \cup \cdots \cup C_{k_1}^\mathbb{C}$ for the connected components of $T_\mathbb{C}$, each of which is a colored double tree $C_\ell = (V_\ell^\mathbb{C}, E_\ell^\mathbb{C})$, and similarly for $T_\mathbb{R} = C_1^\mathbb{R} \cup \cdots \cup C_{k_2}^\mathbb{R}$. We call such a graph a *forest of colored double trees*. It follows that a summand $S_\phi(T)$ only depends on the orderings

$$\psi_\phi^{(\ell)} : [\#(V_\ell^\mathbb{C})] \xrightarrow{\sim} V_\ell^\mathbb{C}, \qquad \ell \in [k_1],$$

on each component $C_\ell^\mathbb{C}$. In particular,

$$S_\phi(T) = \left(\prod_{\ell=1}^{k_1} S_{\psi_\phi^{(\ell)}}(C_\ell^\mathbb{C})\right)\left(\prod_{\ell=1}^{k_2}\prod_{i\in I_\mathbb{R}} \beta_i^{c_i(C_\ell^\mathbb{R})}\right).$$

In this case, for a concatenation of orderings

$$\psi = \times_{\ell=1}^{k_1} \psi_\ell : \bigtimes_{\ell=1}^{k_1} [\#(V_\ell^\mathbb{C})] \xrightarrow{\sim} \bigtimes_{\ell=1}^{k_1} V_\ell^\mathbb{C}$$

with the restrictions

$$\psi_\ell : [\#(V_\ell^\mathbb{C})] \xrightarrow{\sim} V_\ell^\mathbb{C},$$

we write S_ψ for the common value of

$$\{S_\phi(T) : \psi_\phi^{(\ell)} = \psi_\ell \text{ for all } \ell \in [k_1]\}.$$

We may then write

$$\tau^0\big[T(\mathcal{W}_N)\big] = \sum_{\psi:\times_{\ell=1}^{k_1}[\#(V_\ell^\mathbb{C})]\xrightarrow{\sim}\times_{\ell=1}^{k_1} V_\ell^\mathbb{C}} \frac{\sum_{\phi:V\hookrightarrow[N]} \prod_{\ell=1}^{k_1} \mathbb{1}\{\psi_\phi^{(\ell)} = \psi_\ell\}}{N^{\#(V)}} S_\psi(T),$$

where

$$\begin{aligned}\lim_{N\to\infty} \frac{\sum_{\phi:V\hookrightarrow[N]} \prod_{\ell=1}^{k_1} \mathbb{1}\{\psi_\phi^{(\ell)} = \psi_\ell\}}{N^{\#(V)}} &= \int_{[0,1]^V} \prod_{\ell=1}^{k_1} \mathbb{1}\{x_{\psi_\ell(1)} \geq \cdots \geq x_{\psi_\ell(\#(V_\ell^{\mathbb{C}}))}\}\, d\mathbf{x}_V \\ &= \prod_{\ell=1}^{k_1} \int_{[0,1]^{V_\ell^{\mathbb{C}}}} \mathbb{1}\{x_{\psi_\ell(1)} \geq \cdots \geq x_{\psi_\ell(\#(V_\ell^{\mathbb{C}}))}\}\, d\mathbf{x}_{V_\ell^{\mathbb{C}}} \qquad (4.20) \\ &= \frac{1}{\prod_{\ell=1}^{k_1} \#(V_\ell^{\mathbb{C}})!}.\end{aligned}$$

We conclude that

$$\begin{aligned}\lim_{N\to\infty} \tau^0\big[T(\mathcal{W}_N)\big] &= \sum_{\psi:\times_{\ell=1}^{k_1}[\#(V_\ell^{\mathbb{C}})]\xrightarrow{\sim}\times_{\ell=1}^{k_1} V_\ell^{\mathbb{C}}} \left(\frac{1}{\prod_{\ell=1}^{k_1}\#(V_\ell^{\mathbb{C}})!}\left(\prod_{\ell=1}^{k_1} S_{\psi_\ell}(C_\ell^{\mathbb{C}})\right)\left(\prod_{\ell=1}^{k_2}\prod_{i\in I_{\mathbb{R}}} \beta_i^{c_i(C_\ell^{\mathbb{R}})}\right)\right) \\ &= \left(\prod_{\ell=1}^{k_1} \sum_{\psi_\ell:[\#(V_\ell^{\mathbb{C}})]\to V_\ell^{\mathbb{C}}} \frac{1}{\#(V_\ell^{\mathbb{C}})!} S_{\psi_\ell}(C_\ell^{\mathbb{C}})\right)\left(\prod_{\ell=1}^{k_2}\prod_{i\in I_{\mathbb{R}}} \beta_i^{c_i(C_\ell^{\mathbb{R}})}\right) \\ &= \left(\prod_{\ell=1}^{k_1} \lim_{N\to\infty} \tau^0\big[C_\ell^{\mathbb{C}}(\mathcal{W}_N^{\mathbb{C}})\big]\right)\left(\prod_{\ell=1}^{k_2} \lim_{N\to\infty} \tau^0\big[C_\ell^{\mathbb{R}}(\mathcal{W}_N^{\mathbb{R}})\big]\right),\end{aligned}$$

as was to be shown.

Intuitively, we imagine each pair of twin edges $[e]$ imposing a constraint coming from the ordering of its adjacent vertices $\{\mathrm{src}([e]), \mathrm{tar}([e])\}$. We gather these constraints in the ordering ψ_ϕ to carry out the calculation of $S_\phi = S_{\psi(\phi)}$; however, if $\gamma([e]) \in I_{\mathbb{R}}$, the constraint becomes vacuous and we can disregard it, which corresponds to discarding the edge $[e]$ (but keeping the adjacent vertices). In this way, we arrive at the integral (4.20) (and, after discarding the isolated vertices, the forest of colored double trees $T_{\mathbb{C}}$). We return to this notion of a "free" edge $[e]$ in a slightly different context in Section 4.4.

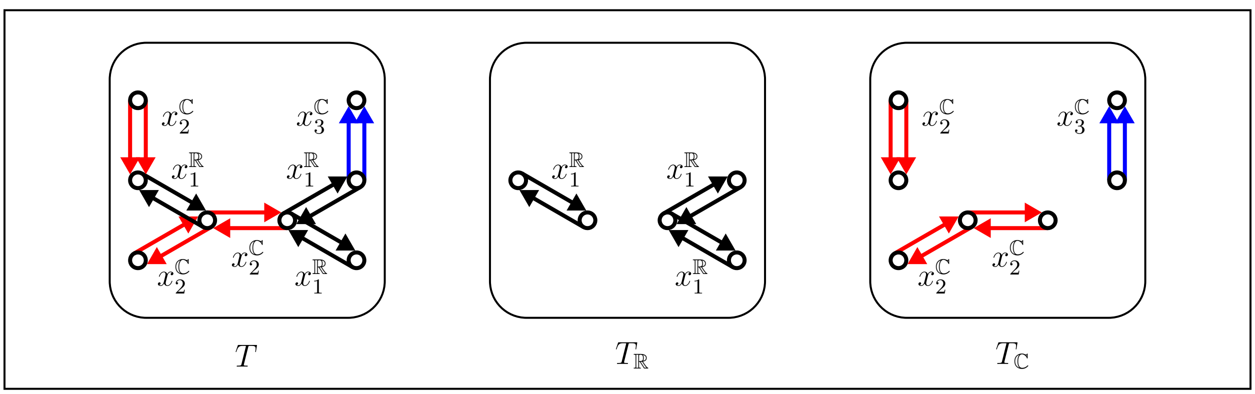


Figure 4.7: An example of the forest subgraph construction starting from a colored double tree T. For simplicity, we label twin edges $[e]$ with a single common indeterminate $\gamma([e])$.

For the second statement (about the lack of asymptotic traffic independence for $\mathcal{W}_N^{\mathbb{C}}$), we give a simple counterexample, namely, for $\beta_2^{\mathbb{C}}, \beta_3^{\mathbb{C}} \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned}\lim_{N\to\infty} \tau^0\left[\cdot \overset{\mathbf{w}_N^{i_2^{\mathbb{C}}}}{\underset{\mathbf{w}_N^{i_2^{\mathbb{C}}}}{\leftleftarrows}} \cdot \overset{\mathbf{w}_N^{i_3^{\mathbb{C}}}}{\underset{\mathbf{w}_N^{i_3^{\mathbb{C}}}}{\rightrightarrows}} \cdot \right] &= \frac{1}{3}(\beta_2^{\mathbb{C}}\beta_3^{\mathbb{C}} + \overline{\beta_2^{\mathbb{C}}\beta_3^{\mathbb{C}}}) + \frac{1}{6}(\beta_2^{\mathbb{C}}\overline{\beta_3^{\mathbb{C}}} + \overline{\beta_2^{\mathbb{C}}}\beta_3^{\mathbb{C}}) \\ &\neq \left(\frac{1}{2}(\beta_2^{\mathbb{C}} + \overline{\beta_2^{\mathbb{C}}})\right)\left(\frac{1}{2}(\beta_3^{\mathbb{C}} + \overline{\beta_3^{\mathbb{C}}})\right) \\ &= \left(\lim_{N\to\infty} \tau^0\left[\cdot \overset{\mathbf{w}_N^{i_2^{\mathbb{C}}}}{\underset{\mathbf{w}_N^{i_2^{\mathbb{C}}}}{\leftleftarrows}} \cdot \right]\right)\left(\lim_{N\to\infty} \tau^0\left[\cdot \overset{\mathbf{w}_N^{i_3^{\mathbb{C}}}}{\underset{\mathbf{w}_N^{i_3^{\mathbb{C}}}}{\rightrightarrows}} \cdot \right]\right).\end{aligned}$$

Yet, we know that free independence describes the asymptotic behavior of the Wigner matrices $\mathcal{W}_N$ regardless of the parameters $(\beta_i)_{i\in I}$. Naturally, we would like to know how to extract this information from the LTD (in particular, how this is consistent with the distinct LTDs (4.8) and (4.19)). Again, we restrict our attention to the joint distribution, the general case of the joint $*$-distribution following from the self-adjointness of our ensembles.

We know that the joint distribution $\mu_{\mathcal{W}_N}$ of $\mathcal{W}_N$ factors through the traffic distribution $\nu_{\mathcal{W}_N}$ of $\mathcal{W}_N$ via

$$\mu_{\mathcal{W}_N} = \nu_{\mathcal{W}_N} \circ \Delta \circ \eta,$$

where η is the embedding (2.4) of the $*$-polynomials into the $*$-graph polynomials. This amounts to computing $\tau\big[C(\mathcal{W}_N)\big]$ for directed cycles $C = (V, E)$. We use the injective traffic state to rewrite this as

$$\tau\big[C(\mathcal{W}_N)\big] = \sum_{\pi\in\mathcal{P}(V)} \tau^0\big[C^\pi(\mathcal{W}_N)\big].$$

In the limit, the only contributions come from (colored) double trees C^π. We claim that if C^π is a double tree, then it can only have opposing twin edges (an *opposing double tree*). Indeed, assume that $\pi \in \mathcal{P}(V)$ identifies the sources $\mathrm{src}(e_1) \overset{\pi}{\sim} \mathrm{src}(e_2)$ and targets $\mathrm{tar}(e_1) \overset{\pi}{\sim} \mathrm{tar}(e_2)$ of two distinct edges $e_1, e_2 \in E$. We write C^ρ for the graph intermediate to C and C^π obtained from C by only making these two identifications. If e_1 and e_2 are consecutive edges in the cycle C, then C^ρ consists of a directed cycle with two loops coming out of a particular vertex ("rabbit ears"). Otherwise, C^ρ consists of two almost disjoint directed cycles overlapping in the twin edge $[e] = \{e_1, e_2\}$ (a "butterfly"as in Figure 3.3). In both cases, we see that no further identifications can possibly result in a double tree C^π. In particular, note that a double tree is a special case of a cactus graph. The butterfly construction then necessitates a crossing partition, which we know fails to produce a cactus grace of Proposition 2.4.6.

Thus, from the perspective of the joint distribution, we need only to consider the behavior of the LTD on opposing colored double trees T. In this case, we see that the LTDs (4.8) and (4.19) agree on the value of

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{W}_N)\big] = 1.$$

Remark 4.2.3. An important application of traffic probability lies in the relationship between traffic independence and free independence. In certain situations, one can actually deduce free independence from traffic independence [Mal, CDM], the advantage being that the traffic setting might be more tractable. Of course, the two notions do not perfectly align, as seen even in the case of the Wigner matrices (Lemma 3.4 in [Mal] gives yet another example). In this case, we see that the traffic distribution specifies the behavior of our matrices in situations that might not be relevant to their joint distribution: in a certain sense, traffic independence asks for too much. Nevertheless, we can still use the traffic framework to make free probabilistic statements, even when a LTD might not exist! In particular, from our work above, we see that if a family of self-adjoint traffics $\mathbf{a}_n = (a_n^{(i)})_{i\in I}$ in a traffic space $(\mathcal{A}_n, \tau_n)$ satisfies

$$\lim_{n\to\infty} \tau_n^0\big[T(\mathbf{a}_n)\big] = \begin{cases} 1 & \text{if } T \text{ is an opposing colored double tree,} \\ 0 & \text{if } T \text{ is an opposing double tree that is not colored,} \\ 0 & \text{if } T \text{ is not a double tree,} \end{cases} \tag{4.21}$$

then $\mathbf{a}_n$ converges in *joint distribution* to a semicircular system $\mathbf{a} = (a_i)_{i\in I}$. Note that we do not specify the behavior of $\tau_n^0\big[T(\mathbf{a}_n)\big]$ on general double trees T (in particular, we do not assume that the limit $\lim_{n\to\infty} \tau_n^0\big[T(\mathbf{a}_n)\big]$ even exists). We will use this criteria in Section 4.4 to treat the case of RBMs of a general parameter $\beta_i \in \mathbb{C}$.

Of course, in the other direction, it is possible to have traffic independence without free independence. We can see this in the context of the traffic CLT (Theorem 2.3.11) by realizing the intermediate values $\alpha \in (0, 1)$.

With the LTD of the Wigner matrices in hand, we can now apply the free product decomposition of the universal enveloping traffic space. In particular, Lemma 3.5 in [Mal] shows that the Wigner matrices satisfy the factorization property (iii) of Proposition 2.4.9. Moreover, since the LTD of $\mathbf{W}_N$ is universal given the parameter β, we can take the GUE ensemble as a representative for $\beta = 0$. In this case, we also have the unitary invariance property (i). The free product structure of the universal enveloping traffic space (Theorem 3.1.1) then proves the if direction of the second statement of Theorem 4.1.3, namely, that $\mathbf{W}_N$, $\mathbf{W}_N^\intercal$, and $\Theta(\mathbf{W}_N)$ are asymptotically free if $\beta = 0$. One can again exploit this trick of universality by taking the GOE as a representative for $\beta = 1$. Up to a normalization, a GOE matrix $\mathbf{W}_N'$ can be written as the sum of a GUE matrix $\mathbf{W}_N$ and its transpose $\mathbf{W}_N^\intercal$, namely,

$$\mathbf{W}_N' = \frac{\mathbf{W}_N + \mathbf{W}_N^\intercal}{\sqrt{2}}.$$

The result for $\beta = 0$ then proves that $(\mathbf{W}_N', \mathbf{W}_N'^\intercal)$ and $\Theta(\mathbf{W}_N')$ are asymptotically free if $\beta' = 1$. The same tricks would seem to fail for general $\beta \in \mathbb{R}$; however, note that the proof of the freeness of $(\mathcal{A}, \mathcal{A}^\intercal)$ and $\Theta(\mathcal{B})$ in Chapter 3 only relies on the cactus structure of the traffic state τ_φ. In particular, the fact that the cacti were oriented only mattered for proving that $\mathcal{A}$ and $\mathcal{A}^\intercal$ are freely independent. Since double trees are a special case of cactus graphs, the same proof shows that $(\mathbf{W}_N, \mathbf{W}_N^\intercal)$ and $\Theta(\mathbf{W}_N)$ are asymptotically free if $\beta \in \mathbb{R}$. In this case, we are bypassing Proposition 2.4.9 altogether and simply working directly with the LTD (4.8). Notably, the multiplicative structure of (4.8′) with respect to the twin edges of the double tree combined with Proposition 2.4.6 supply the analogue of the cactus-cumulant construction (i)-(iii) in the injective traffic state τ_φ^0 of the universal enveloping traffic space (except this now allows for possible contributions from undirected cycles and unoriented cacti, which explains the lack of freeness from the transpose). To see that $\beta = 0$ is necessary for freeness from the transpose, observe that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{W}_N\mathbf{W}_N^\intercal)\right] = \frac{\beta+\overline{\beta}}{2}$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{W}_N\mathbf{W}_N^\intercal\mathbf{W}_N\mathbf{W}_N^\intercal)\right] = \frac{2(\beta^2+\overline{\beta}^2+\beta\overline{\beta})}{3},$$

where we have used (4.19) for a general $\beta \in \mathbb{C}$. If $\mathbf{W}_N$ and $\mathbf{W}_N^\intercal$ are asymptotically free, then both of these quantities must be zero. At the same time,

$$\frac{\beta+\overline{\beta}}{2} = 0 \qquad \Longleftrightarrow \qquad \Re(\beta) = 0,$$

while

$$\frac{2(\beta^2+\overline{\beta}^2+\beta\overline{\beta})}{3} = 0 \qquad \Longleftrightarrow \qquad 4\Re(\beta) = |\beta|^2.$$

This completes the proof of Theorem 4.1.3. Of course, since we know that the $(\mathbf{W}_N^{(i)})_{i\in I}$ are asymptotically free, we can also apply the reasoning from Chapter 3 for the diagonal algebra to conclude that the subalgebras $(\Delta(\mathbf{W}_N^{(i)}))_{i\in I}$ are asymptotically classically independent for real parameters $\beta_i \in \mathbb{R}$.

As we mentioned in introduction, this generalizes the result of Bryc, Dembo, and Jiang on the convergence of the ESD $\mu(\mathbf{M}_N) \xrightarrow{w} \mathcal{SC}(0,1) \boxplus \mathcal{N}(0,1)$ as $N \to \infty$ for a random Markov matrix [BDJ06]. As an interesting aside, we show how the same convergence (but not the asymptotic freeness) can be seen as an instance of the traffic CLT.

The traffic CLT

For simplicity, we restrict our attention to real Wigner matrices $\mathcal{W}_N = (\mathbf{W}_N^{(\ell)})_{\ell\in\mathbb{N}}$ in this section. A classical result of Dykema shows that the matrices $\mathcal{W}_N$ are asymptotically free [Dyk93], thus realizing both the free CLT and the traffic CLT (the latter, for $\alpha = 1$). Yet, Remark 2.3.9 extends the asymptotic traffic independence of $\mathcal{W}_N$ to a much larger class of matrices. In particular, we know that the corresponding family of degree matrices $\mathcal{D}_N = (\mathbf{D}_N^{(\ell)})_{\ell\in\mathbb{N}}$ are also asymptotically traffic independent, where

$$\mathbf{D}_N^{(\ell)} = \underset{\text{in/out}}{\downarrow \mathbf{W}_N} = \frac{1}{2}\Bigg(\underset{\text{in/out}}{\downarrow \mathbf{W}_N}\Bigg) + \frac{1}{2}\Bigg(\underset{\text{in/out}}{\uparrow \mathbf{W}_N}\Bigg) \in \mathbb{C}\mathcal{G}\langle \mathbf{W}_N^{(\ell)}\rangle. \tag{4.22}$$

A simple computation shows that the diagonal matrices $\mathcal{D}_N$ realize the traffic CLT for $\alpha = 0$, in some sense recovering the classical CLT.

Taking linear combinations of the above, we obtain the (p,q)-*Markov matrices*:

$$\mathbf{M}_{N,p,q}^{(\ell)} = p\mathbf{W}_N^{(\ell)} + q\mathbf{D}_N^{(\ell)} \in \mathbb{C}\mathcal{G}\langle \mathbf{W}_N^{(\ell)} \rangle, \qquad \forall p,q \in \mathbb{R}.$$

Recall that the LSD of the Markov matrices is given by the free convolution $\mathcal{SC}(0,1) \boxplus \mathcal{N}(0,1)$. Naively, one may then hope that the interpolation between $\mathbf{W}_N^{(\ell)}$ and $\mathbf{D}_N^{(\ell)}$ given by $\mathbf{M}_{N,p,q}^{(\ell)}$ passes to the traffic CLT, realizing the intermediate values $\alpha \in (0,1)$. We show that this is indeed the case.

Definition 4.2.4 (Stable traffic distribution)**.** Let $\nu : \mathbb{C}\mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle \to \mathbb{C}$ denote the traffic distribution of some family of centered random variables. We say that ν is *stable* if there exists a realization of ν by traffic independent families $\mathbf{a}_1 = (a_1^{(i)})_{i \in I}$ and $\mathbf{a}_2 = (a_2^{(i)})_{i \in I}$ in a traffic space $(\mathcal{A}, \tau)$ such that the sum $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 = (a_1^{(i)} + a_2^{(i)})_{i \in I}$ has the same traffic distribution, up to scale. By this, we mean that

$$\nu = \nu_{\mathbf{a}_1} = \nu_{\mathbf{a}_2}$$

with a scaling parameter $c \in \mathbb{R}_+$ such that

$$\nu_{\mathbf{a}}(T) = c^{\#(E)/2} \nu(T), \qquad \forall T = (V, E, \gamma, \varepsilon) \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle.$$

Lemma 4.2.5. *The families $(\mathcal{M}_N^{(\ell)})_{\ell \in \mathbb{N}} = ((\mathbf{M}_{N,p,q}^{(\ell)})_{p,q \in \mathbb{R}})_{\ell \in \mathbb{N}}$ are asymptotically traffic independent with a stable universal limiting traffic distribution.*

Proof. We need only to prove the stability of the limiting traffic distribution $\nu = \lim_{N \to \infty} \nu_{\mathcal{M}_N^{(1)}}$ as the rest follows from Proposition 4.2.2 and Remark 2.3.9. To this end, we model the limit of our matrices $(\mathcal{M}_N^{(\ell)})_{\ell \in \mathbb{N}}$ within the traffic space $(\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle, \nu)$, where

$$\nu = *_{\ell \in \mathbb{N}} \left(\lim_{N \to \infty} \nu_{\mathbf{W}_N^{(\ell)}} \right) \quad \text{and} \quad x_\ell = x_\ell^*.$$

By the universality of (4.8), the traffic state ν does not depend on the particular choice of Wigner matrices $\mathbf{W}_N^{(\ell)}$. We single out the Gaussian realization $\mathbf{X}_N^{(\ell)}(i,j) \stackrel{d}{=} \mathcal{N}(0, \mathbb{1}\{i \neq j\})$ for the distributional symmetry

$$\mathcal{S}_N^{(k)} = \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} \mathcal{M}_N^{(\ell)} = \left(\frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} \mathbf{M}_{N,p,q}^{(\ell)} \right)_{p,q \in \mathbb{R}} \stackrel{d}{=} (\mathbf{M}_{N,p,q}^{(1)})_{p,q \in \mathbb{R}} = \mathcal{M}_N^{(1)}.$$

This in turn implies the traffic distributional equality

$$\mathcal{S}_N^{(k)} \stackrel{\nu}{=} \mathcal{M}_N^{(1)}. \tag{4.23}$$

By construction, the family $\mathcal{S}_N^{(k)}$ converges in traffic distribution to

$$\mathbf{s}_k = \frac{1}{\sqrt{k}}\sum_{\ell=1}^{k}\mathbf{m}_\ell = \left(\frac{1}{\sqrt{k}}\sum_{\ell=1}^{k} p\left(\underset{\text{out}}{\cdot} \xleftarrow{x_\ell} \underset{\text{in}}{\cdot} \right) + \frac{q}{2}\left(\overset{\cdot}{\downarrow} {\scriptstyle x_\ell} \atop \underset{\text{in/out}}{\cdot} \right) + \frac{q}{2}\left(\overset{\cdot}{\uparrow} {\scriptstyle x_\ell} \atop \underset{\text{in/out}}{\cdot} \right)\right)_{p,q\in\mathbb{R}}$$

in the traffic space $(\mathbb{C}\mathcal{G}\langle \mathbf{x},\mathbf{x}^*\rangle,\nu)$. Passing to the limit, (4.23) becomes

$$\mathbf{s}_k \overset{\nu}{=} \mathbf{m}_1 \subset \mathbb{C}\mathcal{G}\langle x_1, x_1^*\rangle. \tag{4.24}$$

Taking $k=2$ in the above, we have that

$$\mathbf{s}_2 = \frac{1}{\sqrt{2}}(\mathbf{m}_1+\mathbf{m}_2) \overset{\nu}{=} \mathbf{m}_1,$$

where $\mathbf{m}_1$ and $\mathbf{m}_2$ are traffic independent. We conclude that the limiting traffic distribution $\nu = \lim_{N\to\infty}\nu_{\mathcal{M}_N^{(1)}}$ is stable with scaling parameter $c=2$. ■

Corollary 4.2.6. *The ESDs $\mu(\mathbf{M}_{N,p,q}^{(1)})$ converge weakly in expectation to the free convolution $\mu_{p,q} = \mathcal{SC}(0,p^2)\boxplus\mathcal{N}(0,q^2)$.*

Proof. It suffices to prove the result for $p,q\in\mathbb{R}$ of the form $p^2+q^2=1$. Proposition A.3 in [BDJ06] shows that the free convolution $\mu_{1,1}$ is determined by its moments: the same argument applies wholesale to the family of free convolutions $(\mu_{p,q})_{p,q\in\mathbb{R}}$. We may thus proceed by the method of moments.

Using the same notation as before, we know that $\mathbf{M}_{N,p,q}^{(\ell)}$ converges in traffic distribution to the self-adjoint traffic

$$a_{p,q}^{(\ell)} = p\left(\underset{\text{out}}{\cdot} \xleftarrow{x_\ell} \underset{\text{in}}{\cdot} \right) + \frac{q}{2}\left(\overset{\cdot}{\downarrow} {\scriptstyle x_\ell} \atop \underset{\text{in/out}}{\cdot} \right) + \frac{q}{2}\left(\overset{\cdot}{\uparrow} {\scriptstyle x_\ell} \atop \underset{\text{in/out}}{\cdot} \right) \in \mathbf{m}_\ell \subset \mathbb{C}\mathcal{G}\langle x_\ell, x_\ell^*\rangle \subset (\mathbb{C}\mathcal{G}\langle \mathbf{x},\mathbf{x}^*\rangle,\nu).$$

This reduces the problem to showing that the moments of $a_{p,q}^{(1)}$ match those of $\mu_{p,q}$. Now, note that a special case of (4.24) implies that

$$s_{p,q}^{(k)} = \frac{1}{\sqrt{k}}\sum_{\ell=1}^{k} a_{p,q}^{(\ell)} \overset{\nu}{=} a_{p,q}^{(1)}. \tag{4.25}$$

We calculate the mean and variance of $a_{p,q}^{(\ell)}$ using the same Gaussian realization as before:

$$\varphi_\nu\left(\underset{\text{out}}{\cdot} \xleftarrow{a_{p,q}^{(\ell)}} \underset{\text{in}}{\cdot} \right) = \lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{M}_{N,p,q}^{(\ell)})\right] = \lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\left(q\sum_{k\neq j}^{N}\frac{\mathbf{X}_N^{(\ell)}(j,k)}{\sqrt{N}}\right)\right] = 0$$

and

$$\varphi_\nu\left(\underset{\text{out}}{\cdot} \xleftarrow{a_{p,q}^{(\ell)}} \cdot \xleftarrow{a_{p,q}^{(\ell)}} \underset{\text{in}}{\cdot} \right) = \tau\left[\cdot \underset{a_{p,q}^{(\ell)}}{\overset{a_{p,q}^{(\ell)}}{\rightleftarrows}} \cdot \right] = \tau^0\left[\cdot \underset{a_{p,q}^{(\ell)}}{\overset{a_{p,q}^{(\ell)}}{\rightleftarrows}} \cdot \right] + \tau^0\left[\underset{a_{p,q}^{(\ell)}}{\overset{a_{p,q}^{(\ell)}}{\circlearrowleft\!\!\cdot\!\!\circlearrowright}} \right] = \alpha + (1-\alpha).$$

A straightforward calculation then shows that

$$\alpha = \lim_{N\to\infty} \frac{1}{N} \sum_{j\neq k}^{N} \mathbb{E}[\mathbf{M}_{N,p,q}(j,k)^2] = \lim_{N\to\infty} \frac{1}{N} N(N-1)\frac{p^2}{N} = p^2$$

and

$$\begin{aligned} 1-\alpha = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}[\mathbf{M}_{N,p,q}^{(\ell)}(j,j)^2] &= \lim_{N\to\infty} \mathbb{E}[\mathbf{M}_{N,p,q}^{(\ell)}(1,1)^2] \\ = \lim_{N\to\infty} \mathbb{E}\left[\left(q\sum_{j=2}^{N} \frac{\mathbf{X}_N^{(\ell)}(1,j)}{\sqrt{N}}\right)^2\right] &= \lim_{N\to\infty} (N-1)\frac{q^2}{N} = q^2. \end{aligned}$$

Combining (4.25) with the traffic CLT, we obtain the distributional identity

$$a_{p,q}^{(1)} \overset{\nu}{=} s_{p,q}^{(k)} \xrightarrow{d} \mu_{p,q} = \mathcal{SC}(0,p^2) \boxplus \mathcal{N}(0,q^2) \quad \text{as} \quad k\to\infty,$$

as was to be shown. ∎

Taking $(p,q) = (1,-1)$ in the above, we recover the special case of the Markov matrices in [BDJ06]. Corollary 4.2.6 explains this convergence in the context of traffic probability, but it also suggests a far more natural free probabilistic interpretation, namely, the asymptotic freeness of $\mathbf{W}_N^{(1)}$ and $\mathbf{D}_N^{(1)}$, which follows from Theorem 4.1.3.

For convenience, we restricted our attention to real Wigner matrices. One can easily adapt the argument to complex Wigner matrices of a real parameter $\beta_\ell \in \mathbb{R}$ by finding an appropriate complex Gaussian realization. In this case, we must take care to choose an analogue of the degree matrix $\mathbf{D}_N^{(\ell)}$ to ensure that we have a self-adjoint traffic (in particular, we can use the second equality in (4.22) so that $\mathbf{D}_N^{(\ell)}$ now averages the row sums with the column sums). We leave the relatively straightforward details to the interested reader.

Concentration inequalities for graphs of Wigner matrices

For a test graph $T = (V,E,\gamma) \in \mathcal{T}\langle \mathbf{x}\rangle$, we recall the random variable

$$\operatorname{tr}\left[T(\mathcal{W}_N)\right] := \sum_{\phi: V\to[N]} \prod_{e\in E} (\mathbf{W}_N^{(\gamma(e))})(\phi(e)).$$

For natural reasons, we are interested in bounding the deviation of $\mathrm{tr}\left[T(\mathcal{W}_N)\right]$ from its mean. In particular, we would like to emulate the usual approach for the Wigner matrices to show that the variance $\mathrm{Var}(\frac{1}{N}\,\mathrm{tr}\left[T(\mathcal{W}_N)\right]) = O_T(N^{-2})$, which would allow us to upgrade the convergence in Proposition 4.2.2 to the almost sure sense. It turns out that this approach will not work in general, but it will be instructive to see just how it falls short.

For notational convenience, we consider instead the deviation of $\mathrm{tr}\left[T(\mathcal{X}_N)\right]$ (recall that $\mathcal{X}_N = \sqrt{N}\mathcal{W}_N$ are the unnormalized Wigner matrices). To begin,

$$\begin{aligned}\mathrm{Var}(\mathrm{tr}\left[T(\mathcal{X}_N)\right]) &= \mathbb{E}\left[\left|\mathrm{tr}\left[T(\mathcal{X}_N)\right] - \mathbb{E}\,\mathrm{tr}\left[T(\mathcal{X}_N)\right]\right|^2\right] \\ &= \mathbb{E}\left[\left(\mathrm{tr}\left[T(\mathcal{X}_N)\right] - \mathbb{E}\,\mathrm{tr}\left[T(\mathcal{X}_N)\right]\right)\overline{\left(\mathrm{tr}\left[T(\mathcal{X}_N)\right] - \mathbb{E}\,\mathrm{tr}\left[T(\mathcal{X}_N)\right]\right)}\right] \\ &= \sum_{\phi_1,\phi_2:V\to[N]} \mathbb{E}\left[\prod_{\ell=1}^{2}\left(\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e)) - \mathbb{E}\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e))\right)\right], \end{aligned} \tag{4.26}$$

where

$$\mathbf{X}_{N,\ell}^{(i)}(j,k) = \begin{cases}\mathbf{X}_N^{(i)}(j,k) & \text{if } \ell = 1, \\ \mathbf{X}_N^{(i)}(k,j) & \text{if } \ell = 2.\end{cases} \tag{4.27}$$

We again make use of our strong moment assumption (4.1), this time to bound our summands uniformly in ϕ_1, ϕ_2, and N. In particular, our bound only depends on T, i.e.,

$$\mathbb{E}\left[\prod_{\ell=1}^{2}\left(\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e)) - \mathbb{E}\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e))\right)\right] \leq C_T < \infty. \tag{4.28}$$

We are then interested in the number of pairs (ϕ_1, ϕ_2) that actually contribute in (4.26) (i.e., such that the summand (4.28) is nonzero). To this end, note that the maps ϕ_ℓ induce maps $\widetilde{\phi}_\ell : E \to \{\{a,b\} : a, b \in [N]\}$, where

$$e \mapsto \{\phi_\ell(\mathrm{src}(e)), \phi_\ell(\mathrm{tar}(e))\}.$$

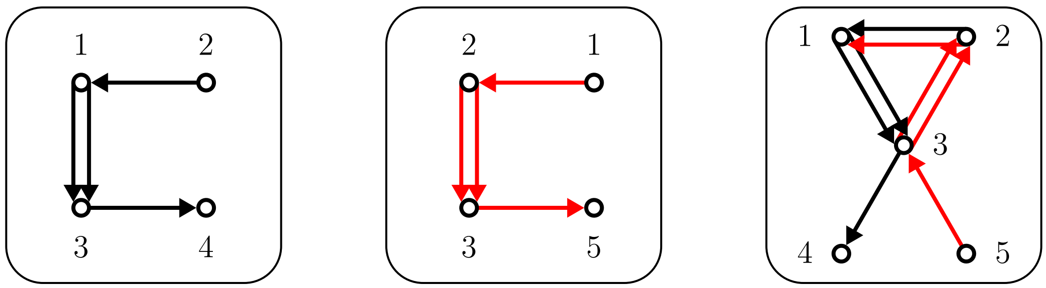
In particular, if $\widetilde{\phi}_1(E) \cap \widetilde{\phi}_2(E) = \emptyset$, then the independence of the $\mathbf{X}_N^{(i)}(j,k)$ implies that the outermost product of (4.28) factors over the expectation, resulting in a zero summand. Thus, we need only to consider so-called *edge-matched* pairs (ϕ_1, ϕ_2). For our purposes, it will be convenient to incorporate the data of such a pair into the graph T itself.

For a pair (ϕ_1, ϕ_2), we construct a new graph $T_{\phi_1\sqcup\phi_2}$ by considering two disjoint copies T_1 and T_2 of T (associated to ϕ_1 and ϕ_2 respectively), reversing the direction of the edges of T_2, and then identifying the vertices according to their images under the maps ϕ_1 and ϕ_2; formally, the vertices of $T_{\phi_1\sqcup\phi_2}$ are then given by

$$V_{\phi_1\sqcup\phi_2} = (\phi_1^{-1}(m) \cup \phi_2^{-1}(m) : m \in [N]).$$

An edge match between ϕ_1 and ϕ_2 then corresponds to an overlay of edges, though not necessarily in the same direction. Note that

$$(\phi_1, \phi_2) \text{ is edge-matched} \quad \Longrightarrow \quad T_{\phi_1 \sqcup \phi_2} \text{ is connected.}$$



(T_1, ϕ_1) (T_2, ϕ_2) $(T_{\phi_1 \sqcup \phi_2}, \phi_1 \sqcup \phi_2)$

Figure 4.8: An example of the construction of the graph $T_{\phi_1 \sqcup \phi_2}$ for an edge-matched pair (ϕ_1, ϕ_2). Here, we omit the edge labels to emphasize the vertex labels $\phi_\ell(v)$. In this case, we use different colors for the edges of the two copies T_1 and T_2 of T to keep track of their origins in the new graph $T_{\phi_1 \sqcup \phi_2}$. Recall that we reverse the direction of the edges of the second copy T_2 before identifying the vertices.

The sum over the set of edge-matched pairs (ϕ_1, ϕ_2) can then be decomposed into a double sum: the first, over the set $\mathcal{S}_T$ of connected graphs $T_\sqcup = (V_\sqcup, E_\sqcup, \gamma_\sqcup)$ obtained by gluing the vertices of two disjoint copies of T with at least one edge overlay (we reverse the direction of the edges of the second copy beforehand, and we keep track of the origin of the edges $E_\sqcup = E_\sqcup^{(1)} \sqcup E_\sqcup^{(2)}$); the second, over the set of injective labelings $\phi_\sqcup : V_\sqcup \hookrightarrow [N]$ of the vertices of $T_\sqcup$. We may then recast (4.26) as

$$\sum_{T_\sqcup \in \mathcal{S}_T} \sum_{\phi_\sqcup : V_\sqcup \hookrightarrow [N]} \mathbb{E}\Big[\prod_{\ell=1}^{2}\Big(\prod_{e \in E_\sqcup^{(\ell)}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e)) - \mathbb{E} \prod_{e \in E_\sqcup^{(\ell)}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e))\Big)\Big]. \tag{4.29}$$

We defined $\mathcal{S}_T$ by reversing the direction of the edges of the second copy of T before gluing in order to write (4.29) without reference to the transposes (4.27). Moreover, by keeping track of the origin of the edges, we ensure that $\mathcal{S}_T$ does not conflate otherwise isomorphic graphs, and so guaranteeing a faithful reconstruction of (4.26) from (4.29). The set $\mathcal{S}_T$ is of course a finite set whose size only depends on T.

We consider a generic $T_\sqcup \in \mathcal{S}_T$, iterating the proof of Proposition 4.2.2. We decompose the set of edges $E_\sqcup = L_\sqcup \cup \mathcal{N}_\sqcup$ as before, and the same for $\widetilde{E}_\sqcup = \widetilde{L}_\sqcup \cup \widetilde{\mathcal{N}}_\sqcup$ (recall that $\widetilde{E}_\sqcup$ denotes the set of equivalence classes in $E_\sqcup$). Suppose that there exists a lone edge

$e_0 \in [e] \in \widetilde{\mathcal{N}}_\sqcup$ with the label $\gamma(e_0) = i_0 \in I$ so that

$$\gamma(e') \neq \gamma(e_0), \qquad \forall e' \in [e] \setminus \{e_0\}.$$

Without loss of generality, we may assume that $e_0 \in E_\sqcup^{(1)}$. We write

$$P_\ell = \prod_{e \in E_\sqcup^{(\ell)}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e)) \quad \text{and} \quad P_1^{(0)} = \prod_{e \in E_\sqcup^{(1)} \setminus \{e_0\}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e)).$$

The independence of the centered random variables $\mathbf{X}_N^{(i)}(j,k)$ and the injectivity of the maps $\phi_\sqcup$ imply that

$$\begin{aligned}\mathbb{E}\big[(P_1 - \mathbb{E}P_1)(P_2 - \mathbb{E}P_2)\big] &= \mathbb{E}\big[(\mathbf{X}_N^{(\gamma_\sqcup(e_0))}(\phi_\sqcup(e_0))P_1^{(0)} - \mathbb{E}\mathbf{X}_N^{(\gamma_\sqcup(e_0))}(\phi_\sqcup(e_0))\mathbb{E}P_1^{(0)})(P_2 - \mathbb{E}P_2)\big] \\ &= \mathbb{E}\big[\mathbf{X}_N^{(\gamma_\sqcup(e_0))}(\phi_\sqcup(e_0))\big]\mathbb{E}\big[(P_1^{(0)} - \mathbb{E}P_1^{(0)})(P_2 - \mathbb{E}P_2)\big] \\ &= 0.\end{aligned}$$

Thus, for $T_\sqcup \in \mathcal{S}_T$ to contribute, each label $i \in I$ present in a class $[e] \in \widetilde{\mathcal{N}}_\sqcup$ must occur with multiplicity

$$m_{i,[e]} \geq 2. \tag{4.30}$$

This in turn implies that

$$\#(\mathcal{N}_\sqcup) \geq 2\#(\widetilde{\mathcal{N}}_\sqcup). \tag{4.31}$$

As before, the underlying simple graph $\underline{T_\sqcup} = (V_\sqcup, \widetilde{\mathcal{N}}_\sqcup)$ is still connected, whence

$$\#(\widetilde{\mathcal{N}}_\sqcup) + 1 \geq \#(V_\sqcup). \tag{4.32}$$

Of course, we also have the inherent bound

$$\#(\mathcal{N}_\sqcup) \leq \#(E_\sqcup) = 2\#(E). \tag{4.33}$$

Recalling the uniform bound (4.28), we arrive at the asymptotic

$$\mathrm{Var}(\mathrm{tr}\,\big[T(\mathcal{X}_N)\big]) = O_T(N^{\max\{\#(V_\sqcup) : T_\sqcup \in \mathcal{S}_T\}}) \leq O_T(N^{\#(E)+1}), \tag{4.34}$$

or, equivalently,

$$\mathrm{Var}\bigg(\frac{1}{N}\,\mathrm{tr}\,\big[T(\mathcal{W}_N)\big]\bigg) = O_T(N^{-1}), \tag{4.35}$$

falling short of our goal. Of course, one might hope that we were overly generous in our bounds and that equality in

$$\max\{\#(V_\sqcup) : T_\sqcup \in \mathcal{S}_T\} \leq \#(E) + 1 \tag{4.36}$$

is not attainable in practice. In fact, in the usual situation of traces of powers

$$\mathrm{tr}\,\big[T(\mathcal{W}_N)\big] = \mathrm{tr}((\mathbf{W}_N^{(i(1))})^{\ell_1} \cdots (\mathbf{W}_N^{(i(m))})^{\ell_m}), \tag{4.37}$$

this is indeed the case; however, in general, (4.34) is tight. In particular, note that if we start with a tree T, we can overlay two disjoint copies T_1 and T_2 of T, the second with reversed edges, to obtain an opposing colored double tree $T_\sqcup$. In this case, we have equality in (4.30)-(4.33). Proposition 4.2.2 then shows that the contribution of $T_\sqcup$ in (4.29) is $\Theta(N^{\#(E)+1})$.

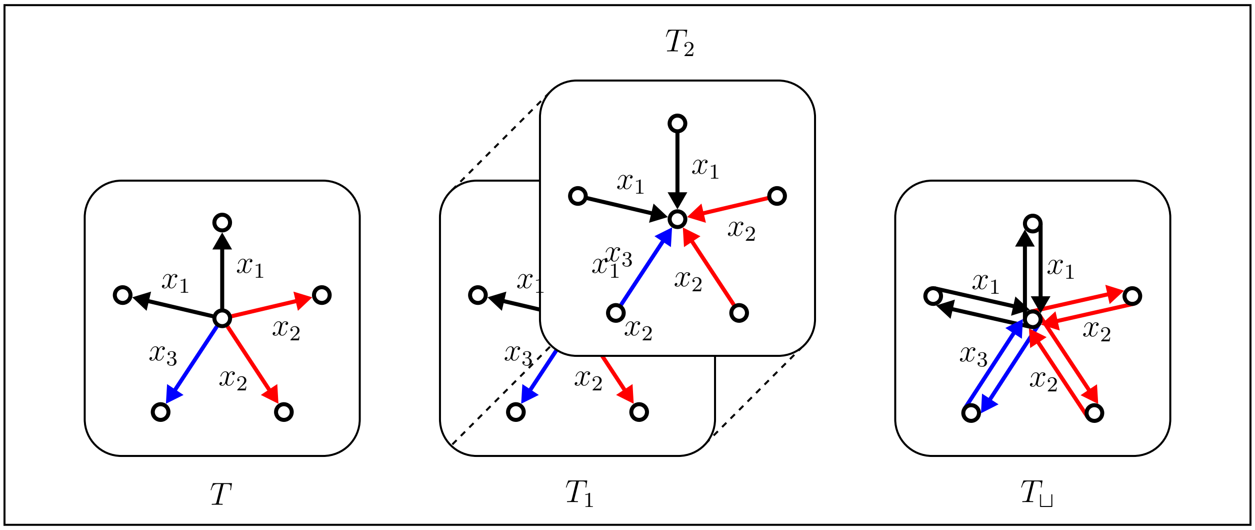


Figure 4.9: An example of an overlay of trees. Here, we consider two copies T_1 and T_2 of the tree T. We depict the second copy T_2 with the direction of its edges already reversed.

Working backwards, we identify the worst case scenario: for (4.30)-(4.33) to hold with equality, we need to glue (not necessarily overlay) disjoint copies T_1 and T_2 of T with at least one edge overlay to obtain a colored double tree $T_\sqcup$ (though T itself need not be a tree in general). In the classical case (4.37), T corresponds to a cycle of length $\ell_1 + \cdots + \ell_m$ and such a gluing does not exist: starting with an edge overlay between two copies of the cycle, we obtain a butterfly as in Figure 3.3, leading to a strict inequality in (4.36) (and hence the usual asymptotic $O(N^{-2})$ in place of (4.35)).

The careful reader will notice that we have actually proven a stronger result in the presence of loops $L \neq \emptyset$: in place of (4.33), we can instead use the tighter bound

$$\#(\mathcal{N}_\sqcup) \leq 2\#(\mathcal{N}).$$

We summarize our findings thus far.

Lemma 4.2.7. *For a family of Wigner matrices $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i \in I}$, we have the asymptotic*

$$\mathrm{Var}(\mathrm{tr}\big[T(\mathcal{X}_N)\big]) = O_T(N^{\#(\mathcal{N})+1}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x} \rangle.$$

The bound is tight in the sense that there exist test graphs T in $\mathbf{x}$ with

$$\mathrm{Var}(\mathrm{tr}\big[T(\mathcal{X}_N)\big]) = \Theta_T(N^{\#(\mathcal{N})+1}).$$

The colored double tree obstruction in Lemma 4.2.7 ramifies into a forest of colored double trees for higher powers, but this construction remains the lone outlier (in particular, things do not get any worse). Drawing inspiration from Proposition 4.15 of [BDJ06], we prove

Theorem 4.2.8. *For a family of Wigner matrices $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$, we have the asymptotic*

$$\mathbb{E}\left[\left|\mathrm{tr}\big[T(\mathcal{X}_N)\big] - \mathbb{E}\,\mathrm{tr}\big[T(\mathcal{X}_N)\big]\right|^{2m}\right] = O_T(N^{m(\#(\mathcal{N})+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x}\rangle.$$

The bound is tight in the sense that there exist test graphs T in $\mathbf{x}$ with

$$\mathbb{E}\left[\left|\mathrm{tr}\big[T(\mathcal{X}_N)\big] - \mathbb{E}\,\mathrm{tr}\big[T(\mathcal{X}_N)\big]\right|^{2m}\right] = \Theta_T(N^{m(\#(\mathcal{N})+1)}).$$

Proof. The concrete case of $m = 2$ contains all of the essential ideas; we encourage the reader to follow through the proof with this simpler case in mind.

To begin, we expand the absolute value as in (4.26) to obtain

$$\sum_{\phi_1,\ldots,\phi_{2m}:V\to[N]} \mathbb{E}\Big[\prod_{\ell=1}^{2m}\Big(\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e)) - \mathbb{E}\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e))\Big)\Big], \tag{4.38}$$

where

$$\mathbf{X}_{N,\ell}^{(i)}(j,k) = \begin{cases} \mathbf{X}_N^{(i)}(j,k) & \text{if } \ell \text{ is odd,} \\ \mathbf{X}_N^{(i)}(k,j) & \text{if } \ell \text{ is even.}\end{cases}$$

Our strong moment assumption (4.1) again ensures that we can bound the summands in (4.38) uniformly in $(\phi_1,\ldots,\phi_{2m})$ and N with a dependence only on T, i.e.,

$$\mathbb{E}\Big[\prod_{\ell=1}^{2m}\Big(\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e)) - \mathbb{E}\prod_{e\in E}\mathbf{X}_{N,\ell}^{(\gamma(e))}(\phi_\ell(e))\Big)\Big] \le C_T < \infty. \tag{4.39}$$

We proceed to an analysis of contributing $2m$-tuples $\Phi = (\phi_1,\ldots,\phi_{2m})$. Using the same notation as before, we say that a coordinate ϕ_ℓ in a $2m$-tuple Φ is *unmatched* if

$$\widetilde{\phi}_\ell(E) \cap \widetilde{\phi}_{\ell'}(E) = \emptyset, \qquad \forall \ell' \neq \ell.$$

Similarly, we say that the distinct coordinates ϕ_ℓ and $\phi_{\ell'}$ (i.e., $\ell \neq \ell'$) are *matched* if

$$\widetilde{\phi}_\ell(E) \cap \widetilde{\phi}_{\ell'}(E) \neq \emptyset.$$

We further say that a $2m$-tuple Φ is *unmatched* if it has an unmatched coordinate ϕ_ℓ; otherwise, we say that Φ is *matched*.

We define an equivalence relation $\sim$ on the coordinates of Φ by matchings; thus,

$$\phi_\ell \sim \phi_{\ell'} \quad \Longleftrightarrow \quad \exists \ell_1, \ldots \ell_k \in [2m] : \phi_{\ell_j} \text{ and } \phi_{\ell_{j+1}} \text{ are matched for } j = 0, \ldots, k,$$

where $\ell(0) = \ell$ and $\ell(k+1) = \ell'$. We write $\widetilde{\Phi}$ for the set of equivalence classes in Φ, in which case (4.39) becomes

$$\prod_{[\widetilde{\phi}] \in \widetilde{\Phi}} \mathbb{E}\Big[\prod_{\phi \in [\widetilde{\phi}]} \Big(\prod_{e \in E} \mathbf{X}_{N,\ell(\phi)}^{(\gamma(e))}(\phi(e)) - \mathbb{E}\prod_{e \in E} \mathbf{X}_{N,\ell(\phi)}^{(\gamma(e))}(\phi(e))\Big)\Big].$$

For an unmatched Φ, this product includes a zero term; henceforth, we only consider matched $2m$-tuples. We incorporate the data of such a tuple into the graph T as before.

For a $2m$-tuple Φ, we construct a new graph $T_{\sqcup\Phi}$ by considering $2m$ disjoint copies $(T_1, \ldots, T_{2m})$ of T (associated to $\Phi = (\phi_1, \ldots, \phi_{2m})$ respectively), reversing the direction of the edges of $(T_2, T_4, \ldots, T_{2m})$, and then identifying the vertices according their images under the maps Φ; formally, the vertices of $T_{\sqcup\Phi}$ are then given by

$$V_{\sqcup\Phi} = (\cup_{\ell=1}^{2m} \phi_\ell^{-1}(m) : m \in [N]).$$

Note that

$$\Phi \text{ is matched} \quad \Longrightarrow \quad T_{\sqcup\Phi} \text{ has} \leq m \text{ connected components.}$$

The sum over the set of matched $2m$-tuples Φ can then be decomposed into a double sum: the first, over the set $\mathcal{S}_T$ of (not necessarily connected) graphs $T_\sqcup = (V_\sqcup, E_\sqcup, \gamma_\sqcup)$ obtained by gluing the vertices of $2m$ disjoint copies of T such that each copy has at least one edge overlay with at least one other copy (we reverse the direction of the edges of the even copies beforehand, and we again keep track of the origin of the edges $E_\sqcup = E_\sqcup^{(1)} \sqcup \cdots \sqcup E_\sqcup^{(2m)}$); the second, over the set of injective labelings $\phi_\sqcup : V_\sqcup \hookrightarrow [N]$ of the vertices of $T_\sqcup$. We write $C(T_\sqcup) = \{C_1, \ldots, C_{d_{T_\sqcup}}\}$ for the set of connected components of $T_\sqcup$. We emphasize that

$$d_{T_\sqcup} \leq m. \tag{4.40}$$

Note that the edges E_p of each connected component C_p consists of a union

$$E_p = E_\sqcup^{(j_p(1))} \sqcup \cdots \sqcup E_\sqcup^{(j_p(k_p))}.$$

We may then recast (4.38) as

$$\sum_{T_\sqcup \in \mathcal{S}_T} \sum_{\phi_\sqcup : V_\sqcup \hookrightarrow [N]} \prod_{p=1}^{d_{T_\sqcup}} \mathbb{E}\Big[\prod_{\ell=1}^{k_p} \Big(\prod_{e \in E_\sqcup^{(j_p(\ell))}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e)) - \mathbb{E}\prod_{e \in E_\sqcup^{(j_p(\ell))}} \mathbf{X}_N^{(\gamma_\sqcup(e))}(\phi_\sqcup(e))\Big)\Big]. \tag{4.41}$$

We consider a generic $T_\sqcup \in \mathcal{S}_T$. Note that our analysis from before applies to each of the connected components $C_p = (V_p, E_p, \gamma_p)$. In particular, using the same notation as before, we know that the components of a contributing $T_\sqcup$ must satisfy

$$m_{i,[e]} = 0 \text{ or } m_{i,[e]} \geq 2, \qquad \forall (i,[e]) \in I \times \widetilde{\mathcal{N}}_p, \tag{4.42}$$

$$\#(\mathcal{N}_p) \geq 2\#(\widetilde{\mathcal{N}}_p), \tag{4.43}$$
$$\#(\widetilde{\mathcal{N}}_p) + 1 \geq \#(V_p). \tag{4.44}$$

Of course, we also have the inherent (in)equalities

$$\sum_{p=1}^{d_{T_\sqcup}} \#(V_p) = \#(V_\sqcup), \qquad \sum_{p=1}^{d_{T_\sqcup}} \#(\mathcal{N}_p) = \#(\mathcal{N}_\sqcup) \leq 2m\#(\mathcal{N}). \tag{4.45}$$

Putting everything together, we arrive at the asymptotic

$$\begin{aligned}\mathbb{E}\left[\left|\operatorname{tr}\left[T(\mathcal{X}_N)\right] - \mathbb{E}\operatorname{tr}\left[T(\mathcal{X}_N)\right]\right|^{2m}\right] &= O_T(N^{\max\{\#(V_\sqcup): T_\sqcup \in \mathcal{S}_T\}}) \\ &\leq O_T(N^{m\#(\mathcal{N})+d_{T_\sqcup}}) \leq O_T(N^{m(\#(\mathcal{N})+1)}).\end{aligned}$$

The tightness of our bound follows much as before. If we start with a tree T, we can overlay pairs of the $2m$-disjoint copies $(T_1, \ldots, T_{2m})$ of T to obtain a forest of $d_{T_\sqcup} = m$ opposing colored double trees. In this case, we have equality in (4.40) and (4.42)-(4.45). Once again, Proposition 4.2.2 shows that the contribution of $T_\sqcup$ in (4.41) is $\Theta(N^{m(\#(\mathcal{N})+1)})$. As was the case for $m = 1$, a forest of m colored double trees $T_\sqcup$ corresponds to the worst case scenario. ■

Reintroducing the standard normalization $\mathcal{W}_N = N^{-1/2}\mathcal{X}_N$, we obtain the asymptotic

$$\mathbb{E}\left[\left|\frac{1}{N}\operatorname{tr}\left[T(\mathcal{W}_N)\right] - \mathbb{E}\frac{1}{N}\operatorname{tr}\left[T(\mathcal{W}_N)\right]\right|^{2m}\right] = O_T(N^{-m(\#(L)+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x} \rangle, \tag{4.46}$$

which bounds the deviation

$$\mathbb{P}\left(\left|\frac{1}{N}\operatorname{tr}\left[T(\mathcal{W}_N)\right] - \mathbb{E}\frac{1}{N}\operatorname{tr}\left[T(\mathcal{W}_N)\right]\right| > \varepsilon\right) = O_{T,m}(N^{-m(\#(L)+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x} \rangle. \tag{4.47}$$

We chose to work with the random variable $\operatorname{tr}\left[T(\mathcal{X}_N)\right]$, but virtually the same proof applies to the injective version

$$\operatorname{tr}^0\left[T(\mathcal{X}_N)\right] = \sum_{\phi: V \hookrightarrow [N]} \prod_{e \in E} (\mathbf{X}_N^{(\gamma(e))})(\phi(e)).$$

In particular, Theorem 4.2.8 holds with $\operatorname{tr}^0\left[T(\mathcal{X}_N)\right]$ in place of $\operatorname{tr}\left[T(\mathcal{X}_N)\right]$, and so too do its implications (4.46) and (4.47). Of course, one could also deduce this from the relations (2.6) and (2.7) between $\operatorname{tr}\left[T(\mathcal{X}_N)\right]$ and $\operatorname{tr}^0\left[T(\mathcal{X}_N)\right]$, which still hold at the level of random variables (i.e., before taking the expectation). This shows that the two results are in fact equivalent. We may then apply the usual Borel-Cantelli machinery to prove the almost sure version of Proposition 4.2.2 (and, as a special case, the almost sure version of Corollary 4.2.6). The results in this section apply just as well to Wigner matrices of a general parameter $\beta_i \in \mathbb{C}$. In this case, we do not need a separate statement for the general situation.

4.3 Classical ensembles beyond Wigner: Ginibre and Wishart-Laguerre

Let $\mathcal{Y}_N = (\mathbf{Y}_N^{(i)})_{i\in I}$ be a family of Ginibre matrices as before. In particular, recall that

$$\mathbb{E}[\mathbf{Y}_N^{(i)}(j,k)^2] = \zeta_i, \qquad \forall j \neq k. \tag{4.48}$$

In contrast to the Wigner matrices, we can treat the general case of $\zeta_i \in \mathbb{D} \subset \mathbb{C}$ without any additional precautions. Indeed, recall that the obstruction in the Wigner ensemble comes from the ordering of adjacent vertices, a consequence of the symmetry class of the matrix. In the Ginibre ensemble, the entries of the matrix are entirely independent, so the ordering of the vertices plays no role in the calculation of twin edges. Otherwise, virtually the same analysis applies to prove the traffic convergence of the normalized family of Ginibre matrices $\mathcal{G}_N = (\mathbf{G}_N^{(i)})_{i\in I}$. Of course, since the Ginibre ensemble is not self-adjoint, we must now consider $*$-test graphs $\mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle$. We state the result below, the proof of which essentially repeats that of Proposition 4.2.2. In particular, we still have convergence to colored double trees, but now with three different twin edge types:

(i)
$$\cdot \underset{x_i}{\overset{x_i}{\leftleftarrows}} \cdot\ ,$$
which we call *congruent*;

(ii)
$$\cdot \underset{x_i^*}{\overset{x_i^*}{\leftleftarrows}} \cdot\ ,$$
which we call $*$-*congruent*;

(iii) and
$$\cdot \underset{x_i^*}{\overset{x_i}{\leftrightarrows}} \cdot\ ,$$
which we call $*$-*opposing*.

We say that a double tree $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle$ is a *Ginibre double tree* if its twin edges each belong to one of the three types above. In particular, we allow for a mixture of different twin edge types. Let $c_i(T)$ (resp., $s_i(T)$) denote the number of pairs of congruent (resp., $*$-congruent) twin edges in T with the common label i. We then have the analogue of Proposition 4.2.2 for the Ginibre ensemble.

Proposition 4.3.1 (ζ-circular traffics). *For any $*$-test graph $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{G}_N)\big] = \begin{cases} \prod_{i\in I} \zeta_i^{c_i(T)}\overline{\zeta}_i^{s_i(T)} & \text{if } T \text{ is a colored Ginibre double tree,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.49}$$

Proof. The result follows from a straightforward modification of the proof of Proposition 4.2.2. ■

As in the Wigner case, we have seemingly left out the contribution of $*$-opposing twin edges. In particular, if we write $o_i(T)$ for the number of pairs of $*$-opposing twin edges in T with the common label i, then we can write the nontrivial part of (4.49) as

$$\prod_{i \in I} \zeta_i^{c_i(T)} \overline{\zeta}_i^{s_i(T)} 1^{o_i(T)}. \tag{4.49$'$}$$

Once again, the multiplicative double tree structure of the LTD (4.49) allows us to appeal to the results of Chapter 3 while bypassing Proposition 2.4.9. We conclude that $(\mathbf{G}_N, \mathbf{G}_N^\intercal)$ and $\Theta(\mathbf{G}_N)$ are asymptotically $*$-free for general ζ. In the case of $\zeta = 0$, we can appeal to Proposition 2.4.9 by taking a standard complex Gaussian Ginibre matrix as a representative. The factorization property (iii) follows much as in the Wigner case [Mal, Lemma 3.5]. This shows that $\mathbf{G}_N$, $\mathbf{G}_N^\intercal$, and $\Theta(\mathbf{G}_N)$ are asymptotically $*$-free if $\zeta = 0$. To see that $\zeta = 0$ is necessary for freeness from the transpose, observe that

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}(\mathbf{G}_N \mathbf{G}_N^\intercal)\right] = \zeta,$$

where we have used (4.49) to compute the limit. Of course, if $\mathbf{G}_N$ and $\mathbf{G}_N^\intercal$ are asymptotically free, then this limit must be equal to zero, and so the necessity follows. Even in the case of $\zeta = 0$, we can bypass Proposition 2.4.9 to prove Theorem 4.1.6. We explain this line of reasoning in Section 4.6 using the cactus-cumulant correspondence.

The traffic distribution of the Ginibre matrices $(\mathbf{G}_N^{(i)})_{i \in I}$ also includes the information of the traffic distribution of the Wishart-Laguerre matrices $(\mathbf{L}_N^{(i)})_{i \in I} = (\mathbf{G}_N^{(i)} \mathbf{G}_N^{(i)*})_{i \in I}$. This already almost proves Theorem 4.1.7. To see that $\zeta = 0$ is necessary for freeness from the transpose, observe that

$$\begin{aligned}
\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}(\mathbf{L}_N \mathbf{L}_N^\intercal)\right] &= \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}(\mathbf{G}_N \mathbf{G}_N^* (\mathbf{G}_N^*)^\intercal \mathbf{G}_N^\intercal)\right] \\
&= \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}\left(\underset{\text{out}}{\cdot} \xleftarrow{\mathbf{G}_N} \cdot \xleftarrow{\mathbf{G}_N^*} \cdot \xrightarrow{\mathbf{G}_N^*} \cdot \xrightarrow{\mathbf{G}_N} \underset{\text{in}}{\cdot} \right)\right] \\
&= 1 + \zeta\overline{\zeta} = 1 + |\zeta|^2,
\end{aligned}$$

where we have again used (4.49) to compute the limit. If $\mathbf{L}_N$ and $\mathbf{L}_N^\intercal$ are asymptotically free, then this limit must be equal to 1, and so the necessity follows.

As an aside, note that we can further use (4.49) to give a quick proof of the Marčenko-Pastur law

$$\mu(\mathbf{L}_N) \xrightarrow{w} \mathcal{MP}(1, 1) \quad \text{as} \quad N \to \infty.$$

In particular, let us compute the limiting moments

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}(\mathbf{L}_N^m)\right] = \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{tr}\left(\underset{\text{out}}{\cdot} \xleftarrow{\mathbf{G}_N} \cdot \xleftarrow{\mathbf{G}_N^*} \cdot \cdots \cdot \xleftarrow{\mathbf{G}_N} \cdot \xleftarrow{\mathbf{G}_N^*} \underset{\text{in}}{\cdot} \right)\right]$$

$$= \lim_{N\to\infty} \tau\big[C_{2m}(\mathbf{G}_N, \mathbf{G}_N^*, \ldots, \mathbf{G}_N, \mathbf{G}_N^*)\big]$$
$$= \lim_{N\to\infty} \sum_{\pi\in\mathcal{P}(V)} \tau^0\big[C_{2m}(\mathbf{G}_N, \mathbf{G}_N^*, \ldots, \mathbf{G}_N, \mathbf{G}_N^*)^\pi\big] =: \mu_m,$$

where $C_{2m} = C_{2m}(\mathbf{G}_N, \mathbf{G}_N^*, \ldots, \mathbf{G}_N, \mathbf{G}_N^*)$ is the directed cycle with $2m$ edges labeled by $\mathbf{G}_N$ and $\mathbf{G}_N^*$ in alternating order. Our formula (4.49) tells us that C_{2m}^π must be a Ginibre double tree to contribute in the limit; moreover, since C_{2m} is directed, we know that

$$C_{2m}^\pi \text{ is a double tree } \implies C_{2m}^\pi \text{ is an opposing double tree}$$

(recall the butterfly obstruction). To contribute, every twin edge of C_{2m}^π must then be of type (iii): $*$-opposing. We consider the possible identifications $\pi \in \mathcal{P}(V)$ that create such a double tree. To this end, it will be convenient to enumerate the vertices $V = (v_i)_{i=1}^{2m}$ and edges $E = (e_i)_{i=1}^{2m}$ of C_{2m}. In particular, we fix the labeling

$$\mathrm{src}(e_i) = v_i, \quad \mathrm{tar}(e_i) = v_{i+1}, \quad \text{and} \quad \gamma(e_i) = \begin{cases} \mathbf{G}_N & \text{if } i \in 2\mathbb{N}+1, \\ \mathbf{G}_N^* & \text{if } i \in 2\mathbb{N}, \end{cases}$$

where $v_{2m+1} = v_1$. We focus on the first edge e_1 with label $\gamma(e_1) = \mathbf{G}_N$. To create a $*$-opposing twin edge with e_1, we need to identify v_1 with a vertex v_{2k+1} and v_2 with a vertex v_{2k} for some $k \in [m]$. But this pinches off our cycle: we have a directed cycle of length $2k-2$ attached at the vertex $v_2 \sim v_{2k}$ and a directed cycle of length $2m-2k$ attached at the vertex $v_1 \sim v_{2k+1}$ (a "dumbbell").

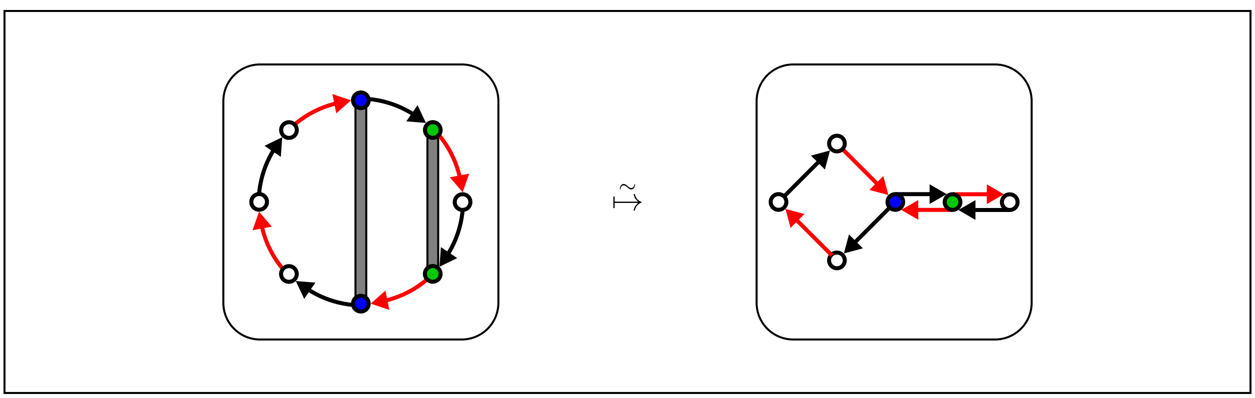


Figure 4.10: An example of an admissible identification pinching off a cycle and creating an uneven dumbbell with shorter cycles attached at each end.

We repeat the same procedure for each of the two cycles created by this identification, which we can think of as C_{2k-2} and C_{2m-2k}. In this way, we arrive at the recurrence

$$\mu_0 = 1 \quad \text{and} \quad \mu_m = \sum_{k=1}^m \mu_{k-1}\mu_{m-k},$$

which is of course the defining recurrence relation of the Catalan numbers $c_k = \frac{\binom{2k}{k}}{k+1}$. Since the moments of the Marčenko-Pastur distribution $\mathcal{MP}(1,1)$ correspond to the Catalan numbers, the result now follows.

Concentration inequalities for graphs of Ginibre matrices

The same adaptations that produce Proposition 4.3.1 from Proposition 4.2.2 also work to prove the Ginibre analogue of Theorem 4.2.8.

Theorem 4.3.2. *For a family of Ginibre matrices $\mathcal{Y}_N = (\mathbf{Y}_N^{(i)})_{i\in I}$, we have the asymptotic*

$$\mathbb{E}\left[\left|\operatorname{tr}\left[T(\mathcal{Y}_N)\right] - \mathbb{E}\operatorname{tr}\left[T(\mathcal{Y}_N)\right]\right|^{2m}\right] = O_T(N^{m(\#(\mathcal{N})+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle.$$

The bound is tight in the sense that there exist $*$-test graphs $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle$ with

$$\mathbb{E}\left[\left|\operatorname{tr}\left[T(\mathcal{Y}_N)\right] - \mathbb{E}\operatorname{tr}\left[T(\mathcal{Y}_N)\right]\right|^{2m}\right] = \Theta_T(N^{m(\#(\mathcal{N})+1)}).$$

Proof. The result follows from a straightforward modification of the proof of Theorem 4.2.8. ■

Reintroducing the normalization $\mathcal{G}_N = N^{-1/2}\mathcal{Y}_N$, we obtain the asymptotic

$$\mathbb{E}\left[\left|\frac{1}{N}\operatorname{tr}\left[T(\mathcal{G}_N)\right] - \mathbb{E}\frac{1}{N}\operatorname{tr}\left[T(\mathcal{G}_N)\right]\right|^{2m}\right] = O_T(N^{-m(\#(L)+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^*\rangle, \quad (4.50)$$

which bounds the deviation

$$\mathbb{P}\left(\left|\frac{1}{N}\operatorname{tr}\left[T(\mathcal{G}_N)\right] - \mathbb{E}\frac{1}{N}\operatorname{tr}\left[T(\mathcal{G}_N)\right]\right| > \varepsilon\right) = O_{T,m}(N^{-m(\#(L)+1)}), \qquad \forall T \in \mathcal{T}\langle \mathbf{x}.\mathbf{x}^*\rangle. \quad (4.51)$$

Once again, this allows us to apply the Borel-Cantelli lemma to upgrade our convergences to the almost sure sense.

4.4 Band matrix variants

Our analysis of the Wigner matrices $\mathcal{W}_N$ in Section 4.2 crucially relies on two important features of our ensemble, namely, the homogeneity of the vertices in our graphs T and the divergence of our normalization $\sqrt{N}$. By the first property, we mean that the label $\phi(v) \in [N]$ of a vertex $v \in V$ does not constrain our choice of a contributing label $\phi(w)$ for an adjacent vertex $w \sim_e v$ (or, in the case of an injective labeling ϕ, does so uniformly in the choice of $\phi(v)$). At the level of the matrices $\mathcal{X}_N$, this corresponds to the fact that any given row (resp., column) of a Wigner matrix looks much the same as any other row (resp., column). For example, if we consider a real Wigner matrix as in Definition 4.1.1 with i.i.d. upper triangular entries, then the rows (resp, columns) each have the same distribution up to a cyclic permutation of the entries. More generally, there exists a permutation invariant realization of our ensemble $\mathcal{X}_N$ iff $\beta_i \in \mathbb{R}$. This property of course does not hold for the random band matrices $\mathbf{\Xi}_N = \mathbf{B}_N \circ \mathbf{X}_N$ (recall Definition 4.1.8): rows (resp, columns) near the top or the bottom (resp., the far left or the far right) of our matrix will in general have fewer nonzero entries. This in turn owes to the asymmetry of the band condition $\mathbf{B}_N$. We can recover the homogeneity of our ensemble by reflecting the band width across the perimeter of the matrix to obtain the so-called periodic random band matrices, providing an intermediate model between the Wigner matrices and the random band matrices. We start with this technically simpler model and work our way up to the RBMs. We summarize the main results at the end of the section on proportional growth RBMs.

Remark 4.4.1. The so-called homogeneity property mentioned above and the corresponding periodization technique first appeared in the work [BMP91] of Bogachev, Molchanov, and Pastur. The authors used this intermediate model to transfer Wigner's semicircle law to random band matrices of slow growth. We employ the same periodization technique to identify the joint limiting traffic distribution of independent random band matrices.

Periodic random band matrices

To begin, we formalize

Definition 4.4.2 (Periodic RBM)**.** Let (b_N) be a sequence of nonnegative integers. We write $\mathbf{P}_N$ for the corresponding $N \times N$ periodic band matrix of ones with band width b_N, i.e.,

$$\mathbf{P}_N(i,j) = \mathbb{1}\{|i-j|_N \leq b_N\},$$

where

$$|i-j|_N = \min\{|i-j|, N - |i-j|\}.$$

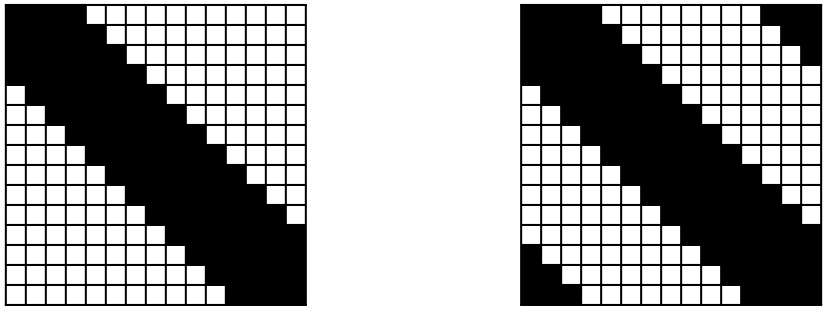
Let $\mathbf{X}_N$ be an unnormalized Wigner matrix. We call the random matrix $\mathbf{\Gamma}_N$ defined by

$$\mathbf{\Gamma}_N = \mathbf{P}_N \circ \mathbf{X}_N$$

an *unnormalized periodic RBM.* Using the normalization $\mathbf{\Upsilon}_N = (2b_N)^{-1/2}\mathbf{J}_N$, we call the random matrix $\mathbf{\Lambda}_N$ defined by

$$\mathbf{\Lambda}_N = \mathbf{\Upsilon}_N \circ \mathbf{\Gamma}_N$$

a *normalized periodic RBM.* We simply refer to periodic RBMs when the context is clear, or when considering the definition altogether.



$(\boldsymbol{\Xi}_N, \boldsymbol{\Theta}_N)$ $(\boldsymbol{\Gamma}_N, \boldsymbol{\Lambda}_N)$

Figure 4.11: An example of the periodization of a random band matrix. Here, we scale the matrix to the unit square $[0,1]^2$. The (i,j)-th entry then corresponds to the subsquare $[\frac{j-1}{N}, \frac{j}{N}] \times [\frac{N-i}{N}, \frac{N-i+1}{N}]$, which we then fill in provided the band width condition $|i-j| \leq b_N$ (resp., $|i-j|_N \leq b_N$) is satisfied.

Let $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$ be a family of unnormalized Wigner matrices as before. We consider a family of divergent band widths $(b_N^{(i)})_{i\in I}$ such that

$$\lim_{N\to\infty} b_N^{(i)} = \infty, \qquad \forall i \in I, \tag{4.52}$$

for which we form the corresponding family of periodic RBMs, unnormalized $\mathcal{R}_N = (\boldsymbol{\Gamma}_N^{(i)})_{i\in I}$ and otherwise $\mathcal{P}_N = (\boldsymbol{\Lambda}_N^{(i)})_{i\in I}$. We identify the LTD of the family $\mathcal{P}_N$ with that of the familiar Wigner matrices $\mathcal{W}_N$ from Proposition 4.2.2.

Lemma 4.4.3. *For any test graph T in $\mathbf{x} = (x_i)_{i\in I}$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{P}_N)\big] = \begin{cases} \prod_{i\in I} \beta_i^{c_i(T)} & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.53}$$

Proof. The proof follows much along the same lines as Proposition 4.2.2 except that we must take care to account for the differing rates of growth in the band widths $b_N^{(i)}$. To begin, suppose that $T = (V, E, \gamma)$. By definition, we have that

$$\begin{aligned} \tau^0\big[T(\mathcal{P}_N)\big] &= \mathbb{E}\Big[\frac{1}{N}\sum_{\phi:V\hookrightarrow[N]}\prod_{e\in E}\boldsymbol{\Lambda}_N^{(\gamma(e))}(\phi(e))\Big] \\ &= \frac{1}{N\prod_{e\in E}\sqrt{2b_N^{(\gamma(e))}}}\sum_{\phi:V\hookrightarrow[N]}\mathbb{E}\Big[\prod_{e\in E}\boldsymbol{\Gamma}_N^{(\gamma(e))}(\phi(e))\Big]. \end{aligned} \tag{4.54}$$

Using our earlier notation, we can recast the sum in (4.54) as

$$\sum_{\phi:V\hookrightarrow[N]}\Bigg(\prod_{[\ell]\in\widetilde{L}}\mathbb{E}\Big[\prod_{\ell'\in[\ell]}\mathbf{\Gamma}_N^{(\gamma(\ell'))}(\phi(\ell'))\Big]\Bigg)\Bigg(\prod_{[e]\in\widetilde{\mathcal{N}}}\mathbb{E}\Big[\prod_{e'\in[e]}\mathbf{\Gamma}_N^{(\gamma(e'))}(\phi(e'))\Big]\Bigg). \tag{4.55}$$

Whereas before the label $\phi(v)$ of a vertex v does not constrain our choice of label $\phi(w)$ for an adjacent vertex $w \sim_e v$ (beyond the injectivity requirement), we note that in this case a summand of (4.55) equals zero if

$$\exists e_0 \in [e] : |\phi(\mathrm{src}(e_0)) - \phi(\mathrm{tar}(e_0))|_N > b_N^{(\gamma(e_0))}.$$

In fact, we see that such a summand equals zero as soon as

$$\exists e_0 \in [e] : |\phi(\mathrm{src}(e_0)) - \phi(\mathrm{tar}(e_0))|_N > \min_{e'\in[e]} b_N^{(\gamma(e'))}.$$

To keep track of these constraints, we define

$$|\phi(e)|_N = |\phi(\mathrm{src}(e)) - \phi(\mathrm{tar}(e))|_N.$$

Note that $|\phi(\cdot)|_N$ is constant on equivalence classes $[e] \in \widetilde{\mathcal{N}}$, and so we further write $|\phi([e])|_N$ for the common value of

$$\{|\phi(e')|_N : e' \in [e]\}.$$

We use the function $|\phi(\cdot)|_N$ to define the band width condition

$$C_{[e]} = \mathbb{1}\{|\phi([e])|_N \leq \min_{e'\in[e]} b_N^{(\gamma(e'))}\},$$

which allows us to rewrite (4.55) as

$$\sum_{\phi:V\hookrightarrow[N]}\Bigg(\prod_{[\ell]\in\widetilde{L}}\mathbb{E}\Big[\prod_{\ell'\in[\ell]}\mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell'))\Big]\Bigg)\Bigg(\prod_{[e]\in\widetilde{\mathcal{N}}}C_{[e]}\mathbb{E}\Big[\prod_{e'\in[e]}\mathbf{X}_N^{(\gamma(e'))}(\phi(e'))\Big]\Bigg) \tag{4.56}$$

in terms of the usual Wigner matrices $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$ (cf. (4.10)). We may then apply our analysis from Proposition 4.2.2 to conclude that a contributing graph T satisfies

$$m_{i,[e]} = 0 \text{ or } m_{i,[e]} \geq 2 \qquad \forall (i,[e]) \in I \times \widetilde{\mathcal{N}}. \tag{4.57}$$

The band width condition

$$|\phi([e])|_N \leq \min_{e'\in[e]} b_N^{(\gamma(e'))}, \qquad \forall [e] \in \widetilde{\mathcal{N}}, \tag{4.58}$$

bounds the number $A_N(T)$ of contributing maps $\phi : V \hookrightarrow [N]$ by

$$A_N(T) \leq N \prod_{[e]\in\widetilde{\mathcal{N}}} \min_{e'\in[e]} 2b_N^{(\gamma(e'))}.$$

Indeed, fixing an arbitrary vertex $v_0 \in V$, we have N choices for $\phi(v_0) \in [N]$; but, having made this choice, we must take into account the band widths in traversing the remaining edges of the simple graph $\underline{T} = (V, \widetilde{\mathcal{N}})$. In fact, we can apply the same reasoning to any spanning tree $\underline{T_0} = (V, \widetilde{\mathcal{N}}_0)$ of $\underline{T}$ since any edge $[e_k] \in \widetilde{\mathcal{N}}$ in a cycle $([e_1], \ldots, [e_k])$ will have already had the admissible range of labels for its incident vertices determined by the band width conditions coming from the other edges $([e_1], \ldots, [e_{k-1}])$. This leads to the refinement

$$A_N(T) \leq N \prod_{[e] \in \widetilde{\mathcal{N}}_0} \min_{e' \in [e]} 2b_N^{(\gamma(e'))}, \tag{4.59}$$

where

$$\#(\widetilde{\mathcal{N}}_0) \leq \#(\widetilde{\mathcal{N}}) \leq \#(\widetilde{E}). \tag{4.60}$$

Recycling the bound (4.14) for the summands of (4.56), we arrive at the asymptotic

$$\begin{aligned} \tau^0\big[T(\mathcal{P}_N)\big] &= O_T\Bigg(\frac{N \prod_{[e] \in \widetilde{\mathcal{N}}_0} \min_{e' \in [e]} 2b_N^{(\gamma(e'))}}{N \prod_{e \in E} \sqrt{2b_N^{(\gamma(e))}}}\Bigg) \\ &= O_T\Bigg(\frac{\prod_{[e] \in \widetilde{\mathcal{N}}_0} \min_{e' \in [e]} 2b_N^{(\gamma(e'))}}{\prod_{e \in \mathcal{N}} \sqrt{2b_N^{(\gamma(e))}} \prod_{\ell \in L} \sqrt{2b_N^{(\gamma(\ell))}}}\Bigg). \end{aligned}$$

For the sake of comparison, we draw the reader's attention to (4.15) for the analogous asymptotic in the case of the Wigner matrices $\mathcal{W}_N$ (note that $\#(\widetilde{\mathcal{N}}_0) = \#(V) - 1$). The divergence (4.52) of the band widths $b_N^{(i)}$ and the inequalities (4.57) and (4.60) then imply that $\tau^0\big[T(\mathcal{P}_N)\big]$ vanishes in the limit unless T is a colored double tree, in which case one clearly obtains the prescribed limit (4.53). ■

Here, the situation for general $\beta_i \in \mathbb{C}$ becomes much different. For a single periodic RBM $\mathbf{\Lambda}_N$ of divergent band width $b_N \to \infty$, the LTD again follows (4.19) as in the Wigner case; however, the joint LTD of $\mathcal{P}_N$ might not exist depending on the fluctuations of the band widths $b_N^{(i)}$. In this case, we need to make additional assumptions on the band widths (e.g., proportional growth) to ensure the existence of an asymptotic proportion for an ordering ψ of the vertices (i.e., the analogue of (4.18)). We comment more on this situation later.

On the other hand, the orderings ψ play no role in the calculation of $\tau^0\big[T(\mathcal{P}_N)\big]$ for an *opposing* colored double tree T. Consequently, we can apply the criteria (4.21) in Remark 4.2.3 to conclude that $\mathcal{P}_N = (\mathbf{\Lambda}_N^{(i)})_{i \in I}$ converges in joint distribution to a semicircular system $\mathbf{a} = (a_i)_{i \in I}$ regardless of $(\beta_i)_{i \in I}$.

Note that a periodic RBM $\mathbf{\Lambda}_N$ with band width $b_N = N/2$ corresponds to a standard Wigner matrix $\mathbf{W}_N$. As such, we can view Lemma 4.4.3 as a generalization of Proposition 4.2.2. We extend the result to include RBMs of slow growth in the next section.

Slow growth

To begin, we partition the index set I of our matrices $\mathcal{X}_N = (\mathbf{X}_N^{(i)})_{i\in I}$ into two camps $I = I_1 \cup I_2$. We consider a class of divergent band widths $(b_N^{(i)})_{i\in I}$ as in (4.52) with the added condition of slow growth for $(b_N^{(i)})_{i\in I_2}$, i.e.,

$$\lim_{N\to\infty} \frac{b_N^{(i)}}{N} = 0, \qquad \forall i \in I_2. \tag{4.61}$$

We form the corresponding family of periodic RBMs as before,

$$\mathcal{R}_N = \mathcal{R}_N^{(1)} \cup \mathcal{R}_N^{(2)} = (\mathbf{\Gamma}_N^{(i)})_{i\in I_1} \cup (\mathbf{\Gamma}_N^{(i)})_{i\in I_2}, \qquad \mathcal{P}_N = \mathcal{P}_N^{(1)} \cup \mathcal{P}_N^{(2)} = (\mathbf{\Lambda}_N^{(i)})_{i\in I_1} \cup (\mathbf{\Lambda}_N^{(i)})_{i\in I_2}.$$

For $i \in I_2$, we also form the corresponding family of slow growth RBMs (see Definition 4.1.8),

$$\mathcal{S}_N^{(2)} = (\mathbf{\Xi}_N^{(i)})_{i\in I_2} = (\mathbf{B}_N^{(i)} \circ \mathbf{X}_N^{(i)})_{i\in I_2}, \qquad \mathcal{O}_N^{(2)} = (\mathbf{\Theta}_N^{(i)})_{i\in I_2} = (\mathbf{\Upsilon}_N^{(i)} \circ \mathbf{\Xi}_N^{(i)})_{i\in I_2}.$$

Lemma 4.4.4. *The family $\mathcal{M}_N = \mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)}$ converges in traffic distribution to the limit*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{M}_N)\big] = \begin{cases} \prod_{i\in I} \beta_i^{c_i(T)} & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.62}$$

Proof. In view of Lemma 4.4.3, it suffices to show that

$$\lim_{N\to\infty} \Big| \tau^0\big[T(\mathcal{P}_N)\big] - \tau^0\big[T(\mathcal{M}_N)\big] \Big| = 0, \qquad \forall T \in \mathcal{T}\langle \mathbf{x} \rangle. \tag{4.63}$$

Of course, the only difference between the families $\mathcal{P}_N$ and $\mathcal{M}_N$ comes from the periodization of the slow growth RBMs $\mathcal{S}_N^{(2)}$. Equation (4.63) then asserts that the contribution of the additional entries arising from this periodization becomes negligible in the limit.

For convenience, we write $\mathcal{U}_N = (\mathbf{U}_N^{(i)})_{i\in I}$ for the unnormalized version of $\mathcal{M}_N$ so that

$$\mathbf{U}_N^{(i)} = \begin{cases} \mathbf{\Gamma}_N^{(i)} & \text{if } i \in I_1, \\ \mathbf{\Xi}_N^{(i)} & \text{if } i \in I_2. \end{cases}$$

Expanding $\tau^0\big[T(\mathcal{M}_N)\big]$, we obtain the analogue of (4.54),

$$\frac{1}{N \prod_{e\in E} \sqrt{2b_N^{(\gamma(e))}}} \sum_{\phi: V \hookrightarrow [N]} \mathbb{E}\Big[\prod_{e\in E} \mathbf{U}_N^{(\gamma(e))}(\phi(e))\Big].$$

Our notation works just as well in this case to produce the analogue of (4.55) for our sum,

$$\sum_{\phi: V\hookrightarrow [N]} \Bigg(\prod_{[\ell]\in \widetilde{L}} \mathbb{E}\Big[\prod_{\ell'\in[\ell]} \mathbf{U}_N^{(\gamma(\ell'))}(\phi(\ell'))\Big]\Bigg)\Bigg(\prod_{[e]\in \widetilde{\mathcal{N}}} \mathbb{E}\Big[\prod_{e'\in[e]} \mathbf{U}_N^{(\gamma(e'))}(\phi(e'))\Big]\Bigg).$$

Naturally, we then look for the analogue of (4.56). Note that the corresponding version of the band width condition (4.58) must now take into account the index $\gamma(e') \in I_1 \cup I_2$ of $e' \in [e]$. We partition the equivalence classes $[e] = [e]_1 \cup [e]_2$ in $\widetilde{\mathcal{N}}$ accordingly, where

$$[e]_j = [e] \cap \gamma^{-1}(I_j).$$

For an edge $e \in \mathcal{N}$, we define

$$|\phi(e)| = |\phi(\mathrm{src}(e)) - \phi(\mathrm{tar}(e))|.$$

As before, $|\phi(\cdot)|$ is constant on equivalence classes $[e] \in \widetilde{\mathcal{N}}$, and so we write $|\phi([e])|$ for the common value of

$$\{|\phi(e')| : e' \in [e]\}.$$

More specifically, we write $|\phi([e]_2)|$ for the common value of

$$\{|\phi(e')| : e' \in [e]_2\}.$$

Note that $[e]_2$ may be empty, in which case we define $|\phi(\emptyset)| = 0$. We use the same convention for $|\phi([e]_1)|_N$ to define the band width condition

$$C'_{[e]} = \mathbb{1}\{|\phi([e]_1)|_N \leq \min_{e' \in [e]_1} b_N^{(\gamma(e'))}\}\mathbb{1}\{|\phi([e]_2)| \leq \min_{e' \in [e]_2} b_N^{(\gamma(e'))}\}, \qquad \forall [e] \in \widetilde{\mathcal{N}}.$$

We may then write the analogue of (4.56) for our family $\mathcal{M}_N$ as

$$\sum_{\phi : V \hookrightarrow [N]} \Big(\prod_{[\ell] \in \widetilde{L}} \mathbb{E}\Big[\prod_{\ell' \in [\ell]} \mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell')) \Big] \Big) \Big(\prod_{[e] \in \widetilde{\mathcal{N}}} C'_{[e]} \mathbb{E}\Big[\prod_{e' \in [e]} \mathbf{X}_N^{(\gamma(e'))}(\phi(e')) \Big] \Big). \tag{4.64}$$

Of course, the inherent inequality $|\cdot|_N = \min\{|\cdot|, N - |\cdot|\} \leq |\cdot|$ implies that

$$C'_{[e]} \leq \mathbb{1}\{|\phi([e])|_N \leq \min_{e' \in [e]} b_N^{(\gamma(e'))}\} = C_{[e]}, \qquad \forall [e] \in \widetilde{\mathcal{N}},$$

which bounds the number $B_N(T)$ of maps $\phi : V \hookrightarrow [N]$ satisfying the band width condition

$$|\phi([e]_1)|_N \leq \min_{e' \in [e]_1} b_N^{(\gamma(e'))} \quad \text{and} \quad |\phi([e]_2)| \leq \min_{e' \in [e]_2} b_N^{(\gamma(e'))}, \qquad \forall [e] \in \widetilde{\mathcal{N}} \tag{4.65}$$

by

$$B_N(T) \leq A_N(T). \tag{4.66}$$

Recall that $A_N(T)$ is the number of maps $\phi : V \hookrightarrow [N]$ satisfying the weaker condition

$$|\phi([e])|_N \leq \min_{e' \in [e]} b_N^{(\gamma(e'))}, \qquad \forall [e] \in \widetilde{\mathcal{N}} \tag{4.67}$$

present in Lemma 4.4.3. In view of (4.66), our work in this previous case implies that

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{M}_N)\big] = 0$$

unless T is a colored double tree. Thus, it remains to prove (4.63) for such a test graph T.

Comparing the two equations (4.56) and (4.64), we arrive at the asymptotic

$$\Big|\tau^0\big[T(\mathcal{P}_N)\big] - \tau^0\big[T(\mathcal{M}_N)\big]\Big| = O_T\Bigg(\frac{D_N(T)}{N\prod_{e\in E}\sqrt{2b_N^{(\gamma(e))}}}\Bigg), \tag{4.68}$$

where $D_N(T) = A_N(T) - B_N(T)$ is the number of maps $\phi : V \hookrightarrow [N]$ that satisfy the band width condition (4.67) but not the stronger condition (4.65). This formalizes the observation that we made at the beginning of the proof about the only difference between the families $\mathcal{P}_N$ and $\mathcal{M}_N$. In particular, for $i \in I_2$, note that the periodic version $\mathbf{\Gamma}_N^{(i)}$ of a slow growth RBM $\mathbf{\Xi}_N^{(i)}$ only differs in the entries within band width's distance of the perimeter; otherwise, the two matrices are identical. For a map $\phi : V \hookrightarrow [N]$, this means that if ϕ stays sufficiently far away from the endpoints of the interval $[N]$, then the two conditions (4.65) and (4.67) are actually equivalent. In particular, this holds if

$$\phi(V) \subset [1 + \max_{e\in E_2} b_N^{(\gamma(e))}, N - \max_{e\in E_2} b_N^{(\gamma(e))}],$$

where $E_2 = \gamma^{-1}(I_2)$ is of course a finite set. In this case, we have the bound

$$D_N(T) = A_N(T) - B_N(T) \le A_N^*(T),$$

where $A_N^*(T)$ is the number of maps $\phi : V \hookrightarrow [N]$ satisfying (4.67) with range

$$\phi(V) \not\subset [1 + \max_{e\in E_2} b_N^{(\gamma(e))}, N - \max_{e\in E_2} b_N^{(\gamma(e))}]. \tag{4.69}$$

We give a simple bound on $A_N^*(T)$ as follows: set aside a vertex $v_0 \in V$ (for which there are $\#(V)$ choices) to satisfy (4.69) (for which there are $2\max_{e\in E_2} b_N^{(\gamma(e))}$ choices) and pick the labels $\phi(v)$ of the remaining vertices according to (4.67) (for which there are at most $\prod_{[e]\in\widetilde{E}} \min_{e'\in[e]} 2b_N^{(\gamma(e'))}$ choices) to see that

$$A_N^*(T) = O_T\Bigg(\max_{e\in E_2} b_N^{(\gamma(e))} \prod_{[e]\in\widetilde{E}} \min_{e'\in[e]} 2b_N^{(\gamma(e'))}\Bigg). \tag{4.70}$$

We may then recast (4.68) as

$$\Big|\tau^0\big[T(\mathcal{P}_N)\big] - \tau^0\big[T(\mathcal{M}_N)\big]\Big| = \frac{\max_{e\in E_2} b_N^{(\gamma(e))}}{N} O_T\Bigg(\frac{\prod_{[e]\in\widetilde{E}} \min_{e'\in[e]} 2b_N^{(\gamma(e'))}}{\prod_{e\in E}\sqrt{2b_N^{(\gamma(e))}}}\Bigg). \tag{4.71}$$

T being a colored double tree, we know that

$$\frac{\prod_{[e]\in\widetilde{E}} \min_{e'\in[e]} 2b_N^{(\gamma(e'))}}{\prod_{e\in E}\sqrt{2b_N^{(\gamma(e))}}} = 1.$$

Moreover, since $\#(E_2) < \infty$, the slow growth (4.61) still holds for the maximum over E_2,

$$\max_{e\in E_2} b_N^{(\gamma(e))} = o(N). \tag{4.72}$$

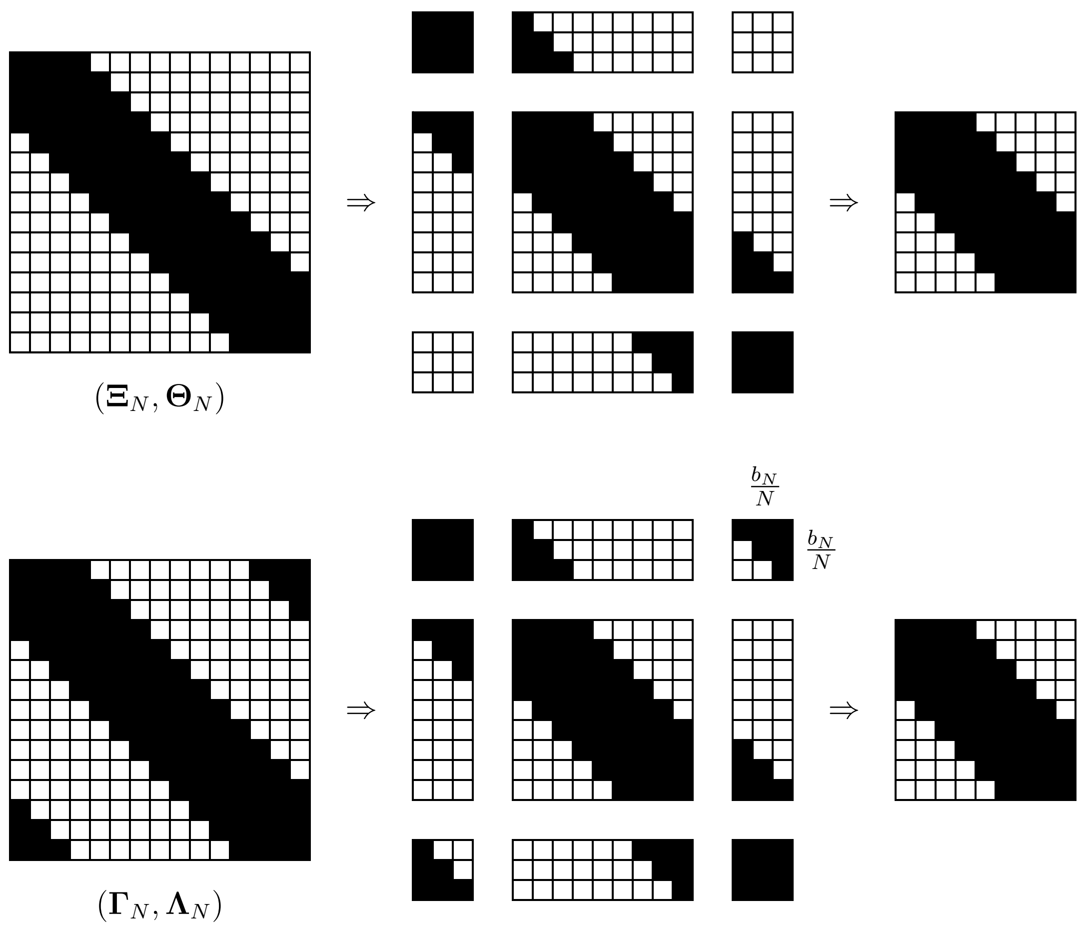


Figure 4.12: An illustration of the "interior" region of a random band matrix (resp., periodic random band matrix) at band width's distance $\frac{b_N}{N} = o(1)$ from the perimeter. Here, we cut off the boundary to see that the two interior regions are indeed identical.

Equations (4.70)-(4.72) formalize our intuition from before: the periodic version of a RBM only differs within band width's distance of the perimeter; for a slow growth RBM, one then needs to be very close to the perimeter to realize this difference; as such, the corresponding interior region accounts for the bulk of the calculations. The result now follows. ■

Remark 4.4.5. If we think of choosing a map $\phi : V \hookrightarrow [N]$ satisfying (4.65) as starting at an arbitrary vertex v_0, making a choice $\phi(v_0) \in [N]$, and then choosing the labels of the remaining vertices in a manner compatible with the band width conditions, then each choice of $\phi(v)$ after $\phi(v_0)$ can be thought of as an incremental walk of distance at most $\min_{e' \in [e]} b_N^{(\gamma(e'))}$ for some $[e] \in \widetilde{\mathcal{N}}$. If $I = I_2$, then starting from a "deep" vertex

$$\phi(v_0) \in [1 + \#(E) \max_{e \in E} b_N^{\gamma(e))}, N - \#(E) \max_{e \in E} b_N^{(\gamma(e))}],$$

the walk never has a chance to loop across the perimeter of the matrix. This line of reasoning can be used to give a more intuitive geometric proof of Lemma 4.2.1 in the simpler case of $I = I_2$. This notion of a deep vertex originates in the work [BMP91].

If $I \neq I_2$, then we need to account for the possibility of the band widths of the periodic RBMs being large enough to bring us close to the perimeter so that the walk crosses over with a step from a periodized version of a slow growth RBM. Taking inspiration from the simpler case of $I = I_2$, our analysis shows that a generic walk stays within a region in which the slow growth RBMs and their periodized versions are identical.

We encounter the same problem from before when considering general $\beta_i \in \mathbb{C}$: without further assumptions on the band widths $b_N^{(i)}$, their fluctuations could possibly preclude the existence of a joint LTD. In general, we must again settle for the convergence of $\mathcal{M}_N = (\mathbf{\Lambda}_N^{(i)})_{i \in I_1} \cup (\mathbf{\Theta}_N^{(i)})_{i \in I_2}$ in joint distribution to a semicircular system $\mathbf{a} = (a_i)_{i \in I}$.

Recall that the Wigner matrices $\mathcal{W}_N$ are asymptotically traffic independent iff $\beta_i \in \mathbb{R}$, and that a permutation invariant realization of our ensemble $\mathcal{W}_N$ exists iff $\beta_i \in \mathbb{R}$. In view of Theorem 2.3.10, one might then expect that permutation invariance is also a necessary condition for matricial asymptotic traffic independence; however, we see that this is not the case. In particular, one cannot find a permutation invariant realization of the periodic RBMs (except in the trivial case of $b_N \sim N/2$), nor of the slow growth RBMs. Instead, we relied on the aforementioned homogeneity property and the divergence of our normalization. Taken alone, neither of these two properties suffices, as we shall see in the proportional growth regime (which lacks homogeneity) and the fixed band width regime (which has a fixed normalization).

Proportional growth

Not surprisingly, the periodization trick from the previous section fails for proportional growth RBMs unless $c = 1$ (recall that $c = \lim_{N \to \infty} \frac{b_N}{N} \in (0, 1]$). In the case of *proper* proportion $c \in (0, 1)$, the entries in the matrix introduced by reflecting the band width across the perimeter now account for an asymptotically nontrivial region in the unit square and so no longer represent a negligible contribution to the calculations. Nevertheless, we can adapt our work from before to prove the existence of a joint LTD supported on colored double trees T, though in general the value of this limit will depend on the degree structure of T.

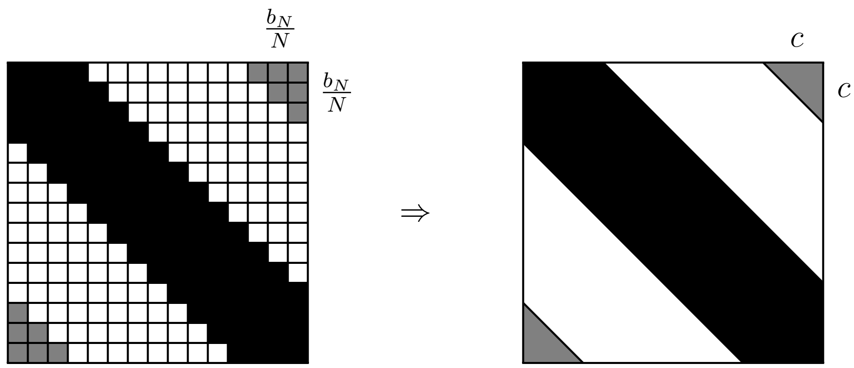


Figure 4.13: An illustration of the limit shape of our scaled matrix in the unit square $[0,1]^2$. Here, we distinguish the periodized version of our matrix with the additional grey area. In the limit, the shape corresponds to the banded region $|x-(1-y)| \leq c$ (resp., the periodic banded region $\min(|x-(1-y)|, 1-|x-(1-y)|) \leq c$). In contrast to slow growth regime, we see a nontrivial contribution from the periodization due to the nonvanishing scale of the band width $\lim_{N\to\infty} \frac{b_N}{N} = c \in (0,1)$.

To formalize our result, we now split the index set $I = I_1 \cup I_2 \cup I_3 \cup I_4$ into four camps. We consider a class of divergent band widths $(b_N^{(i)})_{i\in I}$ as in (4.52) with the added conditions of slow growth for $\mathbf{b}_N^{(2)} = (b_N^{(i)})_{i\in I_2}$, full proportion for $\mathbf{b}_N^{(3)} = (b_N^{(i)})_{i\in I_3}$, and proper proportion for $\mathbf{b}_N^{(4)} = (b_N^{(i)})_{i\in I_4}$ so that

$$\begin{aligned}
\lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= 0, && \forall i \in I_2\\
\lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= c_i = 1, && \forall i \in I_3,\\
\lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= c_i \in (0,1), && \forall i \in I_4.
\end{aligned}$$

For $i \in I_1 \cup I_2$, we form the corresponding families of periodic RBMs and slow growth RBMs as before,

$$\begin{aligned}
\mathcal{R}_N &= \mathcal{R}_N^{(1)} \cup \mathcal{R}_N^{(2)} = (\mathbf{\Gamma}_N^{(i)})_{i\in I_1} \cup (\mathbf{\Gamma}_N^{(i)})_{i\in I_2}, && \mathcal{P}_N = \mathcal{P}_N^{(1)} \cup \mathcal{P}_N^{(2)} = (\mathbf{\Lambda}_N^{(i)})_{i\in I_1} \cup (\mathbf{\Lambda}_N^{(i)})_{i\in I_2};\\
\mathcal{S}_N^{(2)} &= (\mathbf{\Xi}_N^{(i)})_{i\in I_2} = (\mathbf{B}_N^{(i)} \circ \mathbf{X}_N^{(i)})_{i\in I_2}, && \mathcal{O}_N^{(2)} = (\mathbf{\Theta}_N^{(i)})_{i\in I_2} = (\mathbf{\Upsilon}_N^{(i)} \circ \mathbf{\Xi}_N^{(i)})_{i\in I_2}.
\end{aligned}$$

For $i \in I_3 \cup I_4$, we form the corresponding families of proportional growth RBMs,

$$\mathcal{F}_N^{(3)} = (\boldsymbol{\Xi}_N^{(i)})_{i\in I_3} = (\mathbf{B}_N^{(i)} \circ \mathbf{X}_N^{(i)})_{i\in I_3}, \qquad \mathcal{O}_N^{(3)} = (\boldsymbol{\Theta}_N^{(i)})_{i\in I_3} = (\boldsymbol{\Upsilon}_N^{(i)} \circ \boldsymbol{\Xi}_N^{(i)})_{i\in I_3};$$
$$\mathcal{C}_N^{(4)} = (\boldsymbol{\Xi}_N^{(i)})_{i\in I_4} = (\mathbf{B}_N^{(i)} \circ \mathbf{X}_N^{(i)})_{i\in I_4}, \qquad \mathcal{O}_N^{(4)} = (\boldsymbol{\Theta}_N^{(i)})_{i\in I_4} = (\boldsymbol{\Upsilon}_N^{(i)} \circ \boldsymbol{\Xi}_N^{(i)})_{i\in I_4}.$$

We start with the simpler case of the single family $\mathcal{O}_N^{(4)}$ of (proper) proportional growth RBMs. In this case, the LTD of $\mathcal{O}_N^{(4)}$ only depends on the band widths $\mathbf{b}_N^{(4)}$ up to the limiting proportions

$$\mathbf{c}_4 = (c_i)_{i\in I_4}.$$

Lemma 4.4.6. *For any test graph T in $\mathbf{x}_4 = (x_i)_{i\in I_4}$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{O}_N^{(4)})\big] = \begin{cases} p_T(\mathbf{c}_4)\prod_{i\in I}\beta_i^{c_i(T)} & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise,}\end{cases} \tag{4.73}$$

where $p_T(\mathbf{c}_4) > 0$ only depends on the test graph T and the proportions $\mathbf{c}_4 = (c_i)_{i\in I_4}$.

Proof. As usual, we begin by expanding

$$\tau^0\big[T(\mathcal{O}_N^{(4)})\big] = \frac{1}{N^{1+\frac{\#(E)}{2}}\prod_{e\in E}\sqrt{2c_{\gamma(e)} - c_{\gamma(e)}^2}} \sum_{\phi: V\hookrightarrow [N]} \mathbb{E}\Big[\prod_{e\in E}\boldsymbol{\Xi}_N^{(\gamma(e))}(\phi(e))\Big]$$

and rewriting the sum as

$$\sum_{\phi:V\hookrightarrow[N]}\bigg(\prod_{[\ell]\in\widetilde{L}}\mathbb{E}\Big[\prod_{\ell'\in[\ell]}\mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell'))\Big]\bigg)\bigg(\prod_{[e]\in\widetilde{\mathcal{N}}}\mathbb{1}\{|\phi([e])| \le \min_{e'\in[e]} b_N^{(\gamma(e'))}\}\mathbb{E}\Big[\prod_{e'\in[e]}\mathbf{X}_N^{(\gamma(e'))}(\phi(e'))\Big]\bigg).$$

At this point, we can already conclude the second half of (4.73). Hereafter, T denotes a colored double tree. In this case, we have the equality

$$\begin{aligned}\tau^0\big[T(\mathcal{O}_N^{(4)})\big] &= \frac{C_N(T)}{N^{1+\#(\widetilde{E})}\prod_{[e]\in\widetilde{E}}(2c_{\gamma([e])} - c_{\gamma([e])}^2)}\prod_{i\in I}\beta_i^{c_i(T)} \\ &= \frac{C_N(T)}{N^{\#(V)}}\frac{1}{\prod_{[e]\in\widetilde{E}}(2c_{\gamma([e])} - c_{\gamma([e])}^2)}\prod_{i\in I}\beta_i^{c_i(T)},\end{aligned}$$

where $C_N(T)$ is the number of maps $\phi : V \hookrightarrow [N]$ satisfying the band width condition

$$|\phi([e])| \le b_N^{(\gamma([e]))}, \qquad \forall [e] \in \widetilde{E}. \tag{4.74}$$

We may think of the ratio

$$\frac{C_N(T)}{N^{\#(V)}} \sim \frac{C_N(T)}{N^{\underline{\#(V)}}}$$

as the proportion of admissible maps $\phi : V \hookrightarrow [N]$. Unfortunately, the vertices of our graph T lack the homogeneity property from before due to the asymmetry of the band condition (4.74). This makes the task of computing $C_N(T)$ extremely tedious (and highly dependent on T). Nevertheless, we can give an integral representation of the limit of this ratio much as in [BMP91]. In particular, a straightforward weak convergence argument shows that

$$\lim_{N\to\infty} \frac{C_N(T)}{N^{\#(V)}} = \int_{[0,1]^V} \prod_{[e]\in\widetilde{E}} \mathbb{1}\{|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{\gamma([e])}\}\, d\mathbf{x}_V. \tag{4.75}$$

The remaining term in (4.73) follows as

$$p_T(\mathbf{c}_4) = \frac{\int_{[0,1]^V} \prod_{[e]\in\widetilde{E}} \mathbb{1}\{|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{\gamma([e])}\}\, d\mathbf{x}_V}{\prod_{[e]\in\widetilde{E}} (2c_{\gamma([e])} - c_{\gamma([e])}^2)} > 0.$$

■

Remark 4.4.7. For general $\beta_i \in \mathbb{C}$, we must again keep track of the orderings ψ of the vertices. In this case, we combine the integrands of (4.18) and (4.75) to define

$$p_T(\mathbf{c}_4, \psi) = \frac{\int_{[0,1]^V} \mathbb{1}\{x_{\psi(1)} \geq \cdots \geq x_{\psi(\#(V))}\} \prod_{[e]\in\widetilde{E}} \mathbb{1}\{|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{\gamma([e])}\}\, d\mathbf{x}_V}{\prod_{[e]\in\widetilde{E}} (2c_{\gamma([e])} - c_{\gamma([e])}^2)},$$

which replaces the $\frac{1}{\#(V)!}$ term in (4.19). In particular, we can then write the LTD of $\mathcal{O}_N^{(4)}$ as

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{O}_N^{(4)})\big] = \begin{cases} \displaystyle\sum_{\psi:[\#(V)]\tilde{\to}V} p_T(\mathbf{c}_4, \psi) S_\psi(T) & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise.} \end{cases}$$

Naturally, we are interested in the behavior of $p_T(\mathbf{c}_4)$ as the proportions $\mathbf{c}_4$ approach the boundary values $\{0, 1\}$. To this end, we fix some notation. Recall that $T = (V, E, \gamma, \mathrm{src}, \mathrm{tar})$ is a colored double tree. We record the labels $L(\widetilde{F})$ appearing in any subset $\widetilde{F} \subset \widetilde{E}$ of twin edges so that

$$L(\widetilde{F}) = \{\gamma([e]) : [e] \in \widetilde{F}\} \subset I_4.$$

We write $\{\mathrm{src}([e]), \mathrm{tar}([e])\}$ for the pair of vertices adjacent to twin edges $[e] = \{e, e'\}$, which allows us to further record the vertices $V(\widetilde{F})$ appearing in $\widetilde{F}$ as

$$V(\widetilde{F}) = \{\mathrm{src}([e]), \mathrm{tar}([e]) : [e] \in \widetilde{F}\}.$$

For any collection of real numbers $\mathbf{r} = (r_j)_{j\in J}$ in $[0, 1]$ with $L(\widetilde{F}) \subset J$, we define the function

$$\mathrm{Cut}_{\widetilde{F},\mathbf{r}} : [0, 1]^{V(\widetilde{F})} \to [0, 1]$$

by the product

$$\mathrm{Cut}_{\widetilde{F},\mathbf{r}}(\mathbf{x}_{V(\widetilde{F})}) = \prod_{[e]\in\widetilde{F}} \mathbb{1}\{|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq r_{\gamma([e])}\}.$$

We note that $\mathrm{Cut}_{\widetilde{F},\mathbf{r}}$ is simply the indicator on the banded region cut out of the hypercube $[0,1]^{V(\widetilde{F})}$ by the constraints $|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq r_{\gamma([e])}$. For example, our notation allows us to succinctly write the integral

$$\mathrm{Int}_T(\mathbf{c}_4) = \lim_{N\to\infty} \frac{C_N(T)}{N^{\#(V)}} = \int_{[0,1]^V} \mathrm{Cut}_{\widetilde{E},\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_V.$$

Similarly, we group the normalizations coming from the twin edges $\widetilde{F} \subset \widetilde{E}$ with

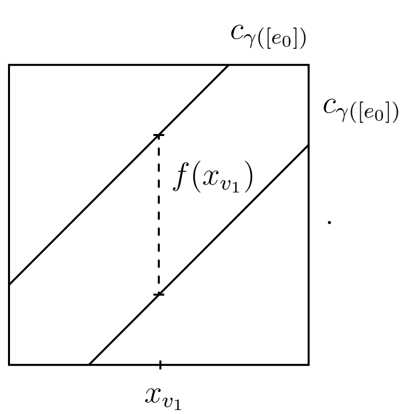
$$\mathrm{Norm}_{\widetilde{F}}(\mathbf{c}_4) = \prod_{[e]\in\widetilde{F}} (2c_{\gamma([e])} - c_{\gamma([e])}^2). \tag{4.76}$$

If $\widetilde{F} = \widetilde{E}$, we write $\mathrm{Cut}_{T,\mathbf{r}} = \mathrm{Cut}_{\widetilde{E},\mathbf{r}}$ (resp., $\mathrm{Norm}_T(\mathbf{c}_4) = \mathrm{Norm}_{\widetilde{E}}(\mathbf{c}_4)$). In this case,

$$p_T(\mathbf{c}_4) = \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)}.$$

We will need some simple bounds on the integral $\mathrm{Int}_T(\mathbf{c}_4)$. We start with an easy upper bound. Consider a leaf vertex v_0 of our colored double tree T. Let $v_1 \sim_{[e_0]} v_0$ denote the unique vertex v_1 adjacent to v_0. We compute the diameter $f(x_{v_1})$ of a cross section in the banded strip of the unit square $[0,1]^2$ defined by $|x_{v_0} - x_{v_1}| \leq c_{\gamma([e_0])}$,

$$\begin{aligned} f(x_{v_1}) &= \int_0^1 \mathbb{1}\{|x_{\mathrm{src}([e_0])} - x_{\mathrm{tar}([e_0])}| \leq c_{\gamma([e_0])}\}\, dx_{v_0} \\ &= \int_0^1 \mathbb{1}\{|x_{v_0} - x_{v_1}| \leq c_{\gamma([e_0])}\}\, dx_{v_0} \\ &= \begin{cases} x_{v_1} + c_{\gamma([e_0])} & \text{if } x_{v_1} \in [0, c_{\gamma([e_0])} \wedge (1 - c_{\gamma([e_0])})] \\ 2c_{\gamma([e_0])} \wedge 1 & \text{if } x_{v_1} \in [c_{\gamma([e_0])} \wedge (1 - c_{\gamma([e_0])}), c_{\gamma([e_0])} \vee (1 - c_{\gamma([e_0])})] \\ 1 + c_{\gamma([e_0])} - x_{v_1} & \text{if } x_{v_1} \in [(c_{\gamma([e_0])} \vee (1 - c_{\gamma([e_0])})), 1] \end{cases} \end{aligned} \tag{4.77}$$

$=$ 

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In particular,

$$c_{\gamma([e_0])} \leq f(x_{v_1}) \leq 2c_{\gamma([e_0])} \wedge 1.$$

It follows that

$$\begin{aligned}
\mathrm{Int}_T(\mathbf{c}_4) &= \int_{[0,1]^V} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V) d\mathbf{x}_V \\
&= \int_{[0,1]^{V\setminus\{v_0\}}} \mathrm{Cut}_{\widetilde{E}\setminus\{[e_0]\},\mathbf{c}_4}(\mathbf{x}_{V\setminus\{v_0\}}) \bigg(\int_0^1 \mathbb{1}\{|x_{v_0} - x_{v_1}| \leq c_{\gamma([e_0])}\}\, dx_{v_0} \bigg) d\mathbf{x}_{V\setminus\{v_0\}} \\
&\leq \int_{[0,1]^{V\setminus\{v_0\}}} \mathrm{Cut}_{\widetilde{E}\setminus\{[e_0]\},\mathbf{c}_4}(\mathbf{x}_{V\setminus\{v_0\}}) \Big(2c_{\gamma([e_0])} \wedge 1 \Big) d\mathbf{x}_{V\setminus\{v_0\}} \\
&= (2c_{\gamma([e_0])} \wedge 1)\, \mathrm{Int}_{T\setminus[e_0]}(\mathbf{c}_4),
\end{aligned}$$

where $T \setminus [e_0]$ is the colored double tree obtained from T by removing the leaf v_0 and its adjacent twin edges $[e_0]$. Iterating this construction, we obtain the upper bound

$$\mathrm{Int}_T(\mathbf{c}_4) \leq \prod_{[e]\in\widetilde{E}} (2c_{\gamma([e])} \wedge 1).$$

The same reasoning of course shows that

$$\mathrm{Int}_T(\mathbf{c}_4) \geq c_{\gamma([e_0])}\, \mathrm{Int}_{T\setminus[e_0]}(\mathbf{c}_4) \geq \cdots \geq \prod_{[e]\in\widetilde{E}} c_{\gamma([e])},$$

but we can do much better for small proportions $\mathbf{c}_4$. In particular, assume that

$$\hat{c} = \max_{[e]\in\widetilde{E}} c_{\gamma([e])} < \frac{1}{2}.$$

Then

$$\begin{aligned}
\mathrm{Int}_T(\mathbf{c}_4) &= \int_{[0,1]^V} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_V \geq \int_{[\hat{c},1-\hat{c}]^V} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_V \\
&= \int_{[\hat{c},1-\hat{c}]^{V\setminus\{v_0\}}} \mathrm{Cut}_{\widetilde{E}\setminus\{[e_0]\},\mathbf{c}_4}(\mathbf{x}_{V\setminus\{v_0\}}) \bigg(\int_{\hat{c}}^{1-\hat{c}} \mathbb{1}\{|x_{v_0} - x_{v_1}| \leq c_{\gamma([e_0])}\}\, dx_{v_0} \bigg) d\mathbf{x}_{V\setminus\{v_0\}} \\
&= \int_{[\hat{c},1-\hat{c}]^{V\setminus\{v_0\}}} \mathrm{Cut}_{\widetilde{E}\setminus\{[e_0]\},\mathbf{c}_4}(\mathbf{x}_{V\setminus\{v_0\}}) \Big((1-2\hat{c})2c_{\gamma([e_0])} \Big) d\mathbf{x}_{V\setminus\{v_0\}} \\
&= \cdots = (1-2\hat{c})^{\#(\widetilde{E})} \prod_{[e]\in\widetilde{E}} 2c_{\gamma([e])}.
\end{aligned}$$

Thus, for $\hat{c} < \frac{1}{2}$, we have the bounds

$$\frac{(1-2\hat{c})^{\#(\widetilde{E})} \prod_{[e]\in\widetilde{E}} 2c_{\gamma([e])}}{\prod_{[e]\in\widetilde{E}} (2c_{\gamma([e])} - c_{\gamma([e])}^2)} \leq \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)} \leq \frac{\prod_{[e]\in\widetilde{E}} 2c_{\gamma([e])}}{\prod_{[e]\in\widetilde{E}} (2c_{\gamma([e])} - c_{\gamma([e])}^2)},$$

which imply that

$$\lim_{\hat{c}\to 0^+} p_T(\mathbf{c}_4) = \lim_{\hat{c}\to 0^+} \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)} = 1. \tag{4.78}$$

We view the limit $\hat{c} \to 0^+$ as approaching the slow growth regime. In view of (4.78), we see that the LTD (4.73) of the proportional growth RBMs behaves accordingly (in particular, we have convergence to the LTD (4.62) of the slow growth RBMs).

In an easier direction, we can also consider the limit

$$\underline{c} = \min_{[e]\in\widetilde{E}} c_{\gamma([e])} \to 1^-.$$

One then clearly has

$$\lim_{\underline{c}\to 1^-} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V) = 1, \qquad \forall \mathbf{x}_V \in [0,1]^V. \tag{4.79}$$

We can push this limit through the integral by dominated convergence to obtain

$$\lim_{\underline{c}\to 1^-} \mathrm{Int}_T(\mathbf{c}_4) = \int_{[0,1]^V} \lim_{\underline{c}\to 1^-} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_V = 1. \tag{4.80}$$

Of course, the same convergence also holds for the normalizations (4.76),

$$\lim_{\underline{c}\to 1^-} \mathrm{Norm}_{\widetilde{F}}(\mathbf{c}_4) = 1, \qquad \forall \widetilde{F} \subset \widetilde{E}, \tag{4.81}$$

and so

$$\lim_{\underline{c}\to 1^-} p_T(\mathbf{c}_4) = \lim_{\underline{c}\to 1^-} \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)} = 1. \tag{4.82}$$

We view the limit $\underline{c} \to 1^-$ as approaching the usual Wigner matrices $\mathcal{W}_N$, or, more generally, the full proportion RBMs. Again, our limit (4.82) shows that the LTD (4.73) behaves accordingly (in particular, we have convergence to the LTD (4.8) of the Wigner matrices).

Up to now, our analysis of the integral $\mathrm{Int}_T(\mathbf{c}_4)$ essentially follows [BMP91]. We take care to account for possibly different band widths by grouping them in the min $\underline{c}$ or the max $\hat{c}$, but in both cases we indiscriminately send the proportions to a single boundary value $\{0, 1\}$. From this point of view, we fail to perceive any differences in the limits

$$\lim_{\hat{c}\to 0^+} p_T(\mathbf{c}_4) = 1 = \lim_{\underline{c}\to 1^-} p_T(\mathbf{c}_4); \tag{4.83}$$

yet, the two cases actually differ quite considerably. To see this, we will need to refine our analysis of $p_T(\mathbf{c}_4)$ to consider sending only a subset of the proportions $\mathbf{c}_4$ to possibly different boundary values. The results will greatly inform our treatment of the joint LTD of the combined families $\mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)}$.

We start with the simpler case of sending the band width c_{i_0} of a single label $i_0 \in I_4$ in our colored double tree T to 1^-. We write $T_{i_0} = (V_{i_0}, E_{i_0})$ for the subgraph of T with edge labels in i_0. In general, T_{i_0} is a forest of colored double trees (in the single "color" i_0). We define $\widetilde{T}_{i_0} = (V_{i_0}, \widetilde{E}_{i_0})$ as before. We remove the twin edges $\widetilde{E}_{i_0}$ from T to obtain a forest of colored double trees $T \setminus \widetilde{E}_{i_0}$ (say, with connected components $T_1, \ldots, T_k$). We emphasize that we only remove the edges $\widetilde{E}_{i_0}$; in particular, we keep any resulting isolated vertices. We then have the analogues of (4.79)-(4.81):

$$\lim_{c_{i_0} \to 1^-} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V) = \mathrm{Cut}_{\widetilde{E} \setminus \widetilde{E}_{i_0}, \mathbf{c}_4}(\mathbf{x}_V) = \prod_{\ell=1}^{k} \mathrm{Cut}_{T_\ell, \mathbf{c}_4}(\mathbf{x}_{V_\ell}), \qquad \forall \mathbf{x}_V \in [0,1]^V, \tag{4.84}$$

$$\begin{aligned} \lim_{c_{i_0} \to 1^-} \mathrm{Int}_T(\mathbf{c}_4) &= \int_{[0,1]^V} \lim_{c_{i_0} \to 1^-} \mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_V \\ &= \prod_{\ell=1}^{k} \int_{[0,1]^{V_\ell}} \mathrm{Cut}_{T_\ell,\mathbf{c}_4}(\mathbf{x}_{V_\ell})\, d\mathbf{x}_{V_\ell} = \prod_{\ell=1}^{k} \mathrm{Int}_{T_\ell}(\mathbf{c}_4), \end{aligned} \tag{4.85}$$

and

$$\lim_{c_{i_0} \to 1^-} \mathrm{Norm}_T(\mathbf{c}_4) = \mathrm{Norm}_{\widetilde{E} \setminus \widetilde{E}_{i_0}}(\mathbf{c}_4) \lim_{c_{i_0} \to 1^-} \mathrm{Norm}_{\widetilde{E}_{i_0}}(\mathbf{c}_4) = \prod_{\ell=1}^{k} \mathrm{Norm}_{T_\ell}(\mathbf{c}_4). \tag{4.86}$$

It follows that

$$\lim_{c_{i_0} \to 1^-} p_T(\mathbf{c}_4) = \lim_{c_{i_0} \to 1^-} \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)} = \frac{\prod_{\ell=1}^{k} \mathrm{Int}_{T_\ell}(\mathbf{c}_4)}{\prod_{\ell=1}^{k} \mathrm{Norm}_{T_\ell}(\mathbf{c}_4)} = \prod_{\ell=1}^{k} p_{T_\ell}(\mathbf{c}_4). \tag{4.87}$$

Of course, if T_ℓ consists of an isolated vertex, then $p_{T_\ell}(\mathbf{c}_4) = 1$. One can then effectively discard the isolated vertices of $T \setminus \widetilde{E}_{i_0}$ and just consider the resulting forest of nontrivial colored double trees. We choose to keep these vertices in writing a simple, consistent formula for our limit.

The reader will no doubt be easily convinced of (4.87), but we give here some intuition for the sake of comparison later. We imagine each vertex v as a country in a league of allied nations V. Each value $x_v \in [0,1]$ represents a proposed amount of aid to be sent by country v to every other country. To avoid showing favoritism, the same amount of aid x_v is sent to each ally $w \neq v$; however, to ensure goodwill, a country can opt to cap the disparity in the amount of aid they exchange with a given ally. We view these restrictions as coming from the edges $\widetilde{E}$, where an edge $v \sim_{[e]} w$ corresponds to a bound $|x_v - x_w| \leq c_{\gamma([e])}$.

We can then interpret the integral $\mathrm{Int}_T(\mathbf{c}_4)$ as the percentage of universally acceptable proposals $\mathbf{x}_V \in [0,1]^V$. Each term in our normalization

$$\mathrm{Norm}_T(\mathbf{c}_4) = \prod_{[e] \in \widetilde{E}} (2c_{\gamma([e])} - c_{\gamma([e])}^2) = \prod_{[e] \in \widetilde{E}} \int_0^1 \int_0^1 \mathbb{1}\{|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{\gamma([e])}\}\, dx_{\mathrm{src}([e])} dx_{\mathrm{tar}([e])}$$

corresponds to the local situation of a single pair of constrained allies $\{\mathrm{src}([e]), \mathrm{tar}([e])\}$. Of course, each such pair must agree to a proposal $\mathbf{x}_V$ for it to be universally acceptable, though in general this is not sufficient. We can then think of the ratio

$$p_T(\mathbf{c}_4) = \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)}$$

as conditioning on the proposals that, at the very least, pass at the local level (though it is possible for $p_T(\mathbf{c}^{(4)}) > 1$). In the limit $c_{i_0} \to 1^-$, the twin edges $[e] \in \widetilde{E}_{i_0}$ with label i_0 represent negotiations between increasingly amicable nations, insomuch that they no longer care to keep track of the disparity in the aid exchanged between them. Here, we again encounter the notion of a free edge. In this case, the proposal $\mathbf{x}_V$ need only to satisfy the constraints coming from the remaining edges $\widetilde{E} \setminus \widetilde{E}_{i_0}$, which explains the limit (4.87).

Of course, there is nothing special about only sending one of the band widths $c_{i_0} \to 1^-$. In fact, the same argument clearly applies to any collection of labels $i_0, \ldots, i_j$ in a colored double tree T. We state the full result later once we have also considered the behavior of $p_T(\mathbf{c}_4)$ for band widths $c_{i_0} \to 0^+$, but first we must introduce some more notation.

For any pair of subsets $W \subset V$ and $\widetilde{F} \subset \widetilde{E}$, we define the conditional expectation

$$\mathrm{Int}_{\widetilde{F}}(\mathbf{c}_4|W) : [0,1]^W \to [0,1]$$

by

$$\mathrm{Int}_{\widetilde{F}}(\mathbf{c}_4|W)(\mathbf{x}_W) = \int_{[0,1]^{V\setminus W}} \mathrm{Cut}_{\widetilde{F},\mathbf{c}_4}(\mathbf{x}_V)\, d\mathbf{x}_{V\setminus W}.$$

For example, the reader can easily verify that

$$\int_{[0,1]^W} \mathrm{Int}_T(\mathbf{c}_4|W)(\mathbf{x}_W)\, d\mathbf{x}_W = \mathrm{Int}_T(\mathbf{c}_4).$$

As before, we start with a single label $i_0 \in I_4$ in T, for which we now consider the limit $c_{i_0} \to 0^+$. To simplify the argument, we first assume that there is a unique pair of twin edges $[e_{i_0}]$ with the label $\gamma([e_{i_0}]) = i_0$. For notational convenience, we write

$$\{a, b\} = \{\mathrm{src}([e_{i_0}]), \mathrm{tar}([e_{i_0}])\}.$$

We condition on the vertices $\{a, b\}$ to obtain

$$\begin{aligned} p_T(\mathbf{c}_4) = \frac{\mathrm{Int}_T(\mathbf{c}_4)}{\mathrm{Norm}_T(\mathbf{c}_4)} &= \int_{[0,1]^V} \frac{\mathrm{Cut}_{T,\mathbf{c}_4}(\mathbf{x}_V)}{\mathrm{Norm}_T(\mathbf{c}_4)}\, d\mathbf{x}_V \\ &= \int_{[0,1]^2} \frac{\mathrm{Int}_{\widetilde{E}\setminus\{[e_{i_0}]\}}(\mathbf{c}_4|\{x_a, x_b\})(x_a, x_b)}{\mathrm{Norm}_{\widetilde{E}\setminus\{[e_{i_0}]\}}(\mathbf{c}_4)} \left(\frac{\mathbb{1}\{|x_a - x_b| \leq c_{i_0}\}}{2c_{i_0} - c_{i_0}^2}\, dx_a dx_b \right) \\ &= \int_{[0,1]^2} f(x_a, x_b)\, \mu_{c_{i_0}}(dx_a, dx_b), \end{aligned} \tag{4.88}$$

where

$$f(x_a, x_b) = \frac{\mathrm{Int}_{\widetilde{E}\setminus\{[e_{i_0}]\}}(\mathbf{c}_4|\{x_a, x_b\})(x_a, x_b)}{\mathrm{Norm}_{\widetilde{E}\setminus\{[e_{i_0}]\}}(\mathbf{c}_4)}$$

is a bounded continuous function that does not depend on c_{i_0} and

$$\mu_{c_{i_0}}(dx_a, dx_b)$$

is the uniform (probability) measure on the banded strip in unit square $[0,1]^2$ defined by $|x_a - x_b| \leq c_{i_0}$. In the limit, we have the weak convergence

$$\mu_{c_{i_0}} \xrightarrow{w} \mu_\Delta \quad \text{as} \quad c_{i_0} \to 0^+,$$

where μ_Δ is the uniform measure on the diagonal $\{(x, x) : x \in [0,1]\} \subset [0,1]^2$. In particular, this implies that

$$\begin{aligned}\lim_{c_{i_0}\to 0^+} p_T(\mathbf{c}_4) &= \lim_{c_{i_0}\to 0^+} \int_{[0,1]^2} f(x_a, x_b)\, \mu_{c_{i_0}}(dx_a, dx_b)\\ &= \int_{[0,1]^2} f(x_a, x_b)\, \mu_\Delta(dx_a, dx_b) = \int_0^1 f(x,x)\, dx = p_{T/[e_{i_0}]}(\mathbf{c}_4),\end{aligned}$$

where $T/[e_{i_0}]$ is the colored double tree obtained from T by contracting the twin edges $[e_{i_0}]$ (i.e., we remove the edges $[e_{i_0}]$ and merge the vertices $\{a, b\}$). We note the contrast to the situation in (4.87) in the limit $c_{i_0} \to 1^-$, where we remove the edges but do not otherwise modify the vertices.

We must take care if the label i_0 appears in more than one set of twin edges. In any case, we can always identify the subgraph T_{i_0} of T with edge labels in i_0. In general, $T_{i_0} = (V_{i_0}, E_{i_0})$ is a forest $T_1 \sqcup \cdots \sqcup T_k$ of colored double trees $T_\ell = (V_\ell, E_\ell)$ (in the single color i_0). Conditioning on the vertices $V_{i_0} = V_1 \sqcup \cdots \sqcup V_k$ of T_{i_0}, we obtain

$$p_T(\mathbf{c}_4) = \int_{\times_{\ell=1}^k [0,1]^{V_\ell}} f(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k}) \prod_{\ell=1}^k \left(\frac{\mathrm{Cut}_{T_\ell, c_{i_0}}(\mathbf{x}_{V_\ell})}{\mathrm{Norm}_{T_\ell}(c_{i_0})}\, d\mathbf{x}_{V_\ell} \right) \tag{4.89}$$

where

$$f(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k}) = \frac{\mathrm{Int}_{\widetilde{E}\setminus\widetilde{E}_{i_0}}(\mathbf{c}_4|V_{i_0})(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k})}{\mathrm{Norm}_{\widetilde{E}\setminus\widetilde{E}_{i_0}}(\mathbf{c}_4)}$$

is again a bounded continuous function that does not depend on c_{i_0}. In this case, we cannot immediately write (4.89) in terms of probability measures

$$\mu_{c_{i_0}}^{(\ell)}(d\mathbf{x}_{V_\ell}) = \frac{\mathrm{Cut}_{T_\ell, c_{i_0}}(\mathbf{x}_{V_\ell})}{\mathrm{Norm}_{T_\ell}(c_{i_0})}\, d\mathbf{x}_{V_\ell}$$

as we did in (4.88) since, in general,

$$\mathrm{Int}_{T_\ell}(c_{i_0}) = \int_{[0,1]^{V_\ell}} \mathrm{Cut}_{T_\ell, c_{i_0}}(\mathbf{x}_{V_\ell})\, d\mathbf{x}_{V_\ell} \neq (2c_{i_0} - c_{i_0}^2)^{\#(\widetilde{E}_\ell)} = \mathrm{Norm}_{T_\ell}(c_{i_0});$$

however, our work (4.78) from before shows that

$$\lim_{c_{i_0}\to 0^+} \frac{\mathrm{Int}_{T_\ell}(c_{i_0})}{\mathrm{Norm}_{T_\ell}(c_{i_0})} = 1.$$

Thus, we can instead write

$$p_T(\mathbf{c}_4) = \delta(c_{i_0}) \int_{\times_{\ell=1}^k [0,1]^{V_\ell}} f(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k}) \Big(\otimes_{\ell=1}^k \mu_{c_{i_0}}^{(\ell)}(d\mathbf{x}_{V_\ell}) \Big), \tag{4.90}$$

where $\delta(c_{i_0})$ is a real number depending on c_{i_0} such that

$$\lim_{c_{i_0}\to 0^+} \delta(c_{i_0}) = 1$$

and $\mu_{c_{i_0}}^{(\ell)}$ is the uniform measure on the banded region $R_\ell \subset [0,1]^{V_\ell}$ defined by the constraints

$$|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{i_0}, \qquad \forall [e] \in \widetilde{E}_\ell.$$

As before, we note that

$$\lim_{c_{i_0}\to 0^+} \mu_{c_{i_0}}^{(\ell)} = \mu_\Delta^{(\ell)},$$

where $\mu_\Delta^{(\ell)}$ is the uniform measure on the diagonal $\{(x, \ldots, x) : x \in [0,1]\} \subset [0,1]^{V_\ell}$. It follows that

$$\begin{aligned} \lim_{c_{i_0}\to 0^+} p_T(\mathbf{c}_4) &= \lim_{c_{i_0}\to 0^+} \int_{\times_{\ell=1}^k [0,1]^{V_\ell}} f(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k}) \prod_{\ell=1}^k \left(\frac{\mathrm{Cut}_{T_\ell, c_{i_0}}(\mathbf{x}_{V_\ell})}{\mathrm{Norm}_{T_\ell}(c_{i_0})} \, d\mathbf{x}_{V_\ell} \right) \\ &= \int_{\times_{\ell=1}^k [0,1]^{V_\ell}} f(\mathbf{x}_{V_1}, \ldots, \mathbf{x}_{V_k}) \Big(\otimes_{\ell=1}^k \mu_\Delta^{(\ell)}(d\mathbf{x}_{V_\ell}) \Big) \\ &= \int_{[0,1]^k} f(x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k) \, dx_1 \cdots dx_k = p_{T/T_{i_0}}(\mathbf{c}_4), \end{aligned}$$

where T/T_{i_0} is the colored double tree obtained from T by contracting the edges of T_{i_0} (i.e., for each $\ell \in [k]$, we remove the edges $\widetilde{E}_\ell$ and merge the vertices V_ℓ into a single vertex).

We can easily adapt our argument to accommodate multiple band widths $c_{i_0}, \ldots, c_{i_j}$ in the limit $\max(c_{i_0}, \ldots, c_{i_j}) \to 0^+$. In this case, we replace T_{i_0} with $T_{\mathbf{i}}$, the subgraph of T with edge labels in $\mathbf{i} = \{i_0, \ldots, i_j\}$; otherwise, the same argument goes through just as well.

Returning to our intuition from before, we think of the limit $c_{i_0} \to 0^+$ as representing negotiations between increasing acrimonious nations, insomuch that they become completely intransigent and insist on absolute parity in the aid exchanged between them. Negotiations along such an edge $\gamma([e]) = i_0$ then stall a proposal $\mathbf{x}_V$ unless $|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| = 0$. In this case, we can effectively consider the two countries $\mathrm{src}([e])$ and $\mathrm{tar}([e])$ as a single entity sending the aid $x_{\mathrm{src}([e])} = x_{\mathrm{tar}([e])}$ to the remaining allies. Our normalization then allows us to recast the problem as the proportion of acceptable proposals in this new world order.

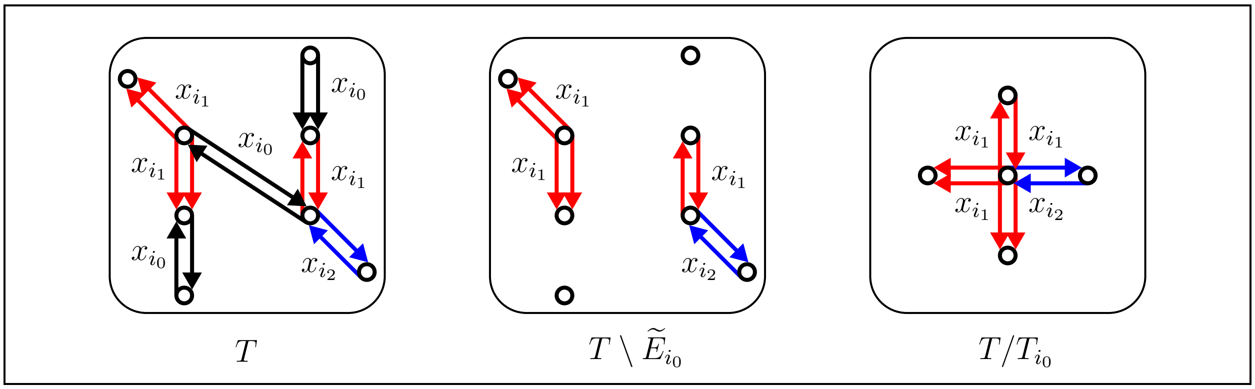


Figure 4.14: A comparison of the resulting graphs in the limits $c_{i_0} \to 1^-$ and $c_{i_0} \to 0^+$ respectively. Here, we start with a colored double tree T and remove (resp., contract) the edges with label x_{i_0} to obtain the limit graph $T \setminus \widetilde{E}_{i_0}$ (resp., T/T_{i_0}). In particular, we note that the two operations can produce substantially different graphs.

At this point, we see how the limits (4.83) come about in different ways: in the limit $\underline{c} \to 0^+$, we contract all of the edges, leaving a single isolated vertex; in the limit $\hat{c} \to 1^-$, we remove all of the edges, leaving $\#(V)$ isolated vertices. Finally, the result for a collection of band widths sent to possibly different boundary values should come as no surprise. We combine our work in the two previous cases taking care to account for parts moving simultaneously in different directions.

To begin, let J_0 (resp., J_1) denote the collection of labels in our colored double tree T whose band widths are to be sent to 0^+ (resp., 1^-). We define

$$\mathbf{c}_0 = (c_i)_{i \in J_0}, \qquad \mathbf{c}_1 = (c_i)_{i \in J_1};$$
$$c_0 = \max_{i \in J_0} c_i, \qquad c_1 = \min_{i \in J_1} c_i,$$

and write $\mathbf{c}_2 = \mathbf{c}_4 \setminus (\mathbf{c}_0 \cup \mathbf{c}_1)$ for the remaining band widths. We are then interested in the limit

$$\lim_{(c_0, c_1) \to (0^+, 1^-)} p_T(\mathbf{c}_4).$$

We decompose our graph as before. We write T_{0^+} for the subgraph of T with edge labels in J_0. In general, $T_{0^+} = (V_{0^+}, E_{0^+})$ is a forest $T_{0^+} = T_1^+ \sqcup \cdots \sqcup T_k^+$ of colored double trees $T_\ell^+ = (V_\ell^+, E_\ell^+)$ (except now possibly with multiple colors). Similarly, we write $T_{1^-} = (V_{1^-}, E_{1^-})$ for the subgraph of T with edge labels in J_1. Finally, we write $E_2 = E \setminus (E_0 \cup E_1)$ for the remaining edges.

Conditioning on the vertices $V_{0^+} = V_1^+ \sqcup \cdots \sqcup V_k^+$ of T_{0^+}, we obtain the analogue of (4.90),

$$p_T(\mathbf{c}_4) = \delta(\mathbf{c}_0) \int_{\times_{\ell=1}^k [0,1]^{V_\ell^+}} f_{\mathbf{c}_1}(\mathbf{x}_{V_1^+}, \ldots, \mathbf{x}_{V_k^+}) \Big(\otimes_{\ell=1}^k \mu_{\mathbf{c}_0}^{(\ell)}(d\mathbf{x}_{V_\ell^+}) \Big),$$

where $\delta(\mathbf{c}_0)$ is a real number depending on $\mathbf{c}_0$ such that

$$\lim_{c_0 \to 0^+} \delta(\mathbf{c}_0) = 1$$

and $\mu_{\mathbf{c}_0}^{(\ell)}$ is the uniform measure on the banded region R_ℓ in $[0,1]^{V_\ell^+}$ defined by the constraints

$$|x_{\mathrm{src}([e])} - x_{\mathrm{tar}([e])}| \leq c_{\gamma([e])} \in \mathbf{c}_0, \qquad \forall [e] \in \widetilde{E}_\ell^+.$$

Despite considering multiple band widths $\mathbf{c}_0$, we still have the weak convergence

$$\lim_{c_0 \to 0^+} \mu_{\mathbf{c}_0}^{(\ell)} = \mu_\Delta^{(\ell)}.$$

As before,

$$\begin{aligned} f_{\mathbf{c}_1}(\mathbf{x}_{V_1^+}, \ldots, \mathbf{x}_{V_\ell^+}) &= \frac{\mathrm{Int}_{\widetilde{E} \setminus \widetilde{E}_{0^+}}(\mathbf{c}_4 | V_{0^+})(\mathbf{x}_{V_1^+}, \ldots, \mathbf{x}_{V_k^+})}{\mathrm{Norm}_{\widetilde{E} \setminus \widetilde{E}_{0^+}}(\mathbf{c}_4)} \\ &= \int_{[0,1]^{V \setminus V_{0^+}}} \frac{\mathrm{Cut}_{\widetilde{E}_{1^-}, \mathbf{c}_1}(\mathbf{x}_V)}{\mathrm{Norm}_{\widetilde{E}_{1^-}}(\mathbf{c}_1)} \frac{\mathrm{Cut}_{\widetilde{E}_2, \mathbf{c}_2}(\mathbf{x}_V)}{\mathrm{Norm}_{\widetilde{E}_2}(\mathbf{c}_2)} \, d\mathbf{x}_{V \setminus V_{0^+}} \end{aligned}$$

is a bounded continuous function that does not depend on $\mathbf{c}_0$; however, $f_{\mathbf{c}_1}$ does depend on $\mathbf{c}_1$. In particular, the function

$$\mathrm{Cut}_{\widetilde{E}_{1^-}, \mathbf{c}_1} : [0,1]^V \to [0,1]$$

is monotonic in $\mathbf{c}_1$ with

$$\lim_{c_1 \to 1^-} \mathrm{Cut}_{\widetilde{E}_{1^-}, \mathbf{c}_1}(\mathbf{x}_V) = 1, \qquad \forall \mathbf{x}_V \in [0,1]^V.$$

Since

$$\lim_{c_1 \to 1^-} \mathrm{Norm}_{\widetilde{E}_{1^-}}(\mathbf{c}_1) = 1,$$

it follows that

$$f(\mathbf{x}_{V_1^+}, \ldots, \mathbf{x}_{V_k^+}) = \lim_{c_1 \to 1^-} f_{\mathbf{c}_1}(\mathbf{x}_{V_1^+}, \ldots, \mathbf{x}_{V_k^+}) = \int_{[0,1]^{V \setminus V_0^+}} \frac{\mathrm{Cut}_{\widetilde{E}_2, \mathbf{c}_2}(\mathbf{x}_V)}{\mathrm{Norm}_{\widetilde{E}_2}(\mathbf{c}_2)} \, d\mathbf{x}_{V \setminus V_{0^+}}.$$

The monotonicity of $\mathrm{Cut}_{\widetilde{E}_{1^-},\mathbf{c}_1}$ in the proportions $\mathbf{c}_1$ then allows us to conclude that

$$\begin{aligned}\lim_{(c_0,c_1)\to(0^+,1^-)} p_T(\mathbf{c}_4) &= \lim_{(c_0,c_1)\to(0^+,1^-)} \int_{\times_{\ell=1}^k [0,1]^{V_\ell^+}} f_{\mathbf{c}_1}(\mathbf{x}_{V_1^+},\ldots,\mathbf{x}_{V_k^+}) \prod_{\ell=1}^k \left(\frac{\mathrm{Cut}_{\widetilde{E}_\ell^+,\mathbf{c}_0}(\mathbf{x}_{V_\ell^+})}{\mathrm{Norm}_{T_\ell^+}(\mathbf{c}_0)}\, d\mathbf{x}_{V_\ell^+} \right) \\ &= \int_{\times_{\ell=1}^k [0,1]^{V_\ell^+}} f(\mathbf{x}_{V_1^+},\ldots,\mathbf{x}_{V_k^+}) \left(\otimes_{\ell=1}^k \mu_\Delta^{(\ell)}(d\mathbf{x}_{V_\ell^+}) \right) \\ &= \int_{[0,1]^k} f(x_1,\ldots,x_1,\ldots,x_k,\ldots,x_k)\, dx_1\cdots dx_k = p_F(\mathbf{c}_2) = \prod_{r=1}^s p_{T_r}(\mathbf{c}_2),\end{aligned}$$

where F is the forest of colored double trees $F = T_1 \sqcup \cdots \sqcup T_s$ obtained from T by removing the edges E_{1^-} and contracting the edges E_{0^+}.

Our treatment of $p_T(\mathbf{c}_4)$ suggests the following form for the joint LTD of the matrices $\mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)}$. We leave the by-now familiar details of the proof to the diligent reader.

Theorem 4.4.8. *For any test graph T in $\mathbf{x}_2 \cup \mathbf{x}_3 \cup \mathbf{x}_4 = (x_i)_{i\in I_2\cup I_3\cup I_4}$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)})\big] = \begin{cases} p_F(\mathbf{c}_4) \prod_{i\in I} \beta_i^{c_i(T)} & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.91}$$

where $F = T_1 \sqcup \cdots \sqcup T_s$ is the forest of colored double trees obtained from T by contracting the edges with labels in I_2 and removing the edges with labels in I_3 and

$$p_F(\mathbf{c}_4) = \prod_{r=1}^s p_{T_r}(\mathbf{c}_4). \tag{4.92}$$

Corollary 4.4.9. *The full proportion RBMs $\mathcal{O}_N^{(3)}$ and the proper proportion RBMs $\mathcal{O}_N^{(4)}$ are asymptotically traffic independent, as are the full proportion RBMs $\mathcal{O}_N^{(3)}$ and the slow growth RBMs $\mathcal{O}_N^{(2)}$. The slow growth RBMs $\mathcal{O}_N^{(2)}$ and the proper proportion RBMs $\mathcal{O}_N^{(4)}$ are not asymptotically traffic independent, nor are independent proper proportion RBMs $\mathcal{O}_N^{(4)} = (\mathbf{\Theta}_N^{(i)})_{i\in I_4}$.*

Proof. The statements about asymptotic traffic independence follow from the calculation of F from our colored double tree T (we simply remove the edges with labels in I_3) and the multiplicativity of (4.92). For the statements about non-asymptotic traffic independence, we give a simple counterexample, namely, for $i_2 \in I_2$ and $i_4, j_4 \in I_4$ with $0 < c_{i_4} \le c_{j_4} < 1$,

$$\begin{aligned}\lim_{N\to\infty} \tau^0\left[\cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot \overset{\mathbf{\Theta}_N^{(i_2)}}{\underset{\mathbf{\Theta}_N^{(i_2)}}{\leftrightarrows}} \cdot \overset{\mathbf{\Theta}_N^{(j_4)}}{\underset{\mathbf{\Theta}_N^{(j_4)}}{\leftrightarrows}} \cdot \right] &= \lim_{N\to\infty} \tau^0\left[\cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot \overset{\mathbf{\Theta}_N^{(j_4)}}{\underset{\mathbf{\Theta}_N^{(j_4)}}{\leftrightarrows}} \cdot \right] \\ &= \lim_{N\to\infty} \tau^0\big[S(\mathbf{\Theta}_N^{(i_4)}, \mathbf{\Theta}_N^{(i_4)}, \mathbf{\Theta}_N^{(j_4)}, \mathbf{\Theta}_N^{(j_4)})\big] = p_S(\{c_{i_4}, c_{j_4}\}),\end{aligned}$$

where

$$p_S(\{c_{i_4}, c_{j_4}\}) = \begin{cases} \dfrac{-\frac{1}{3}c_{i_4}^3 - c_{i_4}^2 c_{j_4} - 2c_{i_4}c_{j_4}^2 + 4c_{i_4}c_{j_4}}{(2c_{i_4} - c_{i_4}^2)(2c_{j_4} - c_{j_4}^2)} & \text{if } c_{i_4} \leq c_{j_4} \leq \dfrac{1}{2}, \\ \dfrac{\frac{1}{3}c_{j_4}^3 - c_{i_4}c_{j_4}^2 - c_{i_4}^2 - c_{j_4}^2 + 2c_{i_4}c_{j_4} + c_{i_4} + c_{j_4} - \frac{1}{3}}{(2c_{i_4} - c_{i_4}^2)(2c_{j_4} - c_{j_4}^2)} & \text{if } 1 - c_{j_4} \leq c_{i_4} \leq \dfrac{1}{2}, \\ \dfrac{-\frac{1}{3}c_{i_4}^3 - c_{i_4}^2 c_{j_4} - 2c_{i_4}c_{j_4}^2 + 4c_{i_4}c_{j_4}}{(2c_{i_4} - c_{i_4}^2)(2c_{j_4} - c_{j_4}^2)} & \text{if } c_{i_4} \leq 1 - c_{j_4} \leq \dfrac{1}{2}, \\ \dfrac{\frac{1}{3}c_{j_4}^3 - c_{i_4}c_{j_4}^2 - c_{i_4}^2 - c_{j_4}^2 + 2c_{i_4}c_{j_4} + c_{i_4} + c_{j_4} - \frac{1}{3}}{(2c_{i_4} - c_{i_4}^2)(2c_{j_4} - c_{j_4}^2)} & \text{if } \dfrac{1}{2} \leq c_{i_4} \leq c_{j_4}. \end{cases}$$

In particular,

$$\begin{aligned} p_S(\{c_{i_4}, c_{j_4}\}) \neq 1 &= \Big(\lim_{N\to\infty} \tau^0\Big[\cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot\Big]\Big)\Big(\lim_{N\to\infty} \tau^0\Big[\cdot \overset{\mathbf{\Theta}_N^{(j_4)}}{\underset{\mathbf{\Theta}_N^{(j_4)}}{\leftrightarrows}} \cdot\Big]\Big) \\ &= \Big(\lim_{N\to\infty} \tau^0\Big[\cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot\Big]\Big)\Big(\lim_{N\to\infty} \tau^0\Big[\cdot \overset{\mathbf{\Theta}_N^{(i_2)}}{\underset{\mathbf{\Theta}_N^{(i_2)}}{\leftrightarrows}} \cdot\Big]\Big)\Big(\lim_{N\to\infty} \tau^0\Big[\cdot \overset{\mathbf{\Theta}_N^{(j_4)}}{\underset{\mathbf{\Theta}_N^{(j_4)}}{\leftrightarrows}} \cdot\Big]\Big), \end{aligned}$$

which covers both statements. ■

Remark 4.4.10. One can also deduce the lack of asymptotic traffic independence for independent proper proportion RBMs $\mathcal{O}_N^{(4)} = (\mathbf{\Theta}_N^{(i)})_{i\in I_4}$ of the same proportion $c_i \equiv c$ from the traffic CLT. Indeed, if the family $\mathcal{O}_N^{(4)}$ were asymptotically traffic independent, then we could adapt the traffic CLT argument from Section 4.2 to identify the LSD of a single proper proportion RBM $\mathbf{\Theta}_N^{(i)}$ as a free convolution $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$ of the form $p^2 + q^2 = 1$. On the contrary, the actual LSD is known to be non-semicircular and of bounded support [BMP91], which simultaneously implies that both $q^2 \neq 0$ and $q^2 = 0$ respectively.

The careful reader will notice that the periodic RBMs $\mathcal{P}_N^{(1)}$ are conspicuously absent in Theorem 4.4.8. Again, we have the familiar obstruction: without any further assumptions on the band widths $\mathbf{b}_N^{(1)} = (b_N^{(i)})_{i\in I_1}$, their fluctuations could preclude the existence of a joint LTD. For example, if a periodic band width $b_N^{(i)}$ has a subsequence of slow growth and another subsequence of proportional growth, then the LTDs along these two subsequences will be different. If we assume that the band widths $\mathbf{b}_N^{(1)} = (b_N^{(i)})_{I_1'} \cup (b_N^{(i)})_{i\in I_1''}$ fall into one of these two regimes, slow growth or proportional growth respectively, then we can prove the extension of Theorem 4.4.8 to $\mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)}$. In this case, the LTD essentially follows (4.91) except that we must now also contract the edges with labels in I_1' and remove the edges with labels in I_1'' (regardless of the limiting proportions $\lim_{N\to\infty} \frac{b_N^{(i)}}{N}$ for $i \in I_1''$).

The contraction of the edges with labels in I_1' should come as no surprise given Lemma 4.4.4, where we saw that periodizing a slow growth RBM does little to affect the calculations.

Just as we contract the labels in I_2, we should then also expect to contract the labels in I_1'. On the other hand, as we noted before, periodizing a proportional growth RBM changes the situation entirely. Formally, we need to work with the periodic absolute value

$$|x|_p = \min(x, 1-x), \qquad \forall x \in [0,1]$$

in our integral to account for the edges with labels in I_1''; however, the analogue of (4.77) does not depend on where we measure the diameter of our cross section,

$$g(x_{v_1}) = \int_0^1 \mathbb{1}\{|x_{v_0} - x_{v_1}|_p \leq c_{\gamma([e_0])}\}\, dx_{v_0} = 2c_{\gamma([e_0])}, \qquad \forall x_{v_1} \in [0,1].$$

This balances out perfectly with the normalization of the periodic RBMs,

$$\mathbf{\Lambda}_N^{(\gamma([e_0]))} = \mathbf{\Upsilon}_N^{(\gamma([e_0]))} \circ \mathbf{\Gamma}_N^{(\gamma([e_0]))} = \frac{1}{\sqrt{2b_N^{(\gamma([e_0]))}}}\mathbf{\Gamma}^{(\gamma([e_0]))},$$

so we can integrate out the vertices that are only adjacent to edges with labels in I_1'' without changing the value of the integral. This of course corresponds to simply removing the edges with labels in I_1'' when calculating $p_F(\mathbf{c}_4)$. In this case, we then know that the periodic RBMs $\mathcal{P}_N^{(1'')} = (\mathbf{\Lambda}_N^{(i)})_{i \in I_1''}$ and the proportional growth RBMs $\mathcal{O}_N^{(4)}$ are asymptotically traffic independent, whereas the periodic RBMs $\mathcal{P}_N^{(1')} = (\mathbf{\Lambda}_N^{(i)})_{i \in I_1'}$ and the proportional growth RBMs $\mathcal{O}_N^{(4)}$ are not.

For general $\beta_i \in \mathbb{C}$, we must again settle for convergence in joint distribution.

Theorem 4.4.11. *Assume that the band widths $(b_N^{(i)})_{i \in I_1}$ of the periodic RBMs fall into one of two categories $I_1 = I_1' \cup I_1''$ as before. For general $\beta_i \in \mathbb{C}$, the families $\mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)}$ converge in joint distribution to a family*

$$\mathbf{a} = (a_i)_{i \in I} = (a_i)_{i \in I_1'} \cup (a_i)_{i \in I_1''} \cup (a_i)_{i \in I_2} \cup (a_i)_{i \in I_3} \cup (a_i)_{i \in I_4} = \mathbf{a}_{1'} \cup \mathbf{a}_{1''} \cup \mathbf{a}_2 \cup \mathbf{a}_3 \cup \mathbf{a}_4.$$

The family $\mathbf{a}_{1'} \cup \mathbf{a}_{1''} \cup \mathbf{a}_2 \cup \mathbf{a}_3$ is a semicircular system; the families $\mathbf{a}_{1''}$, $\mathbf{a}_3$, and $\mathbf{a}_4$ are free; the families $\mathbf{a}_2$ and $\mathbf{a}_4$ are not free, nor are the families $\mathbf{a}_{1'}$ and $\mathbf{a}_4$; finally, the family $\mathbf{a}_4 = (a_i)_{i \in I_4}$ is not free.

Proof. The convergence in joint distribution follows from a modified version of the criteria (4.21) in Remark 4.2.3. In particular, we do not actually need to know the value of

$$\lim_{N \to \infty} \tau^0\big[T(\mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)})\big]$$

for an opposing colored double tree T, just that it exists. In this case, we know that the value of this limit is equal to $p_F(\mathbf{c}_4)$, which in turn is equal to 1 if there are no edges with labels in I_4. This proves the first statement, about $\mathbf{a}_{1'} \cup \mathbf{a}_{1''} \cup \mathbf{a}_2 \cup \mathbf{a}_3$.

For the second statement, about $\mathbf{a}_{1''} \cup \mathbf{a}_3 \cup \mathbf{a}_4$, it suffices to prove that $\mathbf{a}_3$ and $\mathbf{a}_4$ are free. Indeed, this follows from the calculation of $p_F(\mathbf{c}_4)$: edges with labels in either $I_{1''}$ or I_3 are both treated just the same and simply removed. In particular, this implies that the joint distributions $\mu_{\mathbf{a}_{1''}\cup\mathbf{a}_3\cup\mathbf{a}_4}$ and $\mu_{\mathbf{a}_{3''}\cup\mathbf{a}_3\cup\mathbf{a}_4} = \mu_{\mathbf{b}_3\cup\mathbf{a}_4}$ are identical, where $\mathbf{a}_{3''}$ is the limit of the full proportion RBMs $\mathcal{O}_N^{(3'')} = (\mathbf{\Theta}_N^{(i)})_{i\in I_1''}$ and $\mathbf{b}_3 = \mathbf{a}_{3''} \cup \mathbf{a}_3$ is simply the limit of a larger family of independent full proportion RBMs. Now, since the joint distribution $\mu_{\mathbf{a}_3\cup\mathbf{a}_4}$ is universal independent of the parameters β_i, we can calculate $\mu_{\mathbf{a}_3\cup\mathbf{a}_4}$ via a unitarily invariant realization of $\mathcal{O}_N^{(3)}$. The standard techniques then apply to show that $\mathbf{a}_3$ and $\mathbf{a}_4$ are free [Voi91].

Similarly, the joint distributions $\mu_{\mathbf{a}_2\cup\mathbf{a}_4}$ and $\mu_{\mathbf{a}_{1'}\cup\mathbf{a}_4}$ are also identical, so we need only to consider the families $\mathbf{a}_2$ and $\mathbf{a}_4$. Let $a_{i_2} \in \mathbf{a}_2$ and $a_{i_4} \in \mathbf{a}_4$. If a_{i_2} and a_{i_4} were free, then

$$\varphi(a_{i_4}^2 a_{i_2} a_{i_4}^2 a_{i_2}) = \varphi(a_{i_4}^2)^2 \varphi(a_{i_2}^2) = 1;$$

however, one can easily calculate

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}\left((\mathbf{\Theta}_N^{(i_4)})^2 \mathbf{\Theta}_N^{(i_2)} (\mathbf{\Theta}_N^{(i_4)})^2 \mathbf{\Theta}_N^{(i_2)}\right)\right] &= p_T(c_{i_4}) \\ &= \begin{cases} \dfrac{8c_{i_4}^2(\frac{1}{2} - c_{i_4}) + \frac{14}{3}c_{i_4}^3}{(2c_{i_4} - c_{i_4}^2)^2} & \text{if } c_{i_4} \leq \dfrac{1}{2}, \\ \dfrac{2c_{i_4} - 1 + \frac{2}{3}(1 - c_{i_4}^3)}{(2c_{i_4} - c_{i_4}^2)^2} & \text{if } c_{i_4} \geq \dfrac{1}{2} \end{cases} \\ &\neq 1\end{aligned}$$

for $c_{i_4} \in (0, 1)$, where

$$T(\mathbf{\Theta}_N^{(i_4)}, \mathbf{\Theta}_N^{(i_4)}, \mathbf{\Theta}_N^{(i_4)}, \mathbf{\Theta}_N^{(i_4)}) = \cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot \overset{\mathbf{\Theta}_N^{(i_4)}}{\underset{\mathbf{\Theta}_N^{(i_4)}}{\leftrightarrows}} \cdot \,.$$

Finally, suppose that $a_{i_4} \neq a_{j_4} \in \mathbf{a}_4$ with $0 < c_{i_4} \leq c_{j_4} < 1$. If a_{i_4} and a_{j_4} were free, then

$$\varphi(a_{i_4}^2 a_{j_4}^2) = \varphi(a_{i_4}^2)\varphi(a_{j_4}^2) = 1;$$

however, one can again show that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}\left((\mathbf{\Theta}_N^{(i_4)})^2 (\mathbf{\Theta}_N^{(j_4)})^2\right)\right] = p_S(\{c_{i_4}, c_{j_4}\}) \neq 1,$$

where $p_S(\{c_{i_4}, c_{j_4}\})$ is as in the proof of Corollary 4.4.9. ■

Remark 4.4.12. We need the assumption on the band widths $(b_N)_{i\in I_1}$ of the periodic RBMs to handle the interaction with the proper proportional growth RBMs $\mathcal{O}_N^{(4)}$. The families $\mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)}$ converge in joint distribution to a semicircular system regardless, even without this assumption.

Finally, the same considerations that allow us to translate Proposition 4.2.2 to Theorem 4.4.8 also work to prove the RBM version of the concentration inequalities in Theorem 4.2.8. Here, we do not make any assumptions on the band widths $(b_N^{(i)})_{i\in I_1}$ beyond their divergence (4.52), nor on the parameters $\beta_i \in \mathbb{C}$.

Theorem 4.4.13. *Let $\mathcal{Q}_N = \mathcal{P}_N^{(1)} \cup \mathcal{O}_N^{(2)} \cup \mathcal{O}_N^{(3)} \cup \mathcal{O}_N^{(4)}$. For any test graph T in $\mathbf{x}$,*

$$\mathbb{E}\left[\left|\frac{1}{N}\mathrm{tr}\big[T(\mathcal{Q}_N)\big] - \mathbb{E}\frac{1}{N}\mathrm{tr}\big[T(\mathcal{Q}_N)\big]\right|^{2m}\right] = O_T(N^{-m}).$$

The bound is tight in the sense that there exist test graphs T such that

$$\mathbb{E}\left[\left|\frac{1}{N}\mathrm{tr}\big[T(\mathcal{Q}_N)\big] - \mathbb{E}\frac{1}{N}\mathrm{tr}\big[T(\mathcal{Q}_N)\big]\right|^{2m}\right] = \Theta_T(N^{-m}).$$

As before, we can use Theorem 4.4.13 to upgrade the convergence in Theorems 4.4.8 and 4.4.11 to the almost sure sense.

Remark 4.4.14. In the case of a diverging band width $b_N \to \infty$, the bottleneck (4.65) can even compensate for a lack of independence. For example, let $b_N^{(1)}$ be a band width of slow growth or proportional growth, and suppose that $b_N^{(2)} \to \infty$ satisfies $b_N^{(2)} = o(b_N^{(1)})$. For independent Wigner matrices $\mathbf{X}_N^{(1)}$ and $\mathbf{X}_N^{(2)}$ of the same parameter $\beta \in \mathbb{R}$, we form the normalized RBMs

$$\mathbf{\Theta}_N^{(1)} = \mathbf{\Upsilon}_N^{(1)} \circ \mathbf{B}_N^{(1)} \circ \mathbf{X}_N^{(1)}, \quad \mathbf{\Theta}_N^{(2)} = \mathbf{\Upsilon}_N^{(2)} \circ \mathbf{B}_N^{(2)} \circ \mathbf{X}_N^{(2)}, \quad \text{and} \quad \mathbf{\Theta}_N^{(1,2)} = \mathbf{\Upsilon}_N^{(2)} \circ \mathbf{B}_N^{(2)} \circ \mathbf{X}_N^{(1)}.$$

In particular, note that $\mathbf{\Theta}_N^{(1)}$ and $\mathbf{\Theta}_N^{(1,2)}$ are not independent: we use the same Wigner matrix $\mathbf{X}_N^{(1)}$, but with different band widths $b_N^{(1)}$ and $b_N^{(2)}$. Since $b_N^{(2)} = o(b_N^{(1)})$, the band width conditions (4.65) show that a twin edge with mixed labels in $\mathbf{\Theta}_N^{(1)}$ and $\mathbf{\Theta}_N^{(1,2)}$ does not contribute in the limit. Indeed, the minimum of the band widths will be $b_N^{(2)} = o(b_N^{(1)})$, but we have carried the cost of the normalization of the larger band width in $\mathbf{\Theta}_N^{(1)}$. In this case, we cannot have twin edges with mixed labels in $\mathbf{\Theta}_N^{(1)}$ and $\mathbf{\Theta}_N^{(1,2)}$, but this is precisely the limiting condition for the independent RBMs $\mathbf{\Theta}_N^{(1)}$ and $\mathbf{\Theta}_N^{(2)}$. It follows that $(\mathbf{\Theta}_N^{(1)}, \mathbf{\Theta}_N^{(1,2)})$ and $(\mathbf{\Theta}_N^{(1)}, \mathbf{\Theta}_N^{(2)})$ have the same LTD. The heuristic is that *most* of the entries of $\mathbf{\Theta}_N^{(1)}$ are independent from the entries of $\mathbf{\Theta}_N^{(1,2)}$, so the calculation goes through as usual (the nonzero entries of $\mathbf{\Theta}_N^{(1,2)}$ form a vanishingly small proportion of the entries of $\mathbf{\Theta}_N^{(1)}$ since $b_N^{(2)} = o(b_N^{(1)})$).

An almost Gaussian degree matrix

As an application of Theorem 4.4.8, we consider the analogue of the random Markov matrix problem for proportional growth RBMs. In particular, we are interested in the LSD of the degree matrix $\mathbf{D}_N = \mathrm{rDeg}(\mathbf{\Theta}_N)$ of a proportional growth RBM $\mathbf{\Theta}_N$, as well as the joint distribution of $(\mathbf{\Theta}_N, \mathbf{D}_N)$. Here, we find that the free product decomposition of Chapter 3 cannot be extended to the proportional growth regime (in contrast to the periodic regime and the slow growth regime).

For simplicity, we restrict our attention to real Wigner matrices $\mathbf{X}_N$. As before, we form the corresponding proportional growth RBMs, unnormalized $\mathbf{\Xi}_N$ and otherwise $\mathbf{\Theta}_N$. Let $c \in (0,1]$ denote the limiting proportion of the band width b_N, i.e.,

$$\lim_{N\to\infty} \frac{b_N}{N} = c.$$

The entries of the degree matrix $\mathbf{D}_N = \mathrm{rDeg}(\mathbf{\Theta}_N)$ can then be written as

$$\begin{aligned}\mathbf{D}_N(i,j) &= \mathbb{1}\{i=j\} \sum_{k=1}^{N} \mathbf{\Theta}_N(i,k) \\ &= \mathbb{1}\{i=j\} \sum_{k=1}^{N} \frac{\mathbf{\Xi}_N(i,k)}{\sqrt{N}\sqrt{2c-c^2}} = \mathbb{1}\{i=j\} \sum_{k=1}^{N} \frac{\mathbb{1}\{|i-k| \le b_N\}\mathbf{X}_N(i,k)}{\sqrt{N}\sqrt{2c-c^2}}.\end{aligned}$$

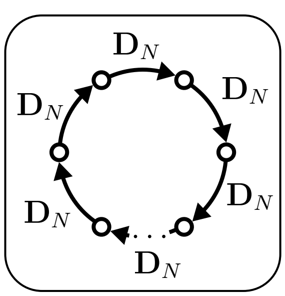
One can then use the asymptotics of partial sums of falling factorials to compute the limits

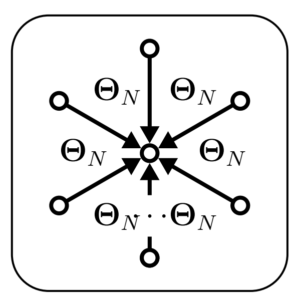
$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{tr}(\mathbf{D}_N^m)\right], \qquad \forall m \in \mathbb{N},$$

for example, by choosing a convenient realization of the random variables $\mathbf{X}_N(i,k)$ and then appealing to the universality of (4.91); however, one can avoid such a tedious calculation and obtain the answer from (4.91) directly. In particular, we can factor the expected moments of the spectral distribution $\mu_{\mathbf{D}_N}$ through the traffic distribution of $\mathbf{\Theta}_N$ via

$$\mathbb{E}\left[\frac{1}{N}\mathrm{tr}(\mathbf{D}_N^m)\right] = \tau\big[C_m(\mathbf{D}_N, \ldots, \mathbf{D}_N)\big] = \tau\big[S_m(\mathbf{\Theta}_N, \ldots, \mathbf{\Theta}_N)\big],$$

where C_m is the directed cycle with m edges and $S_m = (V, E)$ is the inward facing directed m-star graph, i.e.,

$C_m(\mathbf{D}_N, \ldots, \mathbf{D}_N) =$ 

and $S_m(\mathbf{\Theta}_N, \ldots, \mathbf{\Theta}_N) =$ 

.

Here, we have made the substitution

$$\mathbf{D}_N = \underset{\text{in/out}}{\overset{\cdot}{\underset{\cdot}{\downarrow}}}\,\mathbf{\Theta}_N \ .$$

We rewrite this in terms of the injective traffic state to obtain

$$\tau\big[S_m(\mathbf{\Theta}_N,\dots,\mathbf{\Theta}_N)\big] = \sum_{\pi\in\mathcal{P}(V)} \tau^0\big[S_m^\pi(\mathbf{\Theta}_N,\dots,\mathbf{\Theta}_N)\big].$$

In the limit, (4.91) tells us that the only contributions come from double trees. For odd m, this is not possible since a double tree has an even number of edges, while S_m has m edges. This implies that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{D}_N^m)\right] = 0 \quad \text{if } m \text{ is odd.} \tag{4.93}$$

Henceforth, we assume that $m = 2\ell$. Let $v_1,\dots,v_{2\ell}$ denote the leaf vertices of $S_{2\ell}$ with the internal node v_0. We see that

$$S_{2\ell}^\pi \text{ is a double tree} \iff \pi = \{\{v_0\}\}\cup\rho,$$

where ρ is a pair partition of $\{v_1,\dots,v_{2\ell}\}$. In particular, each such π produces the same double tree $T_\ell(\mathbf{\Theta}_N,\dots,\mathbf{\Theta}_N) = S_{2\ell}^\pi(\mathbf{\Theta}_N,\dots,\mathbf{\Theta}_N)$, where T_ℓ is the inward facing double ℓ-star graph. It follows that

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{D}_N^{2\ell})\right] &= \lim_{N\to\infty}\sum_{\pi\in\mathcal{P}(V)} \tau^0\big[S_{2\ell}^\pi(\mathbf{\Theta}_N,\dots,\mathbf{\Theta}_N)\big] \\
&= \#(\mathcal{P}_2(2\ell))p_{T_\ell}(c) = (2\ell-1)!!\frac{\mathrm{Int}_{\mathrm{T}_\ell}(\mathrm{c})}{\mathrm{Norm}_{T_\ell}(c)} \\
&= (2\ell-1)!!\frac{\int_{[0,1]^{\ell+1}}\prod_{k=1}^{\ell}\mathbb{1}\{|x_0-x_k|\le c\}\,dx_\ell\cdots dx_0}{(2c-c^2)^\ell} \\
&= (2\ell-1)!!\frac{\int_0^1\left(\int_0^1 \mathbb{1}\{|x_0-x_1|\le c\}\,dx_1\right)^\ell dx_0}{(2c-c^2)^\ell} \\
&= (2\ell-1)!!\frac{\frac{2}{\ell+1}((2c\wedge 1)^{\ell+1}-c^{\ell+1}) + |2c-1|(2c\wedge 1)^\ell}{(2c-c^2)^\ell},
\end{aligned} \tag{4.94}$$

where we have made use of (4.77) in the last equality. We recognize the double factorial $(2\ell-1)!!$ as the 2ℓ-th moment of the standard normal distribution. In view of Theorem 4.4.13, the limits (4.93) and (4.94) then show that $\mu_{\mathbf{D}_N}$ converges weakly almost surely to a symmetric distribution ν_c of unit variance with *almost* Gaussian moments (if $c = 1$, then these moments are precisely Gaussian). In particular, we can compute the limits

$$\lim_{c\to 0^+}\frac{\frac{2}{\ell+1}((2c\wedge 1)^{\ell+1}-c^{\ell+1}) + |2c-1|(2c\wedge 1)^\ell}{(2c-c^2)^\ell} = 1, \qquad \forall \ell\in\mathbb{N},$$

and

$$\lim_{c\to 1^-} \frac{\frac{2}{\ell+1}((2c\wedge 1)^{\ell+1}-c^{\ell+1})+|2c-1|(2c\wedge 1)^{\ell}}{(2c-c^2)^{\ell}} = 1, \qquad \forall \ell \in \mathbb{N},$$

both of which are special cases of (4.83). The moments (4.94) further imply that the limiting spectral distribution ν_c has unbounded support.

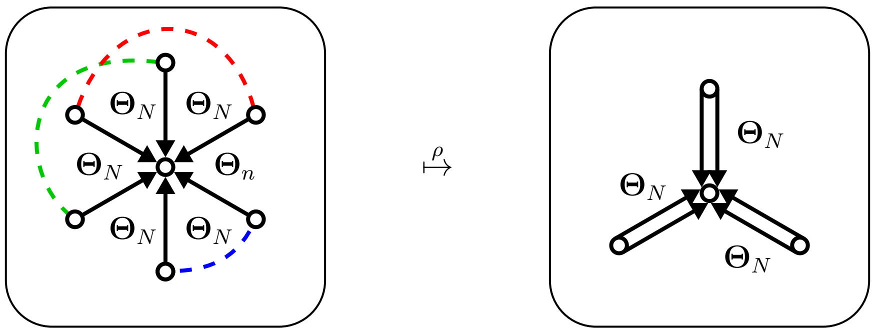


Figure 4.15: An example of a pair partition ρ of the leaf vertices of $S_{2\ell}$ giving rise to an inward facing double ℓ-star graph T_ℓ for $\ell = 3$. Here, we use different colors for the different blocks of the pair partition. Note that any pair partition of the leaf vertices gives rise to the same double tree T_ℓ.

We note that $\mathbf{\Theta}_N$ and $\mathbf{D}_N$ are asymptotically free iff $c = 1$. Indeed, this follows from the calculation

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{\Theta}_N^2\mathbf{D}_N^2)\right] &= \lim_{N\to\infty} \tau^0\left[\ \cdot \underset{\mathbf{\Theta}_N}{\overset{\mathbf{\Theta}_N}{\leftrightarrows}} \cdot \underset{\mathbf{\Theta}_N}{\overset{\mathbf{\Theta}_N}{\leftleftarrows}} \cdot\ \right] \\
&= \frac{2((2c\wedge 1)^3 - c^3) - 3|2c-1|(2c\wedge 1)^2}{(2c-c^2)^2} \\
&\neq 1 = \left(\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{\Theta}_N^2)\right]\right)\left(\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{D}_N^2)\right]\right)
\end{aligned} \tag{4.95}$$

unless $c = 1$. In this case, we see that the free product decomposition of Chapter 3 cannot be extended to the proper proportional growth regime.

Fixed band width

We have much less to say in the fixed band width regime. For starters, we cannot work in the generality of the Wigner matrices of Definition 4.1.1. Instead, we must further assume that the off-diagonal entries (resp., the diagonal entries) of $\mathbf{X}_N$ are identically distributed, independent of N; otherwise, in general, the LSD of even a single fixed band width RBM $\mathbf{\Theta}_N = \mathbf{\Upsilon}_N \circ \mathbf{\Xi}_N = \mathbf{\Upsilon}_N \circ (\mathbf{B}_N \circ \mathbf{X}_N)$ might not exist, never mind the LTD. We assume hereafter that any fixed band width RBM arises from this restricted setting.

Assuming a symmetric distribution for the entries of $\mathbf{X}_N$, Section 6 in [BMP91] proves the existence of a symmetric non-universal LSD μ_b for a real symmetric RBM $\mathbf{\Theta}_N$ of fixed band width $b_N \equiv b$. The authors further prove that the distribution μ_b converges weakly to the standard semicircle distribution $\mu_{\mathcal{SC}}$ in the limit $b \to \infty$. We consider the joint LTD of independent fixed band width RBMs (real and complex) without this symmetry assumption and prove the analogous convergence to the semicircular traffic distribution in the large band width limit.

To formalize our result, we consider a class of fixed band widths $\mathbf{b} = (b_N^{(i)})_{i\in I} = (b_i)_{i\in I}$. We form the corresponding family of fixed band width RBMs

$$\mathcal{J}_N = (\mathbf{\Xi}_N^{(i)})_{i\in I} = (\mathbf{B}_N^{(i)} \circ \mathbf{X}_N^{(i)})_{i\in I}, \qquad \mathcal{O}_N = (\mathbf{\Theta}_N^{(i)})_{i\in I} = (\mathbf{\Upsilon}_N^{(i)} \circ \mathbf{\Xi}_N^{(i)})_{i\in I}.$$

We write μ_i (resp., ν_i) for the distribution of the strictly upper triangular entries $\mathbf{X}_N^{(i)}(j,k)$ (resp., the diagonal entries $\mathbf{X}_N^{(i)}(j,j)$) so that

$$\mu_i = \mathcal{L}(\mathbf{X}_N^{(i)}(j,k)) \quad \text{and} \quad \nu_i = \mathcal{L}(\mathbf{X}_N^{(i)}(j,j)), \qquad \forall j < k.$$

In contrast to the previous sections, our fixed normalizations $\mathbf{\Upsilon}_N^{(i)} = (2b_i+1)^{-1/2}\mathbf{J}_N$ force us to also consider non-tree-like test graphs T in the limit $N \to \infty$.

Theorem 4.4.15. *The family of fixed band width RBMs $\mathcal{O}_N$ converges in traffic distribution; moreover, for any test graph $T = (V, E, \gamma)$ in $\mathbf{x}$, we have the bound*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{O}_N)\big] = O_{T,\boldsymbol{\mu},\boldsymbol{\nu}}\bigg(\frac{\prod_{[e]\in\widetilde{\mathcal{N}}_0} \min_{e'\in[e]} 2b_{\gamma(e')}}{\prod_{e\in E}\sqrt{2b_{\gamma(e)}+1}}\bigg), \tag{4.96}$$

where $(V, \widetilde{\mathcal{N}}_0)$ is any spanning tree of $(V, \widetilde{\mathcal{N}})$ and

$$\boldsymbol{\mu} = (\mu_i)_{i\in I}, \qquad \boldsymbol{\nu} = (\nu_i)_{i\in I}.$$

Proof. We have the familiar expansion

$$\tau^0\big[T(\mathcal{O}_N)\big] = \frac{1}{N\prod_{e\in\mathcal{N}}\sqrt{2b_{\gamma(e)}+1}} \sum_{\phi: V\hookrightarrow[N]} \mathbb{E}\Big[\prod_{e\in E} \mathbf{\Xi}_N^{(\gamma(e))}(\phi(e))\Big], \tag{4.97}$$

where the sum can be written as

$$\sum_{\phi:V\hookrightarrow[N]}\bigg(\prod_{[\ell]\in\widetilde{L}}\mathbb{E}\Big[\prod_{\ell'\in[\ell]}\mathbf{X}_N^{(\gamma(\ell'))}(\phi(\ell'))\Big]\bigg)\bigg(\prod_{[e]\in\widetilde{\mathcal{N}}}\mathbb{1}\{|\phi([e])| \le \min_{e'\in[e]} b_{\gamma(e')}\}\mathbb{E}\Big[\prod_{e'\in[e]}\mathbf{X}_N^{(\gamma(e'))}(\phi(e'))\Big]\bigg).$$

Note that an injective map $\phi : V \hookrightarrow [N]$ satisfying the band width condition

$$|\phi([e])| \leq \min_{e' \in [e]} b_{\gamma(e')}, \qquad \forall [e] \in \widetilde{\mathcal{N}}$$

might not exist (e.g., if $\mathcal{O}_N$ consists of a single RBM $\mathbf{\Theta}_N$ of fixed band width b and T is a star graph S_k with $k > 2b$); however, we can certainly bound the number of such maps by

$$N \prod_{[e] \in \widetilde{\mathcal{N}}_0} \min_{e' \in [e]} 2b_{\gamma(e')},$$

where $(V, \widetilde{\mathcal{N}}_0)$ is any spanning tree of $(V, \widetilde{\mathcal{N}})$. Here, we are simply recycling the bound (4.59). Our strong moment assumption (4.1) then already proves (4.96).

As before, we see that $\tau^0\big[T(\mathcal{O}_N)\big]$ vanishes unless

$$m_{i,[e]} = 0 \text{ or } m_{i,[e]} \geq 2, \qquad \forall (i,[e]) \in I \times \widetilde{\mathcal{N}}.$$

Unfortunately, our fixed normalizations $\sqrt{2b_i + 1}$ allow $\tau^0\big[T(\mathcal{O}_N)\big]$ to survive in the limit for test graphs T with $m_{i,[e]} > 2$. In this case, the assumption that $\beta_i \in \mathbb{R}$ no longer suffices to spare us the consideration of the ordering $\psi_\phi : [\#(V)] \xrightarrow{\sim} V$ on the vertices. Nevertheless, our i.i.d. assumption ensures that if $\phi_1 : V \hookrightarrow [N_1]$ and $\phi_2 : V \hookrightarrow [N_2]$ satisfy the band width condition and induce the same ordering $\psi_{\phi_1} = \psi_{\phi_2}$, then the corresponding summands of (4.97) are equal, i.e.,

$$S_{\phi_1}(T) = \mathbb{E}\bigg[\prod_{e \in E} \mathbf{\Xi}_{N_1}^{(\gamma(e))}(\phi_1(e))\bigg] = \mathbb{E}\bigg[\prod_{e \in E} \mathbf{\Xi}_{N_2}^{(\gamma(e))}(\phi_2(e))\bigg] = S_{\phi_2}(T).$$

For an ordering $\psi : [\#(V)] \xrightarrow{\sim} V$, we can again write S_ψ for the common value of

$$\{S_\phi : \psi_\phi = \psi \text{ and } |\phi([e])| \leq \min_{e' \in [e]} b_{\gamma(e')} \text{ for all } [e] \in \widetilde{\mathcal{N}}\}.$$

This allows us to rewrite (4.97) as

$$\tau^0\big[T(\mathcal{O}_N)\big] = \sum_{\psi : [\#(V)] \xrightarrow{\sim} V} \frac{p_N^{(\psi)}}{\prod_{e \in E} \sqrt{2b_{\gamma(e)} + 1}} S_\psi(T) = \sum_{\psi : [\#(V)] \xrightarrow{\sim} V} q_N^{(\psi)} S_\psi(T),$$

where

$$p_N^{(\psi)} = \frac{\sum_{\phi : V \hookrightarrow [N]} \Big(\mathbb{1}\{\psi_\phi = \psi\} \prod_{[e] \in \widetilde{\mathcal{N}}} \mathbb{1}\{|\phi([e])| \leq \min_{e' \in [e]} b_{\gamma(e')}\}\Big)}{N}.$$

We note the contrast to the situation in (4.17). In particular, we cannot use the same weak convergence argument to give an integral representation of $\lim_{N \to \infty} p_N^{(\psi)}$ as in (4.18) due to the vanishing scales $\lim_{N \to \infty} \frac{b_i}{N} = 0$. Instead, we must opt for a discrete approach.

Let $(a_N^{(\psi)})$ denote the sequence defined by the numerator of $p_N^{(\psi)}$ so that

$$a_N^{(\psi)} = \sum_{\phi:V\hookrightarrow[N]} \Big(\mathbb{1}\{\psi_\phi = \psi\} \prod_{[e]\in\widetilde{\mathcal{N}}} \mathbb{1}\{|\phi([e])| \leq \min_{e'\in[e]} b_{\gamma(e')}\} \Big).$$

By considering a map $\phi_1 : V \hookrightarrow [N]$ (resp., $\phi_2 : V \hookrightarrow [M]$) as a map $\Phi_1 : V \hookrightarrow [N+M]$ (resp., $\Phi_2 : V \hookrightarrow [N+M]$) of the form

$$\Phi_1(v) = \phi_1(v) \qquad (\text{resp., } \Phi_2(v) = \phi_2(v) + N),$$

we see that the sequence $(a_N^{(\psi)})$ is superadditive:

$$a_{N+M}^{(\psi)} \geq a_N^{(\psi)} + a_M^{(\psi)}.$$

Fekete's lemma then implies that

$$p_\psi = \lim_{N\to\infty} p_N^{(\psi)} = \sup_N \frac{a_N^{(\psi)}}{N} \leq \prod_{[e]\in\widetilde{\mathcal{N}}} \min_{e'\in[e]} 2b_{\gamma(e')},$$

which proves the convergence

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{O}_N)\big] = \sum_{\psi:[\#(V)]\tilde{\to}V} \frac{p_\psi}{\prod_{e\in E}\sqrt{2b_{\gamma(e)}+1}} S_\psi(T) = \sum_{\psi:[\#(V)]\tilde{\to}V} q_\psi S_\psi(T). \tag{4.98}$$

■

Note that our bound (4.96) implies the convergence

$$\lim_{\underline{b}\to\infty} \sum_{\psi:[\#(V)]\tilde{\to}V} q_\psi S_\psi(T) = \begin{cases} \prod_{i\in I} \beta_i^{c_i(T)} & \text{if } T \text{ is a colored double tree,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.99}$$

where

$$\underline{b} = \min_{e\in E} b_{\gamma(e)}.$$

Theorem 4.4.15 still holds for general $\beta_i \in \mathbb{C}$: in fact, since we already keep track of the orderings ψ, the same proof goes through just as well (except with different values for $S_\psi(T)$). In this case, the limit (4.99) might not exist depending on the relative rates of growth in the band widths b_i. If we assume that the band widths grow at the same rate in the limit $\underline{b} \to \infty$, then the proportions $q_N^{(\psi)}$ will tend to $\frac{1}{\#(V)}$ as in (4.19), but one can skew these proportions along different subsequences to create an obstruction. One can also periodize the fixed band width RBMs without affecting the calculations (a fixed band width is in some sense the slowest growth possible, and so we can adapt the techniques from the slow growth case).

At this point, we can combine everything into a result for the joint (traffic) distribution of periodic RBMs, slow growth RBMs, proportional growth RBMs, and fixed band width RBMs; however, the result is not much more interesting than what is already known from the previous section due to the form of the LTD (4.98). In particular, we do not have any interesting asymptotic independences arising between the fixed band width RBMs and those of the previously considered regimes, nor amongst the fixed band width RBMs themselves (except in the trivial case $b_i = 0$ of the diagonal matrices, which are permutation invariant and satisfy the conditions of Theorem 2.3.10).

Banded Ginibre matrices

The same analysis applies equally well to prove the analogue of Theorem 4.4.8 for banded Ginibre matrices. In particular, let $\mathcal{Y}_N = (\mathbf{Y}_N^{(i)})_{i\in I}$ be a family of unnormalized Ginibre matrices as in Definition 4.1.4. For a family of band widths $(b_N^{(i)})_{i\in I}$ such that

$$\begin{aligned} \lim_{N\to\infty} b_N^{(i)} &= \infty, && \forall i \in I,\\ \lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= 0, && \forall i \in I_1,\\ \lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= 1, && \forall i \in I_2,\\ \lim_{N\to\infty} \frac{b_N^{(i)}}{N} &= c_i \in (0,1), && \forall i \in I_3, \end{aligned}$$

we form the normalized banded Ginibre matrices

$$\mathcal{H}_N = (\mathbf{H}_N^{(i)})_{i\in I} = (\mathbf{\Upsilon}_N^{(i)} \circ \mathbf{B}_N^{(i)} \circ \mathbf{Y}_N^{(i)})_{i\in I},$$

where $I = I_1 \cup I_2 \cup I_3$ and we use the same normalizations as in Definition 4.1.8.

Theorem 4.4.16. *For any $*$-test graph $T \in \mathcal{T}\langle \mathbf{x}, \mathbf{x}^* \rangle$,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathcal{H}_N)\big] = \begin{cases} p_F(\mathbf{c}) \prod_{i\in I} \zeta_i^{c_i(T)} \overline{\zeta}_i^{s_i(T)} & \text{if } T \text{ is a colored Ginibre double tree,}\\ 0 & \text{otherwise,} \end{cases} \tag{4.100}$$

where $F = T_1 \sqcup \cdots \sqcup T_s$ is the forest of colored Ginibre double trees obtained from T by contracting the edges with labels in I_1 and removing the edges with labels in I_2 and

$$p_F(\mathbf{c}) = \prod_{r=1}^{s} p_{T_r}(\mathbf{c}). \tag{4.101}$$

Proof. The result follows from a straightforward modification of the proof of Theorem 4.4.8. ■

Here, we use the notation $\mathbf{c} = (c_i)_{i\in I_3}$ with the same interpretation for $p_F(\mathbf{c})$ as before. In particular, we see that a banded Ginibre matrix of slow growth has the same LTD as a regular Ginibre matrix. Of course, since the matrices $\mathcal{H}_N$ are non-normal, this falls short of establishing the circular law in the slow growth regime. At the same time, it already *disproves* the circular law in the proportional growth regime. In particular, if $\mathbf{H}_N$ is a banded Ginibre matrix of limiting proportion c, then Theorem 4.4.16 tells us that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{tr}(\mathbf{H}_N\mathbf{H}_N^*\mathbf{H}_N\mathbf{H}_N^*)\right] = 2\frac{\frac{2}{3}((2c\wedge 1)^3 - c^3) + |2c-1|(2c\wedge 1)^2}{(2c-c^2)^2}.$$

At the same time, a simple computation shows that

$$2\frac{\frac{2}{3}((2c\wedge 1)^3 - c^3) + |2c-1|(2c\wedge 1)^2}{(2c-c^2)^2} = 2 \quad\Longleftrightarrow\quad c = 1.$$

We can use the LTD (4.100) for banded Wishart-Laguerre matrices to greater effect: for example, it follows that $\mu(\mathbf{H}_N\mathbf{H}_N^*) \xrightarrow{w} \mathcal{MP}(1,1)$ as $N\to\infty$ if $\mathbf{H}_N$ is of slow growth. The convergence of the ESD $\mu(\mathbf{H}_N\mathbf{H}_N^*)$ in this single matrix case is already known from the work [JS17] of Jana and Soshnikov via Stieltjes transform methods; however, our result extends to the full traffic distribution in the multi-matrix case much as in Section 4.3. For example, this proves the analogue of Theorem 4.1.7 for banded Wishart-Laguerre matrices of slow growth. To see this in action, the reader should compare Figure 4.2 and Figure 4.4.

4.5 Haar distributed orthogonal random matrices

Let $\mathbf{O}_N$ denote an $N\times N$ Haar distributed orthogonal random matrix, for which we compute the limiting traffic distribution. Our proof derives from the analogous result for the unitary case [Mal, Proposition 3.7]. We commit the formal details here for comparison. As usual, we restrict our attention to test graphs $T\in\mathcal{T}\langle x\rangle$. The general case of a $*$-test graph $T=(V,E,\gamma,\varepsilon)$ follows from the relation $\mathbf{O}_N^* = \mathbf{O}_N^\intercal$, which allows us to freely interchange any edge e with $*$-label $\varepsilon(e) = *$ with an edge e' with $*$-label $\varepsilon(e') = 1$ in the opposite direction, i.e.,

$$(\operatorname{src}(e),\operatorname{tar}(e)) = (\operatorname{tar}(e'),\operatorname{src}(e')).$$

In this case, we suppress the map γ since there is only one indeterminate x in consideration.

Definition 4.5.1 (Orthogonal cactus)**.** For a test graph $T=(V,E)\in\mathcal{T}\langle x\rangle$, we write $\mathring{T} = (V,\mathring{E})$ for the underlying undirected multigraph. We further write $P: E\to\mathring{E}$ for the canonical projection onto the undirected edge set. As before, we say that T is a *cactus* if each edge $\mathring{e}$ of $\mathring{T}$ belongs to a unique simple cycle $C_{\mathring{e}}$. We further say that T is an *orthogonal cactus* if T is a cactus such that each cycle $C_{\mathring{e}}$ corresponds to an anti-directed cycle $P^{-1}(C_{\mathring{e}})$ in T. By an anti-directed cycle, we mean that $P^{-1}(C_{\mathring{e}}) = (e_1,\ldots,e_k)$ alternates in direction (as opposed to a directed cycle), i.e.,

$$\exists j\in[k]: \operatorname{tar}(e_j) = \operatorname{tar}(e_{j+1}),\ \operatorname{src}(e_{j+1}) = \operatorname{src}(e_{j+2}),\ \operatorname{tar}(e_{j+2}) = \operatorname{tar}(e_{j+3}),\ldots \qquad (4.102)$$

where $e_{k+1} = e_1$, $e_{k+2} = e_2$, and so on.

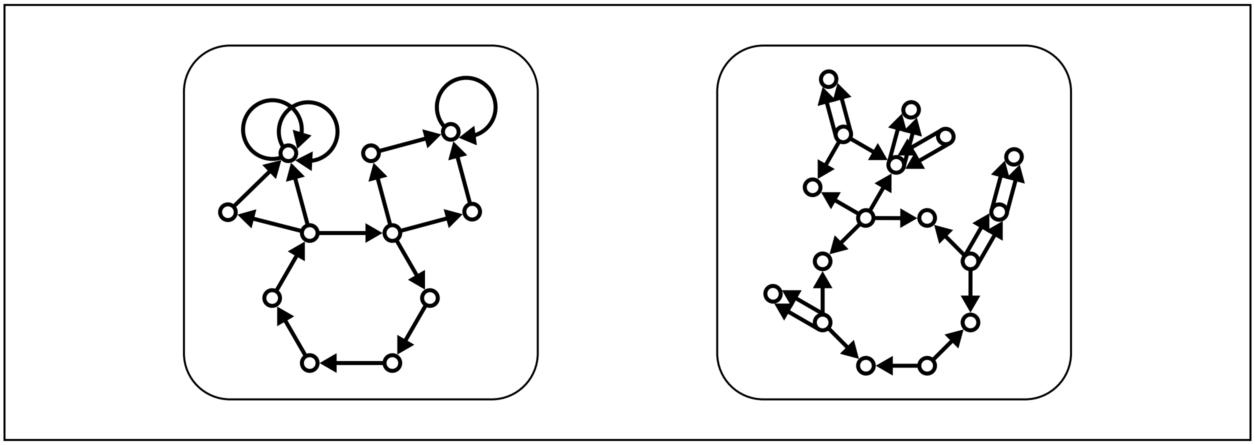


Figure 4.16: For comparison, examples of a cactus and an orthogonal cactus respectively.

For a cactus T, we record the length $\#(C)$ of each of its simple (undirected) cycles C in $\mathring{T}$. By a slight abuse of notation, we also write C for the corresponding pullback $P^{-1}(C)$ in T. For an orthogonal cactus, we know that $\#(C) \in 2\mathbb{N}$ for each such cycle C due to the anti-directedness (4.102)

Recall that we can reconstruct a cactus T from its simple cycles (or "pads") by starting with an arbitrary simple cycle C of T (level 0), reintroducing the simple cycles that share a common vertex with C (level 1), reintroducing the simples cycles that share a common vertex with a simple cycle from level 1 (level 2), and so on. We imagine this process as "growing" the cactus T.

Theorem 4.5.2. *For any test graph T in x,*

$$\lim_{N\to\infty} \tau^0\big[T(\mathbf{O}_N)\big] = \begin{cases} \prod_{C\in \mathrm{Pads}(T)} (-1)^{\frac{\#(C)}{2}-1} c_{\frac{\#(C)}{2}-1} & \text{if } T \text{ is an orthogonal cactus,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.103}$$

where the product is over the pads $\mathrm{Pads}(T)$ *of T and $c_k = \frac{\binom{2k}{k}}{k+1}$ is the kth Catalan number.*

Proof. We start with the usual expansion of the injective trace

$$\begin{aligned}\tau^0\big[T(\mathbf{O}_N)\big] &= \frac{1}{N}\sum_{\phi:V\hookrightarrow[N]} \mathbb{E}\Big[\prod_{e\in E}\mathbf{O}_N(\phi(e))\Big] \\ &= \frac{1}{N}\sum_{\phi:V\hookrightarrow[N]} \mathbb{E}\Big[\prod_{(v,w)\in E}\mathbf{O}_N(\phi(w),\phi(v))\Big],\end{aligned}$$

where we now consider E as a multiset to do away with the source and target functions. In particular, $\mathrm{src}((v,w)) = v$ and $\mathrm{tar}((v,w)) = w$. Note that the distributional invariance of $\mathbf{O}_N$ under conjugation by the permutation matrices implies that the value of a summand

$$S_\phi(T) = \mathbb{E}\Big[\prod_{(v,w)\in E} \mathbf{O}_N(\phi(w),\phi(v))\Big] = \mathbb{E}\Big[\prod_{\ell=1}^{\#(E)} \mathbf{O}_N(\phi(w_\ell),\phi(v_\ell))\Big]$$

does not depend on the particular choice of labeling $\phi : V \hookrightarrow [N]$ of the vertices. In this case, we can fix a labeling $\phi_0 : V \hookrightarrow [N]$ for all large N (for example, by enumerating the vertices $V = (u_r)_{r=1}^s$ and defining $\phi_0(u_r) = r$) to obtain

$$\begin{aligned}\tau^0\big[T(\mathbf{O}_N)\big] &= \frac{N^{\#(V)}}{N}\mathbb{E}\Big[\prod_{\ell=1}^{\#(E)} \mathbf{O}_N(\phi_0(w_\ell),\phi_0(v_\ell))\Big] \\ &\sim N^{\#(V)-1}\mathbb{E}\Big[\mathbf{O}_N(i_1,j_1)\cdots\mathbf{O}_N(i_m,j_m)\Big], \end{aligned} \tag{4.104}$$

where $(i_\ell, j_\ell) = (\phi_0(w_\ell),\phi_0(v_\ell))$ and $m = \#(E)$. The string $\mathbf{i} = (i_1,\dots,i_m)$ defines a partition $\ker(\mathbf{i})$ of $[m]$ by

$$\ker(\mathbf{i}) = \{\{\ell' : i_\ell = i_{\ell'}\} : \ell \in [m]\},$$

and similarly for $\mathbf{j} = (j_1,\dots,j_m)$. The orthogonal Weingarten calculus (in the form of [CŚ06, Corollary 3.4]) tells us that the expectation in (4.104) equals 0 if m is odd; otherwise, $m = 2k$ and

$$\mathbb{E}\Big[\mathbf{O}_N(i_1,j_1)\cdots\mathbf{O}_N(i_{2k},j_{2k})\Big] = \sum_{p_1,p_2\in\mathcal{P}_2(2k)} \delta_{\mathbf{i}}(p_1)\delta_{\mathbf{j}}(p_2)\langle p_1, \mathrm{Wg}_N(p_2)\rangle, \tag{4.105}$$

where $\mathcal{P}_2(2k)$ is the set of pair partitions of $[2k]$, Wg_N is the $N\times N$ orthogonal Weingarten function, and

$$\delta_{\mathbf{k}}(p) = \begin{cases} 1 & \text{if } p \le \ker(\mathbf{k}), \\ 0 & \text{otherwise.}\end{cases}$$

Here, we use the usual reversed refinement order $\le$ on the set of partitions $\mathcal{P}(2k)$.

Of course, the injectivity of the map ϕ_0 implies that

$$\begin{aligned} i_\ell = i_{\ell'} \quad &\iff \quad w_\ell = w_{\ell'}, \\ j_\ell = j_{\ell'} \quad &\iff \quad v_\ell = v_{\ell'}. \end{aligned}$$

We use this correspondence to interpret a pair partition

$$p_1 = \{\{a_\ell, b_\ell\} : \ell\in[k]\} \in \mathcal{P}_2(2k) \qquad (\text{resp., } p_2 = \{\{\alpha_\ell,\beta_\ell\} : \ell\in[k]\}\in\mathcal{P}_2(2k))$$

such that $\delta_{\mathbf{i}}(p_1) = 1$ (resp., $\delta_{\mathbf{j}}(p_2) = 1$) as a pair partition

$$\pi_1 = \{\{(v_{a_\ell}, w_{a_\ell}), (v_{b_\ell}, w_{b_\ell})\} : \ell\in[k]\} \qquad (\text{resp., } \pi_2 = \{\{(\nu_{\alpha_\ell},\omega_{\alpha_\ell}),(\nu_{\beta_\ell},\omega_{\beta_\ell})\} : \ell\in[k]\})$$

of the edges E such that the two edges

$$(v_{a_\ell}, w_{a_\ell}) \text{ and } (v_{b_\ell}, w_{b_\ell}) \qquad (\text{resp., } (\nu_{\alpha_\ell}, \omega_{\alpha_\ell}) \text{ and } (\nu_{\beta_\ell}, \omega_{\beta_\ell}))$$

in any block of the partition have a common target $w_{a_\ell} = w_{b_\ell}$ (resp., a common source $\nu_{\alpha_\ell} = \nu_{\beta_\ell}$). We further interpret the pair partition π_1 as a permutation of the edges E by considering each block $\{(v_{a_\ell}, w_{a_\ell}), (v_{b_\ell}, w_{b_\ell})\}$ as a transposition $((v_{a_\ell}, w_{a_\ell})\ (v_{b_\ell}, w_{b_\ell}))$. In this case, π_1 corresponds to a product of disjoint transpositions

$$\pi_1 = \prod_{\ell=1}^{k} ((v_{a_\ell}, w_{a_\ell})\ (v_{b_\ell}, w_{b_\ell})),$$

and similarly for

$$\pi_2 = \prod_{\ell=1}^{k} ((\nu_{\alpha_\ell}, \omega_{\alpha_\ell})\ (\nu_{\beta_\ell}, \omega_{\beta_\ell})).$$

A pair (p_1, p_2) such that $\delta_{\mathbf{i}}(p_1) = \delta_{\mathbf{j}}(p_2) = 1$ then partitions the edges of T into anti-directed cycles

$$\mathfrak{C}(\pi_1, \pi_2) = \{(e, \pi_2(e), \pi_1\pi_2(e), \pi_2\pi_1\pi_2(e), \ldots) : e \in E\}, \tag{4.106}$$

where we of course assume that cycles are only defined up to a cyclic ordering of the edges. We note that a cycle $C \in \mathfrak{C}(\pi_1, \pi_2)$ need not be simple.

As a sanity check, one can verify the following equivalent construction of $\mathfrak{C}(\pi_1, \pi_2)$. We consider a partition $p \in \mathcal{P}(2k)$ as an element of the symmetric group $\mathfrak{S}_{2k}$ by associating a block $b = \{\ell_1 < \cdots < \ell_{q(b)}\}$ with the cycle $(\ell_1\ \cdots\ \ell_{q(b)})$. A pair (p_1, p_2) as before then partitions the edges of T into anti-directed cycles

$$\begin{aligned}\mathfrak{C}(\pi_1, \pi_2) = \{((v_\ell, w_\ell), &(v_{p_2(\ell)}, w_{p_2(\ell)}), \\ &(v_{p_1p_2(\ell)}, w_{p_1p_2(\ell)}), (v_{p_2p_1p_2(\ell)}, w_{p_2p_1p_2(\ell)}), \ldots) : \ell \in [2k]\}.\end{aligned} \tag{4.107}$$

Note that the cycle decomposition of the permutation $p_1p_2 \in \mathfrak{S}_{2k}$ further splits each cycle C in (4.107) into a pair

$$(w_\ell, w_{p_1p_2(\ell)}, w_{(p_1p_2)^2(\ell)}, \ldots) \quad \text{and} \quad (v_{p_2(\ell)}, v_{p_2p_1(p_2(\ell))}, v_{(p_2p_1)^2(p_2(\ell))}, \ldots).$$

In terms of (4.106), this corresponds to the cycle decomposition of the permutation $\pi_1\pi_2$ of the edges, namely,

$$(e, \pi_1\pi_2(e), (\pi_1\pi_2)^2(e), \ldots) \quad \text{and} \quad (\pi_2(e), (\pi_2\pi_1)\pi_2(e), (\pi_2\pi_1)^2\pi_2(e), \ldots).$$

This implies that

$$\frac{\#(p_1p_2)}{2} = \frac{\#(\pi_1\pi_2)}{2} = \#(\mathfrak{C}(\pi_1, \pi_2)), \tag{4.108}$$

where $\#(p_1p_2)$ denotes the number of cycles of p_1p_2. We assume hereafter that the partitions p_1 and p_2 satisfy $\delta_{\mathbf{i}}(p_1) = \delta_{\mathbf{j}}(p_2) = 1$.

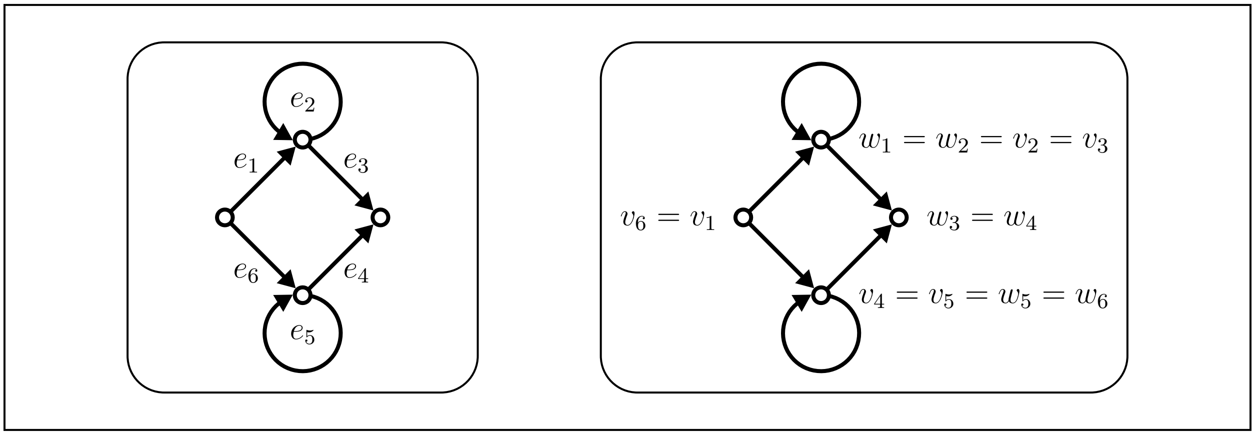


Figure 4.17: An example of the construction of $\mathfrak{C}(\pi_1, \pi_2)$. Here, we start with a test graph T that itself is already a (non-simple) anti-directed cycle $C = (e_1, \ldots, e_6)$, where $e_\ell = (v_\ell, w_\ell)$. Any injective labeling $(i_\ell, j_\ell) = (\phi(w_\ell), \phi(v_\ell))$ of the vertices then generates the partitions $\ker(\mathbf{i}) = \{\{1,2\},\{3,4\},\{5,6\}\}$ and $\ker(\mathbf{j}) = \{\{1,6\},\{2,3\},\{4,5\}\}$. In this case, there is a unique pair partition $p_1 \leq \ker(\mathbf{i})$, namely $p_1 = \ker(\mathbf{i})$, and similarly for $p_2 \leq \ker(\mathbf{j})$. One can then easily verify the corresponding permutation of the edges $\pi_1 = (e_1\ e_2)(e_3\ e_4)(e_5\ e_6)$ (resp., $\pi_2 = (e_6\ e_1)(e_2\ e_3)(e_4\ e_5)$), from which it follows that $\mathfrak{C}(\pi_1, \pi_2) = \{C\}$.

Strictly speaking, we should consider a pair partition $p \in \mathcal{P}_2(2k)$ as a basis element of the Brauer algebra (see, e.g., [HR05]); however, we will only need the very basics of this structure. In particular, we consider a partition p as a graph on $2k$ vertices. We arrange the vertices into two evenly distributed rows, the first of which we consider as given by $1, 2, \ldots, k$; the second by $k+1, k+2, \ldots, 2k$. We then connect the vertices in a given block of p with a line. In this way, we obtain a graph with k connected components, each of size two. For two pair partitions $p_1, p_2 \in \mathcal{P}_2(2k)$, we define $p_1 \circ p_2$ as the graph obtained by overlaying the two graphs corresponding to p_1 and p_2 respectively, which we can again interpret as a partition $p_1 \circ p_2 \in \mathcal{P}(2k)$. The correspondence (4.106) and (4.107) between the pairs (p_1, p_2) and (π_1, π_2) pushes forward to a correspondence between the blocks of $p_1 \circ p_2$ and the anti-directed cycles $\mathfrak{C}(\pi_1, \pi_2)$. In particular, we have a cardinality-preserving bijection

$$\text{blocks}(p_1 \circ p_2) \cong \mathfrak{C}(\pi_1, \pi_2), \qquad b \mapsto C_b, \tag{4.109}$$

where $\#(b) = \#(C_b)$. Indeed, we construct this bijection as follows. For the partition p_1 (resp., p_2), we imagine the vertices $\ell \in [2k]$ in its graph as the vertices $w_\ell \in V$ (resp., $v_\ell \in V$). In this way, a block b of $p_1 \circ p_2$ then naturally corresponds to a cycle $C \in \mathfrak{C}(\pi_1, \pi_2)$ in the form of (4.107). We encourage the reader to work out a special case of this correspondence for the example in Figure 4.17, where $\#(\text{blocks}(p_1 \circ p_2)) = \#(\mathfrak{C}(\pi_1, \pi_2)) = 1$.

Finally, we need to understand the asymptotics of the Weingarten term $\langle p_1, \mathrm{Wg}_N(p_2)\rangle$ in

(4.105). Theorem 3.13 in [CŚ06] shows that

$$\langle p_1, \mathrm{Wg}_N(p_2)\rangle = \left(N^{-2k+\frac{\#(p_1p_2)}{2}} \prod_{b\in\mathrm{blocks}(p_1\circ p_2)} (-1)^{\frac{\#(b)}{2}-1} c_{\frac{\#(b)}{2}-1}\right) + O(N^{-2k+\frac{\#(p_1p_2)}{2}-1}).$$

We can rewrite this in terms of $\mathfrak{C}(\pi_1,\pi_2)$ grace of (4.108) and (4.109) to obtain the equivalent asymptotic

$$\langle p_1, \mathrm{Wg}_N(p_2)\rangle = \left(N^{-2k+\#(\mathfrak{C}(\pi_1,\pi_2))} \prod_{C\in\mathfrak{C}(\pi_1,\pi_2)} (-1)^{\frac{\#(C)}{2}-1} c_{\frac{\#(C)}{2}-1}\right) + O(N^{-2k+\#(\mathfrak{C}(\pi_1,\pi_2))-1}).$$

At this point, we reintroduce this asymptotic for our matrix integral (4.105) back into the injective trace (4.104). This reduces the problem to computing

$$S_{(\pi_1,\pi_2)} = \lim_{N\to\infty} N^{\#(V)-1-2k+\#(\mathfrak{C}(\pi_1,\pi_2))}\left(\prod_{C\in\mathfrak{C}(\pi_1,\pi_2)} (-1)^{\frac{\#(C)}{2}-1} c_{\frac{\#(C)}{2}-1} + O(N^{-1})\right) \tag{4.110}$$

for a given pair (π_1,π_2) as before. To this end, we introduce the bipartite multigraph $\mathfrak{G} = (\mathfrak{V},\mathfrak{E})$, where $\mathfrak{V} = V \cup \mathfrak{C}(\pi_1,\pi_2)$ is the union of the vertices of our original graph T and the anti-directed cycle partition $\mathfrak{C}(\pi_1,\pi_2)$ of the edges E of T. We draw an edge between a vertex $v\in V$ and a cycle $C\in\mathfrak{C}(\pi_1,\pi_2)$ if v is a vertex in the cycle C, in which case the edge comes with multiplicity equal to the number of occurrences of v in C as an undirected cycle. For example, if C is a simple cycle, then we only draw one edge between v and C.

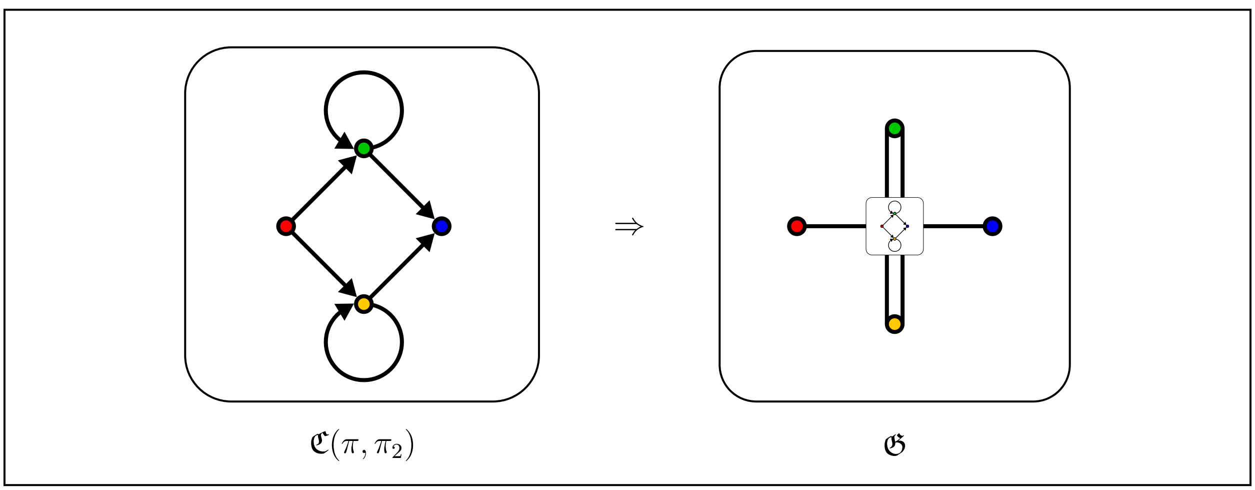


Figure 4.18: An example of the construction of the graph $\mathfrak{G} = (\mathfrak{V},\mathfrak{E})$. Here, we start with the anti-directed cycle $\mathfrak{C}(\pi_1,\pi_2) = \{C\}$ from Figure 4.17. We color the vertices to clarify the construction.

By construction,

$$\#(\mathfrak{V}) = \#(V) + \#(\mathfrak{C}(\pi_1,\pi_2)) \quad \text{and} \quad \#(\mathfrak{E}) = \#(E) = 2k.$$

Moreover, the graph $\mathfrak{G}$ is clearly connected (by virtue of the connectedness of T), whence

$$\#(\mathfrak{V}) \leq \#(\mathfrak{E}) + 1.$$

This allows us to recast (4.110) as

$$\begin{aligned} S_{(\pi_1,\pi_2)} &= \lim_{N\to\infty} N^{\#(\mathfrak{V})-(\#(\mathfrak{E})+1)}\Bigg(\prod_{C\in\mathfrak{C}(\pi_1,\pi_2)} (-1)^{\frac{\#(C)}{2}-1} c_{\frac{\#(C)}{2}-1} + O(N^{-1})\Bigg) \\ &= \mathbb{1}\{\mathfrak{G} \text{ is a tree}\} \prod_{C\in\mathfrak{C}(\pi_1,\pi_2)} (-1)^{\frac{\#(C)}{2}-1} c_{\frac{\#(C)}{2}-1}. \end{aligned}$$

Assume that $\mathfrak{G}$ is a tree. Of course, in this case, $\mathfrak{G}$ cannot have any multi-edges, which implies that each cycle $C \in \mathfrak{C}(\pi_1, \pi_2)$ is simple. In fact, the treeness of $\mathfrak{G}$ implies that T is an orthogonal cactus. Indeed, the tree $\mathfrak{G}$ contains all of the information for how to properly grow the cactus T from the simple anti-directed cycles $\mathfrak{C}(\pi_1, \pi_2)$. We describe this algorithm, as suggested at the beginning of the section. Start with an arbitrary pad $C_0 \in \mathfrak{C}(\pi_1, \pi_2)$ (level 0) and grow (i.e., attach) the pads $C_1 \in \mathfrak{C}(\pi_1, \pi_2)$ at distance two away from C_0 in $\mathfrak{G}$. Note that the pads introduced at level 1 cannot intersect outside of C_0 (this would contradict the treeness of $\mathfrak{G}$). We then introduce the pads $C_2 \in \mathfrak{C}(\pi_1, \pi_2)$ at distance four away from C_0 in $\mathfrak{G}$ (level 2). Each pad at level 2 is only attached to a single pad at level 1 and can only intersect another pad at level 2 in a vertex of a pad C_1. We continue this process until we run out of pads. If we imagine rooting the graph $\mathfrak{G}$ at the vertex C_0 and orienting the rest of the graph upwards, then this process simply amounts to contracting the edges of $\mathfrak{V}$ as we move up.

On the other hand, if T is an orthogonal cactus, then there is a unique pair of pair partitions (p_1, p_2) such that $\delta_{\mathbf{i}}(p_1) = \delta_{\mathbf{j}}(p_2) = 1$ in (4.105). The associated pair of partitions (π_1, π_2) will then correspond precisely to the cycles of T. In this way, we finally arrive at the prescribed limit (4.103). ■

Naturally, one can ask the same question for a family of independent $N \times N$ Haar distributed orthogonal random matrices $(\mathbf{O}_N^{(i)})_{i\in I}$. We can use the same approach to prove the existence of a joint LTD, now supported on *colored orthogonal cacti* (i.e., cacti with anti-directed pads such that each pad is of a uniform color). We leave the details to the interested reader. Instead, we note that the same result can be obtained via Theorem 2.3.10. One need only to prove the factorization property (iv) for $\mathbf{O}_N$, which now follows as in the unitary case [Mal, Proposition 3.7]. In particular, we note that the family $(\mathbf{O}_N^{(i)})_{i\in I}$ is asymptotically traffic independent.

As before, the cactus structure of (4.103) allows us to apply our reasoning from Chapter 3 to prove the asymptotic $*$-freeness of $\mathbf{O}_N$ and $\Theta(\mathbf{O}_N)$. In particular, note that

$$\left(\mathbb{E}\mu\Bigg(\ \overset{\displaystyle\cdot}{\underset{\text{in/out}}{\downarrow}}\,\mathbf{O}_N\Bigg), \mathbb{E}\mu\Bigg(\ \overset{\displaystyle\cdot}{\underset{\text{in/out}}{\uparrow}}\,\mathbf{O}_N\Bigg)\right) \xrightarrow{w} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) \quad \text{as} \quad N \to \infty,$$

where we have applied (4.103) to compute the joint moments in the limit.

Once again, we see that the real case precludes freeness from the transpose $\mathbf{O}_N^\intercal = \mathbf{O}_N^*$, whereas Haar distributed unitary random matrices $\mathbf{U}_N$ are asymptotically $*$-free from the transpose $\mathbf{U}_N^\intercal$ [MP16, Mal, CDM].

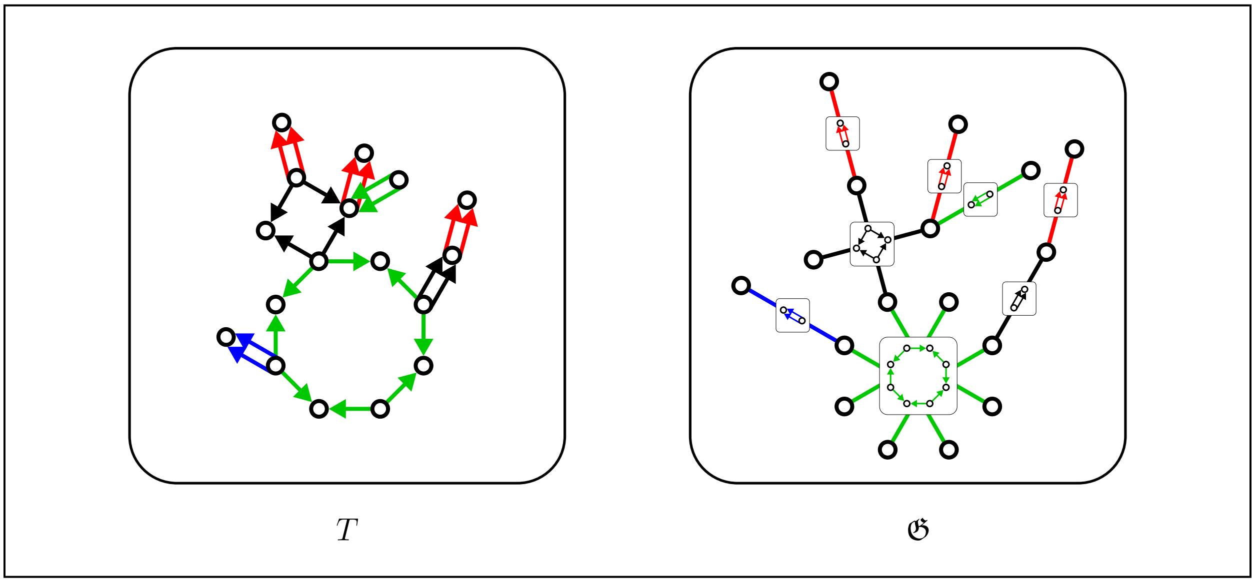


Figure 4.19: An example of the construction of the graph $\mathfrak{G}$ for a colored version T of the orthogonal cactus in Figure 4.16. We color the edges of the tree $\mathfrak{G}$ to clarify the construction.

4.6 The cactus-cumulant correspondence

The reader will no doubt notice that many our formulas for the traffic distribution resemble well-known free cumulant formulas in free probability. For example, if $s \in (\mathcal{A}, \varphi)$ is a standard semicircular random variable, then

$$\kappa_n(s, \dots, s) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.111}$$

At the same time, the LTD (4.8) of the Wigner matrices only allows for *twin* edges, which, in the opposing direction, give a contribution of 1. Moreover, recall that the distribution of a random variable a corresponds to the information of the traffic state on directed cycles with edges labeled by a, and that a double tree constructed from a directed cycle can only have opposing twin edges (see the butterfly obstruction in Figure 3.3). We further note that (4.8) is *multiplicative* with respect to the twin edges of a double tree, where the contribution of opposing twin edges can be seen in (4.8′).

Similarly, a *circular random variable* c is defined as the sum

$$c = \frac{1}{\sqrt{2}}(s_1 + is_2),$$

where s_1 and s_2 are freely independent standard semicircular random variables. In this case,

$$\kappa_n(c^{\varepsilon(1)}, \ldots, c^{\varepsilon(n)}) = \begin{cases} 1 & \text{if } n = 2 \text{ and } \varepsilon(1) \neq \varepsilon(2), \\ 0 & \text{otherwise,} \end{cases} \tag{4.112}$$

where $\varepsilon : [n] \to \{1, *\}$ denotes the $*$-label. Again, we note the resemblance to the LTD (4.49), which only allows for twin edges. Moreover, the only opposing twin edges in a Ginibre double tree are $*$-*opposing*, which give a contribution of 1. As before, we note that (4.49) is multiplicative with respect to the twin edges of a double tree, where the contribution of $*$-opposing twin edges can be seen in (4.49′).

Our last example is that of a *Haar unitary random variable*. Recall that a Haar unitary random variable u in a $*$-probability space $(\mathcal{A}, \varphi)$ is a unitary element such that

$$\varphi(u^m) = 0, \qquad \forall m \in \mathbb{Z}_{\neq 0}.$$

The only nontrivial cumulants of a Haar unitary random variable are those that alternate equally in u and u^*:

$$\kappa_n(u^{\varepsilon(1)}, \ldots, u^{\varepsilon(n)}) \begin{cases} (-1)^{\frac{n}{2}-1} c_{\frac{n}{2}-1} & \text{if } n \in 2\mathbb{N} \text{ and } \varepsilon \text{ alternates,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.113}$$

where we recall that $c_k = \frac{\binom{2k}{k}}{k+1}$ denotes the kth Catalan number. Of course, this closely resembles the LTD (4.103) of a Haar distributed orthogonal random matrix: the pads of an orthogonal cactus must be alternating (and hence of even length). Once again, we note that the formula (4.103) is multiplicative, this time in the pads of the cactus.

The careful reader will anticipate this correspondence based on Proposition 2.4.6. More precisely, Remark 11.19 in [NS06] provides a simple criteria for recognizing the free cumulants of random variables $(a_i)_{i \in I}$. For the convenience of the reader, we quote the result:

> *Assume we are given some complex numbers* $\tilde{\kappa}_\pi[a_{i(1)}, \ldots, a_{i(n)}]$ *for all* $n \in \mathbb{N}$, $\pi \in NC(n)$, $i(1), \ldots, i(n) \in I$ *such that:*
>
> *(i) the* $\tilde{\kappa}_\pi$ *are multiplicative in the sense*
>
> $$\tilde{\kappa}_\pi[a_{i(1)}, \ldots, a_{i(n)}] = \prod_{V \in \pi} \tilde{\kappa}(V)[a_{i(1)}, \ldots, a_{i(n)}],$$
>
> *where, for* $V = (r_1 < \cdots < r_s) \in \pi$, *we use the notation (11.2),*
>
> $$\tilde{\kappa}(V)[a_{i(1)}, \ldots, a_{i(n)}] := \tilde{\kappa}_{1_s}(a_{i(r_1)}, \ldots, a_{i(r_s)});$$

(ii) we can write the moments of $(a_i)_{i\in I}$ *as*

$$\varphi(a_{i(1)}\cdots a_{i(n)}) = \sum_{\pi\in NC(n)} \tilde{\kappa}_\pi[a_{i(1)},\ldots,a_{i(n)}]$$

for all $n\in\mathbb{N}$ *and all* $i(1),\ldots,i(n)\in I$.

Then these $\tilde{\kappa}$ *are the cumulants of* $(a_i)_{i\in I}$*, i.e.,*

$$\kappa_\pi[a_{i(1)},\ldots,a_{i(n)}] = \tilde{\kappa}_\pi[a_{i(1)},\ldots,a_{i(n)}]$$

for all $n\in\mathbb{N}$ *and* $\pi\in NC(n)$.

If the random variables $(a_i)_{i\in I}$ belong to a traffic space $(\mathcal{A},\tau)$, then we can compute the expectation

$$\varphi_\tau(a_{i(1)}\cdots a_{i(n)}) = \varphi_\tau\left(\underset{\text{out}}{\cdot} \xleftarrow{a_{i(1)}} \cdots\cdots \xleftarrow{a_{i(n)}} \underset{\text{in}}{\cdot} \right) = \tau\big[C_n(a_{i(1)},\ldots,a_{i(n)})\big],$$

where $C_n = C_n(a_{i(1)},\ldots,a_{i(n)}) = (V,E,\gamma)$ is a directed cycle of length n in the usual sense: we enumerate the vertices $V = (v_k)_{k=\overline{1}}^{\overline{n}}$ and the edges $E = (e_k)_{k=1}^{n}$, where

$$\mathrm{src}(e_k) = v_{\overline{k+1}}, \quad \mathrm{tar}(e_k) = v_{\overline{k}}, \quad \text{and} \quad \gamma(e_k) = a_{i(k)}.$$

We think of a partition $\pi\in\mathcal{P}(V)\cong\mathcal{P}(\overline{n})$ (and vice versa) as convenient. Similarly, we think of a partition of $\sigma\in\mathcal{P}(E)\cong\mathcal{P}(n)$ (and vice versa) as convenient. For $\pi\in\mathcal{NC}(\overline{n})\cup\mathcal{NC}(n)$, we can define the usual Kreweras complement $K(\pi)\in\mathcal{NC}(n)\cup\mathcal{NC}(\overline{n})$ as before, but we now consider the interlacing

$$\overline{1} < 1 < \cdots < \overline{n} < n$$

for $[\overline{n}+n] = \{\overline{1},1,\ldots,\overline{n},n\}$ in a slight modification of the argument preceding Proposition 2.4.6. In particular, if $\pi\in\mathcal{NC}(V)$ (resp., $\sigma\in\mathcal{NC}(E)$), then $K(\pi)\in\mathcal{NC}(E)$ (resp., $K(\sigma)\in\mathcal{NC}(V)$).

If the injective traffic distribution of the $(a_i)_{i\in I}$ is supported on cacti (not necessarily oriented cacti), then Proposition 2.4.6 tells us that

$$\tau\big[C_n(a_{i(1)},\ldots,a_{i(n)})\big] = \sum_{\pi\in\mathcal{P}(V)} \tau^0\big[C_n^\pi(a_{i(1)},\ldots,a_{i(n)})\big] = \sum_{\pi\in\mathcal{NC}(V)} \tau^0\big[C_n^\pi(a_{i(1)},\ldots,a_{i(n)})\big].$$

If we further assume that the injective traffic distribution of the $(a_i)_{i\in I}$ is multiplicative with respect to the pads of a cacti, then Proposition 2.4.6 further tells us that for $\pi\in\mathcal{NC}(V)$,

$$\tau^0\big[C_n^\pi(a_{i(1)},\ldots,a_{i(n)})\big] = \prod_{B\in K(\pi)} \tau^0\big[C_{\#(B)}(a_{i(j_1)},\ldots,a_{i(j_{\#(B)})})\big],$$

where on the right-hand side we think of π as an element of $\mathcal{NC}(\overline{n})$ so that $K(\pi) \in \mathcal{NC}(n)$ and $B \in K(\pi)$ is a block of the form $B = (j_1 < \cdots < j_{\#(B)})$. In that case, Remark 11.19 in [NS06] allows us to conclude that

$$\kappa_\sigma[a_{i(1)}, \ldots, a_{i(n)}] = \tau^0\big[C_n^{K(\sigma)}(a_{i(1)}, \ldots, a_{i(n)})\big],$$

where on the left-hand side we think of $\sigma \in \mathcal{NC}(n)$ and on the right-hand side we think of $\sigma \in \mathcal{NC}(E)$ so that $K(\sigma) \in \mathcal{NC}(V)$.

The LTDs (4.8), (4.49), and (4.103) then recover all of the well-known cumulant formulas (4.111)-(4.113) from before. At the same time, this approach also reveals new relationships for such cactus-type random variables. For example, let s_β be a β-semicircular traffic in a traffic space $(\mathcal{A}, \tau)$, where $\beta \in \mathbb{R}$ (i.e., s_β has the traffic distribution (4.8)). We can then compute

$$\kappa_\sigma[s_\beta^{\hat{\intercal}(1)}, \ldots, s_\beta^{\hat{\intercal}(n)}] = \tau^0\big[C_n^{K(\sigma)}(s_\beta^{\hat{\intercal}(1)}, \ldots, s_\beta^{\hat{\intercal}(n)})\big], \qquad \forall \sigma \in \mathcal{NC}(n),$$

where $\hat{\intercal} : [n] \to \{1, \intercal\}$ denotes the transpose label. We can rewrite this in terms of a cycle with edge labels only in s_β by reversing the direction of the edges whose transpose label $\hat{\intercal}(i) = \intercal$ (recall our proof of Proposition 2.4.7). If we write $\widetilde{C}$ for this modified cycle, then

$$\kappa_\sigma[s_\beta^{\hat{\intercal}(1)}, \ldots, s_\beta^{\hat{\intercal}(n)}] = \tau^0\big[\widetilde{C}_n^{K(\sigma)}(s_\beta, \ldots, s_\beta)\big].$$

The traffic distribution (4.8) tells us that only cycles of length two, or twin edges, contribute. It follows that

$$\kappa_n(s_\beta^{\hat{\intercal}(1)}, \ldots, s_\beta^{\hat{\intercal}(n)}) = \begin{cases} 1 & \text{if } n = 2 \text{ and } \hat{\intercal}(1) = \hat{\intercal}(2), \\ \beta & \text{if } n = 2 \text{ and } \hat{\intercal}(1) \neq \hat{\intercal}(2), \\ 0 & \text{otherwise.} \end{cases} \tag{4.114}$$

We invite the reader to compare this to the usual free cumulant formula (4.111) for a standard semicircular random variable. The β-semicircular cumulant formula implies that $(s_\beta, s_\beta^\intercal)$ is a semicircular family of covariance $\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$, which generalizes the transpose relation for Wigner matrices of a real parameter $\beta \in \mathbb{R}$ (and proves Corollary 4.1.12).

The same idea allows us to compute the free cumulants of a ζ-circular traffic c_ζ and its transpose $c_\zeta^\intercal$ for general $\zeta \in \mathbb{C}$. In particular,

$$\kappa_n((c_\zeta^{\hat{\intercal}(1)})^{\varepsilon(1)}, \ldots, (c_\zeta^{\hat{\intercal}(n)})^{\varepsilon(n)}) = \begin{cases} 1 & \text{if } n = 2,\ \hat{\intercal}(1) = \hat{\intercal}(2), \text{ and } \varepsilon(1) \neq \varepsilon(2), \\ \zeta & \text{if } n = 2,\ \hat{\intercal}(1) \neq \hat{\intercal}(2), \text{ and } \varepsilon(1) = \varepsilon(2) = 1, \\ \overline{\zeta} & \text{if } n = 2,\ \hat{\intercal}(1) \neq \hat{\intercal}(2), \text{ and } \varepsilon(1) = \varepsilon(2) = *, \\ 0 & \text{otherwise.} \end{cases} \tag{4.115}$$

Similarly, this implies that $(c_\zeta, c_\zeta^\intercal)$ is a circular family with covariance matrix

$$\begin{pmatrix} \kappa_2[c_\zeta, c_\zeta^*] & \kappa_2[c_\zeta, (c_\zeta^\intercal)^*] \\ \kappa_2[c_\zeta^\intercal, c_\zeta^*] & \kappa_2[c_\zeta^\intercal, (c_\zeta^\intercal)^*] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and pseudo-covariance matrix

$$\begin{pmatrix} \kappa_2[c_\zeta, c_\zeta] & \kappa_2[c_\zeta, c_\zeta^\intercal] \\ \kappa_2[c_\zeta^\intercal, c_\zeta] & \kappa_2[c_\zeta^\intercal, c_\zeta^\intercal] \end{pmatrix} = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix},$$

which generalizes the transpose relation for Ginibre matrices of a general parameter $\zeta \in \mathbb{C}$. In fact, a simple calculation shows that we can realize such a circular family as

$$(c_\zeta, c_\zeta^\intercal) = \left(\frac{1}{\sqrt{2}}(s_{\beta_1} + is_{\beta_2}), \frac{1}{\sqrt{2}}(s_{\beta_1}^\intercal + is_{\beta_2}^\intercal)\right),$$

where s_{β_1} and s_{β_2} are traffic independent β-semicircular traffics with $\beta_1, \beta_2 \in \mathbb{R}$ iff

$$\zeta = \beta_1 = -\beta_2.$$

Indeed,

$$\kappa_2\left[\frac{1}{\sqrt{2}}(s_{\beta_1} + is_{\beta_2}), \left(\frac{1}{\sqrt{2}}(s_{\beta_1}^\intercal + is_{\beta_2}^\intercal)\right)^*\right] = \frac{1}{2}(\beta_1 + \beta_2),$$

whereas

$$\kappa_2\left[\frac{1}{\sqrt{2}}(s_{\beta_1} + is_{\beta_2}), \frac{1}{\sqrt{2}}(s_{\beta_1}^\intercal + is_{\beta_2}^\intercal)\right] = \frac{1}{2}(\beta_1 - \beta_2).$$

For example, in the case of $\zeta = 0$, we can construct a standard complex Gaussian Ginibre matrix $\mathbf{G}_N$ as the sum

$$\mathbf{G}_N = \frac{1}{\sqrt{2}}(\mathbf{W}_N^{(1)} + i\mathbf{W}_N^{(2)}),$$

where $\mathbf{W}_N^{(1)}$ and $\mathbf{W}_N^{(2)}$ are independent GUE matrices ($\beta_1 = \beta_2 = 0$). Similarly, in the case of $\zeta = 1$, we can construct a standard real Gaussian Ginibre matrix $\mathbf{G}_N$ as the sum

$$\mathbf{G}_N = \frac{1}{\sqrt{2}}(\mathbf{W}_N^{(1)} + i\mathbf{W}_N^{(2)}),$$

where $\mathbf{W}_N^{(1)}$ and $\mathbf{W}_N^{(2)}$ are now independent Wigner matrices with

$$\mathbf{W}_N^{(1)}(j,k) \stackrel{d}{=} \mathcal{N}(0,1) \quad \text{and} \quad \mathbf{W}_N^{(2)}(j,k) \stackrel{d}{=} \begin{cases} \mathcal{N}(0,1) & \text{if } j = k \\ i\mathcal{N}(0,1) & \text{if } j \neq k, \end{cases}$$

which corresponds to $\beta_1 = -\beta_2 = 1$.

In the case of Haar unitary elements, we distinguish between Haar distributed orthogonal random traffics o (4.103) and Haar distributed unitary random traffics u [Mal, Proposition 3.7]. In particular, the traffic distribution in the latter case is only supported on oriented cacti with edges alternating in u and u^*. This implies that

$$\kappa_n[(u^{\hat{\intercal}(1)})^{\varepsilon(1)}, \ldots, (u^{\hat{\intercal}(n)})^{\varepsilon(n)}] = \begin{cases} (-1)^{\frac{n}{2}-1} c_{\frac{n}{2}-1} & \text{if } n \in 2\mathbb{N},\ \hat{\intercal} \text{ is constant, and } \varepsilon \text{ alternates,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.116}$$

whereas

$$\kappa_n[(o^{\hat{\intercal}(1)})^{\varepsilon(1)}, \ldots, (o^{\hat{\intercal}(n)})^{\varepsilon(n)}] = \begin{cases} (-1)^{\frac{n}{2}-1} c_{\frac{n}{2}-1} & \text{if } n \in 2\mathbb{N} \text{ and } \hat{\intercal} \times \varepsilon \text{ cross-alternates,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.117}$$

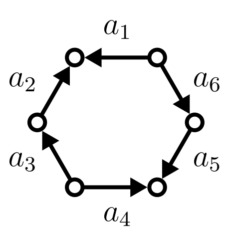
where we say that $\hat{\intercal} \times \varepsilon : [n] \to \{1, \intercal\} \times \{1, *\}$ *cross-alternates* if adjacent pairs $(\hat{\intercal}(i), \varepsilon(i))$ and $(\hat{\intercal}(i+1), \varepsilon(i+1))$ differ in exactly one coordinate for each $i \in [n]$ with the convention that $(\hat{\intercal}(n+1), \varepsilon(n+1)) = (\hat{\intercal}(1), \varepsilon(1))$. This gives another proof of the asymptotic $*$-freeness of a Haar distributed unitary random matrix $\mathbf{U}_N$ from its transpose $\mathbf{U}_N^\intercal$ [MP16, Mal, CDM], which certainly does not hold in the orthogonal case $\mathbf{O}_N^\intercal = \mathbf{O}_N^*$.

Our cumulant formulas allow us to bypass Proposition 2.4.9 for traffic random variables $(a_i)_{i \in I}$ whose injective traffic distribution is supported on cacti and is further multiplicative with respect to the pads. In particular, note that we can define the injective traffic distribution of such random variables using a slight generalization of the same process that defines the injective traffic distribution of the universal enveloping traffic space. For a cycle (possibly undirected), we simply return the free cumulant of the edge labels after choosing an arbitrary orientation with which to read off the cycle: if an edge goes in the direction of our orientation, it comes with a transpose label; if an edge goes against the direction of our orientation, it comes without a transpose label. For example

$$\tau^0\left[\ \cdot \underset{b}{\overset{a}{\leftleftarrows}} \cdot\ \right] = \kappa_2(a, b^\intercal) = \kappa_2(a^\intercal, b),$$

$$\tau^0\left[\ \cdot \underset{b}{\overset{a}{\leftrightarrows}} \cdot\ \right] = \kappa_2(a, b) = \kappa_2(b^\intercal, a^\intercal),$$

and

$\tau^0\Big[$ 

$\Big] = \kappa_6(a_1^\intercal, a_2, a_3, a_4^\intercal, a_5, a_6) = \kappa_6(a_6^\intercal, a_5^\intercal, a_4, a_3^\intercal, a_2^\intercal, a_1).$

Of course, the choice of orientation does not matter since

$$\kappa_n(a_1^{\hat{\intercal}(1)}, \ldots, a_n^{\hat{\intercal}(n)}) = \kappa_n((a_n^{\hat{\intercal}(n)})^\intercal, \ldots, (a_1^{\hat{\intercal}(1)})^\intercal),$$

which follows from the fact that $\intercal : \mathcal{A} \to \mathcal{A}$ defines an involutive anti-isomorphism of $*$-probability spaces

$$\varphi_\tau\left(\underset{\text{out}}{\cdot} \overset{a}{\leftarrow} \underset{\text{in}}{\cdot} \right) = \tau\left[\overset{a}{\circlearrowleft} \right] = \varphi_\tau\left(\underset{\text{out}}{\cdot} \overset{a}{\rightarrow} \underset{\text{in}}{\cdot} \right).$$

For a cactus, we simply multiply the contribution from each pad. Finally, for all other test graphs, we assign the value 0. We invite the reader to compare this to the construction (i)-(iii) of the injective traffic state in the universal enveloping traffic space. The assumptions on our random variables $(a_i)_{i\in I}$ tell us that this process reconstructs the injective traffic distribution of the $(a_i)_{i\in I}$. Thus, given such *cactus-type* traffic random variables $(a_i)_{i\in I}$, the information of the free cumulants

$$\left(\kappa_n((a_{i(1)}^{\hat{\intercal}(i(1))})^{\varepsilon(1)},\dots,(a_{i(n)}^{\hat{\intercal}(i(n))})^{\varepsilon(n)}) \;\middle|\; n\in\mathbb{N},\ \hat{\intercal}:[n]\to\{1,\intercal\},\ \varepsilon:[n]\to\{1,*\}\right) \tag{4.118}$$

is a determining sequence for the injective traffic distribution. We can then apply our analysis from Chapter 3 to the $(a_i)_{i\in I}$ to prove the $*$-freeness of $(a_i)_{i\in I}\cup(a_i^\intercal)_{i\in I}$ and $\Theta((a_i)_{i\in I})$, whereas freeness from the transpose can be judged from the determining sequence (4.118).

We have already encountered a number of examples of cactus-type random variables: namely, β-semicircular traffics of a real parameter $\beta\in\mathbb{R}$; ζ-circular traffics of a general parameter $\zeta\in\mathbb{C}$; and Haar distributed orthogonal random traffics. Let us work through the construction for a β-semicircular traffic. Since such a random variable is self-adjoint, equation (4.114) contains all of the information that we need. In particular, since the only non-vanishing cumulants are of order two, every cycle in our cactus must be a twin edge (a double tree). The contribution from such a twin edge is then given according to the orientation. In this way, we recover the original traffic distribution (4.8) from (4.114).

It is important to note when this line of reasoning also fails. For a strictly complex parameter $\beta\in\mathbb{C}$, a β-semicircular traffic is still supported on cacti; however, the formula (4.19) is *not* multiplicative with respect to the pads. For the same reason, RBMs of proper proportional growth, even with a real parameter, do not qualify as cactus-type. This can already be seen from our calculation of the almost Gaussian degree matrix (4.95). On the other hand, RBMs of slow growth with a real parameter do qualify as cactus-type, and so we can state the analogous asymptotic freeness results for this regime as well.

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