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# Some Results on Tight Stationarity

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by

**William Chen**

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Abstract of the Dissertation

# Some Results on Tight Stationarity

by


**William Chen**

Doctor of Philosophy in Mathematics

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Professor Itay Neeman, Chair


Fix an increasing sequence of regular cardinals $\langle \kappa_n : n < \omega \rangle$. Mutual and tight stationarity are properties akin to the usual notion of stationarity, but defined for sequences $\langle S_n : n < \omega \rangle$ with $S_n \subseteq \kappa_n$. This work focuses particularly on tight stationarity, providing a new characterization for it and comparing it to other concepts of stationarity.

Starting from a pcf-theoretic scale, we define a transfer function mapping sequences of subsets to a single subset of a certain regular cardinal, the length of the scale. The transfer function preserves stationarity, in the sense that a sequence is tightly stationary if and only if it is mapped to a stationary subset.

Using this characterization, we explore the question of whether it is consistent that there exists a sequence of cardinals for which every stationary sequence (i.e., a sequence of subsets, each of which is stationary in the corresponding cardinal) is tightly stationary, and prove some results which give a negative answer in certain cases. We prove that adding Cohen reals introduces stationary sequences which are not tightly stationary, and in the extension by adding uncountably many Cohen reals, every sequence of cardinals has a stationary but not tightly stationary sequence. From a tree-like scale we construct a sequence of stationary sets that is not tightly stationary in a strong way, namely, its image under the transfer function is empty.

Investigating this question in the Prikry model, we define the notion of a forgetful se-

quence and prove that every forgetful sequence of cardinals has a stationary, not tightly stationary sequence. Along the way, we will analyze the scales which appear in the Prikry model.

Then we consider the question of Cummings, Foreman, and Magidor of whether it is consistent that there is a sequence of cardinals on which every mutually stationary sequence is tightly stationary. We prove that it is consistent that there is no such sequence of cardinals. This uses a supercompact version of a construction adapted from Koepke which ensures that every stationary sequence is mutually stationary, provided that there is enough space between successive cardinals of the underlying sequence. Furthermore, this property of the model is indestructible under further Prikry forcing, which suggests that it is difficult to obtain a positive answer to the CFM question. The results in this section were obtained jointly with Itay Neeman.

Finally, we explore the combinatorics of tight stationarity. This leads to the notion of a careful set, which is a strengthening of being in the range of the transfer function. We produce a model where there is a singular cardinal for which all subsets of the successor are careful, which suffices to prove a splitting result for tightly stationary sequences. Using a version of the diagonal supercompact Prikry forcing, we obtain such a model where the singular cardinal is strong limit. These results start from a model with a continuous tree-like scale on the singular cardinal.

The dissertation of William Chen is approved.


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*To Mom*

## Table of Contents

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# Vita

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# Publications

Tight stationarity and tree-like scales. *Annals of Pure and Applied Logic* (2015), 166 (10), 1019-1036.

(with I. Neeman) Square principles with tail-end agreement. *Archive for Mathematical Logic* (2015), 54 (3), 439-452.

(with C.Y. Ku) An analogue of the Gallai-Edmonds Structure Theorem for non-zero roots of the matching polynomial. *Journal of Combinatorial Theory, Series B* (2010), 100 (2), 119–127.

# CHAPTER 1

# Introduction

## 1.1 Background

Investigation of strongly compact and supercompact cardinals led to the study of the combinatorics of $[\lambda]^{<\kappa}$, where $\kappa \leq \lambda$. In the analogy with the consistency-wise weaker notions of weakly compact and measurable cardinals, the set $[\lambda]^{<\kappa}$ under the inclusion ordering corresponds to a cardinal $\kappa$ under the usual ordering on ordinals. In fact, in the context of the nonstationary ideal, the latter can be thought of as the special case of the former where $\lambda = \kappa$.

Jech [Jec73] extended the fundamental concepts of *closed unbounded* and *stationary* to subsets of $[\lambda]^{<\kappa}$; we will use the definitions due to Shelah, which produce slightly different notions than Jech's. An *algebra* on $\lambda$ is a structure on $\lambda$ with countably many function symbols. A subset $S$ of $[\lambda]^{<\kappa}$ is club if there is an algebra on $\lambda$ so that $S$ is the set of all substructures (more precisely, their underlying sets) of size $< \kappa$ which are closed under the functions in the algebra. Usually, the algebra is taken to have symbols for all Skolem functions in its language, and therefore a substructure which is closed under the functions in the language of the algebra is an elementary substructure. $S$ is called nonstationary if its complement contains a club, and the collection of all nonstationary sets forms a $\kappa$-complete normal ideal.

Questions about splitting stationary sets and saturation have been the focus of much of the work since the 1980s. The saturation of an ideal $I$ on a set $Z$ is the least cardinal $\mathrm{sat}(I)$ so that every antichain of $P(Z)/I$ has cardinality less than $\mathrm{sat}(I)$. Recall some of the results for the classical case, $\lambda = \kappa$. Solovay proved that any stationary subset of $\kappa$ can be split into $\kappa$ many disjoint stationary sets, and therefore the nonstationary ideal on $\kappa$ is not $\kappa$-saturated. And while it is consistent relative to large cardinals that the nonstationary ideal on $\omega_1$ is $\omega_2$-saturated (see [FMS88]), Shelah proved that if $\kappa > \omega_1$, then the nonstationary ideal on $\kappa$ is never $\kappa^+$ saturated.

Foreman and Magidor [FM01], building from Shelah's result and work of Burke and Matsubara [BM97], proved that the nonstationary ideal on $[\lambda]^{<\kappa}$ cannot be $\lambda^+$-saturated unless $\lambda = \kappa$. To solve the case where $\kappa = \omega_1$ and $\lambda$ is singular, they introduced mutual stationarity. Fix a cardinal $\theta \geq \lambda$. Let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be a increasing sequence of regular cardinals with supremum $\lambda$, informally, we will call such sequences *products*. Suppose $S_\xi \subseteq \lambda_\xi$ for all $\xi < \mathrm{cf}(\lambda)$.

**Definition 1.1.1.** The sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\lambda) \rangle$ is *mutually stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$, there is $M \prec \mathcal{A}$ such that $\sup(M \cap \lambda_\xi) \in S_\xi$ for all $\xi < \mathrm{cf}(\kappa)$ (we say that $M$ *meets* $\vec{S}$).

Compare this to an alternative definition of stationary for a regular cardinal $\lambda$, which defines a $S \subseteq \lambda$ to be stationary if for any algebra $\mathcal{A}$ on $H(\theta)$, there is $M \prec \mathcal{A}$ such that $\sup(M \cap \lambda) \in S$. An interesting point of view found in [For02] is that mutual stationarity gives a notion of a stationary subset of a singular cardinal, even one of countable cofinality, since a subset $S \subseteq \lambda$ can also be thought of as a sequence $\langle S \cap \lambda_\xi : \xi < \mathrm{cf}(\lambda) \rangle$.

Different choices of $\theta$ yield equivalent definitions, and it is convenient to take $\theta$ large enough so that $H(\theta)$ includes objects needed in various constructions. Note also that mutual stationarity of $\vec{S}$ entails that every $S_\xi$ is stationary in $\lambda_\xi$. We call sequences with this property *stationary sequences.*

In the case that $S_\xi \subseteq \mathrm{Cof}(< \kappa)$ for every $\xi < \mathrm{cf}(\lambda)$, another equivalent definition for $\vec{S}$ to be mutual stationarity is that the set of $M \in [\lambda]^{<\kappa}$ so that $M$ meets $\vec{S}$ is stationary in the sense of Shelah above. This focuses attention on a class of subsets of $[\lambda]^{<\kappa}$ of a specific shape, one which is flexible enough to split $[\lambda]^{<\kappa}$, yet easy to work with since each subset is specified by a small number of parameters. For motivation, we sketch the argument of [FM01] that the nonstationary ideal on $[\lambda]^{<\omega_1}$ is not $\lambda^+$-saturated. Choose $\langle \lambda_\xi : \xi < \mathrm{cf}(\lambda)\rangle$ cofinal in $\lambda$. For each $\xi$, partition the set of ordinals of countable cofinality below $\lambda_\xi$ into $\lambda_\xi$-many disjoint stationary pieces. Using a tagged tree argument, it can be shown that any stationary sequence where each element consists just of countable cofinality ordinals is in fact mutually stationary. Therefore, each of the $\lambda^{\mathrm{cf}(\lambda)}$-many functions which choose one stationary piece at each $\xi$ yields a mutually stationary sequence, and these mutually stationary sequences give disjoint stationary subsets of $[\lambda]^{<\kappa}$.

The situation is more complicated if the stationary subsets concentrate on ordinals of uncountable cofinality. To start with, if the sequence $\vec{S}$ has the property that $S_\xi$ consists of points of cofinality $\kappa_\xi$ and the $\kappa_\xi$ are not eventually constant, then it is not even clear that taking $S_\xi = \mathrm{Cof}(\kappa_\xi) \cap \lambda_\xi$ yields a mutually stationary sequence; see Baumgartner [Bau91], Liu, Liu–Shelah [LS97]. Foreman [For02] observed that mutual stationarity of some sequences like this entails instances of Chang's Conjecture or that the singular cardinal is Jonsson. To avoid these issues, which are orthogonal to our interests here, we restrict $S_\xi$ to concentrate on points of some fixed uncountable cofinality.

The main focus of this dissertation is the notion of tight stationarity, a more tractable strengthening of mutual stationarity introduced in [FM01]. We say $M$ is *tight* if $M$ contains $\langle \lambda_\xi \rangle$ and $M \cap \prod_{\xi<\mathrm{cf}(\lambda)} \lambda_\xi$ is cofinal in $\prod_\xi (M \cap \lambda_\xi)$: this isolates a key property of the internally approachable structures defined in [FMS88], and in fact, Section 6 of [CFM04] shows that tightness is close to internal approachability under the assumption of uniform cofinality. Pick $\theta$ large enough to contain $\prod_{\xi<\mathrm{cf}(\lambda)} \lambda_\xi$ as an element.

**Definition 1.1.2.** The sequence $\vec{S}$ is *tightly stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$, there is a tight $M \prec \mathcal{A}$ such that $\sup(M \cap \lambda_\xi) \in S_\xi$ for all $\xi < \mathrm{cf}(\lambda)$.

The definition is the same as that for mutual stationarity, except that we require the witnessing structure to be tight.

As shown in [CFM04], there is a close connection between tight structures and pcf theory when we require the tight structures to have uniform uncountable cofinality in the product. In Chapter 2, we will use this connection to define a transfer function $\mu$ in the presence a pcf-theoretic scale on $\langle \lambda_\xi : \xi < \mathrm{cf}(\lambda) \rangle$, modulo the ideal of bounded subsets of $\mathrm{cf}(\lambda)$. This transfer function takes a sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\lambda) \rangle$ to a subset of the length of the scale, a regular cardinal. The key property of $\mu$ is that it preserves stationarity in the sense that $\vec{S}$ is tightly stationary if and only if $\mu(\vec{S})$ is stationary (this requires certain assumptions, see Lemma 2.2.4 for a precise statement).

An immediate relationship between the various stationarity notions on sequences of subsets is:

$$\text{tightly stationary} \quad \Longrightarrow \quad \text{mutually stationary} \quad \Longrightarrow \quad \text{stationary},$$

where, as mentioned above, a stationary sequence is simply one in which every coordinate is a stationary subset of the corresponding regular cardinal.

It is natural to ask if more can be said about the relationship between these three notions. For example,

*Question* 1.1.1. Is it consistent that there is a sequence of cardinals $\langle \lambda_n : n < \omega \rangle$ so that every mutually stationary sequence on $\Pi_{i<\omega}\lambda_n$ is tightly stationary?

The question was originally asked in [CFM06] for the sequence $\langle \omega_n : n < \omega \rangle$. The work in that paper produced models where $\langle \omega_n : n < \omega \rangle$ have mutually stationary sequences which are not tightly stationary.

One of our main results, proven in Chapter 5, improves this to a consistency result of a global nature.

**Theorem 5.1.2** (Chen–Neeman). *If there is a proper class of supercompact cardinals, then there is a class forcing extension so that every increasing $\omega$-sequence of regular cardinals has a mutually stationary sequence on cofinality $\omega_1$ which is not tightly stationary.*

Moreover, this property is absolute to a class of forcing extensions satisfying a certain Prikry-type property, the natural candidates for forcing that every mutually stationary sequence is tightly stationary on some sequence of regular cardinals, suggesting that a positive answer to Question 1.1.1 would be difficult to obtain.

The proof of Theorem 5.1.2 involves separately ensuring that the following two properties hold in the final model:

1. Every $\omega$-sequence of regular cardinals carries a stationary sequence which is not tightly stationary.

2. Every stationary sequence is mutually stationary, provided the regular cardinals which carry the sequence are spaced sufficiently far apart.

Whether property (1) is just a theorem of $\mathsf{ZFC}$ is an interesting question that, to the author's knowledge, remains open. There is some partial progress in Chapter 3 towards a positive answer, where we give some conditions for products under which there is a stationary but not tightly stationary sequence, and use forcing to build a model where every sequence of regular cardinals has such a sequence. In Chapter 4 we show that there is always a stationary not tightly stationary sequence on many sequences of regular cardinals associated with Prikry forcing, which we call *forgetful sequences*.

**Theorem 4.2.2.** *Suppose $\langle \mu_n : n < \omega \rangle \in V[E]$ be forgetful. Then there is a stationary but not tightly stationary sequence $\langle S_n : n < \omega \rangle$.*

As part of this analysis, we will develop a general framework for understanding the scales in the Prikry extension.

After understanding more about the relationship between mutual and tight stationarity, the focus shifts to the combinatorics of tightly stationary sequences. Foreman and Magidor in [FM01] were able to prove tight stationary versions of Fodor's lemma and Solovay's splitting theorem (whether those results hold for mutual stationarity is an open problem, see [For02]). In the presence of a scale on the product, the transfer function $\mu$ is a useful tool for proving combinatorial properties of tightly stationary sequences, and its existence is by itself enough to obtain some connections between stationarity in the length of the scale (a regular cardinal above $\lambda$) and tight stationarity at $\lambda$. For example, it can be used to derive the version of Fodor's lemma previously obtained by Foreman-Magidor [FM01] for tight stationarity at $\lambda$ as a consequence of the usual Fodor's lemma for regular cardinals; see Proposition 6.1.1.

But for other applications, we want to have an inverse for $\mu$, in the following strong sense: for each subset $A$ of the length of the scale, we want to have a sequence $\vec{S}$ so that $\mu(\vec{S}) = A$ and $\mu(\vec{S}')$ is the complement of $A$, where $\vec{S}'$ is the sequence $S'_\xi = \lambda_\xi \setminus S_\xi$. Call $A$ *careful* if there exists such a sequence $\vec{S}$. The notion of carefulness can be thought of as a symmetrical strengthening of being in the range of $\mu$ — for example, Boolean operations on sequences corresponding to careful sets commute with $\mu$, although this is not generally true for sets which are just in the range of $\mu$. Consequently, $\mu$ gives a particularly useful connection between careful subsets of the length of the scale and sequences in $\Pi_{\xi<\mathrm{cf}(\lambda)}\lambda_\xi$.

If every subset of the length of the scale is careful, then we can transfer Solovay's splitting theorem on this regular cardinal to the context of tight stationarity on $\lambda$. Under this assumption, we obtain a new splitting result for tightly stationary sets (Proposition 6.1.4). We remark that Proposition 6.1.4 differs from the splitting theorem obtained by Foreman and Magidor in [FM01].

Although there are some situations in which there exists a non-careful subset, we show that it is actually consistent for every subset of the length of the scale to be careful. In Chapter 7, we use forcing to construct a model where every subset of the successor of the singular cardinal is careful. The main result here is:

**Corollary 7.1.3.** *Let $\kappa$ be a singular cardinal of countable cofinality with a tree-like scale of length $\kappa^+$. There is a c.c.c. forcing extension in which every subset of $\kappa^+$ is careful.*

The notion of tree-like scale, studied by Pereira [Per08], plays a key role in this construction.

Chapter 7 continues by modifying the above construction so that in the extension, the singular cardinal $\kappa$ is a strong limit of countable cofinality and every subset of $\kappa^+$ is careful. This requires a supercompact cardinal. Additionally, collapses can be interleaved into the construction so that $\kappa$ is the least cardinal fixed point (i.e., the least $\kappa$ with $\kappa = \aleph_\kappa$). All of this uses ideas from the diagonal supercompact Prikry forcing of Gitik–Sharon [GS08].

# CHAPTER 2

# The transfer function

In this chapter, we define the transfer function $\mu$ and a dual notion $\nu$ that play a crucial role in the remainder of the dissertation. The main point is Lemma 2.2.4, that the transfer function relates tight stationarity to the usual notion of stationarity on regular cardinals.

The transfer function can be directly applied in case the sequence of cardinals under consideration carries a *scale*.

## 2.1 Preliminaries

The background for our work draws primarily [FM01], [CFM04] and [CFM06]. In the interest of making this dissertation self-contained, we state the definitions and basic results we will need.

### 2.1.1 Scales and pcf theory

Scales are the objects which witness the "true cofinality" of a reduced product of regular cardinals in Shelah's pcf theory, see [She94] for information beyond the scope of this section.

Let $\kappa$ be a singular cardinal, $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a sequence of regular cardinals cofinal in $\kappa$, and $I$ be an ideal on $\mathrm{cf}(\kappa)$. Define a partial order $<_I$ on $\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ to be $f <_I g$ if and only if $\{\xi : f(\xi) \geq g(\xi)\} \in I$.

A *scale on* $\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ *modulo* $I$ *of length* $\lambda$ is a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ from $\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ which is $<_I$-increasing and $<_I$-cofinal. We will suppress mention of the ideal $I$ if it is just the ideal of bounded subsets of $\mathrm{cf}(\kappa)$, and in this case we will denote the order $<_I$ by $<^*$.

Define $\mathrm{pcf}(\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle)$ to be the set of $\lambda$ for which there exists an ideal $I$ and a scale on $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ modulo $I$ of length $\lambda$. A useful fact is that $\mathrm{pcf}(\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle)$ has a maximum element.

The basic result on the existence of scales is the following:

**Fact 2.1.1.** If $\kappa$ is a singular cardinal, then there is a sequence $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ of regular cardinals on which there is a scale of length $\kappa^+$ (modulo the ideal of bounded subsets of $\mathrm{cf}(\kappa)$).

A $<_I$-increasing sequence of functions $\vec{f} \subseteq \prod_\xi \kappa_\xi$ has an *exact upper bound* $h$ if $h$ is a $<_I$-upper bound $\vec{f}$ and for any $g$ with $g <_I h$, there is $f \in \vec{f}$ with $g <_I f$, i.e., $\vec{f}$ is $<_I$-cofinal in $\prod_\xi h(\xi)$, ignoring coordinates $\xi$ for which $f(\xi) \geq h(\xi)$ for each $f \in \vec{f}$. A scale $\langle f_\alpha : \alpha < \lambda\rangle$ is *continuous* if for every $\beta < \lambda$ so that there exists an exact upper bound for $\langle f_\gamma : \gamma < \beta\rangle$, $f_\beta$ is an exact upper bound for this sequence. A continuous scale can be easily obtained from an arbitrary one by replacing scale functions at limit ordinals with exact upper bounds. A *good point* of the scale is an ordinal $\beta < \lambda$ with $\mathrm{cf}(\beta) > \omega$ so that there is an exact upper bound $h$ for $\langle f_\gamma : \gamma < \beta\rangle$ so that $\mathrm{cf}(h(\xi)) = \mathrm{cf}(\beta)$ for all $\xi < \mathrm{cf}(\kappa)$. Let Good denote the set of good points of a scale.

Suppose $\langle f_\alpha : \alpha < \lambda\rangle$ and $\langle g_\alpha : \alpha < \lambda\rangle$ are scales of the same length. Then there is a closed unbounded set $C$ so that for any $\alpha \in C$, $\alpha$ is good for $\langle f_\alpha : \alpha < \lambda\rangle$ if and only if $\alpha$ is good for $\langle g_\alpha : \alpha < \lambda\rangle$, and $f_\gamma : \gamma < \alpha$ is $<_I$-cofinally interleaved with $g_\gamma : \gamma < \alpha$ (so that for any $\gamma < \alpha$ there is $\gamma' < \alpha$ so that $f_\gamma <_I g_{\gamma'}$). If in addition both scales are continuous, then $f_\alpha =_I g_\alpha$ for all such $\alpha$.

A crucial concept for the constructions in this dissertation is a tree-like scale. This concept appears in [She94] (see II Conclusion 3.5) and was isolated and further studied by Pereira in [Per08].

**Definition 2.1.2.** A scale $\langle f_\alpha : \alpha < \lambda\rangle$ is *tree-like* if whenever $f_\alpha(\xi) = f_\beta(\xi)$, then $f_\alpha \restriction \xi = f_\beta \restriction \xi$.

If $\mathrm{cf}(\kappa) = \omega$, then any product which carries a scale also carries a tree-like scale, but that scale is not necessarily continuous, as we require. Pereira described a forcing notion in [Per08] which produces a continuous tree-like scale and preserves cardinals, and hence also the approachability property at $\kappa$ (a principle which implies that every scale on $\kappa$ is good) if it holds in the ground model.

If $I$ is dual to an ultrafilter, then $<_I$ is a linear order so there is a scale modulo $I$. Unfortunately, not every product carries a scale modulo the ideal of bounded subsets. The theory of pcf generators gives information about what happens in between.

Let $K = \langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ with $\mathrm{cf}(\kappa) < \kappa_0$. For a cardinal $\lambda$, let $J_{<\lambda}$ be the ideal of subsets $A \subseteq K$ so that for any ultrafilter $U$ with $A \in U$, there is a scale of length $< \lambda$ modulo the dual ideal. Shelah showed that for each $\lambda \in \mathrm{pcf}(K)\rangle)$, the ideal $J_{<\lambda^+}$ is generated by a single set over $J_{<\lambda}$, so there is a set $B_\lambda$ with the property that $A \setminus B_\lambda \in J_{<\lambda}$ for all $A \in J_{<\lambda^+}$. Fix a sequence $\langle B_\lambda : \lambda \in \mathrm{pcf}(K) \rangle$.

**Fact 2.1.3.** For every $\lambda \in \mathrm{pcf}(K)$, there is a continuous scale $\langle f^\lambda_\alpha : \alpha < \lambda \rangle$ on $\prod B_\lambda$ modulo $J_{<\lambda}$.

In [CFM04], it was shown that universes are similar in a certain way must have the same pcf structure. We use only the special case below.

**Fact 2.1.4** (Theorem 7.1 [CFM04])**.** Let $\mathbb{P}$ be a c.c.c. forcing, with $G$ generic for $\mathbb{P}$ over $V$, and let $K \in V$ be a sequence of regular cardinals with $K < \min(K)$. Then

1. $\mathrm{pcf}(K)^V = \mathrm{pcf}(K)^{V[G]}$.

2. $J^V_{<\lambda} = J^{V[G]}_{<\lambda} \cap V$ for all $\lambda$.

3. A pcf generator $B_\lambda$ for $K$ in $V$ remains a pcf generator for $K$ in $V[G]$.

### 2.1.2 Tight structures

Recall from the introduction that a structure $M$ is *tight for* $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ if $M$ contains $\langle \kappa_\xi \rangle$ and $M \cap \prod_{\xi < \mathrm{cf}(\kappa)} \kappa_\xi$ is cofinal in $\prod_\xi (M \cap \kappa_\xi)$.

For any set $M$, and sequence $K = \langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ of regular cardinals, define the *characteristic function* of $M$ in $K$ to be $\chi_M^K : \mathrm{cf}(\kappa) \to \kappa$ so that $\chi_M^K(\xi) = \sup(M \cap \kappa_\xi)$. We will drop the subscript $K$ if it is understood from context.

Because we are mainly restricted to sequences of subsets on a fixed uncountable cofinality $\eta$ in our analysis of tight stationarity, most of the tight structures we consider here are uniform on $K$, i.e., there is some uncountable cardinal $\eta$ so that $\mathrm{cf}(M \cap \kappa_\xi) = \eta$ for all $\kappa_\xi \in K$ (where $K$ is the relevant sequence of cardinals).

In [CFM04], tightness for uniform structures was characterized in terms of pcf theory. Let $\mathcal{A}$ be the structure $(H(\theta); \in, K, \langle B_\lambda \rangle, \langle f_\alpha^\lambda \rangle)$ for some regular $\theta$ large enough so that $H(\theta)$ contains all of the pcf-theoretic objects.

**Fact 2.1.5** (Theorem 5.2 of [CFM04])**.** Let $K$ be a sequence of regular cardinals. Suppose $M$ is in the club of structures elementary in $\mathcal{A}$, $\mathrm{pcf}(K) \subseteq M$, and $M$ is uniform on $K$.

Then $M$ is tight for $K$ if and only if for every $\lambda \in \mathrm{pcf}(K)$, $\sup(M \cap \lambda)$ is a good point of cofinality $\eta$ for $\langle f_\alpha^\lambda : \alpha < \lambda \rangle$ and $f^\lambda_{\sup(M \cap \lambda)} =_{J_{<\lambda}} \chi_M^{B_\lambda}$.

From Theorems 5.2 and 7.1 of [CFM04], we can derive a useful result stating that in a c.c.c. forcing extension, tight structures have characteristic functions in the ground model. Versions of this lemma were used in [CFM04] and [CFM06].

**Lemma 2.1.6.** *Suppose $2^{\aleph_0} < \aleph_\omega$, $K = \langle \kappa_n : n < \omega \rangle$ is a sequence of regular cardinals with supremum $\kappa$, and $\mathbb{P}$ is a c.c.c. poset. Let $G$ be generic for $\mathbb{P}$ over $V$. In $V[G]$, there is a club $C$ of substructures of $H(\theta)^{V[G]}$ so that if $M \in C$ and $M$ is tight and uniform, then $\chi_M \in V$.*

*Proof.* By Theorem 7.1 of [CFM04], a sequence of pcf generators $\langle B_\lambda : \lambda \in \mathrm{pcf}(K) \rangle$ in $V$ remains a sequence of pcf generators in $V[G]$, and the ground model scales $\langle f_\alpha^\lambda \rangle$ on the $B_\lambda$ modulo $J_{<\lambda}$ remain scales in $V$.

In $V[G]$, let $\mathcal{B}$ be the structure $(H(\theta)^{V[G]}; \in, K, \langle B_\lambda \rangle, \langle f_\alpha^\lambda \rangle)$, and $C$ be the club of elementary substructures of $\mathcal{B}$. Suppose $M \in C$ is tight for $K$ and uniform.

We want to work with a model which contains $\mathrm{pcf}(K)$ as a subset. Since $2^{\aleph_0} < \aleph_\omega$, there are fewer than $\aleph_\omega$ possible choices for the pcf generators, so $\rho = |\mathrm{pcf}(K)| < \aleph_\omega$. Let $N$ be the Skolem hull in $\mathcal{B}$ of $M \cup \rho$, so $\mathrm{pcf}(K) \subseteq N$. Letting $n_0$ be such that $\kappa_m > \rho$, we have

$$\chi_M \restriction [n_0, \omega) = \chi_N \restriction [n_0, \omega)$$

since for any term $t \in M$ and $n_0 \leq n < \omega$, $M$ correctly computes the supremum of $\mathrm{range}(t \restriction \rho) \cap \kappa_n$, and $N \cap \kappa_n$ consists only of ordinals of the form $t(\gamma)$ for $\gamma < \rho$. By a similar argument, $N$ is tight, since $M$ correctly computes the pointwise supremum of $\mathrm{range}(t \restriction \rho) \cap \prod K$.

$N$ is uniform on a tail of $K$, so by Theorem 5.2, $\chi_N^{B_\lambda} =_{J_{<\lambda}} f^\lambda_{\sup(N \cap \lambda)}$ for all $\lambda \in \mathrm{pcf}(K)$.

We will construct cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{m^*}$ and corresponding pcf generators $B'_m$ for $\lambda_m$ so that $\chi_N(n) = f^\lambda_{\sup(N \cap \lambda)}(n)$ for $\kappa_n \in B'_m$.

If $\lambda_j$, $B'_j$ have already been constructed for all $j < m$, then let $\lambda_m = \max \mathrm{pcf}(K \setminus \bigcup_{i \leq m} B'_i))$, and let $B'_m$ be the set of cardinals $\kappa_n \in B_{\lambda_m}$ so that $\sup(N \cap \kappa_n)$ is equal to $f^{\lambda_m}_{\sup(N \cap \lambda_m)}(n)$. Since $B'_m$ differs from $B_{\lambda_0}$ by a set in $J_{<\lambda_0}$, $B'_m$ is also a generator for $\lambda_0$, so $\max \mathrm{pcf}(K \setminus B'_{\lambda_m}) < \lambda_m$. The process must terminate after finitely many stages since the $\lambda_m$ are strictly decreasing.

A priori, the $B'_m$ need not be in $V$. For each $m$, let $B''_m = B_{\lambda_m} \setminus \bigcup_{i > m} B_{\lambda_i}$. Since $B_{\lambda_i} \in J_{<\lambda_m}$ for each $i > m$, it follows that $B''_m$ is a pcf generator for $\lambda_m$. By induction starting from $m^*$ and working down, $\{\kappa_n \in B''_m : \sup(N \cap \kappa_n) \neq f^{\lambda_m}_{\sup(N \cap \lambda_m)}(n)\} \subseteq \bigcup_{i > m} B''_i$. The $B''_m$ are disjoint and $\bigcup_m B''_m = \bigcup_m B_{\lambda_m} \supseteq K$, so

$$\sup(N \cap \kappa_n) = f^{\lambda_m}_{\sup(N \cap \lambda_m)}(n)$$

for all $\kappa_n \in B''_m$.

The characteristic function of $M$ on $K$ differs from that of $N$ in only finitely many places, and $\chi_N$ can be reconstructed from the finitely many functions $f^{\lambda_m}_{\sup(N \cap \lambda_m)}$ and sets $B''_m$, all of which are in $V$. $\square$

## 2.2 The transfer function

It will be convenient to use alternative definitions of mutual and tight stationarity, as in [CFS09]. Fix an increasing sequence of regular cardinals $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ cofinal in $\kappa$, and some sufficiently large regular $\theta$. We say that a sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ is *mutually stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$, there are $\xi_0 < \mathrm{cf}(\kappa)$ and $M \prec \mathcal{A}$ such that $\sup(M \cap \lambda_\xi) \in S_\xi$ for all $\xi_0 \leq \xi < \mathrm{cf}(\kappa)$. Tight stationarity is defined similarly, requiring the structure $M$ to be tight, as defined in the introduction. These definitions are equivalent to those from the introduction, provided that $\mathrm{cf}(\kappa) = \omega$ and each $S_\xi \subseteq \kappa_\xi$ is stationary and contains only points of cofinality less than $\kappa_0$, as will be the case in the situations that we consider; see [FM01] for the proof of this equivalence.

Given any function $f : \mathrm{cf}(\kappa) \to \kappa$ and sequence $\vec{S}$, we say that $f$ *meets* $\vec{S}$ if $f(\xi) \in S_\xi$ for all but boundedly many $\xi < \mathrm{cf}(\kappa)$. We now define the transfer function.

**Definition 2.2.1.** Suppose $S_\xi \subseteq \kappa_\xi$ for each $\xi < \mathrm{cf}(\kappa)$. Then define

$$\mu(\vec{S}) = \{\alpha : f_\alpha \text{ meets } \vec{S}\}.$$

Let $S'_\xi = \kappa_\xi \setminus S_\xi$. Then define $\nu(\vec{S}) = \kappa^+ \setminus \mu(\langle S'_\xi \rangle)$.

Another way to think of $\nu(\vec{S})$ is $\{\alpha : f_\alpha(\xi) \in S_\xi \text{ for unboundedly many } \xi\}$. This function will be important in Chapter 6 when defining the key notion of carefulness.

We start by listing some straightforward properties of $\mu$.

**Proposition 2.2.2.** *Let $S_\xi, T_\xi \subseteq \kappa_\xi$ for $\xi < \mathrm{cf}(\kappa)$. Then*

$$\mu(\langle S_\xi \cap T_\xi \rangle) = \mu(\vec{S}) \cap \mu(\vec{T})$$

*and*

$$\mu(\langle S_\xi \cup T_\xi \rangle) \supseteq \mu(\vec{S}) \cup \mu(\vec{T}).$$

**Proposition 2.2.3.** *If $S_\xi$ is club in $\kappa_\xi$ for all $\xi < \mathrm{cf}(\kappa)$, then $\mu(\vec{S})$ contains a club. If $S_\xi$ is nonstationary in $\kappa_\xi$ for unboundedly many $\xi < \mathrm{cf}(\kappa)$, then $\mu(\vec{S})$ is nonstationary.*

The following lemma is the key point relating tight stationarity to the $\mu$ of Definition 2.2.1. We will work in the case where there is some regular cardinal $\eta < \kappa_0$ so that $S_\xi \subseteq \mathrm{Cof}(\eta)$. This avoids some difficulties of getting mutual stationarity with the cofinalities of ordinals in the stationary sets varying among the cardinals on the sequence (this is connected to versions of Chang's Conjecture and Jonsson cardinals, see Section 4 of [For02]). The fixed cofinality assumption also gives the uniformity we need to apply results about the characteristic functions of tight structures from Cummings–Foreman–Magidor [CFM04]. To get a flavor for this assumption, we remark that Section 6 of [CFM04] contains some results to the effect that a tight structure with uniform cofinality is very close to internally approachable.

**Lemma 2.2.4.** *Let $\eta$ be an uncountable regular cardinal in the interval $(\mathrm{cf}(\kappa), \kappa_0)$. Suppose $S_\xi \subseteq \kappa_\xi \cap \mathrm{Cof}(\eta)$. Then $\vec{S}$ is tightly stationary iff $\mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$ is stationary in $\kappa^+$.*

*Proof.* If $\vec{S}$ is tightly stationary, then for any algebra $\mathfrak{A}$ on $H(\theta)$ there is a tight $N \prec \mathfrak{A}$ which meets $\vec{S}$ and contains the scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ as an element. Let $\alpha := \sup(N \cap \kappa^+)$. By Theorem 5.2 of [CFM04], $\chi_N =^* f_\alpha$ and $\alpha$ is a good point of cofinality $\eta$, so $\alpha \in \mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$.

For the converse, suppose that $B = \mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$ is stationary in $\kappa^+$. Let $C \subseteq [H(\theta)]^{<\eta^+}$ be an arbitrary club so that every member of $C$ is an elementary submodel of $(H(\theta); \in, \langle f_\alpha : \alpha < \kappa^+ \rangle, \vec{S})$. Then construct $\langle M_x : x \in [\kappa^+]^{<\eta} \rangle \subseteq C$ so that:

1. $x \subseteq M_x$,

2. if $y$ end-extends $x$, then $M_x \subseteq M_y$ and $M_y$ contains some $\alpha < \kappa^+$ so that $\chi_{M_x} <^* f_\alpha$.

Define $g : [\kappa^+]^{<\eta} \to \kappa^+$ by sending $x$ to the least $\alpha$ so that $\chi_{M_x} <^* f_\alpha$. Consider the set $D = \{\alpha \in \kappa^+ : g``[\alpha]^{<\eta} \subseteq \alpha\}$, and let $E \subseteq \kappa^+$ be its closure. Then $E \cap \mathrm{Cof}(\eta) = D \cap \mathrm{Cof}(\eta)$, so there exists $\gamma \in B \cap D \cap \mathrm{Cof}(\eta)$. Since $\gamma$ is good, there are $\langle \gamma_i : i < \eta \rangle$ cofinal in $\gamma$ and $\xi^* < \mathrm{cf}(\kappa)$ so that for all $\xi \geq \xi^*$, $\langle f_{\gamma_i}(\xi) : i < \eta \rangle$ is strictly increasing. Since $\gamma \in D$, we can further assume that $g(\langle \gamma_j : j < i \rangle) < \gamma_{i+1}$ for all $i < \eta$.

For convenience, let $N_i$ denote the substructure $M_{\langle \gamma_j : j<i \rangle}$. Put $M = \bigcup_{i<\eta} N_i$. Then $M \in C$ (since it is the increasing union of members of $C$) and $M$ is tight (which follows from clause (2) in the construction of the $M_x$). The argument is finished by showing that $\chi_M =^* f_\gamma$.

By clause (1) of the construction of the $M_x$, we have that the range of $f_{\gamma_i}$ is contained in $N_{i+1}$ for all $i < \eta$, and therefore $f_{\gamma_i} < \chi_{N_{i+1}}$. Since $g(\langle \gamma_j : j < i \rangle) < \gamma_{i+1}$, we also have that $\chi_{N_i} <^* f_{\gamma_{i+1}}$. Putting the inequalities together with the fact that $\eta > \mathrm{cf}(\kappa)$, we have $\chi_M =^* \sup_{i<\eta} \chi_{N_i} =^* \sup_{i<\eta} f_{\gamma_i} =^* f_\gamma$. □

# CHAPTER 3

## Stationary but not tightly stationary sequences

In the next few chapters, we will examine the relationship between the different stationarity properties of sequences. The most basic question, perhaps, is whether every product carries a stationary sequence which is not tightly stationary. The question of whether every stationary sequence can be mutually stationary has been the focus of much of the work in this area since the seminal paper [FM01]. The interest here is that while mutual and tight stationarity seem to be subtle notions, stationary sequences are easy to construct and work with.

First we recall the previous work. In one direction, Foreman and Magidor [FM01] showed that every stationary sequence where the subsets consist of ordinals of countable cofinality is mutually stationary. Starting from a measurable cardinal, Cummings–Foreman–Magidor [CFM06] showed that it is consistent that there is a product on which every sequence of stationary sets (uniform cofinality or not) is mutually stationary.

In the other direction, [FM01] showed that in the constructible universe $L$ there is a stationary sequence of subsets of ordinals of cofinality $\omega_1$ which is not mutually stationary. Koepke and Welch [KW06] improved this and showed that consistency of the proposition that all stationary sequences with fixed cofinality $\omega_1$ on some product are mutually stationary requires the existence of a measurable cardinal, so the [CFM06] result is an equiconsistency. They also showed that getting this with the product where $\lambda_n = \omega_{n+2}$ requires more, and this is not yet known to be consistent (relative to any large cardinal axiom).

This shows that a negative answer to our basic question would require large cardinals. We start off with an attempt towards this direction, then quickly change course to give situations where a positive answer holds. One of our results, Theorem 3.2.1, says that after forcing with $\mathrm{Add}(\omega, \omega_1)$, every product carries a stationary sequence which is not tightly stationary. This will be a component for the results of Chapter 7. Finally, we show that there are stationary not tightly stationary sequences at products with continuous tree-like scales.

## 3.1 A "negative" example

At singular cardinals above supercompacts, there is are filters so that any sequence of subsets where each subset is chosen from the positive sets of the corresponding filter is tightly stationary. At first glance, this appears close to having every stationary sequence be tightly stationary.

Suppose $\kappa$ is a $\kappa^{+\omega+2}$-supercompact cardinal, witnessed by an embedding $j : V \to M$ with ${}^{\kappa^{+\omega+2}}M \subseteq M$. Suppose there is a scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^{+\omega+1} \rangle$ on $\prod_{n<\omega} \kappa^{+n}$. For each $n < \omega$, let $U_n$ be the $\kappa$-complete ultrafilter defined by $X \in U_n$ if $\sup(j``\kappa^{+n}) \in j(X)$.

**Proposition 3.1.1.** *Suppose $\langle S_n : n < \omega \rangle$ is a sequence of stationary sets so that $S_n \in U_n$. Then $\langle S_n : n < \omega \rangle$ is tightly stationary.*

*Proof.* Let $\mathcal{A}$ be an algebra which is an expansion of $(H(\kappa^{+\omega+2}); \in, \kappa, \vec{f})$. Let $\mathcal{B}$ be the substructure of $j(\mathcal{A})$ with underlying set equal to $j``H(\kappa^{+\omega+1})$. Then $\mathcal{B} \in M$ and $\mathcal{B} \prec j(\mathcal{A})$. We have that $\mathcal{B}$ meets $\langle j(S_n) : n < \omega \rangle$, since $\sup(j``\mathcal{A} \cap \kappa^{+n}) = \sup(j``\kappa^{+n}) \in j(S_n)$, using the fact that $S_n \in U_n$.

It remains to check that $\mathcal{B}$ is tight in $M$. For each $\alpha$, $j(f_\alpha) \in \mathcal{B}$, and $\langle j(f_\alpha) : \alpha < \kappa^{+\omega+1} \rangle$ is cofinal in $\prod_n j``\kappa^{+n}$ since this product is isomorphic to $\prod_n \kappa^{+n}$, and $\langle j(f_\alpha) : \alpha < \kappa^{+\omega+1} \rangle$ is the pointwise image of the scale $\vec{f}$ under this isomorphism. $\square$

The construction above is deficient in several ways for the purpose of this chapter. First, the characteristic function of the structure does not have a uniform cofinality, and therefore this example does not fall under the case for which the results of the previous chapter apply. Also, $U_n$ seems quite far from being the club filter on $\kappa^{+n}$, as it is just $\kappa$-complete for any $n$.

## 3.2 Positive examples

We collect some examples of situations for which there are stationary, not tightly stationary sequences concentrating on a fixed uncountable cofinality.

Under favorable cardinal arithmetic, namely a large value of the continuum, we can find stationary sequences which are not tightly stationary on certain sequences of regular cardinals.

**Proposition 3.2.1.** *Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals which carry a scale $\langle f_\alpha : \alpha < \lambda \rangle$, and $2^{\aleph_0} > \lambda$. Then there is a stationary sequence in $\langle \mu_n : n < \omega \rangle$ which is not tightly stationary.*

*Proof.* Choose a sequence $\langle (S_n^0, S_n^1) : n < \omega \rangle$ where $(S_n^0, S_n^1)$ is a partition of $\mu_n \cap \mathrm{Cof}(\eta)$ into stationary sets for each $n < \omega$. For each $\alpha < \lambda$, let $x_\alpha \in {}^\omega 2$ be defined by setting $x_\alpha(n)$ equal to the unique $i$ so that $f_\alpha(n) \in S_n^i$ if it exists, and 0 otherwise. Let $E_0$ be the equivalence relation of eventual agreement on ${}^\omega 2$, i.e., $x \equiv y$ iff there is some $m$ so that $x(n) = y(n)$ for all $n \geq m$. The classes of $E_0$ are countable, so there is some $x \in {}^\omega 2$ so that $x$ is not $E_0$-equivalent to $x_\alpha$ for any $\alpha < \lambda$.

The sequence $\langle S_n^{x(n)} : n < \omega \rangle$ is not tightly stationary since $\mu(\langle S_n \rangle) = \emptyset$ by the choice of $x$. $\square$

The basic idea of Proposition 3.2.1 can be used to force every $\omega$-sequence of regular cardinals to have a stationary sequence which is not tightly stationary.

Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals cofinal in $\mu$ which carries a scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$. If a forcing poset $\mathbb{P}$ has the $\mu$-c.c., it is easy to see that any member of $(\prod_n \mu_n)^{V[G]}$ is $<^*$-below a member of $(\prod_n \mu_n)^V$, so $\langle f_\alpha : \alpha < \kappa^+ \rangle$ remains a scale in $V[G]$. Furthermore, stationary subsets of regular cardinals $> \mu$ are preserved. So a tightly stationary sequence in the ground model must remain tightly stationary in the extension.

Recall that the Cohen forcing $\mathrm{Add}(\omega, \lambda)$ is the poset of partial functions $\lambda \times \omega \to 2$ with finite domain ordered by reverse inclusion.

**Proposition 3.2.2.** *Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals and $2^{\aleph_0} < \mu_0$. Then forcing with* $\mathrm{Add}(\omega, 1)$ *adds a stationary sequence in $\langle \mu_n : n < \omega \rangle$ which is not tightly stationary.*

*Proof.* Choose in $V$ a sequence $\langle (S_n^0, S_n^1) : n < \omega \rangle$ where $(S_n^0, S_n^1)$ is a partition of $\mu_n \cap \mathrm{Cof}(\eta)$ into stationary sets for each $n < \omega$. Let $x : \omega \to 2$ be the real added by $\mathrm{Add}(\omega, 1)$. Then the sequence $\langle S_n^{x(n)} : n < \omega \rangle$ is stationary, since $\mathrm{Add}(\omega, 1)$ is c.c.c. and hence preserves all cardinals and the stationarity of subsets of uncountable regular cardinals.

Recall

**Lemma 2.1.6.** *Suppose $2^{\aleph_0} < \mu_0$. If $\mathbb{P}$ is c.c.c. and $G$ is generic for $\mathbb{P}$ over $V$. If $M \in V[G]$ is tight for $\langle \mu_n : n < \omega \rangle$ and $\mathrm{cf}(M \cap \mu_n) = \omega_1$ for all $n < \omega$, then $\chi_M \in V$.*

By a density argument, for any $f \in V \cap \prod_n \mu_n$, there are infinitely many $n < \omega$ so that $f(n) \notin S_n^{x(n)}$. Therefore, no tight structure can meet $\langle S_n^{x(n)} : n < \omega \rangle$, and hence it is not tightly stationary. □

**Theorem 3.2.1.** *Suppose $2^{\aleph_0} < \aleph_\omega$. In the forcing extension by* $\mathrm{Add}(\omega, \omega_1)$, *for any sequence of cardinals $\langle \mu_n : n < \omega \rangle$, there is a stationary sequence on $\langle \mu_n : n < \omega \rangle$ which is not tightly stationary.*

*Proof.* Let $\langle \dot{\mu}_n : n < \omega \rangle$ be a name for an increasing sequence of regular cardinals below $\kappa$, and suppose that there is a condition $p \in \mathrm{Add}(\omega, \omega_1)$ which forces that $\langle \dot{\mu}_n : n < \omega \rangle$ has no stationary, not tightly stationary sequence.

For each $n < \omega$, let $A_n = \{a_n^i : i < \omega\}$ be a maximal antichain in $\mathrm{Add}(\omega, \omega_1)$ so that each $a_n^i$ forces a value for $\mu_n$. Since $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., each $A_n$ is countable and hence $\gamma := \sup(\bigcup_{n,i} \mathrm{dom}(a_n^i)) < \omega_1$. So $\langle \mu_n : n < \omega \rangle \in V^{\mathrm{Add}(\omega, \gamma)}$, where $\mathrm{Add}(\omega, \gamma)$ is thought of as an initial segment of $\mathrm{Add}(\omega, \omega_1)$. Since $\mathrm{Add}(\omega, \omega_1)$ factors as $\mathrm{Add}(\omega, \gamma) \times \mathrm{Add}(\omega, 1) \times \mathrm{Add}(\omega, \omega_1)$, the result follows from Proposition 3.2.2. □

We remark that the previous result is quite indestructible. For example, if $H$ is generic for $\mathrm{Add}(\omega, \omega_1)$ and $\mathbb{P}$ is Prikry forcing over $V[H]$, then the Levy–Solovay theorem says that the normal ultrafilter $U$ used by $\mathbb{P}$ is in fact generated by a normal ultrafilter $\bar{U}$ in $V$. It is then easy to check that over $V$, $\mathrm{Add}(\omega, \omega_1) * \mathbb{P}$ is forcing equivalent to $\bar{\mathbb{P}} \times \mathrm{Add}(\omega, \omega_1)$, where $\bar{\mathbb{P}}$ is Prikry forcing over $V$ using $\bar{U}$, and therefore in the further Prikry extension, every sequence of regular cardinals carries a stationary sequence which is not tightly stationary.

This gives another method to obtain mutually stationary but not tightly stationary sets, since on any Prikry sequence, every stationary sequence is mutually stationary (Theorem 5.4 of [CFM06], using our slightly weaker definition for mutual stationarity). A similar method works for increasing sequences of measurable cardinals. Compared to the examples of mutually stationary but not tightly stationary sequences in [CFM06], these examples have the disadvantage of requiring large cardinals. However, they are more flexible in the ways that they can be iterated, and this flexibility can be used to obtain results of a global nature, as we will do in Theorem 5.1.2.

## 3.3 With a continuous tree-like scale

We will give a negative answer to this question under the assumption of a continuous tree-like scale. This result, and its proof, are similar to Theorem 3 of [CFS09]. There are not many examples known of sequences of regular cardinals which have no tree-like scales. For one example, see Gitik [Git08], where such a sequence is identified in the generic extension by a Prikry-type forcing. This result, and its proof, are similar to Theorem 3 of [CFS09].

**Theorem 3.3.1.** *Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be continuous and tree-like, and let $\eta < \kappa_0$ be a regular cardinal. There is a sequence $\langle S_n : n < \omega \rangle$, $S_n \subseteq \kappa_n \cap \mathrm{Cof}(\eta)$, such that $\nu(\vec{S}) = \emptyset$ (equivalently $\mu(\langle \kappa_n \setminus S_n \rangle) = \kappa^+$) and $S_n$ is stationary in $\kappa_n$ for all $n$.*

*Proof.* Consider the tree $\mathcal{T}$ of initial segments of members of $\vec{f}$. So a node on level $n$ is a sequence of ordinals of length $n$ where the $m$th term is $< \kappa_m$, but because $\vec{f}$ is tree-like there is no ambiguity to identify the node by its last (i.e., $(n-1)$st) term, so if $\beta \in \kappa_{n-1}$ is a node on the $n$th level of the tree, let $s_n(\beta)$ be sequence identified with it. Let $<_{\mathcal{T}}$ denote the tree order.

The main point of the proof is that this tree can be thinned to be stationarily branching after some point, a general fact about continuous tree-like scales which may be of independent interest.

**Lemma 3.3.1.** *There is a subtree $\mathcal{T}'$ of $\mathcal{T}$ such that there is some $\gamma \in \mathcal{T}'$ compatible with every element of $\mathcal{T}'$ ($\gamma$ is called the stem), and for every $\alpha$ on the nth level of $\mathcal{T}'$, where $n \geq$ level($\gamma$), the set $\{\beta \in \kappa_n : \alpha <_{\mathcal{T}'} \beta\} \cap \mathrm{Cof}(\eta)$ is stationary.*

*Proof.* We first define a game. On the $n$th turn, player I plays $A_n \subset \kappa_n$ nonstationary and player II plays $\alpha_n \in \kappa_n$. In addition player II plays $N \in \omega$ on the 0th turn. Player II wins if $\alpha_n <_{\mathcal{T}} \alpha_{n+1}$ for all $n$ and $\alpha_n \in \mathrm{Cof}(\eta) \setminus A_n$ for all $n > N$. Otherwise player I wins.

We will show that II has a winning strategy. This game is open, hence determined, so towards a contradiction assume that I has a winning strategy $\sigma$. Let $M \prec (H(\kappa^+), \sigma)$, where $M$ is internally approachable of length $\eta$. Then $\chi_M =^* f_\alpha$ for some $\alpha \in \kappa^+$, and set $\alpha_n = f_\alpha(n)$. Choose $N$ so that $\chi_M(n) = f_\alpha(n)$ has cofinality $\eta$ for all $n > N$. We show that II can play the $\alpha_n$ and $N$ against $\sigma$ and win, a contradiction. For each $n > N$, let $B_n = \bigcup_{\beta \in \kappa_{n-1}} \sigma(s_{n-1}(\beta))$ be the union of all possible plays of I according to $\sigma$, where the union ranges over all $\beta$ on the $n$th level of $\mathcal{T}$. Each $\sigma(s)$ is a nonstationary subset of $\kappa_n$, so this union is nonstationary in $\kappa_n$. Furthermore, since $\sigma \in M$, we have that $B_n \in M$, so its complement is a club $C_n$ in $\kappa_n$ which is a member of $M$. Therefore $\alpha_n \in C_n$ for all $n > N$. By the definition of $C_n$, $\alpha_n \notin \sigma(s_{n-1}(\alpha_{n-1}))$.

Let $\tau$ be a winning strategy for II. We may assume that II's 0th move according to $\tau$ does not depend on I's, since by the definition of the game, I's 0th move is meaningless. So let $N$ be the 0th move that II plays. We may also assume that II's first $N$ moves according to $\tau$ do not depend on I's moves. Then the subtree of plays according to $\tau$ in $\mathcal{T}$ is stationarily branching in cofinality $\eta$ with stem of length $N$, since otherwise at a nonstationarily branching play, I could block by playing all successors. This completes the proof of the lemma. $\square$

Now we fix ordinals $\langle \beta_n : N < n < \omega \rangle$ such that $\beta_n \in \mathrm{Cof}(\eta)$ is on the $(n-1)$st level of

$\mathcal{T}'$ and the $\beta_n$ form an antichain in $\mathcal{T}'$. Then let $S_n \subseteq \kappa_n$ be the successors of $\beta_n$ for each $n < \omega$. By stationary branching, $S_n$ is stationary in $\kappa_n \cap \mathrm{Cof}(\eta)$. Since the scale is tree-like, for any $\alpha \in \kappa^+$ there is at most one $n$ such that $f_a(n) \in S_n$, so $\nu(\vec{S}) = \emptyset$. Thus, the theorem is proved.

$\square$

By Lemma 2.2.4, the stationary sequence constructed in the theorem is not tightly stationary.

**Corollary 3.3.2.** *If there is a continuous tree-like scale on* $\prod_{n<\omega} \kappa_n$ *and* $\eta < \kappa_0$*, then there is a stationary sequence* $\langle S_n : n < \omega \rangle$*,* $S_n \subseteq \kappa_n \cap \mathrm{Cof}(\eta)$ *which is not tightly stationary.*

# CHAPTER 4

## Scales in Prikry extensions

There are perhaps three natural approaches for forcing a positive answer to Question 1.1.1: (1) destroying the mutual stationarity of a sequence which is not tightly stationary, (2) making a mutually stationary sequence tightly stationary, or (3) forcing to add a new sequence of regular cardinals for which every mutually stationary sequence is tightly stationary.

The first approach appears to be quite difficult. For example, suppose $\vec{S}$ is a mutually stationary sequence on a sequence of regular cardinals with limit $\mu$, and $\mathbb{P}$ is a $\mu$-c.c. forcing notion. Since $\mu$ is singular, in fact $\mathbb{P}$ is $\nu$-c.c. for some $\nu < \mu$. So for any function $F : [\mu]^{<\omega} \to \mu$ in the extension by $\mathbb{P}$, there is a function $\hat{F} : [\mu]^{<\omega} \to \mu$ in the ground model so that any set closed under $\hat{F}$ which contains $\nu$ as a subset is also closed under $F$. Therefore $\vec{S}$ remains mutually stationary after forcing with $\mathbb{P}$.

Since tight stationarity is connected with the continuous scales on the sequence of regular cardinals, the second approach would involve forcing so that ground model scales are not cofinal (or even just adding new sequences of regular cardinals altogether, as in the third approach). Variants of Prikry forcing are essentially the only techniques known for achieving this. Analysis of the scales on certain products after Prikry-type forcing was done by Jech [Jec90] (ordinary Prikry forcing), Cummings–Foreman [CF10] (supercompact diagonal forcing), and Lambie-Hanson [Lam14] (supercompact diagonal forcing). Below, we give a slight generalization of the result in [Jec90] for the ordinary Prikry forcing.

## 4.1 Prikry forcing preliminaries

Let $\kappa$ be a measurable cardinal, and $j : V \to M$ the ultrapower embedding by a normal measure $U$ on $\kappa$. Let $\mathbf{Pr}$ be the Prikry forcing using $U$. Recall that this is the set of all pairs $(s, A)$ with $s$ a finite increasing sequence from $\kappa$ and $A \in U$, ordered by $(s, A) \leq (s', A')$ if and only if $s \supseteq s'$, $A \subseteq A'$ and $s \setminus s' \subseteq A'$. There is also an auxiliary ordering $\leq^*$ defined by $(s, A) \leq^* (s', A')$ if and only if $s = s'$ and $A \subseteq A'$. We abuse notation and identify a generic $E$ for $\mathbf{Pr}$ with the $\omega$-sequence $\langle \zeta_n : n < \omega \rangle$ it adds. Let LP (for "lower part") be the set of all finite increasing sequences from $\kappa$.

There is one crucial feature of Prikry forcing, which can be used to prove many properties of the generic extension, for example that $\mathbf{Pr}$ does not add bounded subsets of $\kappa$.

**Fact 4.1.1.**

1. For every dense open $D \subseteq \mathbf{Pr}$ and $(s, A) \in \mathbf{Pr}$, there is $n < \omega$ and $(s, A^*) \leq^* (s, A)$ so that any $(s', A') \leq (s, A^*)$ with $|s'| \geq n$ is in $D$.
2. As a consequence, for any $(s, A) \in \mathbf{Pr}$ and any statement $\varphi$ in the forcing language, there is $(s, A^*) \leq^* (s, A)$ so that $\varphi$ is decided by $(s, A^*)$.

These are proven using a diagonal intersection argument. A particularly convenient way of taking diagonal intersections in Prikry forcing can be expressed when there is a sequence of measure one sets $\langle A_s : s \in \mathrm{LP} \rangle$, and we define the diagonal intersection to be

$$\Delta_s A_s = \Delta_\xi \bigcap_{s : s(|s|) - 1) = \xi} A_s.$$

The following characterization of genericity for $\mathbf{Pr}$ is due to Mathias.

**Fact 4.1.2.** A sequence $\langle \zeta_n : n < \omega \rangle$ is generic for $\mathbf{Pr}$ if and only if for each $A \in U$, there is $n_A < \omega$ so that $\zeta_n \in A$ for all $n \geq n_A$.

We will make use of the iterated ultrapowers $M_n$ for $n \leq \omega$. These are defined recursively, together with a commuting system of elementary embeddings $i_{m,n} : M_m \to M_n$ for $m < n$.

1. $M_0 = V$ and $i_{0,1} = j$.

2. $M_{n+1} = \mathrm{Ult}(M_n, i_{0,n}(U))$ and $i_{n,n+1}$ is the ultrapower embedding.

3. $i_{m,n+1} = i_{n,n+1} \circ i_{m,n}$ for $m < n$.

4. $M_\omega$ is the direct limit of the system of ultrapowers $\langle M_n, (i_{m,n})\rangle$, and the maps $i_{m,\omega}$ are the direct limit embeddings.

We list some basic facts about the iterated ultrapowers. For notational simplicity, we write $\kappa_n := i_{0,n}(\kappa)$.

**Fact 4.1.3.** Let $\langle M_n, (i_{m,n})\rangle$ be the system of iterated ultrapowers defined above.

1. $i_{0,n} = j^n$ (that is, $j$ composed with itself $n$ times).

2. Every member of $M_n$ can be written as $i_{0,n}(G)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1})$ for some $G : [\kappa]^n \to V$.

Bukovský and Dehornoy [Deh78] independently proved a connection between the generic extension by Prikry forcing and the iterated ultrapowers.

**Fact 4.1.4.** 1. The sequence $\langle \kappa_0, \kappa_1, \ldots\rangle$ is generic for $i_{0,\omega}(\mathbf{Pr})$ over $M_\omega$.

2. The generic extension $M_\omega[\langle \kappa_0, \kappa_1, \ldots\rangle]$ is equal to the intersection of the $M_n$, $n < \omega$.

Motivated by this result, we write $N_\omega := \bigcap_{n<\omega} M_n$. We will use the fact that $N_\omega$ has the same $\omega$-sequences of ordinals as $V$.

**Definition 4.1.5.** If $c$ is a $\mathbf{Pr}$-name, define $[\![c]\!]$ to be the evaluation of $i_{0,\omega}(c)$ using the $M_\omega$-Prikry-generic sequence $\langle \kappa_0, \kappa_1, \ldots\rangle$.

The definition also relativizes to $\mathbf{Pr}$ below some condition $(s, A)$. Note that $N_\omega$ is also the Prikry extension of $M_\omega$ using the generic sequence $s^\frown\langle \kappa_0, \kappa_1, \ldots\rangle$.

**Definition 4.1.6.** If $s \in \mathrm{LP}$, then for a $\mathbf{Pr}$-name $c$, define $[\![c]\!]_s$ to be the evaluation of $i_{0,\omega}(c)$ using the $M_\omega$-Prikry-generic sequence $s^\frown\langle \kappa_{|s|}, \kappa_{|s|+1}, \ldots\rangle$.

**Lemma 4.1.7.** *Let $\phi$ be a formula in the language of set theory, and $\{c_i : i < n\}$ be* **Pr**-*names. If for every $s \in \mathrm{LP}$ we have*

$$N_\omega \models \phi([\![c_0]\!]_s, \ldots, [\![c_n]\!]_s),$$

*then $\Vdash_{\mathbf{Pr}} \phi(c_0, \ldots, c_n)$.*

*Proof.* For each $s \in \mathrm{LP}$, there is a set $A_s \in U$ so that $(s, A_s)$ decides $\phi(c_0, \ldots, c_n)$. In fact, $(s, A_s) \Vdash \phi(c_0, \ldots, c_n)$; otherwise $(s, A_s) \Vdash \neg\phi(c_0, \ldots, c_n)$ so

$$(s, i_{0,\omega}(A_s)) \Vdash \neg\phi(i_{0,\omega}(c_0), \ldots, i_{0,\omega}(c_n)).$$

But $\kappa_m \in i_{0,\omega}(A_s)$ for each $m < \omega$, so $(s, i_{0,\omega}(A_s))$ is compatible with the $M_\omega$-Prikry-generic sequence $s^\frown\langle \kappa_{|s|}, \kappa_{|s|+1}, \ldots\rangle$, so $N_\omega \models \neg\phi([\![c_0]\!]_s, \ldots, [\![c_n]\!]_s)$, contradiction.

Take $A^* = \Delta_s A_s$. Any condition $(t, B)$ can be strengthened to $(t, B \cap A^*)$, and any further strengthening of this must be compatible with $(t, A_t)$, so we have shown that the set of conditions forcing $\phi(c_0, \ldots, c_n)$ is dense. $\square$

## 4.2 Products and scales

Suppose that $\langle \mu_n : n < \omega\rangle$ is an increasing sequence of regular cardinals in $V[E]$ cofinal in $\kappa$. Using Fact 4.1.1, we can find a name for $\langle \mu_n : n < \omega\rangle$ of a particularly nice form.

**Lemma 4.2.1.** *In $V$, there is a name $\langle \dot\mu_n : n < \omega\rangle$ for $\langle \mu_n : n < \omega\rangle$ so that:*

1. *There are $\sigma : \omega \to \omega$ (the arity function) and $F^n : [\kappa]^{\sigma(n)} \to \kappa$ so that for any $n$, if $|s| = \sigma(n)$, then $\langle s, \kappa\rangle$ forces $\dot\mu_n = F^n(s)$.*

2. *Every ordinal in the image of $F^n$ is a regular cardinal.*

3. *There is a non-decreasing, unbounded function $\rho : \omega \to \omega \cup \{-1\}$ so that*

$$F^n(\xi_0, \ldots, \xi_{\sigma(n)-1}) > \xi_{\rho(n)}$$

*for all $(\xi_0, \ldots, \ldots, \xi_{\sigma(n)-1}) \in [\kappa]^{\sigma(n)}$, where we define $\xi_{-1} = 0$.*

We will call such a name *normal*.

For each $n$, $0 < n < \omega$, and each $\gamma < \kappa_n$, fix functions $G^n_\gamma : [\kappa]^{\sigma(n)} \to \kappa$ such that $i_{0,n}(G^n_\gamma)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1})) = \gamma$, i.e., $G^n_\gamma$ represents $\gamma$ in the $n$th iterated ultrapower.

Define a function $\tau$ on $\kappa_\omega$ by setting $\tau(\gamma)$ to be largest so that $\gamma \in \text{image}(i_{0,\tau(\gamma)})$.

**Lemma 4.2.2.** *Suppose $G : [\kappa]^n \to \kappa$ such that $i_{0,n}(G^n_\gamma)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) \in \text{image}(i_{0,m})$. Then there is $A \in U$ such that $G$ restricted to $A$ does not depend on the first $m$ coordinates of the input, i.e., for all sequences $x_1 < \ldots < x_{m-1}$ and $y_1 < \ldots < y_{m-1}$, and $z_m < \ldots < z_n$ with $x_{m-1}, y_{m-1} < z_m$,*

$$G(x_1, \ldots, x_{m-1}, z_m, \ldots, z_{n-1}) = G(y_1, \ldots, y_{m-1}, z_m, \ldots, z_{n-1}).$$

So by changing the functions $G^n_\gamma$ on a measure zero set, we may assume that $G$ does not depend on the first $\tau(\gamma)$ coordinates of the input. From now on, we enforce this assumption.

For certain sequences of regular cardinals, scales on $\kappa$ in the Prikry extension are closely related to ground model scales on (ground model) singular cardinals. Note that the Prikry forcing cannot add new scales to singular cardinals other than $\kappa$: it does not add any $\omega$-sequences bounded below $\kappa$, and it has the $\kappa$-c.c. so cannot add an $\omega$-sequence unbounded by ground model $\omega$-sequences in the product of regular cardinals above $\kappa$.

**Definition 4.2.3.** A normal name given by $\langle F^n : n < \omega \rangle$ with $\text{dom}(F^n) = [\kappa]^{\sigma(n)}$ is *forgetful* if there is a non-decreasing, unbounded function $\tau : \omega \to \omega$ so that $F^n$ does not depend on the first $\tau(n)$ coordinates.

Forgetfulness will be applied through the following straightforward lemma.

**Lemma 4.2.4.** *Suppose $\langle F^n : n < \omega \rangle$ is a forgetful normal name and $s \in \text{LP}$. Then there is $n_0 < \omega$ so that for all $n \geq n_0$,*

$$[\![F^n]\!]_s = [\![F^n]\!].$$

Suppose that $\langle \mu_n : n < \omega \rangle \in V[E]$ is an increasing sequence of regular cardinals cofinal in $\kappa$. Let $F^n$, $\sigma$, $\rho$ be as in Lemma 4.2.1, and $\tau$ be as in Definition 4.2.3. We will focus on the special case where $\langle \dot\mu_n : n < \omega \rangle$ is forgetful. The theorem will show that in these cases, the pcf structure of $\prod_n \mu_n$ reflects that of $\prod_n [\![\mu_n]\!]$.

Define $k(n) := \min\{\tau(n), \rho(n), \sigma(n)\}$. Since $\tau$, $\rho$, and $\sigma$ are non-decreasing and unbounded in $\omega$, so is $k$.

**Lemma 4.2.5.** *For each $n$, the set of ordinals in* $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot\mu_n]\!]$ *is $< \kappa$-closed and unbounded.*

*Proof.* Fix $G^n < F^n$. Define $H^n$ to be the function

$$(\xi_0, \ldots, \xi_{\sigma(n)-1}) \mapsto \sup_{\alpha_0,\ldots,\alpha_{k(n)-1} < \xi_{k(n)}} G^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}).$$

Clearly $G^n \leq H^n$.

We now show

$$H^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}) < F^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}).$$

Since $k(n) \leq \tau(n)$, $F^n$ doesn't depend on the first $k(n)$ coordinates, and since $k(n) \leq \rho(n)$, $\xi_{k(n)} < F^n(\xi_0, \ldots, \xi_{\sigma(n)-1})$, so $H^n < F^n$ as $F^n(\xi_0, \ldots, \xi_{\sigma(n)-1})$ is a regular cardinal. So $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot\mu_n]\!]$ is unbounded in $[\![\dot\mu_n]\!]$.

Since $j$ is continuous at points of cofinality different from $\kappa$, $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot\mu_n]\!]$ is $< \kappa$-closed. $\square$

For each $n$, let $C_n = \mathrm{image}(i_{0,k(n)}) \cap [\![\dot\mu_n]\!]$.

**Theorem 4.2.1.** *Suppose $\langle \dot\mu_n : n < \omega \rangle$ is an $\omega$-sequence of regular cardinals in $V[E]$ with a forgetful normal name. If $\prod_n C_n$ carries a scale $\langle f_\alpha : \alpha < \lambda \rangle$ in $V$, then in $V[E]$ there is a scale $\langle g_\alpha : \alpha < \lambda \rangle$ on $\prod_n \mu_n$ defined by*

$$g_\alpha(n) = G^{\sigma(n)}_{f_\alpha(n)}(\zeta_0, \ldots, \zeta_{\sigma(n)-1}).$$

*If $\langle f_\alpha : \alpha < \lambda \rangle$ is continuous at a point $\delta$, then so is $\langle g_\alpha : \alpha < \lambda \rangle$.*

*Proof.* For each $\alpha < \lambda$, let $\dot{g}_\alpha$ be the forgetful normal name for $g_\alpha$ given by

$$\langle G^{\sigma(n)}_{f_\alpha(n)}(\zeta_0, \ldots, \zeta_{\sigma(n)-1}) : n < \omega \rangle.$$

For any $s \in \mathrm{LP}$,

$$[\![\dot{g}_\alpha]\!] =^* [\![\dot{g}_\alpha]\!]_s$$

by Lemma 4.2.4. Therefore

$$[\![\dot{g}_\beta]\!]_s <^* [\![\dot{g}_\beta]\!]_s.$$

By Lemma 4.1.7, $g_\alpha <^* g_\beta$ in $V[E]$.

Now we check that $\langle g_\alpha : \alpha < \lambda \rangle$ is cofinal in $\prod_n \mu_n$. Suppose that $h$ is a function in $\prod_n \mu_n$ in $V[E]$, and let $\dot{h}$ be a name for $h$ which is forced to be in $\prod_n \dot{\mu}_n$. Fix $s \in \mathrm{LP}$ arbitrary. By Lemma 4.1.7, $[\![h(n)]\!]_s < [\![\dot{\mu}_n]\!]_s$ for all $n < \omega$. By Lemma 4.2.4, $[\![\dot{\mu}_n]\!] =^* [\![\mu_n]\!]_s$. Therefore $[\![h(n)]\!]_s < [\![\dot{\mu}_n]\!]$ for all but finitely many $n$. Since $\langle f_\alpha : \alpha < \lambda \rangle$ is a scale on $\prod_n [\![\dot{\mu}_n]\!]$, there is some $\alpha < \lambda$ so that $[\![h]\!]_s <^* f_\alpha = [\![\dot{g}_\alpha]\!]$. But $[\![\dot{g}_\alpha]\!] =^* [\![\dot{g}_\alpha]\!]_s$, so $[\![h]\!]_s <^* [\![\dot{g}_\alpha]\!]_s$. As $s$ was arbitrary, $h <^* g_\alpha$ in $V[E]$.

The proof for the continuity part of the theorem is similar. Suppose that $h$ is a function in $\prod_n g_\delta(n)$ in $V[E]$, and let $\dot{h}$ be a name for $h$ which is forced to be in $\prod_n \dot{g}_\delta(n)$. Fix $s \in \mathrm{LP}$ arbitrary. Applying Lemma 4.1.7 and Lemma 4.2.4 as before, $[\![h(n)]\!]_s < [\![\dot{g}_\delta(n)]\!]$ for all but finitely many $n$. Since $\langle f_\alpha : \alpha < \lambda \rangle$ is continuous at $\delta$, there is some $\alpha < \delta$ so that $[\![h]\!]_s <^* [\![\dot{g}_\alpha]\!] =^* [\![\dot{g}_\alpha]\!]_s$, so $[\![h]\!]_s <^* [\![\dot{g}_\alpha]\!]_s$. This implies that $h <^* g_\alpha$ in $V[E]$. □

Finally, we can apply the analysis of these scales to tight stationarity.

**Theorem 4.2.2.** *Suppose $\langle \mu_n : n < \omega \rangle \in V[E]$ is given by a forgetful normal name satisfying the hypotheses of Theorem 4.2.1. Then there is a stationary but not tightly stationary sequence $\langle S_n : n < \omega \rangle$.*

*Proof.* Let $\langle F^n \rangle$, $\sigma$, $\tau$, and $k$ be defined for the forgetful normal name for $\langle \mu_n : n < \omega \rangle \in V[E]$ as in Lemma 4.2.1 and Definition 4.2.3. Let $\eta$ be the fixed uncountable cofinality to which we are restricted. Fix a name $\langle \dot{g}_\alpha : \alpha < \lambda \rangle$ for a scale on $\prod_n \mu_n$ as in Theorem 4.2.1.

We will construct a name for $S_n$ for each $n < \omega$. By restricting to a final segment, we may assume that $k(n) > 0$. For each $(\xi_{\rho(n)}, \ldots, \xi_{\sigma-1}) \in [\kappa]^{\sigma(n)-k(n)}$, partition $F^n(\xi_0, \ldots, \xi_{\sigma-1}) \cap \mathrm{Cof}(\eta)$ into $\xi_{k(n)}$ disjoint stationary sets, noting that $F^n$ does not depend on its first $k(n)$ arguments and $\xi_{k(n)} \leq F(\xi_0, \ldots, \xi_{\sigma-1})$. Let $T_n^{(\xi_{kn}, \ldots, \xi_{\sigma(n)-1})}$ be an injection from $\xi_{k(n)}$ into the collection of these stationary sets. Define $S_n : [\kappa]^{\sigma(n)} \to \kappa$ by

$$S_n(\xi_0, \ldots, \xi_{\sigma(n)-1}) = T_n^{(\xi_{k(n)}, \ldots, \xi_{\sigma(n)-1})}(\xi_{k(n)-1}).$$

This function gives a name for the set $S_n \in V[E]$.

We will show that in $V[E]$, $\nu(\langle S_n \rangle) = \emptyset$, which is stronger than required by Lemma 2.2.4 to show that $\langle S_n : n < \omega \rangle$ is not tightly stationary. Suppose otherwise, so there is $(s, A) \in \mathbf{Pr}$ and $\alpha < \lambda$ so that $(s, A)$ forces $\alpha \in \nu(\langle \dot{S}_n \rangle)$. By extending $(s, A)$ we may assume that there is some fixed $n < \omega$ with $k(n) > |s|$ so that $(s, A)$ forces $\dot{g}_\alpha(n) \in \dot{S}_n$.

Take $\xi_0, \ldots, \xi_{\sigma(n)-1} \in [A]^{\sigma(n)-k(n)}$ so that $\xi_i = s(i)$ for $i < |s|$ and $A \cap (\xi_{k(n)-1}, \xi_{k(n)}) \neq \emptyset$. Choose $\xi'_{k(n)-1} \in A \cap (\xi_{k(n)-1}, \xi_{k(n)})$. Let us write $\bar{\xi}$ for $\xi_0, \ldots, \xi_{\sigma(n)-1}$ and $\bar{\xi}'$ for the sequence obtained from $\bar{\xi}$ by replacing $\xi_{k(n)-1}$ with $\xi'_{k(n)-1}$.

Since $(s, A)$ forces $\dot{g}_\alpha(n) \in \dot{S}_n$,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) \in S_n(\bar{\xi}).$$

By the same reasoning,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}') \in S_n(\bar{\xi}').$$

Since $\bar{\xi}$ and $\bar{\xi}'$ differ only in the $k(n) - 1$ coordinate, and $G^{\sigma(n)}_{f_\alpha(n)}$ does not depend on its first $k(n)$ arguments, $G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) = G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}')$. Therefore,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) \in S_n(\bar{\xi}) \cap S_n(\bar{\xi}'),$$

but this is impossible as $S_n(\bar{\xi})$ and $S_n(\bar{\xi}')$ were chosen to be disjoint. $\square$

Still, we do not know what happens when the product is not forgetful.

## 4.3 Examples

In this section we give some examples illustrating the main ideas of this chapter. The following basic fact about the elementary embedding $j$ will be useful.

**Fact 4.3.1.** For any $\lambda$, if $\mathrm{cf}(\lambda) > \kappa$, then $\mathrm{cf}(j(\lambda)) = \mathrm{cf}(\lambda)$.

*Proof.* The set $j``\lambda$ has cofinality $\lambda$ and is unbounded in $j(\lambda)$, since any function $\kappa \to \lambda$ representing an ordinal less than $j(\lambda)$ must have range bounded in $\lambda$. $\square$

The first two examples can be found in Jech [Jec90]. Let $\langle \zeta_n : n < \omega \rangle$ denote the Prikry sequence.

*Example* 1. In the Prikry extension $V[G_{\mathbf{Pr}}]$, $\prod_n \zeta_n$ carries a scale of length $\mathrm{cf}(j(\kappa))$.

*Proof.* For $\langle \zeta_n : n < \omega \rangle$, the normal name is given by $\sigma(n) = n + 1$ and $F^n(s) = s_n$ for $s$ of length $n + 1$. So

$$[\![\dot{\mu}_n]\!] = i_{0,\omega}(\mathrm{id})(\kappa_n) = \kappa_n = j^n(\kappa).$$

Now $\mathrm{cf}(j(\kappa)) > \kappa$, so $\mathrm{cf}(j^n(\kappa)) = \mathrm{cf}(j(\kappa))$ for all $1 \leq n < \omega$.

We check that there is a scale in $\prod_n C_n$: for each $n > 0$ there is a sequence $\langle \kappa_{n,\alpha} : \alpha < \mathrm{cf}(j(\kappa)) \rangle \subseteq C_n$ cofinal in $C_n$. For each $\alpha$, define $f_\alpha$ by $f_\alpha(n) = \kappa_{n,\alpha}$ if $n > 0$, and 0 otherwise. The sequence $\langle f_\alpha : \alpha < \mathrm{cf}(j(\kappa)) \rangle$ is cofinal in $\prod_n C_n$. $\square$

*Example* 2. Let $\gamma$ be an ordinal with $\kappa < \gamma < j(\kappa)$, and $G_\gamma : \kappa \to \kappa$ represent $\gamma$ in the ultrapower by $U$. In the Prikry extension $V[G_{\mathbf{Pr}}]$, $\prod_n G_\gamma(\zeta_n)$ carries a scale of length $\mathrm{cf}(\gamma)$.

*Proof.* The normal name is given by $\sigma(n) = n+1$ and $F^n(s) = G_\gamma(s_n)$ for $s$ of length $n+1$. So

$$[\![\dot{\mu}_n]\!] = i_{0,\omega}(G_\gamma)(\kappa_n) = i_{n+1,\omega}(i_{0,n}(G_\gamma)(\kappa_n)) = i_{0,\omega}(G_\gamma)(\kappa) = \gamma.$$

So there is a scale of length $\mathrm{cf}(\gamma)$ in $\prod_n C_n$. $\square$

The last example does not quite fall into either of the previous two cases.

*Example* 3. Suppose that $j(\kappa) > \kappa^{+\omega}$. Let $m : \omega \to \omega$ be so that in $V$ there is a scale of length $\kappa^{+\omega+1}$ on $\prod_n \kappa^{+m(n)}$. In the Prikry extension $V[G_{\mathbf{Pr}}]$, $\prod_n G_{\kappa^{+m(n)}}(\zeta_n)$ carries a scale of length $\kappa^{+\omega+1}$.

In Chapter 7, we will encounter a similar example using a more complicated Prikry-type forcing.

# CHAPTER 5

# Mutually stationary but not tightly stationary sequences

In this chapter, we will construct a model where every increasing $\omega$-sequence of regular cardinals has a mutually stationary sequence which is not tightly stationary.

First, we will show that for a sequence of regular cardinals with interleaved supercompacts, every stationary sequence is mutually stationary. Our argument is a supercompact version of the proof of Theorem 5.2 in [CFM06].

Suppose $\kappa$ is $\lambda$-supercompact, and $U$ is a normal, fine ultrafilter on $[\lambda]^{<\kappa}$.

For any $n$ and $x, y \in [\lambda]^{<\kappa}$, say that $x \precneqq y$ if $|x| < |y \cap \kappa|$. Say that $x < y$ if $x \precneqq y$ and $x \in \mathrm{Sk}(y)$, where the Skolem hull is computed in the structure $(H(\lambda), \in, \lhd)$. Here $\lhd$ is a fixed well-ordering of $H(\theta)$ (this is a standard device useful for making things definable without parameters).

Supercompactness measures satisfy the following partition property (for a reference, see Kanamori [Kan08]):

**Fact 5.0.1.** For any $n < \omega$ and $f : ([\lambda]^{<\kappa})^n \to 2$, there is $Y \in U$ homogeneous for $f$, i.e., there is $i \in 2$ so that $f(x_0, x_1, \ldots, x_{n-1}) = i$ for any $x_0 < x_1 < \cdots < x_{n-1}$ all from $Y$.

**Proposition 5.0.2.** *Suppose $\langle \lambda_n : n < \omega \rangle$ is an increasing sequence of regular cardinals and $\langle \kappa_n : n < \omega \rangle$ is a sequence of cardinals so that for each $n < \omega$,*

1. *$\kappa_n$ is $\lambda_n$-supercompact,*
2. *$\lambda_{n-1} < \kappa_n \leq \lambda_n$,*

*3.* $\zeta^\xi < \lambda_n$ *for cardinals* $\xi < \kappa_n$ *and* $\zeta < \lambda_n$.

*Then any sequence* $\langle S_n : n < \omega \rangle$ *with* $S_n \subseteq \lambda_n \cap \mathrm{Cof}(< \kappa_n)$ *is mutually stationary.*

*Proof.* For each $0 < n < \omega$, let $U_n$ be a normal, fine ultrafilter on $[\lambda_n]^{<\kappa_n}$.

Let $\mathcal{A}$ be an arbitrary expansion of $(H(\theta); \in, \lhd, \langle \lambda_n, \kappa_n, U_n : n < \omega \rangle)$ for $\theta = \sup_n \lambda_n$. For each $n$ with $0 < n < \omega$, $U_n$ concentrates on the closed unbounded set $X_n$ of structures $x$ so that

1. $\lambda_{n-1} \subseteq x$,
2. $x \cap \kappa_n \in \kappa_n$,
3. $\mathrm{Sk}^{\mathcal{A}}(x) \cap \lambda_n = x$.

Notice that this condition implies that $X_m \cap X_n = \emptyset$ for $m \neq n$.

Given a finite $<$-increasing $\bar{x} \subset \bigcup_i X_i$, define the *type* of $\bar{x}$ to be the function $n \mapsto |\bar{x} \cap X_n|$. If $a \subseteq \omega$, then we define $t \restriction a$ to be the type which is equal to $t$ on $a$, and takes value 0 elsewhere.

We will construct $\langle Y_n : 0 < n < \omega \rangle$ so that:

1. For each $n$, $Y_n \subseteq X_n$ and $Y_n \in U_n$,
2. (Indiscernibility) If $\varphi$ is a formula in the language of $\mathcal{A}$ using only ordinal parameters $\bar{c} \subseteq \sup_n \lambda_n$, and $\bar{x}$, $\bar{y}$ are finite $<$-increasing sequences from $\bigcup_i Y_i$ of the same type which fit as the free variables of $\varphi$ so that

   $$\{x \in \bar{x} : \bar{c} \not\in x\} = \{y \in \bar{y} : \bar{c} \not\in y\},$$

   then

   $$\varphi(\bar{c}, \bar{x}) \text{ iff } \varphi(\bar{c}, \bar{y}).$$

Fix a formula $\varphi$ and parameters $\bar{c}$ and a type $t$ with $\sum_i t(i)$ equal to the number of free variables of $\varphi$. We will construct $\langle Y_n^{\varphi,t,\bar{c}} : 0 < n < \omega \rangle$ that works for $\varphi$, $t$, and $\bar{c}$, and then set

$Y_n = \bigcap_{\varphi,t} \Delta_{\bar{c}} Y_n^{\varphi,t,\bar{c}}$ (we abuse notation in this diagonal intersection by identifying $\bar{c}$ with its ordinal code under some coding of finite tuples definable over $\mathcal{A}$ and closed at inaccessible cardinals).

Let $n_t$ be the least $n < \omega$ so that $t(n') = 0$ for all $n' \geq n$. If $n > n_t$, then set $Y_n^{\varphi,t,\bar{c}}$ equal to $X_n$.

Now by induction on $m < n_t$ we define $Y_{n_t-m}^{\varphi,t,\bar{c}} \in U_{n_t-m}$ so that if $\bar{x}, \bar{y}$ are of type $t$ with:

1. $\bar{x} \cap X_n \subseteq Y_n^{\varphi,t,\bar{c}}$ for all $n \geq n_t - m$
2. $\bar{x} \cap \bigcup_{i<n_t-m} X_i = \bar{y} \cap \bigcup_{i<n_t-m} X_i$,

then $\varphi(\bar{c}, \bar{x})$ iff $\varphi(\bar{c}, \bar{y})$.

So fix $m < n_t$ and suppose that $Y_n^{\varphi,t,\bar{c}}$ has already been defined for $n > n_t - m$. For each $\bar{v} \subseteq \bigcup_{n<n_t-m} X_n$ of type $s\restriction(n_t - m)$, let $Y_{n_t-m}^{\varphi,t,\bar{c},\bar{v}} \in U_{n_t-m}$ be homogeneous for the function mapping a length $t(n_t-m)$-sequence $\bar{u}$ in $[\lambda_{n_t-m}]^{<\kappa_{n_t-m}}$ to the truth value of $\varphi(\bar{c}, \bar{v}, \bar{u}, \bar{u}_{\text{upper}})$, where $\bar{u}_{\text{upper}}$ is a sequence of the appropriate type from $\bigcup_{n>n_t-m} Y_n^{\varphi,t,\bar{c}}$. By the induction hypothesis, this does not depend on the choice of $\bar{u}_{\text{upper}}$. Then define $Y_{n_t-m}^{\varphi,t,\bar{c}} = \bigcap_{\bar{v}} Y_{n_t-m}^{\varphi,t,\bar{c},\bar{v}}$. This completes the construction.

Suppose $n < \omega$. For each $x \in [\lambda_n]^{<\kappa_n}$ and $\xi < \lambda_n$, define $y(x, \xi)$ to be the $\lhd$-least structure $y > x$ in $Y_n$ containing $\xi$ as an element. By our cardinal arithmetic assumptions, the function $\xi \mapsto \sup_{x\subseteq\xi}\{\sup\, y(x, \xi)\}$ maps $\lambda_n$ into $\lambda_n$, so the closure points form a club $C_n$. Therefore, we can take $I_n \subseteq Y_n$ which is $<$-increasing of limit order-type so that $\gamma_n := \sup \bigcup I_n \in S_n$ for each $0 < n < \omega$.

Finally set $W = \mathrm{Sk}^{\mathcal{A}}(\bigcup_i I_i)$. It remains to check that $\sup(W \cap \lambda_n) = \gamma_n$ for each $n$. Suppose $\delta \in W \cap \lambda_n$ for some $n$. Then $\delta = t(\bar{z})$ for some $\mathcal{A}$-term $t$ and finite $\bar{z} \subseteq \bigcup_i I_i$.

Let $\bar{z}' = \bar{z} \cap \bigcup_{i\leq n} X_i$. By indiscernibility, for any $<$-increasing sequence $\bar{u}$ from $\bigcup_n Y_n$ of the same type as $\bar{z} \setminus \bar{z}'$,

$$\delta = t(\bar{z}', \bar{u}).$$

Therefore $\delta$ can be defined over $\mathcal{A}$ with parameter $\bar{z}'$ as "the unique ordinal for which there exist measure one sets so that $\delta = t(\bar{z}', \bar{u})$ whenever $\bar{u}$ is an increasing sequence of the right type taken from those measure one sets." Now take $x \in I_n$ with $x > z$ for every $z \in \bar{z}'$. Since $\mathrm{Sk}^{\mathcal{A}}(x)$ contains $\bar{z}'$, $\delta \in x$, so $\delta < \sup(x) < \sup \bigcup I_n$. $\square$

## 5.1 The main construction

Koepke [Koe07] adapted the argument from [CFM06] to work for cardinals which were formerly measurable but have been collapsed by forcing. Thus, he was able to force to get a mutual stationarity property, for example, on the sequence $\langle \aleph_{2n+1} : n < \omega \rangle$ (note that there is a gap between successive members of this sequence). We can adapt our Proposition 5.0.2 using his methods, and combine this with Theorem 3.2.1 to get a global result on the existence of mutually stationary, not tightly stationary sequences.

The gap between successive cardinals in the sequence seems to be crucial to this argument. For example, Koepke and Welch [KW06] showed that:

1. the existence of a measurable cardinal is equiconsistent with the statement that there is some sequence of regular cardinals where every stationary sequence concentrating on ordinals of cofinality $\omega_1$ is mutually stationary, and

2. to have that every sequence of stationary sets on $\langle \aleph_n : n < \omega \rangle$ concentrating on ordinals of cofinality $\omega_1$ is mutually stationary requires an inaccessible limit of measurable cardinals (and no upper bound is currently known).

The first theorem of this section is a prototype for those which follow.

**Theorem 5.1.1.** *If there is a proper class of supercompact cardinals, then there is a class forcing extension so that for every increasing $\omega$-sequence of regular cardinals $\langle \lambda_n : n < \omega \rangle$ so that for each $n$ the interval $(\lambda_n, \lambda_{n+1})$ contains at least three cardinals, any stationary sequence on cofinality $\omega_1$ is mutually stationary.*

*Proof.* The basic strategy of the proof will be to start from a proper class of supercompact cardinals and force with collapsing posets preserving only

1. cardinals from the given class,
2. limits of cardinals from the class,
3. and ground model successors of cardinals of either type (1) or (2).

We will use an argument adapted from [Koe07] to show that for any increasing $\omega$-sequence $\langle \lambda_n : n < \omega \rangle$ of regular cardinals in the extension so that for each $n < \omega$ there is a formerly supercompact cardinal $\kappa_n$ so that $\lambda_{n-1} < \kappa_n < \lambda_n$, every stationary sequence of cofinality $\omega_1$ ordinals is mutually stationary. Koepke's result is an equiconsistency with the existence of a measurable cardinal, but certain aspects of his proof simplify in our case using the stronger large cardinal assumptions.

By doing some preliminary forcing if necessary, assume Martin's Axiom $\mathsf{MA}(\aleph_1)$, $2^{\aleph_0} < \aleph_\omega$, and $\mathsf{GCH}$ above $\omega_2$. Force so that there is a proper class of indestructibly supercompact cardinals and $\mathsf{GCH}$ is preserved at these supercompact cardinals (this also preserves Martin's Axiom). Let $S = \langle \mu_\xi : \xi < \mathsf{ON} \rangle$ be a continuous increasing sequence so that $\mu_0 = \omega_1$ and $\mu_\xi$ is one of these indestructibly supercompacts for every successor ordinal $\xi$. Let $\mathbb{P}$ be the class length Easton support product of the posets $\mathbb{Q}_{\mu_\xi} := \mathrm{Col}^V(\mu_\xi^+, < \mu_{\xi+1})$ for each $\mu_\xi \in S$. The final model will be a model of $\mathsf{ZFC}$, see Jech [Jec13] for details on class forcing.

Let $G$ be generic for $\mathbb{P}$. Write $G \restriction \mu$ for $G \cap \mathbb{P} \restriction \mu$. In the final model the uncountable cardinals are all of the form $\mu_\xi$ or $\mu_\xi^+$ for an ordinal $\xi$.

Suppose $\langle \lambda_n : n < \omega \rangle$ is an increasing sequence of regular cardinals of $V[G]$ with limit $\lambda = \sup_n \lambda_n$. Assuming that there are at least three $V[G]$ cardinals in each interval $(\lambda_n, \lambda_{n+1})$, we can find a sequence of $V$-supercompact cardinals $\langle \kappa_n : n < \omega \rangle$ so that $\kappa_n = \mu_{\xi_n+1}$ (i.e., $\kappa_n$ has a successor index in the sequence of supercompacts), and letting $\kappa_n' = \mu_{\xi_n}$,

$$\lambda_{n-1} \leq \kappa_n' < \kappa_n < \lambda_n$$

for all $0 < n < \omega$.

Let $\mathbb{P}^{(n)} = \mathbb{Q}_{\kappa_n'}$, and $\mathbb{R}^{(n)} = \mathbb{P}\restriction\kappa_n' \times \mathbb{P} \setminus \kappa_n$ be the quotient of $\mathbb{P}$ by $\mathbb{P}^{(n)}$. Each $\kappa_n$ remains supercompact after forcing with $\mathbb{R}^{(n)}$, since for each $n$ this poset factors into the product of a poset which is $\kappa_n$-directed closed and a poset of size $< \kappa_n$.

For $p \in \mathbb{P}^{(n)}$ and $\alpha < \kappa_n$, we will use the notation $p\restriction\alpha$ to denote the condition given by the restriction of $p$ to domain $\mathrm{dom}(p) \cap (\alpha \times (\kappa_n')^+)$.

We will show that in $V[G]$, any sequence of stationary subsets of $\langle \lambda_n : n < \omega \rangle$ concentrating on cofinality $\omega_1$ is mutually stationary. Let $g_n, H_n$ be the generics for $\mathbb{P}^{(n)}, \mathbb{R}^{(n)}$, respectively, determined by $G$. Fix $\dot{F}$ a $\mathbb{P}\restriction\lambda$-name for a function $F : [\lambda]^{<\omega} \to \lambda$ and $\langle \dot{S}_n : n < \omega \rangle$ a $\mathbb{P}$-name for a sequence of stationary sets of cofinality $\omega_1$ points in $\langle \lambda_n : n < \omega \rangle$.

We will find $W \subseteq \lambda$ so that

$$\sup(\dot{F}``[W]^{<\omega} \cap \lambda_n) \leq \sup(W \cap \lambda_n) \in S_n$$

for each $n < \omega$. Since the choice of $F$ was arbitrary, this suffices for mutual stationarity of $\langle S_n : n < \omega \rangle$. We will abuse notation slightly to write $\dot{F}(\bar{x})$ for $\dot{F}(\sup(x_0), \ldots, \sup(x_n))$, where $\bar{x} = (x_0, x_1, \ldots, x_n)$. Let $X_n$ be defined as in the proof of Proposition 5.0.2, using $\kappa_n, \lambda_n$.

In $V[G]$, let $\theta$ be $(2^\lambda)^+$ and

$$\mathcal{A} = (H(\theta); \in, \lhd, G, \langle \lambda_n, \kappa_n, U_n : n < \omega \rangle, \mathbb{P}, \dot{F}, p)$$

and take $\tilde{N} \prec \mathcal{A}$. Define $N = \tilde{N} \cap V$, so $\tilde{N} = N[G]$.

Since $N$ is countable, $N \in V$. Fix a sequence $\langle e_k : k < \omega \rangle$ so that for every $e < \omega$, there are infinitely many $k$ with $e = e_k$. The number $e_k$ will correspond to the arity of a function in the $k$th step of our construction. We will say that $\langle \sigma_n^k : k < \omega \rangle$ is a *system with stem* $p \in \mathbb{P}^{(n)}$ *and domain* $Y \in U_n$ if $\sigma_n^k : [Y']^{e_k} \to \mathbb{P}^{(n)}$ for some $Y' \supseteq Y$, and $p \leq \sigma_n^k(\bar{x})\restriction(\min(\bar{x}) \cap \kappa_n)$.

Define $Y_n = \bigcap(U_n \cap N[H_n])$, the intersection of all measure one sets in $N[H_n]$. Since $N[H_n]$ is countable, $Y_n \in U_n$.

Let $\langle F_n^j : j < \omega \rangle$ be an enumeration of the functions $[\bigcup_{i \leq n} X_i]^{<\omega} \to \lambda_n$ in $N[G\restriction\lambda_n]$.

**Claim 5.1.1.** *For every $n$, there are $p \in g_n$ and $\sigma^k$ a system with stem $p$ and domain $Y_n$ satisfying the following properties:*

1. *For every $\tilde{k} < \omega$, every $t > e_{\tilde{k}}$, and any $a_0, a_1 \in [t]^{e_{\tilde{k}}}$, there is $k > \tilde{k}$ so that if $\sigma^{\tilde{k}}(\bar{x} \restriction a_0)$ and $\sigma^{\tilde{k}}(\bar{x} \restriction a_1)$ are compatible, then*

$$\sigma^k(\bar{x}) \leq \sigma^{\tilde{k}}(\bar{x} \restriction a_0), \sigma^{\tilde{k}}(\bar{x} \restriction a_1).$$

2. *For every $\tilde{k}$, every $t > e_{\tilde{k}}$, and any $a \in [t]^{e_{\tilde{k}}}$, there is $k > \tilde{k}$ so that $e_k = t$ and $\sigma^k(\bar{x}) \leq \sigma^{\tilde{k}}(\bar{x} \restriction a)$.*

3. *$\sigma^k(\bar{x})$ forces values for $F_n^j(\bar{y}, \bar{z})$ for all $j < k$, all $\bar{y}, \bar{z}$ of the appropriate type with $\bar{y} \subseteq \bigcup_{i<n} X_i$ and $\bar{z} \subseteq \bar{x}$.*

4. *For each $k$, $\sigma^k \in N[H_n]$.*

*Proof of Claim 5.1.1.* The construction is by induction on $k$. Fix in advance a suitable bookkeeping so that for every $\tilde{k}, t, a_0, a_1$ as in (1) or (2), there is $k > \tilde{k}$, a stage with the correct arity where we construct to satisfy the corresponding clause. For each $k < \omega$, we will construct in $N[H_n]$ the following objects:

- $p^k \in g_n$,
- $Y^k \in U_n$,
- $\langle \sigma^k(\bar{x}) : |\bar{x}| = e_k \rangle$ which satisfies the conditions required by the bookkeeping and that for all increasing $\bar{x}$ from $Y^k$,

$$\sigma^k(\bar{x}) \restriction (\min(\bar{x}) \cap \kappa_n) = p^k.$$

Start with some $p^0 \in g_n$ satisfying the demands of (3); this is possible since $\mathbb{P}^{(n)}$ is $\lambda_{n-1}$-distributive over $N[H_n]$. Suppose we have already constructed $p^i$, $Y^i$, and $\sigma^i$ for $i \leq k$. We will describe the construction for stage $k+1$.

Working below an arbitrary condition $q \leq p^k$, build conditions $\sigma^{k+1,q}(\bar{x})$ for each $\bar{x}$ with $|\bar{x}| = e_{k+1}$ satisfying the demands of the bookkeeping and of property (3). There are $p^{k+1,q} \in \mathbb{P}^{(n)}$ and $Y^{k+1,q} \in U_n$ so that for any $<$-increasing $\bar{x} \in [Y^{k+1,q}]^{e_{k+1}}$, $\sigma^{k+1,q}(\bar{x})\restriction(\min(\bar{x}) \cap \kappa_n) = p^{k+1,q}$. The set $D^{k+1} = \{p^{k+1,q} : q \leq p^k\}$ is dense below $p^k$ and a member of $N[H_n]$, so there is $p^{k+1} \in g_n \cap D^{k+1} \cap N$, and we can choose $Y^{k+1}$ and $\sigma^{k+1}$ to be the corresponding objects in $N[H_n]$.

Let $p_n := \bigcup_k p^k$. By definition of $Y_n$, $Y_n \subseteq \bigcap_k Y^k$, so $p_n$ is a stem for the system restricted to domain $Y_n$. This completes the construction. $\square$

Using an argument similar to the proof of Proposition 5.0.2, $Y_n$ satisfies the following indiscernibility property: if $\varphi$ is a formula in the language of $\mathcal{A}$ using only ordinal parameters $\bar{c} \subseteq \lambda$, and $\bar{x}$, $\bar{y}$ are finite increasing sequences from $Y_n$ of the same type which fit as the free variables of $\varphi$ so that

$$\{x \in \bar{x} : \bar{c} \notin x\} = \{y \in \bar{y} : \bar{c} \notin y\},$$

then

$$\varphi(\bar{x}) \text{ iff } \varphi(\bar{y}).$$

Define $C_n \subseteq \lambda_n$ to be the club of closure points of the function mapping $\beta < \lambda_n$ to the supremum of the values of $F^\ell_n(\bar{z})$, $\ell < \omega$ and $\sup(\max(\bar{z})) \leq \beta$, forced by some $\sigma^k_m(\bar{x})$ with $m \leq n$, $k < \omega$, and $\sup(\max(\bar{x})) \leq \beta$.

Suppose $I \subseteq Y_n$ is $<$-increasing of order-type $\omega_1$ with $\sup(\bigcup I) \in C_n$. Define $\mathbb{P}_{n,I}$ to be the subposet of $\mathbb{P}^{(n)}$ of conditions of the form $\sigma^k_n(\bar{x})$ for some $k < \omega$ and finite $<$-increasing $\bar{x} \subseteq I$.

**Claim 5.1.2.** *The poset $\mathbb{P}_{n,I}$ is c.c.c.*

*Proof of Claim 5.1.2.* Suppose otherwise, so there is an uncountable antichain $A$ in $\mathbb{P}_{n,I}$. We can thin to assume that there is a single $k < \omega$ so that $A = \{\sigma^k_n(\bar{z}_i) : i < \omega_1\}$, where the collection of $\bar{z}_i$ forms a $\Delta$-system with root $r$. There is a uncountable $B \subseteq \omega_1$ so that for every $i \in B$,

$$\max(r \cap X_n) < \min((\bar{z}_i \setminus r) \cap X_n).$$

Pick some $i_0 \in B$. There is an uncountable $B' \subseteq B$ so that for every $i \in B'$, and every $n < \omega$

$$\max(\bar{z}_{i_0} \cap X_n) < \min((\bar{z}_i \setminus r) \cap X_n).$$

Pick some $i_1 \in B'$. Then we claim that $\sigma^k(\bar{z}_{i_0})$ and $\sigma^k(\bar{z}_{i_1})$ are compatible in $\mathbb{P}_I$. By (1) in the construction of $\langle \sigma^k : k < \omega \rangle$, it suffices to check that they are compatible in $\mathbb{P}^{(n)}$.

Let $\beta_0 := \min((\bar{z}_{i_0} \setminus r) \cap X_n) \cap \kappa_n$ and $\beta_1 := \min((\bar{z}_{i_1} \setminus r) \cap X_n) \cap \kappa_n$. Since $\kappa_n$ is inaccessible, $\sigma^k_n(\bar{z}_{i_0})$ has support bounded in $\kappa_n$. There is some $z \in Y_n$ above every member of $I$ so that the support of $\sigma^k_n(\bar{z}_{i_0})$ is a subset of $z \cap \kappa_n$. By indiscernibility, the support of $\sigma^k_n(\bar{z}_{i_0})$ must be a subset of $\beta_1$.

Let $\rho_n$ be the $\lhd$-least bijection $\mathbb{P}^{(n)} \to \kappa_n$ with the property that if $p \in \mathbb{P}^{(n)}$ has support which is bounded in an inaccessible $\beta$, then $\rho_n(p) < \beta$. Now

$$\gamma := \rho_n(\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0) < \beta_0,$$

so $\rho_n(\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0) \in x$ for all $x \in ((\bar{z}_{i_0} \cup \bar{z}_{i_1}) \setminus r) \cap X_n$.

Let $\phi(\bar{z})$ be the statement

$$\gamma = \rho_n(\sigma^k_n(\bar{z}) \restriction \beta_{\bar{z}}),$$

where $\beta_{\bar{z}} = \min((\bar{z} \setminus r) \cap X_n) \cap \kappa_n$. The statement holds for $\bar{z} = \bar{z}_{i_0}$. By indiscernibility it then holds also for $\bar{z} = \bar{z}_{i_1}$. So

$$\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0 = \sigma^k_n(\bar{z}_{i_1}) \restriction \beta_1.$$

Therefore, $\sigma^k_n(\bar{z}_{i_0}) = \sigma^k_n(\bar{z}_{i_0}) \restriction \beta_1$ is compatible with $\sigma^k_n(\bar{z}_{i_1})$. $\square$

For each $\bar{x} \subseteq I, j < \omega$ define

$$D_{n,\bar{x},j} := \{p \in \mathbb{P}_{n,I} : p \text{ forces a value for } \dot{F}^j_n(\bar{x})\}.$$

The set $D_{n,\bar{x},j}$ is dense in $\mathbb{P}_{n,I}$, since for any $\sigma_n^k(\bar{z}) \in \mathbb{P}_{n,I}$, there is $k' \geq j$ so that $|\bar{z} \cup \bar{x}| = e_{k'}$ and $\sigma_n^{k'}(\bar{z} \cup \bar{x}) \leq \sigma_n^k(\bar{z})$ by condition (2) in the construction of $\langle \sigma_n^k : k < \omega \rangle$, and $k''$ so that $e_{k''} = e_{k'}$ and $\sigma_n^{k''}(\bar{z} \cup \bar{x}) \leq \sigma^{k'}(\bar{z} \cup \bar{x})$ forces a value for $\dot{F}_n^j(\bar{x})$ by condition (3).

Using Martin's Axiom, there is a generic filter $G_{n,I} \subseteq \mathbb{P}_{n,I}$ which meets each of the $\aleph_1$ many subsets $D_{n,\bar{x},j}$, $\bar{x} \subseteq I$ increasing and $j < \omega$. Define $p_{n,I} := \bigcup G_{n,I}$.

Recall that $C_n$ is the club in $\lambda_n$ of closure points of the function taking an ordinal $\xi$ to the supremum of values forced by the system applied to $[\xi]^{<\kappa_n}$.

**Claim 5.1.3.** *For each $n < \omega$, there is $I_n \subseteq Y_n$ which is $<$-increasing of order-type $\omega_1$ so that $p_{n,I_n} \in g_n$ and $\gamma_n := \sup(\bigcup I_n) \in C_n \cap S_n$.*

*Proof of Claim 5.1.3.* Since $|\mathbb{P}^{(n)}| < \lambda_n$, $S_n$ has a stationary subset $S'_n$ in $V[H_n]$ (as there must be a single condition in $g_n$ forcing stationary many ordinals into it). We will show that the set of $p_{n,I}$, $I$ varying among $<$-increasing subsets of $Y_n$ of order-type $\omega_1$ with $\sup(\bigcup I) \in C_n \cap S'_n$, is predense below $p_n$ in $\mathbb{P}^{(n)}$. Since this set belongs to $V[H_n]$, it follows that it is met by $g_n$, and this implies the claim.

Let $q \leq p_n$ in $\mathbb{P}^{(n)}$ be arbitrary. The elements of $\mathbb{P}^{(n)}$ have domains of size $< \kappa_n$, so there is $\beta < \kappa_n$ with $\mathrm{dom}(q) = \mathrm{dom}(q \restriction \beta)$.

Choose $I \subseteq Y_n$ with $\min(I) \cap \kappa_n > \beta$ and $\sup(\bigcup I) \in S'_n$, which is possible since $Y_n \in U_n$. Since $p_n$ is the stem of the system, $p_n \leq p \restriction (\min(I_n) \cap \kappa_n)$ for any $p \in \mathbb{P}_{n,I}$. So $p_{n,I} \restriction \beta \subseteq p_{n,I} \restriction (\min(I) \cap \kappa_n) \geq p_n$, and therefore $p_{n,I}$ is compatible with $q$. $\square$

Let $W = \bigcup_n I_n$.

For any $m, n < \omega$, the restriction $F \cap ([\bigcup_{i \leq n} X_n]^{<\omega} \times \lambda_m)$ is a member of $N[G \restriction \lambda_n]$. It therefore suffices to prove that for every $m, n < \omega$, every function $f : [\bigcup_{i \leq n} X_n]^{<\omega} \to \lambda_m$ in $N[G \restriction \lambda_n]$ and every $\bar{z} \in [\bigcup_{i \leq \omega} I_n]^{<\omega}$ in the domain of $f$, $f(\bar{z}) < \gamma_m$.

Suppose otherwise, and find a counterexample of such $m, n, t, f, \bar{z}$, minimizing first $m$ and then $n$. Write $\bar{z} = (\bar{z}^0, \ldots, \bar{z}^n)$, with $\bar{z}^i \in I_i^{<\omega}$. Using condition (3) of the construction of the system and introducing more variables to $f$ if necessary, choose $k < \omega$ so that $p_{n,I_n} \leq \sigma_n^k(\bar{z}^n)$

and the value of $f(\bar{z}^0, \ldots, \bar{z}^n)$ is forced over $V[H_n]$ by $\sigma_n^k(\bar{z}^n)$ of the system. If $m \geq n$, then the value of $f(\bar{z}^0, \ldots, \bar{z}^n)$ forced by $\sigma_n^k(\bar{z}^n)$ must be below $\gamma_n$, since $\gamma_n$ is a member of $C_n$. Therefore $m < n$.

Now define $f' : [\bigcup_{i<n} X_n]^{<\omega} \times [X_n]^{<\omega} \to \lambda_m$ so that $f'(\bar{x}^0, \ldots, \bar{x}^n)$ is the value of $f(\bar{x}^0, \ldots, \bar{x}^n)$ forced over $V[H_n]$ by $\sigma_n^k(\bar{x}^n)$, if $\sigma_n^k$ is defined on $\bar{x}^n$ and the value it forces is less than $\lambda_m$, and 0 otherwise. For $\bar{x}^0, \ldots, \bar{x}^{n-1}$, there are $Y \in U_n$ and $\delta < \lambda_m$ so that $f'(\bar{x}^0, \ldots, \bar{x}^{n-1}, \bar{w}) = \delta$ for all increasing $\bar{w} \subseteq Y$ of the right length. By intersecting such $Y$ for all possible $\bar{x}^0, \ldots, \bar{x}^{n-1}$, we can take $Y$ in $N[H_n]$ independent of the choice of $\bar{x}^0, \ldots, \bar{x}^{n-1}$. Let $h(\bar{x}^0, \ldots, \bar{x}^{n-1})$ be this $\delta$. By elementarity, $h \in N[H_n]$ since $\sigma_n^k, U_n \in N[H_n]$. Since $h$ has domain which is a subset of $[\bigcup_{i<n} X_n]^{<\omega}$, by $\lambda_{n-1}^+$-closure of $\mathbb{P}\restriction[\kappa_n, \lambda_n)$ over $V$, $h$ belongs to $V[G\restriction\lambda_{n-1}]$ and maps into $\lambda_m$.

Take $\bar{x}^0 = \bar{z}^0, \ldots, \bar{x}^{n-1} = \bar{z}^{n-1}$, and let $Y$ be as above. Since $Y \in N[H_n]$, $\bar{z}^n \subseteq Y_n \subseteq Y$ and therefore

$$h(\bar{z}^0, \ldots, \bar{z}^{n-1}) = f(\bar{z}^0, \ldots, \bar{z}^n) \geq \gamma_m.$$

The existence of $h$ contradicts the minimality of $n$. □

**Theorem 5.1.2.** *If there is a proper class of supercompact cardinals, then there is a class forcing extension so that every increasing $\omega$-sequence of regular cardinals has a mutually stationary sequence on cofinality $\omega_1$ which is not tightly stationary.*

*Proof.* Force with the poset $\mathbb{P}$ from the previous theorem to obtain $V[G]$, and then force with $\mathrm{Add}(\omega, \omega_1)$. Let $K$ be generic for $\mathrm{Add}(\omega, \omega_1)$ over $V[G]$. Let $\langle \lambda_i : i < \omega \rangle$ be an increasing $\omega$-sequence of regular cardinals greater than $\omega_1$ in $V[G * K]$, and $\lambda = \sup_n \lambda_n$. Let $\langle \lambda_{i_n} : n < \omega \rangle$ be a subsequence of $\langle \lambda_i : i < \omega \rangle$ so that for each $n$, there are at least three cardinals of $V[G * K]$ between $\lambda_{i_n}$ and $\lambda_{i_{n+1}}$. Note that since $\mathbb{P}$ is $\omega_2$-closed, the poset $\mathrm{Add}(\omega, \omega_1)$ we use is actually a member of the ground model $V$.

As before, there is a sequence of $V$-supercompact cardinals $\langle \kappa_n : n < \omega \rangle$ so that $\lambda_{i_{n-1}} \leq \kappa_n' < \kappa_n < \lambda_{i_n}$ for all $0 < n < \omega$, where $\kappa_n'$ is the predecessor of $\kappa_n$ in the sequence of supercompact cardinals. Let $\mathbb{P}^{(n)} = \mathbb{Q}_{\kappa_n'}$ and $\mathbb{R}^{(n)} = \mathbb{P}\restriction\kappa_n' \times \mathbb{P} \setminus \kappa_n$ be the quotient of $\mathbb{P}$ by

$\mathbb{P}^{(n)}$, with corresponding generics $g_n$ and $H_n$, respectively. In the extension by $\mathbb{R}^{(n)}$, $\mathsf{MA}(\aleph_1)$ holds and $\kappa_n$ remains supercompact.

Using Theorem 3.2.1, let $\langle S_{i_n} : n < \omega \rangle$, $S_{i_n} \subseteq \lambda_{i_n} \cap \mathrm{Cof}(\omega_1)$ for each $n$, be a stationary sequence in $V[G * K]$ that is not tightly stationary. In $V[G]$, there is a stationary $S'_n \subseteq S_{i_n}$ since $\mathrm{Add}(\omega, \omega_1)$ has size $< \lambda_n$.

Let $F : [\lambda]^{<\omega} \to \lambda$ be a function in $V[G * K]$. Because $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., there is $F' : \omega \times [\lambda]^{<\omega} \to \lambda$ in $V[G]$ so that for any $\bar{x} \subseteq \lambda$, there is $j < \omega$ so that $F'(j, \bar{x}) = F(\bar{x})$. Work as in Theorem 5.1.1, using $F'$ instead of $F$ in the structure $\mathcal{A}$ and using $\lambda_{i_n}$ and $S'_{i_n}$. Note that although the sequences $\langle \lambda_{i_n} : n < \omega \rangle$ and $\langle S'_n : n < \omega \rangle$ are not necessarily in $V[G]$, each of their finite initial segments are. This, together with the fact that $\{\lambda_{i_n} : n < \omega\}$ is contained in a countable set in $V$, suffices in the proof of Theorem 5.1.1. The proof produces a structure $M$ closed under $F'$, and hence under $F$, with $\sup(M \cap \lambda_{i_n})$ in $S'_n$ and hence in $S_{i_n}$. We conclude that $\langle S_{i_n} : n < \omega \rangle$ is mutually stationary in $V[G * K]$.

Trivially extend $\langle S_{i_n} : n < \omega \rangle$ to a stationary sequence $\langle S_i : i < \omega \rangle$ on $\langle \lambda_i : i < \omega \rangle$ so that $S_i = \lambda_i \cap \mathrm{Cof}(\omega_1)$ if $\lambda_i \notin \{\lambda'_{i_n} : n < \omega\}$. The sequence $\langle S_i : i < \omega \rangle$ is mutually stationary but not tightly stationary. $\square$

## 5.2 Indestructibility under further Prikry forcing

As discussed in Chapter 4, the only plausible strategies to force positive answers to our questions seem to involve Prikry-type forcing. The goal of this section is to show that the property of the model of the previous section that all mutually stationary sequences are tightly stationary is indestructible under Prikry-type forcing.

We first check that this model can have the measurable cardinals necessary to support Prikry-type forcings.

**Lemma 5.2.1.** *Suppose that $\kappa$ is a limit point of the sequence of supercompact cardinals $\langle \mu_\xi : \xi \in \mathsf{ON} \rangle$ and $\kappa$ is measurable in $B$ with $2^\kappa = \kappa^+$. Then $\kappa$ remains measurable in*

*$V[G]$, where $G$ is generic for $\mathbb{P}$ of Theorem 5.1.1, and in $V[G*K]$, where $K$ is generic for $\mathrm{Add}(\omega,\omega_1)$ over $V[G]$.*

*Proof.* That the addition of $K$ preserves measurability follows from the Levy–Solovay Theorem. Forcing with the $\kappa^+$-distributive poset $\mathbb{P}\restriction[\kappa,\infty)$ preserves the fact that $\kappa$ is measurable. Let $H$ be generic for $\mathbb{P}\restriction[\kappa,\infty)$. It remains to check that forcing with $\mathbb{P}\restriction\kappa$ over $V[H]$ preserves measurability.

In $V[H]$, let $j : V[H] \to M$ be the ultrapower by a normal ultrafilter on $\kappa$ and let $G\restriction\kappa$ be generic for $\mathbb{P}\restriction\kappa$ over $V[H]$. Our aim is to lift $j$ to $V[G]$, where $G = G\restriction\kappa \times H$. In order to do this, we need to find an $M$-generic filter in $V[G]$ for $j(\mathbb{P}\restriction\kappa)$. We can factor $j(\mathbb{P}\restriction\kappa) = \mathbb{P}\restriction\kappa \times \mathbb{R}$, where $\mathbb{R} = j(\mathbb{P})\restriction[\kappa, j(\kappa))$.

First, $G\restriction\kappa$ is $M$-generic for $\mathbb{P}\restriction\kappa$. Now in $V[G]$ we construct an $M[G\restriction\kappa]$-generic for $\mathbb{R}$. Let $\langle \dot{D}_\alpha : \alpha < \kappa^+\rangle$ be an enumeration of names for all of the open dense subsets of $\mathbb{R}$ which are in $M[G\restriction\kappa]$, with each name $\dot{D}_\alpha \in M$. There are only $\kappa^+$ many of them since $\mathbb{R}$ has size $j(\kappa)$, and in $V[H]$

$$|P^M(j(\kappa))| = |j(2^\kappa)| = |j(\kappa^+)| = (\kappa^+)^\kappa = \kappa^+.$$

Let $\langle p_i : i < \kappa\rangle$ be an enumeration of $\mathbb{P}\restriction\kappa$. The poset $\mathbb{R}$ is $\kappa^+$-closed in $M$. Using this closure, we inductively define $\langle q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\rangle$ an array of conditions in $\mathbb{R}$ (running through the lexicographic order on the index).

Suppose we have completed the construction up to some index $(\alpha, i)$. Using the fact that $\dot{D}_\alpha$ is forced to be dense in $\mathbb{R}$, pick $q_{\alpha,i}$ to be some $q \in \mathbb{R}$ so that

1. $q$ is stronger than the previously defined conditions in the array;
2. For some $p \le p_i$ in $\mathbb{P}\restriction\kappa$, $p \Vdash q \in \dot{D}_\alpha$.

Since ${}^\kappa M \subseteq M$, all proper initial segments of $\langle q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\rangle$ belong to $M$, so the construction can proceed just using the $\kappa^+$-closure of $\mathbb{R}$ in $M$.

For each $\alpha < \kappa^+$, we have ensured that there is a dense set in $M$ of $p \in \mathbb{P}\restriction\kappa$ which force $q_{\alpha,i} \in \dot{D}_\alpha$ for some $i < \kappa$, and $G\restriction\kappa$ meets each of these dense sets. Therefore in $V[G]$, the set $\{q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\}$ generates an $M[G \restriction \kappa]$-generic filter for $\mathbb{R}$, and so the embedding lifts. $\square$

Now we prove the main theorem of this section. The statement is motivated by doing some kind of Prikry forcing at $\kappa$, having $\lambda$ as the number of possible lower parts.

**Theorem 5.2.1.** *Let $(\mathbf{Pr}, \leq, \leq^*)$ be a forcing notion in $V[G * K]$ and $\kappa, \lambda$ are cardinals satisfying the following properties:*

- *$(\mathbf{Pr}, \leq, \leq^*)$ satisfies the Prikry condition (see Fact 4.1.1)*
- *Forcing with $\mathbf{Pr}$ does not add bounded subsets of $\kappa$ over $V[G * K]$,*
- *$\mathbf{Pr}$ is forced to be $< \lambda$-centered, i.e., it is the union of $< \lambda$ subsets, each of which is centered (has the property that any finite subset has a common lower bound).*

*In the extension $V[G * K * G_{\mathbf{Pr}}]$, where $G_{\mathbf{Pr}}$ is $V[G * K]$-generic for $\mathbf{Pr}$, every increasing $\omega$-sequence of regular cardinals with limit not in the interval $(\kappa, \lambda)$ has a mutually stationary sequence on cofinality $\omega_1$ which is not tightly stationary.*

*Remark* 5.2.2.

- If $\lambda \leq \kappa^{+\omega}$, then the interval $(\kappa, \lambda)$ does not contain any singular cardinals and therefore the theorem applies to all increasing $\omega$-sequence of regular cardinals in the extension.
- The usual Prikry forcing with a normal ultrafilter at $\kappa$ satisfies the properties required of the $\mathbf{Pr}$ of Theorem 5.2.1 with $\lambda = \kappa^+$, since any finitely many conditions with the same lower part have a common lower bound, and there are only $\kappa$ many lower parts.

*Proof.* Let $\mathfrak{c}(\mathbf{Pr})$ be the least cardinal $\mathfrak{c}$ so that $\mathbf{Pr}$ is $\mathfrak{c}$-centered, and work below a condition where this value is forced.

We will show that $V[G * G_{\mathbf{Pr}}]$ satisfies the mutual stationary property of the model in Theorem 5.1.1 to focus on the effect of the Prikry forcing. It is not difficult to adapt these arguments as in Theorem 5.1.2 to factor in $\mathrm{Add}(\omega, \omega_1)$ for every $\langle \lambda_n : n < \omega \rangle$. From this point, the outline of the argument is as before: given an increasing sequence of regular cardinals of $V[G*K*G_{\mathbf{Pr}}]$, there is a sequence $\langle S_n : n < \omega\rangle$ of stationary subsets on cofinality $\omega_1$ which is not tightly stationary. By the mutual stationarity property in $V[G*G_{\mathbf{Pr}}]$, there is a subsequence of the $\lambda_n$ for which the restriction of $\langle S_n : n < \omega\rangle$ is mutually stationary—arguments parallel to those in the proof of Theorem 5.1.2 show that this holds even though the sequence is not in $V[G*G_{\mathbf{Pr}}]$. Finally, this can be extended to a mutually stationary sequence which is not tightly stationary.

As before, it is enough to show that there is a mutually stationary but not tightly stationary sequence on each sequence of regular cardinals $\langle \lambda_n : n < \omega\rangle$ so that there are at least three $V[G]$ cardinals between $\lambda_n$ and $\lambda_{n+1}$ for each $n$, and let $\langle \kappa_n : n < \omega\rangle$ be formerly supercompact cardinals with $\lambda_{n-1} \le \kappa'_n < \kappa_n < \lambda_n$, where $\kappa'_n$ is the predecessor of $\kappa_n$ in the sequence of supercompact cardinals. Let $F : [\kappa]^{<\omega} \to \kappa$ be a function in $V[G*G_{\mathbf{Pr}}]$. We will find $W \subseteq \sup_n \lambda_n$ in $V[G*G_{\mathbf{Pr}}]$ so that $\sup(F``[W]^{<\omega} \cap \lambda_n) \le \sup(W\cap\lambda_n) \in S_n$ for every $n<\omega$.

For each $n < \omega$, let $\mathbb{R}^{(n)} = \mathbb{P}\restriction\kappa'_n \times \mathbb{P}\setminus\kappa_n$ and $\mathbb{P}^{(n)} = \mathrm{Col}^V((\kappa'_n)^+, <\kappa_n)$. Let $g_n$ and $H_n$ be the generics obtained from $G$ for the posets $\mathbb{P}\restriction\lambda_n, \mathbb{P}^{(n)}, \mathbb{R}^{(n)}$, respectively.

**Case 1:** $\sup_n \lambda_n < \kappa$. Forcing with **Pr** does not add bounded subsets of $\kappa$ so we do not change the situation from that of Theorem 5.1.1.

**Case 2:** $\sup_n \lambda_n = \kappa$. By indestructibility, $\kappa_n$ is supercompact in $V[H_n]$, so there is a normal fine ultrafilter $U_n$ on $[\lambda_n]^{<\kappa_n}$ in $V[H_n]$.

In $V[G*G_{\mathbf{Pr}}]$, let $\theta$ be $(2^\kappa)^+$ and take

$$\tilde{N} \prec (H(\theta)^{V[G*G_{\mathbf{Pr}}]}; \in, G, G_{\mathbf{Pr}}, \langle \kappa_n, \lambda_n, U_n : n<\omega\rangle, F).$$

Define $N = \tilde{N}\cap V$, so $\tilde{N} = N[G*G_{\mathbf{Pr}}]$.

The main difference from Theorem 5.1.1 in this case is that $N \notin V$. We can replace all uses of $N$ in the previous proof by $N_n = N \cap V_{\lambda_n+6}$. Since forcing with **Pr** does not add sets of von Neumann rank bounded below $\kappa$, $N_n \in V$. Note that $\langle \kappa_i, \lambda_i : i \le n\rangle \in N_n$.

The rest of the proof goes through as in Theorem 5.1.1, constructing the system at level $n$ to decide values of functions in $N_n[G\restriction\lambda_n]$. We should be careful to ensure that the function $h$ defined at the end of the proof is in $N_{n-1}[G\restriction\lambda_{n-1}]$, while the proof of Theorem 5.1.1 only showed that it was in $N[G\restriction\lambda_{n-1}]$. This holds because there must be a canonical name in $\tilde{N}$ for $h$, and this must be in $N$ and have rank at most $\lambda_{n-1}+5$, and is therefore a member of $N_{n-1}$.

**Case 3:** $\sup_n \lambda_n > \mathfrak{c}(\mathbf{Pr})$. By ignoring an initial segment of the $\lambda_n$, we may assume that $\kappa_0 > \mathfrak{c}(\mathbf{Pr})$. Suppose $\dot{\tau}$ is a $\mathbf{Pr}$ name for an ordinal in $V[G]$. There are at most $\mathfrak{c}(\mathbf{Pr})$ many possibilities for the value of $\dot{\tau}[G_{\mathbf{Pr}}]$. Hence there is some $\tilde{F} \in V[G]$ so that for any $W \subseteq \sup_n \lambda_n$ with $\mathfrak{c}(\mathbf{Pr}) \subseteq W$, $F``[W]^{<\omega} \subseteq \tilde{F}``[W]$. For each $n$, there is a $\lambda_n$-supercompactness measure $U_n$ for $\kappa_n$ in $V[H_n]$.

Let $\langle T_\alpha : \alpha < \mathfrak{c}(\mathbf{Pr})\rangle$ be an enumeration of the centered pieces of $\mathbf{Pr}$ in $V[G]$. Now argue similarly as in Theorem 5.1.1, using the same notation as in that proof but working with $\tilde{F}$ instead of $F$, and at stage $n$, replacing $N$ (a countable Skolem hull of all relevant objects in $V$) with $N_n$, the Skolem hull of $N$ together with some $\alpha_n < \mathfrak{c}(\mathbf{Pr})$ so that there is $s \in T_{\alpha_n} \cap G_{\mathbf{Pr}}$ which forces values for $\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle$ over $V[G]$. Note that these values depend only on $\alpha_n$, since the members of $T_{\alpha_n}$ are pairwise compatible, and since $\mathfrak{c}(\mathbf{Pr}) < \kappa_n$ there is a condition in $N \cap H_n$ which forces a value for the function that takes $\alpha_n$ to $\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle$. With this, $\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle \in N_n[H_n]$.

As before, for each $n$ let $Y_n = \bigcap U_n' \cap N_n[H_n]$ and construct a system $\langle \sigma_n^k : k < \omega\rangle$ with domain $Y_n \in U_n'$. Using Martin's Axiom we can define $p_{n,I}$.

Let $C_n$ be the club in $\lambda_n$ of closure points of the function taking an ordinal $\xi$ to the supremum of values forced by the system applied to $[\xi]^{<\kappa_n}$. A version of Claim 5.1.3 holds, but the proof must be modified as we cannot find in $V[H_n]$ a stationary subset $S_n'$ of $S_n$.

**Claim 5.2.3.** *For each $n < \omega$, there is $I_n \subseteq Y_n$ which is $<$-increasing of order-type $\omega_1$ so that $p_{n,I_n} \in g_n$ with $\gamma_n := \sup(\bigcup I_n) \in C_n \cap S_n$.*

*Proof of Claim 5.2.3.* Work in $V[H_n]$. Consider the set of $(p, \dot{r}) \in \mathbb{P}^{(n)} * \mathbf{Pr}$ for which there is some $I \subseteq Y_n$ which is $<$-increasing of order-type $\omega_1$ so that:

1. $p \leq p_{n,I}$.

2. $(p, \dot{r}) \Vdash \sup(\bigcup I) \in C_n \cap \dot{S}_n$.

We will show that this set is dense below $(p_n, 1_{\mathbf{Pr}})$, and therefore intersects $g_n * G_{\mathbf{Pr}}$.

Let $q \leq p_n$ in $\mathbb{P}^{(n)}$ and $\dot{s}$ be a $\mathbb{P}^{(n)}$-name for an element of $\mathbf{Pr}$. In $V[G]$, there is a stationary $S^*_n \subseteq \lambda_n$ so that for any $\gamma \in S^*_n$, there is some $s' \leq s$ with $s' \Vdash \gamma \in \dot{S}_n$. Since $|\mathbb{P}^{(n)}| < \lambda_n$, there are $q' \leq q$ and $S'_n \in V[H_n]$ so that $S'_n$ is stationary and $q' \Vdash S'_n \subseteq \dot{S}^*_n$. In total, we have defined $q'$ and $S'_n$ so that for any $\gamma \in S'_n$, there is some $\dot{s}'$ so that $q' \Vdash \dot{s}' \leq \dot{s}$ and $(q', \dot{s}') \Vdash \gamma \in \dot{S}_n$.

The elements of $\mathbb{P}^{(n)}$ have domains of size $< \kappa_n$, so there is $\beta < \kappa_n$ with $\mathrm{dom}(q') = \mathrm{dom}(q' \restriction \beta)$. Pick $I \subseteq Y_n$ to be $<$-increasing of order-type $\omega_1$ with $\min(I) \cap \kappa_n > \beta$ and $\sup(\bigcup I) \in S'_n$.

Choose $\dot{r}$ a name for a condition in $\mathbf{Pr}$ so that $q' \Vdash \dot{r} \leq \dot{s}$ and $(q', \dot{r}) \Vdash \sup(\bigcup I) \in \dot{S}_n$. Since $p_n$ is the stem of the system, $p_n$ extends the restriction to $\min(I) \cap \kappa_n$ of any member of $\mathbb{P}_{n,I}$. So $p_{n,I} \restriction \beta \subseteq p_{n,I} \restriction (\min(I) \cap \kappa_n) \geq p_n$, and therefore $p_{n,I}$ is compatible with $q'$. Choosing $p \leq p_{n,I}, q'$ gives the result. $\square$

Take $W = \bigcup_n I_n$ and $\gamma_n = \sup(W \cap \lambda_n)$ for each $n < \omega$.

Let $\lambda = \sup_n \lambda_n$. For any $m, n < \omega$, consider the function which maps $\bar{x} \in [\bigcup_{i \leq n} X_n]^{<\omega}$ to the supremum over all $\bar{y} \subseteq \lambda$ of the values of $\tilde{F}(\bar{y}, \bar{x})$ which are less than $\lambda_m$. This function is a member of $N_n[G \restriction \lambda_n]$. Therefore it suffices to show that for every $m, n < \omega$, every function $f : [\bigcup_{i \leq n} X_n]^{<\omega} \to \lambda_m$ in $N_n[G \restriction \lambda_n]$ and every $\bar{z} \in [\bigcup_{i \leq \omega} I_n]^{<\omega}$ in the domain of $f$, $f(\bar{z}) < \gamma_m$.

Suppose otherwise, and fix a counterexample $m, n, f$, and $\bar{z} = (\bar{z}^0, \ldots, \bar{z}^n)$ with $\bar{z}^i \in I_i^{<\omega}$, minimizing first $m$ and then $n$. As in Section 5.1, we have $m < n$ and the value of $f(\bar{z}^0, \ldots, \bar{z}^n)$ is forced over $V[H_n]$ by some condition $\sigma^k_n(\bar{z}^n)$ of the system where $p_{n,I_n} \leq \sigma^k_n(\bar{z}^n)$.

Define $f' : [\bigcup_{i<n} X_n]^{<\omega} \times Y_n \to \lambda_m$ so that for $\bar{x}^0, \ldots, \bar{x}^n$ of the appropriate type, $f'(\bar{x}^0, \ldots, \bar{x}^n)$ is the value of $f(\bar{x}^0, \ldots, \bar{x}^n)$ forced over $V[H_n]$ by $\sigma_n^k(\bar{x}^n)$ (if such exists, and 0 otherwise). There is a $Y \in U_n \cap N[H_n]$ so that for every $\bar{x}^0, \ldots, \bar{x}^{n-1}$ there is some $\delta < \lambda_m$ so that

$$f'(\bar{x}^0, \ldots, \bar{x}^{n-1}, \bar{w}) = \delta$$

for all increasing $\bar{w} \subseteq Y$ of the appropriate length. Let $h(\bar{x}^0, \ldots, \bar{x}^{n-1})$ be this fixed value.

Now $Y_n \subseteq Y$ and therefore

$$f(\bar{z}^0, \ldots, \bar{z}^{n-1}, \bar{z}^n) = h(\bar{z}^0, \ldots, \bar{z}^{n-1}).$$

Using the $\lambda_{n-1}^+$-closure of $\mathbb{P}^{(n)}$, we have $h \in N_n[G{\restriction}\lambda_{n-1}]$. Since $N_n[G{\restriction}\lambda_{n-1}]$ is the Skolem hull of $N_{n-1}[G{\restriction}\lambda_{n-1}]$ together with $\alpha_n$, there is a function $h' : \mathbf{Pr} \times [\bigcup_{i\leq n} X_i]^{<\omega} \to \lambda_m$ in $N_{n-1}[G{\restriction}\lambda_{n-1}]$ so that $h'(\alpha_n, \bar{x}) = h(\bar{x})$ for all $\bar{x}$ in the domain of $h$. In $N_{n-1}[G{\restriction}\lambda_{n-1}]$, we can define $h''(\bar{x}) = \sup_{\alpha<\mathfrak{c}(\mathbf{Pr})} h'(\alpha, \bar{x})$. Since $\lambda_m$ has cofinality larger than $\mathfrak{c}(\mathbf{Pr})$, we have that $h''(\bar{z}^0, \ldots, \bar{z}^{n-1}) < \lambda_m$ and

$$\begin{aligned} h''(\bar{z}^0, \ldots, \bar{z}^{n-1}) &= h'(\alpha_n, \bar{z}^0, \ldots, \bar{z}^{n-1}) \\ &= h(\bar{z}^0, \ldots, \bar{z}^{n-1}) \\ &\geq f(\bar{z}^0, \ldots, \bar{z}^{n-1}, \bar{z}^n) \\ &\geq \gamma_m. \end{aligned}$$

As $h'' \in N_{n-1}[G{\restriction}\lambda_{n-1}]$, this contradicts the minimality of $n$.

$\square$

# CHAPTER 6

# Careful sets

## 6.1 Definition and basic technique

In this chapter, we aim to apply the transfer function to prove combinatorial properties about tightly stationary sequences, leading naturally to the idea of *careful subsets*.

Suppose that $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ carries a scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$. Recall the defintions of $\mu$ and $\nu$ from Chapter 2.

**Definition 2.2.1.** *Suppose $S_\xi \subseteq \kappa_\xi$ for each $\xi < \mathrm{cf}(\kappa)$. Then define*

$$\mu(\vec{S}) = \{\alpha : f_\alpha \textit{ meets } \vec{S}\}.$$

*Let $S'_\xi = \kappa_\xi \setminus S_\xi$. Then define $\nu(\vec{S}) = \kappa^+ \setminus \mu(\langle S'_\xi \rangle)$.*

As a first illustration, we give another proof of the version of Fodor's Lemma for tightly stationary sets on $\aleph_\omega$ proved in [FM01], which shows the relationship with the usual Fodor's Lemma on regular cardinals.

**Proposition 6.1.1.** *Suppose $\langle S_n : k < n < \omega \rangle$ is tightly stationary and $S_n \subseteq \mathrm{Cof}(\omega_k)$ for some $k < \omega$. If $f : \aleph_\omega \to \aleph_\omega$ satisfies $f(\gamma) < \gamma$ for all $\gamma$, then there is a function $g \in \prod_{n\in\omega} \aleph_n$ such that the sequence $\langle S_n^g : k < n < \omega \rangle$ defined by $S_n^g = \{\gamma \in S_n : f(\gamma) < g(n)\}$ is tightly stationary.*

*Proof.* Let $A = \mu(\vec{S}) \cap \{\alpha : \alpha \text{ is a good point of cofinality } \omega_k\}$. This is stationary by Lemma 2.2.4 since $\vec{S}$ is tightly stationary. Define $\bar{F} : A \to \aleph_{\omega+1}$ to be $\bar{F}(\alpha) =$ least $\beta < \alpha$ such that $f \circ f_\alpha <^* f_\beta$. Such exists since $f$ is regressive and any $\alpha \in A$ is a good point. Then $\bar{F}$ is a regressive function on $A$, hence by the usual Fodor's lemma, is constant on a stationary set $A'$, say with constant value $\beta_0$. Put $g = f_{\beta_0}$. Consider

$$S_n^g = \{\gamma \in S_n : f(\gamma) < g(n)\}.$$

We now show that $A' \subseteq \mu(\langle S_n^g \rangle)$, hence by Lemma 2.2.4 that $\langle S_n^g \rangle$ is tightly stationary. For any $\alpha \in A'$, we have $f \circ f_\alpha <^* g$ by choice of $A'$ and $\beta_0$. This means there is $i \in \omega$ such that for all $n \geq i$ we have $f_\alpha(n) \in S_n^g$, or in other words, $\alpha \in \mu(\langle S_n^g \rangle)$. $\square$

The theme illustrated by the proof is that the $\mu$ function allows us to take tightly stationary sequences and map them to stationary subsets of $\kappa^+$, the length of the scale. We can then perform some construction on the stationary set in $\kappa^+$, and then hope to transfer the construction back to the sequences. For example, we would like to say something about splitting tightly stationary sequences (see [FM01], [Mag77] for more on this). After transferring a given tightly stationary sequence to a stationary set and then splitting the stationary set, we would like to transfer the pieces back to the sequences. This requires each piece to be in the range of $\mu$. Carefulness is a useful symmetrical strengthening of this.

Now we define the crucial notion of carefulness.

**Definition 6.1.2.** A sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ with $S_\xi \subseteq \kappa_\xi$ for all $n < \omega$ is *careful* if $\mu(\vec{S}) = \nu(\vec{S})$. A set $A \subseteq \kappa^+$ is *careful* if there is a careful sequence $\vec{S}$ with $\mu(\vec{S}) = A$.

So a careful set $A$ is in the range of the $\mu$ function, witnessed by a sequence $\vec{S}$ which does not intersect scale functions indexed by members of $\kappa^+ \setminus A$ too much. Careful sequences behave nicely under finite coordinatewise intersections and unions. By the results of Chapter 3, any sequence of regular cardinals with a tree-like scale admits a careful sequence of stationary co-stationary subsets (Theorem 3.3.1), as do many sequences of regular cardinals associated with Prikry forcing (Theorem 4.2.2). So a careful set $A$ is in the range of the $\mu$ function, witnessed by a sequence $\vec{S}$ which does not intersect scale functions indexed by members of $\kappa^+ \setminus A$ too much. Careful sequences behave nicely under finite coordinatewise intersections and unions. By the results of Chapter 3, any sequence of regular cardinals with a tree-like scale admits a careful sequence of stationary co-stationary subsets (Theorem 3.3.1), as do many sequences of regular cardinals associated with Prikry forcing (Theorem 4.2.2).

**Proposition 6.1.3.** *Let $A, B$ be careful, witnessed by the sequences $\langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ and $\langle T_\xi : \xi < \mathrm{cf}(\kappa)\rangle$, respectively. Then*

$$\mu(\langle S_\xi \cap T_\xi\rangle) = \nu(\langle S_\xi \cap T_\xi\rangle) = A \cap B$$

*and*

$$\mu(\langle S_\xi \cup T_\xi\rangle) = \nu(\langle S_\xi \cup T_\xi\rangle) = A \cup B.$$

Our theme gives the following easy splitting result:

**Proposition 6.1.4.** *Suppose every subset of $\kappa^+$ is careful. Then for any tightly stationary sequence $\vec{S}$, there are $\vec{T}^\alpha = \langle T^\alpha_\xi : \xi < \omega\rangle$ for $\alpha < \kappa^+$ such that*

- *$T^\alpha_\xi \subseteq S_\xi$ for all $\xi < \mathrm{cf}(\kappa)$, $\alpha < \kappa^+$,*
- *$\vec{T}^\alpha$ is tightly stationary for all $\alpha < \kappa^+$,*
- *$\nu(\langle T^\alpha_\xi \cap T^\beta_\xi\rangle) = \emptyset$ for all $\alpha \neq \beta < \kappa^+$.*

*Proof.* Let $A = \mu(\vec{S}) \cap \mathrm{Good}$, which is stationary in $\kappa^+$ by Lemma 2.2.4. Then $A$ can be split into $\kappa^+$ many pairwise disjoint stationary subsets of $\kappa^+$, say $\langle A_\alpha : \alpha < \kappa^+\rangle$. Each $A_\alpha$ is careful, so let $\vec{T}^\alpha$ be the corresponding sequence, which is tightly stationary by Lemma 2.2.4. By intersecting with $\vec{S}$, we may assume that $T^\alpha_\xi \subseteq S_\xi$ for all $\xi < \mathrm{cf}(\kappa)$, $\alpha < \kappa^+$. By Proposition 2.2.2, condition (3) holds. □

In the next chapter, we will show that the hypothesis of Proposition 6.1.4 is consistent.

Finally, we use a careful sequence to give a stationary, not tightly stationary sequence, generalizing some results of Chapter 3.

**Proposition 6.1.5.** *Suppose there is a careful sequence of stationary, co-stationary subsets. Then there is a sequence of stationary sets which is not tightly stationary.*

*Proof.* Let $\vec{S}$ be a careful sequence of stationary, co-stationary subsets. Take $X \subseteq \omega$ which is infinite and whose complement is also infinite. Define $\vec{T} = \langle T_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ by $T_\xi = S_\xi$ if $\xi \in X$ and $T_\xi = \kappa_\xi \setminus S_\xi$ otherwise. Then $T_\xi$ is a stationary sequence.

Now we check that $\mu(\vec{T}) = \emptyset$. Suppose $\alpha \in \mu(\vec{T})$. Then $f_\alpha(\xi) \in S_\xi$ for cofinally many $\xi$, so $\alpha \in \nu(\vec{S})$. Similarly, we get $\alpha \in \nu(\langle \kappa_\xi \setminus S_\xi : \xi < \mathrm{cf}(\kappa) \rangle)$, but this is impossible. $\square$

## 6.2 $d$ and $d^*$

From a scale, one can define two-place functions $[\kappa^+]^2 \to \mathrm{cf}(\kappa)$ which will help describe how the $\mu$ function works.

**Definition 6.2.1.** Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a scale, and suppose $\alpha < \beta$. Then $d(\alpha, \beta) = \sup\{\xi + 1 : f_\alpha(\xi) \geq f_\beta(\xi)\}$, and $d^*(\alpha, \beta) = \sup\{\xi + 1 : f_\alpha(\xi) = f_\beta(\xi)\}$.

The function $d$ was used by Shelah in [She94], for example, to prove $\kappa^+ \not\to [\kappa^+]^2_{\mathrm{cf}\kappa}$ for singular $\kappa$.

The next lemma gives a combinatorial criterion for carefulness which involves the $d^*$ function.

**Lemma 6.2.2.** *Suppose $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is continuous, $A \subseteq \kappa^+$ and there is $F : \kappa^+ \to \omega$ such that*

$$d^*(\alpha, \beta) \leq \max\{F(\alpha), F(\beta)\}$$

*for all $\alpha \in A$ and $\beta \notin A$. Then $A$ is careful.*

*Proof.* Define $S_n = \{f_\alpha(n) : \alpha \in A \text{ and } F(\alpha) \leq n\}$. Then $A \subseteq \mu(\vec{S})$ since for any $\alpha \in A$, $f_\alpha(n) \in S_n$ for all $n \geq F(\alpha)$. It remains to show that $\nu(\vec{S}) \subseteq A$. For $\beta \in \kappa^+ \setminus A$, we will show that $f_\beta(n) \notin S_n$ for $n \geq F(\beta)$. Let $n$ be so that $f_\beta(n) \in S_n$. Then $d^*(\alpha, \beta) > n$ for some $\alpha \in A$ with $F(\alpha) \leq n$. Since $n < d^*(\alpha, \beta) < \max\{F(\alpha), F(\beta)\}$, it follows that $n < F(\beta)$. $\square$

*Remark* 6.2.3. This is actually an equivalence if the background scale is tree-like: if $A$ is careful, witnessed by $\vec{S}$, then define $F(\alpha)$ to be the least $n$ such that $f_\alpha(n) \in S_n$ if $\alpha \in A$, and the least $n$ such that $f_\alpha(n) \notin S_n$ if $\alpha \in A$.

The assumption of a tree-like scale has an effect on the $d^*$ function.

**Lemma 6.2.4.** *Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a tree-like scale. For any $\alpha, \beta, \gamma \in \kappa^+$, the smaller two among $d^*(\alpha, \beta), d^*(\beta, \gamma), d^*(\alpha, \gamma)$ are equal.*

*Proof.* Assume without loss of generality that $d^*(\alpha, \beta) \leq d^*(\beta, \gamma) \leq d^*(\alpha, \gamma)$. Fix arbitrary $n < d^*(\beta, \gamma)$. Using the tree-like property and $d^*(\beta, \gamma) \leq d^*(\alpha, \gamma)$, we have $f_\alpha(n) = f_\gamma(n)$. But by definition of $d^*$, $f_\gamma(n) = f_\beta(n)$. Combining the two equations, we get $f_\alpha(n) = f_\beta(n)$, so $d^*(\alpha, \beta) \geq d^*(\beta, \gamma)$. $\square$

In this situation, there is a qualitative difference between $d$ and $d^*$. Our result implies that under the assumption of a continuous tree-like scale, there is a stationary co-stationary careful subset of $\kappa^+$.

**Proposition 6.2.5.** *Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a tree-like scale. Then there are disjoint stationary sets $A, B \subseteq \kappa^+$ such that $d^*(\alpha, \beta)$ is constant on $A \times B$.*

*Proof.* For each $\alpha$, let $D(\alpha) = \{n : d^*(\alpha, \beta) = n \text{ for stationarily many } \beta < \kappa^+\}$. Since $\mathrm{cf}(\kappa^+) > \omega$, each $D(\alpha) \neq \emptyset$.

We will find an $\alpha$ such $|D(\alpha)| > 1$. If this does not exist, then for each $\alpha$ let $n(\alpha)$ be the unique element of $D(\alpha)$. Then $C_\alpha = \{\beta : d^*(\alpha, \beta) = n(\alpha)\}$ must contain a club of $\kappa^+$. Let $C$ be the diagonal intersection of the $C_\alpha$, $\alpha < \kappa^+$. Let $n_0$ be such that $n(\alpha) = n_0$ for $\kappa^+$ many $\alpha \in C$. Then let $E = \{\alpha \in C : n(\alpha) = n_0\}$. If $\alpha < \beta$ are members of $E$, then $d^*(\alpha, \beta) = n(\alpha) = n_0$. This implies that $f_\alpha(n_0), \alpha \in E$, are pairwise distinct, a contradiction.

So fix $\alpha_0$ such that $|D(\alpha_0)| > 1$, and let $m < n$ be elements of $D(\alpha_0)$. Then let $A = \{\alpha : d^*(\alpha_0, \alpha) = m\}$ and $B = \{\beta : d^*(\alpha_0, \beta) = n\}$. By Lemma 6.2.4, $d^*(\alpha, \beta) = m$ for all $\alpha \in A, \beta \in B$. $\square$

On the other hand, Shelah [She94] showed that if $A, B \subseteq \kappa^+$ are unbounded, then for any sufficiently large $n$, there are $\alpha \in A$ and $\beta \in B$ such that $d(\alpha, \beta) = n$.

## 6.3 Examples

A *better* scale is a scale $\langle f_\alpha : \alpha < \kappa^+\rangle$ such that for every limit ordinal $\alpha < \kappa^+$ there is a club $C \subseteq \alpha$ such that for every $\gamma \in C$ there is $N < \omega$ such that $\forall n > N\,(f_\beta(n) < f_\gamma(n))$ for all $\beta < \gamma$ with $\beta \in C$. This is a stronger property than the one that good scales satisfy. The existence of better scales is a consequence of the weak square $\square^*_\kappa$.

We start with the observation that if the background scale is better, then every bounded subset of $\kappa^+$ is careful. The argument follows along the lines of the construction of an ADS-sequence from a better scale by Cummings, Foreman and Magidor in [CFM01].

**Proposition 6.3.1.** *If $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a better scale, then every bounded $A \subset \kappa^+$ is careful.*

*Proof.* In [CFM01], it is proved from a better scale that that for every $\gamma < \kappa^+$, there is a function $G_\gamma : \gamma \to \omega$ such that for any $\alpha < \beta < \gamma$, $d^*(\alpha, \beta) < \max\{G_\gamma(\alpha), G_\gamma(\beta)\}$. Now if $A \subset \kappa^+$ is bounded, then let $\gamma$ be a bound. Set $F(\alpha)$ to be $\max(d(\alpha, \gamma) + 1, G(\alpha))$ if $\alpha < \gamma$, 0 if $\alpha = \gamma$, and $d(\gamma, \alpha) + 1$ if $\alpha > \gamma$. We show that $d^*(\alpha, \beta) \leq \max\{F(\alpha), F(\beta)\}$ for all $\alpha \in A$, $\beta \notin A$. If $\beta < \gamma$, then

$$d^*(\alpha, \beta) < \max\{G(\alpha), G(\beta)\} \leq \max\{F(\alpha), F(\beta)\}.$$

If $\beta = \gamma$, then

$$d^*(\alpha, \beta) < d(\alpha, \beta) \leq F(\alpha).$$

If $\beta > \gamma$, assume towards a contradiction that $d^*(\alpha, \beta) > \max\{F(\alpha), F(\beta)\}$. In particular, this assumption implies that $d^*(\alpha, \beta) > d(\alpha, \gamma), d(\gamma, \beta)$, so $f_\alpha(d^*(\alpha, \beta) - 1) < f_\gamma(d^*(\alpha, \beta) - 1) < f_\beta(d^*(\alpha, \beta) - 1)$, contradicting the definition of $d^*$. $\square$

Starting from a continuous tree-like scale, we will force so that every subset of $\kappa^+$ is careful. We will see below that this poset is c.c.c., and therefore $\langle f_\alpha : \alpha < \kappa^+\rangle$ remains a scale in $V[G]$.

We conclude this chapter by identifying situations where there exist subsets of $\kappa^+$ which are not in the range of $\mu$. Suppose $2^\kappa < 2^{\kappa^+}$ (e.g., when the SCH holds at $\kappa$). Then there are only $2^\kappa$ choices for a sequence $\bar{S}$, so there is a subset of $\kappa^+$ which is not in the range of $\mu$.

We can also add a set which is not in the range of $\mu$ by forcing. This example was inspired by similar arguments of Foreman and Steprāns from Section 4 of [CFM06].

Think of $P = \mathrm{Add}(\omega, \kappa^+)$ as the forcing adding a subset of $\kappa^+$ using finite conditions—if $G$ is generic for $P$, then $\bigcup G$ is a function $\omega \times \kappa^+ \to 2$, and using a bijection $\varphi$ between $\kappa^+$ and $\omega \times \kappa^+$, we obtain a subset $S$ from $G$ (whose characteristic function is $\bigcup G \circ \varphi$). Recall that $P$ is c.c.c., and for any $\delta < \kappa^+$, $P \simeq \mathrm{Add}(\omega, \kappa^+) \times \mathrm{Add}(\omega, \kappa^+)$. Since $P$ is c.c.c., any function in $\prod_{n<\omega} \kappa_n \cap V[G]$ is dominated pointwise by a function in $\prod_{n<\omega} \kappa_n \cap V$. Thus the scale $\vec{f}$ in $V$ remains a scale in $V[G]$.

**Proposition 6.3.2.** *Let $P = \mathrm{Add}(\omega, \kappa^+)$, and $S \subseteq \kappa^+$ as above. Then in $V[G]$, $S$ is not in the range of $\mu$.*

*Proof.* Work in $V[G]$. For any sequence $\langle U_n : n < \omega \rangle$ with $U_n \subseteq \kappa_n$, we claim that the sequence (and hence also every $U_n$) is contained in the generic extension of $V$ by $\mathrm{Add}(\omega, \delta)$ for some $\delta < \kappa^+$. This is because there is a nice name for each $U_n$ (i.e., consisting of pairs $(\check{\alpha}, p)$ where for any given $\check{\alpha}$, $\{p : (\check{\alpha}, p) \in \dot{U}_n\}$ is an antichain), so there is a name for $\langle U_n : n < \omega \rangle$ which uses at most $\kappa$ many elements of $P$.

Factor $V[G] = V[H][G']$ where $H$ is generic for $\mathrm{Add}(\omega, \delta)$ and $\langle U_n : n < \omega \rangle \in V[H]$, and $G'$ is generic for the quotient $\mathrm{Add}(\omega, \kappa^+)$. Now $\mu(\langle U_n \rangle)$ lies in $V[H]$. By a density argument using the construction of $S$ from $G'$, $S \notin V[H]$. $\square$

In the next chapter, we show that it is consistent that every subset of $\kappa^+$ is in the range of $\mu$ (in fact, careful).

# CHAPTER 7

# Models where every set is careful

## 7.1 The forcing construction

**Theorem 7.1.1.** *Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a continuous tree-like scale and $A \subseteq \kappa^+$. Then there is a c.c.c. forcing extension in which $A$ is careful.*

*Remark* 7.1.1. In fact, the proof will show that the poset is $\omega_1$-Knaster.

*Proof.* Given $A$, define $\mathbb{Q}_A$ to be the forcing of finite functions $p : \kappa^+ \to \omega$ such that $d^*(\alpha, \beta) \leq \max\{p(\alpha), p(\beta)\}$ for any $\alpha \in \mathrm{dom}(p) \cap A$ and $\beta \in \mathrm{dom}(p) \setminus A$, ordered by extension.

Now we will show that $\mathbb{Q}_A$ is c.c.c. Towards a contradiction, suppose $\{p_\xi : \xi < \omega_1\}$ is an uncountable antichain. Using the $\Delta$-system lemma, we may assume that the domains of the $p_\xi$ form a $\Delta$-system. The strategy of the proof is to repeatedly thin the antichain by choosing an uncountable subset with certain nice properties, and without loss of generality renaming the thinned antichain by $\{p_\xi : \xi < \omega_1\}$. At the end we will have thinned enough to see that certain members of the antichain were actually compatible.

For any condition $p \in \mathbb{Q}_A$, let the *type of* $p$ be the ordered pair $(m, n)$, where $m = |\mathrm{dom}(p) \cap A|$ and $n = |\mathrm{dom}(p) \cap (\kappa^+ \setminus A)|$. Thin to assume that all members have the same type $(m, n)$, and that the $p_\xi$ agree on the root of the $\Delta$-system. By throwing away the root from the domain of each condition, we may assume the $p_\xi$ have disjoint domains.

Enumerate $\mathrm{dom}(p_\xi) \cap A$ as $\{\alpha^i_\xi : i < m\}$ and $\mathrm{dom}(p_\xi) \setminus A$ as $\{\beta^i_\xi : i < n\}$. By thinning further we may assume that for every $i < m$, $j < n$, there is $k_{ij} < \omega$ (not depending on $\xi$)

such that $\forall \xi < \omega_1(d^*(\alpha^i_\xi, \beta^j_\xi) = k_{ij})$. By thinning yet further we can assume that for every $i < m$, $j < n$, either

$$\forall \xi < \omega_1(p_\xi(\alpha^i_\xi) \geq k_{ij})$$

or

$$\forall \xi < \omega_1(p_\xi(\beta^j_\xi) \geq k_{ij})$$

(i.e., whether it is $\alpha^i$ or $\beta^j$ that satisfies this does not depend on $\xi$).

The goal is to thin the antichain further so that we can find some $i_0 < m$ (or $j_0 < n$) such that $p_\xi(\alpha^{i_0}_\xi) \geq d^*(\alpha^{i_0}_\xi, \beta^j_\zeta)$ for all $\xi, \zeta < \omega_1, j < n$ (or $p_\xi(\beta^{j_0}_\xi) \geq d^*(\alpha^i_\zeta, \beta^{j_0}_\xi)$ for all $\zeta < \omega_1, i < m$). Thus the incompatibility between different members of the antichain cannot come from the elements $\alpha^{i_0}_\xi$ (or $\beta^{j_0}_\xi$) of the domain of each condition, so the property of being an antichain is preserved if we remove these elements from the domain of each condition. Repeating this process, we eventually reach an uncountable antichain where every member is of the same type $(m, 0)$ or $(0, n)$, a contradiction since these would all be compatible in $\mathbb{Q}_A$.

Choose $i_0 < m$ and $j_0 < n$ so that $k_{i_0 j_0} = \max_{i<m, j<n} k_{ij}$, and let $M = k_{i_0 j_0}$. We handle the case $\forall \xi < \omega_1(p_\xi(\alpha^{i_0}_\xi) \geq M)$, the case with $\beta$ is similar. To avoid a mess of sub- and superscripts, we denote $\alpha^{i_0}_\xi$ by $\alpha_\xi$.

We will perform the thinning one $j$ at a time, so fix $j < n$. It suffices to show that there is an uncountable set $Z \subset \omega_1$ such that for all $\xi, \zeta \in Z$, $p_\xi(\alpha_\xi) \geq d^*(\alpha_\xi, \beta^j_\zeta)$.

**Claim 7.1.2.** *For every $\xi, \zeta < \omega_1$, either*

$$d^*(\alpha_\xi, \beta^j_\zeta) \leq k_{i_0 j} \text{ and } d^*(\alpha_\zeta, \beta^j_\xi) \leq k_{i_0 j} \tag{7.1}$$

*or*

$$d^*(\alpha_\xi, \alpha_\zeta) = k_{i_0 j} \text{ and the first case fails.} \tag{7.2}$$

*Proof.* Suppose the first case fails. Without loss of generality, $d^*(\alpha_\xi, \beta^j_\zeta) > k_{i_0 j}$. Since $d^*(\alpha_\zeta, \beta^j_\zeta) = k_{i_0 j}$, Lemma 6.2.4 implies that $d^*(\alpha_\xi, \alpha_\zeta) = k_{i_0 j}$. □

Color $[\omega_1]^2$ in two colors, where $\{\xi, \zeta\}$ is colored according to which case of Claim 7.1.2 holds. Now apply the Dushnik–Miller theorem, $\omega_1 \to (\omega_1, \omega)^2$. In the first possibility, there is an uncountable set $X$ such that (7.1) holds between every $\xi, \zeta \in X$. Then we are done, since by choice of $i_0$, for all $\xi \in X$,

$$p_\xi(\alpha_\xi^{i_0}) \geq M \geq k_{i_0 j}.$$

In the second possibility, there is an infinite set $Y$ such that (7.2) holds between every $\xi, \zeta \in Y$. In particular, (7.1) fails for every $\xi, \zeta \in Y$, so by Ramsey's theorem there is an infinite $Y'$ such that either

$$d^*(\alpha_\xi, \beta_\zeta^j) > k_{i_0 j} \text{ for all } \xi < \zeta \text{ in } Y',$$

or

$$d^*(\alpha_\zeta, \beta_\xi^j) > k_{i_0 j} \text{ for all } \xi < \zeta \text{ in } Y'.$$

Assume that $d^*(\alpha_\xi, \beta_\zeta^j) > k_{i_0 j}$ for $\xi < \zeta$ in $Y'$; the other possibility of Ramsey's theorem would proceed similarly.

Fix $\xi < \zeta < \nu \in Y'$. Then $d^*(\alpha_\xi, \beta_\nu^j) > k_{i_0 j}$ and $d^*(\alpha_\zeta, \beta_\nu^j) > k_{i_0 j}$. By Lemma 6.2.4, we have $d(\alpha_\xi, \alpha_\zeta) > k_{i_0 j}$, but this contradicts (7.2). Theorem 7.1.1 is proved. □

**Corollary 7.1.3.** *There is a c.c.c. forcing extension in which every subset of $\kappa^+$ is careful.*

*Proof.* Iterate the forcing from Theorem 7.1.1 using finite support, with the usual bookkeeping to take care of any sets that were added in the construction. □

The proof of Theorem 7.1.1 relied heavily on the fact that $\mathrm{cf}(\kappa) = \omega$ (and that $\mathbb{P}$ used finite conditions). We can generalize Theorem 7.1.1 to singular cardinals with measurable cofinality, and Corollary 7.1.3 to singular cardinals with supercompact cofinality.

**Theorem 7.1.2.** *Let $\kappa$ be a singular cardinal with $\mathrm{cf}(\kappa) = \theta$ and $\theta < \kappa$ be an indestructibly supercompact cardinal. Let $\langle \kappa_i : i < \theta \rangle$ be a sequence of regular cardinals cofinal in $\kappa$ and*

*$\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a continuous tree-like scale on $\prod_i \kappa_i$. Then there is poset which is $< \theta$-directed closed and $\theta^+$-c.c. forcing that every subset of $\kappa^+$ is careful.*

*Proof.* Given $A \subseteq \kappa^+$, define $\mathbb{Q}_A$ to be the forcing of partial functions $p : \kappa^+ \to \theta$ with $|\mathrm{dom}(p)| < \theta$ such that $d^*(\alpha, \beta) \leq \max\{p(\alpha), p(\beta)\}$ for any $\alpha \in \mathrm{dom}(p) \cap A$ and $\beta \in \mathrm{dom}(p) \setminus A$, ordered by extension. The poset $\mathbb{Q}_A$ is clearly $< \theta$-directed closed.

Iterate the posets $\mathbb{Q}_A$ with supports of size $< \theta$, using a suitable bookkeeping to ensure that for each $A$ in the final model, $\mathbb{Q}_A$ was used at some stage. The indestructibility of the supercompactness of $\theta$ is used in order to ensure that $\theta$ is supercompact in all of the models along the iteration. Let $\mathbb{P}$ denote the iteration poset, and $\dot{Q}_\gamma$ name $\mathbb{Q}_{A^\gamma}$, where $A^\gamma \in V^{\mathbb{P}\restriction\gamma}$ is the set being made careful at stage $\gamma$. It is clear that $\mathbb{P}$ is $< \theta$-directed closed, so it remains to check that $\mathbb{P}$ is $\theta^+$-c.c. Since it is not true in general that an iteration of $\theta^+$-c.c. posets using $< \theta$ supports is $\theta^+$-c.c., we will argue for the whole iteration poset instead of the individual factors.

For contradiction, fix an antichain $\{p_\xi : \xi < \theta^+\}$. By $\theta$-distributivity, there is a dense set of conditions $p$ in the iteration where for each $\gamma \in \mathrm{dom}(p)$, $p\restriction\gamma$ forces the values of $p(\gamma)$ and $\{\alpha \in \mathrm{dom}(p(\gamma)) : \alpha \in A^\gamma\}$ (these are in the ground model). We will assume that the elements of the antichain were taken from this dense set. For $\xi < \theta^+$, let the *type of $p_\xi$ at $\gamma$* be the ordered pair $(m, n)$, where $m = |\mathrm{dom}(p(\gamma)) \cap A^\gamma|$ and $n = |\mathrm{dom}(p(\gamma)) \setminus A^\gamma|$ (by restricting to the dense set, this can be computed in $V$). By judicious thinning, we may assume that the supports of the $p_\xi$ form a $\Delta$-system with root $S$, and for each $\gamma \in S$,

- all of the $p_\xi(\gamma)$ have the same type, so we can enumerate $\mathrm{dom}(p_\xi(\gamma)) \cap A^\gamma$ as $\{\alpha^i_\xi : i < m\}$ and $\mathrm{dom}(p_\xi) \setminus A^\gamma$ as $\{\beta^i_\xi : i < n\}$,
- the domains of the $p_\xi(\gamma)$ form a $\Delta$-system, the $p_\xi(\gamma)$ agree on the root, and the $p_\xi\restriction\gamma$ force the same members of the root into $A^\gamma$.
- for every $i < m$, $j < n$, there is $k_{ij} < \theta$ (not depending on $\xi$) such that $\forall \xi < \omega_1 (d^*(\alpha^i_\xi, \beta^j_\xi) = k_{ij})$,

- for every $i < m$, $j < n$, either

$$\forall \xi < \theta^+ (p_\xi(\gamma)(\alpha^i_\xi) \geq k_{ij})$$

or

$$\forall \xi < \theta^+ (p_\xi(\gamma)(\beta^j_\xi) \geq k_{ij})$$

(i.e., whether it is $\alpha^i$ or $\beta^j$ that satisfies this does not depend on $\xi$).

These assumptions are analogous to ones we made in the proof of Theorem 7.1.1.

For distinct $\xi, \zeta < \theta^+$, let $\gamma(\xi, \zeta)$ be the least $\gamma$ such that $p_\xi(\gamma)$ and $p_\zeta(\gamma)$ are incompatible. Note that $\gamma(\xi, \zeta) \in S$ for every $\xi, \zeta$. By Rowbottom's theorem $\theta \to (\theta)^2_{<\theta}$, there is a subset $C \subseteq \theta^+$ of size $\theta$ and some $\gamma$ such that $\gamma(\xi, \zeta) = \gamma$ for all $\xi, \zeta \in C$. By relabeling the elements of the antichain, we may assume that $C = \theta$. Fix $i < m$ and $j < n$. A version of Claim 7.1.2 holds in this case.

**Claim 7.1.4.** *For every $\xi, \zeta < \theta$, either*

$$d^*(\alpha^i_\xi, \beta^j_\zeta) \leq k_{ij} \text{ and } d^*(\alpha^i_\zeta, \beta^j_\xi) \leq k_{ij} \tag{7.3}$$

*or*

$$d^*(\alpha^i_\xi, \alpha^i_\zeta) = k_{ij} \text{ and the first case fails.} \tag{7.4}$$

Color $[\theta]^2$ in two colors, where $\{\xi, \zeta\}$ is colored according to which case of Claim 7.1.4 holds. Let $U$ be a $\theta$-complete normal ultrafilter on $\theta$. By Rowbottom's theorem, there is $A_{i,j} \in U$ such that either (7.3) holds for all $\xi, \zeta \in A_{i,j}$, or (7.4) holds for all $\xi, \zeta \in A_{i,j}$.

By the same reasoning as in Theorem 7.1.1, the second possibility cannot occur. Let $A = \bigcap_{i,j} A_{i,j}$. For any distinct $\xi, \zeta \in A$,

$$d^*(\alpha^i_\xi, \beta^j_\zeta) \leq k_{ij} \text{ and } d^*(\alpha^i_\zeta, \beta^j_\xi) \leq k_{ij}$$

for all $i < m$, $j < n$. By our thinning assumptions, for any $i < m, j < n$, either

$$p_\xi(\gamma)(\alpha^i_\xi), p_\zeta(\gamma)(\alpha^i_\zeta) \geq k_{ij}$$

or

$$p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\beta^j_\zeta) \geq k_{ij}.$$

In either case, it follows that $d^*(\alpha^i_\xi, \beta^j_\zeta) \leq \max\{p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\beta^j_\zeta)\}$ and that $d^*(\alpha^i_\zeta, \beta^j_\xi) \leq \max\{p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\alpha^i_\zeta)\}$. By the minimality of $\gamma$, $p_\xi\restriction\gamma$ and $p_\zeta\restriction\gamma$ are compatible and any common extension forces that $p_\xi(\gamma)$ and $p_\zeta(\gamma)$ are compatible. □

*Remark* 7.1.5. To prove that the individual posets $\mathbb{P}_A$ as above are $\theta^+$-c.c., it is enough for $\theta$ to be measurable.

## 7.2 All sets careful and $\kappa$ strong limit

In the model produced by the forcing of Theorem 7.1.1, $2^{\aleph_0} > \kappa$. However, in singular cardinal combinatorics, the case where the singular cardinal $\kappa$ is strong limit is of particular interest. Large cardinals are required to obtain a model where every set is careful and $\kappa$ is strong limit, as the SCH would fail at $\kappa$ in such a model. Using a supercompact cardinal, we have the following:

**Theorem 7.2.1.** *Let $\kappa$ be an indestructibly supercompact cardinal and $\mu = \kappa^{+\kappa+1}$. Then there is a forcing poset which preserves cardinals below $\kappa$ and above $\mu$, and adds no bounded subsets of $\kappa$, such that in the extension:*

- *$\kappa$ is a singular strong limit cardinal with countable cofinality, and $\mu = \kappa^+$,*
- *there is a continuous scale on $\kappa$ of length $\mu$ for which every subset of $\mu$ is careful.*

For simplicity of our arguments, assume GCH holds above $\kappa$ in the ground model. By some preliminary forcing using slight modifications of Theorem 1 of Cummings [Cum10], we arrange so that there is a continuous tree-like scale $\langle G_\alpha : \alpha < \mu\rangle$ on $\prod_{\xi<\kappa} \kappa^{+\xi+1}$ (modulo the bounded ideal on $\kappa$). Using Theorem 17 of [CFM01], we can also arrange that $\langle G_\alpha : \alpha < \mu\rangle$ is a good scale.

Our plan is to make every subset of $\mu$ careful relative to $\langle G_\alpha : \alpha < \mu\rangle$, and then use a diagonal Prikry forcing technique from Gitik–Sharon [GS08] to singularize $\kappa$ while reflecting the scale down to $\kappa$ (as in Cummings–Foreman [CF10]). Let $X_\xi$ be the set of $x \in [\kappa^{+\xi+1}]^{<\kappa}$ with $\kappa_x := x \cap \kappa$ an inaccessible cardinal less than $\kappa$ and $\mathrm{ot}(x \cap \kappa^{+\zeta+1}) = \kappa_x^{+\zeta+1}$ for all $\zeta \leq \xi$. Then define LP to be the set of all finite sequences $\langle x_0, \ldots, x_{n-1}\rangle$ satisfying:

- $x_0 \in X_0$.
- For each $i < n$, $x_{i+1} \in X_{\kappa_{x_i}}$.
- $x_i \subseteq x_{i+1}$ and $\mathrm{ot}(x_i) < \kappa_{x_{i+1}}$ (we abbreviate this condition as $x_i \subsetneq x_{i+1}$).

This will be the set of all "lower parts" of conditions in a future Prikry forcing. The posets we define below will be $\kappa$-distributive and therefore all models will compute LP in the same way.

### 7.2.1 Carefulizing forcing

To make every subset of $\mu$ careful, we will define a poset $\mathbb{P}$ akin to those of Theorems 7.1.1 and 7.1.2. One challenge is that in addition to making ground model subsets of $\mu$ careful, we must also anticipate subsets added by the Prikry forcing.

For each family $\vec{A} = \langle A_s : s \in \mathrm{LP}\rangle$, $A_s \subseteq \mu$, define $\mathbb{Q}_{\vec{A}}$ to be the poset of partial functions $P : \mathrm{LP} \times \mu \to \kappa$ such that:

1. $|\mathrm{dom}(P)| < \kappa$, and if $t$ extends $s$ and $(t, \alpha) \in \mathrm{dom}(P)$, then also $(s, \alpha) \in \mathrm{dom}(P)$.
2. If $(s, \alpha)$ and $(s, \beta)$ are in $\mathrm{dom}(P)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then
$$d^*_G(\alpha, \beta) \leq \max\{P(s, \alpha), P(s, \beta)\}.$$
(Here $d^*_G$ is just the $d^*$ function on the scale $\vec{G}$.)
3. If $(t, \alpha) \in \mathrm{dom}(P)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $P(t, \alpha) = P(s, \alpha)$.

4. If $(t, \alpha) \in \mathrm{dom}(P)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $P(t, \alpha) = P(s, \alpha)$.

The ordering on $\mathbb{Q}_{\vec{A}}$ is function extension.

By the usual bookkeeping argument, we can define $\mathbb{P}$, an iteration of posets $\mathbb{Q}_{\vec{A}}$ using supports of size $< \kappa$ so that in the generic extension by $\mathbb{P}$, for each family $\vec{A} = \langle A_s : s \in \mathrm{LP} \rangle$ of subsets of $\mu$ indexed by LP, there is a function $F : \mathrm{LP} \times \mu \to \kappa$ such that:

- If $(s, \alpha)$ and $(s, \beta)$ are in $\mathrm{dom}(F)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then
$$d^*_G(\alpha, \beta) \leq \max\{F(s, \alpha), F(s, \beta)\}.$$
- If $(t, \alpha) \in \mathrm{dom}(F)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $F(t, \alpha) = F(s, \alpha)$.
- If $(t, \alpha) \in \mathrm{dom}(F)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $F(t, \alpha) = F(s, \alpha)$.

We now check that $\mathbb{P}$ does not collapse cardinals. It is easy to see that $\mathbb{P}$ is $< \kappa$-directed closed.

**Lemma 7.2.1.** *$\mathbb{P}$ is $\kappa^+$-c.c.*

*Proof of Lemma 7.2.1.* Suppose $P_\xi : \xi < \kappa^+$ is an antichain. As in the proof of Theorem 7.1.2, we will assume that the supports of the conditions form a $\Delta$-system with root $S$, and that for each $\gamma$, $P \restriction \gamma$ decides the values of $P(\gamma)$ and $\{(\alpha, t) \in \mathrm{dom}(P(\gamma)) : \alpha \in A^\gamma_t\}$, where $\langle A^\gamma_t : t \in \mathrm{LP} \rangle$ is the family used at stage $\gamma$. We will assume that the elements of the antichain were taken from this dense set. We may also assume that for each $\gamma \in S$ the domains of the $P_\xi(\gamma)$ form a $\Delta$-system, and furthermore that the sets $D^\gamma_\xi = \{s \in \mathrm{LP} : \exists \alpha\ (s, \alpha) \in \mathrm{dom}(P_\xi)(\gamma)\}$ form a $\Delta$-system. Let $R^\gamma$ denote the root of the $D^\gamma_\xi$ system. For any condition $P \in \mathbb{P}$ and $s \in \mathrm{LP}$, define the $s$-type of $P$ at $\gamma$ to be the ordered pair $(m, n)$, where $m = |\{\alpha \in A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}|$ and $n = |\{\alpha \notin A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}|$. By thinning the antichain, we may assume that for each $\gamma \in S$:

- If $s \in R^\gamma$, then all of the $P_\xi$ have the same $s$-type $(m_s, n_s)$ at $\gamma$, so we can enumerate $\{\alpha \in A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}$ as $\{\alpha^{s,i}_\xi : i < m_s\}$ and $\{\alpha \notin A_s : (s, \alpha) \in \mathrm{dom}(P)\}$ as $\{\beta^{s,i}_\xi : i < n_s\}$,

- the $P_\xi(\gamma)$ agree on the common parts of their domains,
- for every $s \in R^\gamma$, and every $i < m_s$, $j < n_s$, there is $k^s_{ij} < \kappa$ (not depending on $\xi$) such that $\forall \xi < \kappa^+(d^*(\alpha^{s,i}_\xi, \beta^{s,j}_\xi) = k_{ij})$,
- for every $s \in R^\gamma$, $i < m_s$, $j < n_s$, either

$$\forall \xi < \kappa^+(P_\xi(\gamma)(s, \alpha^{s,i}_\xi) \geq k_{ij})$$

or

$$\forall \xi < \kappa^+(P_\xi(\gamma)(s, \beta^{s,j}_\xi) \geq k_{ij})$$

(i.e., whether it is $\alpha^{s,i}$ or $\beta^{s,j}$ that satisfies this does not depend on $\xi$).

For distinct $\xi, \zeta < \kappa^+$, let $\gamma(\xi, \zeta)$ be the least $\gamma$ such that $P_\xi(\gamma)$ and $P_\xi(\gamma)$ are incompatible. Note that $\gamma(\xi, \zeta) \in S$ for every $\xi, \zeta$. By Rowbottom's theorem, there is a subset $C \subseteq \kappa^+$ of size $\kappa$ and some $\gamma$ such that $\gamma(\xi, \zeta) = \gamma$ for all $\xi, \zeta \in C$.

For $\xi \neq \zeta < \kappa^+$ and every $\gamma \in S$, $P_\xi(\gamma) \cup P_\zeta(\gamma)$ can only fail to be a valid condition in the poset $\mathbb{Q}_{\vec{A}^\gamma}$ by (2) of the definition of the poset: one can check that conditions (3) and (4) are satisfied by using conditions (3) and (4) for $P_\xi(\gamma)$ and $P_\zeta(\gamma)$, together with condition (1) and the fact that the elements of the antichain agree on the common parts of their domains. Therefore we have proven

**Claim 7.2.2.** *For $\xi \neq \zeta < \kappa^+$, there is an $s \in R$ and $\alpha, \beta < \mu$ such that exactly one of $\alpha, \beta$ is in $A_s$, $(s, \alpha) \in \mathrm{dom}(P_\xi(\gamma))$, $(s, \beta) \in \mathrm{dom}(P_\zeta(\gamma))$, and*

$$d^*_G(\alpha, \beta) > \max\{P_\xi(\gamma)(s, \alpha), P_\zeta(\gamma)(s, \beta)\}.$$

Using Rowbottom's theorem, we have a subset of $C$ of size $\kappa$ for which there is a single $s$ that sees the incompatibility between its elements. The proof of the lemma is completed exactly as in Theorem 7.1.2. $\square$

### 7.2.2 Diagonal Prikry forcing

Let $G$ be $\mathbb{P}$-generic, and work in $V[G]$. We now define a version of the supercompact diagonal Prikry forcing $\mathbb{R}$. In $V[G]$, $\kappa$ remains supercompact, so let $U$ be a $\kappa^{+\kappa+1}$-supercompactness measure, i.e., a normal, fine, $\kappa$-complete measure on $[\mu]^{<\kappa}$. For $\xi < \kappa$, define a $\kappa^{+\xi+1}$-supercompactness measure $U_\xi$ by

$$X \in U_\xi \quad \text{iff} \quad \{x \in [\mu]^{<\kappa} : x \cap \kappa^{+\xi+1} \in X\} \in U.$$

The measure $U_\xi$ concentrates on the set $X_\xi$.

Conditions in $\mathbb{R}$ are sequences of the form

$$p = \langle x_0^p, \ldots, x_{n-1}^p \rangle^\frown \langle Y_\xi^p : \kappa_{x_{n-1}^p} \leq \xi < \kappa \rangle$$

for some $n < \omega$ (the *length* of $p$), where $\langle x_0, \ldots, x_{n-1} \rangle \in \mathrm{LP}$, $\xi_p = 0$ if $n = 0$ and $\xi_p = \kappa_{x_{n-1}^p}$ if $n > 0$, and $Y_\xi \in U_\xi$ for each $\xi_p \leq \xi < \kappa$. When $p$ is clear from the context, we will omit the superscript $p$ and use the abbreviation $\kappa_i$ for $\kappa_{x_i}$. We will call $\langle x_0, \ldots, x_{n-1} \rangle$ the *lower part*, and $\langle Y_\xi : \kappa_{n-1} \leq \xi < \kappa \rangle$ the *upper part* of $p$.

A condition $q = \langle x_0^q, \ldots, x_{m-1}^q \rangle^\frown \langle Y_\xi^q : \xi_q \leq \xi < \kappa \rangle$ extends $p$ (written $q \leq p$) if and only if

- $m \geq n$, and $x_i^q = x_i^p$ for all $i < n$.
- For each $n \leq i < m$, $x_i^q \in Y_{\xi_i}^p$, where $\xi_i = \kappa_{x_{i-1}^q}$.
- $Y_\xi^q \subseteq Y_\xi^p$ for each $\xi \geq \xi_q$.

As usual in Prikry-type forcings, $q$ *directly extends* $p$ (written $q \leq^* p$) in case $q \leq p$ and $q$ has the same length as $p$. The underlying set of $\mathbb{R}$ equipped with the $\leq^*$ ordering is $< \kappa$-closed, by the completeness of the ultrafilters.

**Lemma 7.2.3** (Diagonal intersection). *Let $\langle \vec{Y}^s : s \in \mathrm{LP} \rangle$ be a family of upper parts so that $s^\frown \vec{Y}^s \in \mathbb{R}$. Then there is a sequence $\langle Z_\xi : \xi < \kappa \rangle$ such that for every $s \in \mathrm{LP}$, every extension of $s^\frown \langle Z_\xi : \xi_s \leq \xi < \kappa \rangle$ is compatible with $s^\frown \vec{Y}_s$.*

*Proof.* For each $s \in \mathrm{LP}$, write $\vec{Y}^s = \{Y^s_\xi : \xi_s \leq \xi < \kappa\}$. For each $x \in [\kappa^{+\kappa}]^{<\kappa}$ and $\xi \geq \kappa_x$, define $W^x_\xi := \bigcap\{Y^s_\xi : \bigcup s \subseteq x\}$. Since there are fewer than $\kappa$ many $s \in \mathrm{LP}$ with $\bigcup s \subseteq x$ for a given $x \in X_\xi$, all of the $W^x_\xi$ are in $U_\xi$. Now for each $\xi < \kappa$ let $Z_\xi = \{y \in [\kappa^{+\xi+1}]^{<\kappa} : \forall x \in X_\xi\ (x \subsetneq y \to y \in W^x_\xi)\}$, the diagonal intersection of the $W^x_\xi$, $x \in X_\xi$. By normality of $U_\xi$, $Z_\xi \in U_\xi$.

We now check that this works. Suppose $t = \{x_0, \ldots, x_{m-1}\}$ is the lower part of an extension of $s^\frown\langle Z_\xi : \xi < \kappa\rangle$ for some $s \in \mathrm{LP}$. For any $i < m$ greater than the length of $s$, $\bigcup s \subsetneq x_i$, so $x_i \in Y^s_{\xi_i}$. $\square$

In the situation of the lemma, we will call $\langle Z_\xi : \xi < \kappa\rangle$ the *diagonal intersection* of $\langle \vec{Y}^s : s \in \mathrm{LP}\rangle$.

Let $H = \langle x_0, x_1, \ldots\rangle$ be the generic sequence added by $\mathbb{R}$. Note that $x_n \in X_{\xi_n}$, where $\xi_0 = 0$ and $\xi_n = \kappa_{n-1}$ if $n > 0$. The following facts are analogues of the basic properties of the forcing in [GS08].

**Fact 7.2.4.** 1. $\mathbb{R}$ is $\mu$-c.c., and hence preserves all cardinals $\geq \mu$.

2. $\mathbb{R}$ has the Prikry property: if $p \in \mathbb{R}$ and $\sigma$ is a sentence in the forcing language, then there is $q \leq^* p$ which decides $\sigma$, i.e., forces $\sigma$ or $\neg\sigma$.

3. $\mathbb{R}$ adds no bounded subsets of $\kappa$.

4. For any $\langle Y_\xi : \xi < \kappa\rangle$, a sequence of sets with $Y \in U$ and $Y_\xi \in U_\xi$ for all $\xi$, $x_n \in Y_{\xi_n}$ for all sufficiently large $n < \omega$.

5. Forcing with $\mathbb{R}$ changes the cofinality of $\kappa^{+\xi}$ to $\omega$ for all $\xi < \kappa$, and therefore $\mu = \kappa^+$ in the generic extension by $\mathbb{R}$.

*Proof.* (1) follows from the fact that any two conditions with the same lower part are compatible, and there are fewer than $\mu$ many lower parts.

(2) For simplicity, assume that $p$ has length 0. Partition LP into

$$B_0 = \{s \in \mathrm{LP} : \text{ there is } \vec{Y}^s \text{ such that } s^\frown\vec{Y}^s \Vdash \sigma\},$$

$$B_1 = \{s \in \mathrm{LP} : \text{ there is } \vec{Y}^s \text{ such that } s^\frown \vec{Y}^s \Vdash \neg\sigma\},$$

and $B_2 = \mathrm{LP} \setminus (B_0 \cup B_1)$.

We will define a family of LP-indexed upper parts $\vec{Y}^s$. If $s \in B_0 \cup B_1$, take $\vec{Y}^s$ to be an upper part such that $s^\frown \vec{Y}^s$ decides $\sigma$. Otherwise, let $Y^s_{\xi_s} = \{x \in X_{\xi_s} : s^\frown x \in B_2\} \in U_{\xi_s}$ for all $\xi$ and $Y_\xi = X_\xi$ for $\xi_s < \xi < \kappa$. We check that if $s \in B_2$, then $Y^s_{\xi_s} \in U_{\xi_s}$, since otherwise there is $i \in \{0,1\}$ so that $\{x \in X_{\xi_s} : s^\frown x \in B_i\} \in U_{\xi_s}$, which would imply $s \in B_i$.

Take $r$ to be the diagonal intersection of the $\vec{Y}^s$. If the empty lower part is in $B_0 \cup B_1$, then we are done. Otherwise, assume $H = \langle x_0, x_1, \ldots\rangle$ was obtained by forcing below $r$, so by induction $H \restriction n \in B_2$ for all $n$, contradicting the genericity of $H$.

(3) is immediate from (2) and the $< \kappa$-closure of the $\leq^*$ ordering, and (4) is a straightforward density argument.

(5) By a density argument, $\kappa^{+\xi} = \bigcup_{n<\omega}(x_n \cap \kappa^{+\xi})$ for all $\xi < \kappa$. □

### 7.2.3 The final model

For each $\xi < \kappa$ and $\gamma < \kappa^{+\xi+1}$, let $F^\gamma_\xi : X_\xi \to \kappa$ be a function representing $\gamma$ in the ultrapower by $U_\xi$ with $F^\gamma_\xi(y) < \kappa_y^{\xi+1}$ for all $y \in X_\xi$. Define a sequence of functions $\langle f_\alpha : \alpha < \mu\rangle$ of $\prod_{n<\omega} \kappa_n^{+\xi_n+1}$ by

$$f_\alpha(n) = F^{G_\alpha(\xi_n)}_{\xi_n}(x_n).$$

Following Cummings–Foreman [CF10], we prove the following claim:

**Claim 7.2.5.** *In $V[G * H]$, the sequence $\langle f_\alpha : \alpha < \mu\rangle$ is a scale on $\prod_{n<\omega} \kappa_n^{+\xi_n+1}$.*

*Proof of Claim 7.2.5.* It is easy to see that $\langle f_\alpha : \alpha < \mu\rangle$ is $<^*$-increasing.

Suppose $g \in \prod_{n<\omega} \kappa_n^{+\xi_n+1}$. Working in $V[G]$, let $\dot{g}$ be a $\mathbb{R}$-name for $g$, and let $p \in \mathbb{R}$ be arbitrary. We will find $q \leq p$ and $\alpha < \mu$ such that $q \Vdash g <^* f_\alpha$.

For simplicity, assume that $p$ is the trivial condition and forces $\dot{g} \in \prod_{n<\omega} \kappa_n^{+\xi_n+1}$ (otherwise, we would just work below such a condition extending $p$). A lower part $t$ of length

$n+1$ determines the value of $\kappa_n < \kappa$, hence using the Prikry property, we can find an upper part $\vec{Y}^t$ such that $t^\frown \vec{Y}^t$ determines the value of $\dot{g}(n)$, and call this value $h(t)$. Let $q$ be the element of $\mathbb{R}$ with empty lower part and upper part equal to the diagonal intersection of the family $\langle \vec{Y}^t \rangle$, $t \in \mathrm{LP}$, so any condition of length $n+1$ compatible with $q$ with lower part $t$ determines the value of $\dot{g}(n)$ as $h(t)$.

For each $\xi < \kappa$ and $x \in X_\xi$, let

$$H_\xi(x) = \sup\{h(t) + 1 : t \text{ is a lower part with last coordinate } x\}.$$

*Subclaim.* For each $\xi < \kappa$, $H_\xi$ represents an ordinal $\gamma_\xi$ which is less than $\kappa^{+\xi+1}$ in the ultrapower by $U_\xi$.

It suffices to show that for any $\xi < \kappa$ and $U_\xi$-almost every $x \in X_\xi$, there are fewer than $\kappa_x^{+\xi+1}$ many lower parts with last coordinate $x$, and therefore $H_\xi(x) < \kappa_x^{+\xi+1}$. First note that $\{x \in X_\xi : (\forall \zeta \le \xi)\, (\kappa_x^{+\zeta+1})^{<\kappa_x} = \kappa_x^{+\zeta+1}\} \in U_\xi$ by a reflection argument since the GCH holds above $\kappa$ in $V$ and $\mathbb{P}$ does not add new sets of ordinals of size $< \kappa$. Now suppose $x$ is in this set, and $y \in X_\zeta$ appears before $x$ in some lower part, so $\zeta < \xi$. Then $y$ is a subset of $x \cap \kappa_x^{+\zeta+1}$, which has order-type $\kappa_x^{+\zeta+1}$ by the definition of $X_\xi$. The number of subsets of $\kappa_x^{+\zeta+1}$ of size $\kappa_y^{+\zeta+1}$ is equal to $\kappa_x^{+\zeta+1} < \kappa_x^{+\xi+1}$, proving the subclaim.

Since $\langle G_\alpha : \alpha < \mu \rangle$ is a scale, there is $\alpha < \mu$ such that $\gamma_\xi < G_\alpha(\xi)$ for large $\xi$. Therefore, $B_\xi := \{x \in X_\xi : H_\xi(x) < F_\xi^{G_\alpha(\xi)}(x)\} \in U_\xi$ for large enough $\xi$. Let $H = \langle x_0, x_1, \ldots \rangle$ be the $\mathbb{R}$-generic sequence obtained by forcing below $q$. Using Fact 7.2.4 part (4), for sufficiently large $n$, $x_n \in B_{\xi_n}$ and therefore:

$$g(n) = h(H \restriction n+1) < H_\xi(x_n) < F_{\xi_n}^{G_\alpha(\xi_n)}(x_n) = f_\alpha(n).$$

□

**Claim 7.2.6.** *In $V[G * H]$, the scale $\langle f_\alpha : \alpha < \mu \rangle$ is continuous.*

*Proof of Claim 7.2.6.* Let $\beta < \mu$ be a limit ordinal. We will check that $f_\beta$ is an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$. We can assume that $\omega < \mathrm{cf}(\beta)^V < \kappa$, since all other points have cofinality $\omega$ in $V[G * H]$. Working in $V[G]$ and using that $\langle G_\alpha : \alpha < \mu \rangle$ is a good scale, we can find $A$ unbounded in $\beta$ with order-type $\mathrm{cf}(\beta)$ and some $\xi_0 < \omega$ so that $\langle G_\alpha(\xi) : \alpha \in A \rangle$ is strictly increasing for each $\xi > \xi_0$. Therefore, $\xi \mapsto \sup\{G_\alpha(\xi) : \alpha \in A\}$ is an exact upper bound for $\langle G_\alpha : \alpha < \beta \rangle$. Using continuity of $\langle G_\alpha : \alpha < \mu \rangle$, we can pick $\xi_0$ large enough so that $G_\beta(\xi) = \sup\{G_\alpha(\xi) : \alpha \in A\}$ for all $\xi > \xi_0$. For each $\xi > \xi_0$,

$$\{x \in X_\xi : \langle F_\xi^{G_\alpha(\xi)}(x) : \alpha \in A \rangle \text{ is increasing with supremum } F_\xi^{G_\beta(\xi)}(x)\} \in U_\xi.$$

In $V[G * H]$, Fact 7.2.4 part (4) then implies that there is $n_0 < \omega$ so that for every $n \geq n_0$, $\langle f_\alpha(n) : \alpha \in A \rangle$ is strictly increasing with supremum $f_\beta(n)$. For any $h < f_\beta$ and any $n_0 < n < \omega$, there is $\alpha_n \in A$ so that $h(n) < f_{\alpha_n}(n)$. Let $\alpha^* < \beta$ be greater than $\sup_n \alpha_n$. Then $h <^* f_{\alpha^*}$, so $f_\beta$ is an exact upper bound for $\langle f_\alpha : \alpha < \beta \rangle$. $\square$

It remains to check that every $A \subseteq \mu$ is careful in $V[G * H]$. Working in $V[G]$, let $\dot{A}$ be a $\mathbb{R}$-name for $A$. For each $s \in \mathrm{LP}$, let

$$A_s = \{\alpha \in \mu : \text{ there exists an upper part } \vec{Y}^s \text{ such that } s^\frown \vec{Y}^s \Vdash \alpha \in \dot{A}\}.$$

By 7.2.1, there is in $V[G]$ a function $E : \mathrm{LP} \times \mu \to \kappa$ such that:

1. If $(s, \alpha)$ and $(s, \beta)$ are in $\mathrm{dom}(E)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then

$$d^*_G(\alpha, \beta) \leq \max\{E(s, \alpha), E(s, \beta)\}.$$

2. If $(t, \alpha) \in \mathrm{dom}(E)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $E(t, \alpha) = E(s, \alpha)$.

3. If $(t, \alpha) \in \mathrm{dom}(E)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $E(t, \alpha) = E(s, \alpha)$.

In $V[G * H]$, we will find a function $F$ as in Lemma 6.2.2 which shows that $A$ is careful. For any given $\alpha < \mu$, there is $p \in \mathbb{R}$ that either forces $\alpha \in \dot{A}$ or $\alpha \notin \dot{A}$. If $s$ is the lower part of $p$, then set $F_0(\alpha)$ to be the length of $s$. Now either

$$\alpha \in A_t \text{ for all } t \subseteq H \text{ extending } s,$$

or

$$\alpha \notin A_t \text{ for all } t \subseteq H \text{ extending } s,$$

where the measure 1 sets that witness membership (or nonmembership) in $A_t$ come from the upper part of $s$. In either case, the value of $E(t, \alpha)$ is constant for $s \subseteq t \subseteq H$, and let $F_1(\alpha)$ be the least $n$ such that $\xi_n$ is greater than this constant value.

For each $t \in \mathrm{LP}$, $\xi < \kappa$, define $S^t_\xi = \{G_\beta(\xi) : \beta \in A_t \text{ and } E(t, \beta) \leq \xi\}$ (the sequence $\langle S^t_\xi : \xi < \kappa\rangle$ witnesses that $A_t$ is careful on the scale $\langle G_\alpha : \alpha < \mu\rangle$). Then $S^t_\xi$ is a subset of $\kappa^{+\xi+1}$, and since the $U_\xi$ ultrapower is closed under $\kappa^{+\xi+1}$ sequences, $S^t_\xi$ is a member of this ultrapower, and hence is represented in the ultrapower by a function $s^t_\xi$ with domain $X_\xi$.

If $\alpha \in A_t$, then define

$$Y^{\alpha,t}_\xi = \{y \in X_\xi : F^{G_\alpha(\xi)}_\xi(y) \in s^t_\xi(y)\}$$

if $\xi \geq E(t, \alpha)$, and $X_\xi$ otherwise. Then $Y^{\alpha,t}_\xi \in U_\xi$ for each $\xi < \kappa$, since $G_\alpha(\xi) \in S^t_\xi$ in the first case of the definition and it is trivial in the second. If $\alpha \notin A_t$, define $Y^{\alpha,t}_\xi = \{y \in X_\xi : F^{G_\alpha(\xi)}_\xi(y) \notin s^t_\xi(y)\}$ if $\xi \geq E(t, \alpha)$, and $X_\xi$ otherwise. In the first case of the definition, property (1) of $E$ guarantees that $G_\alpha(\xi) \notin S^t_\xi$, so again $Y^{\alpha,t}_\xi \in U_\xi$ for each $\xi < \kappa$.

Let $\langle Y^\alpha_\xi : \xi < \kappa\rangle$ be the diagonal intersection of the $Y^{\alpha,t}_\xi$. By Fact 7.2.4 part (4), there is $N < \omega$ such that $x_n \in Y_{\xi_n}$ for all $n \geq N$; let $F_2(\alpha)$ be such an $N$.

Finally, define $F(\alpha)$ to be the maximum of $F_0(\alpha), F_1(\alpha), F_2(\alpha)$. Now if $\alpha \in A$ and $\beta \notin A$, we must verify that $d^*(\alpha, \beta) < \max\{F(\alpha), F(\beta)\}$ (here $d^*$ denotes the $d^*$ function for the scale $\langle f_\alpha : \alpha < \mu\rangle$). In other words, we must show that for any $n \geq \max\{F(\alpha), F(\beta)\}$, $f_\alpha(n) \neq f_\beta(n)$. Let $t = \langle x_0, x_1, \ldots, x_{n-1}\rangle$ be the initial segment of $H$ of length $n$. Since $n \geq F_0(\alpha), F_0(\beta)$, $\alpha \in A_t$ and $\beta \notin A_t$. Since $n \geq F_2(\alpha), F_2(\beta)$, $x_n \in Y^\alpha_{\xi_n} \cap Y^\beta_{\xi_n}$, and since $n \geq F_1(\alpha), F_1(\beta)$ both $Y^\alpha_{\xi_n}$ and $Y^\beta_{\xi_n}$ were defined using the first cases of their respective definitions. Therefore,

$$f_\alpha(n) = F^{G_\alpha(\xi_n)}_{\xi_n}(x_n) \in s^t_\xi(x_n) \quad \text{and} \quad f_\beta(n) = F^{G_\beta(\xi_n)}_{\xi_n}(x_n) \notin s^t_\xi(x_n)$$

and hence $f_\alpha(n) \neq f_\beta(n)$.

### 7.2.4 Making $\kappa$ into the least cardinal fixed point

Using techniques originating in Magidor [Mag77], collapses can be interleaved into the forcing of Theorem 7.2.1 so that $\kappa$ becomes the least cardinal with $\kappa = \aleph_\kappa$ in the final model. (In [GS08], interleaving collapses in diagonal Prikry forcing was used to turn $\kappa$ into $\aleph_{\omega^2}$). We will roughly sketch this construction. Working in $V[G]$, for each $\xi < \kappa$ let $i_\xi : V[G] \to N_\xi$ be the ultrapower by $U_\xi$. In $N_\xi$, $\mathrm{Col}(\kappa^{+\kappa+2}, i_\xi(\kappa))^{N_\xi}$ has cardinality $i_\xi(\kappa)$ and $i_\xi(\kappa)$-c.c. Back in $V[G]$, $|i_\xi(\kappa)| \leq \kappa^{\kappa^{\xi+1}} = \kappa^{\xi+2}$, so using the $< \kappa^{\xi+2}$-closure of the poset and $N_\xi$, we can find $K_\xi$ which is $\mathrm{Col}(\kappa^{+\kappa+2}, i_\xi(\kappa))^{N_\xi}$-generic over $N_\xi$. Then we can replace $\mathbb{R}$ in the construction of Theorem 7.2.1 by the forcing whose conditions are of the form

$$p = \langle c^p, x_0^p, f_0^p, \ldots, x_{n-1}^p, f_{n-1}^p \rangle^\frown \langle Y_\xi, F_\xi : \xi < \kappa \rangle$$

where

- $\langle x_0, \ldots, x_{n-1} \rangle^\frown \langle Y_\xi : \xi < \kappa \rangle$ is a condition from the diagonal Prikry forcing defined above.
- $c \in \mathrm{Col}(\omega, < \kappa_0)$.
- For all $i < n-1$, $f_i \in \mathrm{Col}(\kappa_i^{+\kappa_i+2}, < \kappa_{i+1})$.
- $f_{n-1} \in \mathrm{Col}(\kappa_{n-1}^{+\kappa_{n-1}+2}, \kappa)$.
- For $\xi \geq \kappa_{n-1}$, $F_\xi$ is a function with domain $Y_\xi$ such that $F_\xi(x) \in \mathrm{Col}(\kappa_x^{\kappa_x+2}, \kappa)$ and $F_\xi$ represents an element of $K_\xi$ in the $U_\xi$ ultrapower.

A condition $q = \langle c^q, x_0^q, f_0^q, \ldots, x_{m-1}^q, f_{m-1}^q \rangle^\frown \langle Y_\xi^q, F_\xi^q : \xi < \kappa \rangle$ extends $p$ if

- $\langle x_0^q, \ldots, x_{n-1}^q \rangle^\frown \langle Y_\xi^q : \xi < \kappa \rangle \leq \langle x_0^p, \ldots, x_{n-1}^p \rangle^\frown \langle Y_\xi^p : \xi < \kappa \rangle$ as conditions from $\mathbb{R}$.
- $c^q \leq c^p$ and $f_i^q \leq f_i^p$ for all $i < n$.
- For all $n \geq i < m$, $f_i^q \leq F_i^p(x_i^q)$.

- For all $\xi \geq \kappa_{m-1}$ and all $x \in Y^q_\xi$, $F^q_\xi(x) \leq F^p_\xi(x)$.

The restriction on the $F_\xi$ is needed to prove the $\mu$-c.c. and the Prikry property.

In the extension, if $\eta < \kappa$, then $\eta < \kappa_n$ for some $n$, and therefore $\aleph_\eta < \aleph_{\kappa_n} \leq \kappa_{n+1} < \kappa$, so $\kappa = \aleph_\kappa$. Furthermore, $\kappa_0$ is collapsed to be $\omega_1$ and for $n > 0$, $\kappa_n$ is the $n$th iteration of the map $\eta \mapsto \eta^{+\eta+3}$ evaluated at $\kappa_0$, so $\kappa$ must be the least cardinal fixed point.

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