Title
Non-classical dynamical equations of thick plateswith complete thermomechanical coupling

Journal
SCIENTIA SINICA Physica, Mechanica & Astronomica, 46(3)

ISSN
1674-7275

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Publication Date
2016-03-01

DOI
10.1360/sspma2015-00341

Peer reviewed

中国科学: 物理学 力学 天文学 2016 年 第 46 卷 第 3 期: 034601
SCIENTIA SINICA Physica, Mechanica & Astronomica phys.scichina.com
《中国科学》杂志社 SCIENCE CHINA PRESS

论 文

热-力双向耦合下平板结构振动的非经典方程

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收稿日期: 2015-07-03; 接受日期: 2015-08-27; 网络出版日期: 2016-01-18
国家自然科学基金(批准号: 51378451, 51276129)和高超声速发动机技术重点实验室开放基金(编号: 20130103007)资助项目

摘要 结构热-力耦合振动支配方程是对结构进行动力学分析与控制设计的基础. 本文基于三维热弹性动力学, 研究了力热双向耦合条件下平板结构的力热耦合动力学问题. 将代数 Vieta 定理与经典算子谱分解方法相结合, 发展了算子谱分解方法在结构振动力学建模中的应用. 选取适当的规范条件, 在时域内首次分别构建了受热平板弯曲振动和拉压振动的精确化方程的具体形式. 给出了力热双向耦合下平板结构中振动模式的频散关系曲线, 并对平板振动的空间和时间演化规律以及结构振动的动态稳定性做了分析和讨论. 本文结果是在没有采用经典假设下得到的, 因此得到控制方程是较精确的. 本文得到的平板力热耦合振动精化方程可用于求解高温下应力场和温度场都是动态变化的耦合问题, 研究高温环境下热-力动态耦合机理、耦合模式以及动态响应.

关键词 受热平板振动精确化理论, 热-力双向耦合, 算子谱分解与复变函数方法, 典型低维结构, 平板弯曲与拉压振动

PACS: 46.40.-f, 46.70.De, 44.10.+I, 02.30.Em, 02.30.Tb, 46.15.Ff, 11.15.-q, 46.25.Hf

现代近空间飞行器技术中有许多关键科学问题需要研究, 其中超高温环境下材料结构中的热-力耦合问题就是人们关心的问题之一, 包括力热耦合机理、模式以及动响应分析计算等[1–4]. 众所周知, 在常温或部件工作温度比较低的情况下, 在力热耦合问题的分析计算中, 常采用单向耦合计算的方法, 即只考虑温度对弹性应力-应变场的影响[5,6]. 这种力热单向耦合的处理方法, 在求解平均温度不是很高的静态力学问题时是可以满足工程精度要求的.

近空间飞行器外部受到很强的气动热作用, 同时发动机内部构件还受到更为复杂的高温流体的作用. 此时不仅需要考虑力热多物理场的耦合作用, 而且应该采用动态力学的观点研究空天材料的耦合动力学行为和结构振动稳定性[7,8]. 在极端环境下, 例如超高温度、超急速加热或结构振动的减缩频率比较高时, 单向耦合分析计算结果可能失实. 在对超高温

引用格式: 胡超, 郑日恒, 王琴, 等. 热-力双向耦合下平板结构振动的非经典方程. 中国科学: 物理学 力学 天文学, 2016, 46: 034601
Hu C, Zheng R H, Wang Q, et al. Non-classical dynamical equations of thick plates with complete thermomechanical coupling (in Chinese). Sci Sin-Phys Mech Astron, 2016, 46: 034601, doi: 10.1360/SSPMA2015-00341

环境下结构部件的分析设计中, 必须采用力热双向耦合的观点, 研究力热过程的相互影响, 才能更好地了解和掌握材料结构的力热耦合动力学行为, 例如结构内力热耦合的频谱特性、力热耦合机理、耦合响应模式等等[9,10].

在对高温热防护材料结构的分析与计算中, 力热耦合下结构振动方程是结构动态力学分析的基础. 但是, 直到目前, 对于典型工程结构, 例如平板, 还没有建立热-力双向耦合作用下平板结构振动的精确化方程[11–13]. 本文将基于三维热弹性动力学, 采用算子谱分解方法和规范场理论, 研究受热平板内力热耦合动力学问题. 在时域内构建受热平板热-力耦合动力学的精确化方程, 研究力热耦合机理、耦合响应模式以及结构动态稳定性等.

1　平板热弹性振动的精确化方程

根据三维热弹性动力学理论, 无体力时 Navier-Cauchy 热弹性方程和热传导方程可写为[14]

$$\mu_M\nabla_0^2\boldsymbol{u}+(\lambda_M+\mu_M)\nabla_0(\nabla_0\cdot\boldsymbol{u})-\beta_M\nabla_0\vartheta=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2},$$
$$K\nabla_0^2\vartheta-\beta_M T_0\frac{\partial}{\partial t}(\nabla_0\cdot\boldsymbol{u})=c_V\rho\frac{\partial\vartheta}{\partial t},\tag{1}$$

其中, $\boldsymbol{u}$ 和 ϑ 分别表示位移矢量和温度升高值, $\boldsymbol{u}=u_1\boldsymbol{e}_1+u_2\boldsymbol{e}_2+u_3\boldsymbol{e}_3$, $\vartheta=\theta-T_0$ 且 $\vartheta/T_0\ll 1.0$, T_0 是无应力条件下材料的平均绝对温度; λ_M 和 μ_M 为材料的 Lame 常数; ρ 为材料的密度; K 和 c_V 分别为材料的导热系数和定容比热; $\beta_M=(3\lambda_M+2\mu_M)\alpha_T$; α_T 是材料的热膨胀系数; $\nabla_0=\boldsymbol{e}_1\frac{\partial}{\partial x}+\boldsymbol{e}_2\frac{\partial}{\partial y}+\boldsymbol{e}_3\frac{\partial}{\partial z}$ 为三维空间 Hamilton 算子, 相应的 Laplace 算子为 $\nabla_0^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$; $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$; t 为时间, 假设在温度变化范围很小的情况下材料的物理参数不变.

方程(1)的热弹性动力学一般解可描述为[14]

$$\boldsymbol{u}=\left(\Box_{01}^2\Box_{02}^2-\varepsilon T_3\nabla_0^2\right)\boldsymbol{G}+\frac{\kappa\beta_M}{\mu_M}\nabla_0\Phi-\left[(1-\kappa)\Box_{03}^2-\varepsilon T_3\right]\nabla_0(\nabla_0\cdot\boldsymbol{G}),\tag{2a}$$

$$\vartheta=\frac{\mu_M}{\beta_M}\varepsilon T_3\Box_{02}^2(\nabla_0\cdot\boldsymbol{G})+\Box_{01}^2\Phi,\tag{2b}$$

其中, $\kappa=\frac{1-2\nu}{2(1-\nu)}$; $\boldsymbol{G}=\sum_{j=1}^{3}\boldsymbol{G}^j$ 是拟 Somigliana 矢量势函数, 满足如下偏微分方程

$$\Box_{01}^2\Box_{02}^2\Box_{03}^2\boldsymbol{G}-\varepsilon T_3\nabla_0^2\Box_{02}^2\boldsymbol{G}=0,\tag{3}$$

而标量势函数 Φ 满足如下方程

$$\Box_{01}^2\Box_{03}^2\Phi-\varepsilon T_3\nabla_0^2\Phi=0,\tag{4}$$

式中, $\Box_{0j}^2=\nabla_0^2-T_j^2=\nabla_0^2-\frac{1}{c_j^2}\frac{\partial^2}{\partial t^2}$ $(j=1,2)$ 为波动微分算子; c_1, c_2 分别为弹性波纵波和横波速度, $c_1^2=\frac{\lambda_M+2\mu_M}{\rho}$, $c_2^2=\frac{\mu_M}{\rho}$; $\Box_{03}^2=\nabla_0^2-T_3$ 是扩散微分算子; $T_3=\frac{1}{a}\frac{\partial}{\partial t}$; $a=\frac{K}{\rho c_V}$ 是热扩散系数; ε 是力热耦合常数[5,16], $\varepsilon=\frac{\kappa a\beta_M^2T_0}{\mu_M K}=\frac{(3-4\kappa)^2\alpha_T^2T_0}{\kappa}\left(\frac{\mu_M}{\rho c_V}\right)$.

将平板振动方程(3)和(4)改写为如下形式[15,16]

$$\left(\nabla_0^2-D_j^2\right)G_i^j=0,\quad(i=1,2,3;\ j=1,2,3),\tag{5a}$$

$$\left(\nabla_0^2-D_j^2\right)\Phi^j=0,\quad(j=1,3),\tag{5b}$$

其中 $D_j(j=1,2,3)$ 是拟时间微分算子, $D_2^2=T_2^2$, 算子 $D_j(j=1,3)$ 满足如下算子代数方程

$$D^4-\left[T_1^2+(1+\varepsilon)T_3\right]D^2+T_1^2T_3=0.\tag{6}$$

根据代数 Vieta 定理, 可有关系式

$$D_1^2+D_3^2=T_1^2+(1+\varepsilon)T_3,\quad D_1^2D_3^2=T_1^2T_3.\tag{7}$$

平板热弹性振动时存在着弯曲和拉伸两种力学状态. 平板拉伸和弯曲分别属于对称运动和反对称运动情况. 弯曲时位移 u_x, u_y 是坐标 z 的奇函数, 而位移 u_z 是 z 的偶函数. 此时, 温度是 z 的奇函数. 拉伸时位移 u_x, u_y 是 z 的偶函数, 而位移 u_z 是 z 的奇函数[12,16]. 此时, 温度是 z 的偶函数.

根据 Taylor 级数展开式, 平板内任意一点的位移和温度可描述为

$$u_k(x,y,z,t)=\exp\left(z\frac{\partial}{\partial z}\right)u_k(x,y,0,t),\ (k=1,2,3),$$
$$\vartheta(x,y,z,t)=\exp\left(z\frac{\partial}{\partial z}\right)\vartheta(x,y,0,t),\tag{8}$$

由式(2)和(8)可知, 力热耦合振动时结构内任意一点的矢量势函数 $\boldsymbol{G}$ 和标量势函数 Φ 分别为

$$\begin{aligned} G_k(x,y,z,t) &= \exp\left(z\frac{\partial}{\partial z}\right)\sum_{j=1}^{3}G_k^j(x,y,0,t) \\ &= 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\exp(\mathrm{i}\,z\square_j)\,g_k^{j1}(x,y,t)\right], \\ \varPhi(x,y,z,t) &= \exp\left(z\frac{\partial}{\partial z}\right)\sum_{j=1,\neq 2}^{3}\varPhi^j(x,y,0,t) \\ &= 2\,\mathrm{Re}\left[\sum_{j=1,\neq 2}^{3}\exp(\mathrm{i}\,z\square_j)\,\varphi^{j1}(x,y,t)\right], \end{aligned} \tag{9}$$

其中, $\square_j=\sqrt{\nabla^2-D_j^2}$, $(j=1,2,3)$ 是复变微分算子. 根据算子谱及其分解理论可有如下表达式[11]

$$\begin{aligned} &\boldsymbol{G}=\sum_{j=1}^{3}\boldsymbol{G}^j=\sum_{j=1}^{3}\sum_{i=1}^{2}\boldsymbol{G}^{ji},\quad \left(\square_j^2+\frac{\partial^2}{\partial z^2}\right)\boldsymbol{G}^j=0, \\ &\left(\frac{\partial}{\partial z}-\mathrm{i}\square_j\right)\boldsymbol{G}^{j1}=0,\quad \left(\frac{\partial}{\partial z}+\mathrm{i}\square_j\right)\boldsymbol{G}^{j2}=0, \\ &\boldsymbol{\varPhi}=\sum_{j=1,\neq 2}^{3}\boldsymbol{\varPhi}^j=\sum_{j=1,\neq 2}^{3}\sum_{i=1}^{2}\boldsymbol{\varPhi}^{ji},\quad \left(\square_j^2+\frac{\partial^2}{\partial z^2}\right)\boldsymbol{\varPhi}^j=0, \\ &\left(\frac{\partial}{\partial z}-\mathrm{i}\square_j\right)\boldsymbol{\varPhi}^{j1}=0,\quad \left(\frac{\partial}{\partial z}+\mathrm{i}\square_j\right)\boldsymbol{\varPhi}^{j2}=0. \end{aligned}$$

在公式推演中总共出现了 11 个未知函数, 需要对函数给出 7 个适当的限制条件, 以消除未知函数的不唯一性[16]. 即消除非物理维数自由度, 同时还会使问题的分析求解适当地简化一些. 针对平板结构的热弹性振动问题, 本文选取了以下规范条件

$$\begin{aligned} &\mathrm{Re}\left(\frac{\partial}{\partial x}g_1^{j1}+\frac{\partial}{\partial y}g_2^{j1}\right)=0,\quad \sum_{j=1,j\neq 2}^{3}\square_j\,\mathrm{Im}(\varphi^{j1})=0, \\ &\sum_{j=1}^{3}(D_j^2-T_2^2)\square_j^2\,\mathrm{Re}(g_3^{j1})=0, \\ &\sum_{j=1}^{3}(D_j^2-T_2^2)\square_j\,\mathrm{Im}(g_3^{j1})=0, \\ &\left[(1-\kappa)(D_3^2-T_3)-\varepsilon T_3\right]D_3^2\square_3\,\mathrm{Im}(g_3^{31}) \\ &+\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,j\neq 2}^{3}D_j^2\,\mathrm{Re}(\varphi^{j1})=0, \end{aligned} \tag{10}$$

式中, $g_k^j\,(j=1,2,3;\ k=1,2)$是一些复变函数.

引进规范条件式(10), 将式(9)代入到式(2), 可得平板结构内振动位移和温度的表达式[14]

$$\begin{aligned} u_k &= 2\,\mathrm{Re}\{[(1-\kappa)(D_2^2-T_3)-\varepsilon T_3]T_2^2\exp(\mathrm{i}\,z\square_2) \\ &\times g_k^{21}\}+2\,\mathrm{Im}\left\{\sum_{j=1}^{3}\left[(1-\kappa)(D_j^2-T_3)-\varepsilon T_3\right]\right. \\ &\left.\times\square_j\frac{\partial}{\partial x_k}\left[\exp(\mathrm{i}\,z\square_j)g_3^{j1}\right]\right\}+2\,\mathrm{Re}\left\{\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\right. \\ &\left.\times\frac{\partial}{\partial x_k}\left[\exp(\mathrm{i}\,z\square_j)\varphi^{j1}\right]\right\},\quad (k=1,2), \end{aligned} \tag{11a}$$

$$\begin{aligned} u_3 &= 2\,\mathrm{Re}\{[(1-\kappa)(D_2^2-T_3)-\varepsilon T_3]T_2^2\exp(\mathrm{i}\,z\square_2) \\ &\times g_3^{21}\}+2\,\mathrm{Re}\left\{\sum_{j=1}^{3}\left[(1-\kappa)(D_j^2-T_3)-\varepsilon T_3\right]\right. \\ &\left.\times\square_j^2\exp(\mathrm{i}\,z\square_j)g_3^{j1}\right\}-2\,\mathrm{Im}\left\{\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\left[\square_j\right.\right. \\ &\left.\left.\times\exp(\mathrm{i}\,z\square_j)\varphi^{j1}\right]\right\}, \end{aligned} \tag{11b}$$

$$\begin{aligned} \vartheta &= -2\frac{\mu_M}{\beta_M}\mathrm{Im}\left[\sum_{j=1}^{3}\varepsilon T_3(D_j^2-T_2^2)\exp(\mathrm{i}\,z\square_j)\square_j g_3^{j1}\right] \\ &+2\mathrm{Re}\left[\sum_{j=1,\neq 2}^{3}(D_j^2-T_1^2)\exp(\mathrm{i}\,z\square_j)\varphi^{j1}\right]. \end{aligned} \tag{11c}$$

由式(11)可得平板中面位移、法线转角、横向正应变、中面温度以及法向温度梯度的表达式

$$\begin{aligned} U_k &= u_k\big|_{z=0} \\ &= 2\left[(1-\kappa)(D_2^2-T_3)-\varepsilon T_3\right]T_2^2\,\mathrm{Re}(g_k^{21}) \\ &+2\sum_{j=1}^{3}\left[(1-\kappa)(D_j^2-T_3)-\varepsilon T_3\right]\square_j\frac{\partial}{\partial x_k} \\ &\times\mathrm{Im}(g_3^{j1})+2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\frac{\partial}{\partial x_k}\mathrm{Re}(\varphi^{j1}), \end{aligned} \tag{12a}$$

$$\begin{aligned} W &= u_3\big|_{z=0} \\ &= 2\left[(1-\kappa)(D_2^2-T_3)-\varepsilon T_3\right]T_2^2\,\mathrm{Re}(g_3^{21}) \\ &+2\sum_{j=1}^{3}\left[(1-\kappa)(D_j^2-T_3)-\varepsilon T_3\right]\square_j^2\,\mathrm{Re}(g_3^{j1}) \\ &-2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\square_j\,\mathrm{Im}(\varphi^{j1}), \end{aligned} \tag{12b}$$

$$\begin{aligned} \psi_k &= -\frac{\partial u_k}{\partial z}\bigg|_{z=0} \\ &= 2\left[(1-\kappa)(D_2^2-T_3)-\varepsilon T_3\right]T_2^2\square_2\,\mathrm{Im}(g_k^{21}) \\ &-2\sum_{j=1}^{3}\left[(1-\kappa)(D_j^2-T_3)-\varepsilon T_3\right]\square_j^2\frac{\partial}{\partial x_k}\mathrm{Re}(g_3^{j1}) \end{aligned}$$

$$+2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\square_j\frac{\partial}{\partial x_k}\mathrm{Im}\left(\varphi^{j1}\right),(k=1,2), \tag{12c}$$

$$\begin{aligned}E&=\left.\frac{\partial u_z}{\partial z}\right|_{z=0}\\&=-2\left[(1-\kappa)\left(D_2^2-T_3\right)-\varepsilon T_3\right]T_2^2\square_2\mathrm{Im}\left(g_3^{21}\right)\\&\quad-2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2-T_3\right)-\varepsilon T_3\right]\square_j^3\mathrm{Im}\left(g_3^{j1}\right)\\&\quad-2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\square_j^2\mathrm{Re}\left(\varphi^{j1}\right),\end{aligned} \tag{12d}$$

$$\begin{aligned}\vartheta^m=\vartheta\big|_{z=0}&=-2\frac{\mu_M}{\beta_M}\sum_{j=1}^{3}\left[\varepsilon T_3\left(D_j^2-T_2^2\right)\square_j\mathrm{Im}\left(g_3^{j1}\right)\right]\\&\quad+2\sum_{j=1,\neq 2}^{3}\left(D_j^2-T_1^2\right)\mathrm{Re}\left(\varphi^{j1}\right),\end{aligned} \tag{12e}$$

$$\begin{aligned}\vartheta^g=\left.\frac{\partial\vartheta}{\partial z}\right|_{z=0}&=-\frac{2\mu_M}{\beta_M}\sum_{j=1}^{3}\left[\varepsilon T_3\left(D_j^2-T_2^2\right)\square_j^2\mathrm{Re}\left(g_3^{j1}\right)\right]\\&\quad-2\sum_{j=1,\neq 2}^{3}\left(D_j^2-T_1^2\right)\square_j\mathrm{Im}\left(\varphi^{j1}\right).\end{aligned} \tag{12f}$$

对中面转角和位移函数作如下形式分解[15,17]:

$$\psi_1=\frac{\partial}{\partial x}F^{(1)}+\frac{\partial}{\partial y}f^{(1)},\quad \psi_2=\frac{\partial}{\partial y}F^{(1)}-\frac{\partial}{\partial x}f^{(1)}, \tag{13a}$$

$$U_1=\frac{\partial}{\partial x}F^{(2)}+\frac{\partial}{\partial y}f^{(2)},\quad U_2=\frac{\partial}{\partial y}F^{(2)}-\frac{\partial}{\partial x}f^{(2)}. \tag{13b}$$

将式(13)代入到式(12)中可得到如下表达式:

$$\mathrm{Im}\left(g_1^{21}\right)=\frac{1}{2}\frac{1}{\square_2\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)}\frac{\partial}{\partial y}f^{(1)},$$

$$\mathrm{Re}\left(g_1^{21}\right)=\frac{1}{2}\frac{1}{\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)}\frac{\partial}{\partial y}f^{(2)},$$

$$\mathrm{Im}\left(g_2^{21}\right)=-\frac{1}{2}\frac{1}{\square_2\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)}\frac{\partial}{\partial x}f^{(1)},$$

$$\mathrm{Re}\left(g_2^{21}\right)=-\frac{1}{2}\frac{1}{\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)}\frac{\partial}{\partial x}f^{(2)},$$

$$\mathrm{Re}\left(g_3^{11}\right)=\frac{1}{2}\frac{\left(\square_2^2W+\nabla^2F^{(1)}\right)}{\square_1^2\left(D_1^2-D_3^2\right)\left(D_1^2-T_2^2\right)},$$

$$\mathrm{Re}\left(g_3^{31}\right)=-\frac{1}{2}\frac{\left(\square_2^2W+\nabla^2F^{(1)}\right)}{\square_3^2\left(D_1^2-D_3^2\right)\left(D_3^2-T_2^2\right)},$$

$$\mathrm{Re}\left(g_3^{21}\right)=\frac{1}{2}\frac{\left(W+F^{(1)}\right)}{\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)},$$

$$\mathrm{Im}\left(\varphi^{11}\right)=-\frac{1}{2}\frac{1}{\square_1\left(D_1^2-D_3^2\right)}\vartheta^g,$$

$$\mathrm{Im}\left(\varphi^{31}\right)=\frac{1}{2}\frac{1}{\square_3\left(D_1^2-D_3^2\right)}\vartheta^g, \tag{14}$$

$$\mathrm{Im}\left(g_3^{11}\right)=\frac{1}{2}\frac{\left(E+F^{(2)}\right)}{\square_1D_1^2\left(D_1^2-T_2^2\right)\left(D_3^2-T_1^2\right)},$$

$$\mathrm{Im}\left(g_3^{21}\right)=\frac{1}{2}\frac{\beta_M}{\mu_M}\frac{1}{\square_2\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)}\vartheta^m-\frac{1}{2}\frac{\left[E+\left(\nabla^2-T_1^2\right)F^{(2)}\right]}{\kappa\square_2\left(D_1^2-T_2^2\right)\left(D_3^2-T_2^2\right)},$$

$$\mathrm{Im}\left(g_3^{31}\right)=-\frac{1}{2}\frac{T_2^2\left(E+\nabla^2F^{(2)}\right)}{\square_3D_1^2\left(D_3^2-T_1^2\right)\left(D_3^2-T_2^2\right)},$$

$$\mathrm{Re}\left(\varphi^{11}\right)=-\frac{1}{2}\frac{\mu_M}{\kappa\beta_M}\frac{\left(D_3^2-T_3\right)^2\left(E+\nabla^2F^{(2)}\right)}{T_3\left(D_1^2-D_3^2\right)\left(D_1^2-T_1^2\right)}+\frac{1}{2}\frac{T_3}{D_1^2\left(D_1^2-D_3^2\right)}\vartheta^m,$$

$$\mathrm{Re}\left(\varphi^{31}\right)=\frac{1}{2}\frac{\mu_M}{\kappa\beta_M}\frac{\left(D_3^2-T_3\right)^2\left(E+\nabla^2F^{(2)}\right)}{T_3\left(D_1^2-D_3^2\right)\left(D_3^2-T_1^2\right)}-\frac{1}{2}\frac{T_3}{D_3^2\left(D_1^2-D_3^2\right)}\vartheta^m.$$

根据 Hooke 定律, 可得应力分量的表达式

$$\begin{aligned}\tau_{3k}&=\mu_M\left(\frac{\partial u_z}{\partial x_k}+\frac{\partial u_k}{\partial z}\right),\quad(k=1,2)\\&=2\mu_M\left\{2\,\mathrm{Re}\left\{\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2-T_3\right)-\varepsilon T_3\right]\square_j^2\frac{\partial}{\partial x_k}\right.\right.\\&\qquad\left.\times\exp\left(\mathrm{i}\,z\square_j\right)g_3^{j1}\right\}\\&\qquad+\mathrm{Re}\left\{\left[(1-\kappa)\left(D_2^2-T_3\right)-\varepsilon T_3\right]\right.\\&\qquad\left.\times T_2^2\exp\left(\mathrm{i}\,z\square_2\right)\frac{\partial}{\partial x_k}g_3^{21}\right\}\\&\qquad-\mathrm{Im}\left\{[(1-\kappa)(D_2^2-T_3)\right.\\&\qquad\left.-\varepsilon T_3]\square_2T_2^2\exp\left(\mathrm{i}\,z\square_2\right)g_k^{21}\right\}-2\frac{\kappa\beta_M}{\mu_M}\\&\qquad\left.\times\mathrm{Im}\left\{\sum_{j=1,\neq 2}^{3}\left[\square_j\exp\left(\mathrm{i}\,z\square_j\right)\frac{\partial}{\partial x_k}\varphi^{j1}\right]\right\}\right\},\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma_z &= \lambda_M \theta + 2\mu_M \varepsilon_z - \frac{E_M}{1-2\nu}\alpha_T \vartheta \\ &= \frac{2\mu_M}{\kappa}\left\{ \mathrm{Im}\left\{ \sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2 - T_3\right) - \varepsilon T_3 \right] \right.\right. \\ &\quad \left. \times \left(D_j^2 - 2\kappa\nabla^2\right)\Box_j \exp\left(\mathrm{i}\, z\Box_j\right) g_3^{j1} \right\} \\ &\quad + \kappa\, \mathrm{Im}\left\{ \sum_{j=1}^{3}\left[\varepsilon T_3 \left(D_j^2 - T_2^2\right)\Box_j \exp\left(\mathrm{i}\, z\Box_j\right) \right.\right. \\ &\quad \left.\left. \times g_3^{j1} \right]\right\} - \mathrm{Im}\left\{ [(1-\kappa)(D_2^2 - T_3) - \varepsilon T_3] T_2^2 \Box_2 \right. \\ &\quad \left. \times \exp\left(\mathrm{i}\, z\Box_2\right) g_3^{21} \right\} + \frac{\kappa\beta_M}{\mu_M}\mathrm{Re}\left\{\left[\left(T_1^2 - 2\kappa\nabla^2\right) \right.\right. \\ &\quad \left.\left.\left. \times \exp\left(\mathrm{i}\, z\Box_j\right)\varphi^{j1} \right]\right\}\right\}. \end{aligned} \tag{16}$$

在应力限制方面, 设板的上下表面 $\left(z=\pm\dfrac{h}{2}\right)$ 承受的切向载荷都为零, 而法向载荷分别为 q 和 0. 将边界条件引入到式(15)和(16)中可得

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{\left[(1-\kappa)\left(D_2^2 - T_3\right) - \varepsilon T_3\right] T_2^2 \left[\cos\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right)\right.\right. \\ &\left. \mp \sin\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)\right] + 2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2 - T_3\right) - \varepsilon T_3\right] \\ &\times \Box_j^2\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right) \mp \sin\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right] - 2\frac{\kappa\beta_M}{\mu_M} \\ &\left.\times \sum_{j=1,\neq 2}^{3}\left\{\Box_j\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(\varphi^{j1}\right) \pm \sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(\varphi^{j1}\right)\right]\right\}\right\} \\ &\quad = \frac{1}{2}\frac{\partial}{\partial y}\left[\cos\left(\frac{h}{2}\Box_2\right) f^{(1)} \pm \Box_2 \sin\left(\frac{h}{2}\Box_2\right) f^{(2)}\right], \end{aligned} \tag{17a}$$

$$\begin{aligned} &\frac{\partial}{\partial y}\left\{\left[(1-\kappa)\left(D_2^2 - T_3\right) - \varepsilon T_3\right] T_2^2 \left[\cos\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right)\right.\right. \\ &\left. \mp \sin\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)\right] + 2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2 - T_3\right) - \varepsilon T_3\right] \\ &\Box_j^2\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right) \mp \sin\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right] - 2\frac{\kappa\beta_M}{\mu_M} \\ &\left.\sum_{j=1,\neq 2}^{3}\left\{\Box_j\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(\varphi^{j1}\right) \pm \sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(\varphi^{j1}\right)\right]\right\}\right\} \\ &\quad = -\frac{1}{2}\frac{\partial}{\partial x}\left[\cos\left(\frac{h}{2}\Box_2\right) f^{(1)} \pm \Box_2 \sin\left(\frac{h}{2}\Box_2\right) f^{(2)}\right], \end{aligned} \tag{17b}$$

$$\begin{aligned} &\sum_{j=1}^{3}\left\{\left[(1-\kappa)\left(D_j^2 - T_3\right) - \varepsilon T_3\right]\left(D_j^2 - 2\kappa\nabla^2\right)\Box_j\right. \\ &\left.\times\left[\pm\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right) + \cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right]\right\} \\ &+\kappa\sum_{j=1}^{3}\left\{\varepsilon T_3\left(D_j^2 - T_2^2\right)\Box_j\left[\pm\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right)\right.\right. \\ &\left.\left. + \cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right]\right\} - \left[(1-\kappa)\left(D_2^2 - T_3\right) - \varepsilon T_3\right] \\ &\times T_2^2\Box_2\left[\pm\sin\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right) + \cos\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)\right] \\ &+\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\left\{\left(T_1^2 - 2\kappa\nabla^2\right)\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(\varphi^{j1}\right)\right.\right. \\ &\left.\left.\mp\sin\left(z\Box_j\right)\mathrm{Im}\left(\varphi^{j1}\right)\right]\right\} = \frac{\kappa}{4\mu_M}\left(\pm q + q\right), \end{aligned} \tag{18}$$

式中, h 是平板结构的厚度.

等式(17a)和(17b)可以看成是一个解析函数的实部和虚部所满足的 Cauchy-Riemann 条件[17], 可有

$$\begin{aligned} &\left[(1-\kappa)\left(D_2^2 - T_3\right) - \varepsilon T_3\right] T_2^2\left[\cos\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right)\right. \\ &\left.\mp\sin\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)\right] + 2\sum_{j=1}^{3}\Box_j^2[(1-\kappa)(D_j^2 - T_3) \\ &-\varepsilon T_3]\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right) \mp \sin\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right] \\ &-2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\left\{\Box_j\left[\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(\varphi^{j1}\right) \pm \sin\left(\frac{h}{2}\Box_j\right)\right.\right. \\ &\left.\left.\times\mathrm{Re}\left(\varphi^{j1}\right)\right]\right\} = 0, \end{aligned} \tag{19}$$

$$\cos\left(\frac{h}{2}\Box_2\right) f^{(1)} \pm \Box_2\sin\left(\frac{h}{2}\Box_2\right) f^{(2)} = 0. \tag{20}$$

将受热平板振动分解为相对于中面的对称和反对称运动, 其中反对称运动表示板的弯曲, 对称运动则描述板的拉伸, 分别由方程(21)和(22)描述

$$\begin{aligned} &2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2 - T_3\right) - \varepsilon T_3\right]\Box_j^2\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right) \\ &+\left[(1-\kappa)\left(D_2^2 - T_3\right) - \varepsilon T_3\right] T_2^2\cos\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right) \\ &-2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\left[\Box_j\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(\varphi^{j1}\right)\right] = 0, \end{aligned} \tag{21a}$$

$$\cos\left(\frac{h}{2}\Box_2\right)f^{(1)}=0, \tag{21b}$$

$$\begin{aligned}&2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2-T_3\right)-\varepsilon T_3\right]\Box_j^2\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\\&+\left[(1-\kappa)\left(D_2^2-T_3\right)-\varepsilon T_3\right]T_2^2\sin\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)\\&+2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\left[\Box_j\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(\varphi^{j1}\right)\right]=0,\end{aligned} \tag{22a}$$

$$\Box_2\sin\left(\frac{h}{2}\Box_2\right)f^{(2)}=0. \tag{22b}$$

由式(18)可以得到分别对应于平板结构的弯曲振动和拉伸振动的方程如下:

$$\begin{aligned}&2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2-T_3\right)-\varepsilon T_3\right]\left(D_j^2-2\kappa\nabla^2\right)\Box_j\\&\times\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right)+2\kappa\sum_{j=1}^{3}\varepsilon T_3\left(D_j^2-T_2^2\right)\Box_j\\&\times\sin\left(\frac{h}{2}\Box_j\right)\mathrm{Re}\left(g_3^{j1}\right)-2\left[(1-\kappa)\left(D_2^2-T_3\right)-\varepsilon T_3\right]\\&\times T_2^2\Box_2\sin\left(\frac{h}{2}\Box_2\right)\mathrm{Re}\left(g_3^{21}\right)-2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\sin\left(\frac{h}{2}\Box_j\right)\\&\times\left(T_1^2-2\kappa\nabla^2\right)\mathrm{Im}\left(\varphi^{j1}\right)=\frac{\kappa}{2\mu_M}q,\end{aligned} \tag{23a}$$

$$\begin{aligned}&2\sum_{j=1}^{3}\left[(1-\kappa)\left(D_j^2-T_3\right)-\varepsilon T_3\right]\left(D_j^2-2\kappa\nabla^2\right)\Box_j\\&\times\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)+2\kappa\sum_{j=1}^{3}\varepsilon T_3\left(D_j^2-T_2^2\right)\Box_j\\&\times\cos\left(\frac{h}{2}\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)-2\left[(1-\kappa)\left(D_2^2-T_3\right)-\varepsilon T_3\right]\\&\times T_2^2\Box_2\cos\left(\frac{h}{2}\Box_2\right)\mathrm{Im}\left(g_3^{21}\right)+2\frac{\kappa\beta_M}{\mu_M}\sum_{j=1,\neq 2}^{3}\cos\left(\frac{h}{2}\Box_j\right)\\&\times\kappa\left(T_2^2-2\nabla^2\right)\mathrm{Re}\left(\varphi^{j1}\right)=\frac{\kappa}{2\mu_M}q.\end{aligned} \tag{23b}$$

热边界条件方面分为两种典型情况: (1) 平板上下表面($z=\pm h/2$)给定温度; (2) 平板上下表面给定热流密度. 一般地, 研究给定材料表面温度情况[18], 由式(11c)中, 可得相应的温度控制方程

$$\begin{aligned}&-2\frac{\mu_M}{\beta_M}\mathrm{Im}\sum_{j=1}^{3}\left[\varepsilon T_3\left(D_j^2-T_2^2\right)\exp\left(\mathrm{i}\frac{h}{2}\Box_j\right)\Box_j g_3^{j1}\right]\\&+2\mathrm{Re}\sum_{j=1,\neq 2}^{3}\left[\left(D_j^2-T_1^2\right)\exp\left(\mathrm{i}\frac{h}{2}\Box_j\right)\varphi^{j1}\right]=\pm\vartheta^d+\vartheta^a,\end{aligned} \tag{24}$$

其中, $\vartheta^a=\frac{1}{2}\left(\vartheta^u+\vartheta^l\right)$; $\vartheta^d=\frac{1}{2}\left(\vartheta^u-\vartheta^l\right)$; ϑ^u 和 ϑ^l 分别为平板结构的上表面和下表面温度.

由式(24)可以得到分别对应于受热平板结构的弯曲振动和拉伸振动时热传导方程如下[14,18]:

$$\begin{aligned}&\sum_{n=1}^{2}(-1)^{n-1}\frac{\sin\left(\frac{h}{2}\Box_{2n-1}\right)}{\left(D_1^2-D_3^2\right)\Box_{2n-1}}\left[\varepsilon T_3\left(\Box_2^2 W+\nabla^2F^{(1)}\right)\right.\\&\left.-\left(D_{2n-1}^2-T_1^2\right)\vartheta^g\right]\frac{\beta_M}{\mu_M}=-\frac{\beta_M}{\mu_M}\vartheta^d,\end{aligned} \tag{25a}$$

$$\begin{aligned}&\sum_{n=1}^{2}(-1)^{n-1}\frac{\cos\left(\frac{h}{2}\Box_{2n-1}\right)}{\left(D_1^2-D_3^2\right)}T_3\left[\varepsilon\left(E+\nabla^2F^{(2)}\right)\right.\\&\left.+\frac{\left(D_{2n-1}^2-T_1^2\right)}{D_{2n-1}^2}\frac{\kappa\beta_M}{\mu_M}\vartheta^m\right]=\frac{\kappa\beta_M}{\mu_M}\vartheta^a.\end{aligned} \tag{25b}$$

根据整函数乘积级数展开公式, 式(21b)和(22b)的算子乘积级数表达式为

$$\cos\left(\frac{h}{2}\Box_2\right)f^{(1)}=\prod_{n=1}^{\infty}\left[1-\frac{h^2\Box_2^2}{(2n-1)^2\pi^2}\right]f^{(1)}, \tag{26a}$$

$$\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}f^{(2)}=\prod_{n=1}^{\infty}\left[1-\frac{h^2\Box_2^2}{4n^2\pi^2}\right]f^{(2)}. \tag{26b}$$

将式(26)的无穷乘积级数做截断, 可得到两个如下形式的 2 阶波动方程

$$\nabla^2 f^{(1)}-\left(\frac{\pi^2}{h^2}+T_2^2\right)f^{(1)}=0, \tag{27a}$$

$$\nabla^2 f^{(2)}-T_2^2 f^{(2)}=0. \tag{27b}$$

式(27a)和(27b)分别表示对应于平板热弹性弯曲和拉压振动的剪切场. 可以看到, 由于热只对膨胀变形起作用, 因此, 方程(27)与没有温度场作用时的平板振动剪切场的控制方程形式是相同的.

将式(21a)和(23a)以及(25a)联立, 可得平板弯曲振动位移势函数 $W,F^{(1)}$ 和温度梯度函数 ϑ^g 的方程

$$\begin{bmatrix}L_{11}&L_{12}&L_{13}\\L_{21}&L_{22}&L_{23}\\L_{31}&L_{32}&L_{33}\end{bmatrix}\begin{bmatrix}W\\F^{(1)}\\\vartheta^g\end{bmatrix}=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}, \tag{28}$$

其中各微分算符的表达式为

$$L_{11}=\frac{1-2\kappa}{16\kappa}h^2\left(\nabla^2-T_2^2\right)+\frac{1}{2\kappa},$$
$$L_{12}=\frac{1}{16\kappa}h^2\left[\left(3-2\kappa\right)\nabla^2-T_2^2\right]-\frac{1}{2\kappa},$$
$$L_{13}=\frac{1}{8}h^2\frac{\beta_M}{\mu_M},$$
$$L_{21}=-\left\{1+\frac{1}{24}h^2\left[\left(1-2\kappa\right)\nabla^2+\kappa T_2^2\right]\right\}\Box_2^2,$$
$$L_{22}=\left\{1-\frac{1}{24}h^2\left[\left(3-2\kappa\right)\nabla^2-\left(2-\kappa\right)T_2^2\right]\right\}\nabla^2,$$
$$L_{23}=\frac{1}{24}\frac{\kappa\beta_M}{\mu_M}h^2\left(T_2^2-2\nabla^2\right),$$
$$L_{31}=-\frac{\mu_M}{48\beta_M}\varepsilon T_3h^3\Box_2^2,$$
$$L_{32}=-\frac{\mu_M}{48\beta_M}\varepsilon T_3h^3\nabla^2,$$
$$L_{33}=\frac{1}{2}h\left\{1-\frac{1}{24}h^2\left[\nabla^2-\left(1+\varepsilon\right)T_3\right]\right\},$$
$$P_1=0,\quad P_2=\frac{1}{\mu_M h}q,\quad P_3=\vartheta^d.$$

根据式(28)算子矩阵的行列式, 可得用广义位移函数 W 表示的受热平板弯曲振动控制方程

$$\begin{aligned}&\nabla^6W-\left[\frac{24}{h^2}+\frac{15-8\kappa}{8\left(1-\kappa\right)}T_2^2+\frac{\left(1+\varepsilon\right)-\kappa}{1-\kappa}T_3\right]\nabla^4W\\&+\frac{3\left(13-8\kappa\right)}{\left(1-\kappa\right)h^2}T_2^2\nabla^2W-\frac{72}{\left(1-\kappa\right)h^4}T_2^2W\\&=-\frac{3}{8\left(1-\kappa\right)\mu_M h}\left\{\frac{192}{h^4}+\left(3-2\kappa\right)\nabla^4\right.\\&\quad-\left\{\frac{16\left(5-3\kappa\right)}{h^2}+T_2^2+\left[3\left(1+\varepsilon\right)-2\kappa\right]T_3\right\}\nabla^2\\&\quad\left.+\frac{8}{h^2}\left[3T_2^2+\left(1+\varepsilon\right)T_3\right]\right\}q+\frac{3}{4\left(1-\kappa\right)h^3}\\&\quad\times\left[\left(32-T_2^2h^2\right)\nabla^2+8T_2^2\right]\frac{\kappa\beta_M}{\mu_M}\vartheta^d.\end{aligned}\tag{29}$$

将式(22a)和(23b)以及(25b)联立, 可得平板拉压振动时位移势函数 $E,F^{(2)}$ 和温度函数 ϑ^m 的控制方程

$$\begin{bmatrix}L_{11}&L_{12}&L_{13}\\L_{21}&L_{22}&L_{23}\\L_{31}&L_{32}&L_{33}\end{bmatrix}\begin{bmatrix}E\\F^{(2)}\\\vartheta^m\end{bmatrix}=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix},\tag{30}$$

其中各微分算符的表达式为

$$\begin{aligned}L_{11}=&\left(1-2\kappa\right)-\frac{1}{24}h^2\left[\left(3-4\kappa\right)\nabla^2\right.\\&\left.-\left(1-2\kappa^2\right)T_2^2+2\kappa\varepsilon T_3\right],\end{aligned}$$
$$\begin{aligned}L_{12}=&\left\{\left(1-2\kappa\right)-\frac{1}{24}h^2\left[\left(3-4\kappa\right)\nabla^2\right.\right.\\&\left.\left.-\left(1-2\kappa^2\right)T_2^2+2\kappa\varepsilon T_3\right]\right\}\nabla^2\\&+\kappa\left(2\nabla^2-T_2^2\right)\left[1-\frac{1}{24}h^2\left(\nabla^2-T_2^2\right)\right],\end{aligned}$$
$$L_{13}=-\left(1-\frac{h^2}{8}\nabla^2+\frac{h^2}{24}T_2^2-\frac{h^2}{12}\kappa T_3\right)\frac{\kappa\beta_M}{\mu_M},$$
$$L_{21}=1+\frac{1}{8}h^2\left[\left(1-2\kappa\right)\nabla^2+T_1^2\right],$$
$$L_{22}=\left[\left(1-2\kappa\right)+\frac{1}{8}h^2\left(\nabla^2-T_1^2\right)\right]\nabla^2,$$
$$L_{23}=-\frac{\kappa\beta_M}{\mu_M}\left(1+\frac{1}{8}h^2\nabla^2\right).$$
$$L_{31}=\frac{1}{8}h^2\varepsilon T_3,$$
$$L_{32}=\frac{1}{8}h^2\varepsilon T_3\nabla^2,$$
$$L_{33}=\frac{\kappa\beta_M}{\mu_M}\left[1-\frac{1}{8}h^2\left(\nabla^2-T_3\right)\right],$$
$$P_1=0,\quad P_2=\frac{\kappa}{2\mu_M}q,\quad P_3=\frac{\kappa\beta_M}{\mu_M}\vartheta^a.$$

根据式(30)算子矩阵的行列式, 可得用广义位移函数 E 表示的受热平板拉伸振动控制方程

$$\begin{aligned}&\nabla^6E-\frac{1}{3}\left[\frac{40}{h^2}+\frac{13-8\kappa^2}{4\left(1-\kappa\right)}T_2^2+2\left(1+\varepsilon\right)T_3\right]\nabla^4E\\&+\left\{\frac{64}{h^4}+\frac{2\left(11-4\kappa^2\right)}{3\left(1-\kappa\right)h^2}T_2^2+\frac{8}{h^2}\left[1+\frac{\left(3-\kappa\right)\varepsilon}{3\left(1-\kappa\right)}\right]T_3\right\}\\&\times\nabla^2E-\frac{16}{\left(1-\kappa\right)h^4}T_2^2E\\&=-\frac{1}{3\left(1-\kappa\right)\mu_M h^2}\left\{2\left(3-\kappa\right)\nabla^4-\left\{\frac{24}{h^2}\left(1+6\kappa\right.\right.\right.\\&\quad\left.\left.\left.-2\kappa^2\right)T_2^2+\left[3\left(1+\varepsilon\right)-2\varepsilon\kappa\right]T_3\right\}\nabla^2+\frac{24}{h^2}T_1^2\right\}q\\&\quad-\frac{2}{3\left(1-\kappa\right)h^2}\left\{4\nabla^4-\left[\frac{48}{h^2}+\left(5-2\kappa\right)T_2^2\right.\right.\end{aligned}$$

$$-2\left(1+\varepsilon-2\kappa\right)T_3\Big]\nabla^2+\frac{24}{h^2}T_2^2\Big\}\frac{\kappa\beta_M}{\mu_M}\vartheta^a. \quad (31)$$

2 热弹性平板振动的频散关系分析

取平板的厚度为结构特征长度, 于是有如下无量纲量: 热弹性波数αh、热弹性波消减系数βh、热-力耦合系数ε和相对热-力耦合长度 h/L. 其中, 热膨胀系数α_T, 平均环境温度 T_0, Poisson 比ν, 热-力耦合参数 $\frac{\mu_M}{\rho c_V}$ 也是无量纲参数. 在算例中选取 C_f/SiC 陶瓷材料的平均参数进行了计算, 参见表 1.

研究问题的定态解, 设 $W=\widetilde{W}e^{i(kx-\omega t)}$, 由式(29)可得到受热平板弯曲振动的频散关系为[19]

$$k^6h^6+\left[24-\frac{15-8\kappa}{8(1-\kappa)}k_2^2h^2-i\left(1+\frac{\varepsilon}{1-\kappa}\right)k_3h^2\right]k^4h^4$$
$$-\frac{3(13-8\kappa)}{1-\kappa}k_2^2k^2h^4-\frac{72}{1-\kappa}k_2^2h^2=0, \quad (32)$$

其中, k 是弹性波复波数, $k=\alpha+i\beta$; $k_2^2h^2=(\alpha h)^2(c/c_2)^2$; $k_3h^2=\alpha h(c/c_2)(h/L)$; $L=\frac{K}{\sqrt{\rho\mu_M}C_V}$ 称为热-力耦合长度.

根据式(32), 可以得到平板弯曲波动的频散关系曲线, 如图 1 和 2 所示. 图 1 中(a)和(b)分别表示在相同参数下平板结构中频散关系和消减系数变化曲线. 以下采用同样方式. 由图 1 和 2 可以看出: 在不同参数下平板中频散关系和消减系数变化曲线的拓扑结构形式变化不大; 图 1(a)中第 2 阶振型和第 3 阶振型与无力热耦合时平板结构弯曲振动的第 1 阶和第 2 阶振型接近; 图 1(a)中第 1 阶振型与无力热耦合时平板中温度变化模式对应, 但是, 考虑力热耦合后, 模式变化很大.

根据图 1(b)可以看到: 平板弯曲振动振幅消减系数的第 1 阶和第 3 阶模式是小于零的, 表示考虑力热耦合影响后振幅是增长的, 但是需要指出的是, 负的消减系数的绝对值很小, 同时只有当平板结构弹性波入射波数很大时, 也就是频率很高时, 消减系数的绝对值才能变得很大, 导致平板弯曲振动产生动态失稳. 因此, 热-力双向耦合作用可能对弯曲振动影响小些.

设 $E=\tilde{E}e^{i(kx-\omega t)}$, 由式(31)可得到受热平板拉伸振动对应的频散关系为[19]

$$k^6h^6+\frac{1}{3}\left[40-\frac{13-8\kappa^2}{4(1-\kappa)}k_2^2h^2-2i(1+\varepsilon)k_3h^2\right]k^4h^4$$
$$+\left\{64-\frac{2(11-4\kappa^2)}{3(1-\kappa)}k_2^2h^2-8i\left[1+\frac{(3-\kappa)\varepsilon}{3(1-\kappa)}\right]k_3h^2\right\}$$
$$\times k^2h^2-\frac{16}{1-\kappa}k_2^2h^2=0. \quad (33)$$

同理, 根据式(33), 可以得到平板拉压波动的频散关系曲线, 如图 3 和 4 所示. 由图 3 和 4 可以看出: 在不同参数下平板中频散关系和消减系数变化曲线的拓扑结构变化不大; 图 2(a)中第 2 阶振型和第 3 阶振型与无力热耦合时平板拉伸振动的第 1 阶和第 2 阶振型接近; 图 2(a)中第 1 阶振型与与无力热耦合时平板中温度变化模式相对应, 但是, 考虑力热耦合后, 耦合模式变化很大.

根据图 2(b)可以看出: 平板拉伸振动振幅消减系数的第 3 阶模式是小于零的, 表示考虑力热耦合影响后振幅是增长的; 消减系数的绝对值还是较大的, 当板结构振动频率比较高时, 消减系数的绝对值可能变得较大, 导致平板拉伸振动产生动态失稳. 因此, 热力双向耦合作用在拉伸振动分析计算中影响作用大些.

表 1 C_f/SiC 材料的热物理和力学参数[20]

Table 1 Thermophysical and mechanical and properties of C_f/SiC materials [20]

材料参数	工作温度 T_0 (K)	
	773	1773
弹性模量(N/m²)	8.79×10^{10}	8.53×10^{10}
Poisson 比	0.32	0.28
材料密度(kg/m³)	1.74×10^3	1.70×10^3
热膨胀系数(1/L)	2.10×10^{-6}	1.90×10^{-6}
热导率(W/(m·K))	2.26	1.71
比热容(J/(kg·K))	800	578

3 平板振动方程和温度方程的讨论

由热弹性平板弯曲振动频散关系曲线可以看出: 板中存在 3 个模式的振动, 第 1 阶模式与平板无力热耦合时非稳态导热模态相对应, 第 2 阶和第 3 阶模态与无力热耦合的平板弯曲振动模式对应; 考虑力热双

向耦合后，第 1 阶耦合振动模式变化很大，第 2 阶和第 3 阶耦合振动模态与平板弯曲振动模式相比区别不大；在不同的平均环境温度和物理参数下，其各阶耦合振动模式拓扑结构变化不大；热-力双向耦合作用对平板拉伸振动的影响要比弯曲振动的影响要大些.

本论文的主要创新之处在于：(1) 首次给出了热-

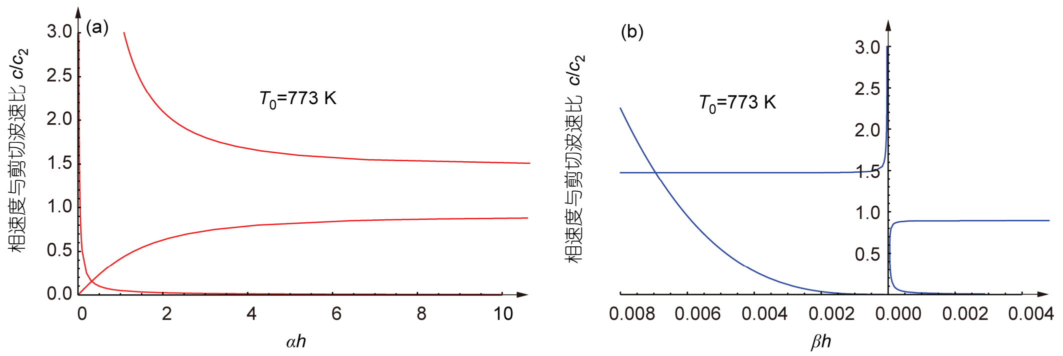

图 1　(网络版彩图) (a) 平板弯曲波动时频散关系；(b) 平板弯曲波动中消减系数变化

Figure 1　(Color online) (a) Dispersion relation of plate bending; (b) attenuation coefficient of plate bending.

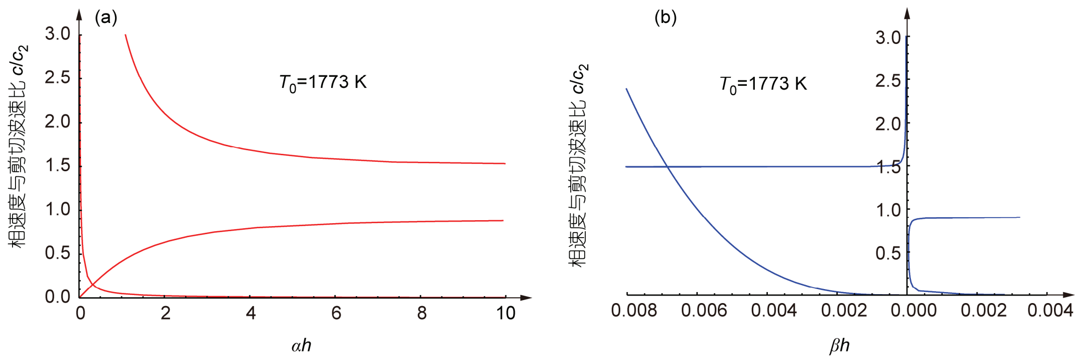

图 2　(网络版彩图)(a)平板弯曲波动时频散关系；(b) 平板弯曲波动中消减系数变化

Figure 2　(Color online) (a) Dispersion relation of plate bending; (b) attenuation coefficient of plate bending.

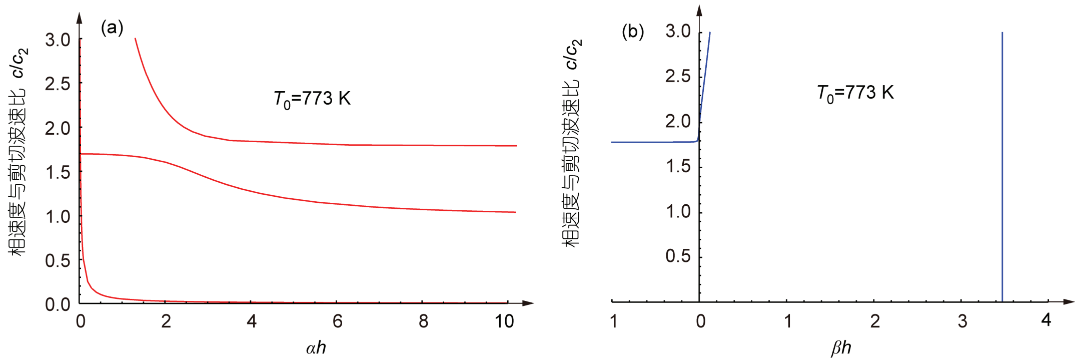

图 3　(网络版彩图)(a)平板拉压波动时频散关系；(b) 平板拉压波动中消减系数变化

Figure 3　(Color online) (a) Dispersion relation of plate stretching; (b) attenuation coefficient of plate stretching.

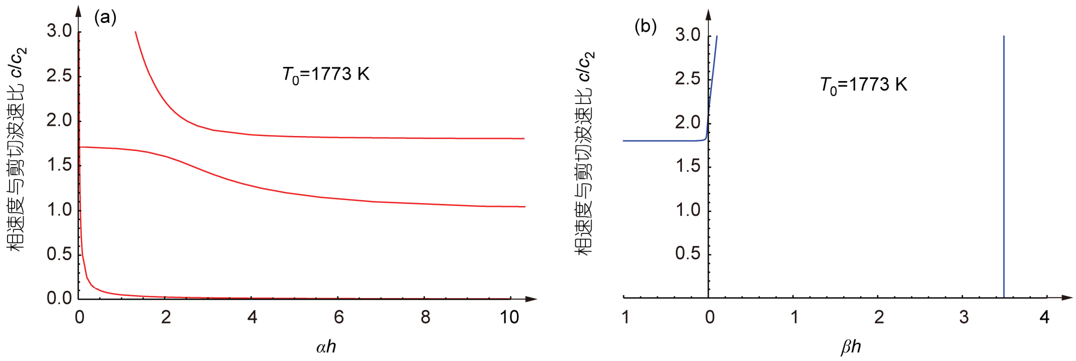

图 4　(网络版彩图)(a)平板拉压波动时频散关系; (b) 平板拉压波动中消减系数变化

Figure 4　(Color online) (a) Dispersion relation of plate stretching; (b) attenuation coefficient of plate stretching.

力双向耦合下平板结构振动的精确化动力学方程; (2) 给出了力热全耦合作用下平板内波模频散关系与分析; (3) 基于本文在时域内给出的平板力热耦合振动方程, 研究了平板结构振动的动态稳定性.

本文没有经典力学假设, 基于力热双向耦合作用的观点, 直接基于弹性动力学以及采用算子谱分解方法构建了受热平板结构振动支配方程. 因此, 本文在时域内给出的平板振动方程可用于研究高温下平板结构力热双向耦合动力学. 分析力热耦合机理、耦合模式以及结构振动的动态稳定性. 本文在时域内给出的平板力热耦合振动方程及其分析方法可望能在近空间飞行器热防护工程中得到应用.

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Non-classical dynamical equations of thick plates with complete thermomechanical coupling

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The dynamical governing equation of structures is the basis for solution of dynamics and control of distributed parameter systems. Until now, the refined dynamic equation of plates including the complete thermomechanical coupling has not been seen in the literature on structural dynamics. With the development of the modern aerospace technology, the refined dynamic equation of plates including the thermomechanical coupling effect is an urgent problem to be solved. But the traditional modeling method described in classical theory of plates and shells cannot be used to obtain the dynamic equation of thick plates under heating conditions because of the limitation of classical theory. In this paper, the coupled dynamical problem of plates with complete thermomechanical coupling is investigated based on the three-dimensional thermoelasticity and modern mathematics. The spectral composition method for modeling the structural dynamics is developed by combining the Vieta theorem of algebra with the classical method of operator spectra. In the time domain the refined dynamical equations of plates, which involve the bending and stretching vibrations under coupled thermomechanical conditions, are constructed by using the spectral decomposition of operators and proper gauge conditions to eliminate the non-uniqueness of unknown potential function without using the classical assumption in the theory of plates and shells. The dispersion relations from the refined equations of thermoelastic plates are presented graphically to test and verify the refined theory of thermoelastic plates. The space and time evolution of wave motion in heated plates and its dynamical stability are analyzed and discussed. We can see that the refined governing equations of thermoelastic plates are accurate, which would be used to solve the dynamics and control of thick plates with complete thermomechanical coupling, for instance, to investigate the coupling mechanism, coupled modes and dynamical response. And the modeling for the coupled dynamics of plates can be used in aerospace engineering for thermal protection system design.

refined dynamic theory of thermoelastic plates, complete thermomechanical coupling, spectral decomposition of operators and complex function method, low-dimensional structure, bending and stretching vibration of plates

PACS: 46.40.-f, 46.70.De, 44.10.+I, 02.30.Em, 02.30.Tb, 46.15.Ff, 11.15.-q, 46.25.Hf

doi: 10.1360/SSPMA2015-00341