University of California
Santa Barbara

Topics in theoretical condensed matter physics

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Physics

by

Chaitanya R. Murthy

Committee in charge:

Professor Chetan Nayak, Co-Chair
Professor Mark Srednicki, Co-Chair
Professor Andrea Young

June 2020

The Dissertation of Chaitanya R. Murthy is approved.

Professor Andrea Young

Professor Mark Srednicki, Committee Co-Chair

Professor Chetan Nayak, Committee Co-Chair

June 2020

Topics in theoretical condensed matter physics

by

Chaitanya R. Murthy

To my parents.

Acknowledgements

I would like to express my deep gratitude to my advisors, Chetan Nayak and Mark Srednicki, for their constant support, encouragement and wisdom. In addition to being amazing physicists, they are dedicated mentors and remarkably kind individuals. I feel lucky to have worked with them. I would also like to thank Andrea Young for being on my thesis committee (and for allowing me to wander into his lab from time to time and bother his students).

It has been a pleasure to work and interact with the wonderful group of people at Station Q. I have learned a great deal from personal discussions with everyone in the group. I am grateful to Michael Freedman for giving me so much of his time, for sharing his profound insights into mathematics, and for many inspiring conversations. Sean Fraer deserves a special mention for his kindness and for being the magician who keeps the whole show running.

I would like to thank the condensed matter theory group at UCSB—especially Matthew Fisher, Leon Balents, Andreas Ludwig, Cenke Xu (and at Q, Roman Lutchyn, Bela Bauer, and Parsa Bonderson)—for taking an active interest in my development and for teaching me so much. I am very grateful to Leon for letting me participate in his group meetings. I am also grateful to all of the condensed matter postdocs and graduate students with whom I overlapped, for our countless enjoyable and edifying discussions over the years.

I have been incredibly fortunate to collaborate and publish papers with many exceptional scientists during my PhD: Bela Bauer, Leonid Glazman, Bernard van Heck, Anna Keselman, Pavel Kurilovich, Vladislav Kurilovich, Chetan Nayak, Dmitry Pikulin, Mark Srednicki, Dominique Laroche, Daniël Bouman, David van Woerkom, Alex Proutski, Ruben van Gulik, Jesper Nygård, Peter Krogstrup, Leo Kouwenhoven, and Attila Geresdi. I am indebted to them for everything that they have taught me. I also thank

my many other physics colleagues, too numerous to list here. I gratefully acknowledge funding from the National Science Foundation and Microsoft.

I thank all of my friends and family; I am truly lucky to have you. Most of all, I thank my parents, for their unwavering and unconditional love and support.

Curriculum Vitæ

Chaitanya R. Murthy

Education

2020	Ph.D. in Physics (Expected), University of California, Santa Barbara.
2017	M.A. in Physics, University of California, Santa Barbara.
2014	M.S. in Nanoengineering, University of California, San Diego.
2011	B.S. in Chemical Engineering (Nanotechnology), University of Southern California.

Publications

1 **C. Murthy**, V. D. Kurilovich, P. D. Kurilovich, B. van Heck, L. I. Glazman, and C. Nayak, *Energy spectrum and current-phase relation of a nanowire Josephson junction close to the topological transition*, Physical Review B **101**, 224501 (2020).

2 **C. Murthy** and C. Nayak, *Almost perfect metals in one dimension*, Physical Review Letters **124**, 136801 (2020).

3 **C. Murthy** and M. Srednicki, *Bounds on chaos from the eigenstate thermalization hypothesis*, Physical Review Letters **123**, 230606 (2019).

4 **C. Murthy** and M. Srednicki, *Structure of chaotic eigenstates and their entanglement entropy*, Physical Review E **100**, 022131 (2019).

5 **C. Murthy** and M. Srednicki, *Relaxation to gaussian and generalized Gibbs states in systems of particles with quadratic hamiltonians*, Physical Review E **100**, 012146 (2019).

6 A. Keselman, **C. Murthy**, B. van Heck, and B. Bauer, *Spectral response of Josephson junctions with low-energy quasiparticles*, SciPost Physics **7**.4, 050 (2019).

7 D. Laroche, D. Bouman, D. J. van Woerkom, A. Proutski, **C. Murthy**, D. I. Pikulin, C. Nayak, R. J. van Gulik, J. Nygård, P. Krogstrup, and L. P. Kouwenhoven, *Observation of the 4π-periodic Josephson effect in indium arsenide nanowires*, Nature Communications **10**.1, 245 (2019).

8 **C. Murthy**, B. Gao, A. R. Tao, and G. Arya, *Dynamics of nanoparticle assembly from disjointed images of nanoparticle-polymer composites*, Physical Review E **93**, 022501 (2016).

9 **C. Murthy**, B. Gao, A. R. Tao, and G. Arya, *Automated quantitative image analysis of nanoparticle assembly*, Nanoscale **7**.21, 9793-9805 (2015).

10 **C. Murthy** and A. M. Armani, *Mass transport effects in suspended waveguide biosensors integrated in microfluidic channels*, Sensors **12**.11, 14327-14343 (2012).

Abstract

Topics in theoretical condensed matter physics

by

Chaitanya R. Murthy

We investigate various topics in theoretical quantum condensed matter physics. First, we show that a one-dimensional quantum wire with as few as 2 channels of interacting fermions can host metallic Luttinger liquid states of matter that are stable against all perturbations up to q^{th}-order in fermion creation/annihilation operators for any fixed finite q. The leading relevant perturbations are thus complicated operators that are expected to modify the physics only at very low energies, below accessible temperatures. The stability of these Luttinger liquid fixed points is due to strong interactions between the channels, which can (but need not) be chosen to be purely repulsive. Our results might enable elementary physical realizations of these phases, and may also serve as a useful paradigm for thinking about higher-dimensional non-Fermi liquids.

Separately, we present an elementary but general description of relaxation to gaussian and equilibrium generalized Gibbs states in lattice models of fermions or bosons with quadratic hamiltonians. Our analysis applies to arbitrary initial states that satisfy a mild condition on clustering of correlations. We obtain quantitive, model-independent predictions for how quickly local quantities relax in such systems. These predictions can be tested in near-term quantum gas experiments.

Finally, we study chaotic many-body quantum systems that obey the eigenstate thermalization hypothesis (ETH). We show that a known bound on the growth rate of the out-of-time-order four-point correlator in such systems follows directly from the general ETH structure of operator matrix elements. This ties together two key paradigms of

thermal behavior in isolated many-body quantum systems. We also consider a bipartition of the system, and study the entanglement properties of an energy eigenstate with nonzero energy density. When the two subsystems have nearly equal volumes, we find a universal correction to the entanglement entropy that is proportional to the square root of the system's heat capacity (or a sum of capacities, if there are conserved quantities in addition to energy).

Contents

Chapter 1

Introduction and outline

The branches of physics are differentiated by the nature and scale of the phenomena that they seek to explain. Condensed matter physics concerns itself with phenomena whose basic length scales range from nanometers to centimeters. These encompass the vast majority of material phenomena—that is, phenomena involving inanimate objects—that we humans encounter in our everyday experience [1]. The fundamental equations governing all ordinary matter are known; these are the equations of electrodynamics and quantum theory (for most purposes one can also ignore relativity and incorporate only its leading corrections, such as spin-orbit coupling). But we cannot solve the equations in general. To quote P. W. Anderson, "the ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe," or—more succinctly and famously—"more is different" [1].

Since the behavior of large and complex "many body" systems is not easily understood by extrapolation of the properties of systems involving a few particles, organizing

[1] I adopt a suitably broad definition of "condensed matter physics" that includes fluid dynamics, materials science, and some parts of chemistry. Research in condensed matter often admits a further coarse classification into either "soft" or "hard" varieties, depending on whether the researchers' preferred value for Planck's constant is $\hbar = 0$ or $\hbar = 1$, respectively. In this dissertation, $\hbar = 1$.

principles are needed [2]. Some of the most important of these, for the physics of many-body systems near equilibrium, are: 1a) symmetry/symmetry breaking, 1b) topological order, 2) the notion of elementary excitations, and 3) the renormalization group and universality. To place the results of this dissertation in a suitable broader context, I now briefly discuss these basic principles in turn.

Symmetry plays a central role in all branches of physics. In condensed matter, it forms the basis for the conventional Landau paradigm for describing phases of matter [2]. In this paradigm, phases are distinguished by a local order parameter that characterizes the various symmetries that the phase breaks. The Landau paradigm is hugely successful, but incomplete: there are phases of matter that it fails to describe. Its completion involves the more modern notion of topological order [3], which may be caricatured mathematically as "homotopy theory in the space of gapped Hamiltonians" or physically as "the study of patterns of entanglement in the ground state" (more on entanglement later). Together, the principles of symmetry breaking and topological order appear to give a satisfactory theoretical classification of most known phases of matter.

The many body problem in full generality is intractable because it involves a large number of particles that interact strongly with one another. In many cases, however, it is possible to describe the low-energy physics of a system of many strongly interacting particles in terms of some other effective degrees of freedom that are *weakly* interacting; the latter are called the elementary excitations of the system [4]. Such a description, when it exists, is hugely powerful. The prototypical example is Landau's Fermi liquid theory, which describes a high-density gas of fermions with strong repulsive interactions in terms of weakly interacting fermionic quasiparticles (defined only near the Fermi surface) and long-wavelength bosonic collective modes [5, 6]. Fermi liquid theory explains the success of the naive Drude-Sommerfeld "free electron" theory of conduction in metals:

the eponymous "free electrons" are really quasielectrons, whose residual interactions are weak enough to be treated in the relaxation time approximation [7]. In addition, many other low-temperature phases can be quantitatively understood as instabilities of the Fermi liquid, most famously in the BCS theory of superconductivity and its various extensions [8]. The modern conception of Fermi liquid theory [9] places it within the general framework of the renormalization group, to which we now turn (historically, Landau arrived at it via different arguments [10]).

The renormalization group (RG) describes changes in the theory of a physical system when the system is viewed at different scales [11]. We start with the observation that a well-defined physical theory describes physics only up to some high-energy "cutoff" scale [2]. Then, one can in principle "integrate out" high-energy modes to get a theory that describes the same system but at lower energy scales. The key insight of the RG is that it is generally better to do this "integrating out" in many small steps, rather than all at once [3]. In the limit of infinitesimal steps, we obtain a continuous flow ("RG flow") in the abstract space of theories. The question of how the physical theory changes as we change energy scale is thereby mapped to a dynamical systems problem in theory space, and can be analyzed as such [12]. In particular, one concentrates on the RG flow in the vicinity of its fixed points; these represent theories that are scale-invariant. Near a fixed point, the RG flow can be linearized [4]. Its eigendirections define a preferred set of *scaling operators* at the fixed point, and its eigenvalues define the *scaling dimensions* of the associated coupling constants. In some directions the RG flow will be directed towards the fixed point, in other directions it may be directed away from it, and in

[2]For the sake of exposition, I consider energy scales, but one could consider length scales instead (the resulting RG transformation may differ), or something else entirely. The basic principle is the same.

[3]In other words, the insight is that "physics is local in scale".

[4]In some cases, one is forced to retain higher-order terms to properly describe the topology of the flow in the vicinity of the fixed point (e.g. the Kosterlitz-Thouless transition). The analysis of such exceptional cases can be systematized using normal form theory [13].

yet others it may stagnate. These directions correspond to irrelevant, relevant, and marginal perturbations of the fixed point theory, respectively. Any fixed point which has few relevant directions and a large basin of attraction [5] is of special importance: the associated *effective theory* captures the *universal* low-energy physics of many distinct physical systems (all those whose microscopic theories lie in the basin of attraction). If an effective theory turns out to be a weakly coupled field theory, then the quanta of its fields may be identified with the elementary excitations discussed earlier [14].

In the RG framework, Fermi liquid theory describes marginal deformations of the Fermi gas fixed point in spatial dimensions $d > 1$. Its great success is indicative of a large basin of attraction. Metallic systems which fall outside the basin—whose properties cannot be reconciled with Fermi liquid theory—are called non-Fermi liquids. The canonical example of a non-Fermi liquid is the interacting electron gas in $d = 1$ dimension [15]. The special kinematics of $d = 1$ preclude well-defined fermionic quasiparticles, but allow for well-defined bosonic collective modes. The universal physics is then controlled by a different manifold of fixed points, collectively called Luttinger liquid theory [16], which is a solvable theory of these bosonic modes.

In Chapter 2, we consider the general question of stability—in the linearized RG sense discussed above—of multi-component Luttinger liquids (LLs). The one-component LL is quite fragile: it can always be destabilized by disorder-induced localization or proximity-induced superconductivity (or both) [15]. We show that this is the exception rather than the rule. For any number of components $N \geq 2$, we find families of LLs that are robust: the perturbations that might destabilize them can be made arbitrarily complicated, and hence are expected not to affect the physics at energy scales of practical interest (one cannot create an infinitely large system and cool it to absolute zero). Our results build (in particular) on earlier work [17] in which it was shown that, in principle, 23-component

[5] More precisely, basin of attraction of a small ball in theory space around the fixed point.

LLs exist that are completely stable (i.e. admit no relevant gap-opening perturbations). The new results described in Chapter 2 of this thesis provide experimentally realizable approximations to this "perfect metal" that may provide a valuable platform to study non-Fermi liquid physics in experiments. The analysis of Chapter 2 is based on the (standard) technique of Abelian bosonization; for completeness, I have included a short review of this technique in the technical appendices following the chapter.

In Chapters 3–5, we move away from relative safe haven of low-energy equilibrium physics and venture into the territory of far-from-equilibrium physics, where significantly less is known. Here, far from the ground state of the system, the powerful organizing principles described earlier can no longer be relied upon. The principles that replace them are still being discovered, and many basic questions remain unanswered. Nevertheless, one can can still hope to find results that are *universal* in the sense of broad applicability and insensitivity to microscopic details. The results described in Chapters 3–5 all have this flavor of universality.

One of the most venerable puzzles in statistical physics, which has occupied physicists and philosophers since the time of Boltzmann, concerns how to properly reconcile microscopic determinism with the obvious tendency of most macroscopic systems to irreversibly approach thermal equilibrium. In my opinion, a satisfactory answer to this question appears only when one realizes that thermalization is a property not of a system alone but of the preferred class of *physical observables* relative to that system [6]. In the case of isolated quantum many-body systems, this leads to the Eigenstate Thermalization Hypothesis (ETH) [18, 19]. The ETH is an ansatz, motivated by quantum chaos theory, that describes the coarse structure of matrix elements, in the energy eigenstate basis, of observables in a chaotic quantum system [20]. It has not been proven rigorously,

[6] This is most obvious in quantum mechanics: projectors onto eigenstates of the Hamiltonian do not evolve with time, so if they were physically observable, the system would never thermalize. It is equally true in classical mechanics.

but it is supported by a variety of analytical arguments and by extensive numerical studies on small systems [21, 22]. The ETH is sufficient to ensure thermalization (in the sense stated above) from generic physical initial conditions [20].

Having understood (at some level) the eventual fate of a generic thermalizing system, we would next like to understand the *dynamics* of thermalization. This problem is difficult, and remains very much unsolved. Dynamics is easier to study in *integrable systems*, but the same integrability that permits their analysis also ensures that they retain too much memory of the initial state to ever truly thermalize [23]. Instead, integrable systems are expected to relax to more complicated equilibrium states, called generalized Gibbs ensembles (GGEs) [24]. In Chapter 3, we provide a complete analysis and description of this relaxation process in the simplest class of integrable models: those that are quadratic in a set of fermion or boson operators. Our analysis is general—we treat arbitrary initial states that satisfy a weak condition on clustering of correlations—and leads to a simple and appealing physical picture of the relaxation process. We also obtain *quantitive, model-independent* predictions for how quickly local quantities relax in such systems. Finally, we extend our analysis to cover periodically driven (Floquet) systems.

In Chapter 4, we return to chaotic quantum systems. A defining property of classical chaos [25] is sensitive dependence on initial conditions, colloquially termed "the butterfly effect". In quantum systems, similar sensitivity can be diagnosed using out-of-time-order four-point correlators (OTOCs), which probe how operator insertions inhibit cancellation between forward and backward time evolution [26]. In some systems with a suitable hierarchy of time scales, the OTOC exhibits exponential behavior $\propto e^{\lambda t}$ at early times, and the exponent λ may be regarded as a quantum analogue of the Lyapunov exponent [27]. Maldecena *et al.* [28] conjectured a sharp bound on how rapidly chaos can develop in such quantum systems, $\lambda \leq 2\pi/\beta$, and were able to prove this "bound

on chaos" under certain assumptions. The bound is saturated in the SYK model, and in large-N conformal field theories with gravitational duals, where it is related to the physics of information scrambling in black holes [29]. In Chapter 4, we show that the bound $\lambda \leq 2\pi/\beta$ can be derived from ETH in systems that obey it (and also have the required hierarchy of time scales). Our proof relies on a quite different set of physical and mathematical assumptions than that of Maldecena *et al.*, and it ties together two important paradigms of thermal behavior in chaotic quantum systems.

In contrast to chaos, which manifests both classically and quantumly, *entanglement* is uniquely quantum mehcanical [30]. In recent decades, entanglement has played an increasingly important role in our understanding of phases of matter and the general low-energy properties of quantum many-body systems [31]. In Chapter 5, we analytically study the entanglement structure of *highly excited* eigenstates (at finite energy density) of lattice models that obey ETH. In the past, Deutsch [32] and others have convincingly argued that the leading term in the bipartite entanglement entropy of a typical eigenstate of a chaotic quantum system equals the thermodynamic entropy of the smaller subsystem at the same energy density [7]. In Chapter 5, we show that there is a universal correction to this result that appears when the two subsystems have nearly equal size, and which is proportional to $\sqrt{C}$, the square root of the heat capacity of the system. In odd spatial dimensions, this correction therefore scales with a half-integer power of the linear dimensions of the system (whereas naively, one would expect all corrections to the entanglement entropy to scale with *integer* powers of the system size). We also calculate corrections to the Rényi entropies, obtaining new results for Rényi index $n < 1$. In addition, we generalize all of the above results to the case of generic systems that have a finite number of conserved quantities in addition to the energy.

[7] In a system with bounded Hilbert space, the infinite-temperature limit of this result agrees with the well known fact that randomly chosen states in a bipartite Hilbert space are typically maximally entangled between the subsystems, up to an order-one correction [33].

In closing, Chapter 6 contains some thoughts regarding promising directions for future research, based on the work presented in each of the earlier chapters.

1.1 Permissions and Attributions

The content of Chapter 2 is the result of a collaboration with Chetan Nayak, and has previously appeared in Physical Review Letters [34]. Excluding Appendix 2.A, it is © 2020 American Physical Society. The content of Chapters 3, 4, and 5 is the result of a collaboration with Mark Srednicki; Chapter 4 has previously appeared in Physical Review Letters [35], while Chapters 3 and 5 have previously appeared in Physical Review E [36, 37]. The content of these chapters is © 2019 American Physical Society. All content is reproduced here with the permission of the American Physical Society (`https://journals.aps.org/copyrightFAQ.html`).

Chapter 2

Almost perfect metals in one dimension

In this chapter, we show that a one-dimensional quantum wire with as few as 2 channels of interacting fermions can host metallic Luttinger liquid states of matter that are stable against all perturbations up to q^{th}-order in fermion creation/annihilation operators for any fixed finite q. The leading relevant perturbations are thus complicated operators that are expected to modify the physics only at very low energies, below accessible temperatures. The stability of these Luttinger liquid fixed points is due to strong interactions between the channels, which can (but need not) be chosen to be purely repulsive. Our results might enable elementary physical realizations of these phases.

2.1 Introduction

Metallic states of matter are gapless and often unstable to either insulating behavior or superconductivity. This is especially true in one-dimensional systems, where the localizing effects of disorder are particularly strong [38]. For a single channel (i.e. a single

propagating mode of each chirality at the Fermi energy), disorder-induced localization can only be avoided when the interaction is strongly attractive, while proximity-induced superconductivity can only be avoided when it is strongly repulsive [15]. The situation is more complicated—and much more interesting—when there are multiple channels. We will show that, surprisingly, even for $N = 2$ channels, it is possible to have a metallic state that is stable against all perturbations up to q^{th}-order in fermion creation/annihilation operators for any fixed finite q (but not $q = \infty$), which we call *(absolute) q-stability*.

Gapless phases of interacting fermions in one dimension are described at low energies by Luttinger liquid (LL) theory [16]. They exhibit a remarkable and universal phenomenology that distinguishes them from Fermi liquids, but this is often obscured in experiments due to dimensional crossover, ordering, or localization [15]. Thus, a physically realizable stable LL is not only interesting as a matter of principle, but also for the practical reason that it would provide a useful experimental platform to study non-Fermi liquid physics.

For $N = \infty$, it was shown two decades ago in Refs. [39, 40, 41, 42, 43, 44, 45, 46] that there exist "sliding Luttinger liquid" phases which are stable against many, but not all, low-order perturbations; it was argued that the relevant perturbations are likely to have small bare values [1]. More recently, it was discovered that it is possible for a one-dimensional metal to be stable against *all* non-chiral perturbations (without restriction on the order) [17]. An explicit construction was given for $N = 23$ which exploited the properties of integral quadratic lattices. In this chapter, we show that a slight relaxation of the condition of complete stability to the weaker condition of q-stability brings the required number of channels down from 23 to 2, thereby greatly increasing the chances of experimental realization.

[1] We restrict attention in this chapter to systems with short-ranged interactions. Long-ranged interactions can also stabilize a Luttinger liquid against a $2k_F$ potential and disorder [47]

The basic observation underlying the results of this chapter and of Ref. [17] is that the possible perturbations of an N-channel LL can be represented as lattice points in a fictitious $2N$-dimensional space equipped with two different metrics: the mixed-signature (N, N) metric $\mathrm{diag}(-\mathbb{I}_N, \mathbb{I}_N)$ and the Euclidean metric $\mathbb{I}_{2N}$, where $\mathbb{I}_N$ is the $N \times N$ identity matrix. The mixed-signature interval from the origin to a lattice point measures the chirality of the associated perturbation, while the Euclidean interval measures its scaling dimension; points sufficiently far from the origin are irrelevant in the renormalization group (RG) sense. The lattice is naturally graded into "shells" consisting of perturbations of a given order; low-order perturbations belong to the inner shells. The effect of interactions is to deform the lattice by an $SO(N, N)$ transformation [2]. For $N = 1$, the deformation is a Lorentz boost that is "aligned" with the lattice; such a boost unavoidably pulls one of the innermost lattice points closer to the origin, enhancing the susceptibility of the system to either localization or induced superconductivity. For $N \geq 2$, on the other hand, the boosts can be "misaligned" with the lattice planes in such a way that all lattice points in the innermost q shells are pushed away from the origin, making the corresponding perturbations irrelevant.

Remarkably, these absolutely q-stable phases can occur even for purely repulsive interactions. Two-channel repulsive LLs can occur in a number of different contexts. One simple example, with sufficient generality to permit the phases described here, is a single-spinful-channel quantum wire with strong spin-orbit coupling. In this case, the two Fermi points of each chirality have different Fermi momenta and velocities, and the interactions between the densities at the different Fermi points are not excessively constrained by symmetries. Our construction shows that, for any fixed finite q, there exist purely repulsive local interactions for which such a metallic state is absolutely q-stable.

[2] The Lie group $SO(N, N)$ consists of all matrices $A \in \mathbb{R}^{2N \times 2N}$ that satisfy $AKA^T = K$ and $\det A = 1$, where $K = \mathrm{diag}(-\mathbb{I}_N, \mathbb{I}_N)$.

2.2 Model and Definitions

Consider a system of interacting fermions in a 1D quantum wire. At low energies, the effective theory of the system involves $2N$ chiral spinless Dirac fermions ψ_I, where $\psi_I^\dagger$ ($\psi_{I+N}^\dagger$) creates a right-moving (left-moving) excitation about the Fermi point $k_{F,I}$ ($k_{F,I+N}$), with Fermi velocity $v_I > 0$ ($v_{I+N} < 0$). The index I distinguishes different bands, accounting for both spin and quantization of the transverse motion. The effective action is given by $S_{\text{eff}} = S_0 + S_{\text{int}} + S_{\text{pert}}$, where

$$S_0 + S_{\text{int}} = \int dt\, dx \Big[\psi_I^\dagger i(\partial_t + v_I \partial_x)\psi_I - U_{IJ}\rho_I \rho_J\Big]. \tag{2.1}$$

Here, the indices I, J are implicitly summed from 1 to $2N$, $\rho_I \equiv \psi_I^\dagger \psi_I$, and the real symmetric $2N \times 2N$ matrix U parametrizes all density-density interactions. All other interaction terms, as well as any quadratic terms accounting for dispersion nonlinearities, are packaged into S_{pert}. If the system is perturbed in any way, for instance by introducing disorder or by proximity-coupling the wire to an external 3D superconductor, the appropriate terms are also included in S_{pert}.

The first part of the action, $S = S_0 + S_{\text{int}}$, describes a gapless N-channel Luttinger liquid. S can be treated non-perturbatively via the method of bosonization [48]. For completeness, we review it in Appendix 2.A. Introducing a chiral boson ϕ_I for each chiral fermion ψ_I, we obtain the bosonic representation

$$S = \frac{1}{4\pi}\int dt\, dx \Big[K_{IJ}\partial_t \phi_I \partial_x \phi_J - V_{IJ}\partial_x \phi_I \partial_x \phi_J\Big], \tag{2.2}$$

with $K = \text{diag}(-\mathbb{I}_N, \mathbb{I}_N)$ and $V_{IJ} = |v_I|\delta_{IJ} + \frac{1}{\pi}U_{IJ}$. The fermion operators are given in terms of the bosons by $\psi_I^\dagger = (2\pi a)^{-1/2} e^{\mp i\phi_I}\gamma_I$, where the sign is $-$ $(+)$ for $I \le N$ $(I > N)$, a is a short-distance cutoff, and the Klein factors γ_I satisfy $\gamma_I\gamma_J = -\gamma_J\gamma_I$ for

$I \neq J$.

The LL action (2.2) is a fixed point under RG flow, parameterized by the symmetric positive definite $2N \times 2N$ matrix V. Our results are based on a systematic linear stability analysis of these fixed points $S[V]$. A generic perturbation of $S[V]$ has the form

$$S' = \int dt\, dx \left[\xi(x)\mathcal{O}(t,x) + \xi^*(x)\mathcal{O}^\dagger(t,x)\right], \tag{2.3}$$

where $\mathcal{O}$ is a local bosonic operator and $\xi(x)$ is an appropriate function. It is natural to distinguish three types of perturbation: (i) *global* perturbations, in which $\xi(x) = ge^{i\alpha}$ is constant in space, (ii) *random* ones, in which $\xi(x)$ is a Gaussian random variable with $\overline{\xi(x)} = 0$ and $\overline{\xi^*(x)\xi(x')} = \sqrt{g}\,\delta(x-x')$, and (iii) *local* ones, in which $\xi(x) = ge^{i\alpha}\delta(x-x_0)$ acts only at a point. In each case, the linearized RG equation specifying how the coupling constant g changes with the energy scale Λ is

$$\frac{d\ln g}{d\ln \Lambda} = \Delta - d_{\text{eff}}, \tag{2.4}$$

where $d_{\text{eff}} = 2, \frac{3}{2}, 1$ for global, random, or local perturbations respectively, and where Δ is the scaling dimension of $\mathcal{O}$. The perturbation is relevant if $\Delta < d_{\text{eff}}$, marginal (at tree-level) if $\Delta = d_{\text{eff}}$, and irrelevant if $\Delta > d_{\text{eff}}$.

The quadratic action (2.2) can be destabilized by localization or the opening of a gap, either of which can be caused by a relevant perturbation (2.3) if $\mathcal{O}$ is a *vertex operator* $\mathcal{O}_{\mathbf{m}} \equiv e^{im_I\phi_I}$, where $\mathbf{m} \in \mathbb{Z}^{2N}$ (we suppress cutoff factors for brevity) [3]. The operator

[3]Besides vertex operators, the other local bosonic operators in the theory are the currents $\partial_x\phi_I$. These have dimension $\Delta = 1$, so terms linear in $\partial_x\phi_I$ are relevant. In fermionic language, they correspond to chemical potential terms. However, such terms can be removed from the action via an appropriate x-dependent shift of the fields ϕ_I (correspondingly, a shift of the Fermi wavevectors $k_{F,I}$). Terms quadratic in $\partial_x\phi_I$ are exactly marginal; these are already included in $S[V]$. Cubic and higher terms are irrelevant.

$\mathcal{O}_{\mathbf{m}}$ is bosonic if and only if its *conformal spin*,

$$K(\mathbf{m}) = \tfrac{1}{2}\mathbf{m}^T K \mathbf{m}, \tag{2.5}$$

is an integer. At the fixed point $S[V]$, the scaling dimension of $\mathcal{O}_{\mathbf{m}}$ is

$$\Delta(\mathbf{m}) = \tfrac{1}{2}\mathbf{m}^T M \mathbf{m}, \tag{2.6}$$

where $M = A^T A$, and $A \in SO(N,N)$ diagonalizes the interaction matrix, $AVA^T = \mathrm{diag}(u_i)$; for a derivation of Eq. (2.6), see Appendix 2.A.6. Although V does not uniquely determine A by this criterion, it *does* uniquely determine $M = A^T A$, so the right side of Eq. (2.6) is well-defined (a proof of this assertion, and a detailed characterization of the map from V to M, is given in Appendix 2.B). Given any M, the set of corresponding interaction matrices can be parameterized as

$$V = M^{-1/2} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} M^{-1/2}, \tag{2.7}$$

where X and Y are arbitrary symmetric positive definite $N \times N$ matrices, and $M^{-1/2}$ is the unique positive definite square root of M^{-1} (see Appendix 2.B for details; in particular, Lemma 3 proves the validity of the parameterization (2.7), and Section 2.B.2 provides some intuition for it). This parameterization of V is closely related to, but distinct from, the one used in Refs. [49, 50].

Note that, since $K = AKA^T$ and $\left|\mathbf{z}^T K \mathbf{z}\right| \leq \|\mathbf{z}\|^2$, where $\|\cdot\|$ denotes the Euclidean norm, the quadratic forms $\Delta(\mathbf{m})$ and $K(\mathbf{m})$ obey the inequality

$$\Delta(\mathbf{m}) \geq |K(\mathbf{m})|. \tag{2.8}$$

Thus, only perturbations that involve operators $\mathcal{O}_{\mathbf{m}}$ with conformal spin $K(\mathbf{m}) = 0, \pm 1$ have a chance of being relevant; those with $K(\mathbf{m}) = \pm 2$ can be at most marginal at tree-level, while higher-spin perturbations are irrelevant on the entire manifold of fixed points $S[V]$.

We define two notions of stability of a LL fixed point $S[V]$. We say that it is ∞-*stable* if all non-chiral (i.e. $K(\mathbf{m}) = 0$) perturbations are irrelevant at $S[V]$. We say that it is *absolutely* ∞-*stable* if all chiral (i.e. $K(\mathbf{m}) \neq 0$) perturbations are irrelevant as well [4]. The scaling dimensions are continuous functions of V, so each stable fixed point belongs to a *stable phase*. ∞-stable phases cannot exist when the LL has only $N = 1$ channel. They can be shown to exist—by explicit construction—when $N \geq 23$ [17]. In the intermediate range, $1 < N < 23$, the existence of ∞-stable phases remains an open question at this time. Meanwhile, upper bounds on the density of high-dimensional sphere packings [51] imply that absolutely ∞-stable phases cannot exist with $N < 11$ channels. It is again possible to show—by explicit construction—that they do exist when N is sufficiently large. For completeness, we discuss these matters in more detail in Appendices 2.E and 2.F.

From a physical point of view, however, the notions of stability introduced above are unnecessarily restrictive. If there are physical reasons to expect the bare value g_0 of a relevant coupling to be small, then although this coupling will eventually destabilize the metallic state, this will only happen at very low temperatures $T \sim \Lambda_0 \, g_0^{1/(d_{\text{eff}}-\Delta)}$. We expect g_0 to be small for perturbations that are sufficiently high-order in the fermion fields. This is based on the assumption that such terms are not appreciably generated during RG flow from the underlying microscopic theory (which only has terms up to quartic order) to the effective theory S_{eff} which describes the system at energies $\sim \Lambda_0$.

[4] Chiral perturbations cannot themselves lead to an energy gap, but one might worry that such perturbations, if relevant, will grow large enough to affect the scaling dimensions of nonchiral operators.

Each vertex operator $\mathcal{O}_{\mathbf{m}}$ in the bosonic formulation corresponds to terms that are $|\mathbf{m}|^{\text{th}}$-order in the fermion fields, where $|\mathbf{m}| \equiv \sum_{I=1}^{2N} |m_I|$. We say that the fixed point $S[V]$ is *q-stable* if q is the largest integer such that all non-chiral perturbations of $S[V]$ with $|\mathbf{m}| \leq q$ are irrelevant. We say that it is *absolutely q-stable* if q is the largest integer such that *all* perturbations with $|\mathbf{m}| \leq q$ are irrelevant. Our earlier notions of stability are the limiting cases $q = \infty$. Based on the comments in the previous paragraph, it is plausible that, in any real system, there will be no observable difference between q-stability and ∞-stability at accessible temperatures if q is sufficiently large [5].

2.3 Relation to Integral Quadratic Lattices

As described in the Introduction to this chapter, there is a beautiful geometric picture associated with all of this. To any interaction matrix V diagonalized by $A \in SO(N, N)$, we associate a lattice $A\mathbb{Z}^{2N} \equiv \{A\mathbf{m} \mid \mathbf{m} \in \mathbb{Z}^{2N}\}$ in a fictitious $\mathbb{R}^{2N}$ equipped with two metrics: the mixed-signature (N, N) metric $K = \text{diag}(-\mathbb{I}_N, \mathbb{I}_N)$ and the Euclidean metric $\mathbb{I}_{2N}$. The scaling dimension of an operator is equal to half the Euclidean interval from the origin to the associated lattice point, $\Delta(\mathbf{m}) = \frac{1}{2}\|A\mathbf{m}\|^2$. There are three "spheres of relevance" centered at the origin, with Euclidean radii $\sqrt{2d_{\text{eff}}} = 2, \sqrt{3}, \sqrt{2}$; any lattice point inside these spheres represents a perturbation that is relevant if it is global, random, or local, respectively. The chirality (i.e. conformal spin) of an operator is equal to half the mixed-signature interval from the origin to the associated lattice point; chiral operators correspond to "spacelike" or "timelike" intervals, and non-chiral operators to "lightlike" (null) intervals. The lattice is naturally graded into "shells" of fixed $|\mathbf{m}| \equiv \sum_{I=1}^{2N} |m_I|$, which equals the order of the corresponding perturbation $\mathcal{O}_{\mathbf{m}}$ in

[5] Even if this assumption turns out to be false, a q-stable phase can be expected to exhibit novel and exotic instabilities, since all the usual instabilities correspond to operators with small $|\mathbf{m}|$.

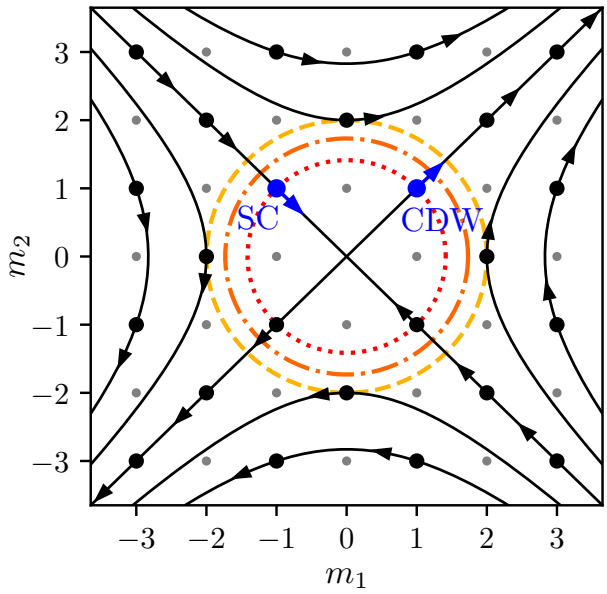


Figure 2.1: Lattice of perturbations for an $N = 1$ channel LL. Large dots are bosonic operators, while small gray dots are fermionic ones; the latter can be ignored. A perturbation is relevant if it falls within the appropriate circle (- - - global, -·-·- random, ····· local). The lattice shown is $\mathbb{Z}^2$, corresponding to the noninteracting fixed point, $\lambda = 0$. With attractive interactions, $\lambda < 0$, the lattice deforms as indicated by the flow field. With repulsive interactions, $\lambda > 0$, the flow is in the opposite direction.

the fermion fields. Bosonic operators have even $|\mathbf{m}|$. The fixed point $S[V]$ is q-stable if no lightlike even lattice point in the innermost q shells falls within the sphere of Euclidean radius 2 centered at the origin. It is absolutely q-stable if the same also holds for spacelike and timelike even lattice points in these shells.

Figure 2.1 illustrates these ideas in the simplest case, that of $N = 1$ channel. The matrix $A \in SO(1,1)$ then describes a boost (hyperbolic rotation) of the plane, and can be parameterized as $A(\lambda) = e^{-(\lambda/2)\sigma_x}$. At the noninteracting fixed point, $\lambda = 0$, the most relevant perturbations couple $\mathcal{O}_{\text{SC}} \equiv \psi_R^\dagger \psi_L^\dagger \sim e^{i(-\phi_1+\phi_2)}$ to an external 3D superconductor, or $\mathcal{O}_{\text{CDW}} \equiv \psi_R \psi_L^\dagger \sim e^{i(\phi_1+\phi_2)}$ to a periodic potential. The corresponding lattice points are $\mathbf{m} = (-1,1)$ and $\mathbf{m} = (1,1)$ respectively. When $\lambda = 0$, both operators have $\Delta = 1$, so both perturbations are relevant; the associated instabilities are induced superconductivity (SC) and a pinned charge density wave (CDW) respectively. When

interactions are turned on, so that $\lambda \neq 0$, the lattice deforms to $A(\lambda)\mathbb{Z}^2$ as indicated in the Figure. Thus, $\lambda < 0$ makes $\mathcal{O}_{\rm CDW}$ less relevant but $\mathcal{O}_{\rm SC}$ more relevant, while $\lambda > 0$ does the opposite. The interaction matrix V can be parametrized as in Eq. (2.7), with $X = u_1 > 0$, $Y = u_2 > 0$, and $M^{1/2} = A(-\lambda) = e^{(\lambda/2)\sigma_x}$; its off-diagonal element is $V_{12} = \frac{1}{2}(u_1 + u_2)\sinh\lambda$. Thus, $\lambda < 0$ ($\lambda > 0$) corresponds to attractive (repulsive) interactions, and we reproduce the well-known phenomenology of the 1-channel Luttinger liquid [15]. Clearly, stability is impossible with just $N = 1$ channel.

2.4 Stable Luttinger Liquids

We now turn to the general case of N channels. Our approach is to study all possible scaling dimension matrices M. After we have identified some M's of interest, we reconstruct the corresponding V's using Eq. (2.7).

A useful structure theorem for $SO(N,N)$, called the *hyperbolic cosine-sine (CS) decomposition* [52], ensures that M can be written as a product of independent boosts in orthogonal planes:

$$M = \begin{bmatrix} Q_1^T & 0 \\ 0 & Q_2^T \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \tag{2.9}$$

where $Q_1, Q_2 \in SO(N)$, $C = \mathrm{diag}(\cosh\lambda_i)$, and $S = \mathrm{diag}(\sinh\lambda_i)$, with $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

The *crucial geometric fact* distinguishing $N \geq 2$ from $N = 1$ is that the boost planes of M can be rotated out of alignment with the lattice planes of $\mathbb{Z}^{2N}$ by suitably chosen Q_i. As a consequence, we have:

Theorem 1. For $N \geq 2$, absolutely q-stable phases exist for any finite q.

Proof. Take $\lambda_i = \lambda$ in the expression (2.9) for M, and let $\mathbf{m} = (\mathbf{m}_R, \mathbf{m}_L)$, with $\mathbf{m}_{R/L} \in$

$\mathbb{Z}^N$. If either $\mathbf{m}_R$ or $\mathbf{m}_L$ vanishes, then $\Delta(\mathbf{m}) = \frac{1}{2}\|\mathbf{m}\|^2 \cosh\lambda > 2$ for $\lambda > \operatorname{arccosh} 2$. If neither $\mathbf{m}_R$ nor $\mathbf{m}_L$ vanishes, we can rewrite the inequality $\Delta(\mathbf{m}) > 2$ as

$$\left|\hat{\mathbf{m}}_R^T Q \hat{\mathbf{m}}_L\right| < f\Big(\frac{\|\mathbf{m}_R\|}{\|\mathbf{m}_L\|}\Big)\coth\lambda - \frac{2\operatorname{csch}\lambda}{\|\mathbf{m}_R\|\|\mathbf{m}_L\|}, \tag{2.10}$$

where $Q \equiv Q_1^T Q_2 \in SO(N)$, $\hat{\mathbf{m}}_\nu \equiv \mathbf{m}_\nu/\|\mathbf{m}_\nu\|$ and $f(x) \equiv \frac{1}{2}(x + x^{-1})$. There are a finite number of vectors $\mathbf{m} \in \mathbb{Z}^{2N}$ that satisfy $|\mathbf{m}| \leq q$, so the unit vectors $\hat{\mathbf{m}}_{R/L}$ in Eq. (2.10) belong to a finite set Ω_q. This set cannot fill the unit sphere densely, so there exists $Q \in SO(N)$ and $\epsilon > 0$ such that $|\hat{\mathbf{m}}_R^T Q \hat{\mathbf{m}}_L| < 1 - \epsilon$ for all $\hat{\mathbf{m}}_{R/L} \in \Omega_q$. But $f(x)\coth\lambda > 1$ for any $x, \lambda > 0$, while $\operatorname{csch}\lambda \to 0$ as $\lambda \to \infty$. Thus the right side of Eq. (2.10) is greater than $1 - \epsilon$ for sufficiently large λ. □

In the $N = 2$ channel case, M is parameterized, according to Eq. (2.9), by two rapidities (λ_1, λ_2) and two angles (θ_1, θ_2, where θ_i is the rotation angle of $Q_i \in SO(2)$). It is convenient to write these as

$$\lambda_{1,2} = \delta \pm \lambda, \qquad \theta_{1,2} = \tfrac{1}{2}(\theta \mp \alpha). \tag{2.11}$$

In the limit $\delta \to 0$, the dependence on α disappears. The full parameterization of M is written down explicitly in Appendix 2.H.1.

We construct an "absolute q-stability phase diagram" for the 2-channel LL by assigning to each point $(\lambda, \delta, \theta, \alpha)$ in the resulting parameter space its absolute q-stability value, q. Figure 2.2 shows the $\delta = 0$ slice of this diagram; other slices may be found in Appendix 2.I. Each point in the phase diagram corresponds to a 6-parameter family of interaction matrices V, which can be obtained using Eq. (2.7). The resulting general expression for V is given in Appendix 2.H.2. Here, we concentrate on the particular case in which the diagonal blocks V_{RR} and V_{LL} are equal, and $\delta = 0$ (see Appendix 2.H.3 for

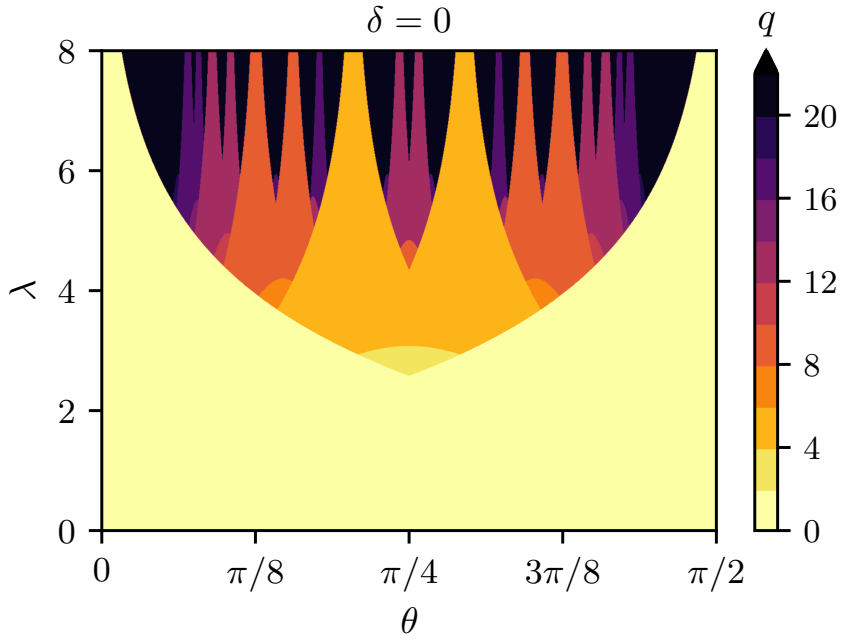


Figure 2.2: A slice of the absolute q-stability phase diagram for the $N = 2$ channel LL. Each point on the plot is assigned the largest integer q such that all perturbations with $|\mathbf{m}| \leq q$ are irrelevant for those parameter values (λ, θ). The diagram is identical for $\theta \mapsto \theta + n\pi/2$.

details). In this case,

$$V = \left[\begin{array}{cc|cc} v_+ & w & c_+ & c_0 \\ w & v_- & c_0 & c_- \\ \hline c_+ & c_0 & v_+ & w \\ c_0 & c_- & w & v_- \end{array}\right], \tag{2.12}$$

where $v_\pm = v \pm u$,

$$c_\pm = (w \sin\theta \pm v_\pm \cos\theta) \tanh\lambda, \tag{2.13a}$$

$$c_0 = v \sin\theta \tanh\lambda. \tag{2.13b}$$

The parameters v, u, and w do not affect scaling dimensions; they can be chosen arbitrarily subject only to the constraint that V must be positive definite, which requires

$v > 0$ and

$$(u \sin\theta - w\cos\theta)^2 \cosh^2\lambda + (u\cos\theta + w\sin\theta)^2 < v^2. \tag{2.14}$$

If in addition $\theta \in [0, \pi]$ and

$$u\cos\theta + w\sin\theta \geq v|\cos\theta|, \tag{2.15}$$

then every entry in the V matrix is nonnegative. Note that the above inequalities *can* be satisfied simultaneously—the first defines the interior of an ellipse in the $(u/v, w/v)$ plane, and the second selects a segment of this ellipse. Thus, we can realize any of the absolutely q-stable phases in Figure 2.2 with purely repulsive interactions.

The 2-channel LLs defined by Eqs. (2.2) and (2.12–2.15) can in principle be realized in a single-spinful-channel quantum wire with either time-reversal or spatial inversion symmetry, but not both; see Appendix 2.C for details. (We hasten to emphasize that these LL phases are q-stable with respect to perturbations that break the symmetry as well.) If the system also has spin-rotation symmetry about some axis, one can reformulate the effective action in terms of non-chiral charge and spin fields [15], as described in Appendix 2.G. In the time-reversal invariant case, the corresponding Hamiltonian takes the form

$$H = \frac{1}{2\pi}\int dx \left[v_c K_c (\pi\Pi_c)^2 + \frac{v_c}{K_c}(\partial_x\varphi_c)^2 + v_s K_s (\pi\Pi_s)^2 + \frac{v_s}{K_s}(\partial_x\varphi_s)^2 \right.$$
$$\left. + d_+(\pi\Pi_c)\partial_x\varphi_s + d_-(\pi\Pi_s)\partial_x\varphi_c \right], \tag{2.16}$$

where $\varphi_{c(s)}$ is the charge (spin) phase field, with conjugate momentum density $\Pi_{c(s)}$. The parameters $v_{c(s)}, K_{c(s)}, d_\pm$ are simple functions of $v_\pm, w, c_\pm, c_0$; the expressions are given in Appendix 2.G. The q-stable phases identified in this work require $d_+ \neq d_-$. Standard

treatments of a spin-orbit-coupled LL, such as Ref. [53], make the additional assumption that interactions are pointlike; this leads to Eq. (2.16) with $d_+ = d_-$. However, $d_+ \neq d_-$ is perfectly consistent with the symmetries of the problem, and appears naturally if one allows for more general short-range interactions.

2.5 Conclusions

As we have seen in this chapter, the 1-channel Luttinger liquid is the exception. For any number of channels $N > 1$—including even $N = 2$—there are parameter regimes in which, for any desired finite q, all instabilities up to q-th order in electron operators are kept at bay. These phases are, in some sense, better examples of non-Fermi liquids than the 1-channel LL since they do not have a tendency to order frustrated only by low dimension. We cannot take $q = \infty$, so these states will eventually be unstable, but this may not occur until unobservably low temperatures. Moreover, it is much more likely that it will be possible to tune the parameters of an $N = 2$ channel Luttinger liquid into the necessary regime in an experiment than it would be for $N = 23$, which appears to be necessary for $q = \infty$. Thus, the work described in this chapter may facilitate the observation of these phases in experiments and may serve as a useful paradigm for thinking about higher-dimensional non-Fermi liquids. Our results can also be translated into statements about stable phases of classical 2-dimensional or layered 3-dimensional systems; it would be interesting to explore the consequences for particular classical systems of experimental interest.

Appendices

2.A Abelian bosonization

In this section, I review the method of abelian bosonization, following the constructive approach described by von Delft and Schoeller [48] (who in turn mostly followed and simplified Haldane [16]). I set $\hbar = 1$ throughout. I work everything out in detail for a single chiral fermion. The generalization to multiple species is a straightforward exercise in bookkeeping, and I merely state the results. Note that my normalization and sign conventions differ from those of Ref. [48] in several places.

2.A.1 Setup of the (fermionic) problem

The starting point is a set of fermion operators c_k indexed by a *discrete* and *unbounded* momentum index $k \in (2\pi/L)\mathbb{Z}$ (this is appropriate for periodic boundary conditions; I ignore most boundary effects). These operators obey the standard anti-commutation relations

$$\{c_k, c_{k'}\} = 0, \tag{2.17a}$$

$$\{c_k, c_{k'}^\dagger\} = \delta_{kk'}. \tag{2.17b}$$

The associated number operators are

$$\hat{n}_k \equiv c_k^\dagger c_k. \tag{2.18}$$

The *fermion vacuum* $|0\rangle$ is defined as the Dirac sea with all $k > 0$ levels empty and all $k \leq 0$ levels occupied:

$$c_k |0\rangle = 0 \quad \text{for} \quad k > 0, \tag{2.19a}$$

$$c_k^\dagger |0\rangle = 0 \quad \text{for} \quad k \leq 0. \tag{2.19b}$$

Note that $|0\rangle$ is the (grand canonical) ground state, at chemical potential $\mu = 0$, of any hamiltonian of the form $H_0 = \sum_k \varepsilon_k \hat{n}_k$, as long as ε_k is a monotonically increasing function of k that changes sign at $k = 0^+$. The fermion Fock space $\mathcal{H}$ is (heuristically) the set of states obtained by acting on $|0\rangle$ with arbitrary combinations of the c_k's and $c_k^\dagger$'s. *Fermion-normal-ordering* is defined with respect to $|0\rangle$; to fermion-normal-order a function of the c_k's and $c_k^\dagger$'s, one moves all c_k's with $k > 0$ and all $c_k^\dagger$'s with $k \leq 0$ to the right of all other operators. The fermion-normal-ordered form of an operator A will be denoted $:A:$. The operator that counts the total number of fermions, relative to the Dirac sea $|0\rangle$, is

$$\hat{N} \equiv \sum_k :\hat{n}_k: \,. \tag{2.20}$$

Here and below, $\sum_k$ means a sum over all $k \in (2\pi/L)\mathbb{Z}$.

The set of all states with $\hat{N}$-eigenvalue N is called the *N-particle Hilbert space* $\mathcal{H}_N$. The full Fock space splits as $\mathcal{H} = \bigoplus_{N\in\mathbb{Z}} \mathcal{H}_N$. The *N-particle ground state* $|N\rangle \in \mathcal{H}_N$ is

defined as:

$$|N\rangle \equiv \begin{cases} c^\dagger_{k_N} \cdots c^\dagger_{k_2} c^\dagger_{k_1} |0\rangle & \text{if} \quad N > 0, \\ |0\rangle & \text{if} \quad N = 0, \\ c_{k_{N+1}} \cdots c_{k_{-1}} c_{k_0} |0\rangle & \text{if} \quad N < 0, \end{cases} \tag{2.21}$$

where $k_n \equiv 2\pi n/L$. Thus, $|N\rangle$ is indeed the N-particle ground state of H_0, with N measured relative to $|0\rangle$.

The fermion field $\psi(x)$ is defined in terms of the c_k's as

$$\psi(x) \equiv \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_k, \tag{2.22}$$

where $x \in [-L/2, L/2]$. The inverse transformation is

$$c_k = \frac{1}{\sqrt{L}} \int dx\, e^{-ikx} \psi(x). \tag{2.23}$$

Here and below, $\int dx$ is shorthand for $\int_{-L/2}^{L/2} dx$. The fermion field obeys the anti-commutation relations

$$\{\psi(x), \psi(x')\} = 0, \tag{2.24a}$$

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x - x'). \tag{2.24b}$$

Note that, with the definitions above, the fermion field $\psi(x)$ is *right-moving*; in the Heisenberg picture of $H_0 = \sum_k vk\, \hat{n}_k$, one has $\psi(x,t) = \psi(x - vt, 0)$.

2.A.2 Bosonic operators and their properties

One can define *bosonic* particle-hole operators indexed by $q \in (2\pi/L)\mathbb{Z}^+$:

$$b_q \equiv -i \left(\frac{2\pi}{Lq}\right)^{1/2} \sum_k c^\dagger_{k-q} c_k. \tag{2.25}$$

Again, these are only defined for $q > 0$. They obey the commutation relations

$$[b_q, \hat{N}] = 0, \tag{2.26a}$$

$$[b_q, b_{q'}] = 0, \tag{2.26b}$$

$$[b_q, b^\dagger_{q'}] = \delta_{qq'}. \tag{2.26c}$$

The first one is obvious, since each term in b_q commutes with $\hat{N}$. The second is straightforward to verify. The third requires some care; one can only cancel terms that are normal-ordered (otherwise each term might be infinite). This is only an issue when $q = q'$:

$$\begin{aligned}[b_q, b^\dagger_q] &= \frac{2\pi}{Lq} \sum_k \left(c^\dagger_k c_k - c^\dagger_{k+q} c_{k+q} \right) \\ &= \frac{2\pi}{Lq} \sum_k \left(:\hat{n}_k: - :\hat{n}_{k+q}: + \langle \hat{n}_k \rangle_0 - \langle \hat{n}_{k+q} \rangle_0 \right).\end{aligned}$$

The normal-ordered terms cancel upon shifting $k \to k - q$ in the second one. The remaining sum gives 1 (since $\langle \hat{n}_k \rangle_0 - \langle \hat{n}_{k+q} \rangle_0 = 1$ for $k \in [-q, 0)$ and 0 otherwise).

Acting on $|N\rangle$, the operator $b^\dagger_q$ creates a linear combination of particle-hole excitations with momentum q. The b_q's annihilate the N-particle ground states:

$$b_q \, |N\rangle = 0. \tag{2.27}$$

Boson-normal-ordering is defined with respect to these states: to boson-normal-order a function of the b_q's and $b_q^\dagger$'s, one moves all b_q's to the right of all $b_q^\dagger$'s. The boson-normal-ordered form of a bosonic operator B will be denoted $:B:$. It is easy to check that an expression consisting purely of boson operators is boson-normal-ordered if and only if it is fermion-normal-ordered (hence the same notation).

Let $\mathcal{F}_{bN}$ denote the set of states obtained by acting on $|N\rangle$ with arbitrary combinations of the $b_q^\dagger$'s. $\mathcal{F}_{bN}$ is spanned by the set of orthonormal basis states

$$|N;\{m_q\}\rangle \equiv \prod_{q>0} \frac{(b_q^\dagger)^{m_q}}{\sqrt{m_q!}} |N\rangle \,. \tag{2.28}$$

Here and below, $\prod_{q>0}$ means a product over all $q \in (2\pi/L)\mathbb{Z}^+$. The justification for the entire bosonization technique rests on the following result (a self-contained proof is given later, in Appendix 2.A.7):

Theorem 2. $\mathcal{F}_{bN} = \mathcal{H}_N$.

To complete the bosonic representation, one needs *Klein factors* $F^\dagger$ and F that raise or lower the total fermion number by one (and thus connect the different $\mathcal{H}_N$ sectors) [6]. These may be defined in the basis (2.28) as:

$$F \equiv \sum_{N\in\mathbb{Z}} \sum_{\{m_q\}} |N-1;\{m_q\}\rangle\langle N;\{m_q\}| \,. \tag{2.29}$$

[6]Note that, in the main text of the chapter, we denoted the Klein factors by γ instead of F.

The Klein factors obey

$$[\hat{N}, F^\dagger] = F^\dagger, \tag{2.30a}$$

$$[b_q, F^\dagger] = [b_q^\dagger, F^\dagger] = 0, \tag{2.30b}$$

$$FF^\dagger = F^\dagger F = 1. \tag{2.30c}$$

At this stage, the bosonic representation has been established in principle. By acting on $|0\rangle$ with suitable combinations of the Klein factors and the $b_q^\dagger$'s, we can reach any state in the fermionic Fock space $\mathcal{H}$. It follows that the c_k's and $c_k^\dagger$'s have bosonic representations. In practice, however, it is more convenient to work with the Fourier transforms (field operators).

The hermitian bosonic field is defined as

$$\phi(x) \equiv \varphi(x) + \varphi^\dagger(x), \tag{2.31}$$

where

$$\varphi(x) \equiv \sum_{q>0} \left(\frac{2\pi}{Lq}\right)^{1/2} e^{iqx - aq/2}\, b_q. \tag{2.32}$$

Here and below, $\sum_{q>0}$ or $\sum_q$ mean a sum over all $q \in (2\pi/L)\mathbb{Z}^+$ or all $q \in (2\pi/L)\mathbb{Z}$, respectively. $a > 0$ is an infinitesimal UV regularization parameter. According to Haldane, a "in no way plays the role of a cut-off length." However, $1/a$ can be interpreted as a kind of "effective band-width" [16, 48].

The boson fields obey the commutation relations

$$[\varphi(x), \varphi(x')] = 0, \tag{2.33a}$$

$$[\varphi(x), \varphi^\dagger(x')] = -\log\bigl(1 - e^{-2\pi\bar{z}/L}\bigr), \tag{2.33b}$$

where $\bar{z}$ is the complex conjugate of

$$z \equiv a + i(x - x'). \tag{2.34}$$

The argument of the logarithm is always in the right half plane; the branch cut can be taken along the negative real axis. As $L \to \infty$,

$$[\varphi(x), \varphi^\dagger(x')] \sim -\log(2\pi\bar{z}/L). \tag{2.35}$$

It follows from Eq. (2.33) that

$$\begin{aligned}
[\phi(x), \phi(x')] &= \log\left(\frac{1 - e^{-2\pi z/L}}{1 - e^{-2\pi \bar{z}/L}}\right) \\
&= \log(z/\bar{z}) - \frac{\pi}{L}(z - \bar{z}) + \cdots \\
&\sim 2i \arctan\left(\frac{x - x'}{a}\right) - \frac{2\pi i}{L}(x - x').
\end{aligned} \tag{2.36}$$

In the limit $a \to 0$, this reduces to

$$[\phi(x), \phi(x')] \sim i\pi \,\mathrm{sgn}(x - x') - \frac{2\pi i}{L}(x - x'). \tag{2.37}$$

Finally, differentiating Eq. (2.36) with respect to x', we obtain

$$[\phi(x), \partial_{x'}\phi(x')] \sim -2\pi i \left[\frac{1}{\pi}\frac{a}{(x - x')^2 + a^2} - \frac{1}{L}\right]. \tag{2.38}$$

As $a \to 0$, the first term in brackets reduces to $\delta(x - x')$. The $1/L$ correction ensures that the commutator is consistent with $\int dx' \, \partial_{x'}\phi(x') = \phi(L/2) - \phi(-L/2) = 0$, as required by periodic boundary conditions. When performing calculations it is always safest to use the commutator formulae with a finite, and to only take $a \to 0$ (if possible) at the end.

Note that, with the definitions above, the boson fields are also *right-moving*; in the Heisenberg picture of $H_0 = \sum_k vk\,\hat{n}_k$, one has $\varphi(x,t) = \varphi(x - vt, 0)$, as can be verified using Eqs. (2.25) and (2.32).

2.A.3 The bosonization formulae

The fundamental "bosonization formula" expresses the fermion field $\psi(x)$ in terms of operators that have simple representations in the bosonic basis (2.28): the boson field $\phi(x)$, Klein factor F, and total number operator $\hat{N}$. Almost as useful (and much easier to derive) is the bosonic representation of the normal-ordered electron density,

$$\rho(x) \equiv :\psi^\dagger(x)\psi(x): = \frac{1}{L}\sum_q e^{iqx} \sum_k :c^\dagger_{k-q} c_k: \tag{2.39}$$

(the summand is already normal-ordered for $q \neq 0$). In terms of the boson operators, one has

$$\rho(x) = i\sum_{q>0} \left(\frac{q}{2\pi L}\right)^{1/2} \left(e^{iqx} b_q - e^{-iqx} b_q^\dagger\right) + \frac{1}{L}\sum_k :n_k:,$$

and thus

$$\rho(x) = \frac{1}{2\pi}\partial_x \phi(x) + \frac{\hat{N}}{L}. \tag{2.40}$$

This is the first "practical formula" of bosonization. It holds for $a \to 0$ on the right side of the equation. In the limit $L \to \infty$, the second term can be dropped (since $\hat{N}$ measures fermion number *relative to the Dirac sea*).

The bosonic representation of $\psi(x)$ is derived by studying its action on arbitrary states in $\mathcal{H}_N$, starting with the N-particle ground state $|N\rangle$. It is easy to verify (from

the definitions) that

$$[b_q, \psi(x)] = \alpha_q(x)\psi(x), \tag{2.41a}$$

$$[b_q^\dagger, \psi(x)] = \alpha_q^*(x)\psi(x), \tag{2.41b}$$

where

$$\alpha_q(x) \equiv i\left(\frac{2\pi}{Lq}\right)^{1/2} e^{-iqx}. \tag{2.42}$$

Since $b_q\,|N\rangle = 0$, this implies

$$b_q\psi(x)|N\rangle = \alpha_q(x)\psi(x)|N\rangle\,. \tag{2.43}$$

Thus $\psi(x)|N\rangle$ is a simultaneous eigenstate of all the b_q's, with eigenvalues $\alpha_q(x)$. It must therefore have a coherent state representation of the form

$$\psi(x)|N\rangle = \lambda_N(x)\,F\,e^{\sum_{q>0}\alpha_q(x)b_q^\dagger}\,|N\rangle\,, \tag{2.44}$$

where $\lambda_N(x)$ is a complex-valued function of N and x, and the Klein factor F is needed to decrease the total number of fermions in the state by one. Projecting the equation onto $|N-1\rangle$, and noting that $\langle N-1|\,F\,e^{\sum_{q>0}\alpha_q b_q^\dagger} = \langle N|$, we obtain

$$\lambda_N(x) = \langle N-1|\psi(x)|N\rangle = \frac{e^{ik_N x}}{\sqrt{L}}, \tag{2.45}$$

where $k_N = 2\pi N/L$. Thus, in the limit $a \to 0$,

$$\psi(x)|N\rangle = \frac{1}{\sqrt{L}}Fe^{i2\pi\hat{N}x/L}\,e^{i\varphi^\dagger(x)}\,|N\rangle\,. \tag{2.46}$$

Here, I commuted the factor $e^{ik_N x}$ past F and wrote it as an operator acting on $|N\rangle$. It

is convenient to define

$$F(x) \equiv F e^{i2\pi \hat{N} x/L} \tag{2.47}$$

and to regard the exponential as the x-dependence of the Klein factor.

Having determined the action of $\psi(x)$ on $|N\rangle$, one can determine its action on an arbitrary state in $\mathcal{H}_N$. By Theorem 2, any state in $\mathcal{H}_N$ can be written as $f(\{b_q^\dagger\})\,|N\rangle$, for some function f. Using Eq. (2.41b), one has

$$\psi(x) b_q^\dagger = (b_q^\dagger - \alpha_q^*(x))\psi(x). \tag{2.48}$$

Therefore,

$$\begin{aligned}\psi(x) f(\{b_q^\dagger\})\,|N\rangle &= f(\{b_q^\dagger - \alpha_q^*(x)\})\,\psi(x)\,|N\rangle \\ &= \frac{1}{\sqrt{L}} F(x) e^{i\varphi^\dagger(x)} f(\{b_q^\dagger - \alpha_q^*(x)\})\,|N\rangle\,.\end{aligned}$$

The second equality holds because $F(x)$ and $\varphi^\dagger(x)$ commute with the $b_q^\dagger$'s. From the definition of $\varphi(x)$, one has, in the limit $a \to 0$:

$$[\varphi(x), b_q^\dagger] = i\alpha_q^*(x), \tag{2.49}$$

and so

$$e^{i\varphi(x)}\, b_q^\dagger\, e^{-i\varphi(x)} = b_q^\dagger - \alpha_q^*(x). \tag{2.50}$$

Thus,

$$\begin{aligned}f(\{b_q^\dagger - \alpha_q^*(x)\})\,|N\rangle &= e^{i\varphi(x)} f(\{b_q^\dagger\})\, e^{-i\varphi(x)}\,|N\rangle \\ &= e^{i\varphi(x)} f(\{b_q^\dagger\})\,|N\rangle\,.\end{aligned}$$

The second equality holds because $\varphi(x)$ annihilates $|N\rangle$. Combining this with the intermediate result above, one gets

$$\psi(x)f(\{b_q^\dagger\})\,|N\rangle = \frac{1}{\sqrt{L}}F(x)e^{i\varphi^\dagger(x)}e^{i\varphi(x)}f(\{b_q^\dagger\})\,|N\rangle\,.$$

This holds for arbitrary states of the form $f(\{b_q^\dagger\})\,|N\rangle$. By Theorem 2, these states span the fermion Fock space $\mathcal{H}$. Thus the *bosonization formula*

$$\psi(x) = \frac{1}{\sqrt{L}}F(x)e^{i\varphi^\dagger(x)}e^{i\varphi(x)} \tag{2.51}$$

holds as an operator identity in $\mathcal{H}$. The right hand side is normal-ordered, and can be evaluated with $a = 0$. There is also the (more common) un-normal-ordered version:

$$\psi(x) = \frac{1}{\sqrt{2\pi a}}F(x)e^{i\phi(x)}. \tag{2.52}$$

This form obviously requires $a > 0$. The equivalence of Eqs. (2.51) and (2.52) follows from the Baker-Campbell-Hausdorff (BCH) formula and Eq. (2.35); $e^{i\varphi^\dagger}e^{i\varphi} = e^{i(\varphi^\dagger+\varphi)-\frac{1}{2}[\varphi^\dagger,\varphi]} = (\frac{L}{2\pi a})^{1/2}e^{i\phi}$. In the limit $L\to\infty$, the x-dependence of the Klein factor can be dropped.

Our earlier formula for the normal-ordered electron density, Eq. (2.40), can be recovered from Eq. (2.52). To do this, we "point-split" the otherwise singular product of operators $\psi^\dagger(x)\psi(x)$ and compute

$$\begin{aligned}
\rho(x) &= \lim_{a\to 0} :\psi^\dagger(x+a)\psi(x): \\
&= \lim_{a\to 0} :\frac{1}{2\pi a}e^{-i\phi(x+a)}e^{i\phi(x)}: \\
&= \lim_{a\to 0} :\frac{1}{2\pi a}e^{-i\phi(x+a)+i\phi(x)}e^{\frac{1}{2}[\phi(x+a),\phi(x)]}: \\
&= \lim_{a\to 0} :\frac{1}{2\pi a}\big(1 - ia\partial_x\phi(x) + \cdots\big)e^{i\pi/2}: = \frac{1}{2\pi}\partial_x\phi(x),
\end{aligned} \tag{2.53}$$

which agrees Eq. (2.40) in the limit $L \to \infty$ (one can also recover the $1/L$ term by being more careful). Here, the second equality uses $F^\dagger F = 1$, the third uses the BCH formula, and the fourth uses Eq. (2.37) for the commutator. Finally, normal ordering gets rid of the infinite constant term $1/2\pi a$.

2.A.4 Bosonization of the free fermion hamiltonian

We could bosonize the free fermion hamiltonian by carefully point-splitting and applying the bosonization identity, Eq. (2.51). However, the algebra involved is somewhat tricky and quite tedious (see Appendix G of Ref. [48] for this derivation). Luckily, there is a simpler and more direct method. First consider the case of linear dispersion,

$$H_0 \equiv \sum_k k \, :c_k^\dagger c_k: = \int dx \, :\psi^\dagger(x)(-i\partial_x)\psi(x): \, . \tag{2.54}$$

Recall that the N-particle ground state $|N\rangle \in \mathcal{H}_N$ has all $k > k_N$ levels empty and all $k \leq k_N$ levels occupied, where $k_N \equiv 2\pi N/L$. Hence, by inspection,

$$H_0 \, |N\rangle = \tfrac{\pi}{L} N(N+1) \, |N\rangle \, . \tag{2.55}$$

Furthermore, a simple calculation using the definition (2.25) of the bosonic operators b_q shows that

$$H_0 b_q^\dagger = b_q^\dagger (H_0 + q). \tag{2.56}$$

Now, the fact that the $b_q^\dagger$'s acting on $|N\rangle$ span the entire N-particle Hilbert space $\mathcal{H}_N$ (Theorem 2) implies that H_0 must have a unique representation in terms of the bosonic

variables. The only bosonic operator that satisfies both of the preceding conditions is

$$H_0 = \sum_q q\, b_q^\dagger b_q + \frac{\pi}{L}\hat{N}(\hat{N}+1) = \frac{1}{4\pi}\int dx\, :(\partial_x\phi(x))^2: + \frac{\pi}{L}\hat{N}(\hat{N}+1). \tag{2.57}$$

The second form is easily seen to be equivalent to the first by using the definition of the boson field $\phi(x)$. In the limit $L \to \infty$, the second term can be dropped (since $\hat{N}$ measures fermion number relative to the Dirac sea).

Nonlinearity of the dispersion relation of the fermions gives rise, in the bosonic hamiltonian, to interaction terms that are proportional to higher powers of $\partial_x\phi$ [16]. For instance, a quadratic term $\propto k^2$ in the fermion dispersion relation gives rise to a $(\partial_x\phi)^3$ interaction, while a cubic nonlinearity $\propto k^3$ gives rise to a $(\partial_x\phi)^4$ interaction. These interaction terms can all be derived by simply repeating the steps above with a general dispersion law, $\varepsilon(k) \sim c_1 k + c_2 k^2 + c_3 k^3 + \cdots$. Very heuristically, a k^n nonlinearity corresponds to $\psi^\dagger(-i\partial_x)^n\psi$, and each derivative acting on $\psi \sim e^{i\phi}$ pulls down an additional factor of $\partial_x\phi$, leading to a term $\propto (\partial_x\phi)^{n+1}$ in the boson hamiltonian. These interaction terms are all irrelevant in the RG sense, and can be neglected for the purposes of this chapter. However, when one considers dynamic quantities these terms are often *dangerously irrelevant* and cannot be handled using naive perturbation theory. For a deeper discussion of dispersion nonlinearities in Luttinger liquids, and how to properly treat them, see Ref. [54]. We will also touch on this topic again briefly in Chapter 3 of this thesis (Section 3.4.4), in the context of understanding relaxation to equilibrium following a quantum quench.

2.A.5 Multiple fermion species

The preceding results may be easily generalized to the case of multiple fermion species, distinguished by an index I. If the species are all (mathematical) right-movers, the

formulae simply acquire an extra species index I. The only nontrivial change occurs in the definition of Klein factors, which must now capture the anti-commutation of fermions belonging to different species. Instead of Eq. (2.29), we have

$$F_I \equiv \sum_{\mathbf{N}} \sum_{\{m_q\}} |\mathbf{N} - 1_I; \{m_q\}\rangle\langle \mathbf{N}; \{m_q\}| \, (-1)^{\sum_{J=1}^{I-1} N_J}. \tag{2.58}$$

These Klein factors obey

$$[\hat{N}_I, F_J^\dagger] = \delta_{IJ} F_J^\dagger, \tag{2.59a}$$

$$[b_{qI}, F_J^\dagger] = [b_{qI}^\dagger, F_J^\dagger] = 0, \tag{2.59b}$$

$$\{F_I, F_J^\dagger\} = 2\delta_{IJ} \quad (\text{with } F_I^\dagger F_I = 1), \tag{2.59c}$$

$$\{F_I, F_J\} = 0 \quad \text{for } I \neq J. \tag{2.59d}$$

We may introduce left-moving fields by simply noting that if $\psi_I(x)$ is a right-mover, then $\tilde{\psi}_I(x) \equiv \psi_I(-x)$ is a left-mover (we should also change $k \to -k$ and $q \to -q$ for the left-movers if we want these quantities to consistently label quasimomentum). The corresponding left-moving bosonic fields are defined as $\tilde{\phi}_I(x) \equiv -\phi_I(-x)$; the extra minus sign in this definition ensures that there is no minus sign in the formula $\tilde{\rho}_I(x) \equiv :\tilde{\psi}_I^\dagger(x)\tilde{\psi}_I(x): = \frac{1}{2\pi}\partial_x \tilde{\phi}_I(x)$. Dropping tildes, we have

$$\psi_I(x) = \frac{1}{\sqrt{2\pi a}} F_I(x) e^{\pm i\phi_I(x)}, \tag{2.60}$$

where the sign is + (−) for right-movers (left-movers). Also, as just discussed,

$$\rho_I(x) \equiv :\psi_I^\dagger(x)\psi_I(x): = \frac{1}{2\pi}\partial_x \phi_I(x) \tag{2.61}$$

holds for both right- and left-movers. The free fermion hamiltonian with linear dispersion,

$$H_0 \equiv \sum_I \sum_k v_I k \, :c_{kI}^\dagger c_{kI}: = \int dx \sum_I :\psi_I^\dagger(x)(-iv_I\partial_x)\psi_I(x):, \tag{2.62}$$

where $v_I > 0$ ($v_I < 0$) for right-movers (left-movers), has bosonic representation

$$H_0 = \frac{1}{4\pi}\int dx \sum_I |v_I| \, :(\partial_x\phi_I(x))^2: + \frac{\pi}{L}\sum_I \hat{N}_I(\hat{N}_I+1). \tag{2.63}$$

Again, nonlinearities in the dispersion relation give rise to $(\partial_x\phi_I)^3$ and $(\partial_x\phi_I)^4$ interaction terms that are irrelevant (although dangerously so for dynamic quantities).

The part of the full hamiltonian describing "forward scattering" interactions between the electron densities ρ_I at the different Fermi points can be immediately bosonized using the identity $\rho_I = \frac{1}{2\pi}\partial_x\phi_I$:

$$H_1 \equiv \int dx \, U_{IJ}\rho_I(x)\rho_J(x) = \frac{1}{4\pi^2}\int dx \, U_{IJ} \, \partial_x\phi_I(x) \, \partial_x\phi_J(x). \tag{2.64}$$

Thus,

$$H \equiv H_0 + H_1 = \frac{1}{4\pi}\int dx \, V_{IJ} \, \partial_x\phi_I(x) \, \partial_x\phi_J(x), \tag{2.65}$$

where $V_{IJ} \equiv |v_I|\delta_{IJ} + \frac{1}{\pi}U_{IJ}$. The associated action is given by $S = \int dt \, (\sum_I \tilde{\Pi}_I\partial_t\phi_I - H)$, where the $\tilde{\Pi}_I$'s are conjugate fields satisfying $[\phi_I(x), \tilde{\Pi}_J(x')] = (i/2)\delta(x-x')$; the extra factor of $1/2$ is because the ϕ_I's are chiral. From Eq. (2.38), we identify $\tilde{\Pi}_I = \mp\frac{1}{4\pi}\partial_x\phi_I$, where the sign is $-$ $(+)$ for right-movers (left)-movers. In this way, we arrive at Eq. (2.2),

$$S = \frac{1}{4\pi}\int dt \, dx \, \Big[K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J\Big], \tag{2.66}$$

where $K_{IJ} = \kappa_I\delta_{IJ}$ and $\kappa_I = -$ $(+)$ for right-movers (left-movers). There are several other

ways to derive this action. One is to demand that $\delta S/\delta\phi_I = 0$ give the correct equations of motion for the densities $\rho_I \propto \partial_x\phi_I$ when $U = 0$: right-movers obey $(\partial_t + \partial_x)\rho_I = 0$, while left-movers obey $(\partial_t - \partial_x)\rho_I = 0$. This fixes the first term in S.

Finally, any other terms in the fermion hamiltonian can also be bosonized by applying the bosonization formula $\psi_I \sim e^{\pm i\phi_I}$. For example, $\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L \sim \cos(\phi_L + \phi_R)$.

2.A.6 Correlation functions and scaling dimensions of vertex operators

As usual in many-body theory, the generating functional of imaginary time-ordered correlation functions of the boson fields at zero temperature may be written as a functional integral,

$$Z[h] = \int [d\phi]\, e^{-S_E[\phi] + \int d^2x\, h_I(\mathbf{x})\phi_I(\mathbf{x})}. \tag{2.67}$$

Here $\mathbf{x} \equiv (\tau, x)$, the integral $\int d^2x$ is over all $\mathbf{x} \in \mathbb{R}^2$, the $h_I(\mathbf{x})$'s are source fields, S_E is the Euclidean version of the action S,

$$S_E[\phi] = \frac{1}{4\pi}\int d^2x \Big[- iK_{IJ}\partial_\tau\phi_I\partial_x\phi_J + V_{IJ}\partial_x\phi_I\partial_x\phi_J\Big], \tag{2.68}$$

and the functional integral $\int[d\phi]$ is over all field configurations $\phi(\mathbf{x})$; the measure is normalized so that $Z[0] = 1$. Recall that $K \equiv \mathrm{diag}(\kappa_j)$ with diagonal entries $\kappa_j = \pm 1$, while the matrix V is positive definite.

We are free to change integration variables in Eq. (2.67) from $\phi_I(\mathbf{x})$ to $\tilde{\phi}_I(\mathbf{x}) = \phi_J(\mathbf{x})(A^{-1})_{JI}$, where A is any invertible matrix with $\det A = 1$; the Jacobian of this transformation is 1. Doing so, we obtain

$$Z[h] = \int [d\tilde{\phi}]\, e^{-\tilde{S}_E[\tilde{\phi}] + \int d^2x\, \tilde{h}_I(\mathbf{x})\tilde{\phi}_I(\mathbf{x})}, \tag{2.69}$$

where $\tilde{h}_I(\mathbf{x}) \equiv A_{IJ} h_J(\mathbf{x})$ and

$$\tilde{S}_E[\tilde{\phi}] = \frac{1}{4\pi} \int d^2x \Big[- i(AKA^T)_{IJ} \partial_\tau \tilde{\phi}_I \partial_x \tilde{\phi}_J + (AVA^T)_{IJ} \partial_x \tilde{\phi}_I \partial_x \tilde{\phi}_J \Big]. \tag{2.70}$$

Assume that we can choose A so that $AKA^T = K$ and $AVA^T = \mathrm{diag}(u_j)$, with $u_j > 0$. Then the action $\tilde{S}_E$ simply describes a collection of free chiral bosons with velocities $\kappa_j u_j$,

$$\tilde{S}_E[\tilde{\phi}] = \frac{1}{4\pi} \int d^2x \sum_j \Big[- i\kappa_j \partial_\tau \tilde{\phi}_j \partial_x \tilde{\phi}_j + u_j (\partial_x \tilde{\phi}_j)^2 \Big]. \tag{2.71}$$

Consequently, Eq. (2.69) yields

$$Z[h] = \exp\left(\frac{1}{2} \int d^2x \int d^2x' \sum_j \tilde{h}_j(\mathbf{x}) \tilde{g}_j(\mathbf{x} - \mathbf{x}') \tilde{h}_j(\mathbf{x}') \right), \tag{2.72}$$

where $\tilde{g}_j(\mathbf{x} - \mathbf{x}')$ is the propagator of the free chiral boson $\tilde{\phi}_j$,

$$\tilde{g}_j(\mathbf{x} - \mathbf{x}') = \frac{\delta^2 Z}{\delta \tilde{h}_j(\mathbf{x}) \delta \tilde{h}_j(\mathbf{x}')} \bigg|_{h=0} = \langle \tilde{\phi}_j(\mathbf{x}) \tilde{\phi}_j(\mathbf{x}') \rangle. \tag{2.73}$$

Computing this propagator in any of the usual ways (e.g. by expanding the field in Fourier modes and explicitly performing the field integral, by returning to the hamiltonian formulation, or by solving the appropriate differential equation), we obtain [48]

$$\tilde{g}_j(\mathbf{x}) = - \log\Big(\frac{2\pi}{L} [(u_j \tau - i\kappa_j x) \operatorname{sgn} \tau + a] \Big). \tag{2.74}$$

Now consider the vertex operator $\mathcal{O}_\mathbf{m}(\mathbf{x}) \equiv \mathcal{N}_\mathbf{m} e^{i m_I \phi_I(\mathbf{x})}$, where $\mathcal{N}_\mathbf{m}$ is a normalization factor. The imaginary time-ordered correlation function of a pair of vertex operators is

$$\langle \mathcal{O}_\mathbf{m}(\mathbf{x}_1) \mathcal{O}_\mathbf{n}(\mathbf{x}_2) \rangle = \int [d\phi]\, e^{-S_E[\phi]}\, \mathcal{O}_\mathbf{m}(\mathbf{x}_1) \mathcal{O}_\mathbf{n}(\mathbf{x}_2). \tag{2.75}$$

Comparing this expression with Eq. (2.67), we notice that

$$\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\mathcal{O}_{\mathbf{n}}(\mathbf{x}_2)\rangle = \mathcal{N}_{\mathbf{m}}\mathcal{N}_{\mathbf{n}}\; Z[h_I(\mathbf{x}) = im_I\delta(\mathbf{x}-\mathbf{x}_1) + in_I\delta(\mathbf{x}-\mathbf{x}_2)]. \tag{2.76}$$

The right side can be easily evaluated using Eq. (2.72) with $\tilde{h}_I(\mathbf{x}) = A_{IJ}h_J(\mathbf{x})$:

$$\begin{aligned}\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\mathcal{O}_{\mathbf{n}}(\mathbf{x}_2)\rangle &= \mathcal{N}_{\mathbf{m}}\mathcal{N}_{\mathbf{n}}\, e^{-\frac{1}{2}\int d^2x\int d^2x' \sum_j [\tilde{m}_j\delta(\mathbf{x}-\mathbf{x}_1)+\tilde{n}_j\delta(\mathbf{x}-\mathbf{x}_2)]\tilde{g}_j(\mathbf{x}-\mathbf{x}')[\tilde{m}_j\delta(\mathbf{x}'-\mathbf{x}_1)+\tilde{n}_j\delta(\mathbf{x}'-\mathbf{x}_2)]} \\ &= \mathcal{N}_{\mathbf{m}}\mathcal{N}_{\mathbf{n}}\prod_j e^{-\frac{1}{2}(\tilde{m}_j^2+\tilde{n}_j^2)\tilde{g}_j(\mathbf{0}) - \frac{1}{2}\tilde{m}_j\tilde{n}_j[\tilde{g}_j(\mathbf{x}_1-\mathbf{x}_2)+\tilde{g}_j(\mathbf{x}_2-\mathbf{x}_1)]},\end{aligned} \tag{2.77}$$

where $\tilde{m}_I \equiv A_{IJ}m_J$ and $\tilde{n}_I \equiv A_{IJ}n_J$. Using Eq. (2.74) for the propagator $\tilde{g}_j(\mathbf{x})$, we get

$$\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\mathcal{O}_{\mathbf{n}}(\mathbf{x}_2)\rangle = \mathcal{N}_{\mathbf{m}}\mathcal{N}_{\mathbf{n}}\prod_j \Big(\frac{2\pi a}{L}\Big)^{\frac{1}{2}(\tilde{m}_j^2+\tilde{n}_j^2)}\Big(\frac{2\pi}{L}[(u_j\tau - i\kappa_j x)\operatorname{sgn}\tau + a]\Big)^{\tilde{m}_j\tilde{n}_j}, \tag{2.78}$$

where $\tau \equiv \tau_1 - \tau_2$ and $x \equiv x_1 - x_2$. A correlation function of physically meaningful operators should not depend on the UV regularization parameter a when the separation between the operators is large compared to a. Therefore, we choose the normalization factors to be

$$\mathcal{N}_{\mathbf{m}} \equiv a^{-\frac{1}{2}\sum_j \tilde{m}_j^2} = a^{-\frac{1}{2}m^TA^TA\,m}. \tag{2.79}$$

With this choice [7], the expression for the correlation function reduces to

$$\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\mathcal{O}_{\mathbf{n}}(\mathbf{x}_2)\rangle = \Big(\frac{2\pi}{L}\Big)^{\frac{1}{2}\sum_j(\tilde{m}_j+\tilde{n}_j)^2}\prod_j\big[(u_j\tau - i\kappa_j x)\operatorname{sgn}\tau + a\big]^{\tilde{m}_j\tilde{n}_j}. \tag{2.80}$$

Clearly the correlation function is nonzero in the limit $L \to \infty$ only if $\tilde{\mathbf{n}} = -\tilde{\mathbf{m}}$ (or equivalently, since A is invertible, only if $\mathbf{n} = -\mathbf{m}$). This is as it should be. The action $S_E[\phi]$ is invariant under a shift $\phi_I(\mathbf{x}) \to \phi_I(\mathbf{x}) + \text{constant}$, so the same is expected of a

[7] Notice that the normalization factor is precisely $\mathcal{N}_{\mathbf{m}} = a^{-\Delta(\mathbf{m})}$, where $\Delta(\mathbf{m})$ is the scaling dimension of $\mathcal{O}_{\mathbf{m}}$; this makes sense dimensionally.

correlation function of two properly normalized exponentials of the boson fields. This is only true for a correlator $\propto \langle e^{im_I\phi_I} e^{in_J\phi_J}\rangle$ if $\mathbf{n} = -\mathbf{m}$. In the limit $L \to \infty$,

$$\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\mathcal{O}_{\mathbf{n}}(\mathbf{x}_2)\rangle = \delta_{\mathbf{n},-\mathbf{m}} \prod_j \left[(u_j\tau - i\kappa_j x)\operatorname{sgn}\tau + a\right]^{-\tilde{m}_j^2}, \tag{2.81}$$

where, as before, $\tau \equiv \tau_1 - \tau_2$ and $x \equiv x_1 - x_2$.

The scaling dimension of the vertex operator $\mathcal{O}_{\mathbf{m}}$, $\Delta(\mathbf{m})$, is encoded in the transformation of its two-point correlation function when we uniformly scale spacetime:

$$\frac{\langle \mathcal{O}_{\mathbf{m}}(\lambda\mathbf{x}_1)\,\mathcal{O}_{-\mathbf{m}}(\lambda\mathbf{x}_2)\rangle}{\langle \mathcal{O}_{\mathbf{m}}(\mathbf{x}_1)\,\mathcal{O}_{-\mathbf{m}}(\mathbf{x}_2)\rangle} = \lambda^{-2\Delta(\mathbf{m})}. \tag{2.82}$$

It follows from Eq. (2.81) that

$$\Delta(\mathbf{m}) = \frac{1}{2}\sum_j \tilde{m}_j^2 = \frac{1}{2} m^T A^T A\, m, \tag{2.83}$$

where the matrix A satisfies $\det A = 1$, $AKA^T = K$ and $AVA^T = \operatorname{diag}(u_j)$. In the case $K = -\mathbb{I}_N \oplus \mathbb{I}_N$, this yields Eq. (2.6) in the main text.

2.A.7 Proof of Theorem 2

Recall that $\mathcal{H}_N$ is the set of states in the fermion Fock space $\mathcal{H}$ with $\hat{N}$-eigenvalue N, while $\mathcal{F}_{bN}$ is the set of states obtained by acting on $|N\rangle$ with arbitrary combinations of the bosonic $b_q^\dagger$'s. The states $|N;\{m_q\}\rangle$ defined in Eq. (2.28),

$$|N;\{m_q\}\rangle \equiv \prod_{q>0} \frac{(b_q^\dagger)^{m_q}}{\sqrt{m_q!}} |N\rangle\,,$$

form an orthonormal basis for $\mathcal{F}_{bN}$. We need to prove that $\mathcal{F}_{bN} = \mathcal{H}_N$. I follow Haldane [16].

It is obvious that $\mathcal{F}_{bN} \subset \mathcal{H}_N$, because the bosonic operators are themselves functions of the fermionic ones, and $[b_q^\dagger, \hat{N}] = 0$. To prove that, in fact, $\mathcal{F}_{bN} = \mathcal{H}_N$, we count states. The overall strategy is as follows. Let Tr_N denote the trace on $\mathcal{H}_N$, and Tr_{bN} the trace on $\mathcal{F}_{bN}$. Given any positive definite operator $\hat{\rho} : \mathcal{H}_N \to \mathcal{H}_N$ whose trace is finite, we can define $Z_{cN} \equiv \mathrm{Tr}_N(\hat{\rho})$ and $Z_{bN} \equiv \mathrm{Tr}_{bN}(\hat{\rho}|_{\mathcal{F}_{bN}})$, where $\hat{\rho}|_{\mathcal{F}_{bN}}$ denotes the restriction of $\hat{\rho}$ to $\mathcal{F}_{bN}$. Since $\hat{\rho}$ is positive definite, it is clear that $Z_{bN} = Z_{cN}$ if and only if $\mathcal{F}_{bN} = \mathcal{H}_N$.

A suitable operator is $\hat{\rho} \equiv e^{-\beta \hat{H}}$, where

$$\hat{H} \equiv \sum_k (k - \tfrac{\pi}{L}) :\hat{n}_k: . \tag{2.84}$$

Let us first compute

$$Z_{bN} \equiv \mathrm{Tr}_{bN}(e^{-\beta \hat{H}}|_{\mathcal{F}_{bN}}). \tag{2.85}$$

Recall that the state $|N\rangle \in \mathcal{H}_N$ has all $k > k_N$ levels empty and all $k \leq k_N$ levels occupied, where $k_N \equiv 2\pi N/L$. Hence,

$$\hat{H}\,|N\rangle = \tfrac{\pi}{L} N^2\,|N\rangle\,.$$

Also note that

$$\hat{H} b_q^\dagger = b_q^\dagger (\hat{H} + q),$$

which follow directly from the definitions (2.84) and (2.25). Therefore, each $|N; \{m_q\}\rangle$

is an eigenstate of $\hat{H}$:

$$\begin{aligned}\hat{H}\,|N;\{m_q\}\rangle &= \left(\prod_{q>0}\frac{(b_q^\dagger)^{m_q}}{\sqrt{m_q!}}\right)\left(\hat{H}+\sum_{q>0}q\,m_q\right)|N\rangle_0 \\ &= \left(\frac{\pi}{L}N^2+\sum_{q>0}q\,m_q\right)|N;\{m_q\}\rangle\,;\end{aligned}$$

the sum is over all $q\in(2\pi/L)\mathbb{Z}^+$. Consequently,

$$\begin{aligned}Z_{bN} &= \sum_{\{m_q\}} e^{-\beta(\frac{\pi}{L}N^2+\sum_{q>0}qm_q)} \\ &= e^{-(\beta\pi/L)N^2}\prod_{q>0}\left(\sum_{m=0}^{\infty}e^{-\beta qm}\right).\end{aligned}$$

Summing the geometric series, we obtain

$$Z_{bN} = w^{N^2}\prod_{n=1}^{\infty}(1-w^{2n})^{-1}, \tag{2.86}$$

where $w\equiv e^{-\beta\pi/L}$.

Next, we compute

$$Z_{cN}\equiv \mathrm{Tr}_N(e^{-\beta\hat{H}}). \tag{2.87}$$

As before, let $k_N\equiv 2\pi N/L$. Consider the fermion occupation number basis for $\mathcal{H}_N$. Each state in this basis is obtained from $|N\rangle$ by creating some number of particles and an equal number of holes. Call this number r. The particles must occupy levels $k_i^+ = k_N+2\pi a_i/L$, where $a_i\in\{1,2,\dots\}$. The holes must deplete levels $k_i = k_N - 2\pi b_i/L$, where $b_i\in\{0,1,2,\dots\}$. The state can be uniquely labeled by the pair of vectors $\mathbf{a}=(a_1,a_2,\dots,a_r)$ and $\mathbf{b}=(b_1,b_2,\dots,b_r)$. Clearly, one must have $a_i\neq a_j$ and $b_i\neq b_j$ for $i\neq j$. Explicitly,

the basis states are

$$|\mathbf{a},\mathbf{b}\rangle \equiv \prod_{i=1}^{r} c^{\dagger}_{k_N+2\pi a_i/L} \prod_{j=1}^{r} c_{k_N-2\pi b_j/L} \, |N\rangle \, .$$

It is easy to see that

$$\hat{H} \, |\mathbf{a},\mathbf{b}\rangle = \left(\tfrac{\pi}{L} N^2 + \tfrac{2\pi}{L} \sum_{i=1}^{r} (a_i + b_i) \right) |\mathbf{a},\mathbf{b}\rangle \, .$$

Therefore,

$$\begin{aligned} Z_{cN} &= \sum_{\mathbf{a},\mathbf{b}} e^{-(\beta\pi/L)[N^2 + 2\sum_{i=1}^{r}(a_i+b_i)]} \\ &= w^{N^2} \sum_{\mathbf{a},\mathbf{b}} w^{2\sum_{i=1}^{r}(a_i+b_i)}, \end{aligned}$$

where, as before, $w \equiv e^{-\beta\pi/L}$. We can write this as

$$Z_{cN} = w^{N^2} \sum_{\ell=0}^{\infty} \alpha(\ell) w^{2\ell}, \tag{2.88}$$

where $\alpha(\ell)$ counts the different ways of expressing the integer $\ell \geq 0$ as

$$\ell = \sum_{i=1}^{r} (a_i + b_i), \qquad \begin{array}{l} 1 \leq a_1 < a_2 < \cdots \, , \\ 0 \leq b_1 < b_2 < \cdots \, , \end{array} \tag{2.89}$$

for some $r \geq 0$.

Miraculously, the sum in (2.88) equals the product in (2.86). This can be proven using some elegant combinatorics, which I now describe.

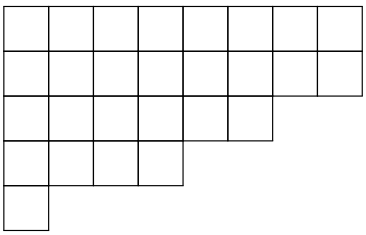
First, some terminology. The *partition number* $p(\ell)$ counts the distinct partitions of

ℓ, where a *partition* is a way of writing ℓ as a sum of positive integers,

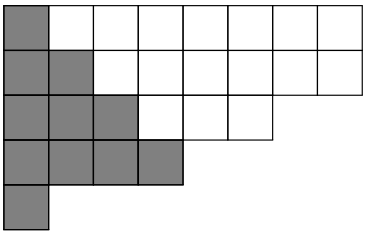
$$\ell = \sum_{j=1}^{\nu} d_j, \quad 1 \leq d_1 \leq d_2 \leq \cdots \tag{2.90}$$

for some $\nu \geq 1$. By convention, $p(0) \equiv 1$.

A simple graphical argument, adapted from Ref. [55], shows that $\alpha(\ell) = p(\ell)$. The argument is as follows. Every partition (2.90) of ℓ is uniquely associated to a *Young diagram*, consisting of ℓ boxes arranged in ν rows of length d_j. For instance, the diagram



represents the partition $27 = 8 + 8 + 6 + 4 + 1$. Now decorate the diagram by coloring all boxes on or below the main diagonal:



The colored set consists of r columns, each with a_i boxes. Because the diagonal is included, $a_i \geq 1$. Because of the slope of the top edge, $a_i \neq a_j$ for $i \neq j$.

The uncolored set consists of r rows, each with b_i boxes. Because the diagonal is excluded, $b_i \geq 0$. Because of the slope of the left edge, $b_i \neq b_j$ for $i \neq j$.

Thus, the colored diagram gives a decomposition of ℓ of the form (2.89). Conversely, given a decomposition (2.89) of ℓ, we can construct its associated colored diagram and then forget the coloring to obtain a partition (2.90) of ℓ. This procedure establishes a

one-to-one correspondence between decompositions of ℓ of the form (2.89) and those of the form (2.90). Thus, $\alpha(\ell) = p(\ell)$, as claimed.

The upshot of this combinatorial interlude is that

$$Z_{cN} = w^{N^2} \sum_{\ell=0}^{\infty} p(\ell) w^{2\ell},$$

where $p(\ell)$ counts the number of distinct partitions of ℓ, with $p(0) \equiv 1$.

Now, every partition of ℓ can be uniquely associated to a vector $\mathbf{m} = (m_1, m_2, \cdots)$, where $m_n \in \{0, 1, 2, \cdots\}$ counts how many times n appears in the partition. For instance, $5 = 2 + 1 + 1 + 1$ corresponds to $(3, 1, 0, \dots)$. Conversely, every such vector $\mathbf{m}$ labels a unique partition of the integer $\ell = \sum_n m_n n$. It follows that

$$\begin{aligned} \sum_{\ell=0}^{\infty} p(\ell) x^{\ell} &= \sum_{\mathbf{m}} x^{\sum_{n=1}^{\infty} m_n n} \\ &= \prod_{n=1}^{\infty} \Big(\sum_{m=0}^{\infty} x^{nm} \Big). \end{aligned}$$

Summing the geometric series, and taking $x = w^2$, we finally obtain

$$Z_{cN} = w^{N^2} \prod_{n=1}^{\infty} (1 - w^{2n})^{-1} = Z_{bN}. \qquad \square$$

Note that, in the literature, equality of Z_{cN} and Z_{bN} is typically established by considering the corresponding grand canonical partition functions Z_c and Z_b, and using the *Jacobi triple product identity* to show that these are equal for any value of the chemical potential μ. The proof given here is longer because it is self contained (in fact, it gives the Jacobi triple product identity as a corollary).

2.B Properties of the map $\varphi : V \mapsto M$ from interaction matrices to scaling dimension matrices

Let $\mathscr{P}_N$ denote the set of real symmetric positive definite $N \times N$ matrices, and let $\mathscr{M}_N \equiv SO(N,N) \cap \mathscr{P}_{2N}$. The map φ from interaction matrices $V \in \mathscr{P}_{2N}$ to "scaling dimension matrices" $M \in \mathscr{M}_N$ is defined as

$$\begin{aligned} &\varphi : \mathscr{P}_{2N} \to \mathscr{M}_N, \\ &\varphi : V \mapsto A^T A, \end{aligned} \tag{2.91}$$

where $A \in SO(N,N)$ and $AVA^T = D$ is diagonal.

2.B.1 General properties

The first and second lemmas below show that φ is well-defined. The third, fourth and fifth lemmas characterize the inverse images $\varphi^{-1}(M)$, and yield the parameterization of V matrices used in the main text of the chapter. All of these results are elementary, but we record them here for completeness.

Lemma 1. If $V \in \mathscr{P}_{2N}$, then there exists $A \in SO(N,N)$ such that $AVA^T = D$ is diagonal and positive definite.

Proof. (by construction). Let $V^{1/2}$ denote the unique symmetric positive definite square root of V, so that

$$V^{1/2} = (V^{1/2})^T, \qquad V^{1/2} > 0, \qquad (V^{1/2})^2 = V, \tag{2.92}$$

and let $V^{-1/2} \equiv (V^{1/2})^{-1} = (V^{-1})^{1/2}$. The matrix $V^{-1/2} K V^{-1/2}$ (where $K = -\mathbb{I}_N \oplus \mathbb{I}_N$) is symmetric, and can therefore be diagonalized by some $Q \in SO(2N)$. Furthermore,

Sylvester's theorem of inertia [56] ensures that $V^{-1/2}KV^{-1/2}$ has N positive and N negative eigenvalues. Thus, Q can be chosen so that

$$QV^{-1/2}KV^{-1/2}Q^T = D^{-1}K, \tag{2.93}$$

where D is diagonal and positive definite (this can be arranged by re-ordering the rows of Q and, if necessary, multiplying one row by -1 to maintain $\det Q = +1$). Taking the determinant of both sides of Eq. (2.93), we have $\det V = \det D$. Therefore $A \equiv D^{1/2}QV^{-1/2}$ satisfies the desired properties: $AKA^T = K$, $\det A = 1$, and $AVA^T = D$. □

Lemma 2. If $A_i \in SO(N,N)$ and $A_iVA_i^T = D_i$ is diagonal for $i = 1, 2$, then $A_1^TA_1 = A_2^TA_2$.

Proof. Note that every $A \in SO(N,N)$ is invertible, with $A^{-1} = KA^TK$, where $K = -\mathbb{I}_N \oplus \mathbb{I}_N$ (this follows immediately from the defining condition for the group, $AKA^T = K$, and the fact that $K^2 = \mathbb{I}_{2N}$.). Thus, to prove the lemma it suffices to prove the equivalent statement that $D_2 = AD_1A^T$ implies $A^TA = \mathbb{I}_{2N}$, where $A \equiv A_2A_1^{-1} \in SO(N,N)$.

Using $A^T = KA^{-1}K$, the equation $D_2 = AD_1A^T$ can be rewritten as

$$D_2K = A(D_1K)A^{-1}. \tag{2.94}$$

Thus, the matrices D_2K and D_1K are similar. But similar diagonal matrices can differ only by a permutation of the diagonal elements. Taking account of the sign structure due to K, one must have $D_2 = PD_1P^{-1}$, with $P = P^{(1)} \oplus P^{(2)}$, where the $P^{(i)}$ are $N \times N$ permutation matrices. Defining $B \equiv P^{-1}A$, Eq. (2.94) reduces to

$$D_1K = B(D_1K)B^{-1}. \tag{2.95}$$

This implies that B preserves each eigenspace of D_1K. Hence it must (at the very least) have the block form $B = B^{(1)} \oplus B^{(2)}$, where the $B^{(i)}$ are $N \times N$ matrices. Since $A = PB$, and P has a similar block structure, one must also have $A = A^{(1)} \oplus A^{(2)}$, where the $A^{(i)}$ are $N \times N$ matrices. Then the condition $AKA^T = K$ implies $A^{(i)} \in O(N)$, so that $A^TA = \mathbb{I}_{2N}$. □

Lemma 3. $V \in \varphi^{-1}(M)$ if and only if

$$V = M^{-1/2} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} M^{-1/2} \tag{2.96}$$

for some $X, Y \in \mathscr{P}_N$, where $M^{-1/2}$ denotes the unique positive definite square root of M^{-1}.

Proof. ($\Longrightarrow$): Assume that $\varphi(V) = M$. Every $M \in \mathscr{M}_N = SO(N,N) \cap \mathscr{P}_{2N}$ has a unique positive definite symmetric square root $M^{1/2} \in \mathscr{M}_N$. Furthermore, any matrix A that satisfies $A^TA = M$ can be written as $A = RM^{1/2}$, for a suitable $R \in O(2N)$. If $A \in SO(N,N)$, then we must have $R \in O(2N) \cap SO(N,N) = O(N) \times O(N)/\mathbb{Z}_2$. Therefore, $\varphi(V) = M$ implies that $(RM^{1/2})V(RM^{1/2})^T = D$ for some diagonal positive definite D and some $R \in O(N) \times O(N)/\mathbb{Z}_2$; equivalently, $V = M^{-1/2}R^TDRM^{-1/2}$, which is of the form indicated.

($\Longleftarrow$): Assume that V has the form indicated. Then there exist $R_1, R_2 \in SO(N)$ that diagonalize X, Y respectively. Let $A \equiv [R_1 \oplus R_2]M^{1/2}$. Then $A \in SO(N,N)$, $AVA^T = D$ is diagonal, and $A^TA = M$. Thus $\varphi(V) = M$. □

The scaling dimension matrix $M \in \mathscr{M}_N$ can, by the hyperbolic CS decomposition,

be written as [Eq. (2.9)]:

$$M = \begin{bmatrix} Q_1^T & 0 \\ 0 & Q_2^T \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \tag{2.97}$$

where $Q_1, Q_2 \in O(N)$, $C = \mathrm{diag}(\cosh\lambda_i)$, and $S = \mathrm{diag}(\sinh\lambda_i)$, with $\lambda_i \geq 0$, $i = 1, 2, \ldots, N$. Note that we can equivalently take $Q_1, Q_2 \in SO(N)$ if we allow one of the λ_i's to be negative, as done in the main text of the chapter.

Lemma 4. $V \in \varphi^{-1}(M)$ if and only if

$$V = \begin{bmatrix} X + F\,Y\,F^T & XF + F\,Y \\ F^T X + Y F^T & F^T X F + Y \end{bmatrix}, \tag{2.98}$$

for some $X, Y \in \mathscr{P}_N$, where

$$F \equiv Q_1^T\,\mathrm{diag}(\tanh(\lambda_i/2))\,Q_2. \tag{2.99}$$

Proof. According to Lemma 3, $V \in \varphi^{-1}(M)$ iff $V = M^{-1/2}[\tilde{X} \oplus \tilde{Y}]M^{-1/2}$ for some $\tilde{X}, \tilde{Y} \in \mathscr{P}_N$. From Eq. (2.97), it follows that

$$M^{-1/2} = \begin{bmatrix} \tilde{C}_1 & \tilde{S} \\ \tilde{S}^T & \tilde{C}_2 \end{bmatrix}, \tag{2.100}$$

where $\tilde{C}_\nu = Q_\nu^T\,\mathrm{diag}(\cosh(\lambda_i/2))\,Q_\nu$ $(\nu = 1, 2)$ and $\tilde{S} = Q_1^T\,\mathrm{diag}(\sinh(\lambda_i/2))\,Q_2$. Thus,

$$V = \begin{bmatrix} \tilde{C}_1\tilde{X}\tilde{C}_1 + \tilde{S}\,\tilde{Y}\tilde{S}^T & \tilde{C}_1\tilde{X}\tilde{S} + \tilde{S}\,\tilde{Y}\tilde{C}_2 \\ \tilde{S}^T\tilde{X}\tilde{C}_1 + \tilde{C}_2\tilde{Y}\tilde{S}^T & \tilde{S}^T\tilde{X}\tilde{S} + \tilde{C}_2\tilde{Y}\tilde{C}_2 \end{bmatrix}. \tag{2.101}$$

Now define $X \equiv \tilde{C}_1 \tilde{X} \tilde{C}_1$ and $Y \equiv \tilde{C}_2 \tilde{Y} \tilde{C}_2$. These maps from $\tilde{X}, \tilde{Y} \in \mathscr{P}_N$ to $X, Y \in \mathscr{P}_N$ are bijections, because $\tilde{C}_\nu \in \mathscr{P}_N$. Noting that $\tilde{S}\tilde{C}_2^{-1} = \tilde{C}_1^{-1}\tilde{S} = F$, we obtain the claimed result, Eq. (2.98). □

We now write the interaction matrix in block form as

$$V = \begin{bmatrix} V_{RR} & V_{RL} \\ V_{LR} & V_{LL} \end{bmatrix}, \tag{2.102}$$

where $V_{RR}, V_{LL} \in \mathscr{P}_N$ and $V_{LR} = V_{RL}^T$.

Lemma 5. $V \in \varphi^{-1}(M)$ if and only if

$$V_{RR} \in \mathscr{P}_N, \tag{2.103a}$$

$$V_{RR} - V_{RL}\, V_{LL}^{-1}\, V_{RL}^T \in \mathscr{P}_N, \tag{2.103b}$$

$$V_{RL} + F\, V_{RL}^T F - V_{RR} F - F V_{LL} = 0, \tag{2.103c}$$

where F is defined in Eq. (2.99) above.

Proof. Equations (2.103a) and (2.103b) are the Schur complement condition for positive definiteness of a symmetric matrix [57]; $V \in \mathscr{P}_{2N}$ iff these equations hold. By Lemma 3, $V \in \varphi^{-1}(M)$ iff $M^{1/2} V M^{1/2} = \tilde{X} \oplus \tilde{Y}$ for some $\tilde{X}, \tilde{Y} \in \mathscr{P}_N$. In the notation of Eq. (2.100), one has

$$M^{1/2} = \begin{bmatrix} \tilde{C}_1 & -\tilde{S} \\ -\tilde{S}^T & \tilde{C}_2 \end{bmatrix}. \tag{2.104}$$

Conjugating the equation $M^{1/2} V M^{1/2} = \tilde{X} \oplus \tilde{Y}$ by the positive definite matrix $\tilde{C}_1^{-1} \oplus \tilde{C}_2^{-1}$, it becomes

$$\begin{bmatrix} \mathbb{I}_N & -F \\ -F^T & \mathbb{I}_N \end{bmatrix} \begin{bmatrix} V_{RR} & V_{RL} \\ V_{RL}^T & V_{LL} \end{bmatrix} \begin{bmatrix} \mathbb{I}_N & -F \\ -F^T & \mathbb{I}_N \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \tag{2.105}$$

where $X \equiv \tilde{C}_1^{-1}\tilde{X}\tilde{C}_1^{-1}$ and $Y \equiv \tilde{C}_2^{-1}\tilde{Y}\tilde{C}_2^{-1}$. These maps from $\tilde{X}, \tilde{Y} \in \mathscr{P}_N$ to $X, Y \in \mathscr{P}_N$ are bijections. Therefore, $V \in \varphi^{-1}(M)$ iff Eq. (2.105) holds for some $X, Y \in \mathscr{P}_N$. The off-diagonal block of Eq. (2.105) yields Eq. (2.103c). The diagonal blocks of Eq. (2.105) are automatically satisfied, because the matrix on the left side is positive definite (it was constructed by conjugating $V \in \mathscr{P}_{2N}$ by other matrices in $\mathscr{P}_{2N}$). □

2.B.2 Intuition for the parameterization (2.7) of V, and its illustration in the case of $N = 1$ channel

To gain some intuition for the parametrization (2.7) of V furnished by Lemma 3,

$$V = M^{-1/2} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} M^{-1/2}, \tag{2.106}$$

first consider the limit $M = \mathbb{I}_{2N}$. Then the interactions encoded in X simply mix the right-movers amongst themselves, leading to new modes with renormalized velocities, while Y does the same with the left-movers. All scaling dimensions (being determined by M alone) remain equal to their values at the free fixed point. Next consider a different limit, $X = Y = \mathbb{I}_N$. Now $V = M^{-1}$ is itself in $SO(N, N)$, and its inverse gives the scaling dimensions directly. To connect these two limits, consider the Euclidean space of all symmetric $N \times N$ matrices, $\mathbb{R}^{N(N+1)/2}$. The positive definite matrices occupy the interior of a convex cone $\mathscr{P}_N \subset \mathbb{R}^{N(N+1)/2}$. The space of interaction matrices is $\mathscr{V} = \mathscr{P}_{2N}$. According to Eq. (2.106), $\mathscr{V}$ should be regarded as a bundle of lower-dimensional convex cones $\mathscr{P}_N \times \mathscr{P}_N$ (parameterized by X, Y) as fibers over the N^2-dimensional submanifold $\mathscr{M}_N \equiv SO(N, N) \cap \mathscr{P}_{2N}$ (parameterized by M). The scaling dimensions $\Delta(\mathbf{m})$, regarded as functions from $\mathscr{V} \to \mathbb{R}$, are then constant on each fiber. Each Luttinger liquid phase, defined in terms of its instabilities (or lack thereof), thus extends over the interior of a

solid cone emanating from the vertex of $\mathscr{V}$.

The $N = 1$ channel case again provides a nice illustration of these general ideas. The set $\mathscr{V} = \mathscr{P}_2$ consists of all 2×2 matrices

$$V = \begin{bmatrix} \alpha + \beta & \gamma \\ \gamma & \alpha - \beta \end{bmatrix} \tag{2.107}$$

with $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ and $\alpha > (\beta^2 + \gamma^2)^{1/2}$. This is quite clearly the interior of a circular cone in $\mathbb{R}^3$. The parameterization (2.106), with $M = e^{-\lambda\sigma_x} \in \mathscr{M}_1$, corresponds to

$$\alpha = \tfrac{1}{2}(x + y)\cosh\lambda, \tag{2.108a}$$

$$\beta = \tfrac{1}{2}(x - y), \tag{2.108b}$$

$$\gamma = \tfrac{1}{2}(x + y)\sinh\lambda, \tag{2.108c}$$

where $x, y > 0$. For fixed λ, the image of the resulting map $(x, y) \mapsto (\alpha, \beta, \gamma)$ is a slice of the cone $\mathscr{P}_2$, in a plane parallel to the β-axis and at an angle $\arctan(\tanh\lambda)$ from the α-axis. Each such slice is the interior of a cone in $\mathbb{R}^2$, with an opening angle that decreases with increasing $|\lambda|$. In terms of stability with respect to clean SC and CDW perturbations, $\mathscr{V} = \mathscr{P}_2$ splits into four regions: $\lambda < -\log 2$ ($\Delta_{\mathrm{SC}} < 2 < \Delta_{\mathrm{CDW}}$), $-\log 2 < \lambda < 0$ ($\Delta_{\mathrm{SC}} < \Delta_{\mathrm{CDW}} < 2$), $0 < \lambda < \log 2$ ($\Delta_{\mathrm{CDW}} < \Delta_{\mathrm{SC}} < 2$), and $\lambda > \log 2$ ($\Delta_{\mathrm{CDW}} < 2 < \Delta_{\mathrm{SC}}$). These regions indeed take the form of solid cones emanating from the vertex of $\mathscr{P}_2$, as illustrated in Figure 2.B.1.

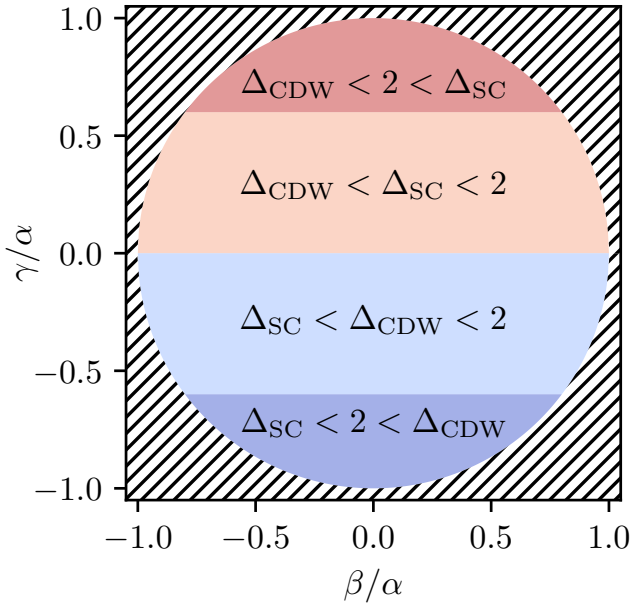


Figure 2.B.1: Phase diagram for the $N = 1$ channel Luttinger liquid, in terms of stability with respect to global SC and CDW perturbations. The hatched region is unphysical (V is not positive definite for these parameter values).

2.C Restrictions on the interaction matrix V imposed by symmetries

Consider the effective theory of a system that is invariant under one or more symmetries that interchange right and left-movers, such as time-reversal ($\mathcal{T}$), and/or spatial inversion ($\mathcal{P}$). On general grounds, $\mathcal{T}$ must be implemented in the effective theory by an anti-unitary operator that squares to -1 when acting on fermionic operators. Spatial inversion $\mathcal{P}$ must be implemented by a unitary operator that squares to $+1$.

2.C.1 $\mathcal{T}$ symmetry but no $\mathcal{P}$ symmetry

First consider the case in which the system has time-reversal symmetry but no inversion symmetry. The chiral boson fields ϕ_I can be chosen to transform as follows under

time-reversal (here the index $I = 1, 2, \ldots N$):

$$\mathcal{T} : \begin{cases} \phi_I(x,t) & \longrightarrow \quad \phi_{I+N}(x,-t), \\ \phi_{I+N}(x,t) & \longrightarrow \quad \pi + \phi_I(x,-t). \end{cases} \tag{2.109}$$

In addition, $\mathcal{T}$ complex conjugates $i \to -i$. Then, $\mathcal{T}$ correctly interchanges right- and left-movers, and squares to -1 when acting on the fermion fields $\psi_I \propto e^{\mp i\phi_I}\gamma_I$. (Alternatively, one could omit the π in Eq. (2.109) and have the Klein factors γ_I transform nontrivially.) In this representation, $\mathcal{T}$ symmetry imposes that the interaction matrix V must satisfy

$$\Sigma V \Sigma = V, \tag{2.110}$$

where $\Sigma \equiv \sigma_x \otimes \mathbb{I}_N$ and σ_x is the usual Pauli matrix. Thus, V must have the block form

$$V = \begin{bmatrix} V_1 & V_2 \\ V_2 & V_1 \end{bmatrix}, \tag{2.111}$$

where $V_i = V_i^T$. Conversely, any $2N \times 2N$ positive definite matrix V of this form can serve as the interaction matrix of a $\mathcal{T}$-symmetric N-channel Luttinger liquid.

2.C.2 $\mathcal{P}$ symmetry but no $\mathcal{T}$ symmetry

Next consider the case in which the system has inversion symmetry but no time-reversal symmetry. The chiral boson fields ϕ_I can be chosen to transform as follows

under spatial inversion (again the index $I = 1, 2, \ldots N$):

$$\mathcal{P} : \begin{cases} \phi_I(x,t) & \longrightarrow \quad -\phi_{I+N}(-x,t), \\ \phi_{I+N}(x,t) & \longrightarrow \quad -\phi_I(-x,t). \end{cases} \tag{2.112}$$

Then, $\mathcal{P}$ correctly interchanges right- and left-movers, and squares to $+1$ when acting on the fermion fields ψ_I. In this representation, $\mathcal{P}$ symmetry imposes that the interaction matrix V must satisfy Eq. (2.110), and hence that it must have the block form (2.111). Conversely, any $2N \times 2N$ positive definite matrix V of the form (2.111) can serve as the interaction matrix of a $\mathcal{P}$-symmetric N-channel Luttinger liquid.

2.C.3 Both $\mathcal{T}$ and $\mathcal{P}$ symmetry

Finally, consider the case in which the system has both time-reversal symmetry and inversion symmetry. The transformation laws (2.109) and (2.112) correspond to *different* representations of the fermion fields in terms of bosons, and hence cannot be used simultaneously. As is well known, symmetry with respect to $\mathcal{PT}$ enforces a twofold degeneracy of the bands at each point in k-space. Therefore, the low-energy effective theory now involves $4N$ chiral spinless Dirac fermions ψ_I, where $I = 1, 2, \ldots, 2N$ labels right-movers and $I = 2N + 1, \ldots, 4N$ labels left-movers. The corresponding chiral boson fields ϕ_I can be chosen to transform as follows under time-reversal and spatial inversion (here the

index $I = 1, 2, \ldots N$):

$$\mathcal{T}: \begin{cases} \phi_I(x,t) & \longrightarrow \quad \phi_{I+2N}(x,-t), \\ \phi_{I+N}(x,t) & \longrightarrow \quad \phi_{I+3N}(x,-t), \\ \phi_{I+2N}(x,t) & \longrightarrow \quad \pi + \phi_I(x,-t), \\ \phi_{I+3N}(x,t) & \longrightarrow \quad \pi + \phi_{I+N}(x,-t), \end{cases} \tag{2.113a}$$

$$\mathcal{P}: \begin{cases} \phi_I(x,t) & \longrightarrow \quad -\phi_{I+3N}(-x,t), \\ \phi_{I+N}(x,t) & \longrightarrow \quad -\phi_{I+2N}(-x,t), \\ \phi_{I+2N}(x,t) & \longrightarrow \quad -\phi_{I+N}(-x,t), \\ \phi_{I+3N}(x,t) & \longrightarrow \quad -\phi_I(-x,t). \end{cases} \tag{2.113b}$$

Now, $\mathcal{T}$ symmetry and $\mathcal{P}$ symmetry respectively impose that the interaction matrix V must satisfy

$$(\mathcal{T}) \qquad \Sigma_1 V \Sigma_1 = V, \tag{2.114a}$$

$$(\mathcal{P}) \qquad \Sigma_2 V \Sigma_2 = V, \tag{2.114b}$$

where $\Sigma_1 \equiv \sigma_x \otimes \mathbb{I}_2 \otimes \mathbb{I}_N$ and $\Sigma_2 \equiv \sigma_x \otimes \sigma_x \otimes \mathbb{I}_N$. Thus, V must have the block form

$$V = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ V_2 & V_1 & V_4 & V_3 \\ V_3 & V_4 & V_1 & V_2 \\ V_4 & V_3 & V_2 & V_1 \end{bmatrix}, \tag{2.115}$$

where $V_i = V_i^T$. Conversely, any $4N \times 4N$ positive definite matrix V of this form can serve as the interaction matrix of a $\mathcal{T}$- and $\mathcal{P}$-symmetric $2N$-channel Luttinger liquid.

2.D Restrictions on the scaling dimension matrix M imposed by symmetries

Let P be any permutation matrix that satisfies $P^2 = \mathbb{I}_{2N}$ and $PKP = -K$. Let $\mathscr{S}_P \equiv \{A \in \mathbb{R}^{2N\times 2N} \mid PAP = A\}$. Then, the following results hold:

Lemma 6. If $V \in \mathscr{P}_{2N} \cap \mathscr{S}_P$ and $M = \varphi(V)$, then $M \in \mathscr{S}_P$.

Proof. Pick some $A \in SO(N,N)$ such that $AVA^T = D$ is diagonal, and define $B \equiv PAP$. It is easy to check that $BKB^T = K$ and $\det B = 1$, so $B \in SO(N,N)$. Also, $BVB^T = PDP$ is diagonal, since P is a permutation matrix. Thus, $M = B^TB = PA^TA\,P = PMP$. □

Lemma 7. Assume $M \in \mathscr{M}_N \cap \mathscr{S}_P$. Then $V \in \varphi^{-1}(M) \cap \mathscr{S}_P$ if and only if $V = M^{-1/2}ZM^{-1/2}$ for some $Z \in (\mathscr{P}_N \times \mathscr{P}_N) \cap \mathscr{S}_P$.

Proof. Lemma 3 implies that $V \in \varphi^{-1}(M)$ iff V has the specified form with $Z \in \mathscr{P}_N \times \mathscr{P}_N$. Note that $PM^{\pm 1/2} = M^{\pm 1/2}P$. Thus V and Z are conjugates of one another by an invertible matrix that commutes with P. It follows that $PVP = V$ iff $PZP = Z$. □

Taking $Z = \mathbb{I}_{2N}$ shows that $\varphi^{-1}(M) \cap \mathscr{S}_P$ is nonempty for any $M \in \mathscr{M}_N \cap \mathscr{S}_P$. Thus, the set of interaction matrices V that satisfy the constraint $PVP = V$ maps (under φ) *onto* the set of scaling dimension matrices M that satisfy the constraint $PMP = M$. The constraints on V derived in Section 2.C are precisely of the form $PVP = V$ (with $P = \Sigma$, Σ_1 or Σ_2). Hence the allowed scaling dimension matrices M for a system with time-reversal ($\mathcal{T}$) and/or spatial inversion ($\mathcal{P}$) symmetry may be characterized as follows.

2.D.1 $\mathcal{T}$ symmetry or $\mathcal{P}$ symmetry, but not both

First consider the case in which the system has either time-reversal symmetry or inversion symmetry, but not both. We choose the $2N$ chiral bosons ϕ_I to transform according to Eq. (2.109) in the former case, and according to Eq. (2.112) in the latter. Then, in either case, the scaling dimension matrix must satisfy

$$\Sigma M \Sigma = M, \tag{2.116}$$

where $\Sigma \equiv \sigma_x \otimes \mathbb{I}_N$ (with no further constraints). Imposing this constraint on the hyperbolic CS decomposition of M, Eq. (2.97), yields the conditions

$$QCQ^T = C, \tag{2.117a}$$

$$QSQ = S, \tag{2.117b}$$

where $Q \equiv Q_1 Q_2^T$. Assume, without loss of generality, that there are m distinct rapidities λ_α with multiplicities N_α (satisfying $N_1 + \cdots + N_m = N$), ordered so that $C = \cosh L$, $S = \sinh L$, and $L = \lambda_1 \mathbb{I}_{N_1} \oplus \lambda_2 \mathbb{I}_{N_2} \oplus \cdots \oplus \lambda_m \mathbb{I}_{N_m}$. Then the conditions above require that

$$Q = R_1 \oplus R_2 \oplus \cdots \oplus R_m, \tag{2.118}$$

where $R_\alpha \in O(N_\alpha)$ and $R_\alpha = R_\alpha^T$ (i.e. each R_α is an $N_\alpha \times N_\alpha$ reflection matrix). In the special case in which all rapidities are equal, one has

$$M = \begin{bmatrix} \mathbb{I}_N \cosh\lambda & -R \sinh\lambda \\ -R \sinh\lambda & \mathbb{I}_N \cosh\lambda \end{bmatrix}, \tag{2.119}$$

where $R \in O(N)$ and $R = R^T$.

2.D.2 Both $\mathcal{T}$ and $\mathcal{P}$ symmetry

In the case that the system has both time-reversal and inversion symmetry, we choose the $4N$ chiral bosons ϕ_I to transform according to Eq. (2.113). Then, the scaling dimension matrix must satisfy

$$(\mathcal{T}) \qquad \Sigma_1 M \Sigma_1 = M, \tag{2.120a}$$

$$(\mathcal{P}) \qquad \Sigma_2 M \Sigma_2 = M, \tag{2.120b}$$

where $\Sigma_1 \equiv \sigma_x \otimes \mathbb{I}_2 \otimes \mathbb{I}_N$ and $\Sigma_2 \equiv \sigma_x \otimes \sigma_x \otimes \mathbb{I}_N$ (with no further independent constraints). We can again impose these constraints on the hyperbolic CS decomposition of M, Eq. (2.97), to obtain an explicit parameterization of all scaling dimension matrices that are consistent with both $\mathcal{T}$ and $\mathcal{P}$ symmetry. However, the general parameterization is somewhat cumbersome, so we will omit it. In the special case in which all rapidities are equal, one finds

$$M = \begin{bmatrix} \mathbb{I}_{2N} \cosh\lambda & -R \sinh\lambda \\ -R \sinh\lambda & \mathbb{I}_{2N} \cosh\lambda \end{bmatrix}, \tag{2.121}$$

where $R \in O(2N)$ has the block form

$$R = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix}, \tag{2.122}$$

with $R_i = R_i^T$ (i.e. R is a $2N \times 2N$ reflection matrix with this particular block form).

2.E Construction of ∞-stable (absolutely ∞-stable) Luttinger liquid phases with $N \geq 23$ ($N \geq 52$), following Plamadeala *et al.* (2014)

In this section we review the approach introduced in Ref. [17] to construct ∞-stable and absolutely ∞-stable phases. Note that this construction, while elegant, is not necessarily optimal, so that ∞-stable or absolutely ∞-stable phases with fewer channels than the ones constructed below may exist.

Consider the field redefinition $\phi_I = W_{IJ}\tilde{\phi}_J$, where $W \in SL(2N, \mathbb{Z})$, the group of $2N \times 2N$ matrices with integer entries and determinant 1. This transformation permutes the integer vectors labelling vertex operators:

$$\mathcal{O}_{\mathbf{m}} = e^{im_I\phi_I} = e^{i\tilde{m}_I\tilde{\phi}_I}, \tag{2.123}$$

where $\tilde{\mathbf{m}} = W^T\mathbf{m} \in \mathbb{Z}^{2N}$. Meanwhile, the fixed-point action S [Eq. (2.2)], written in terms of the $\tilde{\phi}$ fields, reads

$$S = \frac{1}{4\pi}\int dt\, dx \left[\tilde{K}_{IJ}\partial_t\tilde{\phi}_I\partial_x\tilde{\phi}_J - \tilde{V}_{IJ}\partial_x\tilde{\phi}_I\partial_x\tilde{\phi}_J\right], \tag{2.124}$$

where $\tilde{K} = W^TKW$ and $\tilde{V} = W^TVW$. The conformal spin of the operator $\mathcal{O}_{\mathbf{m}} = e^{i\tilde{m}_I\tilde{\phi}_I}$ is easily seen to be

$$K(\mathbf{m}) = \tfrac{1}{2}\tilde{\mathbf{m}}^T\tilde{K}^{-1}\tilde{\mathbf{m}}. \tag{2.125}$$

Its scaling dimension is

$$\Delta(\mathbf{m}) = \tfrac{1}{2}\tilde{\mathbf{m}}^T\tilde{A}^T\tilde{A}\,\tilde{\mathbf{m}}, \tag{2.126}$$

where $\tilde{A} \in GL(2N, \mathbb{R})$ simultaneously diagonalizes $\tilde{K}$ and $\tilde{V}$; $\tilde{A}\tilde{K}\tilde{A}^T = K$, $\tilde{A}\tilde{V}\tilde{A}^T =$

$\mathrm{diag}(\tilde{u}_i)$.

Now assume that $\tilde{K}$ and $\tilde{V}$ are both block-diagonal:

$$\tilde{K} = -\tilde{K}_R \oplus \tilde{K}_L, \tag{2.127a}$$

$$\tilde{V} = \tilde{V}_R \oplus \tilde{V}_L, \tag{2.127b}$$

with $\tilde{K}_\nu$ and $\tilde{V}_\nu$ positive definite ($\nu = R/L$). Then we can take $\tilde{A} = \tilde{Q}_R \tilde{K}_R^{-1/2} \oplus \tilde{Q}_L \tilde{K}_L^{-1/2}$, where $\tilde{Q}_\nu \in SO(N)$ diagonalizes $\tilde{K}_\nu^{-1/2} \tilde{V}_\nu \tilde{K}_\nu^{-1/2}$; it follows that

$$\tilde{A}^T \tilde{A} = \tilde{K}_R^{-1} \oplus \tilde{K}_L^{-1}. \tag{2.128}$$

Thus, if $\tilde{K}$ and $\tilde{V}$ are both block-diagonal, the conformal spin and the scaling dimension are given by

$$K(\mathbf{m}) = \tilde{\Delta}^L_{\mathbf{m}} - \tilde{\Delta}^R_{\mathbf{m}}, \tag{2.129a}$$

$$\Delta(\mathbf{m}) = \tilde{\Delta}^L_{\mathbf{m}} + \tilde{\Delta}^R_{\mathbf{m}}, \tag{2.129b}$$

where

$$\tilde{\Delta}^R_{\mathbf{m}} \equiv \tfrac{1}{2} \tilde{\mathbf{m}}_R^T \tilde{K}_R^{-1} \tilde{\mathbf{m}}_R, \tag{2.130a}$$

$$\tilde{\Delta}^L_{\mathbf{m}} \equiv \tfrac{1}{2} \tilde{\mathbf{m}}_L^T \tilde{K}_L^{-1} \tilde{\mathbf{m}}_L \tag{2.130b}$$

are the *right* and *left scaling dimensions* of the operator. Here, we have split $\tilde{\mathbf{m}} = (\tilde{\mathbf{m}}_R, \tilde{\mathbf{m}}_L)$, with $\tilde{\mathbf{m}}_{R/L} \in \mathbb{Z}^N$.

By construction, $\tilde{K}_\nu$ ($\nu = R/L$) is a positive-definite integer matrix with determinant 1, and so the same is true of its inverse. Thus, $\tilde{K}_\nu^{-1}$ can be regarded as a *Gram matrix* of

an N-dimensional *unimodular integral lattice* $\tilde{\Gamma}_\nu$ with positive definite inner product (see Ref. [58] for the basic terminology of lattices). Concretely, one can take the columns of $\tilde{K}_\nu^{-1/2}$ to form a basis for $\tilde{\Gamma}_\nu$, so that the lattice vectors are $\tilde{\mathbf{v}}_\nu = \tilde{K}_\nu^{-1/2}\tilde{\mathbf{m}}_\nu$, $\tilde{\mathbf{m}}_\nu \in \mathbb{Z}^N$. The right/left scaling dimensions are equal to half the norm-squared of these lattice vectors,

$$(\tilde{\Delta}^R_{\mathbf{m}}, \tilde{\Delta}^L_{\mathbf{m}}) = (\tfrac{1}{2}|\tilde{\mathbf{v}}_R|^2, \tfrac{1}{2}|\tilde{\mathbf{v}}_L|^2). \tag{2.131}$$

Non-chiral operators have $\tilde{\Delta}^R_{\mathbf{m}} = \tilde{\Delta}^L_{\mathbf{m}}$ and hence $|\tilde{\mathbf{v}}_R| = |\tilde{\mathbf{v}}_L|$. Thus, if all nonzero lattice vectors in $\tilde{\Gamma}_R$ or in $\tilde{\Gamma}_L$ have norm-squared > 2 (i.e. if at least one of the two lattices is "non-root"), then the corresponding Luttinger liquid phase is ∞-stable. There are of course *chiral* operators for which only one of $\tilde{\mathbf{v}}_R$ or $\tilde{\mathbf{v}}_L$ is nonzero. Therefore, to obtain an *absolutely ∞-stable* phase, the lattices $\tilde{\Gamma}_{R/L}$ must both have minimum norm-squared > 4.

Unimodular integral lattices are *self-dual*, so $\tilde{K}_\nu$ is also a Gram matrix of $\tilde{\Gamma}_\nu$ (possibly with respect to a different basis). Therefore, $\tilde{K} = -\tilde{K}_R \oplus \tilde{K}_L$ is a Gram matrix of the unimodular integral lattice $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L$ of signature (N, N). Conjugating the Gram matrix $\tilde{K}$ by $W \in SL(2N, \mathbb{Z})$ corresponds merely to a basis change in this lattice. Thus, $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^{2N}$, the signature (N, N) lattice with Gram matrix $K = -\mathbb{I}_N \oplus \mathbb{I}_N$.

Let us summarize what we have accomplished so far. We have reduced the construction of ∞-stable (absolutely ∞-stable) phases of an N-channel Luttinger liquid to the identification of N-dimensional unimodular integral lattices $\tilde{\Gamma}_{R/L}$ with minimum norm-squared > 2 (> 4), subject to the constraint that $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^{2N}$ as a lattice of signature (N, N).

We now make use of two mathematical facts. The first fact is that there is a unique signature (N, N) unimodular lattice of each parity (even/odd), where an integral lattice is *even* if the norm-squared of all lattice vectors is an even integer, and is *odd* otherwise [58].

The lattice $\mathbb{Z}^{2N}$ with Gram matrix $K = -\mathbb{I}_N \oplus \mathbb{I}_N$ is clearly odd. Thus, $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^{2N}$ if (and only if) at least one of $\tilde{\Gamma}_{R/L}$ is odd.

The second fact is that, for any positive integer μ, there exists an N-dimensional positive definite unimodular lattice whose shortest nonzero vector has $|\mathbf{v}|^2 = \mu$ [59]. The required dimension N increases with μ; a theorem of Rains and Sloane [60] states that

$$\mu \leq 2\lfloor N/24 \rfloor + 2, \tag{2.132}$$

unless $N = 23$, in which case $\mu \leq 3$. Here $\lfloor x \rfloor$ denotes the integer part of x (i.e. x rounded down). Thus, to obtain $\mu = 3$ requires $N \geq 23$, and to obtain $\mu = 5$ requires $N \geq 48$.

In $N = 23$ dimensions, the *shorter Leech lattice* Λ_{23} has minimum norm-squared $\mu = 3$. Correspondingly, there is an ∞-stable 23-channel Luttinger liquid with $\tilde{\Gamma}_R = \tilde{\Gamma}_L = \Lambda_{23}$, dubbed the "symmetric shorter Leech liquid" [17]. The "symmetric" modifier distinguishes this phase from the "*asymmetric* shorter Leech liquid" which has $\tilde{\Gamma}_R = \Lambda_{23}$ and $\tilde{\Gamma}_L = \mathbb{Z}^{23}$, and which is also ∞-stable. These phases are discussed in more detail in Ref. [17], and the remarkable transport properties of the latter were analyzed in Ref. [61].

In $N = 52$ dimensions, the lattice G_{52} has $\mu = 5$ [62], and there is a corresponding *absolutely ∞-stable* 52-channel Luttinger liquid with $\tilde{\Gamma}_R = \tilde{\Gamma}_L = G_{52}$.

2.F Sphere packing bounds and the non-existence of absolutely ∞-stable phases for $N < 11$

The *sphere packing problem* [58] is to find the densest possible packing of non-overlapping spheres into $\mathbb{R}^n$. The *density* of a packing is the fraction of space that is contained inside the spheres. Given any lattice $\Gamma \subset \mathbb{R}^n$, we can obtain an associated

sphere packing by placing spheres at each lattice point, with radii equal to half the length of the shortest lattice vector. If Γ has a unit cell of volume Ω and shortest nonzero vector of length $2r$, then the density of the associated packing, d_Γ, equals the volume of an n-ball of radius r, divided by Ω:

$$d_\Gamma = \frac{1}{\Omega} \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)}. \tag{2.133}$$

Hence, upper bounds on the density of sphere packings in $\mathbb{R}^n$ yield upper bounds on the length, $2r$, of the shortest nonzero vector in Γ.

For an N-channel Luttinger liquid, the scaling dimensions of bosonic vertex operators are given by $\Delta(\mathbf{m}) = \frac{1}{2}\|A\mathbf{m}\|^2$, with $A \in SO(N,N)$ and $\mathbf{m} \in D_{2N}$, the "checkerboard lattice" $D_{2N} \equiv \{\mathbf{m} \in \mathbb{Z}^{2N} : |\mathbf{m}| \in 2\mathbb{Z}\}$. D_{2N} has unit cell volume $\Omega = 2$. Since $\det A = 1$, the same holds for the deformed lattice $\Gamma \equiv AD_{2N} \subset \mathbb{R}^{2N}$. Absolute ∞-stability requires every nonzero vector in Γ to have norm-squared > 4, which corresponds to $r > 1$. Thus, the corresponding sphere packing would have density

$$d_\Gamma > \frac{1}{2} \frac{\pi^N}{\Gamma(N+1)}. \tag{2.134}$$

For $N < 11$, this contradicts known upper bounds on the density of sphere packings [51]. Hence, absolutely ∞-stable phases cannot exist with $N < 11$ channels.

2.G Representation of 2-channel Luttinger liquid in terms of charge and spin fields

A single-spinful-channel quantum wire provides the simplest example of a 2-channel Luttinger liquid. Standard treatments of this problem [15] are usually phrased in terms

of non-chiral charge (c) and spin (s) fields, φ_c and φ_s, and their canonical conjugates, $\Pi_c \equiv \frac{1}{\pi}\partial_x \vartheta_c$ and $\Pi_s \equiv \frac{1}{\pi}\partial_x \vartheta_s$ respectively. These fields are related to the slowly varying parts of the charge density, ρ_c, and spin density, ρ_s, via $\rho_c = -\frac{1}{\pi}\partial_x \varphi_c$ and $\rho_s = -\frac{1}{\pi}\partial_x \varphi_s$ (we follow the normalization and sign conventions of Ref. [15]). Our analysis in this chapter, meanwhile, has been phrased in terms of chiral boson fields ϕ_I, which are related to the densities at each Fermi point via $\rho_I = \frac{1}{2\pi}\partial_x \phi_I$.

A rigorous reformulation of the bosonic effective theory in terms of charge and spin fields is possible when the system is invariant under spin rotations about some axis $\hat{\mathbf{n}}$. (Note that such a symmetry, by itself, imposes no constraints on the interaction matrix V.) Then, the component of the spin along $\hat{\mathbf{n}}$ is a good quantum number, and it labels the different Fermi points. One possibility for this labelling is

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (\phi_{R\uparrow}, \phi_{R\downarrow}, \phi_{L\downarrow}, \phi_{L\uparrow}), \tag{2.135}$$

where R/L distinguishes right movers from left movers, and $\uparrow$ / $\downarrow$ denotes the spin component along $\hat{\mathbf{n}}$. With the choice (2.135), the fields transform under time-reversal $\mathcal{T}$ according to Eq. (2.109).

Another possibility is to take

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (\phi_{R\uparrow}, \phi_{R\downarrow}, \phi_{L\uparrow}, \phi_{L\downarrow}). \tag{2.136}$$

With the choice (2.136), the fields transform under spatial inversion $\mathcal{P}$ according to Eq. (2.112).

In either case, one has

$$\begin{bmatrix} \varphi_c \\ \vartheta_c \\ \varphi_s \\ \vartheta_s \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{R\uparrow} \\ \phi_{R\downarrow} \\ \phi_{L\uparrow} \\ \phi_{L\downarrow} \end{bmatrix}, \tag{2.137}$$

and the inverse relation

$$\begin{bmatrix} \phi_{R\uparrow} \\ \phi_{R\downarrow} \\ \phi_{L\uparrow} \\ \phi_{L\downarrow} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_c \\ \vartheta_c \\ \varphi_s \\ \vartheta_s \end{bmatrix}. \tag{2.138}$$

Then, $\rho_s = -\frac{1}{\pi}\partial_x\varphi_s$ is indeed the slowly varying part of the density of excess spin in the $\hat{\mathbf{n}}$-direction.

If spin rotation symmetry is completely broken, on the other hand, one cannot easily reformulate the bosonic effective theory in terms of charge and spin fields. One can of course still use the above formulae to define non-chiral fields φ_s and ϑ_s as linear combinations of the ϕ_I, but in general these non-chiral fields will have nothing to do with the physical spin.

Now consider a 2-channel Luttinger liquid with effective action specified by Eqs. (2.2) and (2.12):

$$S = \frac{1}{4\pi} \int dt\, dx \left[K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J \right], \tag{2.139}$$

where $K = \mathrm{diag}(-\mathbb{I}_2, \mathbb{I}_2)$, and

$$V = \left[\begin{array}{cc|cc} v_+ & w & c_+ & c_0 \\ w & v_- & c_0 & c_- \\ \hline c_+ & c_0 & v_+ & w \\ c_0 & c_- & w & v_- \end{array}\right]. \tag{2.140}$$

To shorten subsequent expressions, let

$$v_\pm \equiv v \pm u, \tag{2.141a}$$

$$c_\pm \equiv c \pm b. \tag{2.141b}$$

As discussed earlier, the V matrix (2.140) describes a system that has either time-reversal ($\mathcal{T}$) symmetry or spatial inversion ($\mathcal{P}$) symmetry, but not both. Assuming that the system also has spin-rotation symmetry about some axis $\hat{\mathbf{n}}$ (as mentioned above, this assumption does not constrain V at all), one can use Eqs. (2.135–2.138) to write down the corresponding effective Hamiltonian H in terms of charge and spin fields.

In the case of $\mathcal{T}$ symmetry, we use Eqs. (2.135) and (2.138). The result is

$$H = \frac{1}{2\pi}\int dx \Big[v_c K_c (\partial_x \vartheta_c)^2 + \frac{v_c}{K_c}(\partial_x \varphi_c)^2 + v_s K_s (\partial_x \vartheta_s)^2 + \frac{v_s}{K_s}(\partial_x \varphi_s)^2 + d_+ \partial_x \vartheta_c \partial_x \varphi_s + d_- \partial_x \vartheta_s \partial_x \varphi_c \Big], \tag{2.142}$$

where

$$v_{c(s)} \equiv [(v \pm w)^2 - (c \pm c_0)^2]^{1/2}, \tag{2.143a}$$

$$K_{c(s)} \equiv \left[\frac{v \pm w \mp c - c_0}{v \pm w \pm c + c_0}\right]^{1/2}, \tag{2.143b}$$

$$d_\pm \equiv -2u \pm 2b. \tag{2.143c}$$

Using $\Pi_c \equiv \frac{1}{\pi}\partial_x \vartheta_c$ and $\Pi_s \equiv \frac{1}{\pi}\partial_x \vartheta_s$, we recover Eq. (2.16).

In the case of $\mathcal{P}$ symmetry, we use Eqs. (2.136) and (2.138) instead. The result is then

$$H = \frac{1}{2\pi}\int dx \left[v_c K_c (\partial_x \vartheta_c)^2 + \frac{v_c}{K_c}(\partial_x \varphi_c)^2 + v_s K_s (\partial_x \vartheta_s)^2 + \frac{v_s}{K_s}(\partial_x \varphi_s)^2 \right.$$
$$\left. + d_+ \partial_x \varphi_c \partial_x \varphi_s + d_- \partial_x \vartheta_c \partial_x \vartheta_s \right], \tag{2.144}$$

where

$$v_{c(s)} \equiv [(v \pm w)^2 - (c \pm c_0)^2]^{1/2}, \tag{2.145a}$$

$$K_{c(s)} \equiv \left[\frac{v \pm w - c \mp c_0}{v \pm w + c \pm c_0}\right]^{1/2}, \tag{2.145b}$$

$$d_\pm \equiv 2u \pm 2b. \tag{2.145c}$$

2.H Explicit parameterization of matrices for the 2-channel Luttinger liquid

2.H.1 Scaling dimension matrix M

Let $Q(\phi)$ denote the $SO(2)$ rotation matrix

$$Q(\phi) \equiv \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}, \tag{2.146}$$

let $R(\phi)$ denote the $O(2)$ reflection matrix

$$R(\phi) \equiv \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}, \tag{2.147}$$

and let

$$L \equiv \begin{bmatrix} \delta+\lambda & 0 \\ 0 & \delta-\lambda \end{bmatrix}. \tag{2.148}$$

The parameterization of $M \in \mathscr{M}_2 \equiv SO(2,2) \cap \mathscr{P}_4$ described in Eqs. (2.9) and (2.11) corresponds to

$$M = \begin{bmatrix} Q^T(\frac{\theta-\alpha}{2}) & 0 \\ 0 & Q^T(\frac{\theta+\alpha}{2}) \end{bmatrix} \begin{bmatrix} \cosh L & -\sinh L \\ -\sinh L & \cosh L \end{bmatrix} \begin{bmatrix} Q(\frac{\theta-\alpha}{2}) & 0 \\ 0 & Q(\frac{\theta+\alpha}{2}) \end{bmatrix}, \tag{2.149}$$

where each entry is a 2×2 matrix. Performing the matrix multiplication, we can write the result as

$$M = \begin{bmatrix} \mathbb{I}_2 \cosh\lambda & -R(\theta)\sinh\lambda \\ -R(\theta)\sinh\lambda & \mathbb{I}_2\cosh\lambda \end{bmatrix} \cosh\delta + \begin{bmatrix} R(\theta-\alpha)\sinh\lambda & -Q(\alpha)\cosh\lambda \\ -Q^T(\alpha)\cosh\lambda & R(\theta+\alpha)\sinh\lambda \end{bmatrix} \sinh\delta, \tag{2.150}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix. The dependence on α disappears in the limit $\delta \to 0$, as stated in the main text. Similarly, the dependence on θ disappears in the limit $\lambda \to 0$.

2.H.1.1 $\mathcal{T}$ symmetry or $\mathcal{P}$ symmetry, but not both

If the system has either time-reversal symmetry or inversion symmetry, but not both, the scaling dimension matrix must satisfy $\Sigma M \Sigma = M$, where $\Sigma \equiv \sigma_x \otimes \mathbb{I}_2$. Applied to Eq. (2.150), this condition requires $\alpha = 0$. Hence, in this case,

$$M = \begin{bmatrix} Q^T(\theta/2) & 0 \\ 0 & Q^T(\theta/2) \end{bmatrix} \begin{bmatrix} \cosh L & -\sinh L \\ -\sinh L & \cosh L \end{bmatrix} \begin{bmatrix} Q(\theta/2) & 0 \\ 0 & Q(\theta/2) \end{bmatrix} \tag{2.151a}$$

$$= \begin{bmatrix} \mathbb{I}_2 \cosh\lambda & -R(\theta)\sinh\lambda \\ -R(\theta)\sinh\lambda & \mathbb{I}_2\cosh\lambda \end{bmatrix} \cosh\delta + \begin{bmatrix} R(\theta)\sinh\lambda & -\mathbb{I}_2\cosh\lambda \\ -\mathbb{I}_2\cosh\lambda & R(\theta)\sinh\lambda \end{bmatrix} \sinh\delta. \tag{2.151b}$$

Note in particular that the presence of $\mathcal{T}$ symmetry or $\mathcal{P}$ symmetry (but not both simultaneously) places *no restrictions* on the allowed values of the parameters λ, δ, and θ. Changing the sign of λ is equivalent to shifting θ by π, so we can assume $\lambda > 0$.

2.H.1.2 Both $\mathcal{T}$ and $\mathcal{P}$ symmetry

If the system has both time-reversal and inversion symmetry, the scaling dimension matrix must satisfy $\Sigma_i M \Sigma_i = M$ for $i = 1, 2$, where $\Sigma_1 \equiv \sigma_x \otimes \mathbb{I}_2$ and $\Sigma_2 \equiv \sigma_x \otimes \sigma_x$. Applied to Eq. (2.150), these conditions require $\alpha = 0$ and $\theta = \pi/2$. Hence, in this case,

$$M = \begin{bmatrix} Q^T(\pi/4) & 0 \\ 0 & Q^T(\pi/4) \end{bmatrix} \begin{bmatrix} \cosh L & -\sinh L \\ -\sinh L & \cosh L \end{bmatrix} \begin{bmatrix} Q(\pi/4) & 0 \\ 0 & Q(\pi/4) \end{bmatrix} \tag{2.152a}$$

$$= \begin{bmatrix} \mathbb{I}_2 \cosh\lambda & -\sigma_x \sinh\lambda \\ -\sigma_x \sinh\lambda & \mathbb{I}_2 \cosh\lambda \end{bmatrix} \cosh\delta + \begin{bmatrix} \sigma_x \sinh\lambda & -\mathbb{I}_2 \cosh\lambda \\ -\mathbb{I}_2 \cosh\lambda & \sigma_x \sinh\lambda \end{bmatrix} \sinh\delta. \tag{2.152b}$$

2.H.2 Interaction matrix V (general expressions)

We parameterize the interaction matrix $V \in \mathscr{P}_4$ using Lemma 4. Let

$$V = \begin{bmatrix} V_{RR} & V_{RL} \\ V_{RL}^T & V_{LL} \end{bmatrix}. \tag{2.153}$$

Lemma 4 states that $V \in \varphi^{-1}(M)$ if and only if

$$V_{RR} = X + F\,Y F^T, \tag{2.154a}$$

$$V_{LL} = Y + F^T X F, \tag{2.154b}$$

$$V_{RL} = XF + F\,Y \tag{2.154c}$$

for some $X, Y \in \mathscr{P}_2$. The 2×2 matrix F corresponding to the M given in Eqs. (2.149) or (2.150) is

$$\begin{aligned} F &= Q^T\left(\tfrac{\theta-\alpha}{2}\right) \begin{bmatrix} \tanh\left(\tfrac{\delta+\lambda}{2}\right) & 0 \\ 0 & \tanh\left(\tfrac{\delta-\lambda}{2}\right) \end{bmatrix} Q\left(\tfrac{\theta+\alpha}{2}\right) \\ &= \frac{\sinh\lambda}{\cosh\lambda+\cosh\delta}\, R(\theta) + \frac{\sinh\delta}{\cosh\lambda+\cosh\delta}\, Q(\alpha), \end{aligned} \tag{2.155}$$

where $Q(\phi)$ and $R(\phi)$ are defined in Eqs. (2.146) and (2.147).

The matrices $X, Y \in \mathscr{P}_2$ can be conveniently parameterized as

$$X = \begin{bmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{bmatrix}, \tag{2.156a}$$

$$Y = \begin{bmatrix} y_0 + y_1 & y_2 \\ y_2 & y_0 - y_1 \end{bmatrix}, \tag{2.156b}$$

where $(x_0, x_1, x_2) \in \mathbb{R}^3$, $x_0 > (x_1^2 + x_2^2)^{1/2}$, and $(y_0, y_1, y_2) \in \mathbb{R}^3$, $y_0 > (y_1^2 + y_2^2)^{1/2}$. Using

Eqs. (4.5) and (2.156) in Eq. (2.154), we obtain

$$V_{RR} = \begin{bmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{bmatrix} + \frac{\sinh^2\lambda}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} y_0 + y_1\cos 2\theta + y_2 \sin 2\theta & y_1 \sin 2\theta - y_2 \cos 2\theta \\ y_1 \sin 2\theta - y_2 \cos 2\theta & y_0 - y_1 \cos 2\theta - y_2 \sin 2\theta \end{bmatrix}$$
$$+ \frac{2\sinh\lambda\sinh\delta}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} y_1\cos(\theta+\alpha) + y_2\sin(\theta+\alpha) + y_0\cos(\theta-\alpha) & y_0\sin(\theta-\alpha) \\ y_0\sin(\theta-\alpha) & y_1\cos(\theta+\alpha) + y_2\sin(\theta+\alpha) - y_0\cos(\theta-\alpha) \end{bmatrix}$$
$$+ \frac{\sinh^2\delta}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} y_0 + y_1\cos 2\alpha + y_2 \sin 2\alpha & y_2 \cos 2\alpha - y_1 \sin 2\alpha \\ y_2 \cos 2\alpha - y_1 \sin 2\alpha & y_0 - y_1 \cos 2\alpha - y_2 \sin 2\alpha \end{bmatrix}, \tag{2.157a}$$

$$V_{LL} = \begin{bmatrix} y_0 + y_1 & y_2 \\ y_2 & y_0 - y_1 \end{bmatrix} + \frac{\sinh^2\lambda}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} x_0 + x_1\cos 2\theta + x_2 \sin 2\theta & x_1 \sin 2\theta - x_2 \cos 2\theta \\ x_1 \sin 2\theta - x_2 \cos 2\theta & x_0 - x_1 \cos 2\theta - x_2 \sin 2\theta \end{bmatrix}$$
$$+ \frac{2\sinh\lambda\sinh\delta}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} x_1\cos(\theta-\alpha) + x_2\sin(\theta-\alpha) + x_0\cos(\theta+\alpha) & x_0\sin(\theta+\alpha) \\ x_0\sin(\theta+\alpha) & x_1\cos(\theta-\alpha) + x_2\sin(\theta-\alpha) - x_0\cos(\theta+\alpha) \end{bmatrix}$$
$$+ \frac{\sinh^2\delta}{(\cosh\lambda + \cosh\delta)^2} \begin{bmatrix} x_0 + x_1\cos 2\alpha - x_2 \sin 2\alpha & x_1 \sin 2\alpha + x_2 \cos 2\alpha \\ x_1 \sin 2\alpha + x_2 \cos 2\alpha & x_0 - x_1 \cos 2\alpha + x_2 \sin 2\alpha \end{bmatrix}, \tag{2.157b}$$

$$V_{RL} = \frac{\sinh\lambda}{\cosh\lambda + \cosh\delta} \begin{bmatrix} (x_1 + y_1 + x_0 + y_0)\cos\theta + (x_2 + y_2)\sin\theta & (x_0 + y_0 + x_1 - y_1)\sin\theta - (x_2 - y_2)\cos\theta \\ (x_0 + y_0 - x_1 + y_1)\sin\theta + (x_2 - y_2)\cos\theta & (x_1 + y_1 - x_0 - y_0)\cos\theta + (x_2 + y_2)\sin\theta \end{bmatrix}$$
$$+ \frac{\sinh\delta}{\cosh\lambda + \cosh\delta} \begin{bmatrix} (x_0 + y_0 + x_1 + y_1)\cos\alpha - (x_2 - y_2)\sin\alpha & (x_2 + y_2)\cos\alpha + (x_1 - y_1 + x_0 + y_0)\sin\alpha \\ (x_2 + y_2)\cos\alpha + (x_1 - y_1 - x_0 - y_0)\sin\alpha & (x_0 + y_0 - x_1 - y_1)\cos\alpha + (x_2 - y_2)\sin\alpha \end{bmatrix}. \tag{2.157c}$$

Equation (2.157) gives a complete and explicit parameterization of the possible interaction matrices $V \in \mathscr{P}_4$ of a 2-channel Luttinger liquid, in terms of the ten real parameters $(\lambda, \delta, \theta, \alpha, x_0, x_1, x_2, y_0, y_1, y_2)$. Of these, only the first four $(\lambda, \delta, \theta, \alpha)$ affect scaling dimensions; they determine the scaling dimension matrix M via Eq. (2.150). The remaining six parameters can be chosen arbitrarily, subject only to the constraints $x_0 > (x_1^2 + x_2^2)^{1/2}$ and $y_0 > (y_1^2 + y_2^2)^{1/2}$ (if either of these inequalities is violated, the resulting V will fail to be positive definite).

2.H.2.1 $\mathcal{T}$ symmetry or $\mathcal{P}$ symmetry, but not both

When symmetries are present, it is convenient to parameterize the interaction matrix $V \in \mathscr{P}_4$ using Lemma 7 instead. In the present case, Lemma 7 gives $V = M^{-1/2}[X \oplus Y]M^{-1/2}$ with $X, Y \in \mathscr{P}_2$ satisfying $\Sigma[X \oplus Y]\Sigma = X \oplus Y$, where $\Sigma \equiv \sigma_x \otimes \mathbb{I}_2$; the condition fixes $Y = X$. Thus, $V = M^{-1/2}[X \oplus X]M^{-1/2}$ with $X \in \mathscr{P}_2$. The scaling dimension matrix M is given by Eq. (2.151). Thus,

$$M^{-1/2} = \begin{bmatrix} Q^T(\theta/2) & 0 \\ 0 & Q^T(\theta/2) \end{bmatrix} \begin{bmatrix} \cosh(L/2) & \sinh(L/2) \\ \sinh(L/2) & \cosh(L/2) \end{bmatrix} \begin{bmatrix} Q(\theta/2) & 0 \\ 0 & Q(\theta/2) \end{bmatrix}. \quad (2.158)$$

The inner factors of $Q(\theta/2)$ in the product $V = M^{-1/2}[X \oplus X]M^{-1/2}$ may be absorbed into X, since the latter is an arbitrary element of $\mathscr{P}_2$. Doing so, we obtain

$$\begin{aligned} V = &\begin{bmatrix} Q^T(\theta/2) & 0 \\ 0 & Q^T(\theta/2) \end{bmatrix} \begin{bmatrix} \cosh(L/2) & \sinh(L/2) \\ \sinh(L/2) & \cosh(L/2) \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \\ &\times \begin{bmatrix} \cosh(L/2) & \sinh(L/2) \\ \sinh(L/2) & \cosh(L/2) \end{bmatrix} \begin{bmatrix} Q(\theta/2) & 0 \\ 0 & Q(\theta/2) \end{bmatrix}. \end{aligned} \quad (2.159)$$

We can parameterize $X \in \mathscr{P}_2$ as

$$X = \zeta \begin{bmatrix} 1+a & -b \\ -b & 1-a \end{bmatrix}, \quad (2.160)$$

where $\zeta > 0$ and $(a, b) \in \mathbb{R}^2$, $a^2 + b^2 < 1$. Performing the matrix multiplications in Eq. (2.159), we obtain

$$V = \zeta \begin{bmatrix} V_1 & V_2 \\ V_2 & V_1 \end{bmatrix} \cosh\delta + \zeta \begin{bmatrix} V_2 & V_1 \\ V_1 & V_2 \end{bmatrix} \sinh\delta, \tag{2.161}$$

where

$$V_1 = \begin{bmatrix} \cosh\lambda + a\cos\theta\cosh\lambda + b\sin\theta & a\sin\theta\cosh\lambda - b\cos\theta \\ a\sin\theta\cosh\lambda - b\cos\theta & \cosh\lambda - a\cos\theta\cosh\lambda - b\sin\theta \end{bmatrix}, \tag{2.162a}$$

$$V_2 = \begin{bmatrix} a + \cos\theta & \sin\theta \\ \sin\theta & a - \cos\theta \end{bmatrix} \sinh\lambda. \tag{2.162b}$$

Equations (2.161) and (2.162) gives a complete and explicit parameterization of the possible interaction matrices $V \in \mathscr{P}_4$ of a 2-channel Luttinger liquid with either time-reversal symmetry or inversion symmetry (but not both), in terms of the six real parameters $(\lambda, \delta, \theta, a, b, \zeta)$. Of these, only the first three $(\lambda, \delta, \theta)$ affect scaling dimensions; they determine the scaling dimension matrix M via Eq. (2.151). The remaining three parameters can be chosen arbitrarily, subject only to the constraints $a^2 + b^2 < 1$ and $\zeta > 0$ (if these inequalities are violated, the resulting V will fail to be positive definite). Note that ζ is an irrelevant overall scale factor.

2.H.2.2 Both $\mathcal{T}$ and $\mathcal{P}$ symmetry

As above, we parameterize the interaction matrix $V \in \mathscr{P}_4$ using Lemma 7. In this case, Lemma 7 gives $V = M^{-1/2}[X \oplus Y]M^{-1/2}$ with $X, Y \in \mathscr{P}_2$ satisfying $\Sigma_i[X \oplus Y]\Sigma_i = X \oplus Y$ for $i = 1, 2$, where $\Sigma_1 \equiv \sigma_x \otimes \mathbb{I}_2$ and $\Sigma_2 \equiv \sigma_x \otimes \sigma_x$. The conditions fix $Y = X = x_0\mathbb{I}_2 + x_1\sigma_x$, with $(x_0, x_1) \in \mathbb{R}^2$, $x_0 > |x_1|$. Thus, $V = M^{-1/2}[X \oplus X]M^{-1/2}$

with X of the form specified. The scaling dimension matrix M is given by Eq. (2.152). Thus,

$$M^{-1/2} = \begin{bmatrix} Q^T(\pi/4) & 0 \\ 0 & Q^T(\pi/4) \end{bmatrix} \begin{bmatrix} \cosh(L/2) & \sinh(L/2) \\ \sinh(L/2) & \cosh(L/2) \end{bmatrix} \begin{bmatrix} Q(\pi/4) & 0 \\ 0 & Q(\pi/4) \end{bmatrix}. \quad (2.163)$$

Writing $X = \zeta(\mathbb{I}_2 + a\sigma_x)$ with $\zeta > 0$, $|a| < 1$, and performing the matrix multiplications in $V = M^{-1/2}[X \oplus X]M^{-1/2}$, we obtain

$$V = \zeta \begin{bmatrix} V_1 & V_2 \\ V_2 & V_1 \end{bmatrix} \cosh\delta + \zeta \begin{bmatrix} V_2 & V_1 \\ V_1 & V_2 \end{bmatrix} \sinh\delta, \quad (2.164)$$

where

$$V_1 = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \cosh\lambda, \quad (2.165a)$$

$$V_2 = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \sinh\lambda. \quad (2.165b)$$

Equations (2.164) and (2.165) gives a complete and explicit parameterization of the possible interaction matrices $V \in \mathscr{P}_4$ of a 2-channel Luttinger liquid with both time-reversal and inversion symmetry, in terms of the four real parameters $(\lambda, \delta, a, \zeta)$. Of these, only the first two (λ, δ) affect scaling dimensions; they determine the scaling dimension matrix M via Eq. (2.152). The remaining two parameters can be chosen arbitrarily, subject only to the constraints $|a| < 1$ and $\zeta > 0$ (if these inequalities are violated, the resulting V will fail to be positive definite). Note that, as before, ζ is an irrelevant overall scale factor.

2.H.3 Interaction matrix V with $\mathcal{T}$ or $\mathcal{P}$ symmetry (but not both), in the special case $\delta = 0$

In the limit $\delta \to 0$, Eqs. (2.161) and (2.162) together yield

$$V = \left[\begin{array}{cc|cc} v_+ & w & c_+ & c_0 \\ w & v_- & c_0 & c_- \\ \hline c_+ & c_0 & v_+ & w \\ c_0 & c_- & w & v_- \end{array}\right], \tag{2.166}$$

where

$$v_\pm = v \pm u, \tag{2.167a}$$

$$v = \zeta \cosh\lambda, \tag{2.167b}$$

$$u = \zeta(a\cos\theta\cosh\lambda + b\sin\theta), \tag{2.167c}$$

$$w = \zeta(a\sin\theta\cosh\lambda - b\cos\theta), \tag{2.167d}$$

$$c_\pm = \zeta(a \pm \cos\theta)\sinh\lambda, \tag{2.167e}$$

$$c_0 = \zeta\sin\theta\sinh\lambda. \tag{2.167f}$$

We assume $\lambda > 0$ (without loss of generality), and identify the values of a, b for which all matrix elements of V are nonnegative. The diagonal elements of a positive definite matrix are necessarily positive, so $v_\pm > 0$ is automatic. $c_0 \geq 0$ requires $\theta \in [0, \pi]$. Nonnegativity of the remaining matrix elements, $c_\pm$ and w, requires

$$a \geq |\cos\theta|, \tag{2.168a}$$

$$a\sin\theta\cosh\lambda \geq b\cos\theta. \tag{2.168b}$$

In fact, Eq. (2.168b) is superfluous, because it follows from Eq. (2.168a) and the positive definiteness condition $a^2 + b^2 < 1$; assuming the latter, we have $|b| < (1 - a^2)^{1/2} \leq |\sin\theta| \leq |\cos\theta \tan\theta| \cosh\lambda \leq a|\tan\theta| \cosh\lambda$, which implies Eq. (2.168b) if $\theta \in [0, \pi]$. We conclude that the V matrix given above is positive definite with all entries nonnegative if $\lambda > 0$, $\theta \in [0, \pi]$, $a^2 + b^2 < 1$, and $a \geq |\cos\theta|$.

These results are completely equivalent to the ones stated in the main text. Indeed, using Eq. (2.167), one can verify that the following linear relations hold:

$$c_\pm = (w \sin\theta \pm v_\pm \cos\theta) \tanh\lambda, \tag{2.169a}$$

$$c_0 = v \sin\theta \tanh\lambda. \tag{2.169b}$$

These are precisely the relations that one obtains by using Eq. (2.166) and $F = Q(\theta)\tanh(\lambda/2)$ in Eq. (2.103c) of Lemma 5. Therefore, we can regard (v, u, w), instead of (ζ, a, b), as the independent variables parameterizing V. From Eq. (2.167), we have

$$\begin{bmatrix} u/v \\ w/v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \operatorname{sech}\lambda \\ \sin\theta & -\cos\theta \operatorname{sech}\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.170}$$

Inverting this linear system,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta \cosh\lambda & -\cos\theta \cosh\lambda \end{bmatrix} \begin{bmatrix} u/v \\ w/v \end{bmatrix}. \tag{2.171}$$

Thus, the conditions $a^2 + b^2 < 1$ and $a \geq |\cos\theta|$ translate to:

$$(u \sin\theta - w \cos\theta)^2 \cosh^2\lambda + (u \cos\theta + w \sin\theta)^2 < v^2, \tag{2.172a}$$

$$u \cos\theta + w \sin\theta \geq v|\cos\theta|. \tag{2.172b}$$

These conditions are stated in the main text of the chapter as Eqs. (2.14) and (2.15).

2.I Absolute q-stability phase diagram for $N = 2$ channel Luttinger liquid

The interaction matrix $V \in \mathscr{P}_4$ of a 2-channel Luttinger liquid depends on ten real parameters; these can be chosen according to Eq. (2.157). Of the ten, only the four parameters $\xi \equiv (\lambda, \delta, \theta, \alpha)$ affect scaling dimensions; they determine the scaling dimension matrix $M(\xi)$ via Eq. (2.150), which in turn determines $\Delta(\mathbf{m}; \xi) \equiv \frac{1}{2}\mathbf{m}^T M(\xi)\mathbf{m}$.

Let $q(\xi)$ denote the absolute q-stability value of a Luttinger liquid with parameters ξ. By definition, $q(\xi)$ is the largest integer such that $\Delta(\mathbf{m}; \xi) > 2$ for all nonzero $\mathbf{m} \in \mathbb{Z}^4$ with $K(\mathbf{m}) \in \mathbb{Z}$ and $|\mathbf{m}| \leq q(\xi)$, where $K(\mathbf{m}) \equiv \frac{1}{2}\mathbf{m}^T K \mathbf{m}$, $K \equiv \mathbb{I}_2 \oplus -\mathbb{I}_2$. Because of the inequality $\Delta(\mathbf{m}) \geq |K(\mathbf{m})|$, one can restrict attention to those $\mathbf{m}$ for which $K(\mathbf{m}) = 0, \pm 1, \pm 2$. Thus, an equivalent definition of $q(\xi)$ to the one given above is: $q(\xi)$ is the smallest positive integer such that $\Delta(\mathbf{m}; \xi) \leq 2$ for some $\mathbf{m} \in \mathbb{Z}^4$ with $K(\mathbf{m}) \in \{0, \pm 1, \pm 2\}$ and $|\mathbf{m}| = q(\xi) + 1$.

We use the second definition to determine the phase diagram numerically. The algorithm is straightforward: at each point ξ, compute $\Delta(\mathbf{m}; \xi)$ for all integer vectors $\mathbf{m}$ with $K(\mathbf{m}) \in \{0, \pm 1, \pm 2\}$ in shells of increasing $|\mathbf{m}|$, until either a vector $\mathbf{m}_*$ is found for which $\Delta(\mathbf{m}_*; \xi) \leq 2$, or $|\mathbf{m}|$ passes a specified cutoff value q_*. Set $q(\xi) = |\mathbf{m}_*| - 1$ in the former case, and $q(\xi) = q_*$ in the latter. The shells of vectors with $|\mathbf{m}| = 2, \ldots, q_*$ can be tabulated in advance, and the matrix $M(\xi)$ only needs to be computed once at each point ξ. Figure 2.2 of the main text and Figure 2.I.1 below were obtained by this method, with cutoff $q_* = 22$.

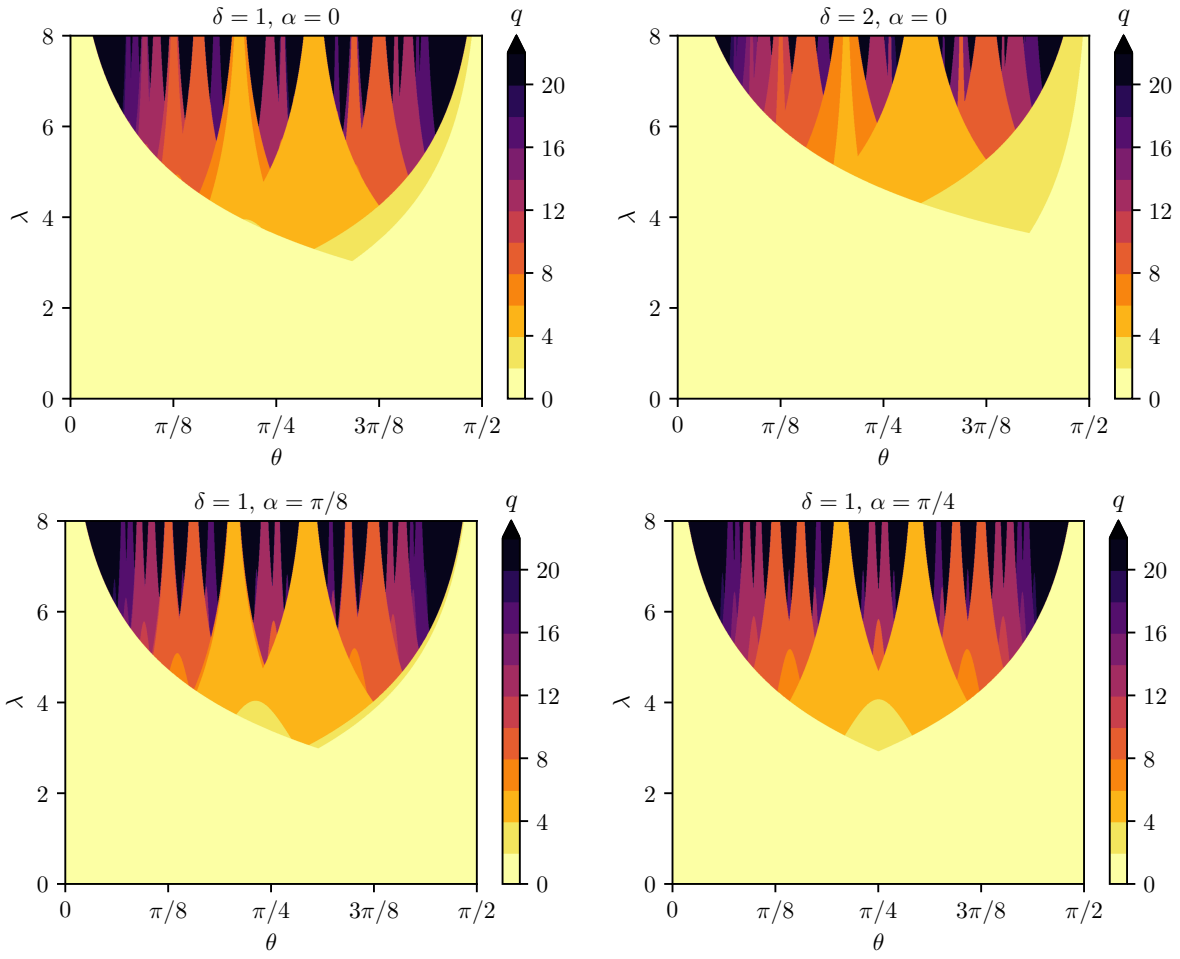


Figure 2.I.1: Various slices of the absolute q-stability phase diagram for the $N = 2$ channel Luttinger liquid (compare Figure 2.2).

Chapter 3

Relaxation to equilibrium in systems with quadratic hamiltonians

In this chapter, we present an elementary, general, and semi-quantitative description of relaxation to gaussian and generalized Gibbs states in lattice models of fermions or bosons with quadratic hamiltonians. Our arguments apply to arbitrary initial states that satisfy a mild condition on clustering of correlations. We also show that similar arguments can be used to understand relaxation (or its absence) in systems with time-dependent quadratic hamiltonians, and provide a semi-quantitative description of relaxation in quadratic periodically driven (Floquet) systems.

3.1 Introduction

In recent years, there has been much work on understanding the nature of the equilibrium state, and the dynamics of the relaxation to this state, in quantum many-body systems with an extensive number of local conservation laws. The motivation for the study of these integrable models is twofold. First, beautiful cold-atom experiments have

successfully realized many such models and studied their nonequilibrium dynamics (for reviews, see Refs. [63, 64, 22, 65]). Second, integrable models are much easier to analyze theoretically than their non-integrable brethren (for reviews, see Refs. [66, 67]).

When prepared in a generic initial state, an integrable system does not thermalize in the usual sense of the word, due to the extensive number of local conserved quantities. Instead, the stationary behavior at late times can be described by an appropriate *generalized Gibbs ensemble* (GGE). For our purposes, the "GGE conjecture" put forth in Ref. [24] and subsequently refined by many authors ([22] and references therein), asserts the following: assuming that local observables in an integrable many-body system relax to stationary values, these values may be computed using the density matrix

$$\hat{\rho}_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_m \lambda_m \hat{I}_m \right), \tag{3.1}$$

where $\{\hat{I}_m\}$ is the set of all local conserved quantities (here locality means that each $\hat{I}_m$ is a sum of local densities), $Z_{\mathrm{GGE}} = \mathrm{Tr}(e^{-\sum_m \lambda_m \hat{I}_m})$ is the partition function needed for normalization, and $\{\lambda_m\}$ are Lagrange multipliers chosen so as to satisfy the constraints $\mathrm{Tr}(\hat{I}_m \, \hat{\rho}_{\mathrm{GGE}}) = \langle \hat{I}_m \rangle (t = 0)$. The density matrix $\hat{\rho}_{\mathrm{GGE}}$ is readily obtained by the general prescription [68] of maximizing the entropy $S = -\mathrm{Tr}(\hat{\rho} \log \hat{\rho})$ subject to these constraints.

In this chapter we consider the simplest class of integrable models: those whose hamiltonians can be expressed as quadratic forms in a set of canonical particle creation and annihilation operators. Such so-called *non-interacting integrable* models describe not only truly non-interacting particles, but also mean-field approximations to models of interacting particles. In view of this, we will refer to the models of interest as "quadratic" rather than "non-interacting integrable". In one dimension, certain other integrable models—some spin chains and systems of hard-core particles—can be mapped to quadratic ones. However, simple observables in the original model often map to complicated operators

in the quadratic model, and one must take this into account when using the mapping to study relaxation [69].

In quadratic models, the local conserved charges $\hat{I}_m$ are also typically quadratic (we demonstrate this explicitly in Section 3.5.1), and therefore the GGE density operator $\hat{\rho}_{\mathrm{GGE}}$ is *gaussian*. It is thus quite natural to divide relaxation in quadratic models into two process: (i) relaxation of the initial state to a gaussian one, and (ii) relaxation of the gaussian state to the appropriate GGE. Our arguments will make clear that these two processes occur for fundamentally distinct physical reasons, and also that the first process often (but not always) occurs faster than the second. We will also show that similar arguments can be used to understand relaxation in systems with quadratic time-periodic hamiltonians. Considering the great recent interest in driven quantum systems ([70, 71] and references therein), we believe that this is a useful synthesis of ideas.

Recently, Gluza et al. [72] have argued that the first process mentioned above (which they term "gaussification") can be understood as a consequence of exponential clustering of correlations in the initial state, together with "delocalizing transport" (the sufficiently rapid suppression of the component of an operator on any given site, due to its "spreading out" over a large area), and have rigorously proven this implication for fermionic lattice systems with quadratic time-independent hamiltonians. The importance of clustering of correlations in the initial state for relaxation had earlier been emphasized by Cramer and Eisert [73], and by Sotiriadis and Calabrese [74], who had shown that it is in fact a necessary and sufficient condition for relaxation to GGE in a broad class of translation-invariant quadratic models. Other early studies of the validity of the GGE in quadratic systems (e.g. [75, 76]) typically assumed gaussian initial states, and hence addressed only the second process described in the previous paragraph. Our results are consistent with, and generalize, the results of these earlier works.

Note that, in the decade since Ref. [24] appeared, the validity of the GGE has been the subject of a large number of theoretical and numerical investigations, of which the works cited above comprise only a handful. For a reasonably complete summary and critical discussion of all these efforts, we refer the reader to one of the recent reviews of the field, e.g. Refs. [22, 66, 67].

Here, we distill the basic ideas present in Ref. [74] and especially in Ref. [72], and give an elementary, general, and semi-quantitative description of relaxation in quadratic lattice models. To state our results, we need to introduce some notation. Let $\hat{\psi}^a_x$ denote the particle creation ($a = +$) or annihilation ($a = -$) operator for the site at position x. Under Heisenberg time-evolution with any time-dependent quadratic hamiltonian $\hat{H}(t)$, these operators evolve linearly:

$$\hat{\psi}^a_x(t) = \sum_{b=\pm} \sum_y G^{ab}_{xy}(t) \hat{\psi}^b_y. \tag{3.2}$$

This equation defines the *single-particle propagator* $G^{ab}_{xy}(t)$, which plays a central role in the following.

For most of this work, we assume **delocalizing dynamics:**

$$\left| G^{ab}_{xy}(t) \right| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.3}$$

The terminology is adapted from Ref. [72]. Equation (3.3) may be interpreted as saying that a particle (or hole) created at x has vanishing probability amplitude to be found at any given y after infinite time, because its wavefunction "spreads out" indefinitely. In the case of bosons, we also require the quasiparticle spectrum of $\hat{H}(t)$ to be uniformly bounded and positive-definite. Note that we *do not* restrict the hamiltonian to be time-independent in general.

We treat *arbitrary initial states* that satisfy a mild condition on **(algebraic) clustering of correlations:**

$$\langle\langle \hat{\psi}^{a_1}_{x_1} \hat{\psi}^{a_2}_{x_2} \cdots \hat{\psi}^{a_n}_{x_n} \rangle\rangle = \; o(|x_i - x_j|^{-(d+\epsilon)}) \quad \text{as} \;\; |x_i - x_j| \to \infty, \tag{3.4}$$

for some $\epsilon > 0$. Here, $\langle\langle \cdots \rangle\rangle$ denotes the *connected correlation function* in the initial state, and d is the spatial dimension. We emphasize that the initial state is not assumed to have any relation whatsoever to the hamiltonian under which the system subsequently evolves. In particular, the initial state may be strongly interacting. We also emphasize that the cluster decomposition property (3.4) is an *extremely weak constraint* on the state; one can expect it to hold for most initial states of interest. In fact, a stronger version of (3.4) has been *rigorously proven* for large classes of states, including ground states of interacting local hamiltonians with a spectral gap [77, 78], as well as thermal states of arbitrary short-ranged fermionic lattice models at sufficiently high temperature [79].

Our emphasis throughout this chapter is on simplicity and physical transparency rather than on mathematical rigor; the reader in search of the latter is encouraged to consult Refs. [72, 80] and similar works in parallel with our treatment. Nevertheless, we believe that most of the arguments presented here can serve as sketches for rigorous proofs.

Our main technical results are summarized below.

1. "Gaussification" under general conditions: In any lattice system of fermions or bosons prepared in an initial state satisfying (3.4) and evolving under a quadratic hamiltonian $\hat{H}(t)$ that leads to delocalizing dynamics (3.3), all local ($n > 2$)-point connected

correlation functions vanish at late times:

$$\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\rangle \to 0 \quad \text{as} \quad t_j \to \infty. \tag{3.5}$$

Following Ref. [72], we refer to this vanishing as "gaussification", because it is equivalent to the dynamical recovery of Wick's theorem—the distinguishing property of a gaussian state. This result significantly generalizes that of Ref. [72], where gaussification was proven for time-independent fermion lattice models and initial states with exponential clustering of correlations.

2. Universal power-law gaussification: In many cases of interest one can identify, for each x, a, and t, a definite volume $\mathcal{V}^a_x(t)$ of y-space in which the single-particle propagator $G^{ab}_{xy}(t)$ is meaningfully supported. In these cases we obtain a more quantitative version of Eq. (3.5):

$$\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\rangle \sim [\mathcal{V}(t)]^{-(n/2-1)} \quad \text{as} \quad t \to \infty, \tag{3.6}$$

where $t \equiv (t_1 + \cdots + t_n)/n$. This asymptotic result holds in the limit $|t_i - t_j| \ll t$, with $\mathcal{V}(t) \approx \mathcal{V}^{a_i}_{x_i}(t_i) \approx \mathcal{V}^{a_j}_{x_j}(t_j)$. Typically, $\mathcal{V}(t)$ grows like a power of t. Thus, Eq. (3.6) gives power-law decay of the connected correlation functions in time, with exponents that depend only on n and on how fast the single-particle propagator of the system spreads.

3. Gaussianity of GGE: We prove that all local conserved quantities of a time-independent quadratic hamiltonian with delocalizing dynamics (3.3) are also quadratic, and therefore that the associated GGE density matrix is indeed gaussian. In past work this property appears simply to have been taken for granted.

4. Relaxation to GGE: For any time-independent hamiltonian that satisfies (3.3), and any initial state that satisfies (3.4) and an additional assumption to be described

below, we show that the system relaxes to the GGE; for any local operator $\hat{\mathcal{O}}$,

$$\langle \hat{\mathcal{O}}(t) \rangle \to \langle \hat{\mathcal{O}} \rangle_{\mathrm{GGE}} \quad \text{as} \quad t \to \infty. \tag{3.7}$$

This result is consistent with and generalizes existing proofs of relaxation to the GGE to a larger class of models and initial states. The additional assumption is formulated precisely in Section 3.5.3 (Eq. (3.134)). Roughly speaking, it excludes situations in which the initial profiles of local conserved densities are inhomogeneous on length scales comparable to the system size. In such cases, the GGE conjecture fails for a trivial reason: it takes infinitely long for the locally conserved density to flow across the whole system in the thermodynamic limit. A mathematically rigorous version of this result has recently been proven in Ref. [80], but under somewhat more stringent conditions than those assumed here.

5. Universal power-law relaxation: Under the conditions of the previous result, we also obtain quantitative estimates for how the local 2-point function relaxes to its GGE value. Consider the instantaneous deviation

$$\delta C^{ab}_{xy}(t) \equiv \langle \hat{\psi}^a_x(t) \hat{\psi}^b_y(t) \rangle - \langle \hat{\psi}^a_x \hat{\psi}^b_y \rangle_{\mathrm{GGE}}. \tag{3.8}$$

Assume, as is often the case, that the density of quasiparticle states of $\hat{H}$ near the band edge has the form $g(\varepsilon) \sim \varepsilon^s$. We show that, *generically*,

$$\delta C^{ab}_{xy}(t) \sim t^{-\alpha(1+s)} \quad \text{as} \quad t \to \infty, \tag{3.9}$$

where $\alpha = 1$ if there is a density wave of one or more of the conserved quantities in the initial state, and $\alpha = 2$ if not. In particular, for *translation-invariant* quadratic

hamiltonians in d dimensions,

$$\delta C^{ab}_{xy}(t) \sim t^{-\alpha d/2} \quad \text{as} \quad t \to \infty, \tag{3.10}$$

where α is defined above. Note that this result holds for *generic* hamiltonians and initial states of the types considered; different exponents can and do occur if the hamiltonian and/or initial state is fine-tuned. Again, a mathematically rigorous version of the result (3.10) was recently proven in Ref. [80], in the form of a bound $\left|\delta C^{ab}_{xy}(t)\right| \leq t^{-\gamma}$, but under somewhat more stringent conditions than those assumed here, and with $\gamma < \alpha d/2$ in general; Eq. (3.10) represents a more accurate asymptotic result.

6. Floquet-GGE: For any *time-periodic* quadratic hamiltonian $\hat{H}(t) = \hat{H}(t+T)$ that satisfies (3.3), we prove that all local conserved quantities of the associated Floquet hamiltonian $\hat{H}_F$ are quadratic, and hence that the Floquet-GGE [81] density matrix, $\hat{\rho}_F$, is gaussian. For any initial state that satisfies the assumptions of result 4 above, we show that the system eventually relaxes to this (time-periodic) Floquet-GGE. In the limit $T \to 0$ of fast driving, we expect to observe power-law relaxation to $\hat{\rho}_F$, with the exponents given by results 2 and 5 above applied to $\hat{H}_F$. In the opposite limit $T \to \infty$ of slow driving, we expect to observe power-law relaxation toward a GGE of the instantaneous hamiltonian $\hat{H}(t)$, followed by much slower exponential relaxation $\sim e^{-t/T}$ toward $\hat{\rho}_F$. Our results are consistent with, and generalize, the original treatment of this problem by Lazarides et al. [82] (in which the initial state was assumed to be gaussian, and no estimates like the ones above were given for the relaxation process itself).

7. Effects/non-effects of localized states: We find that dynamics generated by a quadratic fermion hamiltonian $\hat{H}$ whose quasiparticle spectrum includes discrete localized levels (so that Eq. (3.3) is violated) *will still lead to gaussification and equilibration to an*

appropriate GGE, as long as (i) the initial state has a finite correlation length ξ, and (ii) the spatial distance between any pair of localized levels is large relative to ξ. This result should be viewed as an interesting (but easily understandable; see Section 3.6) exception to the general rule [83, 84] that the GGE fails if the spectrum of $\hat{H}$ contains a pure-point part coming from localized levels. If either of the conditions (i) or (ii) above is violated, we recover the expected failure of gaussification and of the GGE. Thus, our results are fully consistent with those of Ziraldo et al. [83, 84], who considered the case that $\hat{H}$ is disordered; in this case condition (ii) will typically be violated.

Organization of the chapter

In Section 3.2 we introduce our arguments in a simple and concrete setting: a 1d tight-binding model of spinless fermions. In Section 3.3 we define the general problem and fix terminology and notation. In Section 3.4 we present our argument for gaussification in arbitrary quadratic models (of particles), and predict the exponents of the power law decay in time of all higher-point connected functions. In Section 3.5 we describe the manner in which, for time-independent hamiltonians, the gaussian state equilibrates to the GGE. In Section 3.6 we describe how discrete localized levels in the spectrum of the hamiltonian affect relaxation. In Section 3.7 we consider quenches to time-periodic quadratic hamiltonians, and describe relaxation to the Floquet-GGE. In Section 3.8 we briefly comment on the application of our arguments to spin models that can be mapped onto quadratic fermion models. Finally, in Section 5.6 we summarize our results.

3.2 Example: Relaxation in a nearest-neighbor tight-binding chain

In this section we introduce our arguments by working them out carefully in a simple and concrete example, while emphasizing the key ideas. This will also serve to motivate the subsequent general development.

3.2.1 Setup

Consider a tight-binding model of spinless fermions in $d = 1$ dimensions, with nearest-neighbor hopping. The hamiltonian is

$$\hat{H}_0 = -\frac{1}{2}\sum_{x=1}^{L}\left(\hat{c}_x^\dagger \hat{c}_{x+1} + \hat{c}_{x+1}^\dagger \hat{c}_x\right), \tag{3.11}$$

where $\hat{c}_x^\dagger$ and $\hat{c}_x$ respectively create and annihilate a fermion on the site at position x. These operators obey the standard anti-commutation relations

$$\hat{c}_x \hat{c}_y^\dagger + \hat{c}_y^\dagger \hat{c}_x = \delta_{xy}, \tag{3.12a}$$

$$\hat{c}_x \hat{c}_y + \hat{c}_y \hat{c}_x = 0. \tag{3.12b}$$

We have set the lattice spacing equal to 1 and hopping energy equal to 1/2. We will also set $\hbar = 1$. Periodic boundary conditions are assumed (site $L + 1$ is identified with site 1).

Imagine a quench in which the system is prepared in some non-equilibrium initial state, represented by the density operator $\hat{\rho}_0$, at time $t = 0$, and subsequently evolved with the hamiltonian $\hat{H}_0$ of Eq. (3.11). For the majority of this example (up to and including Section 3.2.5), we make only two assumptions about $\hat{\rho}_0$.

The first assumption is very important: $\hat{\rho}_0$ must obey the principle of *cluster decomposition* [85]. Roughly speaking, this principle requires correlations between local operators in the state $\hat{\rho}_0$ to factorize as the operators are taken far apart from one another. We will make this precise in Eq. (3.42) (we use a stronger version of the principle in this section than we do in our general treatment).

The second assumption is *not* important, and we impose it only to simplify the example. We assume that the initial state conserves total particle number:

$$\left[\hat{N},\, \hat{\rho}_0\right] = 0, \tag{3.13}$$

where

$$\hat{N} = \sum_x \hat{c}_x^\dagger \hat{c}_x. \tag{3.14}$$

In the general treatment of Section 3.3 onwards, we make no assumption like Eq. (3.13).

In the last part of this example, Section 3.2.6, we will add a third assumption about $\hat{\rho}_0$, Eq. (3.65). Nothing in Sections 3.2.2 through 3.2.5 relies on this extra assumption; it is only needed for the analysis of Section 3.2.6. Therefore, we do not state it here.

We will study whether and how local observables of the system relax to their values in an appropriate generalized Gibbs ensemble as time progresses. We first discuss the construction of this GGE density operator.

3.2.2 Conserved quantities and GGE density operator

The hamiltonian (3.11) can be diagonalized by introducing quasi-momentum mode operators:

$$\hat{c}_x = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \hat{c}(k), \tag{3.15}$$

where k runs over all integer multiples of $(2\pi/L)$ within the Brillouin zone $(-\pi, \pi]$. In terms of these mode operators,

$$\hat{H}_0 = \sum_k \omega(k)\, \hat{c}^\dagger(k)\hat{c}(k), \qquad \omega(k) = -\cos k. \tag{3.16}$$

The various mode occupation number operators,

$$\hat{n}(k) = \hat{c}^\dagger(k)\hat{c}(k), \tag{3.17}$$

clearly commute with $\hat{H}_0$ and with each other. Furthermore, by forming appropriate linear combinations of them, we can define an extensive set of local conserved quantities in involution:

$$\hat{I}_{2m} = \sum_k \cos(mk)\hat{n}(k) = \frac{1}{2}\sum_{x=1}^{L}\left(\hat{c}_x^\dagger \hat{c}_{x+m} + \hat{c}_{x+m}^\dagger \hat{c}_x\right), \tag{3.18}$$

and

$$\hat{I}_{2m+1} = \sum_k \sin(mk)\hat{n}(k) = \frac{(-i)}{2}\sum_{x=1}^{L}\left(\hat{c}_x^\dagger \hat{c}_{x+m} - \hat{c}_{x+m}^\dagger \hat{c}_x\right). \tag{3.19}$$

where $m = 0, 1, 2, \ldots$. These clearly commute with $\hat{H}_0$ and with one another:

$$\left[\hat{H}_0,\, \hat{I}_m\right] = 0, \tag{3.20a}$$

$$\left[\hat{I}_m,\, \hat{I}_{m'}\right] = 0 \tag{3.20b}$$

(in fact, $\hat{H}_0 = \hat{I}_2$, so the second equation implies the first). They are local because their densities,

$$\propto \left(\hat{c}_x^\dagger \hat{c}_{x+m} \pm \hat{c}_{x+m}^\dagger \hat{c}_x\right),$$

act nontrivially only on finite intervals of length m.

The set of local conserved quantities $\{\hat{I}_m\}$ defined in Eqs. (3.18) and (3.19) has the

further property of being *maximal*: any local conserved quantity $\hat{I}$ that commutes with all of the $\hat{I}_m$ can be expressed as a linear combination of them,

$$\left[\hat{I}_m,\, \hat{I}\,\right] = 0 \quad \forall\, m \quad \Longrightarrow \quad \hat{I} \in \mathrm{span}(\{\hat{I}_m\}), \tag{3.21}$$

where

$$\mathrm{span}(\{\hat{I}_m\}) \equiv \left\{ \sum_m a_m \hat{I}_m \,\middle|\, a_m \in \mathbb{R} \right\}. \tag{3.22}$$

This claim is easy to verify if we assume that $\hat{I}$ is a quadratic operator; the only quadratic operators that commute with $\hat{n}(k)$ for all k are indeed of the form $\hat{I} = \sum_k f(k)\hat{n}(k)$ for some function f. However, once we drop this assumption, the validity of the claim is much less obvious. One can certainly write down many *nonlocal* conserved quantities that violate Eq. (3.21)—products of mode occupation numbers, such as $\hat{n}(k)\hat{n}(k')$—and one might wonder whether it is possible to build a local quantity out of linear combinations of these, à la Eqs. (3.18) or (3.19). We will address this concern later in our general treatment: in Section 3.5.1, we prove that, for a wide class of quadratic hamiltonians (to which $\hat{H}_0$ belongs), all local conserved quantities $\hat{I}$ are themselves quadratic. The claim follows.

Thus, one is tempted to assert that the GGE density operator for the tight-binding chain has the form

$$\hat{\rho}_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_m \lambda_m \hat{I}_m \right) \tag{3.23a}$$

$$= \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_k \mu(k) \hat{n}(k) \right), \tag{3.23b}$$

where the Lagrange multipliers $\{\lambda_m\}$ are fixed by requiring that

$$\mathrm{Tr}\Big(\hat{I}_m\,\hat{\rho}_{\mathrm{GGE}}\Big) = \mathrm{Tr}\Big(\hat{I}_m\,\hat{\rho}_0\Big); \tag{3.24}$$

this in turn fixes the function $\mu(k)$, which is in general unrelated to the function $\omega(k)$ appearing in $\hat{H}_0$.

One may also consider *truncated GGEs* in which only the "most local" conservation laws are taken into account (i.e. only $\hat{I}_m$ with $m \leq 2\ell$ are retained in the density matrix) [86]; this is equivalent to truncating the Fourier series of $\mu(k)$ at order ℓ. More generally, in the limit of infinite system size, $L \to \infty$, one can require that λ_m decay in a certain manner as $m \to \infty$; this is equivalent to placing a smoothness condition on $\mu(k)$. Thus, the GGE (truncated or not) can be defined either in terms of the local charges $\hat{I}_m$ or in terms of the mode occupation numbers $\hat{n}(k)$ [86].

Actually, $\hat{\rho}_{\mathrm{GGE}}$ is not uniquely given by Eq. (3.23) for this model. Although the set $\{\hat{I}_m\}$ defined by Eqs. (3.18) and (3.19) is maximal, it is *not complete*: there exist local conserved quantities $\hat{I}'$ that *cannot* be expressed as linear combinations of the $\hat{I}_m$. A simple example [66] of such a quantity is

$$\hat{I}' = \sum_x (-1)^x \left(\hat{c}_x \hat{c}_{x+1} + \hat{c}^\dagger_{x+1}\hat{c}^\dagger_x\right). \tag{3.25}$$

In k-space, $\hat{I}'$ takes the form

$$\hat{I}' = \sum_k e^{-ik}\,\hat{c}(\pi - k)\hat{c}(k) + \text{h.c.}. \tag{3.26}$$

This quantity is conserved because the mode spectrum $\omega(k) = -\cos k$ of the hamiltonian

$\hat{H}_0$ of Eq. (3.16) has the symmetry

$$\omega(k) = -\omega(\pi - k). \tag{3.27}$$

One can verify that $\hat{I}'$ does not commute with the $\hat{I}_m$, so its existence does not contradict maximality of $\{\hat{I}_m\}$.

Note that the symmetry (3.27) is actually a *degeneracy* of the spectrum $|\omega(k)| = |\cos k|$ of positive-energy quasiparticles of $\hat{H}_0$. In general, the existence of "extra" local conserved quantities such as $\hat{I}'$—and the associated ambiguity in the definition of the GGE—is related to degeneracies in the quasiparticle spectrum of the hamiltonian [87]. We discuss the general relationship in Section 3.5.2.

One way to deal with an incomplete maximal set $\{\hat{I}_m\}$ is to simply *complete* it by adding to $\{\hat{I}_m\}$ additional local conserved quantities, such as $\hat{I}'$. This is the approach advocated by Fagotti [87], who studied this problem in significant detail. The operators comprising the expanded set will no longer be in involution, but one can still assign to each one a Lagrange multiplier and define $\hat{\rho}_{\mathrm{GGE}}$ by maximizing the entropy subject to all constraints. We obtain an expression identical to Eq. (3.23a), but where the index m ranges over the complete set. This maneuver is valid because local conserved quantities satisfy a closed algebra [87] (of which the various maximal sets are maximal abelian subalgebras). The advantage of this approach is that the resulting $\hat{\rho}_{\mathrm{GGE}}$ depends on the initial state only through the Lagrange multipliers $\{\lambda_m\}$. The primary disadvantage is that one can no longer write $\hat{\rho}_{\mathrm{GGE}}$ in terms of a single set of mode occupation numbers, as in Eq. (3.23b).

Our approach to this problem, which we describe in Section 3.5.2, is slightly different. In short, we retain $\hat{\rho}_{\mathrm{GGE}}$ in the original form (3.23), but allow the maximal commuting set $\{\hat{I}_m\}$, or equivalently the set of mode occupation numbers $\{\hat{n}_\alpha\}$, to depend on the

initial state. In this approach, $\hat{\rho}_{\rm GGE}$ always has a mode number representation similar to Eq. (3.23b); however, different classes of initial states lead to *inequivalent* GGEs.

For now, we can ignore these subtleties, because we assumed that the initial state $\hat{\rho}_0$ conserves total particle number (Eq. (3.13)). For this class of initial states, the GGE is correctly given by Eqs. (3.23) and (3.24), with $\hat{I}_m$ defined in Eqs. (3.18) and (3.19), and $\hat{n}(k)$ in Eq. (3.17). We leave the proof of this assertion as an exercise for the reader.

3.2.3 Relaxation of local observables: preliminaries

Having defined the GGE, we turn to the relaxation of local observables. It is convenient to work in the Heisenberg picture. The operators representing observables evolve according to

$$\hat{\mathcal{O}}(t) = e^{i\hat{H}_0 t}\hat{\mathcal{O}}e^{-i\hat{H}_0 t}, \tag{3.28}$$

while the density operator is always $\hat{\rho}_0$. The expectation value of an observable at time t is

$$\langle\hat{\mathcal{O}}(t)\rangle \equiv \mathrm{Tr}\Big(\hat{\mathcal{O}}(t)\,\hat{\rho}_0\Big). \tag{3.29}$$

By a *local observable* we mean any bosonic hermitian operator $\hat{\mathcal{O}}$ that acts nontrivially only on a finite interval (at time $t = 0$). Consider the quantity

$$R_{\mathcal{O}}(t) \equiv \lim_{L\to\infty}\Big[\langle\hat{\mathcal{O}}(t)\rangle - \langle\hat{\mathcal{O}}\rangle_{\rm GGE}\Big], \tag{3.30}$$

where

$$\langle\hat{\mathcal{O}}\rangle_{\rm GGE} \equiv \mathrm{Tr}(\hat{\mathcal{O}}\,\hat{\rho}_{\rm GGE}). \tag{3.31}$$

We say that the system *relaxes (locally) to the GGE* if

$$R_{\mathcal{O}}(t) \to 0 \quad \text{as} \quad t \to \infty \tag{3.32}$$

for every local observable $\hat{\mathcal{O}}$.

Now, any number-conserving local observable has a unique expansion of the form

$$\hat{\mathcal{O}} = \mathcal{O}^{(0)} + \sum_{x,y} \mathcal{O}^{(1)}_{xy} \hat{c}^\dagger_x \hat{c}_y + \sum_{x,x',y,y'} \mathcal{O}^{(2)}_{xx'yy'} \hat{c}^\dagger_x \hat{c}^\dagger_{x'} \hat{c}_y \hat{c}_{y'} + \cdots , \tag{3.33}$$

where locality implies that all sums over positions are restricted to a finite interval, and therefore that the expansion terminates at a finite order (because the space of operators supported on a finite interval in a system of fermions is finite dimensional). Our simplifying assumption (3.13) on the initial state means that we do not need to consider non-number-conserving observables; their expectation values vanish identically.

Thus, it is sufficient to study the relaxation of local *static* $2n$*-point correlation functions*:

$$\langle \hat{c}^\dagger_{x_1}(t) \hat{c}^\dagger_{x_2}(t) \cdots \hat{c}^\dagger_{x_n}(t) \hat{c}_{y_1}(t) \hat{c}_{y_2}(t) \cdots \hat{c}_{y_n}(t) \rangle . \tag{3.34}$$

More generally, one might also consider *dynamic* correlation functions, in which the various t's are allowed to be different. These describe, for instance, the response of the system to an external probe.

For systems with a Lieb-Robinson bound [88], there is a general result [89] which states that, if the system relaxes to a stationary state $\hat{\rho}_{\text{stat}}$ as $t \to \infty$ (as measured by local static correlations), then all local dynamic correlations are also described by $\hat{\rho}_{\text{stat}}$ as $t \to \infty$. In this section, we will simply appeal to this result and concentrate on static correlations. But in fact, most of our arguments apply equally well to dynamic correlations, and we will work directly with the latter in the general treatment from

Section 3.3 onwards. We will do this, despite the result of Ref. [89], for two reasons: firstly, to keep our arguments self-contained, and secondly, because we are interested not just in the limiting behavior of quantities as $t \to \infty$, but also in the manner in which they relax to those limits.

Note that the density operator $\hat{\rho}_{\mathrm{GGE}}$ is gaussian—it is the exponential of a quadratic form in the creation and annihilation operators. Therefore, all correlation functions computed with respect to $\hat{\rho}_{\mathrm{GGE}}$ *Wick factorize* into products of two-point functions, and are determined entirely by the latter:

$$\langle \hat{c}^\dagger_{x_1} \cdots \hat{c}^\dagger_{x_n} \hat{c}_{y_1} \cdots \hat{c}_{y_n} \rangle_{\mathrm{GGE}} = \sum_P \mathrm{sgn}(P) \, \langle \hat{c}^\dagger_{x_1} \hat{c}_{y_{P(n)}} \rangle_{\mathrm{GGE}} \cdots \langle \hat{c}^\dagger_{x_n} \hat{c}_{y_{P(1)}} \rangle_{\mathrm{GGE}}, \tag{3.35}$$

where $\mathrm{sgn}(P)$ is the sign of the permutation

$$P : (1, 2, \ldots, n) \mapsto (P(1), P(2), \ldots, P(n)). \tag{3.36}$$

For instance,

$$\langle \hat{c}^\dagger_{x_1} \hat{c}^\dagger_{x_2} \hat{c}_{x_3} \hat{c}_{x_4} \rangle_{\mathrm{GGE}} = \langle \hat{c}^\dagger_{x_1} \hat{c}_{x_4} \rangle_{\mathrm{GGE}} \, \langle \hat{c}^\dagger_{x_2} \hat{c}_{x_3} \rangle_{\mathrm{GGE}} - \langle \hat{c}^\dagger_{x_1} \hat{c}_{x_3} \rangle_{\mathrm{GGE}} \, \langle \hat{c}^\dagger_{x_2} \hat{c}_{x_4} \rangle_{\mathrm{GGE}}. \tag{3.37}$$

Since the initial state $\hat{\rho}_0$ need not be gaussian, the real correlation functions certainly need not behave in this manner at early times. To show relaxation to the GGE, we therefore need to show that, as time progresses, (i) Wick factorization is recovered, and (ii) two-point correlation functions approach their stationary GGE values.

In any state, such as $\hat{\rho}_0$, one can also define the *connected* $2n$-point correlation function. Roughly speaking, this is the part of the $2n$-point correlation function that *fails* to

factorize into lower-point correlation functions. For instance,

$$\langle\langle \hat{c}^\dagger_{x_1} \hat{c}^\dagger_{x_2} \rangle\rangle = \langle \hat{c}^\dagger_{x_1} \hat{c}^\dagger_{x_2} \rangle, \tag{3.38}$$

and

$$\langle\langle \hat{c}^\dagger_{x_1} \hat{c}^\dagger_{x_2} \hat{c}_{x_3} \hat{c}_{x_4} \rangle\rangle = \langle \hat{c}^\dagger_{x_1} \hat{c}^\dagger_{x_2} \hat{c}_{x_3} \hat{c}_{x_4} \rangle - \langle \hat{c}^\dagger_{x_1} \hat{c}_{x_4} \rangle \langle \hat{c}^\dagger_{x_2} \hat{c}_{x_3} \rangle + \langle \hat{c}^\dagger_{x_1} \hat{c}_{x_3} \rangle \langle \hat{c}^\dagger_{x_2} \hat{c}_{x_4} \rangle. \tag{3.39}$$

The general definition of connected functions is reviewed in Appendix 3.A. The vanishing of all connected $(2n > 2)$-point correlation functions is equivalent to Wick factorization, as is evident from the formulae above:

$$\Big(\langle \hat{c}^\dagger_{x_1} \cdots \hat{c}^\dagger_{x_n} \hat{c}_{y_1} \cdots \hat{c}_{y_n} \rangle \ \text{ Wick factorizes } \forall\ n \Big)$$
$$\iff \quad \Big(\langle\langle \hat{c}^\dagger_{x_1} \cdots \hat{c}^\dagger_{x_n} \hat{c}_{y_1} \cdots \hat{c}_{y_n} \rangle\rangle = 0 \quad \forall\ n \geq 2 \Big). \tag{3.40}$$

Therefore, we are led to study the relaxation of static local connected $2n$-point correlation functions:

$$\langle\langle \hat{c}^\dagger_{x_1}(t) \hat{c}^\dagger_{x_2}(t) \cdots \hat{c}^\dagger_{x_n}(t) \hat{c}_{y_1}(t) \hat{c}_{y_2}(t) \cdots \hat{c}_{y_n}(t) \rangle\rangle. \tag{3.41}$$

These functions, and their dynamic brethren, will be the primary objects of study in this chapter.

We can now state precisely the *cluster decomposition* condition that the initial state $\hat{\rho}_0$ is assumed to satisfy (in this section). We assume that

$$\langle\langle \hat{c}^\dagger_{x_1} \cdots \hat{c}^\dagger_{x_n} \hat{c}_{x_{n+1}} \cdots \hat{c}_{x_{2n}} \rangle\rangle = o(e^{-|x_i - x_j|/\xi}) \quad \text{as} \quad |x_i - x_j| \to \infty, \tag{3.42}$$

for any pair of indices $i, j \in 1, 2, \ldots, 2n$, where ξ is some finite correlation length, and

where "$f(x) = o(g(x))$ as $x \to a$" means that $f(x)/g(x) \to 0$ as $x \to a$. In the general treatment of Section 3.3 onwards, we will significantly weaken this assumption, and require only *algebraic* decay (rather than exponential) of the initial connected correlation functions (Eq. (3.42) will be replaced by Eq. (3.83)).

3.2.4 The single-particle propagator

Because $\hat{H}_0$ is quadratic, the fermion operators

$$\hat{c}_x(t) = e^{i\hat{H}_0 t}\hat{c}_x e^{-i\hat{H}_0 t} \tag{3.43}$$

evolve linearly; they obey

$$\hat{c}_x(t) = \sum_y G_{xy}(t)\hat{c}_y, \tag{3.44}$$

where

$$G_{xy}(t) = \frac{1}{L}\sum_k e^{ik(x-y)+it\cos k}. \tag{3.45}$$

It follows from Eq. (3.44) that

$$\langle 0|\hat{c}_x(t)\hat{c}_y^\dagger(0)|0\rangle = G_{xy}(t), \tag{3.46}$$

where $|0\rangle$ is the fermion vacuum, specified by

$$\hat{c}_x\,|0\rangle = 0 \quad \forall\ x. \tag{3.47}$$

Thus, $G_{xy}(t)$, which is defined as the coefficient appearing in Eq. (3.44), may be identified as the single-particle propagator (the amplitude for a particle added to the vacuum at site y to be found after time t at site x).

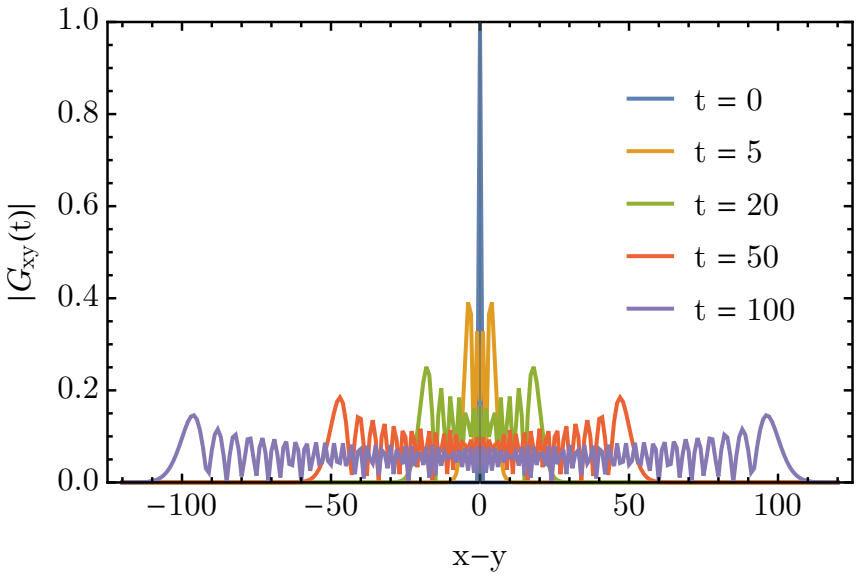


Figure 1: Magnitude of the single-particle propagator $|G_{xy}(t)| = |J_{x-y}(t)|$ for the model described by Eq. (3.11).

Equation (3.44) also implies that

$$\mathrm{Tr}\big(\big[\hat{c}_x(t),\, \hat{c}_y^\dagger(0)\big]_+ \hat{\rho}_0\big) = G_{xy}(t), \tag{3.48}$$

where $\big[\hat{a},\, \hat{b}\big]_+ \equiv \hat{a}\hat{b} + \hat{b}\hat{a}$ denotes the anticommutator. Thus, $G_{xy}(t)$ can also be identified with the retarded single-particle Green's function (if the hamiltonian is quadratic, this quantity is independent of the state $\hat{\rho}_0$).

These interpretations are useful for guessing properties of $G_{xy}(t)$ in situations in which one cannot write down a simple expression for it. We will not need to rely on intuition in this section, however. In the limit $L \to \infty$, one has

$$G_{xy}(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(x-y)k + it\cos k} = i^{x-y} J_{x-y}(t), \tag{3.49}$$

where $J_n(z)$ is the Bessel function of order $n \in \mathbb{Z}$. The magnitude of the propagator,

$|G_{xy}(t)|$, is plotted for various values of t in Figure 1.

We will show in the following that the leading late-time behavior of the connected $(2n \geq 4)$-point functions (3.41) is actually determined by very basic properties of the propagator. Basically, all that matters is how the propagator "spreads out" with time. Let us characterize this "spreading out" more precisely.

Although $|G_{xy}(t)|$ itself is a rapidly oscillating function of $x-y$ at fixed t, its smooth envelope is nonzero and slowly varying inside the "lightcone" $|x-y| < t$, and decays exponentially to zero for $|x-y| > t$. Qualitatively, this can be seen by glancing at Figure 1. More quantitatively, one can apply the method of stationary phase [90] to the integral expression in Eq. (3.49) to obtain

$$|G_{xy}(t)| \sim \left[\frac{4}{\pi^2(t^2-r^2)}\right]^{1/4} \left|\cos\left(\tfrac{\pi}{2}(r-\tfrac{1}{2}) - r\arcsin(r/t) - \sqrt{t^2-r^2}\right)\right| \qquad \text{as} \quad t\to\infty \quad \text{if} \quad \frac{|r|}{t} < 1 - O(t^{-1/3}), \quad (3.50)$$

and

$$|G_{xy}(t)| = o(t^{-n}) \quad \forall\ n \quad \text{as} \quad t\to\infty \quad \text{if} \quad \frac{|r|}{t} > 1 + O(t^{-1/3}), \qquad (3.51)$$

where $r = x-y$. The $|\cos(\cdots)|$ factor in Eq. (3.50) describes the lattice-scale oscillations of $|G_{xy}(t)|$; we replace it with a constant to obtain the smooth envelope.

The two relevant properties of the propagator are that the interval of $|x-y|$ values over which $G_{xy}(t)$ is non-negligible grows linearly with t, and that the matrix elements of $G_{xy}(t)$ inside this interval have a typical magnitude $\propto t^{-1/2}$. The second property can be extracted from Eq. (3.50), but it can also be deduced very simply from the first property, as follows. Unitarity of time-evolution implies that $G(t)$ is a unitary matrix:

$$1 = \sum_y |G_{xy}(t)|^2. \qquad (3.52)$$

We can restrict the sum to the interval over which $|G_{xy}(t)|$ is non-negligible:

$$1 \approx \sum_{y \approx x-t}^{x+t} |G_{xy}(t)|^2. \tag{3.53}$$

Since the envelope of $|G_{xy}(t)|$ is nonzero and slowly varying within this interval, and since the interval grows linearly with t, one must have $|G_{xy}(t)| \sim t^{-1/2}$.

3.2.5 Decay of local connected $(n \geq 4)$-point functions. "Gaussification"

We are now in a position to understand why Wick factorization is recovered as t increases. Consider the equal-time connected 4-point function $\langle\langle \hat{c}^\dagger_{x_1}(t)\hat{c}^\dagger_{x_2}(t)\hat{c}_{x_3}(t)\hat{c}_{x_4}(t)\rangle\rangle$. Equation (3.39) shows that this function measures the extent to which the 4-point function $\langle \hat{c}^\dagger_{x_1}(t)\hat{c}^\dagger_{x_2}(t)\hat{c}_{x_3}(t)\hat{c}_{x_4}(t)\rangle$ fails to Wick factorize. Using Eq. (3.44) and its adjoint to express the operators at time t in terms of operators at time zero,

$$\langle\langle \hat{c}^\dagger_{x_1}(t)\hat{c}^\dagger_{x_2}(t)\hat{c}_{x_3}(t)\hat{c}_{x_4}(t)\rangle\rangle = \sum_{y_1 \cdots y_4} G^*_{x_1 y_1}(t) G^*_{x_2 y_2}(t) G_{x_3 y_3}(t) G_{x_4 y_4}(t)\ \langle\langle \hat{c}^\dagger_{y_1}\hat{c}^\dagger_{y_2}\hat{c}_{y_3}\hat{c}_{y_4}\rangle\rangle. \tag{3.54}$$

We can estimate the magnitude of this quantity by multiplying the number of significant terms in the sum by the typical magnitude of each one. We have already seen that $|G_{xy}(t)|$ is negligible outside the lightcone $|x-y| \sim t$, and that it has typical magnitude $|G_{xy}(t)| \sim t^{-1/2}$ inside. By our assumption (3.42) on exponential clustering of correlations in the initial state, the function $\langle\langle \hat{c}^\dagger_{y_1}\hat{c}^\dagger_{y_2}\hat{c}_{y_3}\hat{c}_{y_4}\rangle\rangle$ is negligible whenever $|y_i - y_j| > \xi$, where ξ is the finite correlation length. As a result, the sum over $\mathbf{y} = (y_1 \cdots y_4)$ in Eq. (3.54) is restricted to a region of size $\sim \xi^3 t$ (this is illustrated in Figure 2):

$$\mathrm{Vol}\big\{\mathbf{y}\big|\big[G^*_{x_1 y_1}(t) G^*_{x_2 y_2}(t) G_{x_3 y_3}(t) G_{x_4 y_4}(t)\ \langle\langle \hat{c}^\dagger_{y_1}\hat{c}^\dagger_{y_2}\hat{c}_{y_3}\hat{c}_{y_4}\rangle\rangle\big] \text{ non-negligible}\big\} \sim \xi^3 t, \tag{3.55}$$

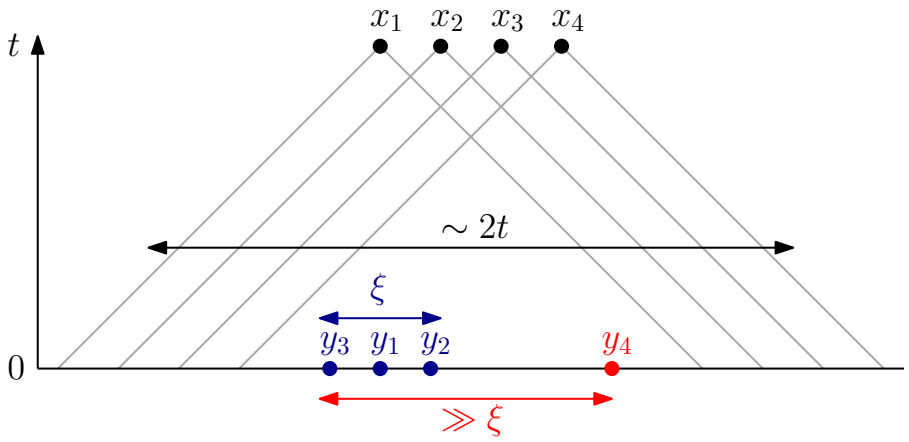


Figure 2: Schematic showing spreading of operators in the model of Eq. (3.11), and how this leads to the decay of connected correlation functions as time t increases. The points x_j are the locations of the operators on the left side of Eq. (3.54); the points y_j are for a representative term in the sum on the right side of this equation. Each y_j must lie inside the backward light-cone of x_j in order for the propagator $G_{x_j y_j}(t)$ to be nonzero. Configurations of the y's in which the distance between any pair is much greater than the correlation length ξ (as is the case in the figure) give negligible contributions due to clustering of correlations in the initial state. This effectively restricts the sum over $y_1 \cdots y_4$ to a region of size $\sim \xi^3 t$ (Eq. (3.55)).

while each term in the sum is of order

$$|G_{xy}(t)|^4 \sim t^{-2}. \tag{3.56}$$

Hence the right hand side of Eq. (3.54) is of order $\sim t^{-1}$ and the connected function on the left vanishes in this manner as $t \to \infty$.

As t increases, there will be additional constructive or destructive interference between different terms in the sum of Eq. (3.54), that we have not taken into account in our crude accounting. Thus, we expect in general that

$$\langle\langle \hat{c}^\dagger_{x_1}(t)\hat{c}^\dagger_{x_2}(t)\hat{c}_{x_3}(t)\hat{c}_{x_4}(t)\rangle\rangle \sim \frac{z(t)}{t} \quad \text{as} \quad t \to \infty, \tag{3.57}$$

where $z(t)$ is some oscillatory function of time.

A similar argument shows that

$$\langle\langle \hat{c}^\dagger_{x_1}(t)\cdots\hat{c}^\dagger_{x_n}(t)\hat{c}_{x_{n+1}}(t)\cdots\hat{c}_{x_{2n}}(t)\rangle\rangle \sim \frac{z_n(t)}{t^{n/2-1}} \tag{3.58}$$

as $t\to\infty$, where the $z_n(t)$ are some other oscillatory functions of time.

Thus, as $t\to\infty$, only the fully disconnected parts of local correlation functions survive (the parts that factorize into products of 2-point functions); in other words, we recover Wick factorization as $t\to\infty$. As mentioned above, this is the defining property of a gaussian density matrix. We conclude that, as $t\to\infty$, the "local state of the system" may be described by a density matrix of the form

$$\hat{\rho}_1(t) = \frac{1}{Z_1(t)}\exp\left(-\sum_{x,y}\hat{c}^\dagger_x K_{xy}(t)\hat{c}_y\right), \tag{3.59}$$

where $K_{xy}(t)$ is chosen such that

$$\mathrm{Tr}\big(\hat{c}^\dagger_x(t)\hat{c}_y(t)\hat{\rho}_1(t)\big) = \langle\hat{c}^\dagger_x(t)\hat{c}_y(t)\rangle \tag{3.60}$$

for all sites x, y with $|x-y|$ finite in the limit $L\to\infty$, and $Z_1(t)$ ensures normalization. As long as $K_{xy}(t)$ is chosen to satisfy this condition at each time t (actually, Eq. (3.60) only needs to hold up to terms of order $\sim t^{-1}$), we have

$$\langle\hat{\mathcal{O}}(t)\rangle \sim \mathrm{Tr}\big(\hat{\mathcal{O}}(t)\hat{\rho}_1(t)\big) + O(t^{-1}) \quad \text{as} \quad t\to\infty, \tag{3.61}$$

for all local observables $\hat{\mathcal{O}}$.

We have shown that the state becomes "locally gaussian" at late times. Following Ref. [72], we refer to this process as "gaussification". In Section 3.4 we describe gaussification in arbitrary quadratic lattice models by generalizing the chain of reasoning leading

from Eq. (3.44) to Eq. (3.58).

3.2.6 Equilibration of the local 2-point function to its GGE value

It remains to compute the local equal-time 2-point function, and to verify that it relaxes to its stationary GGE value. By definition of the GGE, Eq. (3.23), this stationary value is

$$\langle \hat{c}^\dagger_{x_1} \hat{c}_{x_2} \rangle_{\mathrm{GGE}} = \frac{1}{L} \sum_k e^{-ik(x_1 - x_2)} \langle \hat{n}(k) \rangle, \tag{3.62}$$

where $\langle \hat{n}(k) \rangle$ is the expectation of the mode occupation number $\hat{n}(k)$ in the initial state.

The results of this subsection depend on a third assumption about the initial state $\hat{\rho}_0$, in addition to Eqs. (3.13) and (3.42). Roughly speaking, we want to exclude situations in which the initial profiles of local conserved densities are inhomogeneous on length scales comparable to the system size—for instance, an initial state in which sites $x = 1, 2, \ldots, L/2$ are occupied by fermions and the rest are empty. True local equilibration in such cases occurs on timescales of order L, simply because that is how long it takes a locally conserved density to flow across the system.

In order to formulate this assumption precisely, recall that the local conserved quantities (Eqs. (3.18) and (3.19)) are of the form

$$\hat{I}_m = \sum_{x=1}^{L} \hat{\mathcal{I}}_{m,x}, \tag{3.63}$$

where the density $\hat{\mathcal{I}}_{m,x}$ is supported on a finite interval of length $\lfloor m/2 \rfloor$ near site x.

Define the "local excess density"

$$\delta\mathcal{I}_m(x_0; L_0) \equiv \frac{1}{L_0} \sum_{x=x_0-L_0/2}^{x_0+L_0/2} \langle \hat{\mathcal{I}}_{m,x} \rangle - \frac{1}{L} \langle \hat{I}_m \rangle. \tag{3.64}$$

We assume that these excess densities can be made small by taking L_0 sufficiently large (but finite and independent of L as $L \to \infty$):

$$\exists\, L_0 : \ \delta\mathcal{I}_m(x_0; L_0) = O\Big(\frac{1}{L_0}\Big) \ \forall \ x_0, m \quad \text{as} \quad L \to \infty. \tag{3.65}$$

We emphasize that the results of the previous subsections hold even when this assumption is violated. In particular, the system still "gaussifies" as described in Section 3.2.5. Thus, if the initial state violates Eq. (3.65), the natural description of the local state of the system at late times is in terms of a time-dependent gaussian density matrix, given by Eqs. (3.59) and (3.60).

To study relaxation of the 2-point function, we will finally need to use the diagonal form (3.16) of the hamiltonian $\hat{H}_0$, or equivalently, the full form (3.44) of the propagator $G_{xy}(t)$. We may write

$$\langle \hat{c}^\dagger_{x_1}(t) \hat{c}_{x_2}(t) \rangle = \frac{1}{L} \sum_{k_1,k_2} e^{-i(k_1 x_1 - k_2 x_2)} e^{-i(\cos k_1 - \cos k_2)t} \, F(k_1, k_2), \tag{3.66}$$

where

$$F(k_1, k_2) \equiv \langle \hat{c}^\dagger(k_1) \hat{c}(k_2) \rangle. \tag{3.67}$$

We begin by showing that, under the assumptions we have made, the function F

must have the form

$$F(k_1, k_2) = \delta_{k_1,k_2} \langle \hat{n}(k_1) \rangle + \sum_{j=1}^{j_{\max}} \delta_{k_1 - q_j, k_2} \, f_j(k_1) + \frac{1}{L} f(k_1, k_2), \tag{3.68}$$

where each $q_j \neq 0$ remains finite in the limit $L \to \infty$, and where $\langle \hat{n}(k) \rangle$, $f_j(k)$, and $f(k, k')$ are smooth $O(1)$ functions. The various Kronecker deltas contain all of the singular dependence of $F(k_1, k_2)$ on its arguments.

We arrive at Eq. (3.68) as follows. Invert the Fourier transformation and write

$$\begin{aligned} F(k_1, k_2) &= \frac{1}{L} \sum_{y_1, y_2} e^{i(k_1 y_1 - k_2 y_2)} \langle \hat{c}^\dagger_{y_1} \hat{c}_{y_2} \rangle \\ &= \frac{1}{L} \sum_{y_1, y_2} e^{i(k_1 + k_2)(y_1 - y_2)/2} \, e^{i(k_1 - k_2)(y_1 + y_2)/2} \, \langle \hat{c}^\dagger_{y_1} \hat{c}_{y_2} \rangle. \end{aligned} \tag{3.69}$$

The sums over y_1 and y_2 in Eq. (3.69) may be performed with respect to the central coordinate $(y_1 + y_2)/2$ and relative coordinate $(y_1 - y_2)$. Due to clustering of correlations, the sum over the relative coordinate converges absolutely (it is effectively restricted to a finite window $|y_1 - y_2| \lesssim \xi$), and consequently F must be a smooth function of $(k_1 + k_2)$. On the other hand, the central coordinate is summed over the whole system, and so F can depend in a singular manner on $(k_1 - k_2)$. In particular, $F(k_1, k_2)$ is $O(1)$ if and only if the terms in the sum over the central coordinate add constructively. This occurs when $(k_1 - k_2) = 0$ (in which case the phase factor in Eq. (3.67) is independent of the central coordinate), but it may also occur for $(k_1 - k_2) = q \neq 0$ if the initial state has a density wave with wavevector q, so that $\langle \hat{c}^\dagger_{y_1} \hat{c}_{y_2} \rangle \propto e^{-iq(y_1 + y_2)/2}$. Our extra assumption on the initial state, Eq. (3.65), implies that $q \not\to 0$ as $L \to \infty$. This establishes the validity of Eq. (3.68).

Using Eq. (3.68) in Eq. (3.66), we obtain

$$\langle \hat{c}^\dagger_{x_1}(t)\hat{c}_{x_2}(t)\rangle = \frac{1}{L}\sum_k e^{-ik(x_1-x_2)}\langle \hat{n}(k)\rangle + \sum_{j=1}^{j_{\max}} \delta C^{(j)}_{x_1,x_2}(t) + \delta C_{x_1,x_2}(t), \tag{3.70}$$

where

$$\delta C^{(j)}_{x_1,x_2}(t) = \frac{1}{L}\sum_k e^{-ik(x_1-x_2)-iq_jx_2}e^{-i[\cos k-\cos(k-q_j)]t}\, f_j(k), \tag{3.71}$$

and

$$\delta C_{x_1,x_2}(t) = \frac{1}{L^2}\sum_{k_1,k_2} e^{-i(k_1x_1-k_2x_2)}e^{-i(\cos k_1-\cos k_2)t}\, f(k_1,k_2). \tag{3.72}$$

When $t \gg 1$, we may apply the method of stationary phase to estimate the time-dependent pieces. The cleanest way to do this is to first take $L \to \infty$, so that $\frac{1}{L}\sum_k \to \int \frac{dk}{2\pi}$, and only then take t large, and that it what we will do here. However, we note in passing that it is also possible to perform a similar analysis *without* first taking $L \to \infty$; one can use the Poission summation formula to represent $\frac{1}{L}\sum_k$ as a sum of integrals—each integral corresponding to a translated copy of the finite system—and then estimate each of these integrals by stationary phase. As long as $t < L/v_{\max}$, where $v_{\max}$ is the maximal group velocity of particles in the system, the extra translated integrals generate only exponentially small (in t) corrections to the $L \to \infty$ result.

For completeness, let us briefly review the method of stationary phase. This method is described in detail in many standard texts, such as Ref. [90]. A nice heuristic and mathematically elementary treatment may be found in Section 3.3 of Ref. [91]. In the limit $L \to \infty$, Eqs. (3.71) and (3.72) are both of the general form

$$I(t) = \int \frac{d^dk}{(2\pi)^d}\, a(k)\, e^{i\varphi(k)t}, \tag{3.73}$$

where a and φ are smooth functions. The k-integral is one-dimensional in Eq. (3.71)

and two-dimensional in Eq. (3.72). In both cases, the integral is over a compact region without boundary. As $t \to \infty$, the dominant contributions to the $I(t)$ integral come from the vicinity of points k_* at which $\nabla_k \varphi(k_*) = 0$, called *critical points* of φ. A critical point k_* is *nondegenerate* if the Hessian matrix at that point,

$$\mathbf{H}_{ab}(k_*) \equiv \frac{\partial}{\partial k_a} \frac{\partial}{\partial k_b} \varphi(k_*), \tag{3.74}$$

is invertible. Each isolated nondegenerate critical point k_j gives a contribution $I_j(t)$ to $I(t)$ that can be obtained (to leading order in t) by expanding the phase function $\varphi(k)$ up through quadratic order in $(k - k_j)$, extending the limits of the k-integral to infinity, and performing the resulting gaussian integral; the result is

$$I_j(t) = \frac{e^{i(\pi/4)s_j}}{(2\pi t)^{d/2} \left|\det \mathbf{H}(k_j)\right|^{1/2}} a(k_j)\, e^{i\varphi(k_j)t} + \cdots, \tag{3.75}$$

where s_j is the signature (number of positive eigenvalues minus number of negative eigenvalues) of the symmetric matrix $\mathbf{H}(k_j)$. The dots are subleading terms proportional to higher derivatives of $a(k)$ evaluated at k_j. Terms with n derivatives are suppressed relative to the leading term by an additional factor of $t^{-n/2}$.

We obtain $I(t)$ by simply adding up these contributions (assuming φ has no other critical points):

$$I(t) \sim \sum_j I_j(t). \tag{3.76}$$

Thus, whenever the phase function φ has a finite number of critical points, all of which are nondegenerate (and assuming that the amplitude function $a(k)$ does not vanish at all of these points),

$$I(t) \sim t^{-d/2} \quad \text{as} \quad t \to \infty. \tag{3.77}$$

This is the generic situation.

If, however, φ does have *degenerate* critical points, their contributions must also be accounted for. The power of t associated with such a contribution can often be estimated very simply as follows. Assume that k_* is a critical point at which $\varphi(k) - \varphi(k_*)$ has a zero of order m, while $a(k)$ has a zero of order n (that is, the Taylor expansions of these functions about $k = k_*$ start with monomials of order m and n respectively). In spherical coordinates centered at k_*, we would have $\varphi(k_* + k) \approx \varphi(k_*) + |k|^m \Phi(\theta)$ and $a(k_* + k) \approx |k|^n A(\theta)$, where Φ and A are appropriate functions of the angular variables, collectively denoted θ. Thus the leading contribution from the critical point is of the form

$$I_j(t) \sim \int \frac{k^{d-1}\, dk}{(2\pi)^d} \int d\Omega\; k^n A(\theta)\, e^{ik^m \Phi(\theta) t}. \tag{3.78}$$

Scaling t out of the integral by changing integration variables to $p = t^{1/m} k$, we obtain the estimate

$$I_j(t) \sim t^{-(d+n)/m} \quad \text{as} \quad t \to \infty. \tag{3.79}$$

Note that larger m leads to slower decay. Thus, in the (non-generic) case that φ has degenerate critical points, the "most degenerate" of these will typically dominate the $t \to \infty$ behavior of $I(t)$. This concludes our brief mathematical interlude.

For a given $q_j \neq 0$, the phase function $\varphi(k; q_j) = \cos(k - q_j) - \cos(k)$ appearing in Eq. (3.71) has precisely two distinct nondegenerate critical points: $k = k_\pm = \frac{1}{2}(q_j \pm \pi)$. Thus, assuming that $f_j(k)$ does not vanish at these points, the method of stationary phase yields

$$\delta C^{(j)}_{x_1,x_2}(t) \sim t^{-1/2} \quad \text{as} \quad t \to \infty. \tag{3.80}$$

A similar analysis applies to Eq. (3.72). In this case, the phase function $\varphi(k_1, k_2) = \cos k_2 - \cos k_1$ has precisely four distinct nondegenerate critical points: $(k_1, k_2) = (0, 0)$,

$(0, \pi)$, $(\pi, 0)$, and (π, π). Thus, assuming that $f(k_1, k_2)$ does not vanish at these points,

$$\delta C_{x_1,x_2}(t) \sim t^{-1} \quad \text{as} \quad t \to \infty. \tag{3.81}$$

Note that the locations of the critical points in k-space are determined by the dispersion relation of the hamiltonian $\hat{H}_0$, whereas the functions $f_j(k)$ and $f(k_1, k_2)$ are determined by the initial state. Therefore, these functions will only vanish at the critical points for special, fine-tuned, choices of the initial state. We conclude that, for *generic* initial states, as $t \to \infty$,

$$\langle \hat{c}^\dagger_{x_1}(t) \hat{c}_{x_2}(t) \rangle \sim \langle \hat{c}^\dagger_{x_1} \hat{c}_{x_2} \rangle_{\text{GGE}} + R_{x_1 x_2}(t), \tag{3.82}$$

where the remainder $R_{xy}(t)$ is of order $t^{-1/2}$ if the initial state has a density wave, i.e. if $\langle c^\dagger(k) \hat{c}(k - q) \rangle$ is sharply peaked at one or more nonzero wavevectors q, and is of order t^{-1} if not.

We have now explicitly shown that, for any initial state $\hat{\rho}_0$ that satisfies Eqs. (3.13), (3.42) and (3.65), all local observables of the system relax to their values in the GGE (3.23) as $t \to \infty$ under time evolution generated by $\hat{H}_0$. Furthermore, we have obtained the exponents of the power laws governing the relaxation processes. We have shown that if the initial state has a density wave, then we generically expect the system to relax first to a (time-dependent) gaussian state like $\sim t^{-1}$, and then to relax to the GGE like $\sim t^{-1/2}$. In Section 3.4 we describe relaxation of the local 2-point function—and hence relaxation of a gaussified state to the GGE—in arbitrary quadratic models, by generalizing the chain of reasoning leading from Eq. (3.66) to Eq. (3.82).

Although we derived them for the specific model of Eq. (3.11), the relaxation exponents 1/2 and 1 are actually generic for quenches to clean quadratic fermion models in one dimension. Different exponents may be obtained if the final hamiltonian is fine-tuned

(so that the dispersion relation has degenerate critical points) and/or if the initial state is fine-tuned (so that the functions $f_j(k)$ and $f(k_1, k_2)$ vanish at the critical points). For instance, Ref. [92] studied parameter quenches in a dimerized chain and in the Kitaev model of a 1d spinless p-wave superconductor, and obtained parameter-dependent relaxation exponents for the 2-point function. In all cases, however, the exponents can be associated to degenerate critical points and/or to the vanishing of $f_j(k)$ or $f(k_1, k_2)$ at the critical points, and their values agree with the simple estimate (3.79) (the authors of [92] perform a more sophisticated steepest descent analysis to also obtain the prefactors). Moreover, one can easily verify that generic small perturbations of the pre-quench state and post-quench hamiltonian cause the exponents to return to the parameter-independent values 1/2 and 1.

Finally, we briefly comment on relaxation from initial states that violate Eq. (3.65). One might still expect the conclusions of this section to apply *locally*, so that the system relaxes as described above toward a "local GGE" in which the Lagrange multipliers are slowly varying functions of position and time. This "local GGE" would in turn relax—over timescales comparable to the system size—to the global GGE of Eq. (3.23), in a manner consistent with a generalized theory of hydrodynamics [93]. This is certainly a tempting picture, but because one cannot associate a timescale to local power-law relaxation, it is not immediately clear that such a description—based on separation of timescales—is self-consistent. We will not explore these questions further in this thesis.

3.3 General treatment. Setup and basic definitions

3.3.1 System

We consider a lattice system of fermions or bosons in d dimensions, with one orbital per lattice site and N sites in total (the generalization to multiple orbitals per site is straightforward, and merely complicates the bookkeeping). Let $\hat{\psi}_x^-$ and $\hat{\psi}_x^+ = (\hat{\psi}_x^-)^\dagger$ denote the annihilation and creation operators respectively for the site at position x.

Although we work on a lattice, we believe that many of our arguments also apply in the continuum limit, if the symbols in the equations are reinterpreted correctly; in particular $\hat{\psi}_x^\pm$ should be regarded as the operator that creates or destroys a wavepacket at position x. With this in mind, we will also make statements about relaxation in systems of massless particles, etc.

3.3.2 Initial state

At time $t = 0$, the system is prepared in some non-equilibrium initial state represented by the density matrix $\hat{\rho}_0$. For the majority of this chapter, the only condition that we impose on $\hat{\rho}_0$ is that it have the *cluster decomposition* property [85]:

$$\langle\langle \hat{\psi}_{x_1}^{a_1} \hat{\psi}_{x_2}^{a_2} \cdots \hat{\psi}_{x_n}^{a_n} \rangle\rangle = o(|x_i - x_j|^{-(d+\epsilon)}) \quad \text{as} \quad |x_i - x_j| \to \infty \tag{3.83}$$

for any pair of indices $i, j \in 1, 2, \ldots, n$, where $\epsilon > 0$ is some positive real number. Here $\langle\langle \cdots \rangle\rangle$ denotes the *connected correlation function* or *cumulant* of the operators $\hat{\psi}_{x_1}^{a_1} \cdots \hat{\psi}_{x_n}^{a_n}$ in the state $\hat{\rho}_0$ (the definition of connected correlation function is reviewed in Appendix 3.A). Equation (3.83) says that the connected function vanishes at least as rapidly as $|x_i - x_j|^{-(d+\epsilon)}$ when $|x_i - x_j| \to \infty$, for some $\epsilon > 0$. The cluster decomposition property ensures that correlations in the state $\hat{\rho}_0$ factorize as groups of operators

are taken far away from one another, and it is quite reasonable from a physical standpoint. The cluster decomposition property (in fact a stronger exponential version of it) has been *rigorously proven* for large classes of initial states. These include ground states of interacting local hamiltonians with a spectral gap [77, 78], as well as thermal states of arbitrary short-ranged fermionic lattice systems at sufficiently high temperature [79]. We emphasize again that the initial state $\hat{\rho}_0$ need not be related in any way to the hamiltonian of the system. For instance, it can be the ground state or thermal state of some completely different *interacting* hamiltonian; the only requirement is that it satisfy Eq. (3.83).

In Sections 3.5.3, 3.6.4 and 3.7.3, we will require the initial state to satisfy a second condition, in addition to cluster decomposition. This extra assumption is *needed in these three sections and nowhere else*, so we state it when it first becomes relevant, in Section 3.5.3. In the rest of the chapter, only Eq. (3.83) is assumed.

3.3.3 Hamiltonian

For $t > 0$, the evolution of the system is governed by a quadratic, possibly time-dependent, hamiltonian of the form

$$\hat{H}(t) = \sum_{x,y} \Big[\hat{\psi}_x^+ h_{xy}(t) \hat{\psi}_y^- + \tfrac{1}{2} \big(\hat{\psi}_x^+ \Delta_{xy}(t) \hat{\psi}_y^+ + \text{h.c.} \big) \Big], \tag{3.84}$$

where $h_{xy}^* = h_{yx}$ and $\Delta_{xy} = \pm \Delta_{yx}$ for bosons/fermions respectively. This is the most general possible form of a quadratic hamiltonian. The term involving h accounts for hopping and on-site potentials, while the term involving Δ allows for pairing. In the bosonic case, we assume that any linear terms have been eliminated by appropriately shifting the operators, and that the quasiparticle spectrum of $\hat{H}(t)$ is positive-definite.

It is often convenient to organize the annihilation and creation operators into a $2N$-component column vector $\hat{\Psi}$. If one orders the sites in some manner from 1 to N, and temporarily denotes the operators acting on site number j by $\hat{\psi}_j^{\pm}$, then

$$\hat{\Psi} = (\hat{\psi}_1^-, \hat{\psi}_2^-, \cdots, \hat{\psi}_N^-, \hat{\psi}_1^+, \hat{\psi}_2^+, \cdots, \hat{\psi}_N^+)^T. \tag{3.85}$$

The hamiltonian can then be written in the form (column vector times matrix times row vector):

$$\hat{H}(t) = \tfrac{1}{2}\hat{\Psi}^\dagger \mathcal{H}(t) \hat{\Psi} + \text{constant}, \tag{3.86}$$

where

$$\mathcal{H}(t) = \begin{bmatrix} h(t) & \Delta(t) \\ \pm\Delta^*(t) & \pm h^*(t) \end{bmatrix}, \tag{3.87}$$

and where the plus (minus) signs apply to bosons (fermions). $\mathcal{H}(t)$ is a $2N \times 2N$ hermitian matrix whose blocks are the matrices $h = h^\dagger$ and $\Delta = \pm\Delta^T$ with components h_{xy} and Δ_{xy} (ordered to match the operators). In the bosonic case, we require $\mathcal{H}(t)$ to be positive-definite at each t (this is equivalent to requiring the quasiparticle spectrum of $\hat{H}(t)$ to be positive-definite).

In general, we will refer to any $2N \times 2N$ matrix $\mathcal{M}$ as a *canonical hermitian* matrix if it is of the form

$$\mathcal{M} = \begin{bmatrix} X & Y \\ \pm Y^* & \pm X^* \end{bmatrix}, \qquad \mathcal{M} = \mathcal{M}^\dagger. \tag{3.88}$$

We find it preferable to work in the Heisenberg picture throughout our analysis, so that the operators $\hat{\psi}_x^a(t)$ evolve with time t, while the unspecified density matrix $\hat{\rho}_0$ does not.

3.3.4 Observables and relaxation

The observables of interest are *local correlation functions*; by this we mean any n-point function $\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle$ in which $|x_i - x_j| \ll L$ for all pairs of indices $i, j \in 1, 2, \ldots, n$, where L is the physical extent of the system (assumed to be of the same order of magnitude in each spatial direction). This notion of locality can be made precise in the thermodynamic limit $L \to \infty$, by requiring that all distances $|x_i - x_j|$ remain finite.

We say that the system (whose true state in the Heisenberg picture is always given by $\hat{\rho}_0$) *relaxes* to a state described by the density matrix $\hat{\rho}_1(t)$ if the latter reproduces all local correlation functions at late times.

3.3.5 Gaussian density matrices

A density matrix $\hat{\rho}$ is *gaussian* if it is of the form

$$\hat{\rho} = \frac{1}{Z} \exp\Big(-\tfrac{1}{2} \hat{\Psi}^\dagger K \, \hat{\Psi} \Big), \tag{3.89}$$

where K is a $2N \times 2N$ canonical hermitian matrix (that is, it satisfies Eq. (3.88)). The quadratic form $\frac{1}{2} \hat{\Psi}^\dagger K \, \hat{\Psi}$ may be regarded as a "statistical hamiltonian" for the gaussian state (compare Eq. (3.86)).

A density matrix $\hat{\rho}'$ is gaussian if and only if, for each $n \neq 2$, all connected n-point functions with respect to $\hat{\rho}'$ vanish (this is equivalent to Wick's theorem). Any gaussian state is therefore entirely determined by its 2-point functions.

3.4 "Gaussification" of the initial state

We will first study the relaxation, in the sense defined above, of a system prepared in the initial state $\hat{\rho}_0$ and evolving according to the quadratic hamiltonian (3.84), to a state

described by a gaussian density matrix. Following Ref. [72], we refer to this process as "gaussification".

This section can be regarded as generalizing the logic that led from Eq. (3.44) to Eq. (3.59) in Section 3.2.

3.4.1 Spreading of operators. General properties of the propagator

As stated earlier, we work in the Heisenberg picture. Since the hamiltonian (3.84) is quadratic, the Heisenberg equations of motion for $\hat{\psi}^a_x(t)$ yield a system of *linear* ordinary differential equations. These may be written in matrix form, following the notation of Eq. (3.86), as

$$\frac{\partial}{\partial t}\hat{\Psi}(t) = -iM(t)\hat{\Psi}(t), \tag{3.90}$$

where

$$M(t) = \begin{bmatrix} h(t) & \Delta(t) \\ -\Delta^*(t) & -h^*(t) \end{bmatrix}. \tag{3.91}$$

Recall that $h^\dagger = h$ and $\Delta^T = \pm\Delta$ for bosons (fermions). Thus, for fermions, the matrix $M(t) = \mathcal{H}(t)$ is always hermitian, whereas for bosons it is only hermitian if $\Delta = 0$. In either case, one may immediately integrate this matrix differential equation to obtain

$$\hat{\Psi}(t) = G(t)\hat{\Psi}(0), \tag{3.92}$$

which defines the *propagator* $G(t)$; in general $G(t)$ is the time-ordered exponential of the matrix-valued function $M(t)$:

$$G(t) = \begin{bmatrix} G^{--}(t) & G^{-+}(t) \\ G^{+-}(t) & G^{++}(t) \end{bmatrix} = \mathcal{T} e^{-i \int_0^t M(t')dt'}. \tag{3.93}$$

One always has $G^{++}(t) = [G^{--}(t)]^*$ and $G^{-+}(t) = [G^{+-}(t)]^*$. The matrix $G(t)$ is unitary in the case of fermions (or bosons with $\Delta = 0$), since in these cases it is the time-ordered exponential of a hermitian matrix-valued function. For bosons in general, $G(t)$ is instead *pseudo-unitary*; it satisfies $G^\dagger \eta G = \eta$, where $\eta = I_N \oplus -I_N$ and I_N is the $N \times N$ identity matrix. We will at first restrict attention to the cases in which $G(t)$ is unitary, and postpone the discussion of the slightly more subtle case of bosons with nonzero pairing (with $\Delta \neq 0$) to Section 3.4.5.

Equation (3.92) can be written in component form as

$$\hat{\psi}_x^a(t) = \sum_{b=\pm} \sum_y G_{xy}^{ab}(t) \hat{\psi}_y^b, \tag{3.94}$$

where $\hat{\psi}_y^b = \hat{\psi}_y^b(0)$. The components $G_{xy}^{ab}(t)$ of the propagator may be interpreted as giving the amplitude for a particle ($b = -$) or hole ($b = +$) added to the "vacuum" at position y to be found, after time t has elapsed, as a particle ($a = -$) or hole ($a = +$) at position x. $G_{xy}^{ab}(t)$ also equals the retarded single-particle Green's function of the system (both normal and anomalous parts); with a quadratic hamiltonian $\hat{H}(t)$, this Green's function is independent of the state $\hat{\rho}_0$.

Unitarity of the matrix $G(t)$ ensures that

$$\sum_{b=\pm} \sum_y \left| G_{xy}^{ab}(t) \right|^2 = 1. \tag{3.95}$$

In accordance with the interpretation of $G^{ab}_{xy}(t)$ given above, this equation may be understood as expressing conservation of probability of particles along with holes.

Our argument for "gaussification" depends only on very coarse properties of the propagator—on whether and how rapidly it "spreads" as time progresses. Let us make these notions precise. Following the terminology used in Ref. [72], we say that the dynamics are *delocalizing* at (x, a) if

$$|G^{ab}_{xy}(t)| \to 0 \quad \text{as} \quad t \to \infty \quad \forall\ (y, b); \tag{3.96}$$

otherwise we say that the dynamics are *localizing* at (x, a).

If the dynamics are delocalizing at (x, a), then for any $c > 0$, at sufficiently late times t one has $|G^{ab}_{xy}(t)| < c$ for all (y, b). In order to satisfy Eq. (3.95), $|G^{ab}_{xy}(t)|$ must then be nonzero for at least $1/c^2$ pairs (y, b). Thus, "delocalizing dynamics" requires spreading of the propagator. In order to quantify how rapidly this spreading occurs, consider the smooth envelope $\widetilde{G}^{ab}_{xy}(t)$ of $|G^{ab}_{xy}(t)|$, obtained by coarse-graining the latter in x and y (in the example of Section 3.2, for instance, we obtain $\widetilde{G}_{xy}(t)$ by averaging the curves in Figure 1 over their rapid oscillations on the lattice scale). For given position x, index a, time t, and constant $\delta > 0$, define

$$\mathcal{D}^a_x(t; \delta) \equiv \left\{ y \,\middle|\, \widetilde{G}^{a+}_{xy}(t) > \delta \ \text{ or } \ \widetilde{G}^{a-}_{xy}(t) > \delta \right\} \tag{3.97}$$

and

$$\mathcal{V}^a_x(t; \delta) \equiv \mathrm{Vol}_y[\mathcal{D}^a_x(t; \delta)]. \tag{3.98}$$

By choosing δ small enough, we can ensure that, to any desired accuracy,

$$\sum_{y \in \mathcal{D}_x^a(t;\delta)} \sum_{b=\pm} \left| G_{xy}^{ab}(t) \right|^2 =: 1 - \epsilon^2(t,\delta) \approx 1, \tag{3.99}$$

where the first equality defines $\epsilon(t,\delta)$. Thus, whenever $G_{xy}^{ab}(t)$ is present in a sum over y, we may restrict the sum to $y \in \mathcal{D}_x^a(t;\delta)$ while only making an error of order $\epsilon(t,\delta)$. In what follows, we will assume that $\delta = \delta_*(t)$ has been chosen small enough so that the error $\epsilon(t,\delta_*(t))$ is negligible, and suppress it in writing

$$\mathcal{D}_x^a(t) = \mathcal{D}_x^a(t;\delta_*(t)) \tag{3.100}$$

and

$$\mathcal{V}_x^a(t) = \mathcal{V}_x^a(t;\delta_*(t)). \tag{3.101}$$

In many cases of interest, including lattice systems with Lieb-Robinson bounds [88, 94, 72], $\mathcal{V}_x^a(t;\delta)$ depends much more weakly on δ than does $\epsilon(t,\delta)$ in the limit $\delta \to 0$ (a glance back at Figure 1 shows that this is true in the example of Section 3.2). In order to satisfy Eq. (3.99), the non-negligible components $G_{xy}^{ab}(t)$, which belong to the region $y \in \mathcal{D}_x^a(t)$, must then have magnitude

$$|G_{xy}^{ab}(t)| \sim [\mathcal{V}_x^a(t)]^{-1/2} \quad \text{for typical } \; y \in \mathcal{D}_x^a(t). \tag{3.102}$$

If the dynamics are delocalizing, one must have $\mathcal{V}_x^a(t) \to \infty$ as $t \to \infty$.

Usually, the dimension d' of the region $\mathcal{D}_x^a(t)$ equals the dimension d of the ambient space. However, there are also cases in which $d' < d$. For instance, for a system of massless particles with an isotropic dispersion relation, $\mathcal{D}_x^a(t)$ is the $d' = (d-1)$-dimensional surface of a d-dimensional sphere centered at x.

Before proceeding, let us comment on three generic ways in which the dynamics may fail to delocalize:

1. The most obvious one is that $\hat{H}$ describes a system that is Anderson localized [95]; in this case the dynamics are localizing at all points x.

2. More generally, imagine that the quasiparticle spectrum of $\hat{H}$ includes a level whose wavefunction is exponentially localized in space near position x_0. The propagator $G^{ab}_{xy}(t)$ will then include a term, due to the localized state, that does not vanish as $t \to \infty$. However, this contribution will be exponentially small in the distances $|x - x_0|$ or $|y - x_0|$ if either of these is large. Thus, to an excellent approximaiton, the dynamics will only be localizing very near x_0, and will remain delocalizing elsewhere. We will discuss the special effects that arise when the quasiparticle spectrum of $\hat{H}$ contains one or more localized states, in addition to extended states, in Section 3.6. For the remainder of the chapter, we exclude this possibility. Because we define relaxation as a *local* phenomenon, however, our general conclusions also apply to systems *with* localized states, as long as we consider a region of space that is sufficiently far from them.

3. Finally, consider a system of non-interacting particles moving in two dimensions in a constant perpendicular magnetic field. In this case, the dynamics are again localizing; the propagator $G(t)$ is a periodic function of time [96]. This may be inferred from the fact that, in the classical problem, all particles move in circular orbits at the cyclotron frequency $\omega_0 = eB/mc$, regardless of their initial velocity (here m is the mass and e the charge of each particle, B is the magnitude of the magnetic field, and c is the speed of light). Consequently, the wavefunction of a single particle prepared in a wavepacket at some point $\mathbf{r}_0$ simply expands and contracts rhythmically with period $2\pi/\omega_0$.

3.4.2 Decay of connected correlation functions. "Gaussification"

Using Eq. (3.92), any time-dependent connected n-point function can be expressed as a linear combination of connected n-point functions at time zero:

$$\langle\!\langle \hat{\psi}^{a_1}_{x_1}(t_1)\hat{\psi}^{a_2}_{x_2}(t_2)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\!\rangle = \sum_{b_1\cdots b_n=\pm}\sum_{y_1\cdots y_n} G^{a_1b_1}_{x_1y_1}(t_1)\cdots G^{a_nb_n}_{x_ny_n}(t_n)\,\langle\!\langle \hat{\psi}^{b_1}_{y_1}\hat{\psi}^{b_2}_{y_2}\cdots\hat{\psi}^{b_n}_{y_n}\rangle\!\rangle. \tag{3.103}$$

We are interested in *local* correlation functions, so we assume that the x_j's are all close to one another (relative to the size of the system). We can estimate the magnitude of the connected n-point function by simply multiplying the number of significant terms in the sum by the typical magnitude of each one. Based on the discussion in Section 3.4.1, the summand is negligible unless each y_j is contained in the appropriate region $\mathcal{D}^{a_j}_{x_j}(t_j)$. Assume for a moment that the initial state $\hat{\rho}_0$ obeys a strong version of cluster decomposition, and has a finite correlation length ξ such that $\langle\!\langle \hat{\psi}^{b_1}_{y_1}\hat{\psi}^{b_2}_{y_2}\cdots\hat{\psi}^{b_n}_{y_n}\rangle\!\rangle$ is negligible whenever $|y_i - y_j| \gg \xi$. Then, the summand at $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is significant only if $\mathbf{y} \in \mathcal{D}(\{x_i, t_i\})$, where

$$\mathcal{D}(\{x_i,t_i\}) \approx \left\{\mathbf{y} \;\middle|\; y_j \in \mathcal{D}^{a_j}_{x_j}(t_j)\ \ \forall\ j \ \text{ and } \ |y_i - y_j| < \xi\ \ \forall\ i,j\right\}, \tag{3.104}$$

and the number of significant terms in the sum, $\mathcal{N}(t)$, is proportional to the volume, in $\mathbf{y}$-space, of $\mathcal{D}(\{x_i, t_i\})$.

With delocalizing dynamics, each region $\mathcal{D}^{a_j}_{x_j}(t_j)$ grows without bound as $t \to \infty$, so that $\mathcal{V}^{a_j}_{x_j}(t_j) \gg \xi^d$ at late times. In this case, it is easy to see that the number of

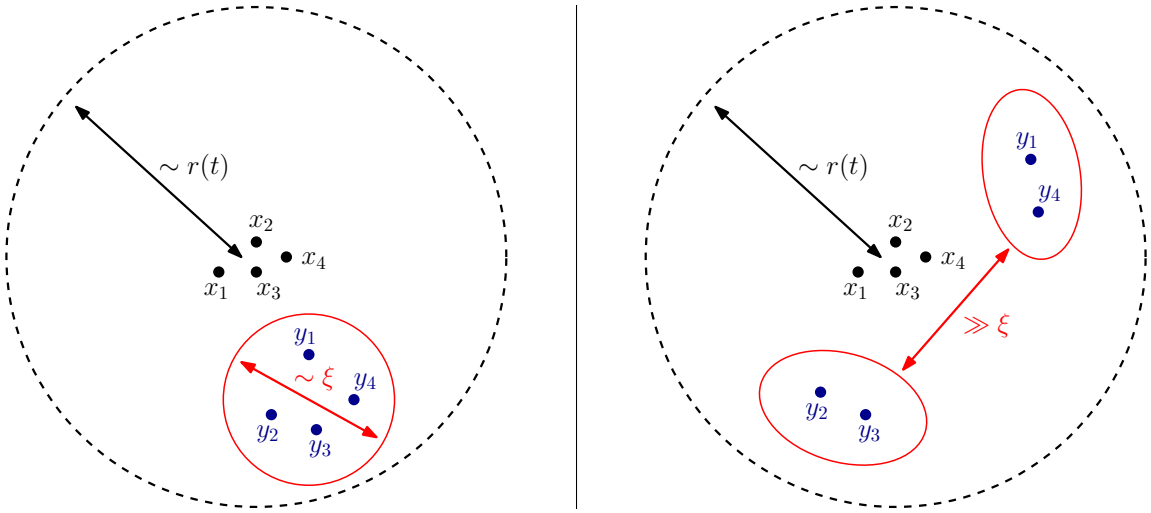


Figure 3: Schematic showing how the spreading of operators in $d = 2$ dimensions causes the connected 4-point function $\langle\langle \hat{\psi}^{a_1}_{x_1}(t)\hat{\psi}^{a_2}_{x_2}(t)\hat{\psi}^{a_3}_{x_3}(t)\hat{\psi}^{a_4}_{x_4}(t)\rangle\rangle$ to decay as time t increases. As in Eq. (3.103), this function is expressed as a weighted sum of connected 4-point functions at time zero, $\langle\langle \hat{\psi}^{b_1}_{y_1}\hat{\psi}^{b_2}_{y_2}\hat{\psi}^{b_3}_{y_3}\hat{\psi}^{b_4}_{y_4}\rangle\rangle$. Cluster decomposition ensures that only configurations of the y's of the form depicted in the left panel contribute to the sum; configurations like that shown in the right panel *do not*, because the connected function $\langle\langle \hat{\psi}^{b_1}_{y_1}\hat{\psi}^{b_2}_{y_2}\hat{\psi}^{b_3}_{y_3}\hat{\psi}^{b_4}_{y_4}\rangle\rangle$ is negligible. This restriction in allowed phase space is ultimately responsible for the power-law decay of all connected 3- and higher-point functions, as explained in the text.

significant terms in the sum is

$$\mathcal{N}(t) \sim \mathcal{V}(t)\xi^{(n-1)d'}, \tag{3.105}$$

where

$$\mathcal{V}(t) = \min\{\mathcal{V}^{a_1}_{x_1}(t_1), \mathcal{V}^{a_2}_{x_2}(t_2), \cdots, \mathcal{V}^{a_n}_{x_n}(t_n)\}. \tag{3.106}$$

The factor of $\mathcal{V}(t)$ comes from a sum over the central coordinate $\bar{y} = \frac{1}{n}(y_1 + y_2 + \cdots + y_n)$, while the $(n-1)$ factors of $\xi^{d'}$ come from sums over the relative y-coordinates; the latter are restricted by cluster decomposition, while the former is not. This straightforward geometric argument is illustrated in Figure 3.

Meanwhile, each $G^{ab}_{xy}(t_j)$ factor in the summand has typical magnitude $\sim [\mathcal{V}^a_x(t_j)]^{-1/2} \lesssim$

$[\mathcal{V}(t)]^{-1/2}$. We conclude that $|\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1)\hat{\psi}^{a_2}_{x_2}(t_2)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\rangle| \lesssim [\mathcal{V}(t)]^{-(n/2-1)}$ as $t\to\infty$. If the various times t_j are comparable (quantitatively, if the time differences $|t_i - t_j|$ are small compared with the average time $\bar{t} = (t_1 + t_2 + \cdots + t_n)/n$), then we expect that $\mathcal{V}^{a_j}_{x_j}(t_j) \approx \mathcal{V}(t)$, and we may boldly promote this bound to an asymptotic estimate of the relaxation rate: $\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1)\hat{\psi}^{a_2}_{x_2}(t_2)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\rangle \sim [\mathcal{V}(t)]^{-(n/2-1)}$ as $t\to\infty$. This estimate ignores all interference between terms in the sum in Eq. (3.103). We briefly comment on some of these neglected interference effects at the end of this section.

With *localizing* dynamics (as in quenches to disordered hamiltonians in $d = 1$ or 2 dimensions), the result depends crucially on the ratio of the localization length $\xi_{\rm loc}$ to ξ. If $\xi_{\rm loc} \gg \xi$, the conclusions of the previous paragraph are essentially unchanged, except that $\mathcal{V}(t) \to (\xi_{\rm loc})^{d'}$ as $t\to\infty$. Thus, the connected functions still relax like $[\mathcal{V}(t)]^{-(n/2-1)}$, but to a finite value of order $\sim (\xi_{\rm loc})^{-(n/2-1)d'}$, rather than to zero, and subsequently oscillate forever. If $\xi_{\rm loc} < \xi$, then the y-sums in Eq. (3.103) are always restricted to regions of size $\sim (\xi_{\rm loc})^{d'}$, cluster decomposition plays no role, and one expects little or no relaxation to occur. One can also consider the intermediate case in which the dynamics has both a localizing *and* a delocalizing component. We study this in some detail in Section 3.6.

A slight refinement of the argument just presented allows us to handle initial states in which the correlation length ξ is infinite, but which nevertheless obey the weaker algebraic form of cluster decomposition (3.83). Thus, assume that $\langle\langle \hat{\psi}^{b_1}_{y_1}\hat{\psi}^{b_2}_{y_2}\cdots\hat{\psi}^{b_n}_{y_n}\rangle\rangle \sim |y_i - y_j|^{-(d+\epsilon)}$ as $|y_i - y_j|\to\infty$, with $\epsilon > 0$. Let ξ now denote the length scale beyond which this power law is valid. Each propagator $G^{ab}_{xy}(t)$ factor in Eq. (3.103) still has typical magnitude $\lesssim [\mathcal{V}(t)]^{-1/2}$. We may rewrite the sum over $y_1\cdots y_n$ as a sum over one central coordinate $\bar{y}$ and $(n-1)$ relative coordinates z_j. The sum over $\bar{y}$ is unrestricted by cluster decomposition, and yields a factor $\sim \mathcal{V}(t)$ as before. In order to estimate the sums over the relative coordinates, assume that each region $\mathcal{D}^{a_j}_{x_j}(t_j)$ is d-dimensional, and

let $r(t)$ denote some typical length scale of these regions. Then,

$$\sum_{z_1 \cdots z_{n-1}} \left| \langle\langle \hat{\psi}^{b_1}_{y+z_1} \hat{\psi}^{b_2}_{y+z_2} \cdots \hat{\psi}^{b_n}_{y-(z_1+z_2+\cdots+z_{n-1})} \rangle\rangle \right| \sim \xi^{(n-1)d} + \left(\int_{\xi}^{r(t)} \frac{|z|^{d-1}\, d|z|}{|z|^{d+\epsilon}} \right)^{n-1}$$

$$\sim \xi^{(n-1)d} + \xi^{-(n-1)\epsilon}/\epsilon, \tag{3.107}$$

where we have retained only the leading terms in the limit $r(t) \gg \xi$ (if the dynamics are delocalizing, $r(t) \to \infty$ as $t \to \infty$, so this limit will be reached at late times). The important point is that this leading term is a constant independent of t. Consequently, our earlier asymptotic estimate of the relaxation rate of the connected n-point function is not modified. If instead the regions $\mathcal{D}^{a_j}_{x_j}(t_j)$ are d'-dimensional (with $d' < d$), the requirement that each y_j lie on the appropriate d'-dimensional manifold places some additional constraints on the z_j's, but this is a detail that does not affect the main conclusion.

Thus, whenever the initial state obeys cluster decomposition, as defined in Eq. (3.83), we expect that

$$\left| \langle\langle \hat{\psi}^{a_1}_{x_1}(t_1) \hat{\psi}^{a_2}_{x_2}(t_2) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\rangle \right| \sim [\mathcal{V}(t)]^{-(n/2-1)}, \tag{3.108}$$

with $\mathcal{V}(t)$ given by Eq. (3.106). Our arguments suggest that this result holds whenever $\mathcal{V}(t) \gg \xi^{d'}$, where $d' \leq d$ is the effective dimension of the regions $\mathcal{D}^{a_j}_{x_j}(t_j)$, and ξ is an appropriate length scale in the initial state (either the correlation length, if this is finite, or the length scale beyond which the initial connected n-point functions exhibit the power law decay required by cluster decomposition).

Notice that Eq. (3.108) does not give any information about the relaxation behavior of the 2-point function, since the exponent of $\mathcal{V}(t)$ vanishes when $n = 2$. This is easily understood. As we saw in the example of Section 3.2, and as we will show later in

generality, the relaxation of the 2-point function is governed by interference between the terms in the sum in Eq. (3.103). This interference was completely ignored in our derivation of Eq. (3.108), which relied only on gross phase space arguments. For $n > 2$, we hypothesize that the neglected interference effects merely lead to an additional oscillatory time-dependence about the power-law decay exhibited in Eq. (3.108), *without* modifying the exponent of the power law itself.

3.4.3 Relaxation power laws

Equation (3.108) gives estimates of the leading time-dependence of all local ($n > 2$)-point connected functions of the system in terms of the coarse spreading behavior of the propagator (as encoded in the function $\mathcal{V}(t)$). Although a detailed study of possible spreading behaviors is beyond the scope of this chapter, we describe some generic types of spreading below.

Typical spreading behaviors fall into two broad classes. In the first class, which we call "volume spreading", the smooth envelope $\widetilde{G}^{ab}_{xy}(t)$ of the propagator (as defined in Section 3.4.1) is non-negligible for most points y inside a d-dimensional region of characteristic size $r(t)$ centered at position x, so that $\mathcal{V}(t) \sim [r(t)]^d$. In the second class, which we call "area spreading", $\widetilde{G}^{ab}_{xy}(t)$ is non-negligible only for points y near the $(d-1)$-dimensional surface of such a region of size $r(t)$ centered at x, so that $\mathcal{V}(t) \sim [r(t)]^{(d-1)}$. In either case, the dynamics are delocalizing if $r(t) \to \infty$ as $t \to \infty$, and localizing if not (with the exception of area spreading in $d = 1$, a case that we discuss separately in Section 3.4.4 below).

We expect behavior of the "volume" type for massive particles in a slowly varying potential (*dispersive* spreading) and of the "area" type for massless particles (*non-dispersive* spreading). In both these cases, $r(t) \sim vt$, where v is the maximum local group veloc-

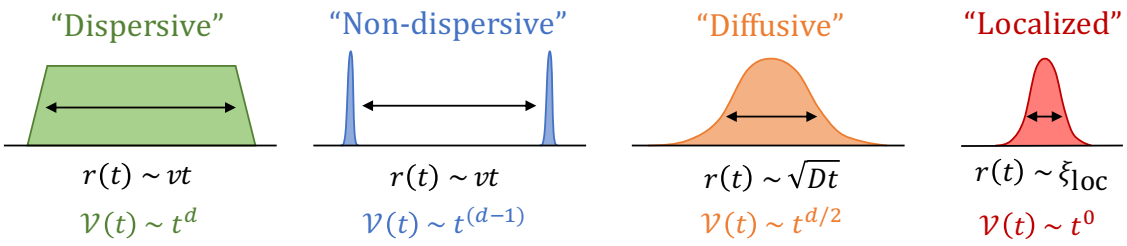


Figure 4: Paradigmatic spreading behaviors of 1-particle propagators. This list is certainly *not* exhaustive, but the spreading behaviors shown may be regarded as "typical".

ity of the particles. In the presence of weak disorder, we again expect behavior of the "volume" type. In $d \geq 3$ dimensions, the expectation is *diffusive* spreading of the form $r(t) \sim \sqrt{Dt}$ (here D is the diffusion constant), while in $d = 1$ and 2 dimensions the expectation is that $r(t)$ saturates at a finite localization length, $r(t) \sim \xi_{\mathrm{loc}}$ as $t \to \infty$ [38]. These four paradigmatic spreading behaviors are depicted schematically in Figure 4. The corresponding relaxation exponents may be easily obtained using Eq. (3.108).

For a time-independent hamiltonian $\hat{H}$ with a Lieb-Robinson bound [88, 94], we expect the propagator to *generically* behave in one of these manners; in the example of Section 3.2, for instance, the propagator exhibited what we are now calling dispersive spreading (of course, there are exceptions, such as the pathological ones noted in Section 3.4.1). More complicated behavior is certainly possible for time-dependent hamiltonians $\hat{H}(t)$, but generically we expect that these will still lead to spreading of either the "volume" or "area" types, with some characteristic size $r(t)$ that must be computed on a case-by-case basis.

If the hamiltonian $\hat{H}$ contains non-local terms, so that there is no Lieb-Robinson bound, we cannot say as much about the envelope of the propagator. However, the unitarity condition (3.95) still relates the typical magnitude of non-negligible matrix elements $G^{ab}_{xy}(t)$ to the volume $\mathcal{V}(t)$ of the region on which the propagator is meaningfully

supported: $G^{ab}_{xy}(t) \sim [\mathcal{V}(t)]^{-1/2}$. Since, in the absence of a Lieb-Robinson bound, we expect $\mathcal{V}(t)$ to grow quite rapidly, the basic argument of Section 3.4.2 still applies, and we expect the system to "gaussify" rapidly (as measured by local operators).

Therefore, if the hamiltonian describes a delocalized system in the sense that $r(t) \to \infty$ as $t \to \infty$, then all local ($n \geq 3$)-point connected functions decay with the power laws obtained above, and the system can be described at late times by a gaussian density matrix. As mentioned earlier, there is one important exception to this result, which we now discuss.

3.4.4 Non-dispersive spreading in $d = 1$ dimension: absence of gaussification

Recently, Sotiriadis [97] has analytically studied the quench dynamics of a massless free bosonic scalar field in one spatial dimension, and has shown that the system always retains significant memory of non-gaussian initial correlations. Thus, the system fails to relax to the corresponding bosonic GGE, which is gaussian. A very similar result was obtained earlier by Ngo Dinh et al. [98].

This result can be understood very easily within the framework that we have established above. The propagator of massless particles is supported entirely along the light cone. In $d = 1$ dimension, at each instant of time, the light cone simply consists of two points. Therefore, unitarity implies that the propagator can never decay; it follows from the analogue of Eq. (3.103) that higher connected correlation functions never relax to zero.

More generally, for any system whose propagator exhibits "area spreading", we have $\mathcal{V}(t) \sim [r(t)]^{(d-1)}$, and so $|G_{xy}(t)| \sim [r(t)]^{-(d-1)/2}$. In $d = 1$ dimension, these factors are constant, implying that higher connected correlation functions fail to relax, and the

system fails to "gaussify". This conclusion is special to 1 dimension; in $d > 1$ dimensions, the same type of system *will* relax to a gaussian state. It is important to keep in mind, however, that the observables for massless particles are typically not correlation functions of the fields themselves, but rather—as discussed in Chapter 2 of this thesis—correlation functions of *vertex operators* (exponentials of the fields) or of derivatives of the fields. Therefore, the precise arguments and decay rates for these systems are slightly different. We will not delve into these details here.

Many properties of (seemingly diverse) gapless systems in one spatial dimension can be obtained within the unifying framework of Luttinger liquid theory [16], which, in its simplest incarnation, can be formulated as a theory of non-interacting massless bosonic fields, as we did in Chapter 2. However, this formulation relies on linearization of the single-particle dispersion relation, and while this is innocuous for most static properties, it is clearly dangerous when considering relaxation behavior: even a slight dispersion nonlinearity will cause a crossover from non-dispersive to dispersive spreading of the propagator at late enough times, and hence lead to the relaxation that is absent in the free massless bosonic field theory. Thus, any consistent description of the quench dynamics of a one-dimensional system (even an exactly integrable one) using Luttinger liquid theory *must* account for dispersion nonlinearities [54, 99], unless the initial state is itself gaussian in terms of the bosonic fields [100].

These points were emphasized by Ngo Dinh et al. [98], and also by Sotiriadis [101] in follow-up work to Ref. [97]. These references contain a comprehensive analysis of relaxation in the Luttinger model, and conclude that any weak nonlinearity of the dispersion would ultimately lead to gaussification, in agreement with the intuitive argument sketched above. We refer the reader to these works for a detailed discussion of most of the issues mentioned in this subsection, and to Ref. [100] for a general pedagogical discussion

of quenches in the Luttinger model.

3.4.5 Bosons with pairing

We now comment on how our "gaussification" results are modified in the case of bosons with pairing. As noted in Section 3.4.1, the propagator for bosons has the general form

$$G(t) = \mathcal{T} e^{-i \int_0^t M(t')dt'}, \tag{3.109}$$

where

$$M(t) = \begin{bmatrix} h(t) & \Delta(t) \\ -\Delta^*(t) & -h^*(t) \end{bmatrix}, \tag{3.110}$$

and where $h^\dagger = h$ and $\Delta^T = \Delta$. When $\Delta \neq 0$, the propagator is not unitary, but rather *pseudo-unitary*; it satisfies $G^\dagger \eta\, G = \eta$, where $\eta = I_N \oplus -I_N$ and I_N is the $N \times N$ identity matrix. Consequently, the right hand side of Eq. (3.95) is no longer simply 1, but rather some function of time:

$$\sum_{b=\pm} \sum_y \left| G^{ab}_{xy}(t) \right|^2 = g^a_x(t) > 0. \tag{3.111}$$

The non-negligible matrix elements of $G^{ab}_{xy}(t)$ thus have typical magnitude $\sim [\mathcal{V}^a_x(t)]^{-1/2} [g^a_x(t)]^{1/2}$. Repeating the phase-space arguments of Section 3.4.2, we obtain the appropriately modified form of Eq. (3.108):

$$\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1) \hat{\psi}^{a_2}_{x_2}(t_2) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\rangle \sim \frac{[g(t)]^{n/2}}{[\mathcal{V}(t)]^{n/2-1}}, \tag{3.112}$$

where

$$g(t) = \left[g^{a_1}_{x_1}(t_1) g^{a_2}_{x_2}(t_2) \cdots g^{a_n}_{x_n}(t_n) \right]^{1/n}. \tag{3.113}$$

In general, $g(t)$ could be a complicated function of time, whose form is difficult to predict without some further knowledge of $M(t)$. However, if the hamiltonian is time-independent, we can easily derive the bound (see Appendix 3.B for details)

$$1 \leq g(t) \leq \left(\frac{\omega_{\max}}{\epsilon_{\min}}\right)^2, \tag{3.114}$$

where $\omega_{\max}$ is the largest boson mode energy, and $\epsilon_{\min} > 0$ is the smallest eigenvalue of the hermitian matrix $\mathcal{H}$ that defines the hamiltonian via Eq. (3.86) (recall that, for bosons, we require $\mathcal{H}$ to be positive-definite). In this case, although the relaxation behavior described by Eq. (3.112) is complicated, it has a power-law envelope determined entirely by $\mathcal{V}(t)$.

Time-evolution in bosonic systems approximately described by unstable or metastable quadratic hamiltonians (those whose mode spectra are *not* positive-definite) has been studied in Ref. [102].

3.5 Equilibration to the GGE

We have shown in Section 3.4 that, if the initial state has the cluster decomposition property (3.83), and if the dynamics are delocalizing in the sense of Eq. (3.96), then, as $t \to \infty$, all local ($n > 2$)-point *connected* correlation functions relax to zero in a manner given by Eq. (3.108). Thus, as $t \to \infty$, local correlation functions themselves Wick factorize and are determined entirely by the local 2-point function $\langle \hat{\psi}^a_x(t)\hat{\psi}^b_y(t)\rangle$, up to corrections of order $1/\mathcal{V}(t)$. All the results of Section 3.4 hold for general time-dependent quadratic hamiltonians $\hat{H}(t)$.

If $\hat{H}$ is time-independent, we can go further—as we do now—and show that the system locally equilibrates to the appropriate GGE. In Sections 3.5.1 and 3.5.2, we construct the

GGE density operator and show that it is gaussian; these sections generalize and complete the discussion in Section 3.2.2. In Section 3.5.3, which generalizes Secton 3.2.6, we study equilibration of the local 2-point function to its GGE value. Combined with the results summarized in the previous paragraph, this analysis proves equilibration to the GGE for a wide class of quadratic lattice models, and also furnishes predictions for the leading time-dependence of local observables as $t \to \infty$. A similar analysis is carried out for time-periodic $\hat{H}(t)$ in Section 3.7.

3.5.1 Conserved quantities

Consider any quadratic time-independent hamiltonian $\hat{H}$ which gives rise to delocalizing dynamics. We begin by showing that in this case all local conserved quantities $\hat{I}_m$ are themselves quadratic in the particle creation and annihilation operators.

By definition of the conserved quantities, we must have $\hat{I}_m(t) = \hat{I}_m(0)$. Without loss of generality, we can take $\hat{I}_m$ to have a definite order n in terms of creation and annihilation operators, because the latter evolve linearly:

$$\hat{I}_m = \sum_{\{x_j\}} \sum_{\{a_j=\pm\}} \mathcal{I}^{a_1 a_2 \cdots a_n}_{x_1 x_2 \cdots x_n} \hat{\psi}^{a_1}_{x_1} \hat{\psi}^{a_2}_{x_2} \cdots \hat{\psi}^{a_n}_{x_n}. \tag{3.115}$$

Locality (recall that this means that the $\hat{I}_m$ are *sums* of local densities) requires that the coefficients $\mathcal{I}^{a_1 \cdots a_n}_{x_1 \cdots x_n}$ vanish unless all $|x_i - x_j| \ll L$. Using Eq. (3.92), we have

$$\hat{I}_m(t) = \sum_{\{x_j, y_j\}} \sum_{\{a_j, b_j=\pm\}} \mathcal{I}^{a_1 \cdots a_n}_{x_1 \cdots x_n} \, G^{a_1 b_1}_{x_1 y_1}(t) \cdots G^{a_n b_n}_{x_n y_n}(t) \, \hat{\psi}^{b_1}_{y_1} \hat{\psi}^{b_2}_{y_2} \cdots \hat{\psi}^{b_n}_{y_n}. \tag{3.116}$$

The conservation condition $\hat{I}_m(0) = \hat{I}_m(t)$ then requires that

$$\mathcal{I}^{b_1\cdots b_n}_{y_1\cdots y_n} = \sum_{\{x_j\}}\sum_{\{a_j=\pm\}} \mathcal{I}^{a_1\cdots a_n}_{x_1\cdots x_n}\, G^{a_1 b_1}_{x_1 y_1}(t)\cdots G^{a_n b_n}_{x_n y_n}(t). \tag{3.117}$$

The same "phase space" arguments that we used in the previous section to show decay of all local connected ($n \geq 3$)-point functions also apply to the right hand side of Eq. (3.117); locality of the coefficients $\mathcal{I}^{a_1\cdots a_n}_{x_1\cdots x_n}$ here plays the role of cluster decomposition. We conclude that for $n \geq 3$, the right hand side of Eq. (3.117) must vanish as $t \to \infty$ if the dynamics are delocalizing. The left hand side, however, is obviously time-independent and finite. This contradiction proves the claim.

Thus, all local conserved quantities of $\hat{H}$ must be of the form

$$\hat{I}_m = \tfrac{1}{2}\hat{\Psi}^\dagger \mathcal{I}_m \hat{\Psi}, \tag{3.118}$$

where $\mathcal{I}_m$ is a canonical hermitian $2N \times 2N$ matrix in which each block is banded to ensure locality.

3.5.2 GGE density operator

A quadratic hamiltonian $\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger \mathcal{H} \hat{\Psi}$ (with $\mathcal{H}$ positive definite in the case of bosons) can always be diagonalized by a Bogolyubov transformation [103]; we can introduce new canonical "quasiparticle" operators $\{\hat{\gamma}^\pm_n\}$ that obey the same (anti)commutation relations as the $\{\hat{\psi}^\pm_x\}$, and are related to the latter by a linear transformation,

$$\hat{\Psi} = S\,\hat{\Gamma}, \tag{3.119}$$

where

$$\hat{\Gamma} = (\hat{\gamma}_1^-, \hat{\gamma}_2^-, \cdots, \hat{\gamma}_N^-, \hat{\gamma}_1^+, \hat{\gamma}_2^+, \cdots, \hat{\gamma}_N^+)^T. \tag{3.120}$$

The transformation S has the block form

$$S = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \tag{3.121}$$

(to preserve adjoints), satisfies $S^\dagger S = I$ for fermions, or $S^\dagger \eta S = \eta$ for bosons, where $\eta = I_N \oplus -I_N$ (to preserve the operator algebra), and is diagonalizing:

$$S^\dagger \mathcal{H} S = \Omega \equiv \mathrm{diag}(\omega_1, \ldots, \omega_N, -\omega_1, \ldots, -\omega_N) \tag{3.122}$$

for fermions, or

$$S^\dagger \mathcal{H} S = \eta\, \Omega \tag{3.123}$$

for bosons. In terms of the quasiparticle operators, we have

$$\hat{H} = E_0 + \sum_{j=1}^{N} \omega_j \hat{n}_j, \tag{3.124}$$

where

$$\hat{n}_j = \hat{\gamma}_j^+ \hat{\gamma}_j^- \tag{3.125}$$

and $\omega_j \geq 0$ (in the case of bosons, $\omega_j > 0$ is required for physical stability).

The mode occupation number operators $\hat{n}_j$ commute with $\hat{H}$ and with one another. If the spectrum $\{\omega_j\}$ is nondegenerate (that is, if $\omega_i = \omega_j$ implies $i = j$), then the set of operators $\{\hat{n}_j\}$ is uniquely defined, and forms a linear basis for the set of all quadratic conserved quantities of $\hat{H}$. We have already shown (in the previous section) that all

local conserved quantities $\hat{I}_m$ of $\hat{H}$ are quadratic if $\hat{H}$ gives rise to delocalizing dynamics. Therefore, in this case we may conclude that the GGE density operator has the form

$$\hat{\rho}_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_m \lambda_m \hat{I}_m \right) \tag{3.126a}$$

$$= \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_j \mu_j \hat{n}_j \right), \tag{3.126b}$$

where only the Lagrange multipliers $\{\lambda_m\}$, or equivalently $\{\mu_j\}$, are left to be determined by the initial state. We emphasize again that this conclusion relies on two assumptions in addition to $\hat{H}$ being quadratic: (i) that the dynamics are delocalizing and (ii) that the mode spectrum $\{\omega_j\}$ is nondegenerate.

If the mode spectrum is degenerate, on the other hand, there is some freedom in the choice of diagonalizing canonical transformation S, and consequently in the mode operators and conserved quantities. For instance, if $\omega_1 = \omega_2$, consider the family of quasiparticle operators defined by

$$\begin{bmatrix} \hat{\alpha}_1^- \\ \hat{\alpha}_2^- \end{bmatrix} = Q \begin{bmatrix} \hat{\gamma}_1^- \\ \hat{\gamma}_2^- \end{bmatrix}, \tag{3.127}$$

where $Q \in U(2)$ is any 2×2 unitary matrix. It is clear that the new number operators

$$\hat{n}'_1 = \hat{\alpha}_1^+ \hat{\alpha}_1^-, \qquad \hat{n}'_2 = \hat{\alpha}_2^+ \hat{\alpha}_2^- \tag{3.128}$$

also commute with the hamiltonian $\hat{H}$. However, they *do not* in general commute with the old $\hat{n}_1$, $\hat{n}_2$ operators:

$$\left[\hat{H},\, \hat{n}'_j\right] = 0 \quad \text{but} \quad \left[\hat{n}_i,\, \hat{n}'_j\right] \neq 0 \quad (i, j = 1, 2). \tag{3.129}$$

Therefore, the primed ($\hat{n}'$) and unprimed ($\hat{n}$) operators yield *inequivalent* sets of conserved quantities. This ambiguity is fundamental [87]—it is present whenever the mode spectrum is degenerate—and it leads, in principle, to additional dependence on the initial state, as we now describe.

Each inequivalent set of conserved quantities gives rise to its own family of GGE density operators (parameterized by the Lagrange multipliers of that set of quantities). Given an initial state $\hat{\rho}_0$, we must chose the canonical transformation S to also diagonalize the correlations within each degenerate subspace; that is, we must choose S so that, for all pairs $i \neq j$ such that $\omega_i = \omega_j$, we have

$$\langle \hat{\gamma}_i^+ \hat{\gamma}_j^- \rangle \equiv \mathrm{Tr}\big(\hat{\gamma}_i^+ \hat{\gamma}_j^- \, \hat{\rho}_0\big) = 0. \tag{3.130}$$

In the case of fermions, we must also choose S to ensure that, whenever $\omega_i = \omega_j = 0$,

$$\langle \hat{\gamma}_i^+ \hat{\gamma}_j^+ \rangle = 0. \tag{3.131}$$

It is always possible to find a canonical transformation S that diagonalizes $\hat{H}$ and also satisfies these conditions. The GGE density operator can then be constructed using the associated mode operators in the usual manner, following Eq. (3.126b).

Thus in general $\hat{\rho}_{\mathrm{GGE}}$, written in the form (3.126b), depends on the initial state $\hat{\rho}_0$ in two distinct ways: (i) the definition of the occupation numbers operators $\{\hat{n}_j\}$ corresponding to degenerate modes $\{\omega_j\}$ of $\hat{H}$, and (ii) the values of the Lagrange multipliers $\{\mu_j\}$.

The general construction of the GGE density operator that we have outlined in this section can be applied to any quadratic hamiltonian $\hat{H}$ that gives rise to delocalizing dynamics; it will indeed yield a density operator $\hat{\rho}_{\mathrm{GGE}}$ that correctly describes all local

observables of the system at late times (as we demonstrate in the next section). However, we have in some sense "cheated" by phrasing our general construction in terms of the mode occupation numbers $\{\hat{n}_j\}$ rather than in terms of the local conserved quantities $\{\hat{I}_m\}$. Since we are studying *local* relaxation, the latter are really the quantities of fundamental importance.

From a more fundamental point of view, then, a set $\{\hat{n}_j\}$ is *admissible* only if, by taking linear combinations of the $\hat{n}_j$, one can construct a maximal set of local conserved quantities $\{\hat{I}_m\}$ (recall that the set $\{\hat{I}_m\}$ is maximal if any local conserved quantity $\hat{I}$ that commutes with all of the $\hat{I}_m$ can be expressed as a linear combination of them). Given a maximal set $\{\hat{I}_m\}$, we can always obtain a corresponding admissible set $\{\hat{n}_j\}$ by finding the Bogolyubov transformation S that simultaneously diagonalizes the $\hat{I}_m$. Sets $\{\hat{n}_j\}$ that are *inadmissible* can—regardless of the initial state—be ignored for the purpose of writing down $\hat{\rho}_{\mathrm{GGE}}$, and one only needs to use initial correlations to distinguish between admissible sets. Thus, the construction outlined in this section, although valid, might overestimate the degree to which the GGE depends on the initial state.

3.5.3 Relaxation of the local 2-point function

Having constructed the GGE density operator, we now study relaxation towards it by analyzing the long-time behavior of the local 2-point function. As in the example of Section 3.2, this part of the analysis requires us to make an additional assumption about the initial state $\hat{\rho}_0$; roughly speaking, we need to exclude situations in which the initial profiles of local conserved densities are inhomogeneous on length scales comparable to the system size. True local equilibration in such cases occurs on timescales of the order of the linear dimension L of the system, simply because that is how long it takes a locally conserved density to flow across the system.

In order to formulate this assumption precisely, recall that the local conserved quantities are of the form

$$\hat{I}_m = \sum_x \hat{\mathcal{I}}_{m,x}, \tag{3.132}$$

where the density $\hat{\mathcal{I}}_{m,x}$ is supported in a finite region centered at position x. Define the "local excess density"

$$\delta\mathcal{I}_m(x_0; r) \equiv \frac{1}{\mathrm{Vol}(B_r)} \sum_{x \in B_r(x_0)} \langle \hat{\mathcal{I}}_{m,x} \rangle \; - \frac{\langle \hat{I}_m \rangle}{\mathrm{Vol(Sys)}}, \tag{3.133}$$

where $B_r(x_0)$ is the d-dimensional ball of radius r centered at x_0, $\mathrm{Vol}(B_r)$ is the volume of this ball, and Vol(Sys) is the volume of the entire system.

We assume that these excess densities can be made small by taking r sufficiently large (but keeping r fixed as $\mathrm{Vol(Sys)} \to \infty$):

$$\exists\, r : \; \delta\mathcal{I}_m(x_0; r) = O(r^{-d}) \;\; \forall \; x_0, m \quad \text{as} \;\; \mathrm{Vol(Sys)} \to \infty. \tag{3.134}$$

We emphasize that the "gaussification" results of Section 3.4 hold even when this assumption is violated (the results of Sections 3.5.1 and 3.5.2 hold as well). Thus, if the initial state violates Eq. (3.134), the natural description of the local state of the system at late times is in terms of a time-dependent gaussian density matrix, of the form

$$\hat{\rho}_1(t) = \frac{1}{Z_1(t)} \exp\Big(-\tfrac{1}{2} \hat{\Psi}^\dagger K(t) \hat{\Psi}\Big), \tag{3.135}$$

where $K(t)$ is a canonical hermitian matrix (it satisfies Eq. (3.88)) that must be chosen so that

$$\mathrm{Tr}\Big(\hat{\psi}^a_x(t) \hat{\psi}^b_y(t)\, \hat{\rho}_1(t)\Big) = \langle \hat{\psi}^a_x(t) \hat{\psi}^b_y(t) \rangle \tag{3.136}$$

for all pairs of indices $a, b = \pm$ and positions x, y with $|x - y|$ finite in the limit of infinite system size.

Also as in the example of Section 3.2, this part of the analysis requires more detailed knowledge of the spectrum of the hamiltonian, or equivalently, of the propagator, than is needed to show "gaussification". Consequently, our treatment will be somewhat schematic.

In terms of the matrix S of the Bogolyubov transformation $\hat{\Psi} = S\,\hat{\Gamma}$ that diagonalizes the hamiltonian $\hat{H}$, the propagator may be written as

$$G(t) = S e^{-i\Omega t}\, S^{-1}, \tag{3.137}$$

where

$$\Omega \equiv \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N, -\omega_1, -\omega_2, \ldots, -\omega_N), \tag{3.138}$$

and $\{\omega_j \geq 0\}$ is the spectrum of quasiparticle excitations. This form of $G(t)$ is valid for both fermions and for bosons; the difference between the two is the unitarity or pseudo-unitarity of the matrix S. It is standard to regard the $2N$ columns of S as eigenvectors of a fictitious single-particle problem whose eigenvalue spectrum is symmetric about zero (while keeping in mind that, for bosons, the eigenvectors are orthonormal with respect to $\eta = I_N \oplus -I_N$ rather than I_{2N}). If we label these eigenvectors by their energy ε ($\varepsilon = \pm\omega_j$) and additional quantum numbers σ, so that (ε, σ) together form a complete set, we can write

$$G^{ab}_{xy}(t) = \sum_{\varepsilon,\sigma} S^a_x(\varepsilon,\sigma)\, e^{-i\varepsilon t}\, (S^{-1})^b_y(\varepsilon,\sigma). \tag{3.139}$$

The equal-time 2-point function is then given by

$$\langle \hat{\psi}_x^{-a}(t)\hat{\psi}_y^b(t)\rangle = \sum_{\varepsilon,\sigma,\varepsilon',\sigma'} [S_x^a(\varepsilon,\sigma)]^* S_y^b(\varepsilon',\sigma')\, e^{-i(\varepsilon-\varepsilon')t}\, F(\varepsilon,\sigma;\varepsilon',\sigma'), \tag{3.140}$$

where

$$F(\varepsilon,\sigma;\varepsilon',\sigma') \equiv \langle \hat{\Gamma}^\dagger(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon',\sigma')\rangle, \tag{3.141}$$

and where

$$\hat{\Gamma}(\varepsilon,\sigma) \equiv \begin{cases} \hat{\gamma}(\varepsilon,\sigma) & \text{if} \quad \varepsilon \geq 0 \\ \hat{\gamma}^\dagger(-\varepsilon,\sigma) & \text{if} \quad \varepsilon < 0 \end{cases}. \tag{3.142}$$

The GGE value of the same 2-point function is

$$\langle \hat{\psi}_x^{-a}\hat{\psi}_y^b\rangle_{\mathrm{GGE}} = \sum_{\varepsilon,\sigma} [S_x^a(\varepsilon,\sigma)]^* S_y^b(\varepsilon,\sigma)\, \langle \hat{\Gamma}^\dagger(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon,\sigma)\rangle. \tag{3.143}$$

Equations (3.140–3.143) are the obvious generalizations of Eqs. (3.66), (3.67) and (3.62).

In the limit of large system size, the spectrum $\{\varepsilon\}$ will in general consist of a continuous part due to spatially extended quasiparticle states and a discrete part due to localized states. For now, we assume that all quasiparticle states are extended. We will discuss what happens when the spectrum includes a discrete part coming from localized states in Section 3.6 (see also the comments in Section 3.4.1).

Since the spectrum is by assumption purely continuous in the limit of large system size, the sums over ε and ε' in Eq. (3.140) become integrals in this limit (whether the other quantum numbers σ are discrete or continuous is less important). The $t \to \infty$ asymptotics of the $(\varepsilon,\varepsilon')$-integral is then determined by the analytic structure of the function F. This structure can in turn be deduced from general arguments of the type

used in Section 3.2. Inverting the Bogoliubov transformation, we have

$$F(\varepsilon,\sigma;\varepsilon',\sigma') = \sum_{a,b=\pm}\sum_{x,y}[(S^{-1})^a_x(\varepsilon,\sigma)]^*(S^{-1})^b_y(\varepsilon',\sigma')\,\langle\hat{\psi}^{-a}_x\hat{\psi}^b_y\rangle. \tag{3.144}$$

The sums over x and y in Eq. (3.144) may be performed with respect to the central coordinate $(x+y)/2$ and relative coordinate $(x-y)$. The sum over the relative coordinate converges absolutely (due to clustering of correlations), whereas the central coordinate is summed over the whole system (because the states are extended). It follows that F can become singular only along "curves" in $(\varepsilon,\sigma;\varepsilon',\sigma')$-space, which we may identify with the zero sets of appropriate functions $\mathcal{C}_j(\varepsilon,\sigma;\varepsilon',\sigma')$. The most obvious such curve is the trivial one, $(\varepsilon',\sigma') = (\varepsilon,\sigma)$, which may be identified with the function $\mathcal{C}_0(\varepsilon,\sigma;\varepsilon',\sigma') \sim (\varepsilon-\varepsilon')(\sigma-\sigma')$; additional curves $\mathcal{C}_j$ can occur if the initial state has an appropriate order (in Section 3.2, for instance, we found that such curves were present if the initial state had a density wave with nonzero wavevector q). We conclude that F has the (highly schematic) general form

$$\begin{aligned} F(\varepsilon,\sigma;\varepsilon',\sigma') &= \delta(\varepsilon-\varepsilon')\delta(\sigma-\sigma')\,\langle\hat{\Gamma}^\dagger(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon,\sigma)\rangle \\ &\quad + \sum_j \delta(\mathcal{C}_j(\varepsilon,\sigma;\varepsilon',\sigma'))\,f_j(\varepsilon,\sigma) \;+\; f(\varepsilon,\sigma;\varepsilon',\sigma'), \end{aligned} \tag{3.145}$$

where the sum in the second line is over a finite number of curves $\mathcal{C}_j$ that, as a consequence of our assumption (3.134), remain distinct from the trivial curve in the limit of infinite system size. The various deltas functions represent all of the possible singular dependence of F on its arguments; $\langle\hat{\Gamma}^\dagger(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon,\sigma)\rangle$, $f_j(\varepsilon,\sigma)$ and $f(\varepsilon,\sigma;\varepsilon',\sigma')$ are smooth functions in the relevant domains of integration.

Taking account of this structure, Eq. (3.140) becomes

$$\langle \hat{\psi}_x^{-a}(t)\hat{\psi}_y^b(t)\rangle = \int_{\varepsilon,\sigma} [S_x^a(\varepsilon,\sigma)]^* S_y^b(\varepsilon,\sigma)\,\langle \hat{\Gamma}^\dagger(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon,\sigma)\rangle + \sum_j [\delta C_j(t)]_{xy}^{ab} + [\delta C(t)]_{xy}^{ab}. \tag{3.146}$$

The first term reproduces the GGE result, Eq. (3.143). The remaining δC_j and δC pieces come from the second and third terms in Eq. (3.145) respectively.

Let us first analyze the δC term,

$$[\delta C(t)]_{xy}^{ab} = \int_{\varepsilon,\varepsilon'}\int_{\sigma,\sigma'} [S_x^a(\varepsilon,\sigma)]^*\, S_y^b(\varepsilon',\sigma')\, e^{-i(\varepsilon-\varepsilon')t}\, f(\varepsilon,\sigma;\varepsilon',\sigma'). \tag{3.147}$$

The behavior of the integral as $t\to\infty$ can be extracted from a straightforward stationary phase analysis (apart from the factor $e^{-i(\varepsilon-\varepsilon')t}$, the integrand is a smooth function of the integration variables). The phase function $\varphi(\varepsilon,\varepsilon') = (\varepsilon-\varepsilon')$ clearly lacks stationary points, so the dominant contribution to the integral as $t\to\infty$ comes from the *corners* of the $(\varepsilon,\varepsilon')$-integration region. Near each corner, the smooth function f can be regarded as a function of σ and σ' alone. The integrals over σ and σ' will then yield factors proportional to the (local) density of states $g(\varepsilon)$ and $g(\varepsilon')$ near the band edges. We are led to conclude that, as $t\to\infty$,

$$\delta C(t) \sim (\#) \times \left| \int d\varepsilon\, g(\varepsilon) e^{-i\varepsilon t} \right|^2. \tag{3.148}$$

If, as is often the case, the density of states near the band edge has the form

$$g(\varepsilon) \sim \varepsilon^s, \tag{3.149}$$

then $\int d\varepsilon\, g(\varepsilon)\, e^{-i\varepsilon t} \sim t^{-(1+s)} \int dz\, z^s e^{-iz}$, and we obtain the estimate

$$\delta C(t) \sim t^{-2(1+s)}. \tag{3.150}$$

This result assumes that f does not vanish at the corners of the $(\varepsilon, \varepsilon')$-integration region. Generically, this will be the case. In special fine-tuned circumstances, in which f does vanish at the corners, the exponent of the power law may be larger (more negative).

We can perform a similar stationary phase analysis of each δC_j term in Eq. (3.146). In this case, the phase function φ is the restriction of $(\varepsilon - \varepsilon')$ to the curve $\mathcal{C}_j$. If φ is nonstationary along this curve, and if the curve terminates at the boundary of the $(\varepsilon, \varepsilon')$ integration region, then the same reasoning that we applied to δC in the previous paragraph yields the estimate

$$\delta C_j(t) \sim t^{-(1+s)}. \tag{3.151}$$

Again, this result may be modified if the initial state or final hamiltonian are fine-tuned. More complicated time-dependence will occur if the phase function φ is stationary somewhere along the curve $\mathcal{C}_j$; such a contribution, if present, will likely dominate the $t \to \infty$ relaxation behavior. However, this must be analyzed on a case-by-case basis, and we will not attempt to make any further statements about the general case.

If the hamiltonian $\hat{H}$ is translation-invariant, then one typically has

$$g(\varepsilon) \sim \varepsilon^{(d/2-1)} \tag{3.152}$$

at each band edge, where d is the dimension of space. In this case the above estimates

become

$$\delta C(t) \sim t^{-d}\,, \tag{3.153a}$$

$$\delta C_j(t) \sim t^{-d/2}. \tag{3.153b}$$

The results obtained in the example of Section 3.2—Eqs. (3.80) and (3.81)—are recovered if one sets $d = 1$.

It is interesting to compare Eq. (3.153), which gives the asymptotic relaxation of the 2-point function in a translation-invariant lattice system, to the asymptotic power law with which such a system should gaussify according to the results of Section 3.4. The latter power law is set by the lowest nonvanishing ($n > 2$)-point connected correlation function. Assuming that this is $n = 4$, Eq. (3.108) suggests that the system gaussifies like $\sim [\mathcal{V}(t)]^{-1}$, where $\mathcal{V}(t)$ is the volume on which the 1-particle propagator is meaningfully supported. In a translation-invariant lattice model, the propagator spreads at the maximal group velocity, so we expect this volume to grow like $\mathcal{V}(t) \sim t^d$. Hence we conclude that the system gaussifies like $\sim t^{-d}$. If there is a density wave of one or more of the conserved quantities in the initial state, then $\delta C_j(t)$ terms are present in the 2-point function; these relax like $\sim t^{-d/2}$ by Eq. (3.153b). Thus the system *first gaussifies like* $\sim t^{-d}$*, and then relaxes to the GGE like* $\sim t^{-d/2}$. If, on the other hand, the initial state lacks such order, then only the $\delta C(t)$ term is present in the 2-point function; this relaxes like $\sim t^{-d}$ by Eq. (3.153a), so *gaussification and relaxation to the GGE both occur with the power law* $\sim t^{-d}$.

Notice that gaussification and relaxation of the 2-point function are controlled (in translation-invariant systems) by fundamentally different aspects of the band structure: gaussification is controlled by the maximal group velocity—typically a property of the middle of the band(s)—whereas relaxation of the 2-point function is controlled by the

density of single-particle levels at the band edge(s).

3.6 Effects due to localized states

Assume now that the quasiparticle spectrum of $\hat{H}$ contains, in the limit of large system size, both discrete localized states and a continuum of extended states. We can write the diagonalizing Bogoliubov transformation as

$$\hat{\psi}_x^a = \int_\varepsilon \int_\sigma S_x^a(\varepsilon,\sigma)\hat{\Gamma}(\varepsilon,\sigma) + \sum_{b=\pm}\sum_j R_{xj}^{ab}\,\hat{\gamma}_j^b, \tag{3.154}$$

where, as before,

$$\hat{\Gamma}(\varepsilon,\sigma) \equiv \begin{cases} \hat{\gamma}(\varepsilon,\sigma) & \text{if} \quad \varepsilon \geq 0 \\ \hat{\gamma}^\dagger(-\varepsilon,\sigma) & \text{if} \quad \varepsilon < 0 \end{cases}. \tag{3.155}$$

The operator $\hat{\gamma}^\dagger(\omega,\sigma)$ creates a quasiparticle in the continuum level with energy $\omega \geq 0$ and additional quantum numbers σ; the operator $\hat{\gamma}_j^+$ creates a quasiparticle in the discrete level j with energy $\omega_j \geq 0$.

3.6.1 The propagator

The propagator splits naturally into two pieces:

$$G(t) = G_{\rm ext}(t) + G_{\rm loc}(t), \tag{3.156}$$

where the first piece $G_{\rm ext}(t)$ involves only the extended states, and the second piece $G_{\rm loc}(t)$ involves only the localized states. For fermions,

$$[G_{\rm ext}(t)]_{xy}^{ab} = \int_\varepsilon \int_\sigma S_x^a(\varepsilon,\sigma)\, e^{-i\varepsilon t}\, [S_y^b(\varepsilon,\sigma)]^* \tag{3.157}$$

and

$$[G_{\mathrm{loc}}(t)]^{ab}_{xy} = \sum_{c=\pm} \sum_{j} R^{ac}_{xj}\, e^{ic\,\omega_j t}\, [R^{bc}_{yj}]^*. \tag{3.158}$$

For bosons, one must multiply the integrand in Eq. (3.157) by $-b\,\mathrm{sgn}(\varepsilon)$, and the summand in Eq. (3.158) by bc.

The dynamics of the propagator $G(t)$, as defined in Section 3.4.1, are thus in general the sum of a *delocalizing* part, due to $G_{\mathrm{ext}}(t)$, and a *localizing* part, due to $G_{\mathrm{loc}}(t)$. We have already discussed general properties of $G_{\mathrm{ext}}(t)$ in Sections 3.4.1 and 3.4.3. Let us now briefly discuss general properties of $G_{\mathrm{loc}}(t)$:

Each level j of the discrete spectrum is exponentially localized near some position x_j; in other words,

$$R^{ac}_{xj} \sim e^{-|x-x_j|/\zeta_j} \quad \text{for} \quad |x-x_j| \gtrsim \zeta_j, \tag{3.159}$$

where $\zeta_j > 0$ is the decay length. It follows from Eq. (3.158) that

$$[G_{\mathrm{loc}}(t)]^{ab}_{xy} \sim e^{-|x-y|/\zeta_x} \quad \text{for} \quad |x-y| \gtrsim \zeta_x, \tag{3.160}$$

where ζ_x is roughly the largest decay length of the states localized near x. Thus, $[G_{\mathrm{loc}}(t)]^{ab}_{xy}$ is negligible whenever $|x-y| \gg \zeta_x$. Given a position x, we may restrict the sum over j in Eq. (3.158) to those levels that are localized within a few decay lengths ζ_j of x, because the remaining levels give negligible contributions. Finally, for fixed x and y, the propagator $[G_{\mathrm{loc}}(t)]^{ab}_{xy}$ oscillates forever without decaying as $t \to \infty$.

3.6.2 Gaussification

Having understood how the propagator is modified, let us study how the localized states affect gaussification and the conclusions of Section 3.4. Consider the time-

dependent connected n-point function. Equation (3.103) becomes

$$\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\rangle = \sum_{\{y_i\}}\sum_{\{b_i=\pm\}}\Big\{[G_{\rm ext}(t_1)+G_{\rm loc}(t_1)]^{a_1b_1}_{x_1y_1}\cdots[G_{\rm ext}(t_n)+G_{\rm loc}(t_n)]^{a_nb_n}_{x_ny_n} \times\langle\langle\hat{\psi}^{b_1}_{y_1}\cdots\hat{\psi}^{b_n}_{y_n}\rangle\rangle\Big\}. \tag{3.161}$$

Write this as

$$\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\rangle = \sum_{k=0}^{n}\langle\langle \hat{\psi}^{a_1}_{x_1}(t_1)\cdots\hat{\psi}^{a_n}_{x_n}(t_n)\rangle\rangle_{k-\rm loc}, \tag{3.162}$$

where $\langle\langle\cdots\rangle\rangle_{k-\rm loc}$ contains all terms in Eq. (3.161) that have k factors of $G_{\rm loc}$ and $(n-k)$ factors of $G_{\rm ext}$.

We have already studied the contribution $\langle\langle\cdots\rangle\rangle_{0-\rm loc}$, in which all the propagators are $G_{\rm ext}$, in detail in Section 3.4.2. As $t\to\infty$, $\langle\langle\cdots\rangle\rangle_{0-\rm loc}$ decays to zero as described by Eq. (3.108). Next, consider the contribution $\langle\langle\cdots\rangle\rangle_{1-\rm loc}$, in which a single propagator is $G_{\rm loc}$. According to the discussion above, this contribution is significant (at any time t) only if one or more of the x_i are located within a few decay lengths of a localized state. The sum over the corresponding y_i is restricted by the propagator $[G_{\rm loc}(t_i)]^{a_ib_i}_{x_iy_i}$ to a region of volume $\sim\zeta^d_{x_i}$ around x_i. Repeating the analysis of Section 3.4.2, the sums over the relative y-coordinates converge absolutely due to cluster decomposition, so the entire y-sum yields a finite, t-independent contribution as $t\to\infty$. Meanwhile, typical matrix elements of the propagators are of order $G_{\rm ext}(t)\sim[\mathcal{V}(t)]^{-1/2}$ and $G_{\rm loc}(t)\sim 1/\zeta^d_{x_i}$. We conclude that, as $t\to\infty$,

$$\langle\langle\cdots\rangle\rangle_{1-\rm loc}\sim[\mathcal{V}(t)]^{-(n-1)/2}. \tag{3.163}$$

This decay is *faster*, by a factor of $[\mathcal{V}(t)]^{-1/2}$, than that of $\langle\langle\cdots\rangle\rangle_{0-\rm loc}$.

By similar reasoning, we conclude that

$$\langle\!\langle \cdots \rangle\!\rangle_{k-\mathrm{loc}} \sim [\mathcal{V}(t)]^{-(n-k)/2} \qquad (k \geq 1). \tag{3.164}$$

Note that, for $k \geq 1$, $\langle\!\langle \cdots \rangle\!\rangle_{(k+1)-\mathrm{loc}}$ decays *slower*, by a factor of $[\mathcal{V}(t)]^{-1/2}$, than does $\langle\!\langle \cdots \rangle\!\rangle_{k-\mathrm{loc}}$. Thus, the leading $t \to \infty$ behavior of the connected ($n > 2$)-point function is

$$\langle\!\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\!\rangle \sim \langle\!\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\!\rangle_{n-\mathrm{loc}} + O([\mathcal{V}(t)]^{-1/2}). \tag{3.165}$$

Only the fully localized contribution $\langle\!\langle \cdots \rangle\!\rangle_{n-\mathrm{loc}}$ survives in the limit $t \to \infty$. Let us analyze this term in more detail:

$$\langle\!\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\!\rangle_{n-\mathrm{loc}} = \sum_{\{y_i\}} \sum_{\{b_i = \pm\}} [G_{\mathrm{loc}}(t_1)]^{a_1 b_1}_{x_1 y_1} \cdots [G_{\mathrm{loc}}(t_n)]^{a_n b_n}_{x_n y_n} \, \langle\!\langle \hat{\psi}^{b_1}_{y_1} \cdots \hat{\psi}^{b_n}_{y_n} \rangle\!\rangle. \tag{3.166}$$

Using Eq. (3.158) or its bosonic version, we have

$$\sum_{b=\pm} \sum_{y} [G_{\mathrm{loc}}(t)]^{ab}_{xy} \, \hat{\psi}^{b}_{y} = \sum_{c=\pm} \sum_{j} R^{ac}_{xj} \, e^{ic\omega_j t} \, \hat{\gamma}^{c}_{j}. \tag{3.167}$$

Consequently,

$$\langle\!\langle \hat{\psi}^{a_1}_{x_1}(t_1) \cdots \hat{\psi}^{a_n}_{x_n}(t_n) \rangle\!\rangle_{n-\mathrm{loc}} = \sum_{\{j_i\}} \sum_{\{c_i = \pm\}} R^{a_1 c_1}_{x_1 j_1} \cdots R^{a_n c_n}_{x_n j_n} \, e^{i \sum_{\ell=1}^{n} c_\ell \omega_{j_\ell} t_\ell} \, \langle\!\langle \hat{\gamma}^{c_1}_{j_1} \cdots \hat{\gamma}^{c_n}_{j_n} \rangle\!\rangle. \tag{3.168}$$

Each sum over j_i in Eq. (3.168) may be restricted to those levels that are localized near x_i, in accordance with our previous discussion. It is evident that the localized contribution is negligible as $t \to \infty$ if and only if $\langle\!\langle \hat{\gamma}^{c_1}_{j_1} \hat{\gamma}^{c_2}_{j_2} \cdots \hat{\gamma}^{c_n}_{j_n} \rangle\!\rangle$ itself is negligible. In bosonic systems prepared in generic initial states, this condition will be violated as soon as there is a single localized level. This is because the nth cumulant of the occupation of this level,

$\langle\langle(\hat{\gamma}^+\hat{\gamma}^-)^n\rangle\rangle$, will be nonzero in general. In a fermionic system, on the other hand, the occupation of a single localized level is characterized entirely by the expectation value $\langle\hat{\gamma}^+\hat{\gamma}^-\rangle$, so these higher cumulants all vanish.

Thus, consider a system of fermions in which the quasiparticle spectrum of $\hat{H}$ contains, in addition to a continuum of extended states, a set of discrete levels $\{j\}$ that are localized near positions $\{x_j^*\}$ with decay lengths $\{\zeta_j\}$. Assume that the initial state $\hat{\rho}_0$ has a finite correlation length ξ, and that for each pair (i,j) of localized levels, $|x_i^* - x_j^*| \gg \xi+\zeta_i+\zeta_j$. Then any connected function involving the operators of two distinct levels $i \neq j$, such as $\langle\langle\hat{\gamma}_i^a\,\hat{\gamma}_j^b\cdots\rangle\rangle$, is negligible. Of course, any connected function involving three or more operators of the *same* level, such as $\langle\langle\hat{\gamma}_j^a\,\hat{\gamma}_j^b\,\hat{\gamma}_j^c\cdots\rangle\rangle$, vanishes identically. It follows that all $\langle\langle\cdots\rangle\rangle_{(k>2)-\mathrm{loc}}$ contributions to the connected n-point function are negligible. Since the $k = 0, 2$ terms decay in the same manner with time, and since the $k = 1$ term decays faster than either of them, we reach the following somewhat surprising conclusion:

Discrete localized levels in the quasiparticle spectrum of a quadratic fermion hamiltonian $\hat{H}$ *have a negligible effect on gaussification* if (i) the initial state has a finite correlation length ξ, and (ii) the spatial distance between any pair of localized levels is large relative to ξ.

3.6.3 GGE density operator

Next, we study how the localized states affect the conclusions of Section 3.5. Before considering equilibration, we must revisit the construction of the GGE density operator itself.

In Section 3.5.1, we showed that, for any quadratic time-independent hamiltonian $\hat{H}$ which gives rise to delocalizing dynamics, all local conserved charges $\hat{I}_m$ are themselves quadratic in the particle creation and annihilation operators. Let us see how this argu-

ment changes with localized states. Equation (3.117) remains valid, but each propagator factor now has an extended piece and a localized piece:

$$\mathcal{I}^{b_1\cdots b_n}_{y_1\cdots y_n} = \sum_{\{x_i\}}\sum_{\{a_i=\pm\}} \mathcal{I}^{a_1\cdots a_n}_{x_1\cdots x_n} \left[G_{\rm ext}(t)+G_{\rm loc}(t)\right]^{a_1 b_1}_{x_1 y_1}\cdots\left[G_{\rm ext}(t)+G_{\rm loc}(t)\right]^{a_n b_n}_{x_n y_n}. \tag{3.169}$$

As we did with the connected n-point function in Section 3.6.2, write this as

$$\mathcal{I}^{b_1\cdots b_n}_{y_1\cdots y_n} = \sum_{k=0}^{n} \left[\mathcal{I}_{k-\rm loc}(t)\right]^{b_1\cdots b_n}_{y_1\cdots y_n}, \tag{3.170}$$

where $\mathcal{I}_{k-\rm loc}(t)$ contains all terms in Eq. (3.169) that have k factors of $G_{\rm loc}(t)$ and $(n-k)$ factors of $G_{\rm ext}(t)$. Repeating the arguments of Section 3.6.2, we conclude that for $n>2$, only the $\mathcal{I}_{n-\rm loc}(t)$ contribution survives as $t\to\infty$. Then, Eq. (3.170) requires that $\mathcal{I}_{n-\rm loc}$ actually be time-independent, and that $\mathcal{I}=\mathcal{I}_{n-\rm loc}$. It is clear that the corresponding local conserved quantities are those that can be built from the quasiparticle operators $\hat\gamma^{\pm}_j$ of the localized levels:

$$\hat I_m \sim \hat\gamma^{c_1}_{j_1}\,\hat\gamma^{c_2}_{j_2}\,\cdots\hat\gamma^{c_n}_{j_n}, \qquad \text{with} \quad \sum_{i=1}^{n} c_i\,\omega_{j_i}=0. \tag{3.171}$$

In addition, the participating levels $\{j_i\}$ must all be localized in the same region of space (otherwise $\hat I_m$ will not be local). Conversely, *all* local conserved charges involving products of $n>3$ creation or annihilation operators must be of this form. Thus, in a bosonic system, the existence of even a single localized level leads to non-quadratic local conserved charges (powers of the occupation of this level, $(\hat\gamma^+\hat\gamma^-)^n$). In a fermionic system, however, one can only construct non-quadratic local conserved charges if there are two or more localized levels close enough to one another in space (how close depends on how local we want the charges to be).

We conclude that, in a quadratic bosonic system, the GGE (defined in terms of local conserved quantities) is gaussian if and only if there are no localized levels at all, whereas in a quadratic fermionic system, the GGE remains gaussian to an excellent approximation even when localized levels do exist, as long as they are located sufficiently far apart in space. In the latter case, the mode occupation numbers $\hat{n}_j = \hat{\gamma}_j^+ \hat{\gamma}_j^-$ of these levels are local conserved charges, and must be included in $\hat{\rho}_{\rm GGE}$. The general analysis of Section 3.5.2 does not require modification.

3.6.4 Equilibration to the GGE

Finally, let us consider equilibration. As in Section 3.5.3, we must make an additional assumption on the initial state, Eq. (3.134), to exclude situations in which the initial profiles of local conserved densities are inhomogeneous on length scales comparable to the system size.

Following Section 3.6.2, we may identify three contributions to the equal-time 2-point function:

$$\langle \hat{\psi}_x^{-a}(t) \hat{\psi}_y^b(t) \rangle = \sum_{k=0}^{2} \langle \hat{\psi}_x^{-a}(t) \hat{\psi}_y^b(t) \rangle_{k-\rm loc}. \tag{3.172}$$

We have already studied the fully extended piece, $\langle \hat{\psi}_x^{-a}(t) \hat{\psi}_y^b(t) \rangle_{0-\rm loc}$, in detail in Section 3.5.3. It generically relaxes to its GGE value as $t \to \infty$ in a manner described by Eq. (3.150) or (3.151), and this relaxation is due to "interference" effects. On the other hand, the $1-$loc piece vanishes as $t \to \infty$ for simpler "phase space" reasons: the results of Section 3.6.2, in particular Eq. (3.163), show that

$$\langle \hat{\psi}_x^{-a}(t) \hat{\psi}_y^b(t) \rangle_{1-\rm loc} \sim [\mathcal{V}(t)]^{-1/2} \quad \text{as} \quad t \to \infty. \tag{3.173}$$

This piece is fully off-diagonal in the quasiparticle basis of $\hat{H}$, so its GGE value is also

zero.

Therefore, we only need to study the fully localized (2−loc) piece. It is given by (compare Eq. (3.168)):

$$\langle \hat{\psi}_x^{-a}(t)\hat{\psi}_y^b(t)\rangle_{2-\mathrm{loc}} = \sum_{j_1,j_2}\sum_{c_1,c_2=\pm} [R_{xj_1}^{ac_1}]^* R_{yj_2}^{bc_2}\, e^{-i(c_1\omega_{j_1}-c_2\omega_{j_2})t}\,\langle \hat{\gamma}_{j_1}^{-c_1}\hat{\gamma}_{j_2}^{c_2}\rangle, \tag{3.174}$$

where the sums over j_1 and j_2 are over all levels in the discrete part of the spectrum. Recall (Section 3.5.2) that the mode operators $\hat{\gamma}_j$ can (and should) be chosen so that, in each degenerate subspace (i.e. when $\omega_{j_1} = \omega_{j_2}$), one has $\langle \hat{\gamma}_{j_1}^{-c_1}\hat{\gamma}_{j_2}^{c_2}\rangle \propto \delta_{c_1c_2}\delta_{j_1j_2}$. This ensures that the infinite time-average of Eq. (3.174) agrees with its GGE value:

$$\langle \hat{\psi}_x^{-a}\hat{\psi}_y^b\rangle_{2-\mathrm{loc}}^{\mathrm{GGE}} = \sum_j\sum_{c=\pm} [R_{xj}^{ac}]^*\, R_{yj}^{bc}\,\langle \hat{\gamma}_j^{-c}\hat{\gamma}_j^{c}\rangle. \tag{3.175}$$

In general, the instantaneous difference

$$[\delta C_{2-\mathrm{loc}}(t)]_{xy}^{ab} \equiv \langle \hat{\psi}_x^{-a}(t)\hat{\psi}_y^b(t)\rangle_{2-\mathrm{loc}} - \langle \hat{\psi}_x^{-a}\hat{\psi}_y^b\rangle_{2-\mathrm{loc}}^{\mathrm{GGE}} \tag{3.176}$$

oscillates forever about zero without relaxing as $t \to \infty$. However, if the localized states are located far enough apart in space that the initial correlations $\langle \hat{\gamma}_{j_1}^{-c_1}\hat{\gamma}_{j_2}^{c_2}\rangle$ between them are negligible (this must hold for all nondegenerate pairs $j_1 \neq j_2$), then it follows that $\delta C_{2-\mathrm{loc}}(t)$ is negligible at all times. We conclude that:

Dynamics generated by a quadratic fermion hamiltonian $\hat{H}$ whose quasiparticle spectrum includes discrete localized levels *will still lead to gaussification and equilibration to the GGE*, as long as (i) the initial state has a finite correlation length ξ, and (ii) the spatial distance between any pair of localized levels is large relative to ξ.

3.7 Time-periodic hamiltonians and the "Floquet-GGE"

Our arguments for gaussification in Section 3.4 were extremely general; they relied only on clustering of correlations in the initial state, and on spreading of the propagator $G(t)$. Thus, they apply to any quadratic hamiltonian $\hat{H}(t)$, as long as it leads to delocalizing dynamics. In this section, we consider the particularly interesting *time-periodic* case:

$$\hat{H}(t) = \tfrac{1}{2}\hat{\Psi}^\dagger \mathcal{H}(t)\hat{\Psi} + \text{constant}, \tag{3.177}$$

where

$$\mathcal{H}(t) = \begin{bmatrix} h(t) & \Delta(t) \\ \pm\Delta^*(t) & \pm h^*(t) \end{bmatrix} = \mathcal{H}(t+T), \tag{3.178}$$

and where, as before, the plus (minus) sign is for bosons (fermions). $\hat{H}(t)$ describes a periodically driven, or "Floquet", closed quantum system.

3.7.1 Floquet theory basics

Let us briefly review some simple facts about this problem [104]. In order to give a complete "stroboscopic" description of the system at times $t = nT$ ($n = 0, 1, 2, \cdots$) one only needs to know the time-evolution operator over a single period,

$$\hat{U}(T) = \mathcal{T} e^{-i\int_0^T \hat{H}(t')dt'} \tag{3.179}$$

(to describe the system at intermediate times, one also needs to know $\hat{U}(t)$ for all $0 < t < T$). Since $\hat{U}(T)$ is unitary, it has a spectral decomposition of the form

$$\hat{U}(T) = \sum_{\alpha} e^{-i\epsilon_\alpha T} |\alpha\rangle\langle\alpha| , \tag{3.180}$$

where $\{|\alpha\rangle\}$ forms a basis for the Hilbert space of the system, and where the "quasienergies" ϵ_α are defined modulo $2\pi/T$. The associated "Floquet hamiltonian"

$$\hat{H}_F \equiv \sum_{\alpha} \epsilon_\alpha |\alpha\rangle\langle\alpha| \tag{3.181}$$

generates

$$\hat{U}(T) = e^{-i\hat{H}_F T} \tag{3.182}$$

by construction. $\hat{H}_F$ is quadratic because $\hat{H}(t)$ is quadratic (quadratic forms in fermion or boson operators form a Lie algebra, and the unitary group is compact). Thus, apart from the subtlety that the quasienergies $\{\epsilon_\alpha\}$ take values on a circle rather than on the real line, the dynamical problem at times $t = nT$ is formally identical to one with a time-independent quadratic hamiltonian

$$\hat{H}_F = \tfrac{1}{2}\hat{\Psi}^\dagger \mathcal{H}_F \hat{\Psi} + \text{constant}, \tag{3.183}$$

where

$$\mathcal{H}_F = \begin{bmatrix} h_F & \Delta_F \\ \pm\Delta_F^* & \pm h_F^* \end{bmatrix} . \tag{3.184}$$

3.7.2 Propagator and gaussification

Recall that the propagator $G(t)$ is defined by the solution of the Heisenberg equations of motion,

$$\hat{\Psi}(t) = G(t)\hat{\Psi}(0). \tag{3.185}$$

Since $\hat{\Psi}(t) \equiv \hat{U}^\dagger(t)\hat{\Psi}(0)\hat{U}(t)$, and since $\hat{U}(T) = \mathcal{T}e^{-i\int_0^T \hat{H}(t')dt'} = e^{-i\hat{H}_F T}$, one obtains two equivalent expressions for the propagator over one period:

$$G(T) = \mathcal{T}e^{-i\int_0^x TM(t')dt'} = e^{-iM_F T}, \tag{3.186}$$

where

$$M(t) = \begin{bmatrix} h(t) & \Delta(t) \\ -\Delta^*(t) & -h^*(t) \end{bmatrix} \tag{3.187}$$

and

$$M_F = \begin{bmatrix} h_F & \Delta_F \\ -\Delta_F^* & -h_F^* \end{bmatrix}. \tag{3.188}$$

As one might expect for a quadratic system, $G(T)$ completely determines $\hat{H}_F$ (modulo shifting the quasienergies by multiples of $2\pi/T$).

The propagator at any time $t = nT + t'$, where $0 \leq t' < T$, is given by

$$G(nT + t') = G(t')[G(T)]^n. \tag{3.189}$$

As $t \to \infty$, the relaxation behavior will be dominated by the $[G(T)]^n$ factor, except possibly in some pathological cases. Therefore, we expect any local connected 3- or higher-point function of the driven system to relax (or fail to relax) with time in exactly the same manner as that of an undriven system with hamiltonian $\hat{H}_F$, up to a

multiplicative periodic factor $f(t) = f(t+T)$ coming from the $G(t')$ part of the propagator. The results of Sections 3.4.3, 3.4.4 and 3.4.5 may thus be applied with only minor modifications.

3.7.3 Relaxation to the Floquet-GGE

Having discussed gaussification in the Floquet context, let us next consider the eventual fate of the effectively gaussian state. Following Section 3.5.2, we may construct a GGE density operator $\hat{\rho}_F$ out of the local conserved charges of the Floquet hamiltonian $\hat{H}_F$; the argument of Section 3.5.1, applied at stroboscopic times $t = nT$, shows that these charges are all quadratic, so that $\hat{\rho}_F$ is indeed gaussian. It is natural to suspect that the system eventually relaxes to a state described by $\hat{\rho}_F$. Note that such a state is a *limit cycle*: for any operator $\hat{\mathcal{O}}$, one has

$$\mathrm{Tr}\Big(\hat{\mathcal{O}}(t)\hat{\rho}_F\Big) = \mathrm{Tr}\Big(\hat{U}(T)\hat{\mathcal{O}}(t+T)\hat{U}^\dagger(T)\hat{\rho}_F\Big) = \mathrm{Tr}\Big(\hat{\mathcal{O}}(t+T)\hat{\rho}_F\Big) \tag{3.190}$$

(the second equality follows from the definition of $\hat{\rho}_F$), but in general

$$\mathrm{Tr}(\hat{\mathcal{O}}(t)\hat{\rho}_F) \neq \mathrm{Tr}(\hat{\mathcal{O}}(t')\hat{\rho}_F). \tag{3.191}$$

This time-periodic limiting state has been called the "periodic Gibbs ensemble (PGE)" or the *Floquet-GGE* [82, 105, 81].

We can generalize the analysis of Section 3.5.3 to study relaxation of the gaussified state to the Floquet-GGE. It is sufficient to study this at stroboscopic times $t = nT$. As in Section 3.5.3, we must make an additional assumption on the initial state, Eq. (3.134), needed to exclude situations in which the initial profiles of local conserved densities are inhomogeneous on length scales comparable to the system size. Equations (3.137–

3.146) are unchanged, except that S must now be understood as the matrix of the Bogoliubov transformation $\hat{\Psi} = S\,\hat{\Gamma}$ that diagonalizes the Floquet hamiltonian $\hat{H}_F$, and ε as a quasienergy defined modulo $2\pi/T$.

First consider the limit of very fast driving, $T \to 0$. In this limit, we expect that we can ignore the periodicity of ε (since the period $2\pi/T \to \infty$), and that the quasiparticle states of $\hat{H}_F$ are organized into one or more well-defined bands. If this is so, the remainder of the analysis in Section 3.5.3 applies, and we conclude that the gaussified state relaxes to $\hat{\rho}_F$ with a power law; in the simplest cases, this power law is given by Eq. (3.150) or (3.151).

Next consider the opposite limit of very slow driving, so that $T \to \infty$ and $\hat{H}(t)$ is a slowly varying function of t. On timescales $t \lesssim T$, we expect, based on an adiabatic approximation and our arguments in the time-independent case, to observe power-law relaxation to a GGE of the instantaneous hamiltonian $\hat{H}(t)$. On much longer timescales $t \gg T$, the Floquet drive becomes important, and we except to eventually observe relaxation to the Floquet GGE, $\hat{\rho}_F$. Stationary phase analysis suggests that this relaxation will be *exponential* in time, $\sim e^{-t/T}$. To see this, consider Eq. (3.147). The quasienergies ε are defined on a circle of radius $2\pi/T \to 0$, so the spectrum is likely to be relatively smooth, without well-defined bands. Therefore

$$\begin{aligned}\delta C(t) &\sim \int_0^{2\pi/T} d\varepsilon \int_0^{2\pi/T} d\varepsilon'\, a(\varepsilon,\varepsilon')\, e^{-i(\varepsilon-\varepsilon')t} \\ &= \frac{1}{T^2}\int_0^{2\pi} dz \int_0^{2\pi} dz'\, a\Big(\frac{z}{T},\frac{z'}{T}\Big) e^{-i(z-z')t/T},\end{aligned} \tag{3.192}$$

where $a(z/T, z'/T)$ is a smooth function of z and z' on the torus $\mathbb{T}$. It follows that $\delta C(t)$ must vanish faster than any power of (t/T) as $(t/T) \to \infty$. Similar arguments apply to $\delta C_j(t)$.

Thus, in the limit $T \to 0$ of fast driving, we expect to observe power-law relaxation to $\hat{\rho}_F$, the (time-periodic) GGE of the Floquet hamiltonian $\hat{H}_F$. In the opposite limit $T \to \infty$ of slow driving, we expect to observe power-law relaxation toward a GGE of the instantaneous hamiltonian $\hat{H}(t)$, followed by much slower exponential relaxation $\sim e^{-t/T}$ toward $\hat{\rho}_F$. It is more difficult to make semi-quantitive general statements about the regime of intermediate driving, and we leave this as an interesting question for future work.

3.8 A comment on spin models mappable to quadratic fermion models

Everything that we have said also applies to any spin system that can be mapped to a quadratic model of fermions (via a Jordan-Wigner transformation or otherwise), assuming (i) that the observables of interest map to local operators in terms of the fermions, and (ii) that the initial state $\hat{\rho}_0$ obeys cluster decomposition *with respect to the fermion operators*. It is by no means obvious that a given physical initial state, which obeys cluster decomposition with respect to spin operators, also does so with respect to the fermions. It would be interesting to identify which states have this property.

3.9 Conclusions

In this chapter, we have presented a general framework for understanding relaxation phenomena in systems described by quadratic fermion or boson hamiltonians that may or may not be time-dependent. We have shown that, as long as the hamiltonian yields delocalizing dynamics, and for any initial state that satisfies a condition on algebraic

clustering of correlations, all local operators of the system relax to values consistent with a gaussian state at late times—the system "gaussifies". Furthermore, we have shown that gaussification can be understood as a simple consequence of the spreading of operators in real space, and that the exponents of the power laws with which quantities gaussify can be extracted from the smooth envelope of the one-particle propagator of the system (which does not depend on the initial state). In this sense, gaussification in quadratic systems appears to be quite universal in character.

Using similar arguments, we have given a simple proof that all local conserved quantities of a quadratic time-independent hamiltonian with delocalizing dynamics are themselves quadratic, and hence that the GGE density operator of such a system is gaussian. We have described how to construct the GGE out of mode occupation numbers in a manner that properly accounts for degeneracies in the mode spectrum. Under an additional assumption on the initial state (needed to avoid having to deal with hydrodynamic timescales comparable to the system size), we have shown that the local 2-point function of the system relaxes to its GGE value with a power law whose exponent can typically be extracted from the local density of single-particle levels at the band edge. Combined with our gaussification results, this proves relaxation to the GGE for a large class of quadratic systems and a large family of initial states, and also gives quantitative information about how local observables relax. We find that, if the initial state has a density wave of some conserved quantity, the system generically relaxes first to a gaussian state, and then, with a smaller inverse power of time, to the GGE. If the initial state is *not* ordered in this sense, "gaussification" and relaxation to the GGE occur with the same powers of time and cannot be distinguished as easily in general.

We have also studied situations in which these conclusions break down, such as the case of free massless bosons in one dimension [97, 101], or when the mode spectrum of

the hamiltonian includes localized levels, and have explained precisely why the breakdown occurs in these cases. We have argued that, perhaps unexpectedly, well-separated localized levels in a system of fermions do not hinder gaussification or relaxation to the GGE. Finally, we have applied our arguments to the case of periodically driven systems, and have shown that the relaxation of such systems to the Floquet-GGE can also be understood semi-quantitatively within our framework.

Appendices

3.A Connected correlation functions

For completeness, in this section we review the standard definition of a connected correlation function [85].

Let $\langle \hat{X} \rangle \equiv \mathrm{Tr}(\hat{X}\hat{\rho})$ denote the expectation of the operator $\hat{X}$ in a given state $\hat{\rho}$. The *connected correlation function* or *cumulant* $\langle\langle \cdots \rangle\rangle$ of a set of operators $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ is defined inductively by the formula

$$\langle \hat{X}_1 \hat{X}_2 \cdots \hat{X}_n \rangle = \sum_P (\pm) \prod_{\alpha \in P} \langle\langle \hat{X}_{\alpha(1)} \hat{X}_{\alpha(2)} \cdots \rangle\rangle, \tag{3.193}$$

where the sum is over all partitions P of the set $\{1, 2, \ldots, n\}$, each element α_j of the partition is ordered so that $\alpha_j(1) < \alpha_j(2) < \cdots$, and the sign is $+$ or $-$ according to whether the rearrangement

$$(1, 2, \cdots, n) \mapsto (\alpha_1(1), \alpha_1(2), \cdots, \alpha_2(1), \alpha_2(2), \cdots, \cdots\cdots) \tag{3.194}$$

involves altogether an even or odd number of exchanges of fermionic operators, respec-

tively. Unpacking the definition for small values of n,

$$\langle \hat{X}_1 \rangle = \langle\langle \hat{X}_1 \rangle\rangle, \tag{3.195a}$$

$$\langle \hat{X}_1 \hat{X}_2 \rangle = \langle\langle \hat{X}_1 \hat{X}_2 \rangle\rangle + \langle\langle \hat{X}_1 \rangle\rangle \langle\langle \hat{X}_2 \rangle\rangle, \tag{3.195b}$$

$$\begin{aligned} \langle \hat{X}_1 \hat{X}_2 \hat{X}_3 \rangle = {} & \langle\langle \hat{X}_1 \hat{X}_2 \hat{X}_3 \rangle\rangle \\ & + \langle\langle \hat{X}_1 \hat{X}_2 \rangle\rangle \langle\langle \hat{X}_3 \rangle\rangle + \langle\langle \hat{X}_1 \rangle\rangle \langle\langle \hat{X}_2 \hat{X}_3 \rangle\rangle \pm \langle\langle \hat{X}_1 \hat{X}_3 \rangle\rangle \langle\langle \hat{X}_2 \rangle\rangle \\ & + \langle\langle \hat{X}_1 \rangle\rangle \langle\langle \hat{X}_2 \rangle\rangle \langle\langle \hat{X}_3 \rangle\rangle, \end{aligned} \tag{3.195c}$$

and so on.

Informally, the cumulant $\langle\langle \hat{X}_1 \hat{X}_2 \cdots \hat{X}_n \rangle\rangle$ equals the correlation function $\langle \hat{X}_1 \hat{X}_2 \cdots \hat{X}_n \rangle$, minus all possible ways of factorizing this function into products of two or more lower-order cumulants (with additional minus signs as needed to account for exchanges of fermionic operators).

3.B Bounds on $g(t)$ for bosons with pairing

Any quadratic time-independent hamiltonian $\hat{H} = \frac{1}{2} \hat{\Psi}^\dagger \mathcal{H} \hat{\Psi}$ for a system of bosons, in which $\mathcal{H}$ is positive-definite, can be diagonalized by a Bogolyubov transformation [103]:

$$\hat{\Psi} = S \, \hat{\Gamma}, \tag{3.196}$$

where

$$\hat{\Gamma} = (\hat{\gamma}_1^-, \hat{\gamma}_2^-, \cdots, \hat{\gamma}_N^-, \hat{\gamma}_1^+, \hat{\gamma}_2^+, \cdots, \hat{\gamma}_N^+)^T. \tag{3.197}$$

The transformation S has the block form

$$S = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}, \tag{3.198}$$

satisfies $S^\dagger \eta S = \eta$, where $\eta = I_N \oplus -I_N$, and diagonalizes the hamiltonian matrix: $S^\dagger \mathcal{H} S = \eta\Omega$, where

$$\Omega \equiv \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N, -\omega_1, -\omega_2, \ldots, -\omega_N). \tag{3.199}$$

Since $\mathcal{H}$ is positive-definite, all $\omega_j > 0$ (by Sylvester's theorem of inertia). The condition $S^\dagger \eta S = \eta$ may be rewritten as $S^\dagger = \eta S^{-1} \eta$. It follows that

$$\eta\mathcal{H} = S\Omega S^{-1}. \tag{3.200}$$

In terms of the quasiparticle operators, we have

$$\hat{H} = E_0 + \sum_{j=1}^{N} \omega_j \hat{n}_j, \tag{3.201}$$

where $\hat{n}_j = \hat{\gamma}_j^+ \hat{\gamma}_j^-$.

It is very important to note that, since S is not in general a unitary transformation, the boson mode energies ω_j are *not* the eigenvalues of the hermitian matrix $\mathcal{H}$. We will denote the eigenvalues of the matrix $\mathcal{H}$ as ϵ_j. We recover $\omega_j = \epsilon_j$ only when all pairing terms in the hamiltonian vanish; in this limit S is indeed unitary.

The propagator $G(t)$ may be written in matrix form as

$$G(t) = e^{-i\eta\mathcal{H}t} = S e^{-i\Omega t} S^{-1} = S e^{-i\Omega t} \eta S^\dagger \eta. \tag{3.202}$$

At each time t, it satisfies $G(t)\eta\, G^\dagger(t) = \eta$. From this fact, we easily obtain the lower bound $1 \leq g_x^a(t)$, as follows (no sum on x, a):

$$\begin{aligned} 1 &= |\eta_{xx}^{aa}| \\ &= \left| \sum_{b=\pm} \sum_y [G(t)]_{xy}^{ab}\, \eta_{yy}^{bb}\, [G^\dagger(t)]_{yx}^{ba} \right| \\ &\leq \sum_{b=\pm} \sum_y \left| G_{xy}^{ab}(t) \right|^2 \\ &= g_x^a(t). \end{aligned} \tag{3.203}$$

More work is required to derive an upper bound on $g_x^a(t)$. We have

$$\begin{aligned} g_x^a(t) &= \sum_{b=\pm} \sum_y \left| G_{xy}^{ab}(t) \right|^2 \\ &= [G^\dagger(t) G(t)]_{xx}^{aa} \qquad \text{(no sum on } x, a). \end{aligned} \tag{3.204}$$

By definition of the operator norm $\|\cdot\|$,

$$g_x^a(t) \leq \left\| G^\dagger(t) G(t) \right\|. \tag{3.205}$$

For bounded operators A and B, one has $\|AB\| \leq \|A\|\|B\|$ and $\|A^\dagger\| = \|A\|$. Since $\|\eta\| = \|e^{-i\Omega t}\| = 1$, it follows that

$$g_x^a(t) \leq \|S\|^4. \tag{3.206}$$

We can derive a bound on $\|S\|$ from the condition

$$S^\dagger \mathcal{H} S = \eta\Omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N, \omega_1, \omega_2, \ldots, \omega_N). \tag{3.207}$$

Since the hermitian matrix $\mathcal{H}$ is positive-definite, it has a unique positive-definite square root, $\mathcal{H}^{1/2}$. Let $R \equiv \mathcal{H}^{1/2}S$, so that $R^\dagger R = \eta\Omega$. The operator norm of R equals the square root of the largest eigenvalue of $R^\dagger R$, so

$$\|R\|^2 = \omega_{\max} \equiv \max\{\omega_1, \omega_2, \cdots, \omega_N\}. \tag{3.208}$$

Since $S = (\mathcal{H}^{1/2})^{-1}R$, it follows that

$$\|S\| \leq \|(\mathcal{H}^{1/2})^{-1}\|\,\|R\|. \tag{3.209}$$

The operator norm of $(\mathcal{H}^{1/2})^{-1}$ equals the square root of the largest eigenvalue of the product $((\mathcal{H}^{1/2})^{-1})^\dagger(\mathcal{H}^{1/2})^{-1} = \mathcal{H}^{-1}$, so

$$\|(\mathcal{H}^{1/2})^{-1}\| = \frac{1}{\sqrt{\epsilon_{\min}}}, \tag{3.210}$$

where $\epsilon_{\min} = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_N\}$ is the smallest eigenvalue of $\mathcal{H}$. Thus,

$$\|S\| \leq \sqrt{\frac{\omega_{\max}}{\epsilon_{\min}}}, \tag{3.211}$$

and we finally obtain the desired upper bound:

$$g_x^a(t) \leq \left(\frac{\omega_{\max}}{\epsilon_{\min}}\right)^2. \tag{3.212}$$

Equations (3.203) and (3.212) together yield Eq. (3.114).

Chapter 4

Bounds on chaos from the eigenstate thermalization hypothesis

In this chapter, we show that a known bound on the growth rate of the out-of-time-order four-point correlator in chaotic many-body quantum systems follows directly from the general structure of operator matrix elements in systems that obey the eigenstate thermalization hypothesis. This ties together two key paradigms of thermal behavior in isolated many-body quantum systems.

4.1 Introduction

In recent years there has been renewed interest in various ways of quantifying the rates of runaway growth processes in quantum chaotic many-body systems. One particular quantity that has been studied extensively is the four-point out-of-time-order (OTO) correlator [106, 27, 28]; for a recent overview, see Ref. [26]. Here, following Ref. [28], we

consider the thermally regulated OTO correlator,

$$F_{\mathrm{OTO}}(t) \equiv \mathrm{Tr}\big[\rho^{1/4}A(t)\rho^{1/4}A(0)\rho^{1/4}A(t)\rho^{1/4}A(0)\big], \tag{4.1}$$

where $A(t) = e^{iHt}Ae^{-iHt}$ is a local operator in the Heisenberg picture, H is the hamiltonian, $\rho \equiv e^{-\beta H}/Z$ is a thermal density operator at inverse temperature β, and $Z \equiv \mathrm{Tr}\, e^{-\beta H}$ is the partition function. (We set $\hbar = k_B = 1$ throughout.) Following Refs. [27, 28], we consider systems with a scrambling time (also called the Ehrenfest time) t_{s} that is large compared to the dissipation time t_{d} that governs the exponential decay rate of the two-point correlator. In this case, for times $t_{\mathrm{d}} \ll t \ll t_{\mathrm{s}}$, we expect

$$F_{\mathrm{OTO}}(t) \propto 1 - e^{\lambda(t-t_{\mathrm{s}})}, \tag{4.2}$$

where λ is a growth rate that is analogous to the Lyapunov growth rate of the deviation of nearby classical trajectories in chaotic systems. The hierarchy of time scales $t_{\mathrm{s}} \gg t_{\mathrm{d}}$ typically arises only in systems that have a small parameter ϵ that determines t_{s} via $t_{\mathrm{s}} \sim \lambda^{-1}\ln(1/\epsilon)$. Examples include $\epsilon \sim 1/N^2$ for the Sachdev–Ye–Kitaev model of $N \gg 1$ Majorana fermions with all-to-all random four-point interactions [107, 27], and for conformal field theories with $N^2 \gg 1$ fields that have gravitational duals [28], and $\epsilon \sim \hbar_{\mathrm{eff}}$, an effective dimensionless Planck's constant, for semiclassical systems such as the kicked rotor [108] and quantized area-preserving maps [109].

Making a set of physical and mathematical assumptions that are plausible in such systems, Maldecena *et al.* [28] argued that the growth rate λ should be bounded by

$$\lambda \leq 2\pi/\beta. \tag{4.3}$$

This bound is saturated in the SYK model, and in large-N conformal field theories with gravitational duals, where it is related to the physics of information scrambling in black holes [29].

Another paradigm that is believed to apply broadly to many-body quantum systems with sufficiently strong interactions and no disorder (or, more generally, disorder that does not result in many-body localization [110]) is the *eigenstate thermalization hypothesis* (ETH) [18, 19, 20, 21, 22], which supposes that the energy eigenstates of such a system cannot be distinguished from a thermal density matrix when probed by local observables.

More precisely, according to ETH, the matrix elements of a local observable A in the energy-eigenstate basis, $H|i\rangle = E_i|i\rangle$, take the form

$$A_{ij} = \mathcal{A}(E)\delta_{ij} + e^{-S(E)/2} f(E,\omega) R_{ij}, \tag{4.4}$$

where $E = (E_i + E_j)/2$ is the average energy of the two eigenstates, $\omega = E_i - E_j$ is the energy difference, $\mathcal{A}(E) = \operatorname{Tr} \rho A$ with β fixed by $E = \operatorname{Tr} \rho H$, $S(E)$ is the thermodynamic entropy (logarithm of the density of states) at energy E, $f(E,\omega)$ is a smooth, real function of its two arguments with $f(E,\omega) = f(E,-\omega)$, and R_{ij} is a hermitian matrix of erratically varying elements, with overall zero mean and unit variance in local ranges of E and ω. It is consistent (as will be seen below) to treat E as an extensive quantity and ω as an intensive quantity.

Our purpose in this chapter is to derive the bound on the OTO correlator growth rate, Eq. (4.3), directly from ETH. In doing so, we relate two important paradigms of thermal behavior in isolated many-body quantum systems.

Our methodology is to use known properties of the ETH matrix elements to put a bound on the Fourier transform of the OTO correlator (more specifically, on its connected part, defined below) at high frequencies. This bound can then be used to infer bounds

on the OTO correlator itself at intermediate times, with some additional dependence on its precise functional form. For a family of functions that includes the OTO correlator for a conformal field in one spatial dimension, as computed by Maldacena *et al.* [111] via the AdS_2 gravity dual, we find that the bound of Eq. (4.3) must hold. For a simpler family of functions that is sometimes used as an approximation to the OTO correlator, we find a stronger bound, indicating that this approximation should be used with care. Our methods do not require the factorization assumption that was used in Ref. [28] (and which we review below). Hence we believe that our result is more general, and that the bound on the exponential growth rate of the OTO correlator holds in any quantum many-body system that obeys ETH and also has the hierarchy of time scales $t_{\rm s} \gg t_{\rm d}$.

Note that we do not claim that ETH is either necessary or sufficient to have exponential growth of the OTO correlator; we claim only that the exponential growth rate, if nonzero, is bounded by Eq. (4.3) in systems that obey ETH. Also, it may also be possible to weaken our assumptions and still prove the bound; for example, our proof would go through if we allowed the envelope function $f(E, \omega)$ to become noisy (rather than smooth) at sufficiently low frequencies ω.

4.2 Bounds on the high-frequency behavior of ETH envelope functions

It will be most convenient to work with an observable A for which $\mathcal{A}(E) = 0$, either due to a symmetry, or simply by subtracting $\operatorname{Tr} \rho A$ from A; we therefore take $\mathcal{A}(E) = 0$ from here on.

We begin by considering a thermally regulated two-point correlator for such an ob-

servable A at inverse temperature β,

$$F_2(t) \equiv \mathrm{Tr}\big[\rho^{1/2} A(t) \rho^{1/2} A(0)\big]. \tag{4.5}$$

Inserting two complete sets of energy eigenstates and using Eq. (4.4) with $\mathcal{A}(E) = 0$, we have

$$F_2(t) = \frac{1}{Z} \sum_{ij} e^{-S(E)-\beta E} |f(E,\omega)|^2 e^{i\omega t} |R_{ij}|^2. \tag{4.6}$$

We replace $|R_{ij}|^2$ with its statistical average 1, and then write each sum as an integral with a suitable density of states, $\sum_i \to \int_0^\infty dE_i\, e^{S(E_i)}$. Using $E_{i,j} = E \pm \omega/2$, we get

$$F_2(t) = \frac{1}{Z} \int_E \int_\omega e^{S(E+\omega/2)+S(E-\omega/2)-S(E)-\beta E}\, |f(E,\omega)|^2 e^{i\omega t}, \tag{4.7}$$

where $\int_E \equiv \int_0^\infty dE$ and $\int_\omega \equiv \int_{-\infty}^{+\infty} d\omega$. We now assume (and later verify) that $f(E,\omega)$ falls rapidly enough at large ω that we can expand the exponent in powers of ω,

$$S(E \pm \tfrac{1}{2}\omega) = S(E) \pm \tfrac{1}{2} S'(E)\omega + \tfrac{1}{8} S''(E)\omega^2 + \cdots, \tag{4.8}$$

which yields

$$F_2(t) = \frac{1}{Z} \int_E e^{S(E)-\beta E} \int_\omega e^{S''(E)\omega^2/4} e^{i\omega t} |f(E,\omega)|^2. \tag{4.9}$$

We do the E integral by Laplace's method; this fixes E to be the solution of $S'(E) = \beta$, which is the usual thermodynamic relation between energy and temperature. We can then also identify $S''(E) = -\beta^2/C$, where C is the heat capacity of the system at inverse temperature β. The remaining integral over E yields a factor of the partition function Z. We therefore find

$$F_2(t) = \int_\omega e^{-\beta^2\omega^2/4C} e^{i\omega t} |f(E,\omega)|^2. \tag{4.10}$$

Next we note that $\mathrm{Tr}(\rho A^2)$ is equal to $F_2(\pm\frac{i\beta}{2})$, and this should be a finite quantity. In the infinite-volume limit, $C \to \infty$, and we have

$$\mathrm{Tr}\big(\rho A^2\big) = \int_\omega e^{\beta\omega/2}|f(E,\omega)|^2. \tag{4.11}$$

For this to be finite, $f(E,\omega)$ must satisfy

$$|f(E,\omega)| \lesssim \exp(-\beta|\omega|/4) \quad \text{as} \quad |\omega| \to \infty. \tag{4.12}$$

We next consider the general four-point correlator for a single observable A at inverse temperature β,

$$F_4(t_1,t_2,t_3) \equiv \mathrm{Tr}\left[\rho^{1/4}A(t_1)\rho^{1/4}A(t_2)\rho^{1/4}A(t_3)\rho^{1/4}A(0)\right]. \tag{4.13}$$

Inserting four complete sets of energy eigenstates, we have

$$F_4 = \frac{1}{Z}\sum_{ijkl} e^{-\beta E}e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)}A_{ij}A_{jk}A_{kl}A_{li}, \tag{4.14}$$

where $E \equiv \frac{1}{4}(E_i+E_j+E_k+E_l)$, $\omega_1 \equiv E_i - E_j$, $\omega_2 \equiv E_j - E_k$, $\omega_3 \equiv E_k - E_l$. We use Eq. (4.4) with $\mathcal{A}(E) = 0$ for A_{ij}. We then replace $R_{ij}R_{jk}R_{kl}R_{li}$ by its statistical average, which, following the general analysis of Foini and Kurchan [112], we take to be

$$\overline{R_{ij}R_{jk}R_{kl}R_{li}} = \delta_{ik} + \delta_{jl} + e^{-S(E)}g(E,\omega_1,\omega_2,\omega_3). \tag{4.15}$$

Here the first two terms account for the fact that for $i = k$ or $j = l$ the left-hand side reduces to the product of the absolute square of two R's, and then the statistical average is 1. The final term accounts for exponentially small correlations between different

R's. Such correlations arise from treating the inner product of energy eigenstates (in a small range of energy) with eigenstates of A (in a small range of its eigenvalues) as a pseudorandom unitary matrix [112]. We view this as a generalized version of Berry's conjecture [113] that the energy eigenstates of a chaotic quantum system can be expressed as superpositions of suitable basis states whose coefficients are pseudorandom numbers with a gaussian distribution; the specific form of the appropriate basis states is system-dependent [114, 115, 116]. Berry's conjecture underlies the formulation of ETH presented in Ref. [19], and so Eq. (4.15) should be viewed as a consequence of ETH.

Returning to Eq. (4.14), we replace the sums by integrals, $\sum_i \to \int_{E_i} e^{S(E_i)}$, expand the entropies to linear order about E, change the integration variables to E and the three ω's, and perform the integral over E by Laplace's method. The final result, in the infinite volume limit, is

$$\begin{aligned} F_4(t_1,t_2,t_3) = F_2(t_1-t_2+\tfrac{i\beta}{4})F_2(t_3+\tfrac{i\beta}{4}) + F_2(t_1-\tfrac{i\beta}{4})F_2(t_3-t_2-\tfrac{i\beta}{4}) \\ + F_{4\mathrm{C}}(t_1,t_2,t_3), \end{aligned} \tag{4.16}$$

where the *connected part* of the four-point function is

$$F_{4\mathrm{C}}(t_1,t_2,t_3) = \int_{\omega_1\cdots\omega_3} e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)} f(\omega_1)f(\omega_2)\, f(\omega_3)f(-\omega_1-\omega_2-\omega_3)\, g(\omega_1,\omega_2,\omega_3). \tag{4.17}$$

Here we have suppressed the E dependence of f and g.

Next we note that

$$\mathrm{Tr}\big(\rho A^4\big) - 2[\mathrm{Tr}\big(\rho A^2\big)]^2 = F_{4\mathrm{C}}(-\tfrac{i\beta}{4}, -\tfrac{i\beta}{2}, -\tfrac{3i\beta}{4}) \tag{4.18}$$

should be a finite quantity. Given Eq. (4.12), convergence of the integral over ω_3 in

Eq. (4.17), with ω_1 and ω_2 fixed, requires that $g(\omega_1, \omega_2, \omega_3)$ must satisfy

$$|g(\omega_1, \omega_2, \omega_3)| \lesssim \exp(-\beta|\omega_3|/4) \quad \text{as} \quad |\omega_3| \to \infty. \tag{4.19}$$

4.3 The OTO correlator and "bound on chaos"

We now turn our attention to the OTO four-point correlator $F_{\text{OTO}}(t) \equiv F_4(t, 0, t)$, which is given by

$$F_{\text{OTO}}(t) = 2\,\text{Re}[F_2(t + \tfrac{i\beta}{4})^2] + F_{4\text{C}}(t, 0, t). \tag{4.20}$$

For times large compared to the dissipation time t_{d}, which itself should be comparable to or larger than β, the first term in Eq. (4.20) will have decayed to a negligible value, and we can replace $F_{\text{OTO}}(t)$ with its connected part $F_{\text{OTOC}}(t) \equiv F_{4\text{C}}(t, 0, t)$.

We will be interested in the Fourier transform of $F_{\text{OTOC}}(t)$, given by

$$\begin{aligned} \widetilde{F}_{\text{OTOC}}(\omega) &\equiv \int_{-\infty}^{+\infty} \frac{dt}{2\pi}\, e^{-i\omega t} F_{\text{OTOC}}(t) \\ &= \int_{\omega_1, \omega_2} f(\omega_1) f(\omega_2) f(\omega - \omega_1) f(-\omega - \omega_2) g(\omega_1, \omega_2, \omega - \omega_1). \end{aligned} \tag{4.21}$$

From the large frequency behavior of f and g specified by Eqs. (4.12) and (4.19), we can infer that $\widetilde{F}_{\text{OTOC}}(\omega)$ must fall off at large $|\omega|$ at least as fast as

$$\widetilde{F}_{\text{OTOC}}(\omega) \sim \exp(-3\beta|\omega|/4). \tag{4.22}$$

To use this information, we need a more complete specification of the OTO correlator than is found in Eq. (4.2), which applies only for intermediate positive times. Assuming an exponential decay at late positive times, a simple model is $F_{\text{OTOC}}(t) \propto 1/(1 + z(t))^{\eta}$,

where

$$z(t) \equiv e^{\lambda(t-t_{\rm s})}, \tag{4.23}$$

and η is a positive real parameter. However this $F_{\rm OTOC}(t)$ is not time-reversal invariant, whereas Eq. (4.1) is. To remedy this, and assuming $\lambda t_{\rm s} \gg 1$, we make the ansatz

$$F_{\rm OTOC}(t) = \mathcal{N} G(z(t)) G(z(-t)), \tag{4.24}$$

where we take $G(z) = 1/(1+z)^{\eta}$; later we will consider other possibilities for $G(z)$. The normalization constant is

$$\mathcal{N} = \mathrm{Tr}\big[(\rho^{1/4} A)^4\big] - 2\big(\mathrm{Tr}\, \rho^{3/4} A \rho^{1/4} A\big)^2. \tag{4.25}$$

From the product form of Eq. (4.24), it follows that the Fourier transform is given by the convolution

$$\widetilde{F}_{\rm OTOC}(\omega) = \mathcal{N} \int_{\omega'} \widetilde{G}(\omega - \omega') \widetilde{G}(\omega'). \tag{4.26}$$

From this it follows that the large-ω behavior of $\widetilde{F}_{\rm OTOC}(\omega)$ is the same as the large-ω behavior of $\widetilde{G}(\omega)$. We find

$$\widetilde{G}(\omega) = e^{-i\omega t_{\rm s}} \frac{\Gamma(\eta + i\omega/\lambda)\Gamma(0^+ - i\omega/\lambda)}{2\pi\lambda\,\Gamma(\eta)} K(\omega), \tag{4.27}$$

where $\Gamma(x)$ is the gamma function, and $K(\omega) = 1$ has been introduced for later convenience. Eq. (4.27) yields $\widetilde{F}_{\rm OTOC}(\omega) \sim \widetilde{G}(\omega) \sim \exp(-\pi|\omega|/\lambda)$ at large $|\omega|$, independent of η. Requiring this fall-off to be at least as fast as Eq. (4.22), we find the bound $\lambda \leq 4\pi/3\beta$, which is more stringent than Eq. (4.3). This shows that for $\lambda = 2\pi/\beta$ (expected in conformal field theories with gravity duals), the form $G(z) = 1/(1+z)^{\eta}$, which has sometimes been used as an approximation (e.g., [117]), is inconsistent with our Eq. (4.22).

Maldacena *et al.* [111] computed the OTO correlator for a conformal field with dimension $\Delta = \eta/2$ via the AdS_2 gravity dual. In our notation, their result is

$$G(z) = \int_0^\infty du\, h(u)(1+uz)^{-\eta} \tag{4.28}$$

with $h(u) = e^{-u}u^{\eta-1}/\Gamma(\eta)$. This yields Eq. (4.27) with

$$K(\omega) = \int_0^\infty du\, h(u) u^{i\omega/\lambda}. \tag{4.29}$$

For the $h(u)$ of Ref. [111], $K(\omega) = \Gamma(\eta + i\omega/\lambda)/\Gamma(\eta) \sim \exp(-\pi|\omega|/2\lambda)$, and hence $\widetilde{F}_{\text{OTOC}}(\omega) \sim \exp(-3\pi|\omega|/2\lambda)$. Requiring this fall-off to be at least as fast as Eq. (4.22), we find the bound $\lambda \leq 2\pi/\beta$, the same as Eq. (4.3).

More generally, the bound $\lambda \leq 2\pi/\beta$ holds if $K(\omega) \sim \exp(-c\pi|\omega|/2\lambda)$ with $c \leq 1$ at large ω. The Paley–Wiener theorem [118] implies that this will be the case if and only if there is a value of $\theta \in [-\pi/2, \pi/2]$ such that

$$\int_0^\infty du\, u|h(e^{i\theta}u)|^2 = \infty. \tag{4.30}$$

For example, this is the case if $h(u) \sim u^a \exp(-bu^\gamma)$ at large u with $a \geq -\frac{1}{2}$, $b \geq 0$ and $\gamma \geq 1$. However, we have not been able to connect the mathematical condition of Eq. (4.30) to a physical property of the system.

4.4 Discussion and conclusions

We can compare our derivation of Eq. (4.3) with that of Ref. [28]. A key assumption used in Ref. [28], their Eq. (23), is an approximate factorization of a different regulariza-

tion of the four-point function at intermediate times $t_{\rm d} \ll t \ll t_{\rm s}$,

$$\mathrm{Tr}\big[\rho^{1/2}A(t)A(0)\rho^{1/2}A(0)A(t)\big] \leq \big(\mathrm{Tr}\,\rho^{1/2}A\rho^{1/2}A\big)^2 + \varepsilon, \tag{4.31}$$

where ε is a small tolerance parameter. In our notation, this is equivalent to

$$|F_2(t-\tfrac{i\beta}{2})|^2 + F_{4\rm C}(t-\tfrac{i\beta}{4}, t, -\tfrac{i\beta}{4}) \leq \varepsilon. \tag{4.32}$$

We expect the first term to be negligible at the relevant intermediate times, and it is plausible that the second term, which is an in-time-order connected four-point function, is also negligible. However, we do not need to assume this in our analysis.

Assuming Eq. (4.31), Ref. [28] then establishes that the rescaled correlator $f(t) \equiv F_{\rm OTO}(t)/[F_2(0)^2+\varepsilon]$ obeys

$$\left|\frac{d}{dt}f(t)\right| \leq \frac{2\pi}{\beta}\coth\left(\frac{2\pi t}{\beta}\right)\frac{1-f(t)^2}{2}. \tag{4.33}$$

Because of the factor of $1-f^2$ on the right-hand side of Eq. (4.33), it provides a meaningful bound on λ in Eq. (4.2) if and only if $f(t) = 1 - e^{\lambda(t-t_{\rm s})}$ for $t_{\rm d} \ll t \ll t_{\rm s}$, where we now have an equality rather than a proportionality. Equivalently, $F_{\rm OTO}(t)$ must be very close to $F_2(0)^2$ for these intermediate times. In our notation, this requires $\mathcal{N} \approx F_2(0)^2$, where $\mathcal{N}$ is the normalization constant of Eq. (4.25). Such a relation would follow from large-N factorization, Eq. (9) in Ref. [28], but is not expected to be true more generally. Our derivation of Eq. (4.3) does not require this assumption.

We note that the scrambling time $t_{\rm s}$ appears as the period of an oscillation in the amplitude of $\widetilde{F}_{\rm OTOC}(\omega)$, cf. Eqs. (4.26,4.27), that must have its origin in a correspoding oscillation in the amplitude of $g(\omega_1,\omega_2,\omega_3)$, cf. Eq. (4.21). The underlying physics of this

sort of oscillation in the four-point correlation of operator matrix elements, cf. Eq. (4.15), is worthy of further exploration.

Recently another measure of chaos was introduced by Parker *et al.* [119]: an operator complexity growth rate α that was shown to be bounded by $\alpha \leq \pi/\beta$. This bound on α is related to the the large-ω behavior of $f(E,\omega)$, and follows from Eq. (4.12). Parker *et al.* conjecture that, very generally, $\lambda \leq 2\alpha$. Our analysis shows that the bound on λ requires information about the large-ω behavior of $g(E,\omega_1,\omega_2,\omega_3)$ in addition to the large-ω behavior of $f(E,\omega)$. A proof that $\lambda \leq 2\alpha$ would therefore imply a further relationship between these two functions, and hence further structure in the ETH matrix elements. This is an interesting topic for further research.

To conclude, we have derived the known bound of Eq. (4.3) on the growth rate of the out-of-time-order four-point correlator from the structure of operator matrix elements that follows from the eigenstate thermalization hypothesis, Eqs. (4.4,4.15). We also needed a mild assumption on the functional form of this correlator, as specified by Eqs. (4.24,4.28,4.30). However, we did not need the assumptions of Eq. (4.31) and of large-N factorization (or its equivalent) that were needed in the analysis of Ref. [28].

We hope that this unification of two key paradigms of thermal behavior in many-body quantum systems will lead to further insights into this important branch of physics.

Chapter 5

Structure of chaotic eigenstates and their entanglement entropy

In this chapter, we consider a chaotic many-body system (i.e., one that satisfies the eigenstate thermalization hypothesis) that is split into two subsystems, with an interaction along their mutual boundary, and study the entanglement properties of an energy eigenstate with nonzero energy density. When the two subsystems have nearly equal volumes, we find a universal correction to the entanglement entropy that is proportional to the square root of the system's heat capacity (or a sum of capacities, if there are conserved quantities in addition to energy). Our conclusions are based on a refined version of a model of a chaotic eigenstate originally due to Deutsch, and analyzed more recently by Lu and Grover.

5.1 Introduction

Consider a macroscopic system of volume V partitioned into two spatial subsystems 1 and 2 with volumes V_1 and $V_2 = V - V_1$. We assume, without loss of generality, that

$V_1 \leq V_2$. We also assume that the hamiltonian of the system is a sum of local terms, and so can be partitioned as

$$H = H_1 + H_2 + H_{12}, \tag{5.1}$$

where H_a acts nontrivially only in region a ($a = 1, 2$) and all terms coupling the two subsystems are contained in H_{12}. We further assume that the system obeys the eigenstate thermalization hypothesis (ETH) [18, 19, 21] for matrix elements of local observables between energy eigenstates corresponding to nonzero energy densities.

To simplify notation, we take all energies to be in units of a fundamental energy scale (e.g., the coefficient of an exchange term in a spin chain) and all lengths, areas, and volumes to be in units of a fundamental length (e.g., the lattice spacing). We also set $k_B = 1$ throughout.

Let $|E\rangle$ denote an eigenstate of H with energy E, with nonzero energy density E/V. For notational convenience, and again without loss of generality, we shift H_{12} by a constant (if necessary) so that

$$\langle E|H_{12}|E\rangle = 0. \tag{5.2}$$

We can write $|E\rangle$ in a basis of tensor products of the eigenstates of H_1 and H_2,

$$|E\rangle = \sum_{i,J} M_{iJ} \, |i\rangle_1 \otimes |J\rangle_2 \, . \tag{5.3}$$

Deutsch [32] conjectured that the coefficient matrix M_{iJ} can be treated as a random matrix with a narrow bandwidth that keeps the sum of the subsystem energies $E_{1i} + E_{2J}$ close to the total system energy E, and using this conjecture showed that the entanglement entropy of the smaller subsystem equals its thermodynamic entropy. More recently, Lu and Grover [120] used this ansatz to calculate the Rényi entropies of the subsystem. Other related work on entanglement entropy at nonzero energy density in chaotic systems

includes Refs. [121, 122, 123, 124, 125]; for a review of basic concepts, see Ref. [31].

Here, we refine the original conjecture by characterizing the coefficient matrix more completely. We further show that at or very near $V_1 = V_2$, there is an extra contribution to the entanglement entropy that scales like $\sqrt{V}$. Specifically, for $V_1 = V_2$ exactly, we find that the entanglement entropy is given by

$$S_{\text{ent}} = \frac{1}{2}S - \sqrt{\frac{C}{2\pi}} + O(A), \tag{5.4}$$

where S is the thermodynamic entropy of the system at energy E, C is its heat capacity, and A is the area of the boundary between the two subsystems. When the system is far from a critical point (which we assume for simplicity), both S and C typically scale like the volume V of the system. We do not compute the coefficient of the $O(A)$ term, since it depends on details of the hamiltonian. The $\sqrt{C}$ and $O(A)$ terms are distinguished by their scaling with system size (except in $d = 2$ spatial dimensions). They are also distinguished by the fact that the latter depends on a property of the boundary between the two subsystems, while the former depends on a property of the system as a whole. The extension of Eq. (5.4) to $V_1 \neq V_2$ is given in Eq. (5.16) below; the $\sqrt{C}$ correction remains significant for $|V_1 - V_2| \lesssim \sqrt{V}$.

A contribution to S_{ent} scaling like $\sqrt{V}$ was found previously by Vidmar and Rigol [126] in a study of a one-dimensional system with one conserved quantum number. Our explanation for the appearance of such a term is essentially the same as theirs, but our formula applies more generally to any system that obeys ETH, and relates the correction to thermodynamic properties of the system. Furthermore, we generalize our result to systems with any finite number of conserved quantities in addition to energy. In such cases, C in Eq. (5.4) becomes the sum of all entries in a matrix of capacities; see Sec. 5.5.

The rest of this chapter is organized as follows. In Section 5.2, we summarize all

of our key results in more precise language, for the simplest case in which only energy is conserved. Sections 5.3–5.5 elaborate on the derivation of the summarized results. The generalization to systems with additional conserved quantities is discussed in the second half of Sec. 5.5, and full details are provided in the Appendix. Section 5.6 has our concluding discussion.

5.2 Summary of results

5.2.1 Structure of the coefficient matrix

Assuming, in line with Refs. [32, 120], that M_{iJ} has the general structure of a random matrix that is sharply banded in total energy, and neglecting any dependence of M_{iJ} on the energy difference $E_{1i} - E_{2J}$, we show that it takes the form

$$M_{iJ} = e^{-S(E_{1i}+E_{2J})/2} F(E_{1i} + E_{2J} - E)^{1/2}\, C_{iJ}, \tag{5.5}$$

where $S(E)$ is the thermodynamic entropy of the full system at energy E (equal to the logarithm of the density of states, and assumed to be a monotonically increasing function of energy, so that temperature is nonnegative), $F(\varepsilon)$ is a window function centered on $\varepsilon = 0$ with a width Δ equal to the quantum uncertainty in the interaction hamiltonian,

$$\Delta = \sqrt{\langle E|H_{12}^2|E\rangle}, \tag{5.6}$$

and C_{iJ} is a matrix of coefficients which, when averaged over narrow bands of energies of each subsystem near E_1 and E_2 (but with each band still containing many subsystem

energy eigenstates), obeys

$$\overline{C_{iJ}} = 0, \qquad \overline{C^*_{iJ} C_{i'J'}} = \delta_{ii'} \delta_{JJ'}, \tag{5.7}$$

where the overbar denotes the dual narrow-band energy averaging. Furthermore, for a system in two or more spatial dimensions, the window function is a gaussian,

$$F(\varepsilon) = \frac{e^{-\varepsilon^2/2\Delta^2}}{\sqrt{2\pi}\Delta}. \tag{5.8}$$

In two or more spatial dimensions, where H_{12} is a sum of local terms along the boundary between the two subsystems, we show that $\Delta \sim \sqrt{A}$, where A is the area of the boundary. For a one-dimensional system, Δ is an order-one quantity (in terms of its scaling with system size). These results are derived in Section 5.3.

5.2.2 Structure of the reduced density matrix

The reduced density matrix $\rho_1 \equiv \mathrm{Tr}_2 \, |E\rangle\langle E|$ of subsystem 1 takes the form

$$(\rho_1)_{ij} = e^{-S(E)+S_2(E-E_1)} \Big[\delta_{ij} + e^{-S_2(E-E_1)/2} e^{-\omega^2/8\Delta^2} R_{ij} \Big], \tag{5.9}$$

where $E_1 \equiv (E_{1i} + E_{1j})/2$ and $\omega \equiv E_{1i} - E_{1j}$, $S_a(E_a)$ is the thermodynamic entropy of subsystem a at energy E_a ($a = 1, 2$), and the R_{ij} are $O(1)$ numbers that vary erratically. We have dropped terms of order $\Delta^2 \sim A$ and smaller in the exponents. Section 5.4 contains details of the derivation.

The diagonal term in Eq. (5.9) is in agreement with Lu and Grover [120], and matches the "subsystem ETH" ansatz of Dymarsky *et al.* [127]. The off-diagonal term, though exponentially smaller than the diagonal term, alters the spectrum of eigenvalues of ρ_1 at

energies $E_1 > E_1^*$, where E_1^* is the solution to

$$S_1(E_1^*) = S_2(E - E_1^*). \tag{5.10}$$

For $E_1 > E_1^*$, the density of states of subsystem 2 is smaller than the density of states of subsystem 1. However, the nonzero eigenvalues of ρ_1 are the same as those of $\rho_2 \equiv \mathrm{Tr}_1 |E\rangle\langle E|$. This effect occurs locally in energy. Hence, in the energy interval $[E_1, E_1 + dE_1]$ for $E_1 > E_1^*$, ρ_1 has approximately $e^{S_2(E-E_1)}dE_1$ nonzero eigenvalues, and each of these nonzero eigenvalues is approximately equal to $e^{-S(E)}e^{S_1(E_1)}$.

5.2.3 Correction to the entanglement entropy

From the discussion above, it follows that

$$\mathrm{Tr}\,\rho_1^n = \frac{\int dE_1\, e^{S_{\min}(E_1)}\left[e^{-S(E)}e^{S_{\max}(E_1)}\right]^n}{e^{-S(E)}\int dE_1\, e^{S_{\min}(E_1)+S_{\max}(E_1)}}, \tag{5.11}$$

where

$$S_{\max}(E_1) \equiv \max[S_1(E_1), S_2(E - E_1)], \tag{5.12a}$$

$$S_{\min}(E_1) \equiv \min[S_1(E_1), S_2(E - E_1)]. \tag{5.12b}$$

The denominator in Eq. (5.11) is the numerator with $n = 1$, and itself equals one up to small corrections; see Sec. 5.3. The entanglement entropy is the $n \to 1$ limit of the nth Rényi entropy,

$$S_{\mathrm{ent}}(E) = \lim_{n\to 1} S_{\mathrm{Ren},n}(E), \tag{5.13}$$

where

$$S_{\mathrm{Ren},n}(E) \equiv \frac{1}{1-n} \log \mathrm{Tr}\, \rho_1^n. \tag{5.14}$$

From Eqs. (5.11)–(5.14), we get

$$S_{\mathrm{ent}}(E) = \frac{\int dE_1\, e^{S_1(E_1)+S_2(E-E_1)}[S(E) - S_{\max}(E_1)]}{\int dE_1\, e^{S_1(E_1)+S_2(E-E_1)}}. \tag{5.15}$$

After performing the integrals over E_1 by Laplace's method, we find

$$S_{\mathrm{ent}}(E) = \min(\bar{S}_1, \bar{S}_2) - \sqrt{\frac{2K}{\pi}}\, \Phi\left(\frac{\bar{S}_2 - \bar{S}_1}{\sqrt{8K}}\right) + O(A), \tag{5.16}$$

where $\bar{S}_1 \equiv S_1(\bar{E}_1)$ and $\bar{S}_2 \equiv S_2(E - \bar{E}_1)$ are the subsystem entropies at the stationary point $\bar{E}_1$, given by

$$S_1'(\bar{E}_1) = S_2'(E - \bar{E}_1), \tag{5.17}$$

$K \equiv C_1 C_2/(C_1+C_2)$ is the harmonic mean of the subsystem heat capacities $C_a \equiv -\beta^2/\bar{S}_a''$ at constant volume and inverse temperature $\beta \equiv S_1'(\bar{E}_1)$, and we have defined the function

$$\Phi(x) \equiv \int_{-\infty}^{+\infty} dy\, e^{-y^2} \big(|y-x| - |x|\big) = \sqrt{\pi}\big(x \,\mathrm{erf}\, x - |x|\big) + e^{-x^2}, \tag{5.18}$$

where $\mathrm{erf}\, x$ is the error function; see Fig. 1. Since $\Phi(x)$ decays to zero exponentially from $\Phi(0) = 1$, this correction is negligible for $|\bar{S}_2 - \bar{S}_1| \gg \sqrt{K}$.

For a uniform system with $V_1 = fV$, $V_2 = (1-f)V$, and $f \le \frac{1}{2}$, we have $\bar{S}_1 = fS(E)$, $\bar{S}_2 = (1-f)S(E)$, $C_1 = fC$, $C_2 = (1-f)C$, and $K = f(1-f)C$, where $C \equiv -\beta^2/S''(E)$ is the heat capacity of the full system. The heat capacity C scales like the volume of the system, so $|\bar{S}_2 - \bar{S}_1| \gg \sqrt{K}$ is equivalent to $|\frac{1}{2} - f| \gg 1/\sqrt{V}$. For $f = \frac{1}{2}$ exactly, we recover Eq. (5.4).

Section 5.5 contains details of the derivation of these results, and also describes their

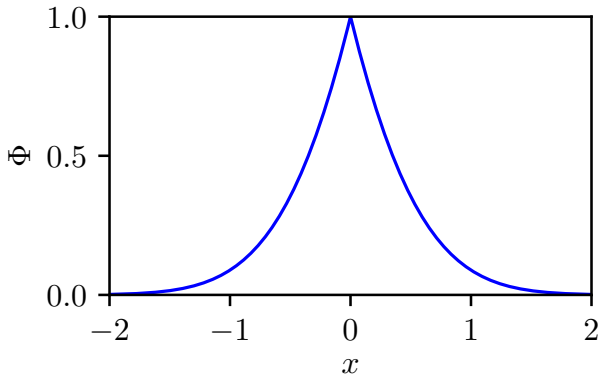


Figure 1: A plot of the function $\Phi(x)$ defined in Eq. (5.18), which parameterizes the correction to the entanglement entropy, Eq. (5.16).

generalization to systems with additional conserved quantities.

5.2.4 Correction to the Rényi entropy for $n < 1$

Evaluating Eq. (5.11) by Laplace's method, and then evaluating the leading terms in Eq. (5.14), we find

$$S_{\mathrm{Ren},n}(E) = \frac{[S_1(\mathcal{E}_1) + nS_2(E - \mathcal{E}_1) - nS(E)]}{1-n}, \tag{5.19}$$

where

$$\mathcal{E}_1 \equiv \min(\bar{E}_1, E_1^*), \tag{5.20}$$

E_1^* is the solution to Eq. (5.10), and $\bar{E}_1$ is the solution to

$$S_1'(\bar{E}_1) = nS_2'(E - \bar{E}_1). \tag{5.21}$$

For $\bar{E}_1 < E_1^*$, Eq. (5.19) coincides with the result of Ref. [120]. For $n > 1$, the convexity of the entropy function (equivalently, positivity of the temperature and the heat capacity) guarantees that $\bar{E}_1 < E_1^*$. However, for $n < 1$, it is possible to have $E_1^* < \bar{E}_1$, and then

Eq. (5.19) differs from the result of Ref. [120]. In particular, for a uniform system split exactly in half, $E_1^* < \bar{E}_1$ for all $n < 1$, and then $S_{\mathrm{Ren},n<1}(E) = S(E)/2$, up to subleading corrections.

5.3 Envelope function of the coefficient matrix

We first establish a useful identity. In the limit of $\Delta \to 0$, we can ignore the energy of the interaction. Then we can compute the density of states $e^{S(E)}$ of the total system at energy E by dividing the energy between the two subsystems, and taking the product of the number of states of each subsystem. This yields

$$e^{S(E)} = \int_0^E dE_1\, e^{S_1(E_1)} e^{S_2(E-E_1)}. \tag{5.22}$$

Note that Eq. (5.22) shows that the denominator in Eq. (5.11) equals one in the $\Delta \to 0$ limit.

Next we warm up by computing $\langle E|E\rangle = 1$. From Eqs. (5.3) and (5.5), we have

$$\langle E|E\rangle = \sum_{iJ} e^{-S(E_{1i}+E_{2J})} F(E_{1i} + E_{2J} - E)\, |C_{iJ}|^2. \tag{5.23}$$

The sums over i and J implement the narrow-band averaging of Eq. (5.7), and can then be replaced by integrals over E_1 and E_2 with factors of the densities of states, yielding

$$\langle E|E\rangle = \int_0^\infty dE_1\, e^{S_1(E_1)} \int_0^\infty dE_2\, e^{S_2(E_2)}\, e^{-S(E_1+E_2)} F(E_1 + E_2 - E). \tag{5.24}$$

In the limit $\Delta \to 0$, $F(\varepsilon) \to \delta(\varepsilon)$, the Dirac delta function. In this limit we have

$$\langle E|E\rangle = e^{-S(E)} \int_0^\infty dE_1\, e^{S_1(E_1)} e^{S_2(E-E_1)} \;=\; 1, \tag{5.25}$$

where the final result follows from Eq. (5.22).

For finite Δ, we take $F(\varepsilon)$ to have the gaussian form of Eq. (5.8), although we only need that $F(\varepsilon)$ be sharply peaked at $\varepsilon = 0$ with width Δ. We then evaluate the integrals in Eq. (5.24) by Laplace's method. The conditions for a stationary point of the exponent are

$$S_1'(E_1) = S'(E_1 + E_2) + (E_1 + E_2 - E)/\Delta^2, \tag{5.26a}$$

$$S_2'(E_2) = S'(E_1 + E_2) + (E_1 + E_2 - E)/\Delta^2. \tag{5.26b}$$

For small Δ, the solution is $E_1 = \bar{E}_1$, $E_2 = \bar{E}_2$, where

$$\bar{E}_1 + \bar{E}_2 = E, \tag{5.27a}$$

$$S_1'(\bar{E}_1) = S_2'(\bar{E}_2) = S'(E) =: \beta, \tag{5.27b}$$

where β is again the inverse temperature of the system as a whole and of each subsystem. Next we Taylor-expand the entropies about the stationary point,

$$S(E_1 + E_2) = S(E) + \beta(E_1 + E_2 - E) - \tfrac{1}{2}(\beta^2/C)(E_1 + E_2 - E)^2 + \cdots, \tag{5.28}$$

$$S_1(E_1) = \bar{S}_1 + \beta(E_1 - \bar{E}_1) - \tfrac{1}{2}(\beta^2/C_1)(E_1 - \bar{E}_1)^2 + \cdots, \tag{5.29}$$

$$S_2(E_2) = \bar{S}_2 + \beta(E_2 - \bar{E}_2) - \tfrac{1}{2}(\beta^2/C_2)(E_2 - \bar{E}_2)^2 + \cdots. \tag{5.30}$$

To leading order in the system volume, Eq. (5.22) implies

$$\bar{S}_1 + \bar{S}_2 = S(E). \tag{5.31}$$

Thus the constant and linear terms all cancel in the combination $S_1(E_1) + S_2(E_2) -$

$S(E_1 + E_2)$ that appears in Eq. (5.24). This cancellation is why it was necessary to have $S(E_1+E_2)$ in the exponent in Eq. (5.5) rather than $S(E)$. The remaining quadratic terms yield gaussian integrals that give an $O(1)$ result for the integral in Eq. (5.24). Adjusting the $O(1)$ terms in the entropies is then necessary to yield the final result of $\langle E|E\rangle = 1$.

The heat capacities in Eqs. (5.28)–(5.30) are proportional to the volumes of the corresponding macroscopic regions, whereas $\Delta^2 \propto A$ (the area of the 1–2 boundary), as will be demonstrated below. Consequently $1/\Delta^2 \gg \beta^2/C, \beta^2/C_1, \beta^2/C_2$, and hence the distribution of $E_1 + E_2$ is controlled by $F(\varepsilon)$. Then we have the following generalization of Eq. (5.24),

$$\langle E|(H_1 + H_2 - E)^n|E\rangle \approx \int_{-\infty}^{+\infty} d\varepsilon\, F(\varepsilon)\varepsilon^n. \tag{5.32}$$

We emphasize that the derivation of Eq. (5.32) from Eqs. (5.5) and (5.7) does not rely on the precise form of $F(\varepsilon)$; it relies only on $F(\varepsilon)$ being sharply peaked at $\varepsilon = 0$ with width Δ that satisfies $\beta^2\Delta^2 \ll C_1, C_2$.

For $n = 1, 2$ in Eq. (5.32), we can replace $H_1 + H_2 - E$ with $H_1 + H_2 - H$, since H will always appear next to either the ket or bra form of its eigenstate. Then using $H_1 + H_2 - H = -H_{12}$, we find for $n = 1$ that

$$\int_{-\infty}^{+\infty} d\varepsilon\, F(\varepsilon)\varepsilon = -\langle E|H_{12}|E\rangle = 0, \tag{5.33}$$

where the second equality follows from our shift of H_{12}. For $n = 2$, we get

$$\int_{-\infty}^{+\infty} d\varepsilon\, F(\varepsilon)\varepsilon^2 = \langle E|H_{12}^2|E\rangle. \tag{5.34}$$

The left-hand side equals Δ^2 by definition, and so Eq. (5.34) verifies Eq. (5.6).

In two or more spatial dimensions, our assumption on the locality of H implies that

H_{12} is a sum of local terms on the boundary B between regions 1 and 2,

$$H_{12} = \sum_{x \in B} h_x. \tag{5.35}$$

We then have

$$\langle E|H_{12}^2|E\rangle = \sum_{x,y \in B} \langle E|h_x h_y|E\rangle. \tag{5.36}$$

Assuming that ETH holds for the bilocal operator $h_x h_y$, the eigenstate expectation value can be replaced by a thermal expectation value at inverse temperature β. We further assume that this thermal correlation function decays rapidly for $|x-y| \gg \xi$ to the disconnected form $\langle h_x\rangle\langle h_y\rangle$, where ξ is an appropriate correlation length [1]. Summing the disconnected form over x and/or y yields zero, by Eq. (5.2). Hence the double sum in Eq. (5.36) effectively becomes a single sum over the boundary, yielding

$$\Delta^2 = \langle E|H_{12}^2|E\rangle \sim A\,\xi^{d-1}\langle h_x^2\rangle, \tag{5.37}$$

where h_x is any one term in H_{12}, and the angle brackets denote either the eigenstate or thermal average, which are equal by ETH. Equation (5.37) shows that $\Delta^2 \sim A$, the boundary area.

We can now generalize this argument to higher powers of H_{12}, again assuming rapid decay of $\langle h_x h_y \cdots\rangle$ whenever an index or group of indices is separated by more than ξ from the others. The multiple sum over $x, y, \ldots$ will then yield approximately zero for odd powers, and be dominated by the factorization into correlated pairs for even powers

[1] Note that the eigenstate expectation value $\langle E|h_x h_y|E\rangle$ must in general differ from the thermal expectation value $\langle h_x h_y\rangle$ by $O(1/V)$, even when $|x-y| \gg \xi$; this is needed to recover $\langle E|(H-E)^2|E\rangle = 0$. However, this difference only contributes an $O(A/V)$ correction to Eq. (5.37), and hence can be neglected.

[2]. This then yields, in accord with the usual combinatorics of Wick's theorem,

$$\langle E|H_{12}^{2n}|E\rangle \approx (2n-1)!!\,\Delta^{2n}, \tag{5.38}$$

characteristic of a gaussian distribution.

Returning to Eq. (5.32), and using

$$H_1 + H_2 - E = H - E - H_{12}, \tag{5.39}$$

we have

$$\begin{aligned}\int_{-\infty}^{+\infty} d\varepsilon\, F(\varepsilon)\varepsilon^{2n} &\approx \langle (H-E-H_{12})^{2n}\rangle \\ &\approx \langle H_{12}^{2n}\rangle - \langle H_{12}(H-E)H_{12}^{2n-2}\rangle + \cdots ,\end{aligned} \tag{5.40}$$

The first term is given by Eq. (5.38), and we would like to show that the remaining terms can be neglected. From Eqs. (5.37) and (5.38), we see that we effectively have $H_{12} \sim \sqrt{A}$, so the terms in Eq. (5.40) with factors of $H-E$ will be suppressed unless $H-E \sim \sqrt{A}$ as well. In each of these terms, H acts on a state of the form $H_{12}^k|E\rangle$. But $H_{12} = \sum_x h_x$, and each h_x is an $O(1)$ operator that can change the energy only by an $O(1)$ amount. Hence, acting with k such operators can change the energy by at most an $O(k)$ amount, which is $O(1)$ in terms of its scaling with A. Summing over x can increase the coefficient of the normalized state, but does not increase the maximum change in energy. Hence $H-E \sim O(1)$, and so the terms with one or more factors of $H-E$ in Eq. (5.40) can be

[2]More precisely, the third and higher cumulants of H_{12}/Δ in the state $|E\rangle$ are suppressed relative to the variance, $\langle E|(H_{12}/\Delta)^2|E\rangle \equiv 1$, by powers of $1/\sqrt{A}$.

neglected. We conclude that, up to corrections suppressed by powers of ξ^{d-1}/A or A/V,

$$\int_{-\infty}^{+\infty} d\varepsilon\, F(\varepsilon)\varepsilon^{2n} = (2n-1)!!\,\Delta^{2n}, \tag{5.41}$$

and therefore that $F(\varepsilon)$ is a gaussian with width Δ, Eq. (5.8).

In one spatial dimension, H_{12} is a single term, rather than a sum of $O(A)$ terms. Hence the combinatoric analysis that led to Eq. (5.38) does not apply, and so we cannot conclude that the shape of $F(\varepsilon)$ is gaussian. However, Eqs. (5.33) and (5.34) are still valid, and so $F(\varepsilon)$ is still sharply peaked at $\varepsilon = 0$ with a width Δ that is given by Eq. (5.6). All of our results for the corrections to the entanglement entropy, including Eqs. (5.4) and (5.16), and their generalizations to multiple conserved quantities via Eqs. (5.57)–(5.59) below, only depend on this sharply-peaked nature of $F(\varepsilon)$, and not on the details of its shape, and so hold for all dimensions, including $d = 1$.

5.4 Reduced density matrix

From Eq. (5.3), the reduced density matrix of subsystem 1 is

$$(\rho_1)_{ij} = \sum_K M_{iK} M^*_{jK}. \tag{5.42}$$

Using Eqs. (5.5), (5.8), and (5.28), assuming $\Delta^2 \ll C/\beta^2$, and neglecting prefactors, we have

$$(\rho_1)_{ij} = e^{-S(E)-\omega^2/8\Delta^2} \sum_K e^{-\beta(E_{2K}+E_1-E)\,-(E_{2K}+E_1-E)^2/2\Delta^2} C_{iK} C^*_{jK}, \tag{5.43}$$

where $E_1 \equiv (E_{1i} + E_{1j})/2$ and $\omega \equiv E_{1i} - E_{1j}$. Taking the statistical average and using Eq. (5.7), only the diagonal term survives, hence $\omega = 0$, and we get

$$\overline{(\rho_1)_{ij}} = e^{-S(E)} \sum_K e^{-\beta(E_{2K}+E_1-E)-(E_{2K}+E_1-E)^2/2\Delta^2} \delta_{ij}. \tag{5.44}$$

We again replace the sum over K with an integral over E_2 weighted by the density of states of subsystem 2, which yields

$$\overline{(\rho_1)_{ij}} = e^{-S(E)} \int_0^\infty dE_2\, e^{S_2(E_2)} e^{-\beta(E_2+E_1-E)-(E_2+E_1-E)^2/2\Delta^2} \delta_{ij}. \tag{5.45}$$

The last exponential factor, arising from the window function $F(\varepsilon)$, forces E_2 to be close to $E - E_1$. Expanding $S_2(E_2)$ about this point, we have

$$S_2(E_2) = S_2(E - E_1) + \beta_{21}(E_2 + E_1 - E) + \cdots, \tag{5.46}$$

where $\beta_{21} \equiv S_2'(E - E_1)$ is the inverse temperature of subsystem 2 when its energy is $E - E_1$. Performing the integral over E_2 in Eq. (5.45) then yields

$$\overline{(\rho_1)_{ij}} = e^{-S(E)+S_2(E-E_1)+\Delta^2(\beta-\beta_{21})^2/2} \delta_{ij}. \tag{5.47}$$

Since $\Delta^2 \sim A$, the last term in the exponent is smaller than the first two, which scale like volume. Additionally, we expect other terms of $O(A)$ to arise from finer structure in the C_{iJ} coefficients that we have neglected. These are necessary to produce the usual "area law" for the entanglement entropy of the ground state (for a review, see Ref. [128]), and we expect such correlations to persist at nonzero energy density.

To estimate the size of the fluctuating off-diagonal elements of ρ_1, we compute the statistical average of the absolute square of $(\rho_1)_{ij}$, $i \neq j$. We neglect any statistical

correlations in the C_{iK} coefficients, and assume that

$$\overline{C_{iK}C^*_{jK}C^*_{iL}C_{jL}} = \delta_{KL}. \tag{5.48}$$

Then we have

$$\overline{|(\rho_1)_{ij}|^2} = e^{-2S(E)-\omega^2/4\Delta^2} \sum_K e^{-2\beta(E_{2K}+E_1-E)-(E_{2K}+E_1-E)^2/\Delta^2}. \tag{5.49}$$

Following the same steps that led to Eq. (5.47), we get

$$\overline{|(\rho_1)_{ij}|^2} = e^{-2S(E)+S_2(E-E_1)-\omega^2/4\Delta^2+\Delta^2(\beta-\beta_{21}/2)^2}. \tag{5.50}$$

As in the case of the diagonal components, we expect additional terms of $O(A)$ to arise from neglected correlations in the C_{iK} coefficients. In the limit that we neglect $O(A)$ corrections, Eqs. (5.47) and (5.50) together yield Eq. (5.9).

5.5 Corrections to the entanglement and Rényi entropies

In the limit that we neglect all subleading corrections, we evaluate the numerator of Eq. (5.11) by Laplace's method, which simply yields the maximum value of the integrand. This gives Eq. (5.19) for the nth Rényi entropy.

We consider subleading corrections only in the case of the entanglement entropy, $n = 1$. In this case we must evaluate Eq. (5.15). We have

$$S_{\max} = \tfrac{1}{2}(S_1 + S_2) + \tfrac{1}{2}|S_1 - S_2|. \tag{5.51}$$

Using Eqs (5.27a) and (5.29)–(5.31), and changing the integration variable from E_1 to

$$u \equiv \beta(E_1 - \bar{E}_1), \tag{5.52}$$

we get

$$S_1 + S_2 = S - u^2/2K, \tag{5.53}$$

$$S_1 - S_2 = \bar{S}_1 - \bar{S}_2 + 2u + O(u^2/K), \tag{5.54}$$

where again $K \equiv C_1 C_2/(C_1 + C_2)$. The factor of $e^{-u^2/2K}$ in the integrand is peaked well away from the lower limit of integration, which can therefore be extended to $-\infty$. When performing the gaussian integral, values of u^2 larger than K are exponentially suppressed; thus the $O(u^2/K)$ term in Eq. (5.54) gives only an $O(1)$ contribution, and can be neglected. Putting all of this together, Eq. (5.15) becomes

$$S_{\text{ent}} = \tfrac{1}{2}S - \frac{\int_{-\infty}^{+\infty} du\, e^{-u^2/2K} \left|\tfrac{1}{2}(\bar{S}_2 - \bar{S}_1) - u\right|}{\int_{-\infty}^{+\infty} du\, e^{-u^2/2K}}. \tag{5.55}$$

Making a final rescaling of $u \to \sqrt{2K}y$, we get Eq. (5.16).

Note that, if we are interested in infinite temperature ($\beta = 0$), then we should also take $C_j \to 0$ so that β^2/C_j remains finite and nonzero. In this limit, $C \to 0$, and so the correction in Eq. (5.4) vanishes. Also note that the precise distribution of eigenvalues of ρ_1 near E_1 makes at most an $O(1)$ correction to the entanglement entropy. For example, the Marčenko-Pastur law [129] gives the Page correction $-e^{S_{\min}}/2e^{S_{\max}}$ to the entanglement entropy of a random state [33, 130]. This can be neglected.

We can also generalize to the case of a system with additional conserved quantities, such as particle number. In the most general case, there are m conserved quantities Q^a

($a = 1, \ldots, m$), including energy, which we take to be Q^1. A quantum state is then labeled by the values of all m quantities. We can then repeat our entire analysis (see Appendix 5.A for details). The thermodynamic entropy of the full system as a function of the Q^a's (near the values that label the state) takes the form

$$S(Q + \delta Q) = S(Q) + \lambda^a \delta Q^a - \tfrac{1}{2}(\mathbf{C}^{-1})_{ab}\, \lambda^a \delta Q^a \lambda^b \delta Q^b + \cdots , \tag{5.56}$$

with $\lambda^1 \equiv \beta$. This generalizes Eq. (5.28); similar generalizations apply to the subsystem entropies. We then ultimately arrive at Eq. (5.55) with $u \equiv \lambda^a \delta Q^a$ and

$$K \equiv \sum_{a,b=1}^{m} \left[(\mathbf{C}_1^{-1} + \mathbf{C}_2^{-1})^{-1}\right]_{ab} \tag{5.57}$$

$$= f(1-f) \sum_{a,b=1}^{m} \mathbf{C}_{ab}, \tag{5.58}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are the capacity matrices for the two subsystems, and the second equality holds for a uniform system with capacity matrix $\mathbf{C}$ and with $f = V_1/V$. Equation (5.16) then holds with K given by Eq. (5.57), and Eq. (5.4) holds with

$$C = \sum_{a,b=1}^{m} \mathbf{C}_{ab}. \tag{5.59}$$

We can now reproduce the results of Ref. [126] for a system with a conserved particle number. There the system was studied near infinite temperature, so that the thermodynamic entropy was taken to be effectively independent of system energy. Hence the problem reduces to the case of a single conserved quantity, the filling fraction n. Then the thermodynamic entropy of the system takes the form

$$S(n) = -L\,[n \ln n + (1-n)\ln(1-n)], \tag{5.60}$$

where L is the linear volume of the one-dimensional system; this is Eq. (13) in Ref. [126]. In the notation of our Eq. (5.56), with a single Q that we identify as n, we have

$$\lambda = S'(n), \tag{5.61}$$

$$\lambda^2 C^{-1} = -S''(n), \tag{5.62}$$

which yields

$$C = L\, n(1-n) \left[\ln\left(\frac{1-n}{n} \right) \right]^2 . \tag{5.63}$$

When used in Eq. (5.4), this reproduces Eq. (17) of Ref. [126] (with $L_A = L/2$).

5.6 Conclusions

We have reconsidered the ansatz of Refs. [32, 120] for an energy eigenstate of a chaotic many-body system that, by assumption, obeys the eigenstate thermalization hypothesis for local observables. This ansatz expresses the energy eigenstate of the full system in the basis of energy eigenstates of two subsystems, each contiguous in space, that interact along their mutual boundary, and is specified by Eqs. (5.3), (5.5), and (5.7).

One of the results of this chapter is that the width Δ of the energy window function $F(\varepsilon)$ is given by Eq. (5.6) in terms of the subsystem interaction hamiltonian, and that (in two or more spatial dimensions) $F(\varepsilon)$ has the gaussian form of Eq. (5.8).

We further showed that the ansatz for the energy eigenstate leads to a reduced density matrix that takes the form of Eq. (5.9). The off-diagonal elements, though exponentially small, are relevant to the calculation of Rényi entropies when the fraction of the energy in the smaller subsystem is large enough to give it a larger entropy than the larger subsystem; this modifies the results of Ref. [120] for $n < 1$.

In the case of equal or nearly equal volume for the two subsystems, there is a universal correction to the entanglement entropy (corresponding to Rényi index $n = 1$) that scales like the square-root of the system volume. In the case of equal subsystem volumes, this correction, displayed in Eq. (5.4), is $\Delta S_{\text{ent}} = -\sqrt{C/2\pi}$, where C is the heat capacity of the whole system. Such a correction was previously found in a specific system by Vidmar and Rigol [126]; our analysis is more general and shows that the effect is generic. We also extended our results to the case of multiple conserved quantities. The correction to the entanglement entropy at equal subsystem volumes is the same, but with C now given by a sum of the elements of a matrix of capacities.

We believe that this work further illuminates the role of the entanglement and Rényi entropies of a subsystem as quantities worthy of study that encode key features of the physical properties of the system as a whole.

Appendices

5.A Multiple conserved quantities

In the most general case, there are m conserved quantities Q^a ($a = 1, \ldots, m$), including energy, which we take to be Q^1. We assume that each Q^a is a sum of local terms, and so can be partitioned as in Eq. (5.1),

$$Q^a = Q_1^a + Q_2^a + Q_{12}^a, \tag{5.64}$$

with

$$\left[Q^a, Q^b\right] = 0, \tag{5.65a}$$

$$\left[Q_1^a, Q_1^b\right] = 0, \tag{5.65b}$$

$$\left[Q_2^a, Q_2^b\right] = 0. \tag{5.65c}$$

Let $|q\rangle$ denote a simultaneous eigenstate of the Q^a, with eigenvalues q^a. Without loss of generality, we shift each Q_{12}^a so that

$$\langle q|Q_{12}^a|q\rangle = 0. \tag{5.66}$$

We can write $|q\rangle$ in a basis of tensor products of the eigenstates of Q_1^a and Q_2^a,

$$|q\rangle = \sum_{i,J} M_{iJ} \, |i\rangle_1 \otimes |J\rangle_2 \,, \tag{5.67}$$

where $Q_1^a \, |i\rangle_1 = q_{1i}^a \, |i\rangle_1$ and $Q_2^a \, |J\rangle_2 = q_{2J}^a \, |J\rangle_2$. The thermodynamic entropy of the full system is now a function $S(q) \equiv S(q^1, \ldots, q^m)$ of all the q^a's. Its Taylor expansion, about the values that label the state, takes the form

$$S(q + \delta q) = S(q) + \lambda^a \delta q^a - \tfrac{1}{2} (\mathbf{C}^{-1})_{ab} \, \lambda^a \delta q^a \lambda^b \delta q^b + \cdots \,, \tag{5.68}$$

with $\lambda^1 \equiv \beta$. This generalizes Eq. (5.28). We assume that $\lambda^a > 0$ and that the capacity matrix $\mathbf{C}$ is positive definite; this generalizes positivity of temperature and heat capacity. Similar generalizations apply to the subsystem entropies.

We can then repeat our entire analysis. The coefficient matrix M_{iJ} takes the form

$$M_{iJ} = e^{-S(q_{1i}+q_{2J})/2} F(q_{1i} + q_{2J} - q)^{1/2} \, C_{iJ}. \tag{5.69}$$

Here $F(z)$ is a window function centered on $z_a = 0$ with second moments given by

$$\int d^m z \, F(z) \, z_a z_b = \mathbf{D}_{ab} \equiv \langle q | Q_{12}^a Q_{12}^b | q \rangle. \tag{5.70}$$

Equations (5.64) and (5.65) together imply

$$\langle q | \big[Q_{12}^a, \, Q_{12}^b \big] | q \rangle = 0, \tag{5.71}$$

so $\mathbf{D}_{ab}$ is symmetric, as it needs to be for the equality in Eq. (5.70) to make sense. The C_{iJ} coefficients obey Eq. (5.7), with the averaging now over narrow bands of all components

of q_1 and q_2. For a system in two or more spatial dimensions, the window function is a multivariate gaussian,

$$F(z) = \frac{e^{-z\cdot\mathbf{D}^{-1}z/2}}{(2\pi)^{m/2}\sqrt{\det\mathbf{D}}}. \tag{5.72}$$

The matrix elements of $\mathbf{D}$ scale like $\mathbf{D}_{ab} \sim A$, where A is the area of the boundary between regions 1 and 2. When $m = 1$ (i.e. only energy is conserved), $\mathbf{D}$ reduces to Δ^2.

The reduced density matrix $\rho_1 \equiv \mathrm{Tr}_2\,|q\rangle\langle q|$ of subsystem 1 takes the form

$$(\rho_1)_{ij} = e^{-S(q)+S_2(q-q_1)}\Big[\delta_{ij} + e^{-S_2(q-q_1)/2}e^{-w\cdot\mathbf{D}^{-1}w/8}R_{ij}\Big], \tag{5.73}$$

where $q_1^a \equiv (q_{1i}^a + q_{1j}^a)/2$ and $w^a \equiv q_{1i}^a - q_{1j}^a$, the R_{ij} are $O(1)$ numbers that vary erratically, and we have dropped terms of order $\mathbf{D}_{ab} \sim A$ in the exponents.

We adopt the notation

$$q_1 \prec q_1^* \quad \text{if} \quad S_1(q_1) < S_2(q-q_1), \tag{5.74a}$$

$$q_1 \succ q_1^* \quad \text{if} \quad S_1(q_1) > S_2(q-q_1), \tag{5.74b}$$

$$q_1 \sim q_1^* \quad \text{if} \quad S_1(q_1) = S_2(q-q_1). \tag{5.74c}$$

In other words, $\succ$ is the order on q_1 induced by the function $S_1(q_1) - S_2(q-q_1)$, and q_1^* is some point at which this function vanishes.

The off-diagonal term in Eq. (5.73), though exponentially smaller than the diagonal term, is relevant for $q_1 \succ q_1^*$. In the small box $[q_1^1, q_1^1+dq_1^1]\times\cdots\times[q_1^m, q_1^m+dq_1^m]$ for $q_1 \succ q_1^*$, ρ_1 has approximately $e^{S_2(q-q_1)}dq_1^1\cdots dq_1^m$ nonzero eigenvalues, each one approximately equal to $e^{-S(E)}e^{S_1(q_1)}$.

Equation (5.19) for the Rényi entropy generalizes to

$$S_{\mathrm{Ren},n}(q) = \frac{[S_1(\mathcal{Q}_1) + nS_2(q-\mathcal{Q}_1) - nS(q)]}{1-n}, \tag{5.75}$$

where $\mathcal{Q}_1$ is the point at which $S_1(q_1)+nS_2(q-q_1)$ attains its maximal value in the region $q_1 \precsim q_1^*$. For $n > 1$, the convexity of the entropy function guarantees that $\mathcal{Q}_1 \prec q_1^*$. In this case, $\mathcal{Q}_1 = \bar{q}_1$, the solution to

$$\nabla S_1(\bar{q}_1) = n\nabla S_2(q-\bar{q}_1). \tag{5.76}$$

However, for $n < 1$, it is possible to have $\mathcal{Q}_1 \sim q_1^* \prec \bar{q}_1$. In particular, for a uniform system split exactly in half, $q_1^* \prec \bar{q}_1$ for all $n < 1$, and then $S_{\mathrm{Ren},n<1}(q) = S(q)/2$, up to subleading corrections.

For the entanglement entropy, we repeat the steps in Sec. 5.5 and arrive at the generalization of Eq. (5.55),

$$S_{\mathrm{ent}} = \tfrac{1}{2}S - \frac{\int d^m v\, e^{-v\cdot\mathbf{K}^{-1}v/2}\left|\frac{1}{2}(\bar{S}_2-\bar{S}_1) - r\cdot v\right|}{\int d^m v\, e^{-v\cdot\mathbf{K}^{-1}v/2}}, \tag{5.77}$$

where $\mathbf{K}^{-1} \equiv \mathbf{C}_1^{-1} + \mathbf{C}_2^{-1}$, $\mathbf{C}_1$ and $\mathbf{C}_2$ are the capacity matrices for the two subsystems, $r \equiv (1,1,\ldots,1)$, $v^a \equiv \lambda^a(q_1^a - \bar{q}_1^a)$, $\bar{S}_1 \equiv S_1(\bar{q}_1)$, $\bar{S}_2 \equiv S_2(q-\bar{q}_1)$, and $\bar{q}_1$ is the solution to

$$\nabla S_1(\bar{q}_1) = \nabla S_2(q-\bar{q}_1). \tag{5.78}$$

The integral in Eq. (5.77) is of the form

$$\begin{aligned} I &= \int d^m v \; e^{-v\cdot \mathbf{K}^{-1} v/2} f(r\cdot v) \\ &= \int du \int d^m v \; e^{-v\cdot \mathbf{K}^{-1} v/2} \delta(u - r\cdot v)\, f(u) \\ &= \int du \int \frac{dk}{2\pi} \int d^m v \; e^{-v\cdot \mathbf{K}^{-1} v/2 + ik(u - r\cdot v)} f(u). \end{aligned} \tag{5.79}$$

Performing all the gaussian integrals,

$$I \propto \int du\, e^{-u^2/(2\, r\cdot \mathbf{K} r)} f(u). \tag{5.80}$$

Thus, Eq. (5.77) reduces to Eq. (5.55) with

$$K = r\cdot \mathbf{K}\, r = \sum_{a,b=1}^{m} \mathbf{K}_{ab}, \tag{5.81}$$

which is equivalent to Eq. (5.57).

Chapter 6

Future directions

The results presented in Chapter 2 on "almost perfect metals" suggest a few natural directions for follow-up work. The first is to try and observe absolutely q-stable Luttinger liquids in experiments. We saw that absolutely q-stable LL phases can occur in one-dimensional quantum wires with as few as 2 channels of strongly interacting fermions. Furthermore, we saw that these phases can occur even for purely repulsive interactions, and even when the unperturbed system has time-reversal or inversion symmetry (but not both). Thus, they can in principle be realized in as simple a system as a single-spinful-channel quantum wire with strong spin-orbit coupling. One could also try to realize them in other experimental platforms that are inherently more tunable, such as gate-defined wires in a two-dimensional electron gas (2DEG), or quantum gas ("cold atom") systems.

To actually realize absolutely q-stable phases, it will be helpful to understand which *microscopic* models (and in particular, which patterns of microscopic interactions) lead to the patterns of *effective* low-energy interactions that we identified in Chapter 2 as required for absolutely q-stability. This is a nontrivial task, and will likely require a combination of physical intuition and extensive numerical work. It will also be important to identify and calculate physically measurable properties, either thermodynamic or transport, that

can serve as experimental signatures of absolutely q-stability (or which can be used to infer the pattern of effective interactions in the experimental system directly, thereby pointing one in the right direction).

On the theoretical side, one could explore whether the geometric ideas described in Chapter 2 also form a useful paradigm for thinking about higher-dimensional non-Fermi liquids, and whether our q-stability results have interesting consequences for classical systems [1] of experimental interest. There are also some lingering mathematical questions that would be nice to resolve, in particular: "what is the minimum number of channels N required to obtain an absolutely ∞-stable (or just ∞-stable) phase?" Abstracting away the physics interpretation, the question is: "what is the smallest positive integer N for which there exists $A \in SO(N,N)$ such that $\|A\mathbf{m}\|^2 > 4$ for all $\mathbf{m} \in \mathbb{Z}^{2N}$ (or just for all $\mathbf{m} \in \mathbb{Z}^{2N}$ that satisfy $\mathbf{m}^T K \mathbf{m} = 0$, where $K \equiv -\mathbb{I}_N \oplus \mathbb{I}_N$) [2] ?" Call the solutions to this question N_1 (N_2).

So far, we only have very loose bounds: $12 \leq N_1 \leq 52$ and $2 \leq N_2 \leq 23$ (as discussed in Chapter 2). For N_1, the problem is to find a lattice in $\mathbb{R}^{2N_1}$ whose shortest vector has length > 2 and for which there exists a generator matrix (that is, a matrix whose columns constitute a basis for the lattice) belonging to $SO(N_1, N_1)$. Motivated by this problem, I have constructed $SO(N,N)$ generator matrices for a few famous lattices ($A_2, D_4, E_8, \Lambda_{24}$) that are known to be optimal [3] in their respective dimensions ($2N = 2, 4, 8, 24$); see Appendix 6.A. It is possible that a similar construction might also yield good lattices in dimension $2N > 24$; this remains to be seen. From a mathematical point of view, the 12-

[1] (via the usual mapping between a quantum system in d dimensions and a classical system in $d+1$ dimensions)

[2] This would ensure stability with respect to *both bosonic and fermionic* perturbations (the latter are physically meaningful if we consider an array of coupled wires). If one is only interested in bosonic perturbations, one can replace $\mathbb{Z}^{2N}$ by $D_{2N} \equiv \{\mathbf{m} \in \mathbb{Z}^{2N} : |\mathbf{m}| \in 2\mathbb{Z}\}$, the "checkerboard lattice".

[3] Given a normalized lattice L of dimension n (normalized means that the unit cell of L has volume 1), define $\mu(L)$ to be the length-squared of the shortest nonzero lattice vector in L. The lattice L is said to be *optimal* if $\mu(L) \geq \mu(L')$ for any other lattice L' of dimension n; we then write $\mu(L) \equiv \mu_n$, which defines the *Hermite constant* μ_n.

channel Luttinger liquid associated to the Leech lattice Λ_{24} (via the above construction) is quite fascinating. At tree level, it has no RG relevant operators, but 196584 marginal ones (one marginal vertex operator for each minimal vector of the Leech lattice, along with the 24 currents $\partial_x \phi_I$)! The one-loop analysis is therefore formidable. However, given the remarkable and mysterious properties of the Leech lattice, and its central role (along with E_8) in the mathematical theory of lattices [58], we can speculate that aspects of the one-loop analysis may have number-theoretic significance; it would be interesting to explore whether such connections indeed exist. For N_2, the problem is further complicated by the fact that the set of lattice vectors satisfying $\mathbf{m}^T K \mathbf{m} = 0$ do not in general form a sublattice. Finally, I note (as mentioned in Chapter 2 in connection with the distinction between stability and absolute stability) that the effect of relevant spin non-zero operators on the IR physics of a Luttinger liquid is still unclear [131, 132]: everyone agrees that they cannot directly open a gap, but a generally accepted picture of how they renormalize the scaling dimensions of spin-zero operators—that *can* open a gap—is lacking. It would be useful to develop such a picture.

The results of Chapter 3 on relaxation to generalized Gibbs ensembles in systems with quadratic hamiltonians also suggest some natural directions for follow-up work. Our analysis there led to several quantitative predictions. In particular, we argued that connected correlation functions of local operators will generically decay to their GGE values according to *model-independent* power laws, and we predicted the exponents of these power laws. These predictions appear to be testable in quantum gas experiments with existing technology. By utilizing Feshbach resonances, it is possible to prepare an ultracold gas of particles in a strongly interacting initial state, and then to quench the interactions to zero and study the subsequent evolution [133, 64]. In addition, the measurement of higher-order correlation functions (up to tenth order!) has already been

demonstrated experimentally in such setups [134]. Thus, we hope that our predictions will be tested experimentally in the near future.

On the theoretical side, the results of Chapter 3 (or their weaker but rigorous analogues, cf. Refs. [72, 80]) may serve as a starting point to develop a satisfactory theory of relaxation to thermal equilibrium in generic weakly interacting (nonintegrable) systems. Here, the standard description of non-equilibrium dynamics involves some sort of quantum Boltzmann equation, with a collision integral accounting for the interactions [135]. Such a description has many desirable properties, most notably physical transparency. However, its mathematical derivation from the underlying *microscopic* dynamics (i.e. the Heisenberg evolution of operators) involves several uncontrolled approximations associated with truncation of the BBGKY hierarchy of evolution equations for connected correlation functions (or corresponding uncontrolled approximations in the Keldysh technique [136]). It is possible that our results on relaxation of correlation functions in the *absence* of interactions can be used to bound the errors introduced by these approximations, and hence to theoretically justify the quantum Boltzmann equation in a wide range of situations.

Finally, there are some—rather more vague and speculative—directions suggested by the results of Chapters 4 and 5. In the former, we established a relation between the "bound on chaos" $\lambda \leq 2\pi/\beta$ and the Eigenstate Thermalization Hypothesis. One might hope that other similar, conjectured, bounds on transport coefficients can also be related to the ETH. In the latter, we characterized the bipartite entanglement structure of finite-energy-density eigenstates in chaotic systems. One might hope to similarly characterize tripartite entanglement, etc. There are two different ways to obtain a given tripartition of a system via successive bipartitions, but both must lead to the same final decomposition of the state in the tripartite tensor product basis. The resulting consistency conditions

may have interesting implications for the entanglement structure of eigenstates; this is worth exploring.

Appendices

6.A Optimal lattices with Gram matrices belonging to $SO(N,N)$

Here, I write down Gram matrices $M \in SO(N,N)$ for the following normalized lattices, known to be optimal in dimensions $2N = 2, 4, 8$ and 24 respectively: A_2 (the hexagonal lattice; $\mu_2 = 2/\sqrt{3}$), D_4 (normalized even sublattice of Z^4; $\mu_4 = \sqrt{2}$), E_8 ($\mu_8 = 2$), and Λ_{24} (Leech lattice; $\mu_{24} = 4$). In all cases, the Gram matrix has the form

$$M = \begin{bmatrix} \mu_{2N}\,\mathbb{I}_N & \sqrt{\mu_{2N}^2 - 1}\,R \\ \sqrt{\mu_{2N}^2 - 1}\,R^T & \mu_{2N}\,\mathbb{I}_N \end{bmatrix}, \tag{6.1}$$

where $R \in O(N)$. An associated generator matrix $A \in SO(N,N)$ is easily obtained as $A = M^{1/2}$ (the unique positive definite square root of M).

For A_2, simply take $R_{A_2} = \pm 1$.

For D_4, take

$$R_{D_4} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{6.2}$$

For E_8, take

$$R_{E_8} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{6.3}$$

For the Leech lattice, the off-diagonal block $\sqrt{15}\ R_{\Lambda_{24}}$ is constructed by finding 12 mutually orthogonal vectors of the form $(\pm 1^{11}, \pm 2)$, and stacking them into a matrix. Explicitly,

$$R_{\Lambda_{24}} = \frac{1}{\sqrt{15}} \begin{bmatrix} -1 & 1 & 1 & -1 & -1 & -1 & -2 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 2 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -2 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 2 & -1 & -1 \\ 1 & -1 & 1 & -1 & -2 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -2 \\ 1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -2 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -2 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 2 & -1 & -1 & 1 \\ -2 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}. \tag{6.4}$$

Bibliography

[1] P. W. Anderson, *More is different*, *Science* **177** (1972), no. 4047 393–396.

[2] P. W. Anderson, *Basic notions of condensed matter physics*. Benjamin-Cummings, 1984.

[3] X.-G. Wen, *Quantum field theory of many-body systems: from the origin of sound to an origin of light and electrons*. Oxford University Press on Demand, 2004.

[4] D. Pines, *Elementary excitations in solids*. Perseus, 1966.

[5] P. Nozières, *Theory of interacting Fermi systems*. Addison-Wesley, 1964.

[6] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of quantum field theory in statistical physics*. Dover, 1975.

[7] N. W. Ashcroft and N. D. Mermin, *Solid state physics*. Harcourt, 1976.

[8] P. G. De Gennes, *Superconductivity of Metals and Alloys*. Westview, 1966.

[9] R. Shankar, *Renormalization-group approach to interacting fermions*, *Reviews of Modern Physics* **66** (1994), no. 1 129.

[10] E. M. Lifshitz and L. P. Pitaevskii, *Statistical physics: theory of the condensed state*, vol. 9. Elsevier, 2013.

[11] J. Cardy, *Scaling and renormalization in statistical physics*, vol. 5. Cambridge university press, 1996.

[12] K. G. Wilson and J. Kogut, *The renormalization group and the ϵ expansion*, *Physics reports* **12** (1974), no. 2 75–199.

[13] A. Raju, C. B. Clement, L. X. Hayden, J. P. Kent-Dobias, D. B. Liarte, D. Z. Rocklin, and J. P. Sethna, *Normal form for renormalization groups*, *Physical Review X* **9** (2019), no. 2 021014.

[14] J. Polchinski, *Effective field theory and the fermi surface*, *arXiv preprint hep-th/9210046* (1992).

[15] T. Giamarchi, *Quantum physics in one dimension*. Clarendon Press, Oxford, UK, 2003.

[16] F. D. M. Haldane, *'Luttinger liquid theory' of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas*, *J. Phys. C* **14** (1981), no. 19 2585–2609.

[17] E. Plamadeala, M. Mulligan, and C. Nayak, *Perfect metal phases of one-dimensional and anisotropic higher-dimensional systems*, *Phys. Rev. B* **90** (2014), no. 24 1–5, [arXiv:1404.4367].

[18] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43** (1991), no. 4 2046–2049.

[19] M. Srednicki, *Chaos and quantum thermalization*, *Phys. Rev. E* **50** (1994), no. 2 888–901, [cond-mat/9403051].

[20] M. Srednicki, *The approach to thermal equilibrium in quantized chaotic systems*, *J. Phys. A* **32** (Jan, 1999) 1163–1175, [cond-mat/9809360].

[21] M. Rigol, V. Dunjko, and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, *Nature* **452** (04, 2008) 854, [arXiv:0708.1324].

[22] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, *Advances in Physics* **65** (may, 2016) 239–362, [arXiv:1509.0641].

[23] T. Kinoshita, T. Wenger, and D. S. Weiss, *A quantum newton's cradle*, *Nature* **440** (2006), no. 7086 900–903.

[24] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, *Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons*, *Physical Review Letters* **98** (feb, 2007) 050405, [cond-mat/0604476].

[25] E. Ott, *Chaos in dynamical systems*. Cambridge university press, 2002.

[26] B. Swingle, *Unscrambling the physics of out-of-time-order correlators*, *Nat. Phys.* **14** (2018), no. 10 988–990.

[27] A. Kitaev, "A simple model of quantum holography." Talks at KITP, April 7, 2015 and May 27, 2015., 2015.

[28] J. Maldacena, S. H. Shenker, and D. Stanford, *A bound on chaos*, *J. High Energy Phys.* **2016** (2016), no. 8 106, [arXiv:1503.0140].

[29] S. H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *J. High Energy Phys.* **2014** (2014), no. 3 67, [arXiv:1306.0622].

[30] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum entanglement*, *Reviews of modern physics* **81** (2009), no. 2 865.

[31] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Entanglement in many-body systems*, *Reviews of Modern Physics* **80** (2008), no. 2 517–576, [quant-ph/0703044].

[32] J. M. Deutsch, *Thermodynamic entropy of a many-body energy eigenstate*, *New Journal of Physics* **12** (2010), no. 7 075021, [arXiv:0911.0056].

[33] D. N. Page, *Average entropy of a subsystem*, *Physical Review Letters* **71** (1993), no. 9 1291–1294, [gr-qc/9305007].

[34] C. Murthy and C. Nayak, *Almost Perfect Metals in One Dimension*, *Phys. Rev. Lett.* **124** (2020), no. 13 136801.

[35] C. Murthy and M. Srednicki, *Bounds on Chaos from the Eigenstate Thermalization Hypothesis*, *Phys. Rev. Lett.* **123** (2019), no. 23 230606.

[36] C. Murthy and M. Srednicki, *Relaxation to Gaussian and generalized Gibbs states in systems of particles with quadratic Hamiltonians*, *Phys. Rev. E* **100** (2019), no. 1 012146.

[37] C. Murthy and M. Srednicki, *Structure of chaotic eigenstates and their entanglement entropy*, *Phys. Rev. E* **100** (2019), no. 2 022131.

[38] P. A. Lee and T. V. Ramakrishnan, *Disordered electronic systems*, *Rev. Mod. Phys.* **57** (1985), no. 2 287–337.

[39] S. A. Kivelson, E. Fradkin, and V. J. Emery, *Electronic liquid-crystal phases of a doped Mott insulator*, *Nature* **393** (1998), no. 6685 550–553, [cond-mat/9707327].

[40] L. Golubović and M. Golubović, *Fluctuations of quasi-two-dimensional smectics intercalated between membranes in multilamellar phases of DNA-cationic lipid complexes*, *Phys. Rev. Lett.* **80** (1998), no. 19 4341–4344.

[41] C. S. O'Hern and T. C. Lubensky, *Sliding columnar phase of DNA-lipid complexes*, *Phys. Rev. Lett.* **80** (1998), no. 19 4345–4348, [cond-mat/9712049].

[42] C. S. O'Hern, T. C. Lubensky, and J. Toner, *Sliding phases in XY models, crystals, and cationic lipid-DNA complexes*, *Phys. Rev. Lett.* **83** (1999), no. 14 2745–2748, [cond-mat/9904415].

[43] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, *Quantum theory of the smectic metal state in stripe phases*, *Phys. Rev. Lett.* **85** (2000), no. 10 2160–2163, [cond-mat/0001077].

[44] A. Vishwanath and D. Carpentier, *Two-dimensional anisotropic non-Fermi-liquid phase of coupled Luttinger liquids*, *Phys. Rev. Lett.* **86** (2001), no. 4 676–679, [cond-mat/0003036].

[45] R. Mukhopadhyay, C. L. Kane, and T. C. Lubensky, *Sliding Luttinger liquid phases*, *Phys. Rev. B* **64** (2001), no. 4 045120, [cond-mat/0102163].

[46] G. Sierra and E. H. Kim, *Renormalization group study of the sliding Luttinger liquids*, *J. Phys. A: Math. Gen.* **36** (2002), no. 2 L37–L44, [cond-mat/0103356].

[47] B. Dóra and R. Moessner, *Luttinger liquid with complex forward scattering: Robustness and Berry phase*, *Phys. Rev. B* **7** (2016), no. 93 075127, [arXiv:1510.0151].

[48] J. von Delft and H. Schoeller, *Bosonization for beginners — Refermionization for experts*, *Annalen Phys.* **7** (1998), no. 4 224–305, [cond-mat/9805275].

[49] J. E. Moore and X.-G. Wen, *Classification of disordered phases of quantum Hall edge states*, *Phys. Rev. B* **57** (1998), no. 16 10138–10156, [cond-mat/9710208].

[50] C. Xu and J. E. Moore, *Stability of the quantum spin Hall effect: Effects of interactions, disorder, and Z2 topology*, *Phys. Rev. B* **73** (2006), no. 4 045322, [cond-mat/0508291].

[51] H. Cohn and N. Elkies, *New upper bounds on sphere packings I*, *Ann. Math.* **157** (2003), no. 2 689–714, [math/0110009].

[52] N. J. Higham, *J-orthogonal matrices: Properties and generation*, *SIAM Rev.* **45** (2003), no. 3 504–519.

[53] A. V. Moroz, K. V. Samokhin, and C. H. W. Barnes, *Spin-orbit coupling in interacting quasi-one-dimensional electron systems*, *Phys. Rev. Lett.* **84** (2000), no. 18 4164–4167, [cond-mat/0003219].

[54] A. Imambekov, T. L. Schmidt, and L. I. Glazman, *One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm*, *Reviews of Modern Physics* **84** (sep, 2012) 1253–1306, [arXiv:1110.1374].

[55] E. M. Wright, *An enumerative proof of an identity of Jacobi*, *Journal of the London Mathematical Society* **1** (1965), no. 1 55–57.

[56] J. J. Sylvester, *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares*, *Philos. Mag.* **4** (1852), no. 23 138–142.

[57] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, Cambridge, UK, 2004.

[58] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices, and groups*. Springer, New York, 1999.

[59] J. Milnor and D. Husemoller, *Symmetric bilinear forms*. Springer, Berlin, 1973.

[60] E. M. Rains and N. J. A. Sloane, *The shadow theory of modular and unimodular lattices*, *J. Number Theory* **73** (1998), no. 2 359–389, [math/0207294].

[61] E. Plamadeala, M. Mulligan, and C. Nayak, *Transport in a one-dimensional hyperconductor*, *Phys. Rev. B* **93** (2016), no. 12 1–18, [arXiv:1509.0428].

[62] P. Gaborit, *Construction of new extremal unimodular lattices*, *Eur. J. Comb.* **25** (2004), no. 4 549–564.

[63] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, *Reviews of Modern Physics* **83** (aug, 2011) 863–883, [arXiv:1007.5331].

[64] T. Langen, R. Geiger, and J. Schmiedmayer, *Ultracold Atoms Out of Equilibrium*, *Annual Review of Condensed Matter Physics* **6** (mar, 2015) 201–217, [arXiv:1408.6377].

[65] T. Langen, T. Gasenzer, and J. Schmiedmayer, *Prethermalization and universal dynamics in near-integrable quantum systems*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (jun, 2016) 064009, [arXiv:1603.0938].

[66] F. H. L. Essler and M. Fagotti, *Quench dynamics and relaxation in isolated integrable quantum spin chains*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (jun, 2016) 064002, [arXiv:1603.0645].

[67] L. Vidmar and M. Rigol, *Generalized Gibbs ensemble in integrable lattice models*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (jun, 2016) 064007, [arXiv:1604.0399].

[68] E. T. Jaynes, *Information theory and statistical mechanics*, *Physical Review* **106** (may, 1957) 620–630.

[69] T. M. Wright, M. Rigol, M. J. Davis, and K. V. Kheruntsyan, *Nonequilibrium Dynamics of One-Dimensional Hard-Core Anyons Following a Quench: Complete Relaxation of One-Body Observables*, *Physical Review Letters* **113** (jul, 2014) 050601, [arXiv:1312.4657].

[70] L. M. Sieberer, M. Buchhold, and S. Diehl, *Keldysh field theory for driven open quantum systems*, *Reports on Progress in Physics* **79** (sep, 2016) 096001, [arXiv:1512.0063].

[71] A. Eckardt, *Colloquium: Atomic quantum gases in periodically driven optical lattices*, *Reviews of Modern Physics* **89** (mar, 2017) 011004, [arXiv:1606.0804].

[72] M. Gluza, C. Krumnow, M. Friesdorf, C. Gogolin, and J. Eisert, *Equilibration via Gaussification in Fermionic Lattice Systems*, *Physical Review Letters* **117** (nov, 2016) 190602, [arXiv:1601.0067].

[73] M. Cramer and J. Eisert, *A quantum central limit theorem for non-equilibrium systems: exact local relaxation of correlated states*, *New Journal of Physics* **12** (may, 2010) 055020, [arXiv:0911.2475].

[74] S. Sotiriadis and P. Calabrese, *Validity of the GGE for quantum quenches from interacting to noninteracting models*, *Journal of Statistical Mechanics: Theory and Experiment* **2014** (2014), no. 7 16, [arXiv:1403.7431].

[75] T. Barthel and U. Schollwöck, *Dephasing and the Steady State in Quantum Many-Particle Systems*, *Physical Review Letters* **100** (mar, 2008) 100601, [arXiv:0711.4896].

[76] M. A. Cazalilla, A. Iucci, and M.-C. Chung, *Thermalization and quantum correlations in exactly solvable models*, *Physical Review E* **85** (jan, 2012) 011133, [arXiv:1106.5206].

[77] B. Nachtergaele and R. Sims, *Lieb-Robinson Bounds and the Exponential Clustering Theorem*, *Communications in Mathematical Physics* **265** (jul, 2006) 119–130, [math-ph/0506030].

[78] M. B. Hastings and T. Koma, *Spectral Gap and Exponential Decay of Correlations*, *Communications in Mathematical Physics* **265** (aug, 2006) 781–804, [math-ph/0507008].

[79] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, *Locality of Temperature*, *Physical Review X* **4** (jul, 2014) 031019, [arXiv:1309.0816].

[80] M. Gluza, J. Eisert, and T. Farrelly, *Equilibration towards generalized Gibbs ensembles in non-interacting theories*, *SciPost Physics* **7** (2019) [arXiv:1809.0826].

[81] R. Moessner and S. L. Sondhi, *Equilibration and order in quantum Floquet matter*, *Nature Physics* **13** (apr, 2017) 424–428, [arXiv:1701.0805].

[82] A. Lazarides, A. Das, and R. Moessner, *Periodic Thermodynamics of Isolated Quantum Systems*, *Physical Review Letters* **112** (apr, 2014) 150401, [arXiv:1401.0164].

[83] S. Ziraldo, A. Silva, and G. E. Santoro, *Relaxation Dynamics of Disordered Spin Chains: Localization and the Existence of a Stationary State*, *Physical Review Letters* **109** (dec, 2012) 247205, [arXiv:1206.4787].

[84] S. Ziraldo and G. E. Santoro, *Relaxation and thermalization after a quantum quench: Why localization is important*, *Physical Review B* **87** (feb, 2013) 064201, [arXiv:1211.4465].

[85] S. Weinberg, *The quantum theory of fields*, vol. 2. Cambridge university press, 1995.

[86] M. Fagotti and F. H. L. Essler, *Reduced density matrix after a quantum quench*, *Physical Review B* **87** (jun, 2013) 245107, [arXiv:1302.6944].

[87] M. Fagotti, *On conservation laws, relaxation and pre-relaxation after a quantum quench*, *Journal of Statistical Mechanics: Theory and Experiment* **2014** (mar, 2014) P03016, [arXiv:1401.1064].

[88] E. H. Lieb and D. W. Robinson, *The finite group velocity of quantum spin systems*, *Comm. Math. Phys.* **28** (1972) 251–257.

[89] F. H. L. Essler, S. Evangelisti, and M. Fagotti, *Dynamical correlations after a quantum quench*, *Physical Review Letters* **109** (2012), no. 24 247206, [arXiv:1208.1961].

[90] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, 1993.

[91] L. Mandel and E. Wolf, *Optical coherence and quantum optics*. Cambridge University Press, 1995.

[92] Y.-H. Jhu, P. Chen, and M.-C. Chung, *Relaxation of the entanglement spectrum in quench dynamics of topological systems*, *Journal of Statistical Mechanics: Theory and Experiment* **2017** (jul, 2017) 073105, [arXiv:1610.0085].

[93] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium*, *Physical Review X* **6** (dec, 2016) 041065, [arXiv:1605.0733].

[94] M. B. Hastings, *Decay of Correlations in Fermi Systems at Nonzero Temperature*, *Physical Review Letters* **93** (sep, 2004) 126402, [cond-mat/0406348].

[95] P. W. Anderson, *Absence of Diffusion in Certain Random Lattices*, *Physical Review* **109** (mar, 1958) 1492–1505.

[96] R. P. Feynman, A. R. Hibbs, and D. F. Styer, *Quantum mechanics and path integrals*. Courier Corporation, 2010.

[97] S. Sotiriadis, *Memory-preserving equilibration after a quantum quench in a one-dimensional critical model*, *Physical Review A* **94** (sep, 2016) 031605(R), [arXiv:1507.0791].

[98] S. Ngo Dinh, D. A. Bagrets, and A. D. Mirlin, *Interaction quench in nonequilibrium Luttinger liquids*, *Physical Review B* **88** (dec, 2013) 245405, [1310.0735].

[99] L. Seabra, F. H. L. Essler, F. Pollmann, I. Schneider, and T. Veness, *Real-time dynamics in the one-dimensional Hubbard model*, *Physical Review B* **90** (dec, 2014) 245127, [arXiv:1410.1670].

[100] M. A. Cazalilla and M.-C. Chung, *Quantum quenches in the Luttinger model and its close relatives*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (jun, 2016) 064004, [arXiv:1603.0425].

[101] S. Sotiriadis, *Equilibration in one-dimensional quantum hydrodynamic systems*, *Journal of Physics A: Mathematical and Theoretical* **50** (oct, 2017) 424004, [arXiv:1612.0037].

[102] L. Hackl, E. Bianchi, R. Modak, and M. Rigol, *Entanglement production in bosonic systems: Linear and logarithmic growth*, *Physical Review A* **97** (mar, 2018) 032321, [arXiv:1710.0427].

[103] J. L. van Hemmen, *A note on the diagonalization of quadratic boson and fermion hamiltonians*, *Zeitschrift für Physik B Condensed Matter* **38** (sep, 1980) 271–277.

[104] M. Bukov, L. D'Alessio, and A. Polkovnikov, *Universal high-frequency behavior of periodically driven systems: from dynamical stabilization to Floquet engineering*, *Advances in Physics* **64** (mar, 2015) 139–226, [arXiv:1407.4803].

[105] A. Russomanno, G. E. Santoro, and R. Fazio, *Entanglement entropy in a periodically driven Ising chain*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (jul, 2016) 073101, [arXiv:1603.0366].

[106] A. I. Larkin and Y. N. Ovchinnikov, *Quasiclassical method in the theory of superconductivity*, *JETP* **28** (1969), no. 6 1200–1205.

[107] S. Sachdev and J. Ye, *Gapless spin fluid ground state in a random, quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339, [cond-mat/9212030].

[108] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, *Lyapunov exponent and out-of-time-ordered correlator's growth rate in a chaotic system*, *Phys. Rev. Lett.* **118** (Feb, 2017) 086801, [arXiv:1609.0170].

[109] I. García-Mata, M. Saraceno, R. A. Jalabert, A. J. Roncaglia, and D. A. Wisniacki, *Chaos signatures in the short and long time behavior of the out-of-time ordered correlator*, *Phys. Rev. Lett.* **121** (2018), no. 21 210601, [arXiv:1806.0428].

[110] R. Nandkishore and D. A. Huse, *Many-Body Localization and Thermalization in Quantum Statistical Mechanics*, *Annu. Rev. Condens. Matter Phys.* **6** (2015), no. 1 15–38, [arXiv:1404.0686].

[111] J. Maldacena, D. Stanford, and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *Prog. Theor. Exp. Phys.* **2016** (2016), no. 12 12C104, [arXiv:1606.0185].

[112] L. Foini and J. Kurchan, *Eigenstate thermalization hypothesis and out of time order correlators*, *Phys. Rev. E* **99** (Apr, 2019) 042139, [arXiv:1803.1065].

[113] M. V. Berry, *Regular and irregular semiclassical wavefunctions*, *J. Phys. A* **10** (Dec, 1977) 2083–2091.

[114] S. Hortikar and M. Srednicki, *Correlations in Chaotic Eigenfunctions at Large Separation*, *Phys. Rev. Lett.* **80** (Feb, 1998) 1646–1649, [chao-dyn/9710025].

[115] S. Hortikar and M. Srednicki, *Random matrix elements and eigenfunctions in chaotic systems*, *Phys. Rev. E* **57** (Jun, 1998) 7313–7316, [chao-dyn/9711020].

[116] I. V. Gornyi and A. D. Mirlin, *Wave function correlations on the ballistic scale: Exploring quantum chaos by quantum disorder*, *Phys. Rev. E* **65** (Jan, 2002) 025202, [cond-mat/0105103].

[117] D. A. Roberts and D. Stanford, *Diagnosing chaos using four-point functions in two-dimensional conformal field theory*, *Phys. Rev. Lett.* **115** (2015), no. 13 131603, [arXiv:1412.5123].

[118] M. Reed and B. Simon, *II: Fourier Analysis, Self-Adjointness*. Methods of Modern Mathematical Physics. Academic Press, 1975.

[119] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, *A Universal Operator Growth Hypothesis*, *Phys. Rev. X* **9** (2019), no. 4 041017, [arXiv:1812.0865].

[120] T. C. Lu and T. Grover, *Renyi entropy of chaotic eigenstates*, *Physical Review E* **99** (2019), no. 3 032111, [arXiv:1709.0878].

[121] L. F. Santos, A. Polkovnikov, and M. Rigol, *Weak and strong typicality in quantum systems*, *Physical Review E* **86** (2012), no. 1 010102, [arXiv:1202.4764].

[122] J. M. Deutsch, H. Li, and A. Sharma, *Microscopic origin of thermodynamic entropy in isolated systems*, *Physical Review E* **87** (2013), no. 4 042135, [arXiv:1202.2403].

[123] W. Beugeling, A. Andreanov, and M. Haque, *Global characteristics of all eigenstates of local many-body Hamiltonians: participation ratio and entanglement entropy*, *Journal of Statistical Mechanics: Theory and Experiment* **2015** (2015), no. 2 P02002, [arXiv:1410.7702].

[124] Y. O. Nakagawa, M. Watanabe, H. Fujita, and S. Sugiura, *Universality in volume-law entanglement of scrambled pure quantum states*, *Nature Communications* **9** (2018), no. 1 1635, [arXiv:1703.0299].

[125] H. Fujita, Y. O. Nakagawa, S. Sugiura, and M. Watanabe, *Page Curves for General Interacting Systems*, *JHEP* **2018** (2018), no. 12 112, [arXiv:1805.1161].

[126] L. Vidmar and M. Rigol, *Entanglement Entropy of Eigenstates of Quantum Chaotic Hamiltonians*, *Physical Review Letters* **119** (2017), no. 22 220603, [arXiv:1708.0845].

[127] A. Dymarsky, N. Lashkari, and H. Liu, *Subsystem eigenstate thermalization hypothesis*, *Physical Review E* **97** (2018), no. 1 012140, [arXiv:1611.0876].

[128] J. Eisert, M. Cramer, and M. B. Plenio, *Colloquium: Area laws for the entanglement entropy*, *Reviews of Modern Physics* **82** (2010), no. 1 277–306, [arXiv:0808.3773].

[129] V. A. Marčenko and L. A. Pastur, *Distribution of eigenvalues for some sets of random matrices*, *Mathematics of the USSR-Sbornik* **1** (1967), no. 4 457–483.

[130] E. Bianchi and P. Donà, *Typical entanglement entropy in the presence of a center: Page curve and its variance*, *Physical Review D* **100** (2019), no. 10 105010.

[131] C. L. Kane, M. P. A. Fisher, and J. Polchinski, *Randomness at the edge: Theory of quantum Hall transport at filling ν= 2/3*, *Phys. Rev. Lett.* **72** (1994), no. 26 4129, [cond-mat/9402108].

[132] F. D. M. Haldane, *Stability of chiral luttinger liquids and abelian quantum hall states*, *Phys. Rev. Lett.* **74** (1995), no. 11 2090–2093, [cond-mat/9501007].

[133] I. Bloch, J. Dalibard, and W. Zwerger, *Many-body physics with ultracold gases*, *Reviews of Modern Physics* **80** (2008), no. 3 885.

[134] T. Schweigler, V. Kasper, S. Erne, I. Mazets, B. Rauer, F. Cataldini, T. Langen, T. Gasenzer, J. Berges, and J. Schmiedmayer, *Experimental characterization of a quantum many-body system via higher-order correlations*, *Nature* **545** (2017), no. 7654 323–326.

[135] L. P. Kadanoff and G. A. Baym, *Quantum Statistical Mechanics: Green's Function Methods in Equilibrium and Nonequilibrium Problems*. Benjamin, 1962.

[136] A. Kamenev, *Field theory of non-equilibrium systems*. Cambridge University Press, 2011.