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MICROSTRUCTURAL MECHANICS OF GRANULAR MEDIA

BY

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Introduction

During the last decade a number of theoretical works appeared in which the problem of developing the new models in mechanics of deformable solids was addressed. These models differ from a conventional classical approach prevalent in the field for a very long time. These new developments were dictated in the first place by the well-known limitations of the classical models of the solids going back to Cauchy and G. Green. According to the latter any continuum representing a solid body consists of an infinite number of material points not separated by any gaps. Since the point mass is an infinitesimal (and we should say more a mathematical than physical) object it has only three degrees of freedom, namely translation along three axes. As a result the deformations in such a body are entirely determined by the 3-D translational vector, $\vec{u}$.

Such an approach fails to take into account the fact that any real body consists of the material particles (physical entities, and not mathematical points!) which have small but finite sizes. These particles are either molecules or larger aggregates, e.g. the bulk solid granules. What is considered as such particles, depends on the solid's structure, goals of a specific study and other facts. But in any case, the particles in question have more than three degrees of freedom (cf. the above mentioned translation vector $\vec{u}$ for point masses). Therefore there may be a need to introduce at least one more three-dimensional vector $\vec{\phi}$ representing the rotations of a particle viewed as a small solid body.

Generally speaking one can introduce a variety of different quantities describing the additional degrees of freedom for an arbitrary particle and taking into account its possible deformations. Such an approach is characteristic for the microstructural theories of mechanics of solids.

The above considerations make clear why it is both timely and important to broaden the development of various microstructural theories, especially for the bodies with strongly pronounced discrete-continuous structure (e.g., bulk solids, or granular media) since their particles can move both translationally and rotationally.

In this paper a new version of a microstructural theory for a granular medium is presented. In contradistinction to the existing theories we develop a new and rather simple approach based upon the ideas borrowed from the solid state physics. Our approach stems from a discrete nature of a granular medium and the contact character of interaction of deformable particles comprising the medium. As a result the structure of the medium, the size of its particles, and two displacement vectors $\vec{u}$ and $\vec{\phi}$ are taken into account.

The proposed theory is characterized by the following features:

1) its assumptions have simple geometrical and obvious physical meaning,

2) it allows one to drastically simplify one of the most difficult problems in the solid mechanics, that is the problem of the constitutive equations, including the non-linear cases,

3) prediction of the stressed states based on particles' sizes (the scaling effect). In particular, this effect is essential for the dynamic situations with perturbations whose length is comparable to the typical particle size,

4) it represents a natural generalization of the well-known conventional theories of elasticity developed by Cauchy (classical model) and Cosserat brothers (model of coupled stresses). These models follow as particular cases from our theory.

1. Microstructure

We consider a granular body as a spatial set H comprised of a large number N of the elastic spherical particles of a small diameter d. The particles are either in contact with each other or separated by thin adhesive layers.

To simplify the problem without loosing its main features we assume that H_o (where "o" denotes the initial state of the set) is a regular set. The centers of particles (which we call the nodes) form the Bravais lattice, Γ.

A couple of the adjacent particles A, $B \in H$ represents a doublet (A, B) with the directed axis described by the vector $\vec{\zeta}^o$ going from the node $a \in A$ to the node $b \in B$ (Fig. 1). The respective unit vector (director) is $\vec{\tau}^o = \frac{\vec{\zeta}^o}{\eta}$, where quantity $\eta \equiv |\vec{\zeta}^o|$ denotes the distance between the nodes a and b. If the particles are in contact then $\eta = d$, if they are separated by thin adhesive layers, $\eta \approx d$ (more exactly, $\eta = d + \Delta d, \Delta d \ll d$, where Δd is layer's thickness).

Let V_o and V'_o denote the volumes of the set H_o as a whole and of a thin surface layer S_o respectively. As a result, every internal node $a \in \Gamma$ in the region $(V_o - V'_o)$ generates the ray $T_m(a)$ comprised of m doublet axes ζ^o_α. Here α denotes the number of an adjacent node $b_\alpha \in B_\alpha$ in a given doublet (A, B_α) (see Fig. 1), and $\alpha = 1,2, \cdots, m$, with $m = 2n$ (n is the valence of the Bravais lattice). For the granular structures made of spherical particles whose nodes $a \in \Gamma$, the valence $n = 3,4,5,6$. In particular, the simple cubic structure (SCS) and the face-centered cubic structure (FCCS) have the valence $n = 3$, and $n = 6$ respectively (Fig. 2). Let us mention that the FCCS is also called a pyramidal structure.

From the inversion symmetry of the Bravais lattice Γ follows that at any internal node $a \in \Gamma$ the set $T_m(a)$ admits the following dichotomy

$$T_m(a) = \hat{T}_n(a) \cup \hat{T}_n^-(a), \; \hat{T}_n(a) \cap \hat{T}_n^-(a) = \phi \tag{1.1}$$

where the subsets $\hat{T}_n$ and $\hat{T}_n^-$ coincide with respect to the reflection about the center of inversion at the node a. This means that these rays are structurally equivalent, and each of them is sufficient for a description of the lattice Γ. For definiteness sake, we consider in what follows only the rays $\hat{T}_n$.

The relations (1.1) lead to the following properties of the set H_o in the region $(V_o - V_o')$:

1) $\hat{T}_n(a) = \hat{T}_n(b) \forall a, b \in \Gamma$, i.e., for any vector $\vec{\tau}^o_\alpha \in \hat{T}_n(a)$ there is always an equal vector $\vec{\tau}^o_\alpha \in \hat{T}_n(b)$ (the structural homogeneity). Thus we obtain for fixed α

$$\zeta^o_\alpha, \vec{\tau}^o_\alpha = const \; \forall a \in \Gamma. \tag{1.2}$$

Here and in what follows the Greek subscripts α, β, etc. $= 1,2, \ldots, n$.

2) The set of the rays $T = \{\hat{T}_n(a) \mid \forall a \in \Gamma\}$ forms the covering of the Bravais lattice Γ. Therefore we can establish the correspondence between a primitive cell of the volume v and respective ray $\hat{T}_n(a)$, which implies

$$V = \sum_{\forall a \in \Gamma} v + kV_o' \; (0 < k < 1). \tag{1.3}$$

Let the particle number $N \to \infty$ and its diameter $d \to 0$. As a result, the values v and V_o' go to zero, and we can consider v as an elementary volume. This procedure allows one to replace summation over all the nodes $\alpha \in \Gamma$ in (1.3) by the integration over the volume V_o. Thus if some function $\Phi(a) = \sum_{\alpha=1}^{n} F_\alpha(a)$ is defined on a doublet ray $\hat{T}_n(a), a \in \Gamma$ then performing the transition to a continuous model one can assume that

$$\sum_{\forall a \in \Gamma} \Phi(a) \equiv \sum_{\forall a \in \Gamma} \sum_{\alpha=1}^{n} F_\alpha(a) = \int_{V_o} \sum_{\alpha=1}^{n} F_\alpha(\vec{r}) dV \tag{1.4}$$

where $\vec{r}$ is the position vector of the node $a \in \Gamma$ in the region V_o including S_o. Since the valence of the Bravais lattice Γ is constant ($n = const\ \forall \vec{r} \in V_o$) then in (1.4) we can interchange the summation and integration:

$$\sum_{\forall a \in \Gamma} \sum_{\alpha=1}^{n} F_\alpha(a) = \sum_{\alpha=1}^{n} \int_{V_o} F_\alpha(\vec{r}) dV. \tag{1.5}$$

This identity forms the basis for the subsequent transition to the continuous description of a discrete Bravais lattice Γ in the volume V_o occupied by the granular body under consideration.

2. Microstrains

When a granular body undergoes a deformation certain microstrains are developed in each of its doublet. We specify three of them:

1) the relative separation (or convergence) of the doublet nodes,
2) the mutual twist of particles about the doublet axis,
3) the slipping of particles past their contacts.

Let us call these doublet microstrains

1) the elongation (the compression),
2) the torsion, and
3) the shear respectively.

The above microstrains are induced by inhomogeneous translations of the nodes and by rotations of the particles in the given granular body. It should be noted that all these displacements are defined only at the nodes of a discrete Bravais lattice Γ. Meanwhile, we have to make a transition to the continuous description of the above displacements throughout the volume occupied by the granular body. But it is clear that this goal can not be achieved exactly, because the continuous volume is not isomorphic to any multitude of its points. Therefore the problem of a transition to a continuum can be solved only approximately.

There are approximate methods for doing this, and they were applied to various microstructural models (see for example, the review[1]). All these methods prove to be rather complicated. In view of this fact we are going to use another, more convenient and simple approach inherent to the solid state physics, especially to the theory of elastic crystals[2] .

We assume that the displacements of particles vary little at the lengths on the order of their diameter d. We can then introduce two smooth continuous functions, the mutually independent vector fields of the translations $\vec{u}(\vec{R},t)$ and rotations $\vec{\phi}(\vec{R},t)$ where $\vec{R}$ is the position vector of an arbitrary point in a region V_o, t is the time. We assume that these two vectors coincide with the real translations and rotations of the granular body particles at the nodes $a \in \Gamma$, i.e., when $\vec{R} = \vec{r}$.

We introduce also two increment vectors $\Delta\vec{u}_\alpha$ and $\Delta\vec{\phi}_\alpha$. The first of them is $\Delta\vec{u}_\alpha = \vec{u}(\vec{r}+\vec{\zeta}^{\,o}_\alpha,t) - \vec{u}(\vec{r},t)$. It represents an increment of the translation vector $\vec{u}$ in a transition from an arbitrary node $a \in A$ to the adjacent node $b_\alpha \in B_\alpha$ (Fig. 3). The second vector $\Delta\vec{\phi}_\alpha$ has the same meaning if we use the rotation vector $\vec{\phi}$.

We assume that the above increment vectors $\Delta\vec{u}_\alpha$, $\Delta\vec{\phi}_\alpha$ may be expanded in convergent Taylor series at the neighborhood of an arbitrary node $a \in \Gamma$ whose position vector is $\vec{r}$. Truncating this series at the M-th term we obtain

$$\left.\begin{matrix} \Delta\vec{u}_\alpha \\ \Delta\vec{\phi}_\alpha \end{matrix}\right\} = \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^\chi}{\chi!} (\vec{\tau}^{\,o}_\alpha \cdot \vec{\nabla})^\chi \left\{\begin{matrix} \vec{u}(\vec{R},t) \\ \vec{\phi}(\vec{R},t) \end{matrix}\right. (when\ \vec{R} = \vec{r}) \tag{2.1}$$

where $\vec{\nabla}$ is the Hamilton operator, and "$\cdot$" denotes the dot product. The value of number M depends on a degree of approximation. The greater is this number the more exact will be a description given by (2.1).

Furthermore, we introduce a stationary orthogonal Cartesian frame of reference $\{x_i\}$ with a basis $\vec{e}_i\ (i = 1,2,3)$. In this frame of reference the above vectors $\vec{\tau}^{\,o}_\alpha, \vec{u}, \vec{\phi}, \vec{r}, \vec{R}$ and operator $\vec{\nabla}$ are

$$\vec{\tau}^o_\alpha = \tau^o_{\alpha i}\vec{e}_i,\ \vec{u} = u_i\vec{e}_i,\ \vec{\phi} = \phi_i\vec{e}_i,\ \vec{r} = x_i^o\vec{e}_i,\ \vec{R} = x_i\vec{e}_i,\ \vec{\nabla} = \vec{e}_i\frac{\partial}{\partial x_i}. \tag{2.2}$$

We adopt the convention that the repeated indices denote summation from 1 to 3. This convention does not cover the Greek subscripts.

In view of expression (2.2), the homogeneity condition (1.2) takes on the form

$$\tau^o_{\alpha i} = const\ \forall \vec{r} \in V_o. \tag{2.3}$$

Equations (2.1)-(2.3) allow one to proceed with a derivation of the basic kinematic relations for the three above-mentioned microstrains.

Let us consider an arbitrary doublet (A, B_α) with axis $\vec{\zeta}^o_\alpha$ in the initial region V_o (see Fig. 1). In actual region V (which reflects the deformation of the body) this axis becomes another axis $\vec{\zeta}_\alpha$ (see Fig. 3):

$$\vec{\zeta}_\alpha = \vec{\zeta}^o_\alpha + \Delta\vec{u}_\alpha. \tag{2.4}$$

According to (2.4), the corresponding director $\vec{\tau}_\alpha$ is

$$\vec{\tau}_\alpha \equiv \frac{\vec{\zeta}_\alpha}{\zeta_\alpha} = \frac{1}{1+\varepsilon_\alpha}(\vec{\tau}^o_\alpha + \frac{\Delta\vec{u}_\alpha}{\eta_\alpha}) \tag{2.5}$$

where $\zeta_\alpha \equiv |\vec{\zeta}_\alpha|$, $\eta_\alpha \equiv |\vec{\zeta}^o_\alpha|$, $\varepsilon_\alpha \equiv \frac{(\zeta_\alpha - \eta_\alpha)}{\eta_\alpha}$ is the unit microelongation (microcompression) or the elongation microstrain of an arbitrary doublet. The elongation occurs if $\varepsilon_\alpha > 0$, and the compression occurs if $\varepsilon_\alpha < 0$.

We also assume that relative displacements of the doublet nodes and the elongation microstrains are small, i.e., $|\Delta\vec{u}_\alpha| \ll \eta_\alpha$ and $\varepsilon_\alpha \ll 1$ respectively. Then taking into account these assumptions and a pair of the identities $\vec{\tau}^o_\alpha \cdot \vec{\tau}^o_\alpha \equiv 1$, $\vec{\tau}_\alpha \cdot \vec{\tau}_\alpha \equiv 1$ one obtains from an exact expression (2.5) the approximate relation

$$\varepsilon_\alpha = \frac{\vec{\tau}^o_\alpha \cdot \Delta\vec{u}_\alpha}{\eta_\alpha}. \tag{2.6}$$

Substituting $\Delta\vec{u}_\alpha$ from (2.1) into (2.6) and using (2.2), (2.3) we have the following formula for the elongation microstrain ε_α of any arbitrary doublet (A, B_α):

$$\varepsilon_\alpha = \tau^o_{\alpha i} \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^{\chi-1}}{\chi!} \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\chi} \frac{\partial^\chi u_i}{\partial x_{k_1} \cdots \partial x_{k_\chi}} \Big|\ x = x^o. \tag{2.7}$$

The equality $x = x^o$ means that after derivation the continuous coordinates $x_{k_1}, \cdots, x_{k_\chi}$ have to be replaced by discrete coordinates of Bravais lattice nodes, viz., $x^o{}_{k_1}, \ldots, x^o{}_{k_\chi}$ respectively. Each subscript of the latter set $\{k_1, \cdots, k_\chi\}$ runs through the integers 1,2,3.

It should be noted that elongation microstrain ε_α of the doublet (A, B_α) is caused by the motion of a node $b_\alpha \in B_\alpha$ away from a node $a \in A$ along the vector-director $\vec{\tau}_\alpha$. Therefore this microstrain can be conveniently represented as the vector $\vec{\varepsilon}_\alpha = \varepsilon_\alpha\vec{\tau}_\alpha$. At the same time, in the above discussion we assumed that $|\Delta\vec{u}_\alpha| \ll \eta_\alpha$ and $\varepsilon_\alpha \ll 1$. It results in the fact that the angle ψ_α between the directors $\vec{\tau}_\alpha$ and $\vec{\tau}^o_\alpha$ is small: $\psi_\alpha \ll 1$ (see Fig. 3). Hence $\vec{\tau}_\alpha \approx \vec{\tau}^o_\alpha$, and instead of $\vec{\varepsilon}_\alpha = \varepsilon_\alpha\vec{\tau}_\alpha$ one can write $\vec{\varepsilon}_\alpha = \varepsilon_\alpha\vec{\tau}^o_\alpha$. Thus using (2.2) we obtain the equality

$$\vec{\varepsilon}_\alpha = \varepsilon_\alpha\vec{\tau}^o_\alpha = \varepsilon_\alpha\tau^o_{\alpha j}\vec{e}_j = \varepsilon_{\alpha j}\vec{e}_j \tag{2.8}$$

where $\varepsilon_{\alpha j} = \varepsilon_\alpha\tau^o_{\alpha j}$ and scalar ε_α is defined by (2.7).

It follows from the expression (2.7) that the first approximation ($M = 1$) for the elongation microstrain has the form

$$\varepsilon_\alpha = \tau^o_{\alpha i}\tau^o_{\alpha j}\varepsilon_{ij}\ \Big|\ x = x^o. \tag{2.9}$$

Here $\varepsilon_{ij} \equiv \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ are the components of the usual linear strain tensor $\hat{\varepsilon} \equiv \varepsilon_{ij}\vec{e}_i\vec{e}_j$. We assume for the moment being that the quantities u_i are the components of a translational velocity. The expression (2.9) then coincides with the similar one obtained earlier for the granular media by V. Nikolaevskii and E. Afanasiev [3].

Now we can turn our attention to the other two microstrains of a doublet, i.e., to its torsion and shear.

In particular it is clear (see Fig. 3) that an elongation ε_α of an arbitrary doublet (A, B_α) is induced by that part of a translation vector increment $\Delta\vec{u}_\alpha$ which is directed along the unit vector $\vec{\tau}^o_\alpha$. Similarly a torsion μ_α of the above doublet (A, B_α) is caused by that part of a rotation vector increment $\Delta\vec{\phi}_\alpha$ which is directed along the same unit vector $\vec{\tau}^o_\alpha$ (Fig. 4). From these facts the simple analogy follows: the formula for a torsion μ_α can be obtained from the expression (2.6) for an elongation ε_α by replacing the quantities ε_α and $\Delta\vec{u}_\alpha$ with μ_α and $\Delta\vec{\phi}_\alpha$, respectively. As a result we obtain

$$\mu_\alpha = \frac{\vec{\tau}^o_\alpha \cdot \Delta\vec{\phi}_\alpha}{\eta_\alpha}. \tag{2.10}$$

Substituting $\Delta\vec{\phi}_\alpha$ from (2.1) in (2.10) and taking into account (2.2), (2.3) we find the following expression for the torsion microstrain μ_α of any arbitrary doublet (A, B_α):

$$\mu_\alpha = \tau^o_{\alpha i} \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^{\chi-1}}{\chi!} \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\chi} \frac{\partial^\chi \phi_i}{\partial x_{k_1} \cdots \partial x_{k_\chi}} \Bigg|_{x = x^o}. \tag{2.11}$$

Combining this with (2.8) one can obtain the similar formula for the torsion microstrain vector

$$\vec{\mu}_\alpha = \mu_\alpha \vec{\tau}^o_\alpha = \mu_\alpha \tau^o_{\alpha j} \vec{e}_j = \mu_{\alpha j} \vec{e}_j \tag{2.12}$$

where $\mu_{\alpha j} = \mu_\alpha \tau^o_{\alpha j}$ and scalar μ_α is defined by (2.11).

Finally, we consider the shear microstrain. To this end we mention that the nodes a and b_α of an arbitrary doublet (A, B_α) have different translations. Therefore the doublet axis $\vec{\zeta}^o_\alpha$ turns by the angle $\vec{\psi}_\alpha$. Because the increment vector $\Delta\vec{u}_\alpha$ is small ($|\Delta\vec{u}_\alpha| \ll 1$) this angle is also small: $|\vec{\psi}_\alpha| \ll 1$ (see Fig. 3). If the independent rotations $\vec{\phi}$ of the granular body particles would be impossible ($\vec{\phi} \equiv 0$), the contact points $a' \in A$ and $b'_\alpha \in B_\alpha$ (see Fig. 1) would remain in contact after the body deformation.

However our case is somewhat different. Owing to the independent rotations the vector $\vec{\phi} \neq 0$, and the doublet particles A and B_α undergo the auxiliary rotations by the small angles $\vec{\theta}_\alpha = \vec{\phi} - \vec{\psi}_\alpha$ and $\vec{\theta}'_\alpha = \vec{\theta}_\alpha + \Delta\vec{\phi}_\alpha$ respectively. Because of that, the above contact points a' and b'_α move (slip) into the opposite directions which are perpendicular to the director $\vec{\tau}^o_\alpha$. This leads to the appearance of the shear of the doublet particles, (i.e., shear microstrain) which can be easily found.

In fact the relative displacements of the contact points a' and b'_α are

$$\Delta\vec{w} = \frac{1}{2}\vec{\theta}_\alpha \times \vec{\zeta}^o_\alpha = \frac{1}{2}(\vec{\phi} - \vec{\psi}_\alpha) \times \vec{\zeta}^o_\alpha, \quad \Delta\vec{w}' = -\frac{1}{2}\vec{\theta}'_\alpha \times \vec{\zeta}_\alpha = -\frac{1}{2}(\vec{\phi} + \Delta\vec{\phi}_\alpha - \vec{\psi}_\alpha) \times \vec{\zeta}^o_\alpha.$$

We assume here that the angle $\vec{\phi}$ (analogous to $\vec{\psi}_\alpha$) is small: $|\vec{\phi}| \ll 1$. Then the difference between $\Delta\vec{w}$ and $\Delta\vec{w}'$, divided by the length η_α of the above doublet (A, B_α), yields its shear microstrain vector $\vec{\gamma}_\alpha$:

$$\vec{\gamma}_\alpha = (\vec{\phi} + \frac{1}{2}\Delta\vec{\phi}_\alpha - \vec{\psi}_\alpha) \times \vec{\tau}^o_\alpha. \tag{2.13}$$

The quantity $\psi_\alpha \equiv |\vec{\psi}_\alpha|$ is the small angle between the directors $\vec{\tau}^o_\alpha$ and $\vec{\tau}_\alpha$ (see Fig. 3). Therefore

$$\vec{\psi}_\alpha = \vec{\tau}^o_\alpha \times \vec{\tau}_\alpha. \tag{2.14}$$

Substituting the vectors $\Delta\vec{\phi}_\alpha$, $\vec{\phi}$, $\vec{\psi}_\alpha$ from (2.1), (2.2), (2.14), respectively into (2.13) and taking into account the expressions (2.3), (2.5) we obtain

$$\vec{\gamma}_\alpha = \gamma_{\alpha i} \vec{e}_i, \tag{2.15}$$

$$\gamma_{\alpha i} = \Bigg[\Big(\phi_j + \frac{1}{2} \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^\chi}{\chi!} \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\chi} \frac{\partial^\chi \phi_j}{\partial x_{k_1} \cdots \partial x_{k_\chi}}\Big) \tau^o_{\alpha p} \varepsilon_{ijp} + (\tau^o_{\alpha i} \tau^o_{\alpha j} - \delta_{ij}) \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^{\chi-1}}{\chi!} \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\chi} \frac{\partial^\chi u_j}{\partial x_{k_1} \cdots \partial x_{k_\chi}} \Bigg]_{x = x^o} \tag{2.16}$$

where ε_{ijp} is the Levi-Civita axial tensor.

3. The Evaluations of the Formulas for Microstrains

Let us consider a particular case in which the translations u_i vary according to the following

$$u_i = f_i(t)\exp(m_j x_j)\ (m_j = const, m_j \equiv l_j^{-1}).$$

From this we obtain

$$\frac{\partial u_i}{\partial x_j} = f_i(t) m_j \exp(m_i x_j) = \frac{u_i}{l_j}. \tag{3.1}$$

In the general case, the above relations are not valid. Nevertheless, we can write the equality (3.1) approximately, as an estimation (symbol "~")

$$\frac{\partial u_i}{\partial x_j} \sim \frac{u_i}{l_j} \tag{3.2}$$

where l_j and u_i are the constants treated usually as the certain average (characteristic) quantities: the halfwavelength and the amplitude of a microstrain, respectively [4].

In view of (3.2), the expression (2.7) takes on the form

$$\varepsilon_\alpha \sim \frac{\tau^o_{\alpha i} u_i}{\eta_\alpha} \sum_{\chi=1}^{M} \frac{(\nu_\alpha)^\chi}{\chi!}. \tag{3.3}$$

Here $\nu_\alpha = \frac{\tau^o_{\alpha j}}{\rho_j}$, $\rho_j = \frac{l_j}{\eta_\alpha}$. The value ρ_j shows by how much the microstrain halfwavelength l_j is larger than the length η_α of a doublet. Below in this section, we assume for simplicity that a granular body consists of identical dry particles, without thin adhesive pellicles. In that case $\eta_\alpha = d = const\ \forall\alpha = 1,2,\ldots,n$. Therefore one would be able to replace the worlds "length η_α of doublet" by the worlds "particle diameter d".

Let us return to the dependence (3.3). When $M \rightarrow \infty$, this estimation reaches the upper limit of its exactness

$$\varepsilon_\alpha \sim \frac{\tau^o_{\alpha i} u_i}{\eta_\alpha} (\exp\nu_\alpha - 1). \tag{3.4}$$

While comparing (3.3) and (3.4), we come to a conclusion that the elongation microstrain ε_α can be determined with an error of 5% by the following approximations of the formula (2.7):

1) the first approximation ($M = 1$), if the microstrain halfwavelength l_j is 27 times the particle diameter d. Let us call this "a low-frequency wave approximation"
2) the second approximation ($M = 2$), if the quantity l_j is 5 times the particle diameter d. This case can be named as " a medium-frequency wave approximation",
3) by the third approximation ($M = 3$), if the quantity l_j is 2.5 times the particle diameter d. We call it " a high-frequency wave approximation"

It should be emphasized that in second and higher approximations, the expression (2.7) involves the scale parameter: the particle diameter d. Therefore we can call any theory corresponding to such an approximation "the scale theory". The others theories correspond then to "the non-scale theories".

The traditional theories of elasticity are non-scale ones since they neglect the solid's microstructure and do not account for the particle sizes. Therefore they are unsuitable for the study of granular body deformations in general case. Their applicability is restricted to the low-frequency wave deformations.

Rigorously speaking, such a case is possible only under uniform (or close to uniform) compression (tension) of the whole body, i.e., under very particular conditions. In other cases any theory leads to the errors which increase with the increases of the inhomogeneity of the deformations in space and time. However the scale theories are in principle much more exact than the traditional theories, especially under the deformations with medium- and high-frequency waves, i.e., in the problems of the dynamics, stress concentration, etc.

The preceding consideration is concerned with elongation microstrain. One can arrive at the same conclusions about the microstrains due to torsion and shear if we make use of the approximate formulas (2.11) and (2.16).

4. Microstresses. Equations of Motion. Boundary Conditions

Let us postulate the existence of internal generalized microforces which will be consistent with internal generalized microdisplacements. We will take the above microstrains $\vec{\varepsilon}_\alpha$, $\vec{\mu}_\alpha$, $\vec{\gamma}_\alpha$ as generalized microdisplacements. As the generalized microforces one then has to adopt the following microstresses:

1) elongation microstresses (consistent with $\vec{\varepsilon}_\alpha$)

$$\vec{p}_\alpha = p_\alpha \vec{\tau}^{\,o}_\alpha = p_\alpha \tau^o_{\alpha i} \vec{e}_i = p_{\alpha i} \vec{e}_i \;\; (p_{\alpha i} = p_\alpha \tau^o_{\alpha i}), \tag{4.1}$$

2) torsion microstresses (consistent with $\vec{\mu}_\alpha$)

$$\vec{m}_\alpha = m_\alpha \vec{\tau}^{\,o}_\alpha = m_\alpha \tau^o_{\alpha i} \vec{e}_i = m_{\alpha i} \vec{e}_i \;\; (m_{\alpha i} = m_\alpha \tau^o_{\alpha i}), \tag{4.2}$$

3) shear microstresses (consistent with $\vec{\gamma}_\alpha$)

$$\vec{T}_\alpha = t_{\alpha i} \vec{e}_i . \tag{4.3}$$

The virtual work of microstresses in each doublet ray $\hat{T}_n(a)$, $a \in \Gamma$, is caused by the corresponding microstrains in an actual (deformed) state of the granular body. Therefore it is

$$\delta A(a) = -\sum_{\alpha=1}^{n} (\vec{p}_\alpha \cdot \delta\vec{\varepsilon}_\alpha + \vec{m}_\alpha \cdot \delta\vec{\mu}_\alpha + \vec{T}_\alpha \cdot \delta\vec{\gamma}_\alpha). \tag{4.4}$$

In all the volume of the granular body such a work is

$$\delta A = \sum_{\forall a \in \Gamma} \delta A(a). \tag{4.5}$$

We can go from the discrete expression (4.5) to the continuous one, using basic identity (1.5). Then, taking into account (4.4), we transform (4.5) into the continuous form

$$\delta A = -\sum_{\alpha=1}^{n} \int_V (\vec{p}_\alpha \cdot \delta\vec{\varepsilon}_\alpha + \vec{m}_\alpha \cdot \delta\vec{\mu}_\alpha + \vec{T}_\alpha \cdot \delta\vec{\gamma}_\alpha) dV. \tag{4.6}$$

Naturally, one assumes here that under granular body deformation, its particles don't mix and the valence n is conserved. In addition, because the microstrains are supposed to be small we don't distinguish in (4.6) the volumes and surfaces of granular body in the actual and initial states. This entails $V = V_o$ and $S = S_o$. Such assumptions are also used later.

Now we use the variational principle of the virtual displacements in the following form [5]

$$\delta A + \int_V (\vec{F} - \rho\vec{a}) \cdot \delta\vec{u} dV + \int_S (\vec{T} \cdot \delta\vec{u} + \vec{M} \cdot \delta\vec{\phi}) dS = 0$$

where $\vec{T} = T_i \vec{e}_i$, $\vec{M} = M_i \vec{e}_i$ are the force and couple on a per-unit basis of the surface S, respectively, $\vec{F} = F_i \vec{e}_i$ is the force on a per-unit basis of the volume V, $\vec{a} = (\frac{\partial^2 u_i}{\partial t^2}) \vec{e}_i$ is the acceleration of an arbitrary granular medium point with position vector $\vec{R} \in V$, ρ is the bulk density of the above medium.

We substitute in (4.7) the expression δA according to (4.6). Then, using the equations (2.7), (2.8), (2.11), (2.12), (2.15), (2.16), (4.1)-(4.3) and using the well-known technique based on Gauss's theorem we obtain for the granular media the following equations in the terms of microstresses (the designation $|x = x^o$ is further omitted):

1. **Differential equations of the motion** (small oscillations) in the volume V:

   a) **"force" equations** (of the linear momentum conservation):

$$\sum_{\alpha=1}^{n} \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^{\chi-1}}{\chi!} \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\chi} \frac{\partial^\chi (t_{\alpha i} - p_{\alpha i})}{\partial x_{k_1} \cdots \partial x_{k_\chi}} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{4.8}$$

b) **"couple" equations** (of the moment of momentum conservation):

$$\sum_{\alpha=1}^{n}\left[\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}\overset{o}{\tau}_{\alpha q}+\sum_{\chi=1}^{M}(-1)^{\chi}\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\cdot\right.$$
$$\left.\cdot\overset{o}{\tau}_{\alpha k_1}\cdots\overset{o}{\tau}_{\alpha k_\chi}\frac{\partial^{\chi}(m_{\alpha i}+\frac{1}{2}\eta_\alpha\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}t_{\alpha q})}{\partial x_{k_1}\cdots\partial x_{k_\chi}}\right]=0. \tag{4.9}$$

2. **Natural boundary conditions** at the surface S:

a) **"force" conditions** :

$$n_k\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha k}\sum_{\chi=r}^{M}(-1)^{\chi}\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\cdot\overset{o}{\tau}_{\alpha k_{r+1}}\cdots\overset{o}{\tau}_{\alpha k_\chi}\frac{\partial^{\chi-r}(t_{\alpha i}-p_{\alpha i})}{\partial x_{k_{r+1}}\cdots\partial x_{k_\chi}}=T_i\delta_{r1}, \tag{4.10}$$

b) **"couple" conditions** :

$$n_k\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha k}\sum_{\chi=r}^{M}(-1)^{\chi}\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\cdot$$
$$\cdot\overset{o}{\tau}_{\alpha k_{r+1}}\cdots\overset{o}{\tau}_{\alpha k_\chi}\frac{\partial^{\chi-r}(m_{\alpha i}+\frac{1}{2}\eta_\alpha\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}t_{\alpha q})}{\partial x_{k_{r+1}}\cdots\partial x_{k_\chi}}=-M_i\delta_{r1}. \tag{4.11}$$

Here $\vec{n}=n_i\vec{e}_i$ denotes the unit vector of an external normal directed towards the surface S. The subscript $r=1,2,\ldots,M-1$ if $M\geq 2$, and $r=1$, if $M=1$. However the value $r+1$ should not exceed the upper limit χ : $r+1\leq\chi$. Otherwise one has to take the product $\overset{o}{\tau}_{\alpha k_{r+1}}\cdots\overset{o}{\tau}_{\alpha k_\chi}=1$ and to ignore the differential operator in (4.10), (4.11), assuming $\partial^{\chi-r}(\cdots)=(\cdots)$.

For example, let $M=3$. Then max $r=M-1=2$, and formula (4.10) gives

a) **for** $r=1$:

$$n_{k_1}\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha k_1}\left[-(\tau_{\alpha i}-p_{\alpha i})+\frac{\eta_\alpha}{2!}\overset{o}{\tau}_{\alpha k_2}\frac{\partial(t_{\alpha i}-p_{\alpha i})}{\partial x_{k_2}}-\right.$$
$$\left.-\frac{(\eta_\alpha)^2}{3!}\overset{o}{\tau}_{\alpha k_2}\overset{o}{\tau}_{\alpha k_3}\frac{\partial^2(t_{\alpha i}-p_{\alpha i})}{\partial x_{k_2}\partial x_{k_3}}\right]=T_i,$$

b) **for** $r=2$:

$$n_{k_2}\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha k_2}\left[\frac{\eta_\alpha}{2!}(t_{\alpha i}-p_{\alpha i})-\frac{(\eta_\alpha)^2}{3!}\overset{o}{\tau}_{\alpha k_3}\frac{\partial(t_{\alpha i}-p_{\alpha i})}{\partial x_{k_3}}\right]=0.$$

5. The Transition from Microstresses to Macrostresses

Let us represent the components T_i and M_i of the above surface vectors of the forces $\vec{T}$ and couples $\vec{M}$ in a usual form [5]

$$T_i=\sigma_{ki}n_k,\ M_i=M_{ki}n_k \tag{5.1}$$

where σ_{ki} and M_{ki} are the components of two tensors: of the force macrostresses $\hat{T}=\sigma_{ij}\vec{e}_i\vec{e}_j$ and the couple macrostresses $\hat{M}=M_{ij}\vec{e}_i\vec{e}_j$, respectively. From comparison (5.1) with (4.10) and (4.11) follows the natural connections between the micro- and macrostresses:

$$\sigma_{k_1 i}{}^{(M)}=\sum_{\alpha-1}^{n}\overset{o}{\tau}_{\alpha k_1}\sum_{\chi=1}^{M}(-1)^{\chi}\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\cdot\overset{o}{\tau}_{\alpha k_2}\cdots\overset{o}{\tau}_{\alpha k_\chi}\frac{\partial^{\chi-1}(t_{\alpha i}-p_{\alpha i})}{\partial x_{k_2}\cdots\partial x_{k_\chi}}, \tag{5.2}$$

$$M_{k_1 i}{}^{(M)}=-\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha k_1}\sum_{\chi=1}^{M}(-1)^{\chi}\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\cdot$$
$$\cdot\overset{o}{\tau}_{\alpha k_2}\cdots\overset{o}{\tau}_{\alpha k_\chi}\frac{\partial^{\chi-1}(m_{\alpha i}+\frac{1}{2}\eta_\alpha\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}t_{\alpha q})}{\partial x_{k_2}\cdots\partial x_{k_\chi}}. \tag{5.3}$$

The superscript M gives here the level of an approximation at which the macrostresses are represented by microstresses.

The formulas (5.2) and (5.3) allow us to show that in general case the macrostresses σ_{ij} and M_{ij} are asymmetric. Indeed, we can write the couple equations of motion (4.9) in the form

$$\sum_{\alpha=1}^{n}\left[\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}t_{\alpha q}-\overset{o}{\tau}_{\alpha l}\frac{\partial m_{\alpha i}}{\partial x_l}+O_i(\eta_\alpha)\right]=0 \tag{5.4}$$

where $O_i(\eta_\alpha)$ denotes the terms which contain the characteristic diameter η_α of the granular particles to the first and higher powers and depend on the subscript i. The equations (5.4) show that $\sum_{\alpha=1}^{n}\varepsilon_{ijq}\overset{o}{\tau}_{\alpha j}t_{\alpha q}\not\equiv 0$ if $m_{\alpha i}\not\equiv const\,(\not\equiv 0)$, $\eta_\alpha\not\equiv 0$ and $O_i(\eta_\alpha)\not\equiv 0$, or simpler

$$\Delta t_{ij}\equiv\sum_{\alpha=1}^{n}(\overset{o}{\tau}_{\alpha j}t_{\alpha i}-\overset{o}{\tau}_{\alpha i}t_{\alpha j})\not\equiv 0(i\neq j). \tag{5.5}$$

In a well-known case of a nonpolar medium ($m_{\alpha i}\equiv 0$) with infinitesimal particles ($\eta_\alpha\to 0, O_i(\eta_\alpha)\to 0$) Eqs. (5.4) imply

$$\Delta t_{ij}\equiv 0. \tag{5.6}$$

One can write the dependence (5.2) similarly to (5.4):

$$\sigma_{ji}^{(M)}=\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha j}\left[t_{\alpha i}-p_{\alpha i}+O'_i(\eta_\alpha)\right]$$

or

$$\sigma_{ij}^{(M)}=\sum_{\alpha=1}^{n}\overset{o}{\tau}_{\alpha i}\left[t_{\alpha j}-p_{\alpha j}+O'_j(\eta_\alpha)\right]$$

whence it appears that

$$\Delta\sigma_{ij}^{(M)}\equiv\sigma_{ij}^{(M)}-\sigma_{ji}^{(M)}=-\Delta t_{ij}+\sum_{\alpha=1}^{n}\left[\overset{o}{\tau}_{\alpha j}p_{\alpha i}-\overset{o}{\tau}_{\alpha i}p_{\alpha j}+O'_{ij}(\eta_\alpha)\right] \tag{5.7}$$

where $O'_{ij}(\eta_\alpha)\equiv\overset{o}{\tau}_{\alpha i}O'_j(\eta_\alpha)-\overset{o}{\tau}_{\alpha j}O'_i(\eta_\alpha)$, $O'_{ij}(\eta_\alpha)\not\equiv O'_{ji}(\eta_\alpha)$, if $i\neq j$.

While using the expression (4.1), we obtain

$$\overset{o}{\tau}_{\alpha j}p_{\alpha i}-\overset{o}{\tau}_{\alpha i}p_{\alpha j}\equiv\overset{o}{\tau}_{\alpha j}p_{\alpha}\overset{o}{\tau}_{\alpha i}-\overset{o}{\tau}_{\alpha i}p_{\alpha}\overset{o}{\tau}_{\alpha j}\equiv 0.$$

In view of this identity, the equality (5.7) reduces to the form

$$\Delta\sigma_{ij}^{(M)}=-\Delta t_{ij}+\sum_{\alpha=1}^{n}O'_{ij}(\eta_\alpha). \tag{5.8}$$

Taking into account the expression (5.5), we obtain from (5.8) the final result: $\Delta\sigma_{ij}^{(M)}\not\equiv 0$, or $\sigma_{ij}^{(M)}\not\equiv\sigma_{ji}^{(M)}$, even if $\eta_\alpha\to 0$ and $O'_{ij}(\eta_\alpha)\to 0$. In other words, the force macrostresses σ_{ij} are asymmetric in any polar media with internal couple microstresses $\vec{m}_\alpha\not\equiv 0$ under any particle sizes η_α considered.

In usual case of nonpolar media with infinitesimal particles we have to use the identity (5.6) and condition $O'_{ij}(\eta_\alpha)\to 0$. Then (5.8) takes on the form $\Delta\sigma_{ij}^{(M)}\equiv 0$, or $\sigma_{ij}^{(M)}\equiv\sigma_{ji}^{(M)}$. Thus, in such (and only in such) media the macrostresses σ_{ij} are symmetric.

It is very interesting to look into an intermediate case, where the granular medium is nonpolar ($\vec{m}\equiv 0$), but the scaling effect ($\eta_\alpha\not\equiv 0$) is considered. Then the equations (5.4) imply that $\Delta t_{ij}=\sum_{\alpha=1}^{n}O_q(\eta_\alpha)$, $q\neq i$, $q\neq j$. Substituting this equality in (5.8), we have

$$\Delta\sigma_{ij}^{(M)}\equiv\sum_{\alpha=1}^{n}\left[O'_{ij}(\eta_\alpha)-O_q(\eta_\alpha)\right]\not\equiv 0.$$

It follows that $\sigma_{ij}^{(M)} \neq \sigma_{ji}^{(M)}$, i.e., the force macrostresses σ_{ij} are asymmetric.

Such an asymmetry of nonpolar and therefore, one would think, conventional medium (but with finite sizes of its particles!) has been discovered we believe for the first time. This result revises a widely-held belief that "in the nonpolar case, that is, when couple stresses, body moments and internal spin are zero, the stress tensor $\hat{T}$ is symmetric"[6].

The similar results can be derived for the couple macrostresses M_{ij}, using the corresponding identities (5.3). We omit this step for the shortness of space.

Let us return to the identities (5.2) and (5.3) having another goal in mind. While comparing them with (4.8) and (4.9), we obtain the equations of motion for a granular medium in the volume V which are expressed in terms of macrostresses:

$$\frac{\partial \sigma_{ij}^{(M)}}{\partial x_i} + F_j = \rho \frac{\partial^2 u_j}{\partial t^2}, \tag{5.9}$$

$$\frac{\partial M_{ij}^{(M)}}{\partial x_i} + \varepsilon_{jik} \sigma_{ik}^{(1)} = 0. \tag{5.10}$$

Taking into account (5.2) and (5.3), the boundary conditions (4.10) and (4.11) for the microstresses on the granular body surface S can be represented in the form of macrostresses, as well:

$$T_j = \sigma_{ij}^{(M)} n_i, \ \mathrm{M}_j = M_{ij}^{(M)} n_i. \tag{5.11}$$

It should be noted that in the first approximation ($M = 1$), the dependences (5.9) and (5.10) are in agreement with the similar differential equations of motion for Cosserat's continuum[5]. When the couple macrostresses are absent in all of the body volume V ($M_{ij} \equiv 0 \forall \vec{R} \in V$), the medium is nonpolar, and for $M = 1$ the dependences (5.9) are reduced to the differential equations of motion used in a classical linear theory of elasticity with symmetric force macrostresses $\sigma_{ij} = \sigma_{ji}$[5]. The couple equations (5.1) and the second boundary conditions (5.11) become identities.

Thus in the first approximation, the equations of motion (5.9) and (5.10) are the same as the corresponding dependencies in the conventional theories of elasticity. In the subsequent approximations ($M = 2,3,\ldots$), this similarity comes to an end because the equations (5.9), (5.10) include a scale parameter η_α and are virtually the motion equations (4.8), (4.9) expressed in microstresses. In fact already at $M = 2$ we have in the first system the macrostresses $\sigma_{ij}^{(2)}$ which drastically differ from the macrostresses $\sigma_{ij}^{(1)}$ in the second system (5.10). This difference is eliminated in a natural way when we return to the equations (4.8) and (4.9).

In that way, if the problem is expected to be related to deformations with medium- or high-frequency waves (see Sec. 3) then one should use the motion equations only in terms of the microstresses, i.e., in the form (4.8) and (4.9). At the same time, even for a first approximation with low-frequency waves of deformations, the proposed theory is not identical to the Cosserat's theory because the former provides an insight into the microstructure of the granular body. This allows us to consider the microscopic interactions of body particles. This in turn gives us the opportunity to simplify one of the most difficult problems in solid mechanics: the formulation of the invariant constitutive (physical) relations for the media when their internal structure is included into consideration (see below sec. 7).

In this respect, the considered theory is also sharply different from the well-known microstructural theories proposed by C. A. Eringen and E. S. Suhubi[7], A. E. Green and R. S. Rivlin[8], R. D. Mindlin[9], R. Stojanovic[10].

6. Thermodynamics of Microstrains

The system of the kinematic equations (2.7), 2.11), (2.16) and the dynamic equations (4.8), (4.9) is not closed. For its closure it is necessary to use the constitutive relations which can be correctly formulated only by including a thermodynamics of the medium deformations[11].

We will consider only elastic (reversible) deformations which follow a thermodynamics of the reversible processes. For such processes, the first and the second principles of thermodynamics for an elementary volume are of the form[12]:

$$dW = -dA + dQ\,, \tag{6.1}$$

$$T\ dS = dQ. \tag{6.2}$$

Here W, A, Q, T and S are the internal energy, work of microstresses, external heat inflow, absolute temperature and entropy respectively. When combined the equations (6.1), (6.2) yield

$$dW = -dA + T\ dS. \tag{6.3}$$

The internal energy W is a function which depends on the variables defining a deformed state thermodynamics of granular media. It is quite naturally to select as such variables the microstrains ε_α, μ_α, $\gamma_{\alpha i}$ and entropy S, i.e., to specify W in the form $W \equiv W(\varepsilon_\alpha, \mu_\alpha, \gamma_{\alpha i}, S)$. It follows

$$dW = \sum_{\alpha=1}^{n} \left(\frac{\partial W}{\partial \varepsilon_\alpha} d\varepsilon_\alpha + \frac{\partial W}{\partial \mu_\alpha} d\mu_\alpha + \frac{\partial W}{\partial \gamma_{\alpha i}} d\gamma_{\alpha i}\right) + \frac{\partial W}{\partial S} dS. \tag{6.4}$$

Besides that, by virtue of (4.4) we have

$$dA = -\sum_{\alpha=1}^{n} (p_\alpha d\varepsilon_\alpha + m_\alpha d\mu_\alpha + t_{\alpha i}\, d\gamma_{\alpha i}). \tag{6.5}$$

The expression (6.5) allows one to write (6.3) in the form

$$dW = \sum_{\alpha=1}^{n} (p_\alpha d\varepsilon_\alpha + m_\alpha d\mu_\alpha + t_{\alpha i}\, d\gamma_{\alpha i}) + T\ dS. \tag{6.6}$$

While comparing (6.6) with (6.4), we obtain the following potential relations (caloric state equations):

$$p_\alpha = \frac{\partial W}{\partial \varepsilon_\alpha},\ m_\alpha = \frac{\partial W}{\partial \mu_\alpha},\ t_{\alpha i} = \frac{\partial W}{\partial \gamma_{\alpha i}},\ T = \frac{\partial W}{\partial S}. \tag{6.7}$$

We consider also the second thermodynamic potential of the free energy $\Phi \equiv W - ST$ that is the function of the above microstrains and temperature T. In such a case, by reasoning along similar lines, we find

$$p_\alpha = \frac{\partial \Phi}{\partial \varepsilon_\alpha},\ m_\alpha = \frac{\partial \Phi}{\partial \mu_\alpha},\ t_{\alpha i} = \frac{\partial \Phi}{\partial \gamma_{\alpha i}},\ S = -\frac{\partial \Phi}{\partial T}. \tag{6.8}$$

Relations (6.7) and (6.8) are used below for a correct derivation of the well-grounded constitutive equations.

7. Linear Elastic Uniform Granular Media. Constitutive Equations

The linear elastic granular media are such that their microstresses depend on the microstrains linearly. According to this definition, we may write the linear dependences between microstresses and microstrains in the most general invariant form

$$p_\alpha = \sum_{\beta=1}^{n} (A_{\alpha\beta}\varepsilon_\beta + B_{\alpha\beta}\mu_\beta + C_{\alpha\beta i}\gamma_{\beta i}) + J_\alpha \Theta, \tag{7.1}$$

$$m_\alpha = \sum_{\beta=1}^{n} (D_{\alpha\beta}\varepsilon_\beta + E_{\alpha\beta}\mu_\beta + F_{\alpha\beta i}\gamma_{\beta i}) + K_\alpha \Theta, \tag{7.2}$$

$$t_{\alpha i} = \sum_{\beta=1}^{n} (G_{\alpha\beta i}\varepsilon_\beta + H_{\alpha\beta i}\mu_\beta + I_{\alpha\beta ij}\gamma_{\beta j}) + L_{\alpha i} \Theta \tag{7.3}$$

where for an uniform granular body, the quantities $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta i}$, $\cdots$, $L_{\alpha i}$ are the scalar, vector or tensor microconstants (micromoduli of thermoelasticity), $\Theta \equiv T - T_o$ is an increment of the temperature, T_o is the absolute temperature of the granular media in an initial state.

The above micromoduli can not be arbitrary inasmuch as the microstresses and microstrains are connected with each other by the caloric equations (6.7) and (6.8). Therefore the invariant equations (7.1)-(7.3) should be examined from the thermodynamic standpoint.

We begin from the microstrains ε_α and ε_β entering the caloric equations (6.7), (6.8). It follows

$$\frac{\partial^2 W}{\partial\varepsilon_\alpha\partial\varepsilon_\beta} = \frac{\partial^2\Phi}{\partial\varepsilon_\alpha\partial\varepsilon_\beta} = \frac{\partial p_\alpha}{\partial\varepsilon_\beta} = \frac{\partial p_\beta}{\partial\varepsilon_\alpha}. \tag{7.4}$$

With the help of (7.1) we obtain from (7.4)

$$A_{\alpha\beta} = A_{\beta\alpha}. \tag{7.5}$$

Analogously one derives

$$E_{\alpha\beta} = E_{\beta\alpha},\ B_{\alpha\beta} = D_{\beta\alpha},\ C_{\alpha\beta i} = G_{\beta\alpha i},\ F_{\alpha\beta i} = H_{\beta\alpha i},\ I_{\alpha\beta ij} = I_{\beta\alpha ji}. \tag{7.6}$$

The caloric relations (6.7), (6.8) can not give us more information than the restrictions (7.5), (7.6). The further examination of the constitutive equations (7.1)-(7.3) have to based on other considerations beyond the bounds of the thermodynamic principles (6.7), (6.8).

Therefore we turn our attention to the symmetric properties of the body's deformations. Let the granular body be subjected to a uniform heating ($\Theta = const\ \forall\vec{R} \in V$). We consider in this body an arbitrary doublet (A, B_α) and draw a plane Π which runs through the contact point C being perpendicular to the director $\vec{\tau}_\alpha^{\,o}$ (Fig. 5). In that plane, the torsion and shear microstresses m_α, $\vec{t}_\alpha$ can act. Since the temperature field is isotropic in the plane Π, any directions of microstresses m_α and $\vec{t}_\alpha$ are equivalent. However, it is clear that under any $m_\alpha \neq 0$ and $|\vec{t}_\alpha| \neq 0$ the certain directions are preferred and the isotropic condition is violated. It follows that microstresses $m_\alpha \equiv \vec{t}_\alpha \equiv 0$. The above reasons hold true for corresponding microstrains μ_α and $\vec{\gamma}_\alpha$, viz., $\mu_\alpha \equiv \vec{\gamma}_\alpha \equiv 0$.

As for the elongation microstress $\vec{p}_\alpha$ and corresponding microstrain $\vec{\varepsilon}_\alpha$ (see Fig. 5) their projections onto the plane of isotropy Π are the points under any $|\vec{p}_\alpha| \neq 0$ and $|\vec{\varepsilon}_\alpha| \neq 0$. Hence these quantities may be arbitrary and under a uniform heating, they are, generally speaking, different from zero.

Thus, we obtain $\mu_\alpha \equiv \gamma_{\alpha i} \equiv m_\alpha \equiv t_{\alpha i} \equiv 0$. Then the expressions (7.2) and (7.3) take the form

$$\sum_{\beta=1}^{n} D_{\alpha\beta}\varepsilon_\beta + K_\alpha\Theta \equiv 0,$$

$$\sum_{\beta=1}^{n} G_{\alpha\beta i}\,\varepsilon_\beta + L_{\alpha i}\,\Theta \equiv 0.$$

Since the variables ε_α and Θ are mutually independent, the above identities mean respectively that

$$D_{\alpha\beta} \equiv K_\alpha \equiv 0,\qquad G_{\alpha\beta i} \equiv L_{\alpha i} \equiv 0. \tag{7.7}$$

With the help of (7.6) these identities imply

$$B_{\beta\alpha} \equiv C_{\beta\alpha i} \equiv 0. \tag{7.8}$$

Now we assume that the torsion microstress m_α is induced by the shear microstrain $\vec{\gamma}_\alpha$. If one looks at the plane Π from above (see Fig. 5), then the vector $\vec{\gamma}_\alpha$ is rotated by the micromoment m_α counterclockwise. If one looks at the plane Π from below than $\vec{\gamma}_\alpha$ is rotated by m_α clockwise. To return to the preceding situation we might have turned the vector $\vec{\gamma}_\alpha$ into the opposite direction (see the dashed arrow at Fig. 5). However, in such a case we would obtain quite a different pattern of deformed state, if only $\vec{\gamma}_\alpha \neq 0$. At the same time, it is clear that the strain field of a medium does not have to be dependent on the observer position. This requirement can be satisfied at $\vec{\gamma}_\alpha \neq 0$, if the induced micromoment $m_\alpha \equiv 0$. Taking into account the constitutive equations (7.2), such a condition means that micromoduli $F_{\alpha\beta i} \equiv 0$. In view of thermodynamic restriction (7.6) this implies $H_{\beta\alpha i} \equiv 0$. Hence, we have the additional identities

$$F_{\alpha\beta i} \equiv H_{\beta\alpha i} \equiv 0. \tag{7.9}$$

In that way, including the identities (7.7)-(7.9), the invariant constitutive equations (7.1)-(7.3) take on the final form

$$p_\alpha = \sum_{\beta=1}^{n} A_{\alpha\beta}\varepsilon_\beta + J_\alpha\Theta, \tag{7.10}$$

$$m_\alpha = \sum_{\beta=1}^{n} E_{\alpha\beta}\mu_\beta, \tag{7.11}$$

$$t_{\alpha i} = \sum_{\beta=1}^{n} I_{\alpha\beta ij}\gamma_{\beta j}. \tag{7.12}$$

We would like to attract attention to the simplicity of these very general physical relations. Together with the kinematic equations (2.7), (2.11), (2.16) and the dynamic relations (4.8), (4.9) these physical relations form the closed system of equations which allow one to define the mechanical fields of displacements u_i, ϕ_i, microstrains ε_α, μ_α, $\gamma_{\alpha i}$ and microstresses p_α, m_α, $t_{\alpha i}$ in the linear elastic uniform granular media under an adiabatic loading. In the different cases, the above mechanical fields are connected with the heat field Θ, and their determination calls for further thermodynamic study. Such a study is carried out in the next section.

In conclusion of this section we briefly consider one important problem about the relations between the macrostresses σ_{ij}, m_{ij}, on the one hand and the displacements u_i, ϕ_i, on the other hand. Let us substitute in the physical equations (7.10)-(7.12) the expressions for microstrains ε_β, μ_β, $\gamma_{\beta i}$ from the kinematic equations (2.7), (2.11), (2.16) and then substitute the expressions for microstresses p_α, m_α, $t_{\alpha i}$ in the macro-micro-equations (5.2), (5.3). In the long run, we obtain the desired expression. For example, in a first approximation ($M = 1$), the force macrostresses $\sigma_{ij}^{(M)}$ are represented in the following form:

$$\sigma_{ij}^{(1)} = A_{ijkl}\varepsilon_{kl} + B_{ijkl}\left(\frac{\partial u_k}{\partial x_l} - \varepsilon_{mlk}\phi_m\right) - \beta_{ij}\Theta + \frac{\eta}{2}C_{ijklp}\varepsilon_{kpm}\frac{\partial \phi_m}{\partial x_l} \tag{7.13}$$

where $\eta \equiv \eta_\alpha = const$ is a diameter of particles, ε_{kl} are the components of a conventional linear strain tensor (see formula (2.9)), the macromoduli of elasticity A_{ijkl}, B_{ijkl}, C_{ijklm} and macromoduli of heat expansion β_{ij} are simply expressed in terms of above micromoduli $A_{\alpha\beta}$, etc.:

$$A_{ijkl} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}(A_{\alpha\beta}\overset{o}{\tau}_{\alpha j} - I_{\alpha\beta jp}\overset{o}{\tau}_{\beta p})\overset{o}{\tau}_{\alpha i}\overset{o}{\tau}_{\beta k}\overset{o}{\tau}_{\beta l}, \tag{7.14}$$

$$B_{ijkl} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} I_{\alpha\beta jk}\overset{o}{\tau}_{\alpha i}\overset{o}{\tau}_{\beta l}, \tag{7.15}$$

$$C_{ijklp} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} I_{\alpha\beta jk}\overset{o}{\tau}_{\alpha i}\overset{o}{\tau}_{\beta l}\overset{o}{\tau}_{\beta p}, \tag{7.16}$$

$$\beta_{ij} = -\sum_{\alpha=1}^{n} J_\alpha \overset{o}{\tau}_{\alpha i}\overset{o}{\tau}_{\alpha j}. \tag{7.17}$$

The similar dependences are also obtained for the couple macrostresses $M_{ij}^{(1)}$, which are not presented here due to shortness of space. In the second and higher approximations ($M \geq 2$), the expressions of the above type become rather complicated.

On the whole the dependences (7.13) are a generalization of the Duhamel's equations for the anisotropic media with the couple stresses and scaling effect. Such equations, as far as we know, are obtained for the first time. It is worthy to mention an extremely natural way in which we derived these dependences.

8. Entropy. Heat Flow Equations

It follows from caloric state equations (6.8) and constitutive equations (7.10)-(7.12) that the specific free energy for the linear elastic uniform granular media is defined by the following expression

$$\Phi = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}(A_{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta + E_{\alpha\beta}\mu_\alpha\mu_\beta + I_{\alpha\beta ij}\gamma_{\alpha i}\gamma_{\beta j}) + \sum_{\alpha=1}^{n} J_\alpha\varepsilon_\alpha\Theta + \Lambda(\Theta). \tag{8.1}$$

According to (6.8) we obtain from (8.1)

$$S = -\frac{\partial \Phi}{\partial T} = -\left(\sum_{\alpha=1}^{n} J_\alpha\varepsilon_\alpha + \frac{\partial \Lambda}{\partial T}\right). \tag{8.2}$$

Hence it is seen that $S \equiv S(\varepsilon_\alpha, T)$, i.e.,

$$dS = \sum_{\alpha=1}^{n} \left[\frac{\partial S}{\partial \varepsilon_\alpha}\right]_T d\varepsilon_\alpha + \left[\frac{\partial S}{\partial T}\right]_{\varepsilon_\alpha} dT. \tag{8.3}$$

The dependence (8.2) can be written as follows

$$dS = -\sum_{\alpha=1}^{n} J_\alpha d\varepsilon_\alpha - \frac{d^2\Lambda}{dT^2} dT. \tag{8.4}$$

Combining (8.3) and (8.4) we find

$$\frac{\partial S}{\partial T} = -\frac{d^2\Lambda}{dT^2}. \tag{8.5}$$

The quantity $c_\varepsilon = T\frac{\partial S}{\partial T}\Big|_{\varepsilon_\alpha}$ is the measure of the heat, which is expended for heating or cooling of the unit volume by one degree at a constant microstrain ε_α. This quantity is called a specific heat. According to Dulong's principle, $c_\varepsilon \approx const$ for any temperature T greater than the characteristic temperature T^* (the Debye temperature). For granular media in realistic situations their temperatures $T > T^*$. Therefore one may set $c_\varepsilon = const$. In view of that, we can multiply the equality (8.5) by T and integrate both sides. As a result we obtain

$$-\frac{d\Lambda}{dT} = c_\varepsilon \ln(1 + \frac{\Theta}{T_o}).$$

Upon substitution of this identity into (8.2) we find

$$S = -\sum_{\alpha=1}^{n} J_\alpha \varepsilon_\alpha + c_\varepsilon \ln(1 + \frac{\Theta}{T_o}). \tag{8.6}$$

The temperature T doesn't usually exceed the value T_o by far which implies $|\frac{\Theta}{T_o}| \ll 1$. Therefore $\ln(1+\frac{\Theta}{T_o}) \approx \frac{\Theta}{T_o}$, and the relation (8.6) may be written as follows

$$S = -\sum_{\alpha=1}^{n} J_\alpha \varepsilon_\alpha + c_\varepsilon \frac{\Theta}{T_o}. \tag{8.7}$$

Formula (8.7) yields the final expression for the specific entropy dependence on the elongation microstrains ε_α and temperature increment Θ.

Let us proceed to the derivation of the heat flow equations. We return to the second principle of thermodynamics (6.2) which connects the external heat flow Q with entropy S and temperature T. In the case of the heat conduction, i.e., under heat transport due to a nonuniform temperature distribution in the body, the elementary heat flow during the time dt is defined by formula [12]

$$dQ = -div\vec{q}dt \tag{8.8}$$

where $\vec{q}$ is the heat flux vector. Taking into account the second principle of thermodynamics (6.2) and an identity $div\vec{q} \equiv \frac{\partial q_i}{\partial x_i}$ the expression (8.8) takes on the form

$$T\frac{\partial S}{\partial t} = -\frac{\partial q_i}{\partial x_i} \tag{8.9}$$

From (8.6) follows

$$T\frac{\partial S}{\partial t} = -T\sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t} + c_\varepsilon \frac{\partial T}{\partial t}. \tag{8.10}$$

Comparing (8.9) and (8.10), we obtain

$$\frac{\partial q_i}{\partial x_i} - T\sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t} + c_\varepsilon \frac{\partial T}{\partial t} = 0. \tag{8.11}$$

According to the Fourier law of heat conduction, the vector q_i satisfies the equations

$$q_i = -\lambda_{ij} \frac{\partial T}{\partial x_j} \tag{8.12}$$

where λ_{ij} are the components of the symmetric tensor of thermal conductivity. Eliminating q_i from equations (8.11) and (8.12), we arrive at the following dependence

$$\lambda_{ij} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} + T\sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t} - c_\varepsilon \frac{\partial \Theta}{\partial t} = 0. \tag{8.13}$$

Eq. (8.13) is nonlinear. Assuming that $\left|\frac{\Theta}{T_o}\right| \ll 1$ (which we did earlier) we may set in (8.13) $T \approx T_o$. This would result in the linearized version of (8.13)

$$\lambda_{ij} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} + T_o\sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t} - c_\varepsilon \frac{\partial \Theta}{\partial t} = 0. \tag{8.14}$$

There may be some heat sources in a granular body. In this case, the heat flow equation (8.14) is transformed to the form

$$\lambda_{ij} \frac{\partial^2 \Theta}{\partial x_i \, \partial x_j} + T_o\sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t} - c_\varepsilon \frac{\partial \Theta}{\partial t} + H = 0 \tag{8.15}$$

where H denotes the density of the heat sources in the body. The term $T_o \sum_{\alpha=1}^{n} J_\alpha \frac{\partial \varepsilon_\alpha}{\partial t}$ connects the temperature field $\Theta(x_i, t)$ with a field of microstrains $\varepsilon_\alpha(x_i, t)$.

9. Basic Equations of Thermoelasticity for Granular Media

In an adiabatic process the entropy $S = const$. Without any loss of generality this constant can be set to be 0. Eq. (8.7) is then reduced to the form

$$\Theta = \frac{T_o}{c_\varepsilon} \sum_{\alpha=1}^{n} J_\alpha \varepsilon_\alpha. \tag{9.1}$$

This equation replaces the heat flow equation (8.15) and relates the temperature increment Θ to microstrains ε_α in adiabatic processes. Substituting (9.1) in (7.10) we obtain

$$p_\alpha = \sum_{\beta=1}^{n} \varepsilon_\beta \left(A_{\alpha\beta} + \frac{J_\alpha T_o}{c_\varepsilon} J_\beta\right). \tag{9.2}$$

The expressions (9.2) replace now the first group, Eq. (7.10), of the constitutive equations (7.10)-(7.12) for adiabatic processes.

Thus we have obtained the closed system of equations for determining the stressed-strained and thermal states of the granular media in both arbitrary and adiabatic processes. This system incorporates the following equations:

1. **For an arbitrary process** :

1.1. **The kinematic equations** (2.7), (2.11), (2.16), relating the microstrains ε_α, μ_α, $\gamma_{\alpha i}$ to the displacements u_i, ϕ_i. In total one has $n+n+3n = 5n$ such equations and $5n+3+3=5n+6$ unknowns.

1.2. **The dynamic equations** (4.8), (4.9), connecting the microstresses p_α, m_α, $t_{\alpha i}$ to the external forces F_i and displacements u_i. In total one has 3+3=6 such equations and $n+n+3n = 5n$ new unknowns.

1.3. **The physical equations** (7.10)-(7.12), relating the microstresses p_α, m_α, $t_{\alpha i}$ to the microstrains ε_β, μ_β, $\gamma_{\beta j}$ and temperature increment Θ. There is only one new unknown here and $n+n+3n = 5n$ equations.

1.4. **The thermal equation** (8.15) that contains no new unknowns.

Thus we have $10n+7$ equations and $10n+7$ unknowns . In addition the mechanical fields and the temperature field are coupled.

2. **For an adiabatic process** :

2.1. **The kinematic equations** remain unchanged (see 1.1).

2.2. **The dynamic equations** also remain unchanged (see 1.2).

2.3. **The physical equations** are Eqs. (7.11), (7.12) and (9.2).

These $10n+6$ equations contain $10n+6$ mechanical unknowns which do not depend on the temperature increment Θ. The latter is evaluated with the help of equation (9.1) after determining the unknowns of mechanical origin ε_α.

The above equations have to be supplemented by the mechanical boundary conditions (4.10), (4.11). The thermal boundary conditions have to be formulated, as well. Furthermore, one mast include into the set of boundary conditions the initial conditions for both mechanical and thermal variables. The formulation of these conditions represents a topic of a specific character and therefore they will be addressed in the sequence to this study "Applications of the theory", Part 2.

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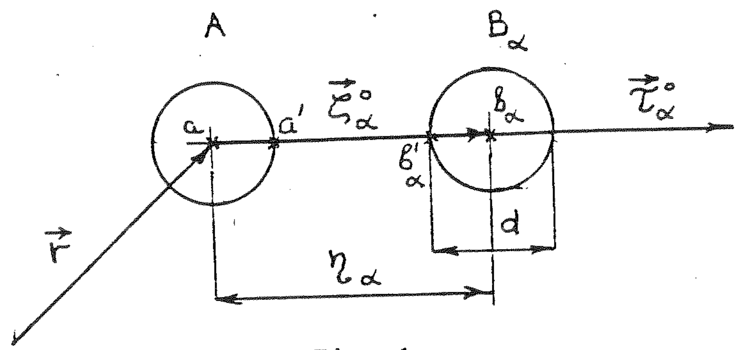

Fig. 1

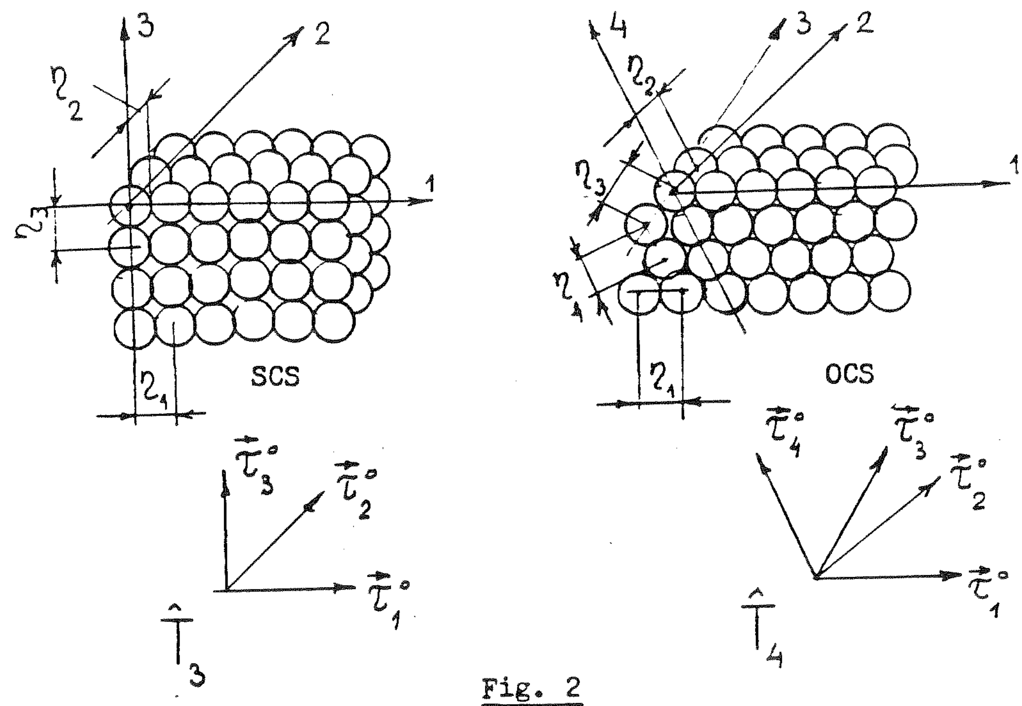

Fig. 2

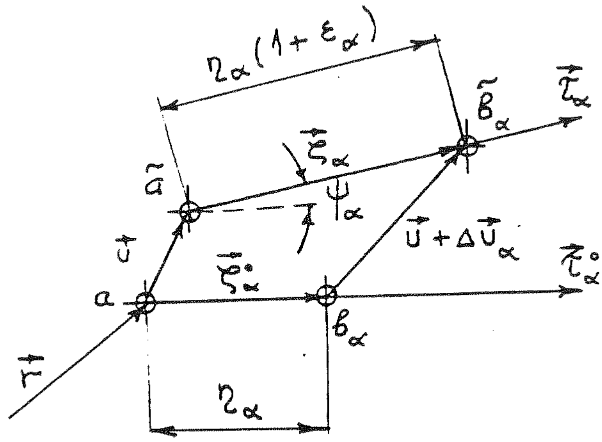

Fig. 3

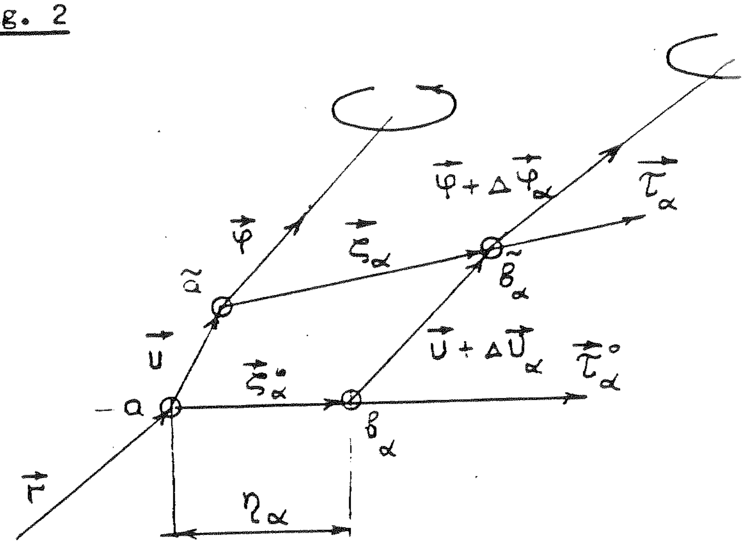

Fig. 4

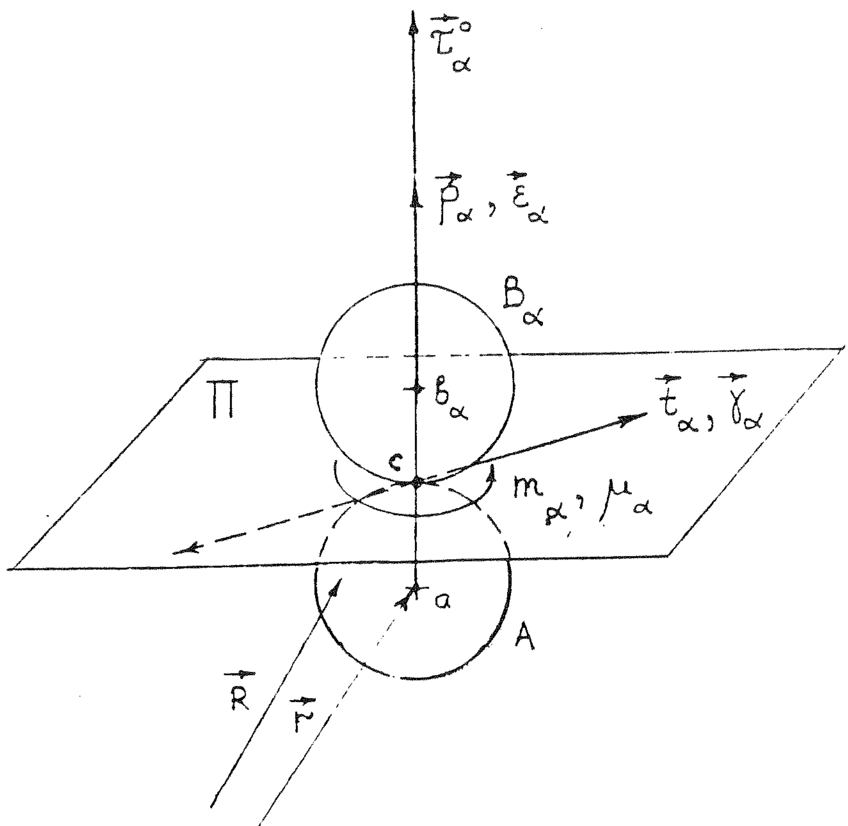
$\vec{\tau}^{\circ}_{\alpha}$
$\vec{p}_{\alpha}$, $\vec{\varepsilon}_{\alpha}$
B_{α}
Π
b_{α}
$\vec{t}_{\alpha}$, $\vec{\gamma}_{\alpha}$
c
m_{α}, μ_{α}
a
A
$\vec{R}$
$\vec{r}$

Fig. 5