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Radial transport of fluctuation energy in a two-field model of drift-wave turbulence

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A theory of spatial propagation of turbulence, referred to as turbulence spreading, is developed for the two-field model of drift wave turbulence. Markovian closure expressions for the flux of kinetic and internal fluctuation energies are systematically derived. Simplified closure expressions are used to obtain two coupled reaction-diffusion equations for kinetic and internal energy. The efficacy of various nonlinear interaction mechanisms for spreading is analyzed systematically. Spreading of internal energy is predicted to "lead" that of kinetic energy. The important role of zonal flow damping in spreading is identified, but zonal flows are shown *not* to be the dominant agents of turbulence spreading. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2180668]

I. INTRODUCTION

A. Motivation

Anomalous transport remains a critical problem for magnetic fusion theory today.[1,2] The traditional approach to the problem of calculating turbulent transport fluxes is based on local stability and local mixing length estimates of saturated fluctuation levels and transport. This paradigm of *local* mixing and transport, first advanced by Kadomtsev,[3] necessarily ties the fluctuation levels and transport at a particular radius to the *local* gradient, which sets the *local* stability criterion. Thus, for example, ion temperature gradient driven (ITG) turbulence is usually expected to appear only in regions where the $\nabla T/T$ exceeds a certain critical local value. However, there are now several observations, in both simulations[4,5] and experiment,[6] of turbulence appearing in regions of the plasma, which are predicted to be *stable*. Such observations of "turbulence spreading" suggest that nonlinear interactions (and possibly linear wave propagation) can transport fluctuation energy in radius, and into locally stable regions (see Fig. 1). This transport of fluctuation energy in turn redistributes the profile of local transport activity by modifying the profile of the effective local transport coefficient, and so is classified as a "nonlocal transport phenomenon." The spatial transport of fluctuation energy has been called turbulence spreading.

Turbulence spreading is one example of a *mesoscale transport process*. Mesoscale or mesoscopic phenomena occur on scales ℓ such that $\Delta r_c < \ell < L$, where Δr_c is the turbulence correlation length and L is a gradient scale length or the system size. Thus, mesoscale phenomena all involve the collective interaction or cooperation of localized sites of turbulence. Turbulence spreading, avalanches [associated with the self-organized criticality (SOC) paradigm], transport barrier advance and retreat, pulse propagation, profile relaxation oscillations and even edge localized modes (ELMs) are *all* examples of mesoscale phenomena. It is particularly instructive to situate turbulence spreading in the taxonomy of mesoscopic phenomena by comparing it to the better known processes of avalanches and transport barrier evolution. Generally speaking, an avalanche is a propagating "pulse" of intense transport with extent in the range of mesoscales. Such a pulse necessarily involves *both* a perturbation in the local gradient and increase in the local fluctuation intensity, each of which are locally of finite duration. Most avalanche and pulse propagation models extend quasilinear approaches to describe the evolution of the *gradient perturbation.* The theory of turbulence spreading is concerned with *fluctuation intensity profile evolution.* Obviously, a description of a physical avalanche requires one to account for *both* effects in the transport pulse. Turbulence spreading theory is concerned with instances where the profiles are relatively stiff, so that intensity transfer is more prominent than the profile perturbation. Similarly, it should be apparent that "turbulence spreading" and "retreat of an internal transport barrier at the back transition" are really one and the same phenomena, since barrier retreat necessarily implies fluctuation advance. Thus, we see that turbulence spreading is an integral part of the zoology of mesoscale phenomena.

A key implication of turbulence spreading theory, and "nonlocality phenomena" in general, is that the *radial profile of the turbulence intensity should be considered as an integral part of the "answer" to the anomalous transport problem.* This is because spreading enters via smearing or delocalizing the relation of *the turbulence intensity profiles* to the *plasma profiles*, such as the profiles of temperature, density, etc. Just as deposition profiles (i.e., heating, fuelling profiles) "drive" the plasma profiles, so the plasma profiles in turn "drive" the intensity profile. In simple terms, turbulence spreading introduces spatial mixing or transport into this relation. Such spatial mixing is due to turbulent transport of fluctuation energy, but also may involve wave propagation

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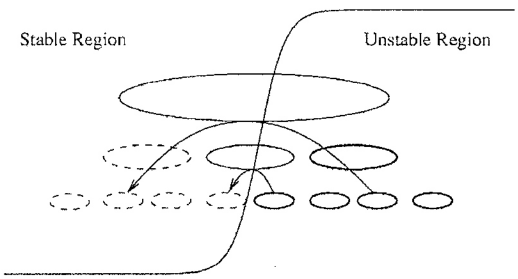

FIG. 1. Spreading can occur via nonlinear mode couplings. Both the inverse cascade and the forward cascade (usually of different quantities) are important for the spreading to be substantial. The inverse cascade of energy in the unstable region may result in radially elongated convective cells that spread the internal energy very effectively. The internal energy then cascades forward and gets damped in the stable region. If the nonlinear transfer rate is faster than the damping rate, turbulence can accumulate in the stable region.

effects etc. In multifield systems transport need not be diffusive but may also involve "pinch" effects. We emphasize, though, that the main effect of turbulence spreading on transport is via alteration of the relation between the local gradients and the intensity profile, and *not* due to wave transport or direct losses of fluctuation field energy.[7]

Turbulence spreading in magnetically confined plasmas was first discussed by Garbet *et al.*,[8] who compared the efficacy of spreading via nonlinear coupling with that via linear coupling of poloidal harmonics due to toroidicity effects. Following a surge of interest in avalanches and self-organized criticality, a nonperturbative bivariate Burger's equation model of spreading was proposed.[9,10] This model described the co-evolution of the turbulence population density, radially (in space), and as a function of the poloidal wave number k_θ (in wave number space), and thus constituted a simple model of turbulence spreading as well as spectral cascade. However, since this model was based on symmetry properties (à la the Ginzburg-Landau theory of second order phase transitions) it was not amenable to qualitative predictions. Recently, the effect of realistic geometry and zonal flows (at the expense of realistic nonlinear couplings) was investigated,[11] which seems to conclude that zonal flows are essential in spreading. Notice that our paper contradicts with this conclusion.

Also, more recently, a Fokker-Planck-type model of intensity transport was applied to the spreading problem.[12–14] This model, which was very much in the vein of a $K-\epsilon$ model of fluid turbulence,[15] described the evolution of $\varepsilon(x,t)$, the turbulence intensity field, using a reaction diffusion equation similar in structure to the well-known Fisher equation.[16,17] Here the "reaction" was spatially profiled growth and nonlinear dissipation (i.e., intensity dependent energy transfer to small scale dissipation), and "diffusion" was nonlinear interaction-induced spatial scattering of intensity [i.e., here $D=D(\varepsilon)$]. The model illustrated many aspects of spreading dynamics, most notably the possibility of nondiffusive front propagation from the unstable region into the stable region. While the predictions of this model correlate well with several simulation results, it had two notable limitations. First, it tacitly assumed quasi-Gaussian fluctuations and the existence of the second moment of the probability distribution function (pdf) of transport event size. The latter is formally required for the applicability of Fokker-Planck theory. Second, the model treated the fluctuation energy $\varepsilon(x,t)$ as a single field, lumped sum and did not distinguish between kinetic and internal fluctuation energy, etc.

In fact, based on experience with passive scalar advection, one may argue that the statistics are ultimately non-Gaussian[18,19] and so one must consider the distribution of flight times and step sizes in order to resolve the power law tails of the probability distribution function. This approach leads to a fractal kinetic description[20,21] of the evolution of turbulence intensity. Such a description is generally nondiffusive. The problem with this approach is that it requires the distributions of flight-times and step-sizes as *input to the calculation*, instead of predicting them *from* the theory. Determination of those distributions is tantamount to solution of the problem, and almost always requires direct numerical simulations (DNS).[22] This "chicken and egg" impasse must be short-circuited if we are to develop any intuition for the spreading process in a complex multifield system. For example we already learned that a diffusion equation with additional terms and a nonlinear diffusion coefficient does not necessarily imply "diffusive" transport of fluctuation energy. It is possible to obtain a multitude of behaviors, including ballistic spreading, from such nonlinear reaction-diffusion equations. From a practical point of view, it is also useful to construct a single length scale and quantify the effect of spreading using this "nonlocality length."[23] In order to make progress in this direction, a model that can *estimate* (even if only roughly) the dominant length scales involved in the problem, rather than one that asks them as input, is required.

Upon proceeding from first principles, one notes that, even for a simple, two-field model of drift wave turbulence (such as the Hasegawa-Wakatani model) and even when only "direct interactions" are calculated in the closure, a full description of nonlocal mode coupling involves *four equations*, with a concomitantly larger number of nonlinear diffusion and drag coefficients. Therefore in practice, this "more accurate" description is nearly intractable. In this paper, we instead give a simple derivation that demonstrates how to reconcile the practical desire for simpler models with the requirement of a description appropriate to the multifield character of the problem. In particular we will consider the model equation

$$\frac{\partial \varepsilon}{\partial t} + v_g \frac{\partial \varepsilon}{\partial x} - \frac{\partial}{\partial x}\left(D_0 \varepsilon^{\alpha}(x,t) \frac{\partial \varepsilon}{\partial x} \right) = \gamma(x)\varepsilon - \gamma_{\mathrm{NL}}(x)\varepsilon^{\alpha+1},$$

and present a direct derivation of this from a simple, two-fluid model of drift wave turbulence. Here ε is the energy density, v_g is the radial group velocity, D_0 is the diffusion coefficient, γ and γ_{NL} are linear growth and nonlinear damping rates. This model describes weak turbulence for $\alpha=1$ and strong turbulence for $\alpha=1/2$. We will show when and how this single equation accurately describes the turbulent spreading of energy. We do this by proposing a *two-field* spreading model, appropriate to the weak turbulence limit of the Hasegawa-Wakatani model. The two-field model extends the single equation model given above and describes the co-evolution of kinetic and internal energy using the coupled reaction-diffusion equations:

$$\frac{\partial}{\partial t}K + v_{gx}\frac{\partial}{\partial x}K - \frac{\partial}{\partial x}\left(D_1 K\frac{\partial}{\partial x}K\right) = \gamma(\beta N + (1-\beta)K) - \gamma_{\rm NL}K^2,$$

$$\frac{\partial}{\partial t}N + v_{gx}\frac{\partial}{\partial x}N - \frac{\partial}{\partial x}\left(D_2 N\frac{\partial}{\partial x}K\right) - \frac{\partial}{\partial x}\left(D_3 K\frac{\partial}{\partial x}N\right) = \gamma(\beta K + (1-\beta)N) - \gamma_{\rm NL}N^2.$$

Here K is the kinetic energy and N is the internal energy, D_i's are various coefficients that define the strengths of various diffusive nonlinear processes in nondimensionalized forms, and α is a coupling coefficient describing the linear coupling between N and K in the Hasegawa-Wakatani model. From the two-field model, we conclude that in almost all cases of interest, the spreading of N "leads" and K follows (slightly behind) for the weak turbulence regime. Also, in most cases of interest, the "nonlocality length" associated with N (the amount of internal energy overshoot in the steady state) is slightly larger than that of K. These are testable predictions, which will ultimately decide the validity and applicability of the suggested model, and the assumptions leading to it. We believe that the tendency of N to spread faster than K is a manifestation of the corresponding spectral transport dynamics of internal and kinetic energy, respectively. Dual cascade, where the kinetic energy couples to larger scales, while the "passive" scalar energy flows to smaller scales is a well known result in 2D fluid turbulence (and the Hasegawa-Wakatani model[24]). This implies that internal energy (i.e., $\langle \tilde{n}^2/n_0^2\rangle$) mixes faster than the kinetic energy or enstrophy. This fact manifests itself in two ways. First, the dissipation of the N field will be dominated by the small scales and thus cause it to behave diffusively, as assumed in the simple model. Second, since the kinetic energy (hence the flow) tends toward large scales, it can "spread" the N field more effectively than simple diffusion (possibly also leading to large scale Levy flights and non-Gaussian statistics). Both of these effects will preferentially spread internal energy, instead of kinetic energy. Notice that both the inverse cascade (and large scale structures as effective spreaders) and the forward cascade (and the small scale dynamics as "spreadees") are needed for effective spreading as depicted in Fig. 1. Notice that unlike previous work,[11,25] the model that we suggest, reduces correctly to the local paradigm in the limit of no spreading. Note also that Naulin *et al.*[25] include the effect of an accumulated $\bar{n}$ as a possible cause of the propagation of the instability boundary (as in penetrative convection), whereas our model deals in the propagation of turbulence itself, once the stability boundary is set (as in turbulence overshoot).

It is important to state why we think the Hasegawa-Wakatani model is a good model for the study of turbulence spreading. The primary reasons are of course its simplicity and the fact that it contains the basic ingredients, including internal instability drive, necessary for the discussion of drift wave turbulence. Another reason is the fact that even though the three-wave interaction driven radial flux of energy or enstrophy can be shown to vanish in the Hasegawa-Mima model (see Sec. V A below), the Hasegawa-Wakatani model removes this degeneracy and allows turbulence spreading via nonlinear wave coupling. In addition, inhomogeneities can be incorporated into the Hasegawa-Wakatani system, in a two-scale sense, by making the background profiles functions of the slow spatial scale ΔX [i.e., $L_n \sim L_n(X)$, $c\sim c(X)$ or $\nu = \nu(X)$ etc.]. This implies that the question of instability in this "locally" Hasegawa-Wakatani system is no longer a question of a global nature since the "global" scale in this two-scale approach is replaced by the "cell" scale ΔX. Here ΔX is the length scale corresponding to the slowly varying spatial scale, which in turn is represented in the final model as an inhomogeneous growth rate (for the adiabatic limit with $\chi \sim \nu$) as

$$\gamma \equiv \gamma(X) \approx \frac{L_{n0}^2}{L_n^2(X)c(X)}\frac{k_y^2k^2}{(1+k^2)^3} - \nu(X)k^2,$$

where L_{n0} is the global average of the scale length corresponding to the background density gradient. This second scale, ΔX, also corresponds to the scale at which the wave envelopes are modulated (with an ordering $L_n \gg \Delta X \gg k^{-1}$). Thus the expansion parameter for the two-scale approach (i.e., $k\Delta X$) is in fact set by the changes in the background profiles.

Notice that depending on the ordering of time scales between the nonlinearities and the large-scale inhomogeneities, global linear eigenmode solutions might possibly be unimportant. Here, since we assume linear growth and nonlinear damping occur at a rate faster than the spreading rate (in other words two spatio-temporal scales corresponding to the wave and the envelope are taken as disparate), the spatial structure of the linear eigenmode is mixed by nonlinear couplings much faster than a global eigenmode forms (in other words $\tau_c > \tau_{\rm spr} > v_{gx}/L_n$). In this limit, the stability condition is effectively local (i.e., in X).

One could of course consider the opposite limit of small box size, where no scale separation is possible. Notice that this case does not correspond to the state in the tokamak where profiles evolve on a large scale. In that case, "spreading" would manifest itself as the excitation of (low k) global eigenmodes, that are regularly damped in the linear theory (i.e., those that are symmetric with respect to the local stability boundary), and have nonvanishing intensity in regions where the local gradient is subcritical. Coupling to such "damped eigenmodes," is a feature of the theory of drift-wave turbulence in the hydrodynamic limit[26] (i.e., low k). We have included the coupling to the damped mode in our calculation of the nonlinear diffusion coefficient in the hydrodynamic limit, though we focused on the two-scale case, as this seems to be of greater interest in the context of MFE.

In this paper, we derive the two-field model (introduced above) using a two-scale weak turbulence theory (i.e., the weak turbulence limit of the two-scale direct interaction approximation[27] developed in order to treat second order moments when the mean flows are either weak or absent), of the

Hasegawa-Wakatani system. The method outlined here can, in principle, be used for any other multifield system within any closure framework (e.g., resonance broadening or EDQNM instead of weak turbulence). We also impose the additional simplifying assumption that the cross-correlation between fields is small, so only linear coupling between N and K results. This leads to simplified coupled equations, which can be solved analytically for a saturated state. The diffusion coefficients given above (such as D_3) are in fact functionals of the entire turbulence spectrum. Therefore, if the turbulence consists of different types of modes (e.g., zonal flows and drift waves), the coefficients include the spreading due to *all* these different types of modes (e.g., both the zonal flows and the drift waves). However one can, in principle, separate various classes of fluctuations and study the "spreading effect" of one type of structure on the other. Notice that it is important to clarify that we consider *only* the net spreading effect, as there are surely other effects. One such case, for example, is the generation of the zonal flow by the drift waves+the damping effect by the zonal flow on the drift waves.[28] These two effects cancel one another when summed over the whole spectrum (since total energy is conserved). Some of these effects may even cause spatial modifications of the turbulence profile. However since the "net" effect of all such couplings vanishes when summed over the entire spectrum, they are neglected. This is so, even when the action of one mode on the other, rather than the total spreading, is considered.

The remainder of the paper is organized as follows. Section II is intended as background. First, the Hasegawa-Wakatani system is introduced and the linear dispersion relation and its solutions are given for important limiting cases. Then, the conservation laws in Poynting's form are given and the kinetic and internal energy fluxes are defined. Section II ends with a brief review of the closure theory and the Markovian assumption. In Sec. III, a method of computing the fluctuation energy fluxes based on a two-scale version of the weak turbulence theory is outlined and a derivation of the energy flux is presented. In Sec. IV, the fluctuation energy fluxes are computed in a general way using this two-scale methodology, and the resulting model is introduced in a general manner. In Sec. V, an extensive study of various limiting cases, such as adiabatic and hydrodynamic limits, and various different types of interactions, such as those between a zonal flow or a streamer and drift waves is performed and the diffusion coefficients for each of these cases are calculated. Section VI discusses the results obtained by numerical integration of the two-field model, and its correspondence to the previous one-field model. Section VII contains results and conclusions.

II. BACKGROUND

The Hasegawa-Wakatani model, which describes the evolution of "dissipative" drift waves can be written as

$$(\partial_t + \hat{\mathbf{z}} \times \nabla\Phi \cdot \nabla)\nabla^2\Phi - \nu\nabla^4\Phi = -\mu^{-1}\nabla_\parallel^2(\Phi - n), \qquad (1a)$$

$$(\partial_t + \hat{\mathbf{z}} \times \nabla\Phi \cdot \nabla)n + \partial_y\Phi - \chi\nabla^2 n = -\mu^{-1}\nabla_\parallel^2(\Phi - n). \qquad (1b)$$

The usual dimensionless drift wave variables are used, so that

$$\Phi \to \frac{e\phi}{\epsilon_* T_e}, \quad n \to \frac{n_{i1}}{\epsilon_* n_0}, \quad t \to \Omega_i \epsilon_* t,$$

$$\mathbf{x} \to \mathbf{x}/\rho_s, \quad \epsilon_* = \frac{\rho_s}{L_n} = \frac{v_*}{c_s}, \quad \mu^{-1} = \frac{\tau_{ei} T_{i0} \rho_s}{m_e c_s \epsilon_*},$$

$$\nu \to \frac{\nu}{\rho_s^2 \Omega_i \epsilon_*}, \quad \chi \to \frac{\chi}{\rho_s^2 \Omega_i \epsilon_*}.$$

Here T_e is the electron temperature, n_0 is the background density, n_{i1} is the fluctuation density of ions, ϕ is the electrostatic potential, Ω_i is the ion cyclotron frequency, ρ_s is the ion Larmour radius at electron temperature, and τ_{ei} is the electron-ion collision frequency, so that μ can be considered as a dimensionless resistivity. Notice that diffusion of the density term (i.e., χ) does not appear in the original set of equations.[29]

Taking the Fourier transforms of (1a) and (1b) yields

$$\left(\partial_t + \nu k^2 + \frac{c}{k^2}\right)\Phi_{\mathbf{k}} - \frac{c}{k^2}n_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}=-\mathbf{k}} \hat{\mathbf{z}} \times \mathbf{q} \cdot \mathbf{p}\frac{(q^2-p^2)}{k^2}\Phi_{-\mathbf{q}}\Phi_{-\mathbf{p}},$$

$$(\partial_t + \chi k^2 + c)n_{\mathbf{k}} + (ik_y - c)\Phi_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}=-\mathbf{k}} \hat{\mathbf{z}} \times \mathbf{q} \cdot \mathbf{p}(n_{-\mathbf{p}}\Phi_{-\mathbf{q}} - \Phi_{-\mathbf{p}}n_{-\mathbf{q}}),$$

where $c \equiv \mu^{-1}k_\parallel^2$. This set of two ordinary differential equations can be written compactly in the form:

$$\partial_t \eta_{\mathbf{k}}^\alpha + H_{\mathbf{k}}^{\alpha\beta}\eta_{\mathbf{k}}^\beta = \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}=-\mathbf{k}} M_{-\mathbf{k},-\mathbf{p},-\mathbf{q}}^{\alpha\beta\gamma}\eta_{-\mathbf{p}}^\beta\eta_{-\mathbf{q}}^\gamma$$

with

$$\eta_{\mathbf{k}}^\alpha \equiv \begin{pmatrix}\Phi_{\mathbf{k}} \\ n_{\mathbf{k}}\end{pmatrix}, \quad H_{\mathbf{k}}^{\alpha\beta} \equiv \begin{pmatrix}\nu k^2 + c/k^2 & -c/k^2 \\ ik_y - c & \chi k^2 + c\end{pmatrix}$$

and

$$M_{\mathbf{k},\mathbf{p},\mathbf{q}}^{\alpha\beta\gamma} \equiv \left[\delta^{\alpha 1}\delta^{\beta 1}\delta^{\gamma 1}\frac{(q^2-p^2)}{k^2} - \delta^{\alpha 2}\epsilon^{\beta\gamma}\right]\hat{\mathbf{z}} \times \mathbf{q} \cdot \mathbf{p},$$

where $\delta^{\alpha\beta}$ is the Kronecker delta and $\epsilon^{12}=-\epsilon^{21}=1$. Notice that M is symmetrical with respect to the joint exchange $(\mathbf{p},\beta) \leftrightarrow (\mathbf{q},\gamma)$ as it should be.

The linear theory of the Hasegawa-Wakatani system is well known, and is contained in the dispersion relation (see the Appendix A for details)

$$\omega^2 + i\omega[c(1+1/k^2) + (\chi+\nu)k^2] - c(\nu k^2 + \chi) - \nu\chi k^4 - ick_y/k^2 = 0. \qquad (2)$$

A useful form for the frequency that passes smoothly and explicitly from the adiabatic limit to hydrodynamic limit is the general expression for the $\chi = \nu$ case:

$$\omega_{\mathbf{k}}^{(r)} = \pm \frac{k_y}{(1+k^2)\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{16k^4k_y^2}{c^2(1+k^2)^4}}}},$$

where the "adiabaticity" depends on whether the term involving c in the above expression is greater or less than 1 (i.e., adiabatic for $c \gg 1$ and hydrodynamic for $c \ll 1$).

III. A TWO SCALE WEAK TURBULENCE THEORY FOR THE SPATIAL EVOLUTION OF SECOND ORDER MOMENTS

Yoshizawa's two scale direct interaction approximation (TSDIA) (Ref. 27) provides a systematic framework for describing the large scale, slow evolution of mean flows in the presence of inhomogeneous turbulence. His method is a two-scale generalization of the direct interaction approximation (DIA),[30,31] based on an assumption of scale separation between turbulent scales and those associated with mean flows. He postulates that direct three wave interactions occur locally, while being modulated by the large scale dynamics, and advected by the mean flows. Here we will consider a Markovian (see Ref. 32 for discussion on realizability) TSDIA, in the limit of weak turbulence, and weak mean flows.

On the other hand, the relevant formalism for large scale, slow evolution of second order moments such as energy or intensity (with little or no mean flow), is the wave kinetic equation or Landau equation:

$$\frac{\partial N(x,k,t)}{\partial t} + \frac{\partial \Omega}{\partial k}\frac{\partial N(x,k,t)}{\partial x} - \frac{\partial \Omega}{\partial x}\frac{\partial N(x,k,t)}{\partial k} = C(N), \quad (3)$$

where $N(x,k,t)$ is the wave action, Ω is some renormalized angular frequency, which includes the effects of mean-flows and coherent nonlinear dampings due to fluctuations as well as the linear frequency. In particular, if one is interested only in spatial evolution, one can integrate out the wave number dependence of the wave kinetic equation and obtain a conservation law for the relevant quantity of the form

$$\frac{\partial N(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\int \frac{\partial \Omega}{\partial k} N(x,k,t)dk\right] = \int C(N)dk.$$

Here $N(x,t)$ is the integrated wave action density. Since this is a two-scale theory however, the k integral has to have a low-wave number cutoff, which might be regarded as a "source term" for this conservation law. Here we assume "separation of scales," i.e., no flux of energy across the boundary between the large scales and the small scales. One is tempted based on these assumptions to write the flux of the conserved quantity as:

$$\Gamma_{\text{coherent}} \sim \int \frac{\partial \Omega}{\partial k} N(x,k,t)dk.$$

However this is not true, as Γ is not complete. The wave kinetic equation in the form (3) was derived based on a "two-scale" approach.[33,34] In this approach, the collision term is usually not expanded. Instead it is commonly stated that collisions respect the conserved quantity, and as such

$$\int C(N)dk = 0.$$

However, if there are significant spatial inhomogeneities, this simple story is no longer true. Collisions in regions where the wave population density is larger may transport energy into regions where it is not. So, *collisions do not conserve wave action at each point in space, but rather do so only in some averaged or integrated sense*, where the average is over the scale of inhomogeneity of N. In this case, all we can say is:

$$\int\int C(N)dkdx = 0$$

which is always true. The subtlety is to ascertain what, in fact, sets the limits of integration over x. This implies, in general, that

$$\int C(N)dk = -\partial_x \Gamma_{\text{collisional}}$$

so that

$$\Gamma = \Gamma_{\text{coherent}} + \Gamma_{\text{collisional}}.$$

This result means that, unless the "collision" term in the wave kinetic equation is confronted, adiabatic theory cannot be used to properly describe turbulence spreading, apart from that which occurs via simple advection. The method outlined in this paper may be viewed as a way of including the effects of the "collision term" in the wave kinetic equation in the calculation of spreading. Here our starting point is the *exact* conservation laws such as (B1) and (B2). Since these are *exact*, they include *all* the effects (i.e., collisions or incoherent noise as well as coherent damping).

When averaged, these conservation laws yield a new set of conservation laws, which describe the evolution of conserved quantities by the long time dynamics and large scale motions. These average conservation laws include the average effects of collisions and damping. However, since they are integrated over the spectrum, only the "net average effect" of all collisions and dampings can be known. In fact it is even not possible to separate coherent damping from incoherent noise in these "exact" expressions, since the spectral dimension of the initial information is lost. It is clear, however, that such an approach is sufficient for a study of strictly spatial spreading.

Also, a rigorous Wigner distribution function treatment of the Hasegawa-Wakatani system, requires evolution equations for all four conserved quantities. Even though this may be necessary in order to determine the asymptotic form of the spectra, it is unnecessarily laborious for an estimate of the transport of turbulent intensity. Here we instead use a formulation that is simpler than the wave action formulation but contains more than a nonrenormalized (hence "weak" turbulence theory) version of the TSDIA formalism (in the sense of dealing with second order moments), with and without weak mean flows. The two scale field equations can be written in general notation as

$$\partial_t \eta_k^\alpha + H_k^{\alpha\beta}\eta_k^\beta = \frac{1}{2}\sum_{p+q=-k}(M^{\alpha\beta\gamma}_{-k,-p,-q}\eta^\beta_{-p}\eta^\gamma_{-q} - iP^{\alpha\beta\gamma}_{-k,-p,-q}\partial_X\eta^\beta_{-p}\eta^\gamma_{-q} - iP^{\alpha\gamma\beta}_{-k,-q,-p}\eta^\beta_{-p}\partial_X\eta^\gamma_{-q}), \tag{4}$$

where

$$P^{\alpha\beta\gamma}_{k,p,q} \equiv \frac{\partial M^{\alpha\beta\gamma}_{k,p,q}}{\partial p_x}.$$

Notice that (4) is simply the Hasegawa-Wakatani system in our abstract notation, where the amplitudes on the right-hand side are assumed to have slow spatial variation in the radial direction. In Eq. (4), **p** and **q** are wave numbers of the fast spatial fluctuation; however **k** is the full wave number, which includes both the fast spatial fluctuations and the modulations induced by envelope dynamics. In other words, $\mathbf{k} = -\mathbf{p}-\mathbf{q}+\Delta k$, where $\Delta\mathbf{k}$ is the wave number of the envelope. Hence the expressions

$$N_k \equiv \langle \sigma^N_k \eta^{(2)}_k \eta^{(2)}_{-k}\rangle = \frac{1}{2}\langle |n_k|^2\rangle, \tag{5}$$

$$K_k \equiv \langle \sigma^K_k \eta^{(1)}_k \eta^{(1)}_{-k}\rangle = \frac{1}{2}\langle (1+k^2)|\Phi_k|^2\rangle, \tag{6}$$

$$\varepsilon \equiv \sum_{k,\alpha}\langle \sigma^{\alpha\alpha}_k \eta^\alpha_k \eta^\alpha_{-k}\rangle = \sum_k (K_k + N_k)$$

are exact. Once the equation for N_k is written, we will also consider a two-scale expansion for **k**, but the nonlinear term will then be independent of **k** and the expansion of the linear part [say of $H_k^{\alpha\beta}$ in (4)] will simply result in a group velocity term and linear diffusion, if χ and ν are nonzero.

Equations for $N \equiv \Sigma_k N_k$ and $K \equiv \Sigma_k K_k$ could also be constructed from (4) using (5) and (6). The results should be nothing but the statistical averages of (B1) and (B2). Three correlations that appear in these equations for nonlinearly conserved quantities will then be computed via the direct interaction approximation (DIA), using the two-scale equations for the beat mode. Thus

$$\delta\eta^\alpha_k = \int R^{\alpha\lambda}_k(t,t')M^{\lambda\beta\gamma}_{-k,-p,-q}\eta^\beta_{-p}(t')\eta^\gamma_{-q}(t')dt' + i\int R^{\alpha\lambda}_k(t,t')[P^{\lambda\beta\gamma}_{-k,-p,-q}\partial_X\eta^\beta_{-p}(t')\eta^\gamma_{-q}(t') + P^{\lambda\gamma\beta}_{-k,-q,-p}\eta^\beta_{-p}(t')\partial_X\eta^\gamma_{-q}(t')]dt'. \tag{7}$$

Of course for a fully renormalized system, one should use the response function $R^{\alpha\lambda}_k(t,t')$, which satisfies an equation of the form

$$\partial_t R^{\alpha\lambda}_k(t,t') + H^{\alpha\beta}_k R^{\beta\lambda}_k(t,t') + \nu_{NL}R^{\alpha\lambda}_k(t,t') = \delta(t-t')\delta^{\alpha\lambda}.$$

Here, however, we intend to focus primarily on the case of weak turbulence, hence the nonlinear damping as well as linear growth or damping are assumed to be small compared to the wave frequency. More rigorously, it is the smallness of the net damping rate as compared to the mismatch that justifies the weak turbulence approximation.

A contrast with what is usually called the Whitham theory of modulations,[35,36] is also somewhat useful here. The basic method of Whitham modulations consists of casting conservation laws in a Poynting's theorem form, as in (B1), and using basic solutions such as

$$\Phi = A\sin(k_y(y-V_+t))\sin k_x x + \bar{\Phi},$$

$$n = -B\sin(k_y(y-V_+t))\sin k_x x + \bar{P}$$

and using the fact that the conservation law applies to the adiabatic, slow evolution of the parameters of this sine wave solution, such as amplitudes of the sine waves [i.e., $A = A(X,T)$ and $B=B(X,T)$] or the Doppler velocity [i.e., $V_+ = V_+(X,T)$]. This gives slow spatio-temporal evolution equations of equal number to the number of independent conservation laws the system has. These can be solved to describe the modulations of the basic solution under the action of slow inhomogeneities (e.g., the effect of sheared flow on amplitude modulations, etc.). The method of Whitham modulations is based mathematically on a variational formulation. Its limitation is that it describes the evolution of "isolated modes" (or isolated solitons if the system permits them as exact solutions, see for instance, Ref. 37) but does not take mode coupling into account. The method we outline here can be considered as a statistical generalization of this method, especially in the sense that evolution due to mode couplings is accounted for.

A. Derivation of the fluctuation energy flux

In order to derive a spatial evolution equation for energy, the statistical averages of Eqs. (B1) and (B2) must be considered. The only major challenge in writing these average conservation laws is the computation of the third order moments in the flux terms. Notice that it does not matter if we first compute the inhomogeneous evolution of the statistically averaged spectrum (à la the wave kinetic equation) and *then* average out the **k** dependence or if we start with (B1) and (B2) and compute the statistical averages later. This independence from the order of operations is guaranteed by the fact that closure approximations invoked here (such as the DIA) respect the conservation laws.

Since averaging and differentiation commute (i.e., $\langle\partial_X\Gamma_X\rangle = \partial_X\langle\Gamma_X\rangle$), we need to compute the average nonlinear flux terms in order to compute the statistically averaged evolution equations for kinetic and internal energies [using (B1) and (B2)], i.e.,

$$\langle\Gamma_N\rangle = -\left\langle \frac{n^2}{2}\partial_Y\Phi\right\rangle = -\frac{i}{6}\sum_{p+q+k=0}(k_y\Phi_k n_p n_q + p_y\Phi_p n_q n_k + q_y\Phi_q n_k n_p) \approx \sum_{k+p+q=0}\mathrm{Im}([p_y n_q\Phi_p + q_y n_p\Phi_q]\delta n_k - [q_y n_q n_p + p_y n_p n_q]\delta\Phi_k) \equiv \sum_{k+p+q=0}\mathrm{Im}(\Lambda(N)^{\alpha\beta\gamma}_{k,p,q}\eta^\beta_p\eta^\gamma_q\delta\eta^\alpha_k), \tag{8}$$

$$\langle \Gamma_K \rangle = -\left\langle \frac{\Phi^2}{2}\partial_Y \nabla^2 \Phi \right\rangle = \frac{i}{6}\sum_{\mathbf{p+q+k=0}} (k_y k^2 + p_y p^2 + q_y q^2)\Phi_{\mathbf{k}}\Phi_{\mathbf{p}}\Phi_{\mathbf{q}} = -\sum_{\mathbf{p+q+k=0}} \mathrm{Im}[(k_y k^2 + p_y p^2 + q_y q^2)\Phi_{\mathbf{p}}\Phi_{\mathbf{q}}\delta\Phi_{\mathbf{k}}] \equiv \sum_{\mathbf{k+p+q=0}} \mathrm{Im}(\Lambda(K)^{\alpha\beta\gamma}_{\mathbf{k,p,q}} \eta^{\beta}_{\mathbf{p}} \eta^{\gamma}_{\mathbf{q}} \delta \eta^{\alpha}_{\mathbf{k}}), \quad (9)$$

where

$$\Lambda(N)^{1\beta\gamma}_{\mathbf{k,p,q}} = \begin{pmatrix} 0 & 0 \\ 0 & -q_y - p_y \end{pmatrix}, \quad \Lambda(N)^{2\beta\gamma}_{\mathbf{k,p,q}} = \begin{pmatrix} 0 & p_y \\ q_y & 0 \end{pmatrix},$$

$$\Lambda(K)^{1\beta\gamma}_{\mathbf{k,p,q}} = \begin{pmatrix} k_y k^2 + p_y p^2 + q_y q^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda(K)^{2\beta\gamma}_{\mathbf{k,p,q}} = 0.$$

Substituting (7) into (8) and (9) and assuming Markovian evolution, we find the general expression for the flux of a nonlinearly conserved second order moment:

$$\langle \Gamma_{N,K} \rangle = \sum_{\mathbf{k+p+q=0}} \Lambda(N,K)^{\alpha\beta\gamma}_{\mathbf{k,p,q}} \mathrm{Im}\left(\theta^{\mu}_{\mathbf{k,p,q}} r^{\alpha\lambda\mu}_{\mathbf{k}} \left[M^{\lambda\sigma\rho}_{\mathbf{-k,-p,-q}} C^{\beta\sigma}_{\mathbf{p}} C^{\gamma\rho}_{\mathbf{q}} + \frac{i}{2} P^{\lambda\sigma\rho}_{\mathbf{-k,-p,-q}} C^{\gamma\rho}_{\mathbf{q}} (\partial_X C^{\beta\sigma}_{\mathbf{p}} + i J^{\beta\sigma}_{\mathbf{p}}) + \frac{i}{2} P^{\lambda\rho\sigma}_{\mathbf{-k,-q,-p}} C^{\beta\sigma}_{\mathbf{p}} (\partial_X C^{\gamma\rho}_{\mathbf{q}} + i J^{\gamma\rho}_{\mathbf{q}}) \right] \right), \quad (10)$$

where

$$J^{\alpha\beta}_{\mathbf{k}} \equiv i[\eta^{\alpha}_{\mathbf{k}} \partial_X \eta^{\beta}_{\mathbf{-k}} - \eta^{\beta}_{\mathbf{-k}} \partial_X \eta^{\alpha}_{\mathbf{k}}] \sim \Delta k C^{\beta\alpha}_{\mathbf{k}}$$

is the "probability current," which is a nonlinear, two-scale correction to the group velocity term and is negligible in most cases that are considered here (since $\Delta k \ll \{k_x p_x, q_x\}$) and

$$\theta^{\mu}_{\mathbf{k,p,q}} \equiv \int_0^t e^{[-i(\omega^{(\mu)}_{\mathbf{k}} - \omega_{\mathbf{-p}} - \omega_{\mathbf{-q}}) - (\nu^{\mathrm{NL}}_{\mathbf{k}} + \nu^{\mathrm{NL}}_{\mathbf{-p}} + \nu^{\mathrm{NL}}_{\mathbf{-q}})](t-t')} dt' = \frac{1 - e^{-i\Delta\omega^{(\mu)}t + \Gamma^{(\mu)}t}}{-i\Delta\omega^{(\mu)} - \Gamma^{(\mu)}}$$

is the triad interaction time. Notice that in the weak turbulence limit, only $\mathrm{Re}[\theta_{\mathbf{k,p,q}}]$ contributes due to resonances, as it approaches a delta function. Notice also that in general $r^{\alpha\lambda\mu}_{\mathbf{k}}$ in (10) is a matrix with complex elements (see Appendix A).

The method outlined in this chapter is essentially a Markovian two-scale DIA, where the mean flows are weak or nonexistent. This form is especially suitable for applications to second moments, and therefore applicable to fluctuation energy flux calculations. The reason we call this closure a "two-scale" closure is that here, the beat mode is assumed to evolve on two separate spatial scales, namely that of the fluctuations and that of mesoscale modulations [i.e., $\hat{\mathbf{z}} \times \mathbf{p}\cdot\mathbf{q} \to \hat{\mathbf{z}} \times (\mathbf{p} - \mathbf{i}\partial_x)\cdot(\mathbf{q} - \mathbf{i}\partial_x)$], which is the slow spatial scale corresponding to the evolution of the fluctuation energy. In practice we will also assume the wave dynamics dominate near saturation and, so we formally take the weak turbulence limit.

IV. THE FULL MODEL

We aim to consider various limiting cases, namely the adiabatic, near-adiabatic and hydrodynamic limits of the Hasegawa-Wakatani system. However the present framework [i.e., Eqs. (7) and (10)] can be applied to any model, and within the framework of any desired closure approximation. Even though our challenge is to derive minimal analytical models, sacrificing simplicity in some cases is necessary to gain further insight into the dynamics of spreading. Initially, we will consider the full model but assume that the cross-correlations are small. This is consistent with a weak turbulence approximation and necessary to justify the neglect of nonlinear terms with any sort of cross-correlation in them, even though linear couplings induced by the cross correlations are retained. Moreover, we take

$$\frac{\lambda^{(2)}_{\mathbf{k}} - \chi k^2 - c}{\lambda^{(1)}_{\mathbf{k}} - \lambda^{(2)}_{\mathbf{k}}} \approx \frac{\lambda^{(2)}_{\mathbf{k}} - \nu k^2 - c/k^2}{\lambda^{(1)}_{\mathbf{k}} - \lambda^{(2)}_{\mathbf{k}}} \approx -\frac{1}{2}$$

which is true either close to the hydrodynamic limit or for $\chi \sim \nu$ and $k^2 \lesssim 1$, since $\lambda^{(1)}_{\mathbf{k}} - \lambda^{(2)}_{\mathbf{k}} = -2\lambda^{(2)}_{\mathbf{k}} - \chi k^2 - c - \nu k^2 - c/k^2$ (see Appendix A). This assumption makes

$$\theta^{\mu}_{\mathbf{k,p,q}} r^{\lambda\lambda\mu}_{\mathbf{k}} \sim \frac{1}{2}[\theta^{(+)}_{\mathbf{k,p,q}} + \theta^{(-)}_{\mathbf{k,p,q}}]$$

where $\theta^{(+)}_{\mathbf{k,p,q}}$ and $\theta^{(-)}_{\mathbf{k,p,q}}$ are triad interaction times for the cases when all the members of the triad are growing modes, and when one of them is damped, respectively. For the adiabatic and near adiabatic cases we will set $\theta^{(-)}_{\mathbf{k,p,q}}=0$ explicitly, as in those cases the damped mode is *very strongly* damped indeed, so that interactions with the damped mode are negligible.

Given these assumptions, the fluxes become:

$$\Gamma_K \approx \sum_{\mathbf{k+p+q=0}} \pi[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)] \frac{(k_y k^2 + p_y p^2 + q_y q^2)}{k^2 p^2 q^2} \times [(q_y(q^2 - p^2) + 2p_x \hat{\mathbf{z}} \times \mathbf{q}\cdot\mathbf{p}) K_{\mathbf{q}} \partial_X K_{\mathbf{p}} + (p_y(p^2 - q^2) + 2q_x \hat{\mathbf{z}} \times \mathbf{p}\cdot\mathbf{q}) K_{\mathbf{p}} \partial_X K_{\mathbf{q}}], \quad (11)$$

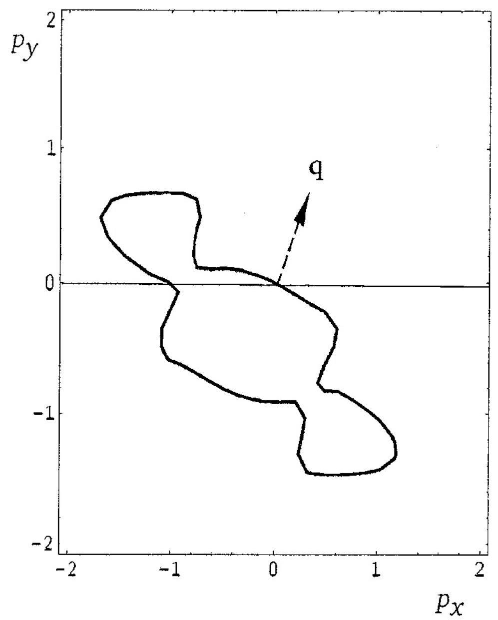

FIG. 2. The resonance manifold for a given value of $\mathbf{q}$ (q_x=0.5, q_y=0.8) and parameters ν=0.2, χ=0.27, and c=1.6 is depicted. The tip of the $\mathbf{p}$ vector that is in resonance with the given $\mathbf{q}$, spans the above curve, where the final wave number of the triad is $\mathbf{k}=-\mathbf{p}-\mathbf{q}$.

$$\Gamma_N \approx \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \pi[\delta(\Delta\omega^+)+\delta(\Delta\omega^-)]\left[\frac{q_y^2}{q^2}K_{\mathbf{q}}\partial_X N_{\mathbf{p}} - \frac{p_y q_y}{q^2}N_{\mathbf{q}}\partial_X K_{\mathbf{p}} + \frac{p_y^2}{p^2}K_{\mathbf{p}}\partial_X N_{\mathbf{q}} - \frac{q_y p_y}{p^2}N_{\mathbf{p}}\partial_X K_{\mathbf{q}}\right]. \quad (12)$$

Notice that Onsager relations do not apply in this case and there is no counterpart of the last term of Eq. (12) in Eq. (11). In order to obtain an answer, we need to compute the sums (or integrals) over the resonance manifold that is defined by the resonance condition $\Delta\omega^{(\pm)}=0$. In general, determining this resonance manifold is a formidable analytical task, except for very simple dispersion relations or various limiting cases. Hence, here we only emphasize that there indeed is such a manifold (see Fig. 2), and that, on this manifold, the "flux coefficients" (i.e., $\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}}$'s) do *not* vanish, in general.

One can then write the flux in the form of a Fick's law

$$\Gamma_K \approx \sum_{\mathbf{p}} D_{\mathbf{p}}^{(KK)}\partial_X K_{\mathbf{p}}, \quad (13)$$

where

$$D_{\mathbf{p}}^{(KK)} \equiv \int\int \frac{(k_y k^2 + p_y p^2 + q_y q^2)}{k^2 p^2 q^2}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)](q_y(q^2-p^2) + 2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p}) \times K_{\mathbf{q}}d^2\mathbf{q}d^2\mathbf{k} \quad (14)$$

and

$$\Gamma_N \approx \sum_{\mathbf{p}} D_{\mathbf{p}}^{(NK)}\partial_X K_{\mathbf{p}} + \sum_{\mathbf{p}} D_{\mathbf{p}}^{(NN)}\partial_X N_{\mathbf{p}}, \quad (15)$$

where

$$D_{\mathbf{p}}^{(NN)} \equiv \int\int \frac{q_y^2}{q^2}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)]K_{\mathbf{q}}d^2\mathbf{q}d^2\mathbf{k}, \quad (16)$$

$$D_{\mathbf{p}}^{(NK)} \equiv -\int\int \frac{q_y p_y}{p^2}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+) + \delta(\Delta\omega^-)]N_{\mathbf{q}}d^2\mathbf{q}d^2\mathbf{k}. \quad (17)$$

In practice, we will further claim [based on Eqs. (13) and (14)] that

$$\Gamma_K \approx D_1 K\partial_X K$$

and based on Eqs. (15)–(17), that

$$\Gamma_N \approx D_2 N\partial_X K + D_3 K\partial_X N.$$

Using these approximations, the general model equations for kinetic and internal energy densities take the form

$$\frac{\partial}{\partial t}K + v_{gx}\frac{\partial}{\partial x}K - \frac{\partial}{\partial x}\left(D_1 K\frac{\partial}{\partial x}K\right) = \gamma(\beta N + (1-\beta)K) - \gamma_{\mathrm{NL}}K^2, \quad (18a)$$

$$\frac{\partial}{\partial t}N + v_{gx}\frac{\partial}{\partial x}N - \frac{\partial}{\partial x}\left(D_2 N\frac{\partial}{\partial x}K\right) - \frac{\partial}{\partial x}\left(D_3 K\frac{\partial}{\partial x}N\right) = \gamma(\beta K + (1-\beta)N) - \gamma_{NL}N^2, \quad (18b)$$

where $D_1\to 0$ as we approach the adiabatic limit, and both D_1 and D_2 vanish for q_y=0. Here, β is a parameter that allows linear coupling between kinetic energy and internal energy (mocking up the effect of the cross-correlation), v_{gx} is the radial group velocity, γ is the linear growth rate ($\gamma \approx 2\langle\gamma_k\rangle$) and γ_{NL} is nonlinear damping. These additional terms are necessary to capture linear dynamics and saturation physics. Notice that the radial group velocity v_{gx} is purely linear here. There are convection-like nonlinear corrections to this term in the general expression (10), however in the limit of weak turbulence, these corrections are neglected.

The physical meanings of the diffusion coefficients can be identified: $D_1 K$, is the diffusion coefficient of kinetic energy, $D_2 N$ is some sort of "stress" on internal energy by the kinetic energy profile, and $D_3 K$ is the diffusion coefficient of internal energy, for which the particle diffusion coefficient D_{GB} can be used as a crude estimate via $D_3 \approx D_{GB}/K$.

This two-field model suggests that near saturation (where $K\approx N$), the one-field model given previously,[12,13] was in fact accurate. Moreover, the fact that we have rigorously derived (18a) and (18b) from the Hasegawa-Wakatani model clarifies the range of validity of these type of spreading models. For instance single field model such as the one given in Ref. 12 is not only valid for the turbulent evolution of a passive scalar (as commonly thought) but also for more complicated reactive systems, such as the Hasegawa-Wakatani system.

This system contains reaction, manifested by its tendency toward local saturation, and diffusion as a result of nonlinear mode coupling, and thus in general is a reaction-diffusion system with nonlinear (and nondiagonal) diffusion.

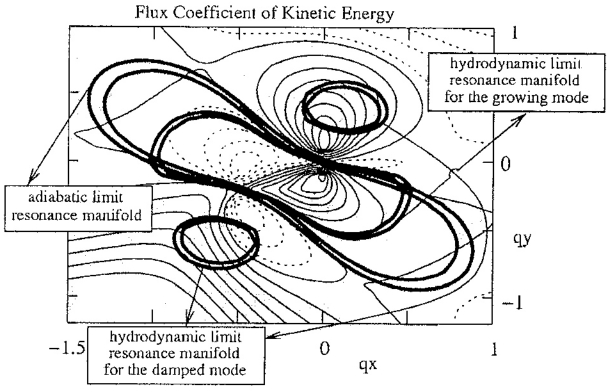

FIG. 3. (Color online) In order to answer the question of sign of D_1, we need to evaluate the integral (14) along the resonance manifold. The figure shows the sign of $(k_yk^2+p_yp^2+q_yq^2)(q_y(q^2-p^2)+2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p})/k^2p^2q^2$ (shaded regions are positive) and the resonance manifolds in the adiabatic limit and the hydrodynamic limit ($c=0.01$) superimposed, for the case $\mathbf{p}=(0.5,0.2)$.

It also has the form of two coupled nonlinear Fisher equations and thus suggests front propagation with more or less constant speed, with possible additional interplay between the two fields. Thus, one might speculate that, if non-Markovian effects are included, the time delay *might* produce cycles, or periodic bursts of transport activity.

Fisher-type reaction-diffusion equations predict that the front propagation begins only after a quasisaturated state is reached. The spreading is subdiffusive initially, before local saturation takes place. Here we do not claim to represent the initial linear growth and nonlinear damping in a quantitatively accurate way, since we model it with a single growth rate and a single nonlinear damping rate. However this guarantees that Eqs. (18a) and (18b) agree with the local saturation paradigm when spreading is neglected. The basis for this is an assumption of modest time scale separation between the time scales associated with local saturation and time scales for which the nonlocal spreading takes place.

One of the essential problems for the model to be useful is the determination of the signs of D_1 and D_2 (as D_3 is positive definite). The crucial point for D_2 is that for any reasonably symmetrical spectral distribution $\langle p_yq_y\rangle<0$, when the average is taken over the resonant modes, thus making $D_2\sim-\langle q_yp_y\rangle>0$. The same is true for the $\langle p_xq_x\rangle$ average as well. However it seems it may be possible to construct asymmetrical distributions where $D_2\lesssim 0$. Similar analyses are not readily available for D_1, which vanishes in most important limits, anyway. See Fig. 3 for a discussion.

V. ENERGY FLUX IN VARIOUS LIMITING CASES

We defined the diffusion coefficient via expressions of the form

$$D_{\mathbf{p}}^{\alpha\beta}\approx\int\int f(\mathbf{p},\mathbf{q},\mathbf{k})d^2\mathbf{q}d^2\mathbf{k}$$

and fluxes via, the Fick's law, where

$$\Gamma_\alpha\approx\sum_{\mathbf{p}}D_{\mathbf{p}}^{\alpha\beta}\partial_XN_{\mathbf{p}}^{\beta}\equiv D^{\alpha\beta}\partial_XN^{\beta}.$$

It is important to symmetrize the diffusion coefficients with respect to $\mathbf{k}$ and $\mathbf{q}$ so that when we fix $\mathbf{q}=\mathbf{q}'$, we also take the $\mathbf{k}=\mathbf{q}'$ contribution into account, as well. This will allow us to restrict $\mathbf{q}$ and $\mathbf{p}$ into regions in spectral space and study spreading processes by the interactions between these spectral regions,

$$D_{\mathbf{p}}^{(KK)}\equiv\int\int\frac{(k_yk^2+p_yp^2+q_yq^2)}{k^2p^2q^2}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+)+\delta(\Delta\omega^-)]\frac{1}{2}[(q_y(q^2-p^2)+2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p})K_{\mathbf{q}}+(k_y(k^2-p^2)-2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p})K_{\mathbf{k}}]d^2\mathbf{q}d^2\mathbf{k},\quad(19)$$

$$D_{\mathbf{p}}^{(NN)}\equiv\int\int\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+)+\delta(\Delta\omega^-)]\times\frac{1}{2}\left(\frac{q_y^2}{q^2}K_{\mathbf{q}}+\frac{k_y^2}{k^2}K_{\mathbf{k}}\right)d^2\mathbf{q}d^2\mathbf{k},\quad(20)$$

$$D_{\mathbf{p}}^{(NK)}\equiv-\int\int\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})[\delta(\Delta\omega^+)+\delta(\Delta\omega^-)]\times\frac{1}{2}\left(\frac{q_yp_y}{p^2}N_{\mathbf{q}}+\frac{k_yp_y}{p^2}N_{\mathbf{k}}\right)d^2\mathbf{q}d^2\mathbf{k},\quad(21)$$

where $N_{\mathbf{p}}^{\beta}=(K_{\mathbf{p}},N_{\mathbf{p}})$, etc. This particular form has the advantage of showing the total contributions of all possible $\mathbf{q}$ and $\mathbf{k}$ modes to the spreading of a single $\mathbf{p}$ mode. Strictly speaking, the energy flux for a particular $\mathbf{p}$ mode is not necessarily equal to the $\mathbf{p}$th component of the above sum. Instead we can write $\Gamma_{\mathbf{p}}^{\alpha}\sim D_{\mathbf{p}}^{\alpha\beta}\partial_XN_{\mathbf{p}}^{\beta}+\delta_{\mathbf{p}}$, where $\delta_{\mathbf{p}}$ denotes "other terms" that vanish when summed over $\mathbf{p}$, and thus do not contribute to the spreading of *total turbulence intensity*. Such terms are neglected even when we talk about the effect of spreading of one part of the turbulence on the other parts. This is because only the flux terms that do not cancel really are the bit that corresponds to spreading. Note that this ansatz is necessary because it is impossible to separate the mechanisms of "spreading" and "nonlinear transfer."

Within this picture, we can ask questions such as, "how much does a $q_y=0$ mode (and a suitably selected 'other mode') spread an arbitrary drift mode?," even though the total diffusion coefficient will include the back-reaction induced spreading by the test modes on the $q_y=0$ modes, as well. Therefore, from this point on, we will refer to $\mathbf{q}$ (and $\mathbf{k}$) "the spreader" and $\mathbf{p}$ "the spreadee," and try to use this convention consistently throughout the paper. Notice that for total spreading, we need ultimately to compute the sum over $\mathbf{p}$. This includes the effects of all the processes (such as a $q_y=0$ mode and a drift wave spreading another drift wave) and their inverses (two drift waves spreading a $p_y=0$ mode).

A. The adiabatic limit

It is well known that the Hasegawa-Wakatani system reduces to the Hasegawa-Mima equation in the adiabatic limit, for which the weak turbulence expression for the flux becomes

$$\langle\Gamma\rangle = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}} \pi\delta(\Delta\omega)(P_{\mathbf{k},\mathbf{p},\mathbf{q}} C_{\mathbf{q}} \partial_X C_{\mathbf{p}} + P_{\mathbf{k},\mathbf{q},\mathbf{p}} C_{\mathbf{p}} \partial_X C_{\mathbf{q}}),$$

where $\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}}$ is either

$$\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}} = k_y k^2 + p_y p^2 + q_y q^2$$

for the energy, or

$$\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}} = -(q_y(1+p^2)(p^2+2\mathbf{q}\cdot\mathbf{p}) + p_y(1+q^2)(q^2 + 2\mathbf{q}\cdot\mathbf{p}))$$

for the enstrophy and

$$P_{\mathbf{k},\mathbf{p},\mathbf{q}} = \frac{q_y(q^2-p^2) + 2p_x\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{p}}{1+k^2}.$$

However the mismatch is

$$\Delta\omega = -\frac{(q_y((1+p^2)(p^2+2\mathbf{q}\cdot\mathbf{p})) + p_y((1+q^2)(q^2+2\mathbf{q}\cdot\mathbf{p})))}{(1+p^2)(1+q^2)(1+k^2)} = 0,$$

so the coefficient of the enstrophy flux in the x direction vanishes *exactly* when the resonance condition is satisfied. In fact,

$$\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}}(W) = (1+p^2)(1+q^2)(1+k^2)\Delta\omega \tag{22}$$

for enstrophy, where $\Delta\omega$ is the wave mismatch.

This is *not* exactly the case for energy. Nevertheless, in the long wavelength limit ($k_\perp\rho_i \ll 1$), which, one might argue, is the most relevant case for drift wave turbulence,

$$\Delta\omega \approx -\frac{(q_y(p^2+2\mathbf{q}\cdot\mathbf{p}) + p_y(q^2+2\mathbf{q}\cdot\mathbf{p}))}{(1+p^2)(1+q^2)(1+k^2)} \approx 0 \tag{23}$$

and

$$\Lambda_{\mathbf{k},\mathbf{p},\mathbf{q}}(\varepsilon) = k_y k^2 + p_y p^2 + q_y q^2 \approx (1+p^2)(1+q^2)(1+k^2)\Delta\omega$$

as well.

These results mean that, energy and enstrophy are conserved "locally" in the Hasegawa-Mima equation, within the framework of weak-turbulence theory. It should be noted that there is also linear dispersion, which can "disperse" a wave packet and destroy its localization. Nonresonant interactions (or resonance broadening) would similarly break down the local conservation constraints. Notice also that introduction of zero-frequency modes, such as zonal flows, or convective cells, might also change this picture. This result is not surprising, and it can be viewed simply as a critique of the application of simple weak turbulence theory to describe the spreading of turbulence in the Hasegawa-Mima model.

B. Near-adiabatic limit

The Hasegawa-Wakatani equation has a special form in the limit of large c (i.e., $c \gtrsim 1$). Assuming $\chi \approx \nu$,

$$\omega_{\mathbf{k}} \approx \frac{k_y}{1+k^2},$$

$$\gamma_{\mathbf{k}}^{(+)} \approx \frac{1}{c}\frac{k_y^2 k^2}{(1+k^2)^3} - \nu k^2 - \nu_{\mathrm{eddy}},$$

$$\gamma_{\mathbf{k}}^{(-)} \approx -c\frac{(1+k^2)}{k^2} - \frac{1}{c}\frac{k_y^2k^2}{(1+k^2)^3} - \nu k^2 - \nu_{\mathrm{eddy}}, \tag{24}$$

where ν_{eddy} is the eddy damping rate, which is an ad hoc coefficient describing the effect of higher order moments on the second order, included here for flexibility. Notice here that even though the growing mode is weakly growing, the damped mode is actually strongly damped ($\gamma_{\mathbf{k}}^{(-)} \sim -c$). This is in contrast to the hydrodynamic limit, where the two modes grow or damp at approximately equal rates. Based on this observation, we will completely neglect the damped mode in the near-adiabatic limit. Since the dispersion relation is still the same, this implies $D_{\mathbf{p}}^{(KK)} = 0$. We will compute the rest of the diffusion coefficients in various limiting cases.

An important point to note is that, as we shall see below, in all the cases considered either $D_{\mathbf{p}}^{(KK)} \sim D_{\mathbf{p}}^{(NK)} \sim 0$ when $D_{\mathbf{p}}^{(NN)}$ is finite, or $D_{\mathbf{p}}^{(NN)} \gg D_{\mathbf{p}}^{(NK)}$. This indicates that the internal energy spreads diffusively, with a diffusion coefficient proportional to the kinetic energy. Thus one can speak of a "spreader" (the flow) and a "spreadee" (the density). This tendency remains prevalent as long as $D_3 > (D_1 - D_2)$, which is very easy to satisfy, since for almost all $\mathbf{p}$: $D_{\mathbf{p}}^{(NN)} > (D_{\mathbf{p}}^{(KK)} - D_{\mathbf{p}}^{(NK)})$.

All these results point to the conclusion that, except for very restricted cases (i.e., when most of the wave numbers are suppressed for example), the internal energy will *lead* the enstrophy or kinetic energy in spreading into a stable region at least by a few linear growth times (since the time scale for the linear coupling between N and K is the linear growth time). As we shall see, this is also consistent with what is observed when (18a) and (18b) are numerically integrated.

C. Resonant interactions

In order to make any practical sense out of expressions such as (16), an understanding of the three wave resonance is necessary. Even though a general calculation is not feasible (in the sense that the results are too complicated), certain limits can be explored. One such case is when one member of the triad is a zonal flow (i.e., $q_y \sim 0$). It is important to note that this will immediately qualify the interaction as a resonant interaction with $k_y = -p_y$ and $k^2 = p^2$. This is one way to see why zonal flows are essential to the dynamics of Hasegawa-Wakatani turbulence. Here we will discuss this case, along with the case when one of the modes is a streamer (i.e., has $q_x \sim 0$).

1. $q_x \sim 0$: The spreader is a small scale streamer

Evaluating (20) and (21) using the resonance condition (23), where q_x is set to zero, yields the diffusion coefficients:

$$D_{\mathbf{p}}^{(NN)} \approx \frac{h(p_x,p_y)}{2}\left[K(0,-p_y(1+p_x^2/3p_y^2)) + \frac{1/9}{p_y^2/p_x^2+1/9}K(-p_x,p_x^2/3p_y)\right] \tag{25a}$$

and

$$D_{\mathbf{p}}^{(NK)} \approx \frac{h(p_x,p_y)}{2}\left[\frac{(p_y^2/p_x^2+1/3)}{1+p_y^2/p_x^2}N(0,-p_y(1+p_x^2/3p_y^2)) - \frac{1/3}{1+p_y^2/p_x^2}N(-p_x,p_x^2/3p_y)\right], \tag{25b}$$

where

$$h(p_x,p_y) = \frac{(p^2+1)(p_y^2(1+p_x^2/9)+p_x^4/9)((p_y^2+p_x^2/3)^2+p_y^2)}{3p_y^4(p_y^2+p_x^2/3)}.$$

Here $D_{\mathbf{p}}^{(NN)}$ is positive definite (note that $|p_y| \geqslant |q_y|$). A particularly interesting limit of this case is when p_x is also zero, for which

$$\frac{D_p^{(NN)}}{K(0,-p_y)} \approx \frac{D_p^{(NK)}}{N(0,-p_y)} \approx \frac{(p_y^2+1)^2}{p_y^2}. \tag{26}$$

Notice that a streamer is actually the radially elongated limit of the convective cell solution. In other words, a streamer has $q_z \sim 0$ as well as $q_x \sim 0$. The mode discussed in this section has finite q_z since we used the adiabatic dispersion relation. Nevertheless it is also common to call those linear solutions streamers. This is in fact the reason we call it a "small scale" streamer ("small scale" denoting the scale of fluctuations in the z direction). In fact, a true streamer should also have $q_z \sim 0$, hence $c \ll 1$. Let us consider this particular case (which also requires treating the damped modes) next.

2. $q_x \approx 0$, $c \ll 1$: The spreader is a large scale streamer

When one of the modes involved in the triad is a $q_x \approx 0$ hydrodynamic mode, the resonance condition becomes

$$\Delta\omega^{(\pm)} = \frac{p_y}{1+p^2} - \frac{k_y}{1+k^2} \pm \sqrt{\frac{c|q_y|}{2q_y^2}} = 0,$$

where $\Delta\omega^{(+)}$ and $\Delta\omega^{(-)}$ are the mismatches for the growing and the damped hydrodynamic modes, respectively. The approximate solution of the resonance condition is

$$q_y = \pm q_y(p_x,p_y) \approx \pm \frac{\mathrm{sign}(1+p_x^2-p_y^2)c^{1/3}(1+p^2)^{4/3}}{2^{1/3}(1+p_x^2-p_y^2)^{2/3}}, \tag{27}$$

and

$$k_y = -p_y \mp q_y(p_x,p_y).$$

This determines the diffusion coefficients to be

$$D_{\mathbf{p}}^{(NN)} \approx \frac{2(1+p^2)}{3|1+p_x^2-p_y^2|}\left[K(0,q_y(p_x,p_y)) + \frac{(-p_y \mp q_y(p_x,p_y))^2}{p_x^2+(-p_y \mp q_y(p_x,p_y))^2}K(-p_x, -p_y \mp q_y(p_x,p_y))\right] \tag{28}$$

and

$$D_{\mathbf{p}}^{(NK)} \approx \frac{2(1+p^2)}{3|1+p_x^2-p_y^2|}\left(\frac{p_y^2}{p^2}N(-p_x,-p_y \mp q_y(p_x,p_y))\right).$$

The diffusion coefficients are both positive definite.

Another interesting observation is that when we set $p_x = 0$ in these expressions, the diffusion coefficients become

$$D_{\mathbf{p}}^{(NN)} \approx \frac{4(1+p_y^2)}{3|1-p_y^2|}K(0,q_y(p_x,p_y))$$

and

$$D_{\mathbf{p}}^{(NK)} \approx \frac{2(1+p_y^2)}{3|1-p_y^2|}N(0,-p_y \mp q_y(p_x,p_y))$$

both of which are singular as $p_y \approx 1$. Notice that $p_y \approx 1$ and $p_x \approx 0$ is usually the most unstable mode for various types of drift instabilities. This means a large scale streamer will "most effectively spread" the most unstable mode. Note that even though $p_y \approx 1$ also causes (27) to become large, it is also proportional to $c^{1/3}$ which may be arbitrarily small. This means if $p_y^2 = 1+\epsilon$, as long as ϵ is small but at the same time large compared to $\sqrt{c}$, interaction between a large scale streamer and a drift wave is feasible. Not surprisingly this is also the most efficient mechanism for spreading of all the cases considered in this study, and is comparable only to the case when all the interacting modes are hydrodynamic.

3. $q_y \approx 0$: Spreader is a zonal flow

Zonal flows, being poloidal flows, do not cause radial transport. This is true for the transport of fluctuation energy as well as particles. However, if a certain "nonlinear model" considers only the couplings via the zonal flows (e.g., Ref. 11) and neglects all direct fluctuation-fluctuation couplings, this simple fact becomes obscured. To the extent that nonlinear interactions drive the transport, it may seem as if the zonal flows were the cause of the transport. For example if one turns-off the zonal flow, since all the nonlinear interaction is also turned-off, no spreading occurs. One might, therefore, be mislead into believing that, zonal flow is the "cause" of spreading. Here we try to demonstrate conclusively that, this is not the case.

The distinction that was introduced for streamers based on their q_z is irrelevant for zonal flows in practice, since $\omega(q_y \to 0) \approx 0$, whether the mode is hydrodynamic or not (in other words, zonal flows are always hydrodynamic). In this case, zonal flows may only "mediate" the interaction, by allowing two oppositely directed drift waves to interact with each other thus letting them also spread one another, while being spread by them. Zonal flows do not cause spreading

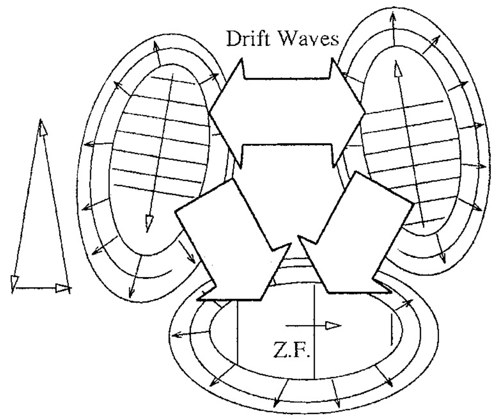

FIG. 4. (Color online) Cartoon of the two-scale direct interaction between a zonal flow and two drift waves of equal and opposite poloidal wave numbers. Drift waves spread one another as well as the zonal flow. Zonal flow allows that only by completing the triangle. Notice that there are many other triangles completed by many other k's. In fact when one of the legs is not a zonal flow, all the legs contribute to spreading unlike the zonal flow, which gets a free ride.

themselves, as the diffusion coefficient is independent of the zonal flow amplitude. The resonance condition with $q_y \approx 0$ gives $q_x = -2p_x$ and

$$D_{\mathrm{p}}^{(NN)} = \frac{(1+p^2)p_y^2}{4|p_x p_y|p^2} K(p_x, -p_y),$$
$$D_{\mathrm{p}}^{(NK)} \sim \frac{(1+p^2)p_y^2}{4|p_x p_y|p^2} N(p_x, -p_y). \tag{29}$$

Here $D_{\mathrm{p}}^{(KK)}$ is already zero, due to the fact that flux coefficient is proportional to mismatch.

Notice that the spreading here is caused by the "other" drift wave, i.e., the $k_y = -p_y$ mode, and not the zonal flow (see Fig. 4). Physically, spreading occurs via inhomogeneous scattering of the drift wave by the zonal flow.

4. $p_y = 0$: *Spreadee is a zonal flow*

Even though zonal flows do not "spread" drift waves, a drift wave can spread a zonal flow, along with the other drift wave in the triad, resulting in spreading of the total turbulence (i.e., the sum of the zonal flows and the fluctuations). In this case the diffusion coefficients become

$$D_{\mathrm{p}}^{(NN)} = \int \frac{(1+q_y^2+p_x^2/4)^2 q_y^2}{2|q_y p_x|(q_y^2+p_x^2/4)} [K(-p_x/2, q_y) + K(p_x/2, -q_y)] dq_y \tag{30}$$

which is positive definite. On the other hand, $D_{\mathrm{p}}^{(NK)}$ vanishes, since for $p_y = 0$,

$$D_{\mathrm{p}}^{(NK)} \sim -\frac{q_y p_y}{p^2} N_{\mathrm{q}} - \frac{k_y p_y}{p^2} N_{\mathrm{k}} \to 0.$$

Therefore turbulence with zonal flows may in fact spread, via nonlinear interactions (between **q** and **k** in this case) "mediated" by the zonal flow (**p** here). However the role of the zonal flow in this type of spreading is *predominantly passive.* The zonal flow allows two drift waves that are oppositely aligned in the y direction to resonantly interact and therefore spread each other. While doing that, they also spread the zonal flow as a side effect. This is neither the dominant interaction, nor do zonal flows play any essential role in the spreading here.

D. Consequences

Before continuing further, it is absolutely necessary to summarize these results and explain what they mean, physically. First and foremost of all, the primary observation is that for all the cases considered here, be it the near-adiabatic or hydrodynamic limits, $D_{\mathrm{p}}^{(NN)} \geqslant D_{\mathrm{p}}^{(NK)} > D_{\mathrm{p}}^{(KK)}$ *consistently*, which implies, $D_3 K \geqslant D_2 N > D_1 K$ for any reasonable ensemble of drift wave turbulence (i.e., containing a variety of modes). *Since the dominant diffusion coefficient D_3K accounts for the diffusion of N by K, we can talk about a "spreader" K and a "spreadee" N. Such an ordering strongly implies that the spreading of N will lead the spreading of K.* This is a concrete observation and a testable prediction.

Second, the process by which the zonal flows are involved in the spreading of turbulence is clarified. Zonal flows do *not* cause spreading directly (none of the diffusion coefficients are proportional to the zonal flow amplitude so long as zonal flow damping is neglected), however they "mediate" spreading by scattering the drift modes, while in fact being spread by them. It is clear from expressions such as (29) and (30), that the drift modes [i.e., $(-p_x/2, q_y)$ and $(-p_x/2, -q_y)$] in fact cause spreading of the zonal flow and each other at the same time. Thus we call the drift modes the "spreaders" and the zonal flow the "spreadee." *Notice that it is in fact the conserved sum of the drift wave and the zonal flow energies that actually spreads.*

When the effect of zonal flow damping is included, it results in broadening of the three wave resonance and thus in a radial energy flux proportional to the zonal flow damping. This is not surprising, because zonal flows inhibit transport and spreading by shearing apart structures with radial extent. Thus, anything that damps the zonal flows will necessarily lead to turbulence spreading. Also, secondary instabilities of the zonal flow (i.e., tertiary instabilities of the drift waves), such as the Kelvin-Helmholtz break-up,[38] would result in a similar outcome. Once again, any process that reduces the inhibitor enables spreading.

It is also important to note that there is a very large number of other modes which could similarly mediate spreading, and in addition contribute to the spreading themselves. The total sum of this large number of modes contributes much more to the spreading than zonal flow mediated spreading process alone. Therefore it is not in any way justifiable to neglect the effects of all these other modes based on arguments about zonal flow interaction being stronger when compared with only one of the other modes. In fact, since $1/c$ does not appear in the expressions Eqs. (29) and (30), even that claim may not be justified.

Another interesting observation concerns the most efficient path to spreading. Finite q_z streamers, just like any other drift mode on the resonance manifold, may contribute to the spreading phenomenon even though they are not particularly effective. On the other hand, streamers as large scale convective cells elongated in the radial direction (and thus having $c \ll 1$), are very interesting from the point of view of being particularly efficient in enabling spreading. First of all, the properties of such structures do not change when they themselves are subjected to spreading. This is essential, because they can keep spreading other modes efficiently without being scattered into other parts of k space. The condition that a large scale streamer interacts with a small scale drift wave seems to be that $1+p_x^2-p_y^2$ must be small, but at the same time large compared to $\sqrt{c}$ (which is already very small as, the streamer is assumed to be hydrodynamic). Pragmatically speaking this allows the diffusion coefficient to be as large as $O(1/\sqrt{c})$. Notice also that the most unstable modes in most drift wave turbulence problems satisfy these conditions. This makes large scale streamers particularly interesting, because they can spread the *most* unstable modes, *most* efficiently.

Similarly hydrodynamic interactions in general are more efficient than near-adiabatic ones, as near the hydrodynamic limit all the diffusion coefficients scale as $1/\sqrt{c}$, which is large. However this may in fact be unphysical, as it may be argued that due to resonance broadening, weak turbulence will not be valid for those types of modes. Here we argued that local saturation is robust enough that a weak turbulence theory (for the purpose of spreading) may be based on a locally saturated state.

VI. SOLUTIONS AND IMPLICATIONS OF THE SPREADING MODEL

$N=K=\gamma/\gamma_{\mathrm{NL}}$ is a fixed point of the two-field model. In fact by letting $N\sim K\sim \varepsilon/2$, the equation for the total energy becomes

$$(\partial_t + v_{gx}\partial_X)\varepsilon = \partial_X(D_0\varepsilon\partial_X\varepsilon) + \gamma\varepsilon - \gamma_{\mathrm{NL}}\varepsilon^2, \tag{31}$$

where $D_0 \approx (D_1+D_2+D_3)/4$. This is the usual nonlinear Fisher equation, which frequently appears in the study of spreading phenomena.[16,17] Thus all the solutions of (31), which were already given in some detail in Ref. 13 are also solutions of (18a) and (18b) near the fixed point. However, since there are more degrees of freedom in the two-field model, various other things may be expected to happen. Even though there is a host of exact analytical solutions (for instance those of the one-field model), it is not clear *a priori*, which of those solutions will actually be realized. However numerical solutions of this 1D, two-field system is quite feasible. Here we use a two-field version of the same numerical method used in Ref. 13 which employs an implicit Crank-Nicolson scheme for the linear terms, a third order Runge-Kutta-Wray scheme (RKW-3) for nonlinear terms, and simple finite differencing approximations for the spatial derivatives. The agreement of this numerical method, with exact analytical solutions was tested and found to be quite accurate. The result, as expected, is a slight enhancement of spreading due to the internal energy dynamics. In fact it is clear that spreading of internal energy "leads" that of kinetic energy by some finite amount in most cases of interest (see Fig. 5).

VII. DISCUSSION AND CONCLUSIONS

In this paper, we developed a theory of turbulence spreading for the two-field Hasegawa-Wakatani system. The principal results of this paper are:

(a) The systematic derivation of Markovian closure expressions for the flux of kinetic and internal fluctuation energy. These expressions may be thought of as a statistical generalization of the Whitham modulation theory which accounts for mode couplings.

(b) The simplification of the closure expressions to obtain two coupled, nonlinear reaction-diffusion equations for the kinetic and internal energy density. These equations reduce to earlier, simpler models in the appropriate limits.

(c) The calculation of the fluctuation energy flux in various limits. Specifically we have systematically studied the efficacy of different interaction mechanisms for turbulence spreading. These results are summarized in Tables I and II.

(d) The conclusion that spreading of internal energy "leads" the spreading of kinetic energy. This prediction is easily testable via numerical simulation.

(e) The conclusion that zonal flows are *not* the predominant agents of spreading.

The model agrees qualitatively with earlier models of spreading, in that it can be reduced to the single equation model with additional assumptions, and it also verifies their validity in the appropriate limit. There are various additional observations that can be made via the two-field model, however.

It is our belief that, the tendency of K to lag behind N is connected to the fact that, in two-dimensional turbulence, kinetic energy tends to inverse cascade, whereas internal energy set by the "passive" scalar, tends to cascade forward. Diffusion coefficients D_1 and D_3 are linked to the concept of eddy viscosity in homogeneous isotropic turbulence. Since it is well known that eddy viscosity is negative (at least for fluid) 2D hydrodynamic turbulence, such a correspondence is not unexpected. Simply put, N spreads faster because it is mixed on small scales.

However as noted in the Introduction, the synergy between forward and inverse cascades possible in a two-field model may facilitate spreading into stable region. Notice that if one measures the $\mathbf{E}\times\mathbf{B}$ flow energy, it is possible to find larger scale structures in the damped region.[39] This may be a manifestation of the fact that the turbulence at the scale of the driver (i.e., mixing scale corresponding to the most unstable mode) is spread most effectively by "streamers" or large scale, radially elongated convective cells, which have low k_z (i.e., $c \ll 1$). This is a manifestation of the fact that we need *both* the large scale structures, which effectively spread the turbulence, and the smaller scale turbulence, which can be spread by the large scale structures. However if one measures the density or the internal energy in the damped region, it should consist mainly of small scale contributions if the spreading is a result of nonlinear mode coupling.

Another point is that the kinetic energy is mixed at a

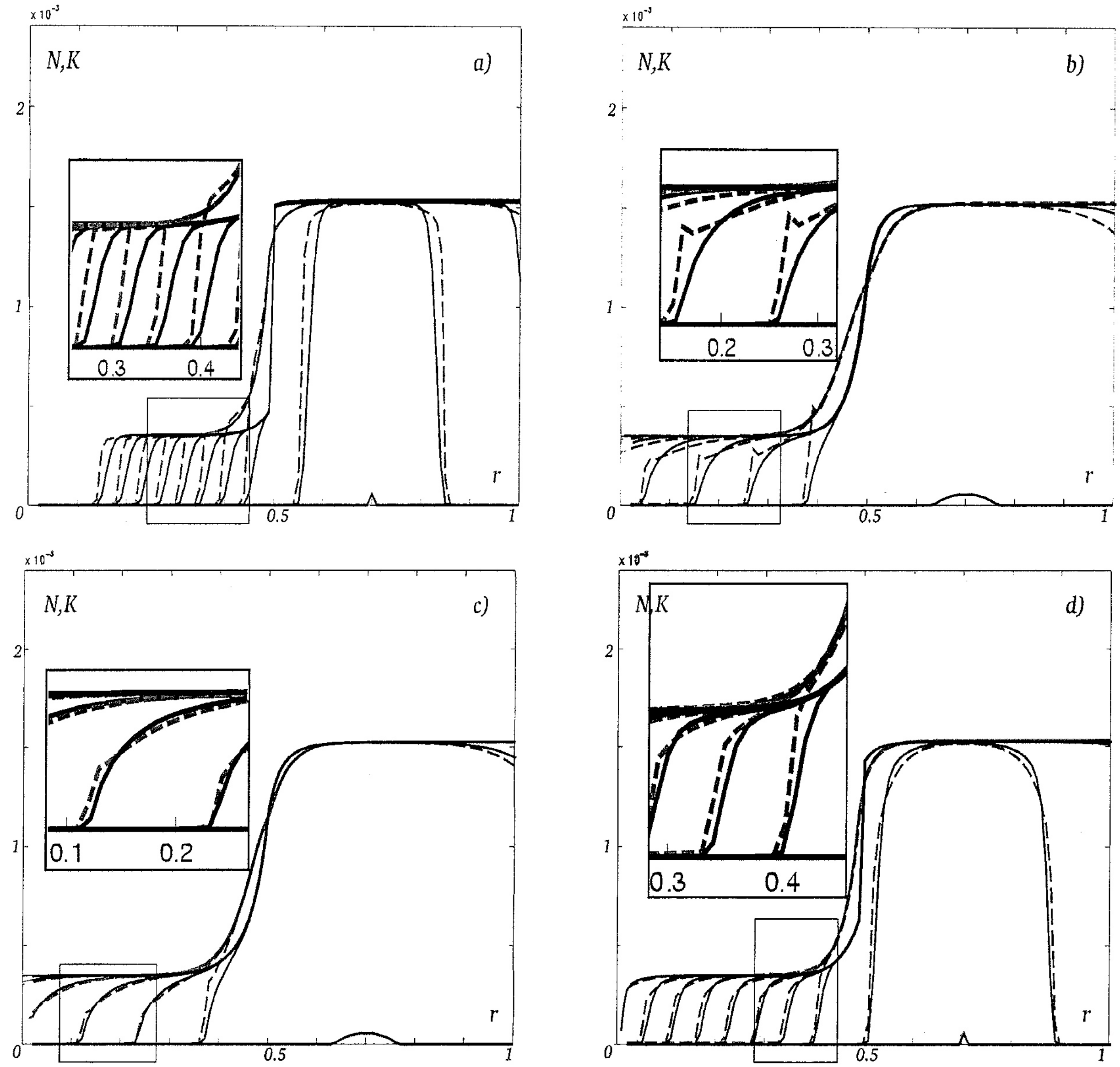

FIG. 5. (Color online) Numerical solutions of the two-field model across the full radius. Here, dotted lines correspond to the internal energy, N and the solid lines to the kinetic energy, K. In all the cases considered with equal, or similar diffusion coefficients, N leads K slightly even though K saturates first. Also there is much more N in the "stable" region than K. Shown here are (a) $D_1 \sim D_2 \sim 0$, $D_3 = D_{GB}$, and $\alpha = 0.2$; (b) $D_1 = D_3 = D_{GB}$ and $\alpha \sim 0.2$; (c) $D_1 = D_2 = D_3 = D_{GB}$ and $\alpha = 0.8$; and (d) $D_1 \sim D_2 \sim 0$, $D_3 = D_{GB}$ and $\alpha = 0.9$, where $D_{GB} = D$.

smaller scale (the scale of the most unstable mode) in the unstable region, which then inverse cascades to larger scales and either gets damped by back-coupling to small scales or accumulates at the largest scale. In any case the process is continuous, so all the scales between the mixing scale and the largest scales exist in the unstable region. However, once the turbulence enters the damped region, it is no longer mixed. The inverse cascade continues as the turbulence propagates in the damped region. However there is no longer instability driven mixing at the small scales, therefore the small scales are not populated. Therefore it is not surprising to find more large scale structures in the damped region than in the unstable region. Again, this argument should be reversed for internal energy, and more small scale turbulence should be observed in the internal energy spectrum in the damped region.

Another important result is on the role played by zonal flows ($q_y = 0$ modes) in the process of spreading. First of all, if the drift waves are "adiabatic" (i.e., consistent with the Hasegawa-Mima dispersion relation), there is effectively *no* flux of energy or enstrophy for resonant interactions. This is true unless zonal flow damping is introduced, which causes resonance broadening. Then the energy diffusion becomes proportional to the zonal flow damping. It should be noted that *it is the total energy of the zonal flow and the fluctuation that spreads*. Hence, *the zonal flow is only one of infinitely many mediators. Removing the zonal flow does not remove spreading, and probably does not even reduce it.* Even

TABLE I. Table of results for the diffusion coefficients in the near-adiabatic limit, giving information about the relative values of the diffusion coefficients. Recall that $D^{KK} \equiv D_{\mathrm{p}}^{KK}$ is the coefficient of $\partial_X K$ in the expression for the flux of K, D^{NN} is the coefficient of $\partial_X N$ in the expression for the flux of N, and D^{NK} is the coefficient of $\partial_X K$ in the expression for the flux of N. D always corresponds to D^{NN} and λ is simply a p dependent coefficient which is always less than 1. In addition to these we have also considered the interactions between a hydrodynamic streamer and two adiabatic drift waves. The result for that case is given in Eqs. (28).

spreader→ spreadee ↓	$q_y=0$ and $k_y=-p_y$ (a zonal flow and a drift wave)	$q_x=0 (c \gg 1)$ and $k_x=-p_x$ (a small scale streamer and a drift wave)	other $\mathbf{q}$ and $\mathbf{k}=\mathbf{p}-\mathbf{q}$ (two drift waves)
$p_y=0$ (i.e., zonal flow)	$D^{KK}=0$ $D^{NN}=D^{NK}=0$	$D^{KK}=0$ $D^{NN}=D^{NK}=0$ no resonance	$D^{KK}=0$ $D^{NN}=D, D^{NK}=0$ see Eq. (30)
$p_x=0$, $c \gg 1$ (i.e., small scale streamer)	$D^{KK}=0$ $D^{NN}=D^{NK}=0$	$D^{KK}=0$ $D^{NN}=D, D^{NK}=D$ see Eq. (26)	$D^{KK}=0$ $D^{NN}=D, D^{NK}\neq 0$ not computed
other $\mathbf{p}$ (i.e., drift wave)	$D^{KK}=0$ $D^{NN}=D^{NK}=D$ see Eq. (29)	$D^{KK}=0$ $D^{NN}=D, D^{NK}=\lambda D$ see Eqs. (25a) and (25b)	$D^{KK}=0$ $D^{NN}=D, D^{NK}\neq 0$ not computed

though the zonal flows automatically satisfy the resonance condition, they also make the coefficient of the kinetic energy flux vanish, effectively removing resonant interactions. Thus, any advantage gained from resonant interaction is lost from the point of view of spreading.

There are various modes on the other hand, for which the resonance condition is satisfied and the kinetic energy flux coefficient does not vanish. At least from the point of view of a weak turbulence analysis, these modes are the most important ones in terms of facilitating spreading. In particular, spreading induced by hydrodynamic streamers (on either other hydrodynamic modes, or drift wave turbulence) is shown to be the most efficient. In fact, other physical effects, such as the shearing of the streamers by the zonal flows, that reduce this tendency should probably be included in order to justify the claim that the resulting diffusion coefficient is quantitatively accurate.

Notice that, we mainly considered weak turbulence and weak zonal flows in this work. One should also consider strong turbulence and strong zonal flow cases separately. Strong zonal or mean flows are usually incorporated into the framework of TSDIA as Doppler shifted Fourier transforms. If the linear eigenmodes have the time to form, this would imply modified eigenmode structure. We also tried to isolate the effects of all kinds of modes individually. Alternatively, one could assume homogeneous, isotropic turbulence (for small scales), calculate the diffusion coefficients under these assumptions and check if introducing (strong) zonal flows (or external shear flows) enhance or reduce spreading. We believe future studies on spreading should tackle these issues.

In accordance with our model, a proper numerical experiment of turbulence spreading in the Hasegawa-Wakatani model requires carefully designed profiles of local dissipa-

TABLE II. Table of results for the diffusion coefficients in the hydrodynamic limit, giving information about the relative values of the diffusion coefficients. Recall that $D^{KK} \equiv D_{\mathrm{p}}^{KK}$ is the coefficient of $\partial_X K$ in the expression for the flux of K, D^{NN} is the coefficient of $\partial_X N$ in the expression for the flux of N, and D^{NK} is the coefficient of $\partial_X K$ in the expression for the flux of N. D always corresponds to D^{NN} and λ and λ' are simply p dependent coefficients which are always less than 1.

spreader→ spreadee ↓	$q_y=0$ and $k_y=-p_y$ (a zonal flow and a drift wave)	$q_x=0$ and $k_x=-p_x$ (a streamer and a drift wave)	other $\mathbf{q}$ and $\mathbf{k}=-\mathbf{p}-\mathbf{q}$ (two drift waves)
$p_y=0$ (i.e., zonal flow)	$D^{KK}=0$ $D^{NN}=D^{NK}=0$	$D^{KK}=0$ $D^{NN}=D^{NK}=0$ no resonance	$D^{KK}=0$ $D^{NN}=D, D^{NK}=0$ see Eq. (C4)
$p_x=0$ (i.e., streamer)	$D^{KK}=0$ $D^{NN}=D^{NK}=0$	$D^{NK}=0$ $D^{NN}=0, D^{NK}=0$ see $p_x \to 0$ limit of Eqs. (C5a)–(C5c)	$D^{KK}=0$ $D^{NN}=D, D^{NK}\neq 0$ not computed
other $\mathbf{p}$ (i.e., drift wave)	$D^{KK}=0$ $D^{NN}=D^{NK}=D$ see Eq. (C3)	$D^{KK}=\lambda' D$ $D^{NN}=D, D^{NK}=\lambda D$ see Eqs. (C5a)–(C5c)	$D^{KK}=0$ $D^{NN}=D, D^{NK}\neq 0$ not computed

tion, parallel collisionality, and density gradient such that there are regions, in which significant fluctuation intensity is excited, and regions, in which it is weak or absent. The scale at which those regions can be distinguished (i.e., ΔX) should be much larger than the characteristic scale of the turbulence, but small compared to the system size, so that it can define an envelope scale for the turbulence which is not confused by interactions with the boundaries. This probably implies that a very large box size is necessary. One way to check if the suggested mechanism of three wave coupling plays an important role in turbulence spreading, is to examine bicoherence/bispectrum[40] around the boundary of excited region [i.e., where $\gamma(X)$ changes most rapidly]. One could of course, pick two neighboring cells (of size ΔX), take windowed Fourier transforms, and compute a nonlocal bispectrum where two of the modes are selected from the unstable region and one from the damped region. This would be a measure of how much intensity is being nonlocally transported via three-wave interactions.

Note that many predictions in this paper are easily testable also by gyrokinetic codes. The major testable prediction is that N leads K in spreading. Another one is the prediction that zonal flows do not play an essential role in spreading and removing the zonal flows would not stop spreading. Finally, the role played by streamers or convective cells should also be testable by simulations. This can be achieved for instance by comparing two different types of turbulence (say ETG and ITG) i.e., one case where the code is known to lead to the formation of streamers and another case where it is known to lead to the formation of zonal flows. Notice that this is insensitive to the fact that ETG may also actually form zonal flows in the long time limit. What is important here is to use a code (and a value of the magnetic shear) that leads to the formation of streamers, and look for the spreading in the presence of these streamers.

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APPENDIX A: LINEAR THEORY OF THE HASEGAWA-WAKATANI SYSTEM

The linear response function to the Hasegawa-Wakatani problem is

$$R_{\mathbf{k}}^{\alpha\beta}(t,t') = r_{\mathbf{k}}^{\alpha\beta\gamma} e^{-\lambda_{\mathbf{k}}^{\gamma}|t-t'|},$$

where

$$r_{\mathbf{k}}^{\alpha\beta 1} = \frac{1}{\lambda_{\mathbf{k}}^1 - \lambda_{\mathbf{k}}^2}\begin{pmatrix} \lambda_{\mathbf{k}}^1 - \chi k^2 - c & -c/k^2 \\ ik_y - c & \lambda_{\mathbf{k}}^1 - \nu k^2 - c/k^2 \end{pmatrix},$$

$$r_{\mathbf{k}}^{\alpha\beta 2} = \frac{-1}{\lambda_{\mathbf{k}}^1 - \lambda_{\mathbf{k}}^2}\begin{pmatrix} \lambda_{\mathbf{k}}^2 - \chi k^2 - c & -c/k^2 \\ ik_y - c & \lambda_{\mathbf{k}}^2 - \nu k^2 - c/k^2 \end{pmatrix},$$

and

$$\lambda_{\mathbf{k}}^{1,2} = i\omega_{\mathbf{k}}^{\pm} \pm \gamma_{\mathbf{k}}^{\pm},$$

where

$$\omega_{\mathbf{k}}^{(r)} = \mp\sqrt{\frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4} - \frac{B^2}{2}}$$

and

$$\gamma_{\mathbf{k}}^{\pm} = -A \pm \sqrt{\frac{B^2}{2} + \frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4}},$$

where

$$B = \frac{1}{2}\left[(\chi k^2 + c) - \left(\frac{c}{k^2} + \nu k^2\right)\right],$$

$$A = \frac{1}{2}\left[(\chi k^2 + c) + \left(\frac{c}{k^2} + \nu k^2\right)\right].$$

Notice that this implies that the eigenmodes for the decaying and the growing modes may be put together in to the form

$$\lambda_{\mathbf{k}}^{(1)} = -i\,\mathrm{sign}(k_y)\sqrt{\frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4} - \frac{B^2}{2}} + A + \sqrt{\frac{B^2}{2} + \frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4}},$$

$$\lambda_{\mathbf{k}}^{(2)} = i\,\mathrm{sign}(k_y)\sqrt{\frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4} - \frac{B^2}{2}} + A - \sqrt{\frac{B^2}{2} + \frac{1}{2}\sqrt{B^4 + c^2k_y^2/k^4}},$$

respectively.

APPENDIX B: CONSERVATION LAWS

If we take (1a) and multiply by Φ and rearrange we get the law of conservation of total kinetic energy:

$$\partial_t\left(\frac{[\nabla\Phi]^2}{2}\right) + \nabla\cdot(-\Phi\partial_t\nabla\Phi + \nu\Phi\nabla\nabla^2\Phi - \nu(\nabla\Phi\cdot\nabla)\nabla\Phi) + \nu\nabla\nabla\Phi{:}\nabla\nabla\Phi - c\Phi(n-\Phi) + \nabla\cdot\left(\frac{\Phi^2}{2}\hat{\mathbf{z}}\times\nabla\nabla^2\Phi\right) = 0. \quad \text{(B1)}$$

Similarly multiplying (1b) by n, and rearranging, we get the law of conservation of total internal energy:

$$\partial_t\left(\frac{n^2}{2}\right) + n\partial_y\Phi + \mathrm{cn}(n-\Phi) - \nabla\cdot(\chi\nabla n) + \chi(\nabla n)^2 + \nabla\cdot\left(\frac{n^2}{2}\hat{\mathbf{z}}\times\nabla\Phi\right) = 0. \quad \text{(B2)}$$

The last terms in each of these equations correspond to the nonlinear spatial flux of the conserved quantity due to the *advection by the* $\mathbf{E}\times\mathbf{B}$ *flow associated with the fluctuating scalar potential* Φ. There is no internal energy analog for the electrostatic field (i.e., Φ^2) since Φ cannot advect itself. However Φ may advect vorticity. This is the reason why the kinetic energy equation has more derivatives.

When the two equations are added, the equation for total energy,

$$\varepsilon \equiv K + N \equiv \frac{[\nabla\Phi]^2}{2} + \frac{n^2}{2}$$

is obtained in Poynting's form

$$\partial_t \varepsilon + \nabla\cdot\Gamma_\varepsilon + Q = 0,$$

where Q is the total dissipation minus the total internal drive and the total flux of energy Γ_ε is the sum of the flux of internal energy and the flux of kinetic energy, i.e.,

$$\Gamma_\varepsilon = \frac{\Phi^2}{2}\hat{\mathbf{z}}\times\nabla\nabla^2\Phi + \frac{n^2}{2}\hat{\mathbf{z}}\times\nabla\Phi \equiv \Gamma_K + \Gamma_N.$$

Since the Hasegawa-Wakatani system is linearly unstable and there is dissipation, Q is nonzero, in general. This means none of these quantities are actually conserved "linearly." However they are still very important as they are "nonlinearly" conserved. In other words, the mode coupling processes respect these quantities.

Notice that for the Hasegawa-Wakatani system there are at least two more similarly conserved (i.e., nonlinearly conserved) quantities, the enstrophy $\langle(\nabla^2\Phi)^2\rangle$ and the "cross-helicity" $\langle n\nabla^2\Phi\rangle$. Here we will not consider the independent evolution of those quantities and instead only point out that they can induce linear coupling between N and K. Notice that similar conservation laws of the gyrokinetic equation (from which the Hasegawa-Wakatani model may be derived) are known to be useful in benchmarking simulation codes.[41]

APPENDIX C: SPREADING IN THE HYDRODYNAMIC LIMIT

The spreading of energy vanishes for the adiabatic limit. Even in the near-adiabatic limit, the spreading is exclusively due to the spreading of N. Even though this tendency prevails in the more general case as well, the diffusion coefficient for kinetic energy is not always strictly zero. In order to demonstrate this, let us consider the hydrodynamic limit ($c \to 0$) of the Hasegawa-Wakatani equations (with $\chi \sim \nu$):

$$(\partial_t + \hat{\mathbf{z}}\times\nabla\Phi\cdot\nabla)\nabla^2\Phi - \nu\nabla^4\Phi = -\,\mathrm{c}n, \tag{C1}$$

$$(\partial_t + \hat{\mathbf{z}}\times\nabla\Phi\cdot\nabla)n + \partial_y\Phi - \nu\nabla^2 n = 0 \tag{C2}$$

which gives the wave frequency

$$\omega_{\mathbf{k}}^{(\pm)} \approx \pm\,\mathrm{sign}(k_y)\sqrt{\frac{ck_y}{2k^2}}$$

and the growth and damping rates

$$\gamma_{\mathbf{k}}^{(\pm)} \approx \pm\,\mathrm{sign}(k_y)\sqrt{\frac{c|k_y|}{2k^2}} - \nu k^2 - \nu_{\mathrm{eddy}}.$$

Notice that in order to justify weak turbulence ($\omega > \gamma$) treatment, we have to assume saturation, i.e., $\gamma_{\mathbf{k}}^{(\pm)} \approx 0^-$, which means that both the growing and the damped modes may contribute to the resonance. Of course, if all the modes are damped one does not expect any nonlinear interaction, therefore we only consider two cases, namely when all the modes are growing and when only one of the modes is damped.

Resonant interactions: Again, a general calculation of the resonance manifold is not feasible. Therefore we will consider various limiting cases.

$q_y=0$ (*Spreader is a zonal flow*): Using the hydrodynamical limit of the dispersion relation the resonance condition for the case $q_y=0$ (i.e., $k_y=-p_y$) is easily solved to give $k_x=p_x$ (i.e., $q_x=-2p_x$). Hence an isosceles triangle with one leg having $q_y=0$ represents a resonant interaction. In this limit

$$D_{\mathbf{p}}^{(NN)} = \sqrt{\frac{2}{c|p_y|}}\frac{p_y^2}{|p_x|}K(p_x,-p_y),$$
$$D_{\mathbf{p}}^{(NK)} \sim \sqrt{\frac{2}{c|p_y|}}\frac{p_y^2}{|p_x|}N(p_x,-p_y). \tag{C3}$$

When one of the modes is a zonal flow, the resonance condition causes the flux coefficient $q_yk^2+p_yp^2+k_yk^2$ to vanish for the hydrodynamic limit as well as the adiabatic limit, resulting in

$$D_{\mathbf{p}}^{(KK)} \approx 0.$$

$p_x=0$ (*Spreadee is a zonal flow*): Similarly, the resonance condition yields $q_x=-p_x/2$, which in turn yields

$$D_{\mathbf{p}}^{(NN)} = \int \sqrt{\frac{2}{c|q_y|}}\frac{(q_y^2+p_x^2/4)^{1/2}q_y^2}{2|p_x|}[K(-p_x/2,q_y)dq_y + K(p_x/2,-p_y-q_y)]dq_y, \quad D_{\mathbf{p}}^{(NK)} \sim 0 \tag{C4}$$

and

$$D_{\mathbf{p}}^{(KK)} \approx 0,$$

as usual since the resonance condition causes the flux coefficient to vanish.

$q_x=0$ (*Spreader is a streamer*): The resonance condition

$$\Delta\omega = \sqrt{\frac{c}{2|q_y|}} - \sqrt{\frac{c|p_y|}{2p^2}} - \sqrt{\frac{c|p_y+q_y|}{2(p^2+q_y^2+2p_yq_y)}} = 0$$

cannot be solved in a simple manner. Therefore we will use the approximate solution

$$q_y = -p_y(1+p_x^4/4p_y^2)$$

yielding

$$D_{\mathbf{p}}^{NN} \approx h(p_x,p_y)\left[K(0,-p_y(1+p_x^4/4p_y^2)) + \frac{p_x^2/4p_y}{1+p_x^6/16p_y^2}N(-p_x,p_x^4/4p_y)\right], \tag{C5a}$$

$$D_{\mathbf{p}}^{NK} \approx h(p_x,p_y)\left[\frac{p_y^2}{p^2}N(0,-p_y(1+p_x^4/4p_y^2)) + \frac{p_x^4}{4p^2}N(-p_x,p_x^4/4p_y)\right] \quad \text{(C5b)}$$

and

$$D_{\mathbf{p}}^{KK} \approx \frac{h(p_x,p_y)p_x^2(3p_x^4/4-1)}{p^2(2p_y^2+(p_x/2)^3)}[((p_x/2)^3+p_x^2p_y^2-6p_y^2)K(0,-p_y(1+p_x^4/4p_y^2)) + (-(p_x/2)^3+p_x^2p_y^2+4p_y^2)\times K(-p_x,p_x^4/4p_y)], \quad \text{(C5c)}$$

$$h(p_x,p_y) \equiv \sqrt{\frac{2}{c|p_y|}}(1+p_x^4/4p_y^2) \times\left(\frac{2p_x^2(p_x^6+16p_y^2)^{3/2}|p||p_y|}{2|p||p_y|(8p_x^4+16p_y-p_x^6)+p_x^2(p_x^6+16p_y^2)^{3/2}}\right).$$

Notice that the first two are positive definite, whereas the sign of the last one uncertain, but usually positive for $1>p_y>p_x$.

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