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Monoidal Extensions of a Locally Quasi-Unmixed Unique Factorization Domain

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ABSTRACT OF THE DISSERTATION

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Let R be a locally quasi-unmixed domain, $a, b_1, \ldots, b_n$ an asymptotic sequence in R, $I = (a, b_1, \ldots, b_n)R$, and $S = R[b_1/a, \ldots, b_n/a] = R[I/a]$. Then S is a locally quasi-unmixed domain, $a, b_1/a, \ldots, b_n/a$ is an asymptotic sequence in S, and there is a one-to-one correspondence between the asymptotic primes $\hat{A}^*(I)$ of I and the asymptotic primes $\hat{A}^*(aS)$ of $aS = IS$. Moreover, if $a, b_1, \ldots, b_n$ is an R-sequence, then that one-to-one correspondence extends between $\mathrm{Ass}_R(R/I)$ and $\mathrm{Ass}_S(S/aS)$.

We give a sufficient condition for the monoidal transform S to be a unique factorization domain, or a Krull domain whose class group is torsion, finite, or finite cyclic. As a corollary, we give a necessary and sufficient condition for R and its monoidal transform to have the same class group.

In the case that R is a unique factorization domain, we examine the height-one prime ideals of S to determine how far S is from unique factorization. In Section 3.2, a complete description is given of which height-one prime ideals P of S are principal or have a prinicpal primary ideal in the case that $\mathrm{ht}(P \cap R) = 1$. In Section 3.3, we show that if the prime divisors of a satisfy a mild condition, we may give a similar description in the case that $\mathrm{ht}(P \cap R) > 1$. We give a necessary and sufficient condition for S to

be a Krull domain with finite cyclic class group in the case that a is a power of a prime element, and we show that this holds for the Rees ring $R[1/t, It]$ as a monoidal transform over $R[1/t]$ as well. Furthermore, if a is a power of a prime element, we show that if $\mathrm{Rad}(I)$ is not prime and p is a height-one prime ideal of R contained in at least one but not all asymptotic prime divisors of I, then the height-one prime ideal $pR[1/a] \cap S$ of S has no principal primary ideal.

Contents

Chapter 1

Introduction

For a ring R and elements $a, b_1, \ldots, b_n \in R$, where a is not a zero divisor, the overring $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$, a subring of the total quotient ring of R, is called a *monoidal transformation*, or *transform* of R [3]. Monoidal transformations arise naturally in Algebraic Geometry and have been studied often since Zariski's foundational paper [33]. In addition to Zariski's initial work, monoidal transformations were used by himself, Abhyankar, and Hironaka on the resolution of singularities of algebraic varieties. In 1965, Ratliff investigated monoidal transforms generated by R-sequences. He established some properties of such a transformation, and showed that $a, \frac{b_1}{a}, \ldots, \frac{b_n}{a}$ is an S-sequence. In 1967, Davis published [3], which laid down some basic properties of monoidal transformations, in particular that they behave nicely when generated by a *strongly analytically independent* set, and that R-sequences are strongly analytically independent. More recently, Heinzer, Li, Ratliff and Rush gave conditions for a monoidal transform of a Noetherian Cohen-Macaulay UFD to be also a Cohen-Macaulay UFD [9]. Also, in 2006, Hetzel and Saydam gave conditions such that if the base ring satisfies ACCP (ascending chain condition on principal ideals) (resp. is a Krull domain, resp. is a

UFD), then the monoidal transform would also satisfy ACCP (resp. be a Krull domain, resp. be a UFD) ([10], [11]). Note that one of the conditions given by Hetzel and Saydam was that the monoidal transform be generated by a strongly analytically independent sequence.

Ratliff has proved many useful results for a class of rings which generalizes properties of Cohen-Macaulay rings: locally quasi-unmixed rings. In 1974 [21] he proved that a Noetherian ring R is locally quasi-unmixed if and only if for each ideal I of the principal class in R all the associated primes of I_a have the same height (that is, I_a is *height unmixed*), which we have restated below as Theorem 2.2.10 for ease of reference. This is an analogue of Nagata's classical result that a Noetherian ring R is Cohen-Macaulay if and only if for each ideal I of the principal class in R all the associated primes of I have the same height, known as the Unmixedness Theorem. Ideals of the principal class (ideals I generated by height(I) elements) are well understood in Cohen-Macaulay rings: they are generated by R-sequences. Rees introduced asymptotic sequences, a generalization of R-sequences, in 1981 [29]. In 1983 [24], Ratliff proved that most well-known results for R-sequences have a valid analogue for asymptotic sequences in Noetherian rings, and that asymptotic sequences relate to locally quasi-unmixed Noetherian rings very much as R-sequences relate to Cohen-Macaulay Noetherian rings.

I will consider monoidal transforms of locally quasi-unmixed Noetherian rings. In Lemma 3.1.5, I give conditions on the sequence $a, b_1, \ldots, b_n$ such that S is a locally quasi-unmixed Noetherian ring if R is. Theorem 3.1.10, the main result of Section 3.1, is a strengthening of an analogous Theorem in [9]. Theorem 3.1.10 gives a sufficient condition for the monoidal transform S to be a unique factorization domain, or a Krull domain whose class group is torsion, finite, or finite cyclic. As a corollary, I give a necessary and sufficient condition for R and its monoidal transform to have the same class group.

Since in a UFD every height-one prime ideal is principal, and since a Krull domain has torsion class group if and only if every height-one prime has a principal primary ideal, we examine the height-one primes of the monoidal transform S, to see how far from a UFD it may be. Section 3.2 deals with the height-one prime ideals P of S such that $p := P \cap R$ has $\mathrm{ht}(p) = 1$. In Theorem 3.2.11 and Corollary 3.2.12, which summarize the results in this section, it is shown that if R is a locally quasi-unmixed UFD, $I = (a, b_1, \ldots, b_n)R$ is height unmixed, $a, b_1, \ldots, b_n$ is an R-sequence, and $S = R[I/a]$, then $P = pS$ if and only if p is not in any prime divisor of I. Also, if p is contained in some prime divisor of I, then $P(= pR[1/a] \cap S)$ has a principal primary ideal if and only if there is a positive integer h and an element $x \in p \cap I^h$ such that either $a, x/a^h$ is an S-sequence or $(a, x/a^h)S = S$. This gives a complete description of whether P is principal or has a principal primary ideal in the case $\mathrm{ht}(p) = 1$.

Section 3.3 then deals with the case where $\mathrm{ht}(p) > 1$. In particular, if R and I are as in Theorem 3.2.11 and the prime factors of a satisfy a mild condition, then P is principal (resp. has a principal primary ideal) if and only if for some prime factor a_i of a, the ideal $(a_i, b_1, \ldots, b_n)R = P \cap R$ (resp. $(a_i, b_1, \ldots, b_n)R$ is $(P \cap R)$-primary).

Chapter 4 treats two special cases: where a is a power of a prime element, and the case of monoidal transformations over the Rees ring $R[1/t, It]$. In Section 4.1 we show that if a is a power of a prime element and R is a locally quasi-unmixed UFD, I is height unmixed and generated by an R-sequence, then S is a Krull domain with torsion class group if and only if S is Krull with finite cyclic class group if and only if $\mathrm{Rad}(I)$ is prime and integrally closed. Also, if $\mathrm{Rad}(I)$ is not prime, then for each height-one prime ideal p contained in at least one but not all prime divisors of I, the height-one prime ideal $pR[1/a] \cap S$ has no principal primary ideals. Section 4.2 shows that the results of the previous section hold for $R[1/t, It]$, since $1/t$ is a prime element and $R[1/t, It]$ is a monoidal transform over $R[1/t]$.

Chapter 2

Preliminary Definitions and Results

2.1 Cohen-Macaulay Rings and R-Sequences

In this section, we establish the definitions and some of the results for which we will use asymptotic analogues in the next section. For more about R-sequences and Cohen-Macaulay rings and modules, the reader may refer to [2], [15], [12], among others. More about associated primes may be found in [1] and [15], among other places. If one is interested in associated primes in non-Noetherian rings, Bourbaki ([1]) is especially helpful.

Definition 2.1.1 Let $a_1, \ldots, a_n \in R$, $A_i = (a_1, \ldots, a_i)R$. We say that the ordered sequence $a_1, \ldots, a_n$ is an *R-sequence* if

1. $A_n \neq R$
2. $a_i \notin Z(R/A_{i-1})$, that is $(A_{i-1} : a_i R) = A_{i-1}$ for $i = 1, \ldots, n$.

Definition 2.1.2 For a local ring $(R, \mathfrak{m})$, $a_1, \ldots, a_r$ is a *system of parameters* if $r = \dim(R) = \mathrm{ht}(\mathfrak{m})$ and if $(a_1, \ldots, a_r)R$ is $\mathfrak{m}$-primary.

Definition 2.1.3 A local (Noetherian) ring R is a *Macaulay local ring* if there is a system of parameters $a_1, \ldots, a_r$ for R that is an R-sequence. Such a system of parameters is called *distinct.* A local ring R is a *regular local ring* if there is a system of parameters $a_1, \ldots, a_r$ such that $(a_1, \ldots, a_r)R = \mathfrak{m}$. In this case $a_1, \ldots, a_r$ is called a *regular system of parameters.* In the non-local case, a Noetherian ring R is called *Cohen-Macaulay* (or *locally Macaulay*) if for every prime ideal P of R, R_P is a Macaulay local ring.

In fact, every regular system of parameters is an R-sequence (this result is stated in Bruns and Herzog's book [2, Proposition 2.2.5], among other places). Hence any regular local ring is Macaulay.

Theorem 2.1.4 [12, Theorem 121] *Let R be a Noetherian ring, I an ideal in R and M a finitely generated R-module. Assume that $M \neq IM$. Then any two maximal R-sequences on M contained in I have the same length.*

Definition 2.1.5 Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade of I on M*, denoted by

$$\mathrm{Grade}(I, M).$$

We say that $\mathrm{Grade}(I, M) = \infty$ if $M = IM$.

Definition 2.1.6 Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finitely generated R-module. Then the grade of $\mathfrak{m}$ on M is called the *depth of M*, denoted

$$\mathrm{depth}(M).$$

The following property is a generalization of Serre's normality condition.

Definition 2.1.7 A Noetherian ring R is said to satisfy (S_i) if for all $P \in \operatorname{Spec}(R)$, $\operatorname{depth}(R_P) \geq \min\{\operatorname{ht}(P), i\}$.

For an integral domain R, (S_2) is equivalent to the property that, for every prime divisor P of a non-zero principal ideal, $\operatorname{ht}(P) = 1$. (See [15, p. 183].) The following theorem shows why (S_2) is called Serre's condition for normality.

Theorem 2.1.8 [15, Corollary to Thm 11.5] *Let R be a Noetherian domain. Then R is normal (integrally closed) if and only if R satisfies (S_2) and R_P is a discrete valuation ring for each height 1 prime ideal P.*

Definition 2.1.9 Let R be a ring and I an ideal. A prime ideal P of R is called an *associated prime ideal* of I if $P = (I :_R x)$ for some $x \in R \setminus I$. The set of associated primes of I is written $\operatorname{Ass}_R(R/I)$. The associated primes of an ideal are also known as the *prime divisors*. The minimal members of this set are known as *isolated* associated primes and are denoted $\operatorname{mAss}_R(R/I)$. (Associated primes of I that are not minimal are known as *embedded* primes.) If all the prime divisors of an ideal I have the same height, then I is said to be *height unmixed*.

2.2 Asymptotic Sequences and Locally Quasi-Unmixed Rings

In this section we review many of the properties of asymptotic sequences and locally quasi-unmixed rings. The relationship between asymptotic sequences and locally quasi-unmixed rings is analogous to that of R-sequences and Cohen-Macaulay rings. Many of the results in this section are examples of reults for locally quasi-unmixed rings or asymptotic sequences that are analogues of well-known results on Cohen-Macaulay rings or R-sequences. For more information on and quasi-unmixed rings, and $\mathfrak{m}$-adic completion of local rings, one may refer to [17]. For more on integral closure of ideals, rings and modules, see [32].

Definition 2.2.1 If $(R, \mathfrak{m})$ is a local ring, let R^* be its completion in the $\mathfrak{m}$-adic topology. Then R is called *quasi-unmixed* if for every minimal prime ideal $z \in \mathrm{mAss}(R^*)$, $\dim(R^*/z) = \dim(R)$.

Definition 2.2.2 A Noetherian ring R is called *locally quasi-unmixed* if for any prime ideal P of R, R_P is quasi-unmixed.

Definition 2.2.3 If I is an ideal in a ring R, then the *integral closure* of I in R (written I_a) is the set of elements $x \in R$ such that for some $b_i \in I^i$ and some $n \in \mathbb{N}$

$$x^n + b_1 x^{n-1} + \ldots + b_n = 0.$$

Then I_a is an ideal of R, and $I \subseteq I_a \subseteq \mathrm{Rad}(I)$. Also, if J is another ideal of R such that $I \subseteq J$, then $I_a \subseteq J_a$. The following property of integral closure is called *persistence* in [32, Remark 1.1.3(7)]: if $\phi : R \to S$ is a ring homomorphism, then $\phi(I_a) \subseteq (\phi(I)S)_a$.

In 1984, Ratliff proved the following theorem, which was an improvement on an earlier result of his from 1976, which required I to have $\mathrm{ht}(I) \geq 1$ ([22, Theorem 2.5]).

Theorem 2.2.4 [26, Theorem 2.4] *Let I be an ideal in a Noetherian ring R and let Q be a prime divisor of $(I^i)_a$ for some $i \geq 1$. Then Q is a prime divisor of $(I^n)_a$ for all $n \geq i$.*

Definition 2.2.5 The ring $\mathfrak{R}(R, I) = R[It, u]$, where t is an indeterminate and $u = 1/t$, is called the *Rees ring of R with respect to I*. $\mathfrak{F}(R, I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is the *form ring of R with respect to I*.

Rees proved in [28, Theorem 2.1] that $\mathfrak{F}(R, I) \cong \mathfrak{R}(R, I)/u\mathfrak{R}(R, I)$, and we will frequently identify the two.

In the next theorem, $\mathfrak{R}'$ denotes the integral closure of $\mathfrak{R}$ (in its total quotient ring).

Theorem 2.2.6 [26, Theorem 2.7] *If I is an ideal in a Noetherian ring R, then the sets $\mathrm{Ass}_R(R/(I^i)_a)$ are equal for all large i. In fact, if $\mathfrak{R} = R[It, u]$ where t is an indeterminate and $u = t^{-1}$, then for all large i,*

$$\mathrm{Ass}_R(R/(I^i)_a) = \left\{p' \cap R \mid p' \in \mathrm{Ass}_{\mathfrak{R}'}(\mathfrak{R}'/u\mathfrak{R}')\right\}$$

$$= \{P \cap R \mid P \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/(u^n\mathfrak{R})_a) \text{ for some } n \geq 1\}.$$

We write this eventual constant value as $\hat{A}^*(I) := \mathrm{Ass}(R/(I^m)_a)$ for large m. The prime ideals $\hat{A}^*(I)$ are known as the *asymptotic primes* of I.

Definition 2.2.7 Let $b_1, \ldots, b_n \in R, B_i = (b_1, \ldots, b_i)R$. For an ideal I of R, we say an element $r \in R$ is *asymptotically prime to I* if $(r, I)R \neq R$ and $((I^m)_a :_R rR) = (I^m)_a$ for all $m \geq 1$. Then we say the ordered sequence $b_1, \ldots, b_n$ is an *asymptotic sequence over I* if

1. $(I, B_n)R \neq R$
2. for each $i = 1, \ldots, n$, $(((I, B_{i-1})^m)_a :_R b_iR) = ((I, B_{i-1})^m)_a$ for all $m \geq 1$.

The elements $b_1, \ldots, b_n$ are an *asymptotic sequence in R* if they are an asymptotic sequence over (0).

We will use asymptotic sequences over an arbitrary ideal for the statement of some preliminary results, but for the main sections of this paper, we will restrict our attention to asymptotic sequences over (0).

Observe that for $I = (0)$, the second condition is equivalent to $b_i \notin \bigcup \hat{A}^*(B_{i-1})$ for $i = 1, \ldots, n$. Indeed, $((B_{i-1}^m)_a : b_i R) = (B_{i-1}^m)_a$ for each $m \geq 1$ is equivalent to b_i is not a zero divisor on $R/(B_{i-1}^m)_a$. The set of zero divisors of $R/(B_{i-1}^m)_a$ is the union of the associated primes, $\mathrm{Ass}(R/(B_{i-1}^m)_a)$. By 2.2.4, $P \in \mathrm{Ass}(R/(B_{i-1}^m)_a)$ for some $m \geq 1$ if and only if $P \in \hat{A}^*(B_{i-1})$.

We will now record several theorems about asymptotic sequences and locally quasi-unmixed rings for later use.

Corollary 2.2.8 [19, Corollary 2.2] *Let R be a quasi-unmixed semi-local ring and let A be an ideal in R. Then R/A is quasi-unmixed if and only if* $\mathrm{ht}(A) = \mathrm{ht}(P)$ *for every prime ideal $P \in \mathrm{mAss}_R(R/A)$.*

Definition 2.2.9 If I is an ideal of a ring R, we say that I is *of the principal class* if I can be generated by $\mathrm{ht}(I)$ elements. We say I is of the principal class m if $\mathrm{ht}(I) = m$.

Theorem 2.2.10 [21, Theorem 2.29] *The following statements are equivalent for a Noetherian ring R:*

1. *R is locally quasi-unmixed.*

2. *For all ideals B of the principal class in R, $(B^i)_a$ is height unmixed, for all $i > 0$.*

3. *For all ideals B of the principal class in R such that* $\mathrm{ht}(M/B) = 1$, *for some maximal ideal M in R, $(B^i)_a : M = (B^i)_a$, for infinitely many $i > 0$.*

Remark 2.2.11 [24, Remark 2.3] Let $b_1, \ldots, b_g$ be elements of a ring R, let $B_i = (b_1, \ldots, b_i)R$ for $i = 1, \ldots, g$ and let $B_0 = (0)$. Then the following statements hold:

1. For $i = 1$, the definition of asymptotic sequence simply says that b_1 is not in any *minimal* prime ideal in R.

2. If R is Noetherian, then the second part of the definition of asymptotic sequence is equivalent to: b_i is not in any $P \in \hat{A}^*(B_{i-1})$ for $i = 1, \ldots g$.

3. If R is a Noetherian ring and $b_1, \ldots, b_g$ are an asymptotic sequence, then $\mathrm{ht}(B_i) = i$ for $i = 0, 1, \ldots, g$.

4. Let M be a maximal ideal in a Noetherian ring R and let $b_1, \ldots, b_g$ be an asymptotic sequence in R. If $b_1, \ldots, b_g$ are a maximal asymptotic sequence and $B_g \subseteq M$, then $M \in \hat{A}^*(B_g)$. The converse is also true.

5. If R is a Noetherian ring and $b_1, \ldots, b_g$ are an R-sequence, then $b_1, \ldots, b_g$ are an asymptotic sequence, but not conversely.

6. If R is a quasi-unmixed local ring and $\mathrm{ht}(B_g) = g$, then every $P \in \hat{A}^*(B_i)$ has height i (for $i = 1, \ldots, g$) and $b_1, \ldots, b_g$ are an asymptotic sequence.

The next remark concerns the passage of asymptotic sequences and asymptotic primes to and from localizations of Noetherian rings.

Remark 2.2.12 [24, Remark 2.9] Let $b_1, \ldots, b_g$ be elements in a Noetherian ring R and let S be a multiplicatively closed subset of R such that $BR_S \neq R_S$, where $B = (b_1, \ldots, b_g)R$. Then:

1. If $b_1, \ldots, b_g$ are an asymptotic sequence in R, then the images of $b_1, \ldots, b_g$ are an asymptotic sequence in R_S. (The proof uses 2.2.11(2) and the fact that $I_a R_S = (IR_S)_a$ for all ideals I in R.)

2. If I is an ideal in R and $P_S \in \mathrm{Spec}(R_S)$, then $P \in \hat{A}^*(I)$ if and only if $P_S \in \hat{A}^*(IR_S)$.

3. If $b_1, \ldots, b_g$ are an asymptotic sequence in R and $P \in \hat{A}^*(B)$, then by 2.2.12(1), 2.2.12(2) and 2.2.11(4), the images of $b_1, \ldots, b_g$ are a maximal asymptotic sequence in R_P.

4. If the b_i are in the Jacobson radical of R, then it follows immediately from 2.2.12(2) and 2.2.11(2) that $b_1, \ldots, b_g$ are an asymptotic sequence in R if and only if their images are an asymptotic sequence in R_M for all maximal ideals M in R.

Corollary 2.2.13 [24, Corollary 2.10] *If $b_1, \ldots, b_s$ are an asymptotic sequence in the Jacobson radical of R, then each permutation of the b_i is an asymptotic sequence in R.*

We now consider ideals of the principal class. As mentioned in the introduction, ideals of the principal class n in Cohen-Macaulay rings can be generated by an R-sequence of length n. Before we classify ideals of the principal class in locally quasi-unmixed rings, we first examine ideals of the principal class in Noetherian rings, as they will also prove useful later.

Definition 2.2.14 *Let R be a commutative ring with identity. A set $\{z_1, \ldots, z_m\} \subset R$ is said to be analytically independent if every homogeneous $f \in R[Z_1, \ldots, Z_m]$ such that $f(z_1, \ldots, z_m) = 0$ has its coefficients in $\mathrm{Rad}((z_1, \ldots, z_m)R)$. We say $\{z_1, \ldots, z_m\}$ is strongly analytically independent if every homogeneous $f \in R[Z_1, \ldots, Z_m]$ such that $f(z_1, \ldots, z_m) = 0$ has its coefficients in the ideal $(z_1, \ldots, z_m)R$ itself.*

Corollary 2.2.15 [3, Corollary 1] *If R is Noetherian and $I = (z_1, \ldots, z_m)R$ is an ideal of the principal class m, then $\{z_1, \ldots, z_m\}$ is analytically independent.*

This is sometimes called the theorem of "analytic independence of systems of parameters" [4]. It is proved in many places, including [3, Corollary 1]. In 1968, Davis proved the converse.

Proposition 2.2.16 [4, Theorem] *If R is Noetherian and $\{z_1, \ldots, z_m\}$ is analytically independent, then $I = (z_1, \ldots, z_m)R$ is of the principal class m.*

Thus in a Noetherian ring, an ideal is of the principal class m if and only if any generating set of m elements is analytically independent. For locally quasi-unmixed rings, we have the following result.

Proposition 2.2.17 [24, Proposition 4.6] *If B is an ideal of the principal class in a locally quasi-unmixed ring R, then B is generated by an asymptotic sequence, and if P is a prime divisor of $(B^n)_a$ for some $n \geq 1$, then $\mathrm{ht}(P) = \mathrm{ht}(B)$.*

Some of the standard results for R-sequences in Noetherian rings do not transfer to asymptotic sequences: in particular, if $b_1, \ldots, b_i$ are an asymptotic sequence in R and $b_{i+1}, \ldots, b_n$ are an asymptotic sequence in $R/((b_1, \ldots, b_i)R)$, it is not necessarily true that $b_1, \ldots, b_n$ is an asymptotic sequence. (See [24, Example 7.1.2].) However, the next result of Ratliff shows that this does hold if R is locally quasi-unmixed.

Theorem 2.2.18 [25, Theorem 3.7] *Let $b_1, \ldots, b_i$ be an asymptotic sequence over an ideal I in a locally quasi-unmixed Noetherian ring R, let $B = (b_1, \ldots, b_i)R$, and let $b_{i+1}, \ldots, b_s$ be elements in R whose images in R/B are an asymptotic sequence over $(I + B/B)$. Then $b_1, \ldots b_s$ is an asymptotic sequence over I.*

2.3 Monoidal Transforms and Rees Rings

In this section we establish properties of monoidal transforms and Rees rings for later use. Indeed, for a Noetherian ring R, an ideal I of R, and an indeterminate u, the Rees ring $R[I/u, u]$ is a monoidal transform over $R[u]$, so results on monoidal transforms are results on Rees rings. Rees rings are an important tool for study because they allow us to view an ideal I in R as a contraction of the principal ideal generated by u in $R[I/u, u]$. For more about Rees rings, one may refer to [32] and others; for monoidal transforms from a commutative algebra standpoint, see [3] and [5]. For monoidal transforms from the perspective of algebraic geometry, see [33].

We recall the following definition given in the Introduction.

Definition 2.3.1 For a ring R and elements $a, b_1, \ldots, b_n \in R$, where a is not a zero divisor, the overring $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$, a subring of the total quotient ring of R, is called a *monoidal transformation* or *transform.*

Lemma 2.3.2 [18, Theorem 2.3] *Let R be a locally Macaulay ring, let $a, b_1, \ldots, b_n$ be an R-sequence, and let $X_1, \ldots, X_n$ be algebraically independent over R. If H is the kernel of the natural homomorphism from $R[X_1, \ldots, X_n]$ onto $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$, then $H = (aX_1 - b_1, \ldots, aX_n - b_n)R[X_1, \ldots, X_n]$.*

Ratliff has given a valid analogue of this result in terms of locally quasi-unmixed rings and asymptotic sequences. Before we state it, some more definitions are in order.

Typically, we define monoidal transforms $R \subset S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$ where a is not a zero divisor. The statement $a \in R$ is an R-sequence is equivalent to saying that a is not a zero divisor on R, so if we want that $a, b_1, \ldots, b_n$ is an R-sequence for certain properties to ascend from R to S, we get that a is not a zero divisor. However, if we wish to work with asymptotic sequences, the statement a is an asymptotic sequence in R is equivalent to a is not in any minimal prime, which means that a may be a zero divisor. So to achieve the full generality, we adopt the following notation:

Let a be an element of R not contained in any minimal prime of R, let $T = \{a^k \mid k \geq 0\}$, and let $Z = \cup\{(0 : a^m R) \mid m \geq 0\}$, then: $R[\frac{1}{a}]$ denotes the ring $(R/Z)_{(T+Z)/Z}$ (or R_T). (Note that a is regular on R/Z, and $R \not\subseteq R[\frac{1}{a}]$ if a is not regular on R.)

Let S denote the subring of $R[\frac{1}{a}]$ generated over R/Z by the elements $\frac{\overline{b_i}}{\overline{a}}$, where $\overline{x}$ denotes residue class modulo Z.

Let f be the natural homomorphism $f : R \to R/Z$. Then for an ideal I in R, IS denotes $f(I)S$, and if J is an ideal in S then $J \cap R$ denotes $f^{-1}(J \cap (R/Z))$. Also, if T is a multiplicative subset of R, then S_T denotes $S_{f(T)}$.

Theorem 2.3.3 [23, Theorem 2.5] *Let $I = (a, b_1, \ldots, b_n)R$ be an ideal in a Noetherian ring R such that $a \notin \mathrm{Rad}(R)$, $B = R[X_1, \ldots, X_n]$, $Y_i = aX_i - b_i$ $(i = 1, \ldots, n)$, $K = (Y_1, \ldots, Y_n)B$ and $H = \ker(B \to S)$. Then*

1. *$K \subseteq H$ and there is a one-to-one correspondence between $z \in \mathrm{Ass}(R)$ such that $a \notin z$ and $P \in \mathrm{Ass}(B/H)$ given by $P \cap R = z$.*

2. *If z and P are corresponding ideals as in (1.) and $z \in \mathrm{mAss}R$, then P is a minimal prime divisor of H, $\mathrm{ht}(P) = n$, $(K + \mathrm{Rad}(B_P))B_P = K_a B_P = H_a B_P = PB_P$, and $L = B_P/\mathrm{Rad}(B_P)$ is a regular local ring and the images in L of the Y_i are a regular system of parameters.*

3. *$\mathrm{Rad}(K) \subseteq \mathrm{Rad}(H)$, and if I is of the principal class ($\mathrm{ht}(I) = n+1$), then $\mathrm{ht}(K) = n = \mathrm{ht}(H)$ and $\mathrm{Rad}(K) = \mathrm{Rad}(H)$.*

4. *If* $z \in \mathrm{mAss}(R)$ *is such that* $\mathrm{ht}(I+z)/z = n+1$, *then* $\mathrm{Rad}(K+zB)/zB$ *is a prime ideal* H^*. *Further,* H^* *is the* H^**-primary component of* $(K+zB)/zB$ *and of* $(H+zB)/zB$, *and* $H^* = \mathrm{Ker}((B/zB) \to (S/z^*))$, *where* $z^* = zR[\frac{1}{a}] \cap S$.

5. *If* $\mathrm{ht}(I) = n+1$ *and* R *is locally quasi-unmixed, then* $K \subseteq H \subseteq K_a = H_a = \mathrm{Rad}(H) = \mathrm{Rad}(K)$.

The next theorem of Ratliff has to do with the ascension of an asymptotic sequence over an ideal I to a certain Rees ring.

Theorem 2.3.4 [27, Theorem 3.3] *Let* $b_1, \ldots, b_s$ *be an asymptotic sequence over an ideal* I *in a Noetherian ring* R *and let* $B_i = (b_1, \ldots, b_i)R$ *for each* $i = 1, \ldots, s$. *Fix* i *and let* $\mathfrak{R} = \mathfrak{R}(R, B_i)$. *Then the following statements hold:*

1. *Let* $d_1, \ldots, d_{s+1}$ *be a permutation of* $u, tb_1, \ldots, tb_i, b_{i+1}, \ldots, b_s$ *of one of the following types:*

   (a) $d_1 = u$ *and if* $d_j = b_k$ *and* $k > i+1$, *then* $b_{k-1} = d_{j-g}$ *for some* $g \geq 1$.

   (b) $d_1 = tb_1, \ldots, d_j = tb_j$ *(for some* j $(1 \leq j \leq i)$*),* $d_{j+1} = u$, *and if* $d_h = b_k$ *and* $k > i+1$, *then* $b_{k-1} = d_{h-g}$ *for some* $g \geq 1$.

   (c) $d_j = tb_j$ $(j = 1, \ldots, i)$, $d_{i+h} = b_{i+h}$ *(for* $h = 1, \ldots, k$ *and with* $1 \leq k \leq s-i$*),* $d_{i+k+1} = u$ *and* $d_m = b_{m-1}$ *(for* $m = i+k+2, \ldots, s+1$*).*

   Then $d_1, \ldots, d_{s+1}$ *are an asymptotic sequence over* $I\mathfrak{R}$.

2. *If every permutation of* $b_1, \ldots, b_s$ *is an asymptotic sequence over* I, *then every permutation of* $u, tb_1, \ldots, tb_i, b_{i+1}, \ldots, b_s$ *is an asymptotic sequence over* $I\mathfrak{R}$.

Note that every permutation of $b_1, \ldots, b_s$ is an asymptotic sequence (over $I = (0)$) if $b_1, \ldots, b_s$ are contained in the Jacobson radical of R by Corollary 2.2.13. In fact, Ratliff shows in [27, Corollary 6.3] that, for an arbitrary ideal I of R, every permutation of $b_1, \ldots, b_s$ is an asymptotic sequence over I when the b_i are in the Jacobson radical.

Using the correspondence $u \leftrightarrow b_k$, $b_h \leftrightarrow b_h$ for $h = i+1, \ldots, s$, and $\frac{b_j}{b_k} \leftrightarrow tb_j$ for $j = 1, \ldots i$, we pass naturally from Rees algebras to monoidal transforms, resulting in the following corollary.

Corollary 2.3.5 [27, Corollary 3.6] *With the notation of* (2.3.4), *fix* k $(1 \leq k \leq i)$ *and let* $S = R[\frac{b_1}{b_k}, \ldots, \frac{b_i}{b_k}]$. *Then each permutation of the images of*

$$b_{i+1}, \ldots, b_s, \frac{b_1}{b_k}, \ldots, \frac{b_{k-1}}{b_k}, \frac{b_{k+1}}{b_k}, \ldots, \frac{b_i}{b_k}, b_k$$

which corresponds to one of the permutations in (2.3.4(1)) *is an asymptotic sequence over* IS.

Chapter 3

Monoidal Transforms over Locally Quasi-Unmixed Domains

3.1 A Sufficient Condition for S to Be a Locally Quasi-Unmixed UFD

This section deals with with some basic results concerning monoidal transforms of locally quasi-unmixed rings, including Lemma 3.1.5, which we shall use throughout chapters 3 and 4. The main result of this section gives a set of conditions for $S = R[I/a]$ to be a unique factorization domain.

It is well known that for a Cohen-Macaulay ring R and an ideal I of the principal class, R/I is also Cohen-Macaulay [18, p. 400]. Our first result is an asymptotic analogue of this.

Theorem 3.1.1 *If R is locally quasi-unmixed and A is an ideal of the principal class and H is any ideal such that* $\mathrm{Rad}(H) = \mathrm{Rad}(A)$*, then R/H is locally quasi-unmixed.*

Proof. Note that $\mathrm{Rad}(A) = \mathrm{Rad}(H)$ implies $\mathrm{mAss}(R/A) = \mathrm{mAss}(R/H)$. Let

M be a maximal ideal of R. If $P \not\subseteq M$ for some prime ideal P of R, $R_M/PR_M = 0$ is trivially quasi-unmixed. Now suppose $P \in \text{mAss}(R/H)$ and $P \subseteq M$ (and thus that $A \subseteq M$ and therefore $\text{mAss}(R_M/HR_M) = \text{mAss}(R_M/AR_M)$).

Since A is an ideal of the principal class in a locally quasi-unmixed ring, it can be generated by an asymptotic sequence by Theorem 2.2.17, and since $A \subseteq M$, the images of that asymptotic sequence in R_M form an asymptotic sequence in R_M by Theorem 2.2.12(1). Then by Remark 2.2.11(3) AR_M is an ideal of the principal class in a quasi-unmixed ring, so $\text{ht}(PR_M) = \text{ht}(AR_M)$ for each $PR_M \in \hat{A}^*(AR_M)$ by Theorem 2.2.10. Further, $\hat{A}^*(AR_M) = \text{mAss}(R_M/AR_M) = \text{mAss}(R_M/HR_M)$.

Then $\text{ht}(PR_M) = \text{ht}(HR_M)$ for each $PR_M \in \text{mAss}(R_M/HR_M)$, so by Corollary 2.2.8, $R_M/HR_M = (R/H)_M$ is quasi-unmixed for each M. ■

Corollary 3.1.2 *If R is locally quasi-unmixed, A is an ideal of the principal class, and H is any ideal of R such that $H_a = A_a$, then R/H is locally quasi-unmixed.*

Proof. If $P \in \text{mAss}(R/H)$, $P \supseteq \text{Rad}(H) \supseteq H_a = A_a$. If P is not minimal over A, there is a prime ideal Q such that $P \supset Q \supseteq \text{Rad}(A) \supseteq A_a = H_a \supseteq H$, contradicting minimality of P over H. Thus $\text{mAss}(R/H) \subseteq \text{mAss}(R/A)$. The opposite inclusion follows similarly, and therefore $\text{Rad}(H) = \text{Rad}(A)$. The result then follows from Theorem 3.1.1. ■

The following lemma from E. Davis [5, Lemma 1] records several facts about monoidal transforms for later use.

Lemma 3.1.3 [5, Lemma 1] *Let $I = (a, b_1, \ldots, b_n)R$, where a is regular on R, $B = R[X_1, \ldots, X_n]$, $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$ and $H = \ker(B \to S)$. If J is an ideal of R such that $H \subseteq JB$, then $JS \cap R = J$, $S/JS \cong (R/J)[X_1, \ldots, X_n]$, and:*

1. *If J is prime (resp. primary), then JS is prime (resp. primary).*

2. *If $J = \bigcap J_i$ for some family of ideals $\{J_i\}$, then $JS = \bigcap J_i S$.*

3. $(J :_R L)S = (JS :_S LS)$ *for any ideal L of R.*

4. *If p is an isolated prime divisor of J, then $l_{R_p}(R_p/JR_p) = l_{S_{pS}}(S_{pS}/JS_{pS})$, where $l_R(M)$ denotes the length of the R-module M.*

In fact, the lemma holds if we only require that a is not in any minimal prime of R (for example, if we wanted to assume that $a, b_1, \ldots, b_n$ is an asymptotic sequence). The additional proof follows from the definitions and the discussion above for S in the case that a is non-nilpotent. However, if $a, b_1, \ldots, b_n$ is an asymptotic sequence it is not in general true that $H \subseteq IB$ (see example below), so the conclusions of the preceding Lemma do not necessarily hold for $I = J$.

Example 3.1.4 We shall exhibit an asymptotic sequence that is not strongly analytically independent.

Recall that a set $\{z_1, \ldots, z_m\} \subset R$ is said to be *analytically independent* if every homogeneous $f \in R[Z_1, \ldots, Z_m]$ such that $f(z_1, \ldots, z_m) = 0$ has its coefficients in $\mathrm{Rad}(z_1, \ldots, z_m)$. A set $\{z_1, \ldots, z_m\} \subset R$ is said to be *strongly analytically independent* if every homogeneous $f \in R[Z_1, \ldots, Z_m]$ such that $f(z_1, \ldots, z_m) = 0$ has its coefficients in $(z_1, \ldots, z_m)R$ itself. Davis shows in [3, Remarks 1.b, 1.b$'$] that these two conditions have equivalent formulations in terms of monoidal transforms, namely that $\{a, b_1, \ldots, b_n\}$ is analytically independent if and only if $H \subseteq \mathrm{Rad}(I)B$, where I, H, and B are as in Lemma 3.1.3; and that $\{a, b_1, \ldots, b_n\}$ is strongly analytically independent if and only if $H \subseteq IB$. Let k be a field, X, Y indeterminates. Set $R = k[[X, Y]]/(X^2, XY) \cong k[[x, y]]$. As associated primes are (prime) annihilators of elements, $(x) = (0 :_R y)$, $(x, y) = (0 :_R$

x) and all other elements are regular (in fact, all elements of R not in $(x, y)R$ are units), R has associated primes $(x) \subset (x, y)$. Then y is not in any minimal (i.e. asymptotic prime of (0)), but $y \in (x, y)$, an associated prime. So y is an asymptotic sequence but not an R-sequence.

Let $f = xZ \in R[Z] \setminus yR[Z]$. Then $f(y) = xy = 0$, and so f is a homogeneous polynomial which is zero on y whose coefficients are not in $yR[Z]$. Therefore $\{y\}$ is not a strongly analytically independent set.

This brings us to the following asymptotic analogue of [9, Proposition 2.2].

Lemma 3.1.5 *Let $a, b_1, \ldots, b_n$ be an asymptotic sequence in a locally quasi-unmixed ring R, $(a, b_1, \ldots, b_n)R = I$, and let $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$. Then*

1. *S is locally quasi-unmixed*
2. *$a, c_1, \ldots, c_n$ is an asymptotic sequence on S for any permutation $c_1, \ldots, c_n$ of $\frac{b_1}{a}, \ldots, \frac{b_n}{a}$*
3. *$\mathrm{Rad}(aS) \cap R = \mathrm{Rad}(IS) \cap R = \mathrm{Rad}(I)$, $S/\mathrm{Rad}(aS) \cong (R/\mathrm{Rad}(I))[X_1, \ldots, X_n]$, and there is a one-to-one correspondence between elements of $\hat{A}^*(aS)$ and elements of $\hat{A}^*(I)$ given by p $(\in \hat{A}^*(I)) = P \cap R$ with $P \in \hat{A}^*(aS)$ and $P = pS$.*
4. *Each $q \in \hat{A}^*(I)$ has height $n + 1$.*

Additionally, if $a, b_1, \ldots, b_n$ is an R-sequence, then (3) becomes $aS \cap R = IS \cap R = I$, $S/aS \cong (R/I)[X_1, \ldots, X_n]$, so there is a one-to-one correspondence between elements of $\mathrm{Ass}_S(S/aS)$ and elements of $\mathrm{Ass}_R(R/I)$ given by p $(\in \mathrm{Ass}_R(R/I)) = P \cap R$ with $P \in \mathrm{Ass}_S(S/aS)$ and $P = pS$.

Proof. For (1), let $B = R[X_1, \ldots, X_n]$, $K = (aX_1 - b_1, \ldots, aX_n - b_n)B$ and $H = \ker(B \to S)$ be as in Theorem 2.3.3. Since I is generated by an asymptotic sequence, it is an ideal of the principal class (by Remark 2.2.12), K is of the principal class by Theorem 2.3.3. Then by Theorem 3.1.1, $B/H \cong S$ is locally quasi-unmixed.

For (2), Ratliff proved this for general Noetherian rings with Corollary 2.3.5.

For (3), observe that $K = (aX_1 - b_1, \ldots, aX_n - b_n)B \subseteq IR[X_1, \ldots, X_n] = IB$. So by Theorem 2.3.3, $H \subseteq \mathrm{Rad}(H) = \mathrm{Rad}(K) \subseteq \mathrm{Rad}(IB)$. Thus most of the conclusions follow from Lemma 3.1.3 and it remains to show the correspondence between $\hat{A}^*(I)$ and $\hat{A}^*(aS)$.

It is clear from above that

$$\{P \in \mathrm{Spec}(B) \mid P \supseteq (\mathrm{Rad}(I)B)\} = \{P \in \mathrm{Spec}(B) \mid P \supseteq IB\}$$

is in one-to-one correpsondence with

$$\{P \in \mathrm{Spec}(S) \mid P \supseteq IS\} = \{P \in \mathrm{Spec}(S) \mid P \supseteq aS\} = \{P \in \mathrm{Spec}(S) \mid P \supseteq \mathrm{Rad}(aS)\}.$$

Since R is locally quasi-unmixed and $a, b_1, \ldots, b_n$ is an asymptotic sequence in R, for any associated prime divisor P of $(I^n)_a$ for some $n \geq 1$, $\mathrm{ht}(P) = \mathrm{ht}(I)$ (Proposition 2.2.17). Therefore P is minimal over I and $\hat{A}^*(I) = \mathrm{mAss}_R(R/I)$. Similarly, S is locally quasi-unmixed and a is an asymptotic sequence in S, so $\hat{A}^*(aS) = \mathrm{mAss}_S(S/aS)$. So suppose $p \in \hat{A}^*(I) = \mathrm{mAss}_R(R/I)$. Then pB is minimal over IB, and so $pS \in \mathrm{mAss}_S(S/aS) = \hat{A}^*(aS)$. For $P \in \mathrm{mAss}_S(S/aS)$, there is a $Q \in \mathrm{mAss}_B((B/IB)$ corresponding to it. Then $Q \cap R = p \in \mathrm{mAss}_R(R/I)$ and $pB = Q$. Thus $pS = P$ and $P \cap R = p$.

(4) is given by Proposition 2.2.17. ■

Recall that an ideal J is *pre-normal* if all large powers of J are integrally closed, and J is *normal* if J^n is integrally closed for all positive integers n. The following two theorems will help provide conditions for S to be integrally closed. The first was proved by Lipman and Mattuck independently, and the second is a theorem of Goto.

Theorem 3.1.6 [14, Lemma 5.2][16, Theorem 1] *Let R be an integrally closed Noetherian domain and J an ideal of R. Then every monoidal transform $R[J/b]$ with respect to J is integrally closed if and only if J is pre-normal, where b is a nonzero element of J.*

Theorem 3.1.7 [8, Theorem 1.1] *If J is an ideal of the principal class in the Noetherian ring R, the following are equivalent:*

1. *J is integrally closed*
2. *J is normal*
3. *For each $p \in \operatorname{Ass}_R(R/J)$, R_p is regular and $l_{R_p}((JR_p + p^2R_p)/p^2R_p) \geq \operatorname{ht}(J) - 1$. When this holds, each $p \in \operatorname{Ass}_R(R/J)$ is minimal over J and J is generated by an R-sequence.*

We use the following extension of [9, Remark 2.3].

Remark 3.1.8 *Let R be an integrally closed Noetherian domain, let I be generated by the analytically independent set $\{a, b_1, \ldots, b_n\}$, and $S = R[I/a]$. Then for the following statements, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5). If $a, b_1, \ldots, b_n$ is an R-sequence, then (4) $\Rightarrow$ (1).*

1. *I is integrally closed*
2. *I is normal*
3. *I is pre-normal*
4. *S is integrally closed*
5. *S_q is integrally closed for each prime divisor q of aS.*

Proof. By [4, Theorem], if R is Noetherian and $\{a, b_1 \ldots, b_n\}$ is analytically independent, then I is an ideal of the principal class $n+1$. Thus the hypotheses of theorems 3.1.6 and 3.1.7 are satisfied.

It is clear that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). The implication (1) $\Rightarrow$ (2) holds by Theorem 3.1.7, and (3) $\Rightarrow$ (4) by Theorem 3.1.6.

Now assume that (5) holds and let q be a prime divisor of a principal ideal in S. If $a \in q$, then S_q is integrally closed, by (5). If $a \notin q$, then $S[\frac{1}{a}] = R[\frac{1}{a}]$ is integrally closed, since R is, and $S_q = S[\frac{1}{a}]_{qS[\frac{1}{a}]}$, so S_q is integrally closed. Therefore S_q is integrally closed for each prime divisor q of a principal ideal in S, so S is integrally closed, by [12, Theorem 54], hence (5) $\Rightarrow$ (4).

Finally, let $a, b_1, \ldots, b_n$ be an R-sequence. Then we have that

$$\ker(R[X_1, \ldots, X_n] \to S) \subseteq IR[X_1, \ldots, X_n] \subseteq I_a R[X_1, \ldots, X_n],$$

so by Lemma 3.1.5 we get that $aS \cap R = I$ and $I_a S \cap R = I_a$. Also, S integrally closed implies that aS is integrally closed [32, Proposition 1.5.2]. Using persistence of integral closure as defined in Definition 2.2.3, $I_a S \subseteq (IS)_a$, so $aS = IS \subseteq I_a S \subseteq (IS)_a = (aS)_a = aS$. Thus $I_a = I_a S \cap R = aS \cap R = I$, so (4) $\Rightarrow$ (1). ■

Remark 3.1.9 If R is a Noetherian domain, $a, b_1, \ldots, b_n$ is an R-sequence, and $S = R[I/a]$ satisfies (S_2), then aS is primary if and only if $\mathrm{Rad}(aS)$ is prime, since principal ideals in domains which satisfy (S_2) have no embedded primes. Moreover, by Lemma 3.1.5, the elements of $\mathrm{Ass}_R(R/I)$ are in one-to-one correspondence with the elements of $\mathrm{Ass}_S(S/aS)$, so I is primary if and only if aS is primary if and only if $\mathrm{Rad}(aS)$ is prime if and only if $\mathrm{Rad}(I)$ is prime.

We will use this fact in two particular cases. If R is an integrally closed

Noetherian domain, $a, b_1, \ldots, b_n$ are an R-sequence, $I = (a, b_1, \ldots, b_n)R$ is integrally closed, and $S = R[I/a]$, then S is integrally closed by Remark 3.1.8, so S satisfies (S_2) by Corollary 2.1.8.

If R is locally quasi-unmixed unique factorization domain, $a, b_1, \ldots, b_n$ is an R-sequence, $I = (a, b_1, \ldots, b_n)R$, and $S = R[I/a]$, then S satisfies (S_2) by [5, Theorem]

We now introduce a sufficient condition for S to be a UFD. For R a Krull domain, we will denote its divisor class group by $\mathrm{Cl}(R)$.

Theorem 3.1.10 [9, cf. Theorem 2.4] *Assume that R is an integrally closed Noetherian domain and that $I = (a, b_1, \ldots, b_n)R$ is of the principal class $n+1$. Then:*

1. *If I is integrally closed, then S is integrally closed, and there is a surjective homomorphism $\phi : \mathrm{Cl}(S) \to \mathrm{Cl}(S[\frac{1}{a}])$ whose kernel is generated by the classes of elements of $\mathrm{mAss}_S(S/aS)$.*

2. *If I is integrally closed, $\mathrm{Rad}(I)$ is prime, and if $\mathrm{Cl}(R)$ is torsion (resp. finite, resp. trivial), then $\mathrm{Cl}(S)$ is torsion (resp. finite, resp. finite cyclic).*

3. *If I is prime and a is a product of prime elements of R, then $aS \in \mathrm{Spec}(S)$ and the divisor class groups $\mathrm{Cl}(R)$ and $\mathrm{Cl}(S)$ are isomorphic.*

4. *If I is prime and R is a UFD, then S is a UFD.*

Proof. By [3, Corollary 1], $\{a, b_1, \ldots, b_n\}$ is analyticaly independent, since I is of the principal class $n+1$. Thus if I is integrally closed, then S is integrally closed, by Remark 3.1.8, so S is a Krull domain. Then [7, Corollary 7.2] tells us that we have a surjection between the class group $\mathrm{Cl}(A)$ of a Krull domain A and the class group $\mathrm{Cl}(A_M)$ of its localization A_M for each multiplicative set M, and that the kernel is generated by the classes of the height one prime ideals which meet M. For our case, $\mathrm{Cl}(S)$ surjects onto $\mathrm{Cl}(S[\frac{1}{a}])$, and the kernel is generated by the classes of the height one primes of S containing a, which necessarily belong to the set of minimal primes of aS. Since a is a regular non-unit in the integrally closed Noetherian domain S, these are exactly the minimal primes of aS, $\mathrm{mAss}_S(S/aS)$.

For (2), if I is integrally closed, then S is a Krull domain, by (1). Also, by Theorem 3.1.7, I is generated by an R-sequence. Therefore, if $\mathrm{Rad}(I)$ is prime, Lemma 3.1.5 gives a one-to-one correspondence between $\mathrm{Ass}_R(R/I)) = \{\mathrm{Rad}(I)\}$ and $\mathrm{mAss}_S(S/aS) = \{\mathrm{Rad}(aS)\}$. Thus $\mathrm{Ker}(\phi)$ is the finite cyclic subgroup of $\mathrm{Cl}(S)$ generated by the class of $(\mathrm{Rad}(aS))$. If $\mathrm{Cl}(R)$ is torsion, so is its image, $\mathrm{Cl}(R[\frac{1}{a}])$. This means that for any $g \in \mathrm{Cl}(S)$, $(\phi(g))^m = (0)$ for some m, or that $g^n \in \mathrm{Ker}(\phi) = \langle(\mathrm{Rad}(aS))\rangle$. R is Noetherian, so there is a k such that $(\mathrm{Rad}(aS))^k \subseteq aS$. But aS is a principal divisorial ideal, so is in the same coset as (0) in $\mathrm{Cl}(S)$, i.e. $(\mathrm{Rad}(aS))$ has finite order, thus $\mathrm{Cl}(S)$ is torsion. If $\mathrm{Cl}(R)$ is finite, $\mathrm{Cl}(S[\frac{1}{a}])$ is finite, and $\mathrm{Cl}(S)/\langle(\mathrm{Rad}(aS))\rangle \cong \mathrm{Cl}(S[\frac{1}{a}])$, so by Lagrange's Theorem,

$$[\mathrm{Cl}(S) : (0)] = [\mathrm{Cl}(S) : \langle(\mathrm{Rad}(aS))\rangle][\langle(\mathrm{Rad}(aS))\rangle : (0)] < \infty.$$

If $\mathrm{Cl}(R)$ is trivial, so is $\mathrm{Cl}(R[\frac{1}{a}])$. Since ϕ is surjective, $\mathrm{Cl}(S) = \mathrm{Ker}(\phi) = \langle(\mathrm{Rad}(aS))\rangle$. Thus $\mathrm{Cl}(S)$ is finite cyclic.

$$\begin{tikzcd}
\mathrm{Cl}(R) \arrow[dr, "\psi"] & & \mathrm{Cl}(S) \arrow[dl, two heads, "\phi"'] \\
& \mathrm{Cl}(R[\frac{1}{a}]) = \mathrm{Cl}(S[\frac{1}{a}]) &
\end{tikzcd}$$

For (3), if I is prime, $I \subseteq I_a \subseteq \mathrm{Rad}(I) = I$. Then since $a, b_1, \ldots, b_n$ is analytically independent, S is integrally closed (and therefore a Krull domain) by Remark 3.1.8, and $\ker(R[X_1,\ldots,X_n] \to S) \subseteq \mathrm{Rad}(IR[X_1,\ldots,X_n]) = IR[X_1,\ldots,X_n]$, so $a, b_1, \ldots, b_n$ is in fact strongly analytically independent. Then by Lemma 3.1.3, aS is prime, so $\mathrm{Ass}_S(S/aS) = \{aS\}$. We then have that $\mathrm{Ker}(\phi)$ is generated by (aS), which is in the equivalence class of (0). Thus ϕ is an isomorphism. Since a is a product of prime elements of R, the canonical surjection ψ is also an isomorphism by [7, Corollary 7.3]. Then $\psi^{-1} \circ \phi$ must also be an isomorphism.

(4) then follows from (3). ■

Remark 3.1.11 It follows from Lemma 3.1.5 that if R, I, and S are as above and, additionally, R is locally quasi-unmixed, then S is locally quasi-unmixed for each of Theorem 3.1.10(1)-(4). In particular, if R is a locally quasi-unmixed UFD and $I = (a, b_1, \ldots, b_n)R$ is of the principal class $n+1$, then S is a locally quasi-unmixed UFD.

Corollary 3.1.12 [9, cf. Corollary 2.7] *Assume R is a locally quasi-unmixed UFD, that $a, b_1, \ldots, b_n$ is a permutable asymptotic sequence, and that $I = (a, b_1, \ldots, b_n)R$ is a prime ideal. Then each of the rings $S_j = R[\frac{I}{b_j}]$ is a locally quasi-unmixed UFD and $b_j S_j \in \mathrm{Spec}(S_j)$.*

Proof. This follows from [18, Theorem 2.4], 3.1.5 and 3.1.10 (4). ■

The following two theorems of Samuel and Li respectively were listed in [9] as special cases of Theorem 2.4 (of which Theorem 3.1.10 is the asymptotic analogue) from same article:

Corollary 3.1.13 [31, Proposition 7.6, p. 28] *If A is an integrally closed Noetherian domain and if $aA \cap bA = abA$, and if aA and $(a, b)A$ are prime ideals, then $A' = A[X]/(aX - b)$ is again integrally closed and the class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(A')$ are canonically isomorphic.*

Corollary 3.1.14 [13, Corollary 2.4] *If A is a Noetherian UFD, $aA \cap bA = abA$, and $(a,b)A$ is a prime ideal, then $B = A[\frac{b}{a}]$ is a UFD.*

Note that for a Noetherian domain R, the conditions $aR \cap bR = abR$, and $(a,b)R$ is a prime ideal imply that a, b is an R-sequence. Since R is a domain it suffices to show that $(aR : bR) = aR$. Suppose $rb \in aR$ for some $r \in R$. Then $rb = ar'$ for some $r' \in R$ and $rb = ar' \in aA \cap bR = abR$. Thus $rb = ar' = abr''$ for some r''. Since R is a domain, $rb = abr''$ implies $r = ar''$, and hence b is not a zero divisor on aR and a, b is an R-sequence. Conversely, if R is an integrally closed Noetherian domain and a, b is an asymptotic sequence in R, then a, b is an R-sequence since integrally closed Noetherian domains satisfy (S_2) (and therefore $\mathrm{Ass}_R(R/aR) = \mathrm{mAss}_R(R/aR) = \hat{A}^*(aR)$). It is clear that $abR \subseteq aR \cap bR$, so suppose $r \in aR \cap bR$. Then $r = ax = by$ for some $x, y \in R$. Therefore $y \in (aR : bR) = aR$, $y = ar'$, and $r = by = abr' \in abR$.

These corollaries are technically not special cases of [9, Theorem 2.4], because, as stated, [9, Theorem 2.4] requires that A be Cohen-Macaulay. However, as the proof of Theorem 3.1.10 (3) shows, we do not need to assume that A is locally quasi-unmixed or Cohen-Macaulay to obtain the isomorphism of class groups.

It is well known that an integral domain is a UFD if and only if each height-one prime ideal is principal [12, Theorem 5]. For a Krull domain A, $\mathrm{Cl}(A)$ is torsion if and only if each height-one prime ideal of A has a principal primary ideal [7, Proposition 6.8]. Therefore, if we want to know how far $S = R[I/a]$ is from a UFD, we want to investigate which height-one primes P of S are principal, and which have a principal primary ideal. We consider two cases: when $\mathrm{ht}(P \cap R) = 1$ (section 3.2) and when $\mathrm{ht}(P \cap R) > 1$ (section 3.3).

The last result in this section is the asymptotic version of [9, Proposition 2.9].

If P is a height-one prime ideal of $S = R[I/a]$, the next lemma shows that if R is locally quasi-unmixed and $a, b_1, \ldots, b_n$ is an asymptotic sequence, there are only two possible values of $\mathrm{ht}(P \cap R)$, and will be helpful in the next two sections as we classify such P.

Lemma 3.1.15 [9, cf. Proposition 2.9] *Let $a, b_1, \ldots, b_n$ be an asymptotic sequence in a locally quasi-unmixed domain R, $I = (a, b_1, \ldots, b_n)R$, and let $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$. For any $P \in \mathrm{Spec}(S)$, let $p = P \cap R$. Then the following hold:*

1. $\mathrm{ht}(P) \leq \mathrm{ht}(p) \leq \mathrm{ht}(P) + n$.

2. *If $a \notin p$, then $\mathrm{ht}(p) = \mathrm{ht}(P)$ and $P = pR[\frac{1}{a}] \cap S$ (so $S_P = R_p$).*

3. *If $\mathrm{ht}(P) = 1$, then $\mathrm{ht}(p)$ is either 1 or $n+1$, and $\mathrm{ht}(p) = n+1$ if and only if $a \in p$ if and only if $p \in \hat{A}^*(I)$. Moreover, if $\mathrm{ht}(p) = n+1$, then $P = pS$.*

Proof. (1) By [20, Theorem 3.6], a Noetherian domain R is locally quasi-unmixed if and only if the dimension equality (also called the altitude formula) holds between R and any finitely generated extension of R that is also a domain. In particular, the dimension formula holds between R and S. Thus for any prime ideal P of S, if we let $p = P \cap R$, we have

$$\mathrm{ht}(P) + \mathrm{tr.deg.}[S/P : R/p] = \mathrm{ht}(p) + \mathrm{tr.deg.}[S : R].$$

But S is algebraic over R, so this reduces to

$$\mathrm{ht}(P) + \mathrm{tr.deg.}[S/P : R/p] = \mathrm{ht}(p).$$

Since S/P is an extension of R/p by n elements (the residues of $b_1/a, \ldots, b_n/a$ modulo P), $\mathrm{ht}(p)$ is at most n larger than $\mathrm{ht}(P)$.

(2) If $a \notin p$, $pR_p \cap S$ is the only prime ideal lying over p, so $P = pR_p \cap S$, and $R_p = S_P$, by [3, Lemma]. Since $a \notin p$, $R_p \supseteq R[\frac{1}{a}]$, and thus

$$P = pR_p \cap S = pR[\frac{1}{a}] \cap S = pS[\frac{1}{a}] \cap S.$$

(3) If $\mathrm{ht}(P) = 1$ and $\mathrm{ht}(p) > 1$, then by (2), $a \in p$. By [3, Lemma], $a \in p$ if and only if $I \subseteq p$. If $I \subseteq p$, $\mathrm{ht}(p) \geq n+1$. But from (1) we have that $\mathrm{ht}(p) \leq \mathrm{ht}(P)+n = 1+n$. Then p is minimal over I and $p \in \hat{A}^*(I)$ By Lemma 3.1.5, $P = pS$. Conversely, if $p \in \hat{A}^*(I)$, $\mathrm{ht}(p) = n+1$ by Lemma 3.1.5 and clearly $a \in I \subseteq p$. ■

Note that if R is a locally quasi-unmixed ring (which may contain zero divisors), for any $z \in \mathrm{mAss}(R)$, R/z is a locally quasi-unmixed domain and satisfies the dimension formula. Let P be a prime ideal of S and let z be a minimal prime of S such that $P \supseteq z$. Let $p = P \cap R$ and $z' = z \cap R$. Then

$$\mathrm{ht}(P/z) + \mathrm{tr.deg.}[S/z/P/z : R/z'/p/z'] = \mathrm{ht}(p/z') + \mathrm{tr.deg.}[S/z : R/z']$$

$$\mathrm{ht}(P/z) + \mathrm{tr.deg.}[S/P : R/p] = \mathrm{ht}(p/z').$$

Thus 3.1.15(1) holds for locally quasi-unmixed rings that may have zero divisors.

3.2 The case where ht$(P \cap R) = 1$

Throughout this section, let R be a Noetherian ring, $(a, b_1, \ldots, b_n)R = I$, $S = R[\frac{b_1}{a}, \ldots, \frac{b_n}{a}]$.

As mentioned above, this section examines height-one prime ideals P of S such that $\mathrm{ht}(P \cap R) = 1$ to discover when P is principal or when P has a principal primary ideal. The first result in this section is an extension of [5, Theorem 2] from Noetherian domains to Noetherian rings.

Lemma 3.2.1 *Let R be a Noetherian ring, let $a, b_1, \ldots, b_n$ be an R-sequence, and let $I = (a, b_1, \ldots, b_n)R$ be height unmixed. Then if R satisfies Serre's condition (S_2), so does $S = R[I/a]$.*

Proof. Let $x \in S$ be a regular non-unit, $P \in \mathrm{Ass}_S(S/xS)$, and $p = P \cap R$. Then x is a maximal S-sequence in P, so $\mathrm{Grade}(xS) = \mathrm{Grade}(P) = 1$. Thus for any other regular element $y \in P$, y is a maximal S-sequence in P, so $P \in \mathrm{Ass}_S(S/yS)$.

Suppose $a \notin P$. Since $R[1/a] = S[1/a]$, we have $PS[1/a] \cap S = PR[1/a] \cap S = P$ and $pR[1/a] \cap R = p$. Thus $pR[1/a] \cap R = p = P \cap R = (PS[1/a] \cap S) \cap R$, and $PS[1/a] \cap (S \cap R) = PS[1/a] \cap R = PR[1/a] \cap R$. So $pR[1/a] = PS[1/a]$. Now $pR[1/a]$ contains a regular element $\frac{y}{a}$, so $\frac{ya}{a} = \frac{y}{1}$ is also a regular non-unit in $pR[1/a]$. Then $\frac{y}{1} \notin Q$ for each $Q \in \mathrm{Ass}(R[1/a])$. Suppose $y \in q$ for some $q \in \mathrm{Ass}(R)$. By the one-to-one correspondence between prime ideals of $R[1/a]$ and prime ideals of R which do not contain a, we must have $a \in q$. But a is regular and q consists of zero divisors. Therefore y is also regular in R. Then $1 = \mathrm{Grade}(P) = \mathrm{Grade}(PS[1/a]) = \mathrm{Grade}(pR[1/a]) \geq \mathrm{Grade}(p) \geq 1$. Thus $p \in \mathrm{Ass}_R(R/yR)$, so by hypothesis $1 = \mathrm{ht}(p) = \mathrm{ht}(pR[1/a]) = \mathrm{ht}(PS[1/a]) = \mathrm{ht}(P)$.

Now suppose $a \in P$. Thus $P \in \mathrm{Ass}_S(S/aS)$, so by Lemma 3.1.3, $p \in \mathrm{Ass}_R(R/I)$. Then hypothesis p is minimal over I and $\mathrm{ht}(p) = n + 1$, so P is minimal over aS. Then by the Prinicpal Ideal Theorem, $\mathrm{ht}(P) = 1$. ■

The following lemma is an extension of [18, Corollaries 3.6, 3.7] which is necessary for Lemma 3.2.3, itself an extension of [9, Lemma 3.2].

Lemma 3.2.2 *Assume that R is a locally quasi-unmixed ring satisfying (S_2). Let $a, b_1, \ldots, b_n$ be an R-sequence and assume that I is height unmixed. Then for each $e \geq 1$ and $k \geq e$, $(I^k :_R a^e R) = I^{k-e}$.*

Proof. Let $R[u, ta, tb_1, \ldots, tb_n] = \mathfrak{R}(R, I) = \mathfrak{R}$. For any ideal B in R let

$B' = BR[u,t] \cap \mathfrak{R}$. For any homogeneous ideal B^* in $\mathfrak{R}$ let $[B^*]_k = \{r \in R \mid rt^k \in B^*\}$. It follows that $[B^*]_k$ is an ideal in R and $I^k \supseteq [B^*]_k$. We may decrease the degree of any element of B^* by multiplying by u. That is, for any $x \in [B^*]_{k+1}$, $xt^{k+1} \in B^*$, so $xt^k = (xt^{k+1})u \in B^*$, and $x \in [B^*]_k$. Similarly, we may increase the degree of an element in B^* by multiplying by a nonzero element of It, so for any $x \in [B^*]_k$, $xt^k \in B^*$, so $xyt^{k+1} = (xt^k)yt \in B^*$ for some $y \in I$, thus $yx \in I[B^*]_k$. Collectively, we see that $I^k \supseteq [B^*]_k \supseteq [B^*]_{k+1} \supseteq I[B^*]_k$ for all integers k (using the convention that $I^k = R$ if $k \leq 0$). Since R is Noetherian, if k is greater than or equal to the maximum degree of the generators of B^*, then $[B^*]_{k+1} = I[B^*]_k$, and therefore $[B^*]_{k+j} = I^j[B^*]_k$. If k is less than or equal to the minimum degree of the generators of B^*, then $[B^*]_{k+1} = [B^*]_k$. Now consider $B = a^eR \subset I^e$ and $B^* = a^et^e\mathfrak{R}$. Clearly $B' = a^eR[u,t] \cap \mathfrak{R} \supseteq B^*$. For $k \leq e$, $[B']_k = a^eR \cap I^k = a^eR = [B^*]_e = [B^*]_k$. For $k > e$, $[B']_k = a^eR \cap I^k \supseteq [B^*]_k = I^{k-e}[B^*]_e = I^{k-e}B = a^eI^{k-e}$. Since $B'R[u,t] = B^*R[u,t] = a^eR[u,t]$, $B^* = B'$ if and only if u is not in any prime divisor of B^*.

Since R satisfies (S_2), $R[u]$ also satisfies (S_2). Let $\mathrm{Ass}_R(R/I) = \{q_1, \ldots, q_m\}$. Then $\mathrm{Ass}_{R[u]}(R[u]/(u,I)R[u]) = \{(u,q_i)R[u]\}_{i=1}^m$, so that $\mathrm{ht}(q_i) = \mathrm{ht}(q_j)$ for each $i, j = 1, \ldots, m$, $\mathrm{ht}(q_iR[u]) = \mathrm{ht}(q_jR[u])$, and thus

$$\mathrm{ht}((u,q_i)R[u]) = \mathrm{ht}(q_iR[u]) + 1 = \mathrm{ht}(q_jR[u]) + 1 = \mathrm{ht}((u,q_j)R[u]).$$

So $(u,I)R[u]$ is height unmixed. Then since $\mathfrak{R} = R[u, ta, tb_1, \ldots, tb_n]$ is a monoidal transform over $R[u]$, by [5, Theorem 2], $\mathfrak{R}$ satisfies (S_2).

From Theorem 2.3.4 we see that at, u is an asymptotic sequence in $\mathfrak{R}$, and since $\mathrm{Rad}(at\mathfrak{R}) = \mathrm{Rad}(a^et^e\mathfrak{R})$ and R is locally quasi-unmixed, $\hat{A}^*(at\mathfrak{R}) = \hat{A}^*(a^et^e\mathfrak{R})$, and therefore a^et^e, u is an asymptotic sequence in $\mathfrak{R}$. Because $\mathfrak{R}$ satisfies (S_2), a^et^e, u is in fact an $\mathfrak{R}$-sequence, so $B' = B^*$. In particular, $[B']_k = a^eR \cap I^k = a^eI^{k-e} = [B^*]_k$

for $k > e$. Finally, since a is not a zero divisor in R,

$$I^{k-e} = (a^e I^{k-e} :_R a^e R) = ((a^e R \cap I^k) :_R a^e R) = (I^k :_R a^e R).$$

■

We use the next lemma frequently, including in the statement of Proposition 3.2.4. The lemma shows that each element $\beta \in S \setminus I$ has a unique representation as $\beta = x/a^h$, where $x \in I^h \setminus (I^{h+1} \cup aR)$. Note that elements of I cannot be written in this form. Later in this section, and in the next, this lemma will help us classify prime ideals with principal primary ideals and prime ideals that are principal.

Lemma 3.2.3 *Assume that R is a locally quasi-unmixed ring satisfying Serre's condition (S_2). Let $a, b_1, \ldots, b_n$ be an R-sequence and assume that I is height unmixed. For each element $\beta \in S \setminus I$ there exists a unique nonnegative integer h and a unique element $x \in I^h \setminus (I^{h+1} \cup aR)$ such that $\beta = x/a^h$. (If $\beta \in S \setminus R$, then $h > 0$.)*

Proof. Every element $\beta \in S \setminus I$ may be written $\beta = y_k(a, b_1, \ldots, b_n)/a^k$ for all large integers k, where $y_k(X_0, X_1, \ldots, X_n) \in R[X_0, X_1, \ldots, X_n]$ is a form of degree k. Then since $a, b_1, \ldots, b_n$ is an R-sequence and hence strongly analytically independent, $y_k(a, b_1, \ldots, b_n) \in I^k \setminus I^{k+1}$. If $y_k \in aR$, then let i be the positive integer such that $y_k \in a^i R \setminus a^{i+1} R$, so $y_k = x a^i$ for some $x \in R$ (and note that $i \leq k$). Then by Lemma 3.2.2, $x \in (I^k :_R a^i R) = I^{k-i}$, so $\beta = x/a^{k-i}$ with $x \in I^{k-i}$, and by choice of k and i we see that $x \notin (I^{k-i+1} \cup aR)$.

Suppose $\beta = \frac{x}{a^h} = \frac{x'}{a^{h'}}$, and without loss of generality, assume $h' \geq h$. Since a is regular, $x a^{h'} = x' a^h$, and $x a^{h'-h} = x'$. If $h' > h$, then $x' \in aR$, which is a contradiction. So $h' = h$, and hence $x = x'$. Finally, it is clear that if $\beta \in S \setminus R$, $h > 0$. ■

Proposition 3.2.4 has an asymptotic version of [9, Proposition 3.3]. The original result on R-sequences did not assume that R is an integral domain, but we retain the hypothesis to make some of the proofs easier and because after this result we will assume that R is a domain for the rest of the chapter and the next. The proposition gives several characterizations for an element $\beta \in S \setminus I$ to have the property that a, β is an S-sequence or $(a, \beta)S = S$; and characterizations for β to have the property that a, β is an asymptotic sequence in S or $(a, \beta)S = S$.

Proposition 3.2.4 *Let R be a locally quasi-unmixed domain, and let $a, b_1, \ldots, b_n$ be an asymptotic sequence in R. Let β be a nonzero nonunit in $S \setminus I$ such that there exist h, a nonnegative integer, and an $x \in I^h \setminus (I^{h+1} \cup aR)$, where $\beta = x/a^h$. Also, let $\mathfrak{R} = \mathfrak{R}(R, I) = R[u, ta, tb_1, \ldots, tb_n]$. Consider the following:*

1. *$(\beta S :_S aS) = \beta S$.*

2. *Either a, β is an S-sequence or $(a, \beta)S = S$.*

3. *$\beta S[\frac{1}{a}] \cap S = \beta S$.*

4. *Either $u, t^h x$ is an $\mathfrak{R}$-sequence or $(u, t^h x)\mathfrak{R} = \mathfrak{R}$ (in which case $h = 0$ and $(I, x)R = R$).*

5. *Either $u, t^h x$ is an $\mathfrak{R}[\frac{1}{ta}]$-sequence or $(u, t^h x)\mathfrak{R}[\frac{1}{ta}] = \mathfrak{R}[\frac{1}{ta}]$.*

6. *$x \in I^h \setminus \bigcup \{qI^h \mid q \in \mathrm{Ass}_R(R/I)\}$.*

7. *$x + I^{h+1}$ is a regular element in the form ring $\mathfrak{F}(R, I)$.*

8. *For all nonnegative integers e and for all $y \in I^e \setminus I^{e+1}$ it holds that $xy \in I^{h+e} \setminus I^{h+e+1}$.*

and their asymptotic analogues:

1.* *$((\beta^m S)_a) :_S aS) = (\beta^m S)_a$ for large m.*

*2.** *Either* a, β *is an asymptotic sequence in* S *or* $(a, \beta)S = S$.

*3.** $(\beta^m S)_a = (\beta^m S[\frac{1}{a}])_a \cap S$ *for large* m.

*4.** *Either* $u, t^h x$ *is an asymptotic sequence in* $\mathfrak{R}$ *or* $(u, t^h x)\mathfrak{R} = \mathfrak{R}$ *(in which case* $h = 0$ *and* $(I, x)R = R$*).*

*5.** *Either* $u, t^h x$ *is an asymptotic sequence in* $\mathfrak{R}[\frac{1}{ta}]$ *or* $(u, t^h x)\mathfrak{R}[\frac{1}{ta}] = \mathfrak{R}[\frac{1}{ta}]$.

*6.** $x \in I^h \setminus \bigcup \left\{ qI^h \mid q \in \hat{A}^*(I) \right\}$.

*7.** $x + I^{h+1}$ *is not contained in any minimal prime of the form ring of* R *with respect to* I, $\mathfrak{F}(R, I) = \bigoplus_{i \geq 0} I^i / I^{i+1}$.

Then

(i.) *the star statements* (1^*)*-*(7^*) *are equivalent.*

(ii.) *We have* $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (5)$, *and* $(4) \Leftrightarrow (7) \Leftrightarrow (8)$.

(iii.) *if* $a, b_1, \ldots, b_n$ *is an* R*-sequence, then in addition, we have* $(4) \Leftrightarrow (6)$.

(iv.) *if* $a, b_1, \ldots, b_n$ *is an* R*-sequence,* R *satisfies Serre's condition* (S_2) *and either* I *is height unmixed or* $I = \mathrm{Rad}(I)$*, then for any* $\beta \in S \setminus R$ *there exist* x, h *such that* h *is a nonnegative integer,* $x \in I^h \setminus (I^{h+1} \cup aR)$*, and* $\beta = x/a^h$*; and* (1) *through* (8) *are equivalent.*

(v.) *Now assume that* $a, b_1, \ldots, b_n$ *is an* R*-sequence,* R *satisfies Serre's condition* (S_2) *and either* I *is height unmixed or* $I = \mathrm{Rad}(I)$*, and that* (1)*-*(8) *hold. If* xR *is a prime (resp., primary) ideal, then* βS *is a prime (resp., primary) ideal, and the converse holds if* $(xR :_R aR) = xR$.

Proof. For (ii): That (1), (2), (3) and (5) are equivalent follows from the proof of [9, Proposition 3.3], as does the equivalence of (4), (7) and (8).

For (i): We proceed to show that the star statements (1^*)-(7^*)are equivalent. Let $\mathfrak{S} = \mathfrak{R}[\frac{1}{at}]$, so $S = R[I/a] \subset R[I/a, ta, 1/(ta)] = R[u, tI, \frac{1}{ta}] = \mathfrak{S}$, since $\frac{b_i}{a} = \frac{tb_i}{ta}$. Observe that $\mathfrak{S} = S[ta, \frac{1}{ta}]$ is a localization of $S[ta]$, a simple transcendental extension of S. Then $u = \frac{a}{ta} \in aS[ta, \frac{1}{ta}]$, and $a = u(ta) \in uS[ta, \frac{1}{ta}]$, so $u\mathfrak{S} = a\mathfrak{S}$. Also note that $t^h x = (ta)^h(\frac{x}{a^h}) \in \frac{x}{a^h}\mathfrak{S}$ and $\frac{x}{a^h} = (\frac{1}{ta})^h t^h x \in t^h x\mathfrak{S}$, so $t^h x\mathfrak{S} = \frac{x}{a^h}\mathfrak{S}$.

(1^*) $\Rightarrow$ (2^*): Assume that $((\beta^m S)_a) :_S aS) = (\beta^m S)_a$ for large m. Recall that this is equivalent to $a \notin P$ for every $P \in \hat{A}^*(\beta S)$. Then either $(a, \beta)S = S$ or β, a is an asymptotic sequence. If the latter, then $\mathrm{ht}((a, \beta)S) = 2$. Thus $\beta \notin Q$ for each $Q \in \hat{A}^*(aS)$. Indeed, if $\beta \in Q$ for some asymptotic prime of aS, $(a, \beta)S \subseteq Q$, whence $\mathrm{ht}(Q) \geq 2$. But a is regular, and thus an asymptotic sequence, so $\mathrm{ht}(Q) = 1$. Therefore, if $(a, \beta)S \neq S$, a, β is an asymptotic sequence.

(2^*) $\Rightarrow$ (1^*): Assume that a, β is an asymptotic sequence in S. Then we have $\mathrm{ht}((a, \beta)S) = 2$. If $a \in P$ for some $P \in \hat{A}^*(\beta S)$, then $(a, \beta)S \subseteq P$, so $\mathrm{ht}(P) \geq 2$. But β is regular, and hence and asymptotic sequence, thus $\mathrm{ht}(P) = 1$ for each asymptotic prime P of $S/\beta S$.

Now assume that $(a, \beta)S = S$. If $a \in P$ for some asymptotic prime P of βS, then $S = (a, \beta)S \subseteq P$, which is a clear contradiction.

(1^*) $\Leftrightarrow$ (3^*): For large integers k and m, $((\beta^m S)_a :_S a^k S) = (\beta^m S)_a S[\frac{1}{a}] \cap S$. Since integral closure behaves well with respect to localization (see [32, Proposition 1.1.4] for reference), $(\beta^m S)_a S[\frac{1}{a}] \cap S = (\beta^m S[\frac{1}{a}])_a \cap S$. Thus $((\beta^m S)_a :_S a^k S) = (\beta^m S)_a$ if and only if $(\beta^m S[\frac{1}{a}])_a \cap S = (\beta^m S)_a$.

(2^*) $\Rightarrow$ (5^*): If $(a, \beta)S = S$, $1 \in (a, \beta)S \subset (a, \beta)\mathfrak{S} = (u, t^h x)\mathfrak{S}$. Now suppose $a, x/a^h$ is an asymptotic sequence in S. Then since $S[ta]$ is a flat S-module, by [24, Proposition 5.1], either $a, x/a^h$ is an asymptotic sequence in $S[ta]$ or $(a, x/a^h)S[ta] = S[ta]$. As the latter implies $(a, x/a^h)\mathfrak{S} = (u, t^h x)\mathfrak{S} = \mathfrak{S}$, we may assume that $a, x/a^h$ is an asymptotic sequence in $S[ta]$. Since $\mathfrak{S}$ is a localization of $S[ta]$, by Remark 2.2.12, either $(a, x/a^h)\mathfrak{S} = (u, t^h x)\mathfrak{S} = \mathfrak{S}$ or $a, x/a^h$ is an asymptotic sequence in $\mathfrak{S}$.

$(5^*) \Rightarrow (2^*)$: If $(u, t^h x)\mathfrak{S} = \mathfrak{S}$, $(a, x/a^h)\mathfrak{S} = \mathfrak{S}$, so there are elements $r_1, r_2 \in \mathfrak{S}$ such that $r_1 a + r_2(x/a^h) = 1$. We can write $r_i = \sum_{j=-n}^{m} r_{ij}(ta)^j$ for $i = 1, 2$, where $r_{ij} \in R$.

$$\sum_{j=-n}^{m} r_{1j} t^j a^{j+1} + \sum_{j=-n}^{m} r_{2j} t^j a^{j-h} = 1$$

We see then that $r_{ij} = 0$ for $j \neq 0$, so this reduces to $r_{10}a + r_{20}(x/a^h) = 1$, whence $(a, x/a^h)R = R$.

Now suppose $u, t^h x$ is an asymptotic sequence in $\mathfrak{S}$, i.e.

$$(u^m\mathfrak{S})_a = (a^m\mathfrak{S})_a = ((a^m\mathfrak{S})_a :_{\mathfrak{S}} (x/a^h)\mathfrak{S}) = ((u^m\mathfrak{S})_a :_{\mathfrak{S}} t^h x\mathfrak{S}) \text{ for each } m \geq 1.$$

So $a, x/a^h$ is an asymptotic sequence in $\mathfrak{S}$. If $x/a^h \in P$ for some $P \in \hat{A}^*(aS)$, then $t^h x\mathfrak{S} \subseteq (x/a^h)\mathfrak{S} \subseteq P\mathfrak{S} \subseteq Q$, where Q is some prime ideal minimal over $P\mathfrak{S}$, and hence over $u\mathfrak{S} = a\mathfrak{S}$, contradicting our assumption that $u, t^h x$ is an asymptotic sequence in $\mathfrak{S}$.

$(5^*) \Rightarrow (4^*)$: Suppose $u, t^h x$ is an asymptotic sequence in $\mathfrak{S}$. If $u, t^h x$ is not an asymptotic sequence in $\mathfrak{R}$, then $t^h x \in P$ for some $P \in \hat{A}^*(u\mathfrak{R})$. Then $\mathrm{ht}(P) = 1$ since $\mathfrak{R}$ is locally quasi-unmixed and u is an $\mathfrak{R}$-sequence. Then $t^h x \in P\mathfrak{S}$. Since $\mathfrak{S}$ is a localization of $\mathfrak{R}$, there is a one-to-one correspondence between prime ideals of $\mathfrak{S}$ and prime ideals of $\mathfrak{R}$ which miss $\{(ta)^k\}_{k=1}^{\infty}$, either $\mathrm{ht}(P\mathfrak{S}) = 1$ and $P\mathfrak{S}$ is prime, or $ta \in P$. But $ta \notin Q$ for any $Q \in \hat{A}^*(u\mathfrak{R})$ since u, ta is an asymptotic sequence in $\mathfrak{R}$ by Theorem 2.3.4. Thus $\mathrm{ht}(P\mathfrak{S}) = 1$. But $u, t^h x$ an asymptotic sequence in $\mathfrak{S}$ means that $(u, t^h x)\mathfrak{S}$ is an ideal of the principal class and therefore has height 2, so since $(u, t^h x)\mathfrak{S} \subseteq P\mathfrak{S}$,

$\mathrm{ht}(P\mathfrak{S}) \geq 2$, which is a contradiction.

Now assume $(u, t^h x)\mathfrak{S} = \mathfrak{S}$ and $(u, t^h x)\mathfrak{R} \neq \mathfrak{R}$. If $t^h x \in P$ for some $P \in \hat{A}^*(u\mathfrak{R})$, then either $ta \in P$ or $P\mathfrak{S}$ is a prime ideal of $\mathfrak{S}$. We see that $ta \notin P$ since u, ta is an asymptotic sequence in $\mathfrak{R}$. We also have that $(u, t^h x)\mathfrak{R} \subseteq P$ implies $\mathfrak{S} = (u, t^h x)\mathfrak{S} \subseteq P\mathfrak{S}$, which contradicts the fact that $P\mathfrak{S}$ is a prime ideal. Thus $t^h x \notin P$ for any $P \in \hat{A}^*(u\mathfrak{R})$ and $u, t^h x$ is an asymptotic sequence in $\mathfrak{R}$.

(4*) $\Rightarrow$ (5*): Suppose that $(u, t^h x)\mathfrak{R} = \mathfrak{R}$. It is clear that $\mathfrak{S} = (u, t^h x)\mathfrak{S} = (a, \frac{x}{a^h})\mathfrak{S}$, so $(a, \frac{x}{a^h})S = S$ as above. So now suppose that $u, t^h x$ is an asymptotic sequence in $\mathfrak{R}$. Since $\mathfrak{S}$ is a localization of $\mathfrak{R}$, by Remark 2.2.12, either $u, t^h x$ is an asymptotic sequence in $\mathfrak{S}$ or $\mathfrak{S} = (u, t^h x)\mathfrak{S}$.

(4*) $\Leftrightarrow$ (6*): Since $(R[u]/(u, I)R[u]) \cong R/I$, there is a one-to-one correspondence between prime ideals of R containing I and prime ideals of $R[u]$ containing $(u, I)R[u]$. In particular, all asymptotic primes of $(u, I)R[u]$ are in one-to-one correspondence with $\hat{A}^*(I)$, and they will be the smallest prime ideals containing $uR[u]$ and $qR[u]$ for each $q \in \hat{A}^*(I)$. That is, $\hat{A}^*((u, I)R[u]) = \left\{(u, q)R[u] \mid q \in \hat{A}^*(I)\right\}$.

Let $L = \ker(R[u][X_0, X_1, \ldots, X_n] \to \mathfrak{R})$. Now $u, a, b_1, \ldots, b_n$ is an asymptotic sequence in $R[u][X_0, X_1, \ldots, X_n]$ by [27, Proposition 3.2], and observe that $\mathfrak{R} = R[u][ta, tb_1, \ldots, tb_n] = R[u][\frac{a}{u}, \frac{b_1}{u}, \ldots, \frac{b_n}{u}]$ is a monoidal transform over $R[u]$, so Lemma 3.1.5 applies. So the asymptotic primes of $(u, I)R[u]$ and of $u\mathfrak{R} = (u, I)\mathfrak{R}$ (this equality holds since $I\mathfrak{R} \subseteq u\mathfrak{R}$) are in one-to-one correspondence. Thus for every $p \in \hat{A}^*(u\mathfrak{R})$,$p = P\mathfrak{R}$ for some $P \in \hat{A}^*((u, I)R[u])$, that is $p = ((u, q)R[u]\mathfrak{R} = (u, q)\mathfrak{R}$, and we have $\hat{A}^*(u\mathfrak{R}) = \left\{(u, q)\mathfrak{R} \mid q \in \hat{A}^*(I)\right\}$.

Say $\hat{A}^*(I) = \{q_1, \ldots, q_m\}$. Then $u, t^h x$ is an asymptotic sequence in $\mathfrak{R}$ if and only if $t^h x \notin \bigcup\{(u, q_i) \mid i = 1, \ldots, m\}$, which is true if and only if $x \notin ((u, q_i)\mathfrak{R})_{[h]} = \left\{r \in R \mid rt^h \in (u, q_i)\mathfrak{R}\right\}$ for $i = 1, \ldots, m$. Then $((u, q_i)\mathfrak{R})_{[h]} = q_i I^h + I^{h+1} = q_i I^h$ since

$I^{h+1} \subseteq q_i I^h$ for all $h \geq 0$.

(4*) $\Rightarrow$ (7*): If $u, t^h x$ is an asymptotic sequence in $\mathfrak{R}$, then $t^h x$ is an asymptotic sequence in $\mathfrak{R}/u\mathfrak{R}$, which by Remark 2.2.11 means that $t^h x$ is not in any minimal prime of $\mathfrak{R}/u\mathfrak{R}$. Since $(u, t^h x)\mathfrak{R} \neq \mathfrak{R}$, there is no element $y \in \mathfrak{R}$ such that $(t^h x + u\mathfrak{R})(y + u\mathfrak{R}) = (t^h xy + u\mathfrak{R}) = 1 + u\mathfrak{R}$.

If $(u, t^h x)\mathfrak{R} = \mathfrak{R}$, then $h = 0$ and $(I, x)R = R$. Thus $i + xr = 1$ for some $i \in I$ and $r \in R$. Thus $xr + I = 1 + I$, or x is a unit in R/I, and hence in $\bigoplus_{i \geq 0} I^i/I^{i+1}$.

(7*) $\Rightarrow$ (4*): If $x + I^{h+1}$ is a nonunit not contained in any minimal prime of $\bigoplus I^i/I^{i+1}$, then $t^h x + u\mathfrak{R}$ is not in any minimal prime of $\mathfrak{R}/u\mathfrak{R}$ under the isomorphism $\bigoplus I^i/I^{i+1} \cong \mathfrak{R}/u\mathfrak{R}$, as given by Rees in [28, Theorem 2.1]. By Theorem 2.2.18, $u, t^h x$ is an asymptotic sequence in $\mathfrak{R}$, since $\mathfrak{R}$ is locally quasi-unmixed and since u is an asymptotic sequence in $\mathfrak{R}$.

If $x + I^{h+1}$ is a unit in $\bigoplus I^i/I^{i+1}$, then by the isomorphism, $t^h x + u\mathfrak{R}$ is a unit in $\mathfrak{R}/u\mathfrak{R}$. Thus for some $y \in \mathfrak{R}$, $(t^h x + u\mathfrak{R})(y + u\mathfrak{R}) = (t^h xy + u\mathfrak{R}) = 1 + u\mathfrak{R}$, so $t^h x + uz = 1 + wu$ for some $z, w \in \mathfrak{R}$. Then $1 = (z - w)u + t^h xy \in (u, t^h x)\mathfrak{R} = \mathfrak{R}$.

For (iii), let us now assume that $a, b_1, \ldots, b_n$ is an R-sequence. (4) $\Leftrightarrow$ (6): There is a one-to-one correspondence between the prime divisors of $(u, I)R[u]$ and the associated primes of I, so

$$\mathrm{Ass}_{R[u]}(R[u]/(u, I)R[u]) = \{(u, q)R[u] \mid q \in \mathrm{Ass}_R(R/I)\}.$$

Now $u, a, b_1, \ldots, b_n$ is an $R[u][X_0, \ldots, X_n]$-sequence, so the additional statement in Lemma 3.1.5 holds and there is a one-to-one correspondence between $\mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$ and $\mathrm{Ass}_{R[u]}(R[u]/(u, I)R[u])$. Thus $\mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R}) = \{(u, q)\mathfrak{R} \mid q \in \mathrm{Ass}_R(R/I)\}$ and the rest follows.

To prove (iv), it suffices to show that (4) $\Leftrightarrow$ (5). Since (4) $\Rightarrow$ (5) is clear, we will simply show that (5) $\Rightarrow$ (4).

Let us also assume that R satisfies (S_2) and that I is height unmixed, so that by [5, Theorem 2], $\mathfrak{R}$ also satisfies (S_2), as in the proof of Lemma 3.2.2. Then Lemma 3.2.3 applies, so for every $\beta \in S \setminus R$ there exist x, h as in Lemma 3.2.3.

Suppose $u, t^h x$ is an $\mathfrak{S}$-sequence. If $u, t^h x$ is not a regular sequence in $\mathfrak{R}$, then $t^h x \in P$ for some $P \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$. Then $\mathrm{ht}(P) = 1$ since $\mathfrak{R}$ satisfies (S_2) and u is an $\mathfrak{R}$-sequence. Then $t^h x \in P\mathfrak{S}$. Since $\mathfrak{S}$ is a localization of $\mathfrak{R}$, there is a one-to-one correspondence between prime ideals of $\mathfrak{S}$ and prime ideals of $\mathfrak{R}$ which miss $\{(ta)^k\}_{k=1}^{\infty}$, either $\mathrm{ht}(P\mathfrak{S}) = 1$ and $P\mathfrak{S}$ is prime, or $ta \in P$. Note that since $\mathfrak{R}$ satisfies (S_2) (and $\mathfrak{R}$ is locally quasi-unmixed), $\mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R}) = \hat{A}^*(u\mathfrak{R})$. Thus u, ta is an asymptotic sequence in $\mathfrak{R}$ if and only if it is an $\mathfrak{R}$-sequence, so it is clear that $ta \notin Q$ for any $Q \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$ since u, ta is an asymptotic sequence in $\mathfrak{R}$. Thus $\mathrm{ht}(P\mathfrak{S}) = 1$. But $u, t^h x$ an $\mathfrak{S}$-sequence means that $(u, t^h x)\mathfrak{S}$ is an ideal of the principal class and therefore has height 2, so since $(u, t^h x)\mathfrak{S} \subseteq P\mathfrak{S}$, $\mathrm{ht}(P\mathfrak{S}) \geq 2$, which is a contradiction.

Now assume $(u, t^h x)\mathfrak{S} = \mathfrak{S}$ and $(u, t^h x)\mathfrak{R} \neq \mathfrak{R}$. If $t^h x \in P$ for some $P \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$, then either $ta \in P$ or $P\mathfrak{S}$ is a prime ideal of $\mathfrak{S}$. We see that $ta \notin P$ since u, ta is an $\mathfrak{R}$-sequence. We also have that $(u, t^h x)\mathfrak{R} \subseteq P$ implies $\mathfrak{S} = (u, t^h x)\mathfrak{S} \subseteq P\mathfrak{S}$, which contradicts the fact that $P\mathfrak{S}$ is a prime ideal. Thus $t^h x \notin P$ for any $P \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$ and $u, t^h x$ is an $\mathfrak{R}$-sequence.

The proof of (v) is the same as the proof of the final statement in [9, Proposition 3.3]. ■

The next proposition is not quite a generalization of [9, Proposition 3.6], since the original result did not require R to be an integral domain. Nonetheless, it allows us to prove Proposition 3.2.7, which does generalize [9, Proposition 3.9]. For the following proposition, if $p = \pi R$ with $\pi \notin I$, the preceding result could be used to obtain a short proof. However, π may be in I, so we give an alternate proof. This result gives three conditions equivalent to when a nonzero principal prime ideal p of R (such that $a \notin p$) extends to a prime ideal in S.

Proposition 3.2.5 *Let R be a locally quasi-unmixed domain, let $a, b_1, \ldots, b_n$ be an R-sequence, $I = (a, b_1, \ldots, b_n)R$ and $S = R[I/a]$. Let p be a height one principal prime ideal of R such that $a \notin p$. Then the following are equivalent:*

1. *pS is a (principal) prime ideal.*
2. *$p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_R(R/I)\}$.*
3. *$(pS :_S aS) = pS$.*
4. *$pS = pR[\frac{1}{a}] \cap S$.*

Proof. Let $\pi \in R$ be the prime element such that $\pi R = p$.

(3) $\Rightarrow$ (2): If there exist $r, r' \in S$ such that $\pi r = ar'$, we must have that $r' \in pS$, or $r' = \pi r''$ for some $r'' \in S$. Then $\pi r = a\pi r''$, and since S is a domain, $r = ar''$, that is $r \in aS$. So $(aS :_S \pi S) = aS$ and π is not a zero divisor on aS. Thus $\pi \notin qS$, and therefore $\pi \notin q$ for each $q \in \mathrm{Ass}_R(R/I)$. (Note that by Lemma 3.1.5, all associated primes of aS are of this form.) Finally, since $\pi \notin q$ for each $q \in \mathrm{Ass}_R(R/I)$, $\pi R = p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_R(R/I)\}$.

(2) $\Rightarrow$ (3): If $\pi \notin q \subset qS$ for every $q \in \mathrm{Ass}_R(R/I)$, then $\pi \notin \bigcup\{qS \mid q \in \mathrm{Ass}_R(R/I)\} = \bigcup \mathrm{Ass}_S(S/aS)$. Thus π is not a zero divisor on aS. It remains to show that a is not a zero divisor on $\pi S = pS$. Suppose $\pi r = ar'$ for some $r, r' \in S$. Then $r = ar''$ for some $r'' \in S$. Because a is not a zero divisor on S, we may cancel a, and $\pi ar'' = ar'$ becomes $\pi r'' = r'$, whence $r' \in \pi S = pS$. Therefore $(pS :_S aS) = pS$.

The rest of the proof follows exactly as in [9, Proposition 3.6]. ■

For the rest of the chapter, our base assumption will be that R is a locally quasi-unmixed unique factorization domain, and that $a, b_1, \ldots, b_n$ are an R-sequence. Let us say that a factors uniquely as follows: $a = a_1^{c_1} \cdots a_d^{c_d}$, where the a_i are non-associate prime elements in R and the c_i are positive integers.

The next result examines the height one prime ideals of R with no prime ideals of S lying over them. That is, prime ideals p of R such that there are no prime ideals P of S where $P \cap R = p$. The remark is essentially the same as [9, Remark 3.8], since as the proof below shows, we need only assume that R is a Noetherian UFD.

Remark 3.2.6 The height one prime ideals p of R such that $pS \cap R \neq p$ are exactly the prime ideals $a_1R, \ldots, a_dR$.

Proof. It is clear that $aR \subseteq a_iR$ and $I \not\subseteq a_iR$ for each i, so by [3, Lemma], there are no primes in S contracting to a_iR. Thus $a_iR \in \{p \in \operatorname{Spec}(R) \mid \operatorname{ht}(p) = 1 \text{ and } pS \cap R \neq p\}$.

To show the reverse inclusion, suppose $p \in \operatorname{Spec}(R)$, $\operatorname{ht}(p) = 1$ and $pS \cap R \neq p$. Then $p \subset pS \cap R \subseteq pS[\frac{1}{a}] \cap R = pR[\frac{1}{a}] \cap R$. Since $p \neq pR[\frac{1}{a}] \cap R$ and since the prime ideals of $R[\frac{1}{a}]$ are in one-to-one correspondence with the prime ideals of R that do not contain a, we have $pR[\frac{1}{a}] \cap R = R$ and $a_1^{c_1} \cdots a_d^{c_d} = a \in p$. This gives $a_iR \subseteq p$ for some i, but a_iR and p are both height one prime ideals, so $a_iR = p$. ■

Proposition 3.2.7 examines height one prime ideals P of S such that P is the radical of a principal ideal xS for some $x \in R$. Naturally this includes P that have a principal primary ideal, and using Remark 3.1.9, we see that if we also assume I is height unmixed, these primes P are the same.

Proposition 3.2.7 *Let R be a locally quasi-unmixed unique factorization domain, let $a, b_1, \ldots, b_n$ be an R-sequence, $I = (a, b_1, \ldots, b_n)R$ and $S = R[I/a]$. Assume that x is* *a nonzero nonunit in R such that xS is a primary ideal. Let $P = \mathrm{Rad}(xS)$, and let $p = P \cap R$.*

1. *If $\mathrm{ht}(p) > 1$, then $a \in \mathrm{Rad}(xR)$, $p \in \mathrm{Ass}_R(R/I)$ and $\mathrm{Rad}(xS) = pS$.*

2. *If $\mathrm{ht}(p) = 1$, then $(xR :_R a^k R)$ is p-primary for $k \gg 0$, $xS = (xR :_R a^k R)S$ for all $k \geq 0$, $P = \mathrm{Rad}(pS)$ and $p \not\subseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$.*

If we also have that I is height unmixed, then for (2), the above holds and in addition, $P = pS$ and $p \not\subseteq \bigcup \{q \mid q \in \mathrm{Ass}_R(R/I)\}$.

Proof. For (1), if $\mathrm{ht}(p) > 1$, then by Lemma 3.1.15 (3), $a \in p \in \mathrm{Ass}_R(R/I)$ and $P = pS$. If $q \in \hat{A}^*(xR)(= \mathrm{mAss}_R(R/xR))$, $\mathrm{ht}(q) = 1$, so if $a \notin q$, then $Q = qR[\frac{1}{a}] \cap S$ has $\mathrm{ht}(Q) = \mathrm{ht}(q)$ by Lemma 3.1.15 (2). Thus Q is a height 1 prime divisor of xS and $Q = P$. But $q = Q \cap R = P \cap R = p$ is a contradiction ($\mathrm{ht}(q) = 1 < \mathrm{ht}(p) = g + 1$). We conclude $a \in \cap \{q \mid q \in \mathrm{mAss}_R(R/xR)\} = \mathrm{Rad}(xR)$.

For (2), P is a minimal prime divisor of xS (which is generated by an S-sequence) in a locally quasi-unmixed domain, so $\mathrm{ht}(P) = 1$. Thus by Lemma 3.1.15, if $\mathrm{ht}(p) = 1$, $a \notin p = P \cap R$, and $P = pR[\frac{1}{a}] \cap S$. Since $PS[\frac{1}{a}] \cap S = P$, $(P :_S aS) = PS[\frac{1}{a}] \cap S = P$. Then since a is not in the only associated prime of xS, it is not a zero divisor on S/xS, and $(xS :_S aS) = xS$. Thus also $xS = xS[\frac{1}{a}] \cap S$. Then $xS \cap R = xS[\frac{1}{a}] \cap S \cap R = xS[\frac{1}{a}] \cap R = (xR :_R a^k R)$ for all large integers k. Since contractions of primary ideals are primary, $xS \cap R$ is $p = P \cap R$-primary. Therefore $(xR :_R a^k R)$ is p-primary. Also, since $xR \subseteq xS \cap R = (xR :_R a^k R) \subset xS$, $(xR :_R a^k R)S = xS$ for $k \geq 0$. Further, $xS \subseteq pS \subseteq P$, so $P = \mathrm{Rad}(xS) \subseteq \mathrm{Rad}(pS) \subseteq \mathrm{Rad}(P) = P$. It follows that P is the only prime ideal of S minimal over pS, and since S is locally quasi-unmixed, that $\mathrm{ht}(P) = 1$. Observe that since R is a UFD, $p = \pi R$ for some prime element $\pi \in R$. Lastly, assume $p \subset q$ for some $q \in \hat{A}^*(I)$. Then $pS \subset qS$ and, more importantly, $\mathrm{Rad}(pS) \subset qS$, since $a \notin \mathrm{Rad}(pS) \cap R = p$. But $qS \in \hat{A}^*(aS)$ and $\mathrm{ht}(qS) = 1$ by Lemma 3.1.5, whence $\mathrm{Rad}(pS)$ is not prime. This is a contradiction, so $p \nsubseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$.

Now assume that I is height unmixed. Then the asymptotic primes of I coincide with the associated primes of I, so $p \nsubseteq \bigcup\{q \mid q \in \mathrm{Ass}_R(R/I)\}$, which is equivalent to $pS = P$ by Proposition 3.2.5. ■

The next remark extends [9, Remark 3.10], and is used in the proof of Proposition 3.2.9.

Remark 3.2.8 Let R be a locally quasi-unmixed unique factorization domain, and let $a, b_1, \ldots, b_n$ an R-sequence, $I = (a, b_1, \ldots, b_n)R$ and $S = R[I/a]$. If $x \in I \setminus \{0\}$ and if xS is P-primary, then $\hat{A}^*(aS) = \{P\}$ and $\mathrm{ht}(P \cap R) = n + 1$. If we also have that I is height unmixed, then I is $P \cap R$-primary and aS is P-primary.

Proof. Let $Q \in \hat{A}^*(aS)$. Then $\mathrm{ht}(Q) = 1$. But $xS \subseteq (x, a)S \subseteq IS = aS \subseteq Q$. So Q is minimal over xS, thus $Q = P$. Therefore $\mathrm{mAss}_S(S/aS) = \hat{A}^*(aS) = \{P\}$. By Lemma 3.1.5, $P \cap R \in \hat{A}^*(I)$ and $\mathrm{ht}(P \cap R) = n + 1$.

Now assume that I is height unmixed. Since R is a UFD, R satisfies (S_2), so by [5, Theorem 2], S also satisfies (S_2). Therefore $\mathrm{Ass}_S(S/aS) = \hat{A}^*(aS) = \{P\}$ and aS is P-primary. By Lemma 3.1.5, each associated prime of I is the contraction of an associated prime of aS, so I must also be $P \cap R$-primary. ■

The following proposition and its corollary are slight generalizations of [9, Proposition 3.11, Corollary 3.12]. These results examine height one prime ideals p of R such that $a \notin p$. Since Proposition 3.2.5 deals with such p when $p \nsubseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$, the next result assumes that $p \subseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$. Proposition 3.2.9 then characterizes when $pR[1/a] \cap S$ has a principal primary ideal, and the following corollary characterizes when $pR[1/a] \cap S$ is principal.

Proposition 3.2.9 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $p = \pi R$ be a (height-one) prime ideal in R, assume $a \notin p \subseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$ and let $\hat{A}^*(I) = \{q_1, \ldots, q_m\}$. Then the following are equivalent:*

1. *$P = pR[\frac{1}{a}] \cap S$ has a principal primary ideal.*
2. *There exist positive integers e, h and non-negative integers $e_1, \ldots, e_d$ such that $\pi^e a_1^{e_1} \cdots a_d^{e_d} \in I^h \setminus (aS \cup q_1 I \cup \cdots \cup q_m I)$, and then $((\pi^e a_1^{e_1} \cdots a_d^{e_d})/a^h)S$ is P-primary.*
3. *There exist a positive integer h and an element $x \in (p \cap I^h)$ such that $(x/a^h)S$ is P-primary.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is clear, so it remains to show (1) $\Rightarrow$ (2). Let βS be P-primary for some $\beta \in S$. If $\beta \in I$, by Remark 3.2.8, $\hat{A}^*(aS) = \{P\}$ and therefore $\operatorname{ht}(P \cap R) = n + 1$. So since $\operatorname{ht}(p) = 1$, $\beta \notin I$. Let x and h be those given by Lemma 3.2.3, such that $\beta = \frac{x}{a^h}$, $x \in I^h \setminus (aR \cup I^{h+1})$. Then $a^h \beta = x \in \pi R = P \cap R$, so since R is a UFD, there is a positive integer e such that $x \in \pi^e R \setminus \pi^{e+1} R$.

We have $aS = IS$ and $\hat{A}^*(aS) = \{q_1 S, \ldots, q_m S\}$ by Lemma 3.1.5. Also S is locally quasi-unmixed, so $\operatorname{ht}(q_i S) = 1$ for each $i = 1, \ldots, m$. Since βS is P-primary and $a \notin p = P \cap R$, it follows that $(\beta S :_S aS) = \beta S$. Then, by Proposition 3.2.4(1) $\Leftrightarrow$ (2) either $(a, \beta)S = S$ or a, β is an S-sequence (and hence an asymptotic sequence in S. Then by Proposition 3.2.4(2*) $\Leftrightarrow$ (6*), $x \in I^h \setminus (aS \cup q_1 S \cup \cdots \cup q_m S)$. Since $\pi \in P = \operatorname{Rad}(\beta S)$, there is a $k_1 > 0$ such that $\pi^{k_1} = \beta\gamma$ for some $\gamma \in S$.

If $\gamma \notin I$, let k_2 be the non-negative integer and y the element in $I^{k_2} \backslash (aS \cup I^{k_2+1})$ such that $\gamma = y/(a^{k_2})$ given by Lemma 3.2.3. Then $x/a^h = \beta = \pi^{k_1}/\gamma = (\pi^{k_1}a^{k_2})/y$, or $xy = \pi^{k_1}a^{h+k_2}$. Using unique factorization of R and the fact that $x \in \pi^e R \setminus \pi^{e+1}R$, we may write $x = u\pi^e a_1^{e_1} \cdots a_d^{e_d}$ for some non-negative integers $e_1, \ldots, e_d$ and some unit $u \in R$ (and since $a \notin xR$, at least one $e_i < g_i$).

In the case $\gamma \in I$, $\pi^{k_1}a^h = x\gamma$, so the argument follows similarly to the previous paragraph and the same conclusion holds. ■

Corollary 3.2.10 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $p = \pi R$ be a height-one prime ideal in R such that $a \notin p \subseteq \bigcup \left\{ q \mid q \in \hat{A}^*(I) \right\}$. Then $P = pR[\frac{1}{a}] \cap S$ is a principal prime ideal if and only if e may be chosen to be 1 in Proposition 3.2.9.*

Proof. Suppose that P is a principal prime ideal, $P = \beta S$. Since $\mathrm{ht}(P \cap R) = 1$, by Remark 3.2.8, we must have $\beta \notin I$. Then let $\beta = \frac{x}{a^h}$ as in Lemma 3.2.3. Following the proof of Proposition 3.2.9, we get $x = \pi^e a_1^{e_1} \cdots a_d^{e_d}$ for some positive integer e and non-negative integers $e_1, \ldots, e_d$. Then $\pi^e S[\frac{1}{a}] = xS[\frac{1}{a}] = \beta S[\frac{1}{a}] = PS[\frac{1}{a}] = pS[\frac{1}{a}] = \pi S[\frac{1}{a}]$, so $e = 1$.

For the converse, let P have a principal primary ideal, βS. Then by Proposition 3.2.9, $\beta = (\pi a_1^{e_1} \cdots a_d^{e_d})/a^h$, so $\beta S[\frac{1}{a}] = ((\pi a_1^{e_1} \cdots a_d^{e_d})/a^h)S[\frac{1}{a}] = \pi S[\frac{1}{a}]$, and $\pi S[\frac{1}{a}] \cap S = \beta S[\frac{1}{a}] \cap S = \beta S$ since βS is P-primary and $a \notin P$. Thus βS is prime. ■

The following theorem and its corollary summarize the results from the section. Theorem 3.2.11 classifies height one prime ideals P of S such that $\mathrm{ht}(P \cap R) = 1$ (so that $a \notin P \cap R$) and P is the radical of a principal ideal. As mentioned above in Remark 3.1.9, the hypotheses of the theorem imply that P in fact has a principal primary ideal.

Theorem 3.2.11 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $P \in \mathrm{Spec}(S)$ have a principal primary ideal, let $p = P \cap R$, and assume that $\mathrm{ht}(p) = 1$ (so $a \notin p$). Then exactly one of the following holds:*

1. *$P = \mathrm{Rad}(\pi S)$ for some prime element $\pi \in R$. (This is true if and only if $p = \pi R$ for some prime element $\pi \in R \setminus \bigcup \{q \mid q \in \mathrm{Ass}_R(R/I)\}$ and $pS = P$.)*

2. *$P = \mathrm{Rad}(\beta S)$ for some $\beta \in S \setminus R$. (This is true if and only if $p = \pi R \subseteq \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$, $P = pR[\frac{1}{a}] \cap S$, and β may be chosen to be $(\pi^e a_1^{e_1} \cdots a_d^{e_d})/a^h$ as in Proposition 3.2.9(2).)*

Proof. Let βS be P-primary. Either $\beta \in R$ or $\beta \in S \setminus R$. Suppose that $\beta \in R$. It then follows from the proof of Proposition 3.2.7(2) that $P = \mathrm{Rad}(\pi S)$ for some prime element $\pi \in R$ and that $p \not\subseteq \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$. Moreover, if we suppose that $p = \pi R$ for some prime element $\pi \in R \setminus \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$, then since $\beta S \cap R \subseteq P \cap R = p = \pi R$, $\beta S \subseteq \pi S \subseteq P$. Therefore $P = \mathrm{Rad}(\beta S) \subseteq \mathrm{Rad}(\pi S) \subseteq \mathrm{Rad}(P) = P$.

For (2), suppose that $\beta \in S \setminus R$. The preceding paragraph showed that $\beta \in R$ if and only if $p \not\subseteq \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$, so we must have $p \subseteq \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$. The rest follows from Proposition 3.2.9(1) $\Leftrightarrow$ (2).

Since I is height unmixed, $\hat{A}^*(I) = \mathrm{Ass}_R(R/I)$, so by Proposition 3.2.5, $pS = P$. ■

This next corollary then classifies height one primes P of S such that $\mathrm{ht}(P \cap R) = 1$ and P is principal.

Corollary 3.2.12 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. If P is a nonzero principal prime ideal in S and $\mathrm{ht}(P \cap R) = 1$, then either:*

1. $P = \pi S$ *for some prime element* $\pi \in R \setminus \bigcup \{q \mid q \in \mathrm{Ass}_R(R/I)\}$*; or,*

2. $P = \beta S$ *for some* $\beta \in S \setminus R$ *as in Theorem 3.2.11(2) with* $e = 1$*.*

Proof. The prime element generating P is either in R or $S \setminus R$. If that element is in R, then by Proposition 3.2.5, $(P \cap R) \not\subseteq \bigcup \{q \mid q \in \mathrm{Ass}_R(R/I)\}$.

If that element is in $S \setminus R$, then by Theorem 3.2.11, $(P \cap R) \subseteq \bigcup \left\{q \mid q \in \hat{A}^*(I)\right\}$, so the rest follows from Corollary 3.2.10. ∎

3.3 The case where ht$(P \cap R) > 1$

This section deals with height-one prime ideals P of S for which $a \in P$. For the best statement of results, we will assume a mild condition on the prime factors $a_1, \ldots, a_d$ of a in Proposition 3.3.6, and for the results which follow it. Each of the results in this section is an extension of the results of section 4 of [9] to the case where R is locally quasi-unmixed and I is height unmixed.

Throughout this section we will continue to assume that R is a locally quasi-unmixed unique factorization domain, that $a, b_1, \ldots, b_n$ are an R-sequence, and that a factors uniquely as $a = a_1^{c_1} \cdots a_d^{c_d}$, where the a_i are non-associate prime elements in R and the c_i are positive integers. We will denote by $J = (b_1, \ldots, b_n)R$, so that $I = (a, J)R$.

Our first result, Proposition 3.3.1, is a variation on Lemma 3.1.5. We will use this variation in later results of this section.

Proposition 3.3.1 *Assume that* $a = a_1^{c_1} \cdots a_d^{c_d}$*, where the* a_i *are non-associate prime elements in* R *and the* c_i *are positive integers. Then for each* $i = 1, \ldots, d$*, either*

1. $(a_i, J)R = R$ *(which holds if and only if* $a_i S = S$*); or*

2. *$(a_i, J)S = a_iS$, $a_iS \cap R = (a_i, J)R$, and $S/a_iS \cong (R/(a_i, J)R)[X_1, \ldots, X_n]$, so there is a one-to-one correspondence between the elements of $\hat{A}^*((a_i, J)R)$ and $\hat{A}^*(a_iS)$ given by $p\ (\in \hat{A}^*(a_i, J)) = P \cap R$ with $P \in \hat{A}^*(a_iS)$ and $P = pS$. Also, each $q \in \hat{A}^*((a_i, J)R)$ has $\mathrm{ht}(q) = n + 1$.*

Moreover, (2) holds for at least one $i = 1, \ldots, d$.

Proof. Observe that $b_j = a_i((\prod_{k \neq i} a_k)\frac{b_j}{a}) \in a_iS$ for each $j = 1, \ldots, d$. Since the reverse inclusion is obvious, $a_iS = (a_i, J)S$.

Clearly $I \subseteq (a_i, J)R$ for each $i = 1, \ldots, d$. Also, since $a, b_1, \ldots, b_n$ is an R-sequence, it is strongly analytically independent, so

$$(*)\ H = \mathrm{Ker}(R[X_1, \ldots, X_n] \to S) \subseteq IR[X_1, \ldots, X_n] \subseteq (a_i, J)R[X_1, \ldots, X_n].$$

Then by Lemma 3.1.3, it follows that $a_iS \cap R = (a_i, J)S \cap R = (a_i, J)R$, and $S/a_iS \cong (R/(a_i, J)R)[X_1, \ldots, X_n]$.

Each a_i is either a unit in S or not. Fix i, and suppose that a_i is a unit in S. Then $a_iS = S$, and $(a_i, J)R = (a_i, J)S \cap R = a_iS \cap R = S \cap R = R$. It is clear that $1 \in (a_i, J)R \subset (a_i, J)S = a_iS$. Now suppose a_i is not a unit in S. Lemma 3.1.3 and $(*)$ give the desired correspondence between $\hat{A}^*((a_i, J)R)$ and $\hat{A}^*(a_iS)$. Moreover, $\hat{A}^*((a_i, J)R) = \left\{p \in \hat{A}^*(I) \mid a_i \in p\right\}$, so $\mathrm{ht}(q) = n + 1$ for each $q \in \hat{A}^*((a_i, J)R)$.

Finally, since $a, b_1, \ldots, b_n$ is an R-sequence, it must have at least one minimal prime p, and $a = a_1^{c_1} \cdots a_d^{c_d} \in p$, so $a_k \in p$ (and thus (2) holds) for some k. ■

Remark 3.3.2 Let $p \in \mathrm{Spec}(R)$ have $\mathrm{ht}(p) = n + 1$. Then the following are equivalent:

1. $p \in \hat{A}^*(I)$;

2. pS is a height-one prime ideal;

3. $\mathrm{ht}(pS) = 1$;

4. there is a prime ideal P of S such that $\mathrm{ht}(P) = 1$ and $P \cap R = p$;

5. $pS \in \hat{A}^*(aS)$ and $p = pS \cap R$.

Proof. (5) $\Rightarrow$ (1) $\Rightarrow$ (2) by Lemma 3.1.5. (2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (4), let P be a height-one prime divisor of pS. Then $p \subseteq pS \cap R \subseteq P \cap R$, and $n+1 = \mathrm{ht}(p) \leq \mathrm{ht}(P \cap R) \leq n+1$. Therefore $p = P \cap R$.

For (4) $\Rightarrow$ (5), we see that by Lemma 3.1.15(3), the fact that there is a height-one prime lying over p tells us that $p \in \hat{A}^*(I)$. So by Lemma 3.1.5, $pS = P \in \hat{A}^*(aS)$ and $pS \cap R = P \cap R = p$. ■

Our next proposition, which is an extension of [9, Proposition 4.5], examines height one prime ideals of R which contain a.

Proposition 3.3.3 *If p is a height-one prime ideal in R such that $a \in p$, then $p = a_iR$ for some $i = 1, \ldots, d$. Also, $pS = a_iS$ is a (height-one) prime (resp., primary) ideal (resp., $= S$) if and only if $(a_i, J)R$ is a (height $n+1$) prime (resp., primary) ideal (resp., $= R$).*

Proof. The first statement follows from Remark 3.2.6. By Proposition 3.3.1, $pS = a_iS = S$ if and only if $(a_i, J)R = R$. If $pS = a_iS$ is prime (resp., primary) then by Proposition 3.3.1 $pS \cap R = a_iS \cap R = (a_i, J)R$ is prime (resp., primary). Note that if pS is prime or primary and $\mathrm{ht}(pS) = 1$, then by Proposition 3.3.1 $\mathrm{ht}((a_i, J)R) = n+1$.

If $(a_i, J)R$ is prime (resp., primary) then a_iS is prime (resp., primary). Also, if $(a_i, J)R$ is prime or primary and $\mathrm{ht}((a_i, J)R) = n+1$, then $\mathrm{ht}(a_iS) = 1$. ■

The following result extends [9, Remark 4.6], and strengthens the conclusions of Proposition 3.2.7(1).

Remark 3.3.4 Let x be a nonzero nonunit in R such that xS is a primary ideal, let $P = \mathrm{Rad}(xS)$, let $P \cap R = p$, and assume that $\mathrm{ht}(p) > 1$. Then there exists an $i \in \{1, \ldots, d\}$ such that $\mathrm{Rad}(a_i S) = \mathrm{Rad}(xS) = P$, $\mathrm{Rad}((a_i, J)R) = p$, and $P \cap R \in \hat{A}^*(I)$. Also,

1. if I is height unmixed, then $(a_i, J)R$ and $a_i S$ are primary.

2. if xS is prime, then $a_i S = P$ and $(a_i, J)R = p$.

Proof. By Lemma 3.1.15(3), $a \in p \in \hat{A}^*(I)$ and $\mathrm{ht}(p) = n + 1$, and $a \in \mathrm{Rad}(xR)$. Further, $x \in P \cap R = p$, so there is an asymptotic prime divisor q of xR such that $xR \subseteq q \subset p$, and then $a_1^{c_1} \cdots a_d^{c_d} = a \in \mathrm{Rad}(xR) \subseteq q$. Therefore $a_i \in q$ for some $i \in \{1, \ldots, d\}$ and $a_i R \subseteq q$. Both $a_i R$ and q are height-one prime ideals, so $a_i R = q$. Therefore $xR \subseteq a_i R \subseteq p$, so $xS \subseteq a_i S \subseteq pS \subseteq P$, and then we have $P = \mathrm{Rad}(xS) \subseteq \mathrm{Rad}(a_i S) \subseteq \mathrm{Rad}(pS) \subseteq P$. In particular, $\mathrm{Rad}(xS) = \mathrm{Rad}(a_i S) = P$. Via the one-to-one correspondence given in Proposition 3.3.1, this means that $(a_i, J)R$ has only one asymptotic prime divisor, p, and $\mathrm{Rad}((a_i, J)R) = p$.

For (2), assume that xS is in fact prime. Then the above holds and $P = xS \subseteq a_i S \subseteq pS \subseteq P$ gives us $a_i S = P$. That $(a_i, J)R = p$ follows from Proposition 3.3.1.

For (1), assume that I is height unmixed. Then since R is a UFD, S satisfies (S_2) by [5, Theorem 2], so $\mathrm{Rad}(a_i S) = P$ implies that $a_i S$ is P-primary and $\mathrm{Rad}((a_i, J)R) = p$ implies $(a_i, J)R$ is p-primary. ■

We use the following definitions from [9]. These definitions will allow us to obtain sharper conclusions for the following results.

Definition 3.3.5 1. With $a_1, \ldots, a_d$ as above, we say that $a_1, \ldots, a_d$ satisfy the *Radical Property* with respect to $J = (b_1, \ldots, b_n)R$ if for each $i = 1, \ldots, d$ it holds that $a_1 \cdots a_{i-1} a_{i+1} \cdots a_d \notin \mathrm{Rad}((a_i, J)R)$.

2. A *product-quotient* of elements $x_1, \ldots x_m$ in R is a product $x_1^{n_1} \cdots x_d^{n_d}$ where at least one of the $n_i > 0$ and possibly some n_j are nonpositive.

Proposition 3.3.6 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let P be a height-one prime ideal in S, let $p = P \cap R$, and assume that* $\mathrm{ht}(p) > 1$ *and that* $P = \mathrm{Rad}(\beta S)$ *for some $\beta \in S$. Then $P = \mathrm{Rad}(\delta S)$ for some product-quotient δ of $a_1, \ldots, a_d$. Moreover, if $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J, then $a_i S$ is P-primary for some $i \in \{1, \ldots, d\}$ and $(a_i, J)R$ is p-primary.*

Proof. If $\mathrm{ht}(p) > 1$, then $a \in p$, $I \subseteq P \cap R$, and $P = pS$, by Lemma 3.1.15(3).

If $\beta \in R$, then by Proposition 3.3.4, $P = \mathrm{Rad}(a_i S)$ for some $i \in \{1, \ldots, d\}$. Therefore assume that $\beta \notin R$. Since $a \in P = \mathrm{Rad}(\beta S)$, there is a positive integer m such that $a^m = \beta\gamma$ for some $\gamma \in S$. Then by Lemma 3.2.3, we may write $\beta = \frac{x}{a^h}$, with $x \in I^h \setminus (I^{h+1} \cup aR)$ and $h > 0$. There are then two cases:

Case 1: $\gamma \in R$. Multiplying both sides of $a^m = \beta\gamma$ by a^h, we obtain $a^{m+h} = a_1^{c_1(h+m)} \cdots a_d^{c_d(h+m)} = x\gamma$ in R. If γ is a unit in R, then $\beta = x/a^h = (a^{h+m}\gamma^{-1})/a^h = a\gamma^{-1} \in R$, which is a contradiction. Thus $x = \omega a_1^{e_1} \cdots a_d^{e_1}$, where ω is a unit in R, $e_j \leq c_j(h+m)$ for each $i = 1, \ldots, d$, $e_l > 0$ for at least one l and $e_k < c_k(h+m)$ for at least one k (since γ is a non-unit). Therefore $\beta = \frac{x}{a^h} = \omega a_1^{e_1 - hc_1} \cdots a_d^{e_d - hc_d}$ and at least one $e_l - hc_l < 0$, since $\beta \notin R$. Reorder the subscripts so that $\beta = \frac{u}{v}$, where $u = \omega a_1^{e_1 - hc_1} \cdots a_i^{e_i - hc_i}$, $v = a_{i+1}^{e_{i+1} - hc_{i+1}} \cdots a_{i+j}^{e_{i+j} - hc_{i+j}}$ (for some i, j such that $1 \leq i < j \leq d$) and the exponents of u and v are all positive. If $a_f S = S$ for $f = i+1, \ldots, j$, then $\beta S = (u/v)S = uS = (a_1^{e_1 - hc_1} \cdots a_i^{e_i - hc_i})S$, so $a_m \in \mathrm{Rad}(\beta S)$ for some $m \in \{1, \ldots, i\}$. Since clearly $\beta \in (a_m S)$, $P = \mathrm{Rad}(\beta S) = \mathrm{Rad}(a_m S)$ as desired.

Therefore we may assume that for at least one $f \in \{i+1, \ldots, i+j\}$, $a_f S \neq$

S, say for $f = i+1$. Then $P = \mathrm{Rad}(\delta S)$ for the product-quotient $\delta = u/v$ of $a_1, \ldots, a_d$. It follows from this that $u = v(u/v) = v\beta \in vS \cap R \subseteq a_{i+1}^{e_{i+1}-hc_{i+1}}S \cap R \subseteq a_{i+1}S \cap R = (a_{i+1}, J)R$, by Proposition 3.3.1. Therefore $u \in (a_{i+1}, J)R$, so $a_1 \cdots a_i \in \mathrm{Rad}((a_{i+1}, J)R)$. It follows that if $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J, this is a contradiction, hence $P = \mathrm{Rad}(a_m S)$ for some $m \in \{1, \ldots, i\}$, and then $\mathrm{Rad}((a_m, J)R) = p$ by Proposition 3.3.1. Additionally, since R is a UFD and I is height unmixed, S satisfies (S_2), thus $a_m S$ is P-primary and by Proposition 3.3.1 so $(a_m, J)R$ is p-primary.

Case 2: $\gamma \in S \setminus R$. Using Lemma 3.2.3, there exists $y \in I^k \setminus (I^{k+1} \cup aR)$ with $k > 0$ such that $\gamma = y/a^k$. Then $a^{h+m+k} = xy \in R$, so the argument follows similarly. ■

The following corollary shows that if $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J, then for each $q \in \hat{A}^*(I)$, we can know whether qS has a principal primary ideal from q itself.

Corollary 3.3.7 *Let R be a locally quasi-unmixed unique factorization domain, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Assume that $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J. Then for each $q \in \hat{A}^*(I)$ the following are equivalent:*

1. *qS has a principal primary ideal;*
2. *$(a_i, J)R$ is q-primary for some $i \in \{1, \ldots, d\}$;*
3. *$a_i S$ is qS-primary for some $i \in \{1, \ldots, d\}$.*

Proof. (1) $\Rightarrow$ (3) by Proposition 3.3.6. (2) $\Leftrightarrow$ (3) by Proposition 3.3.1. (3) $\Rightarrow$ (1) is clear. ■

Corollary 3.3.8 *Let R be a locally quasi-unmixed unique factorization domain, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Assume that $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J and that P is a height-one prime ideal of S such that* $\mathrm{ht}(P \cap R) > 1$*. Then the following are equivalent:*

1. *P is a principal prime ideal;*
2. *$(a_i, J)R = P \cap R$ is prime;*
3. *$a_i S = P$.*

Proof. (2) $\Leftrightarrow$ (3) by Proposition 3.3.3. (3) $\Rightarrow$ (1) is clear. (1) $\Rightarrow$ (3) by Remark 3.3.4 and the proof of Proposition 3.3.6. $\blacksquare$

Our last result in this section considers asymptotic prime divisors of πS, where π is a prime element of R. We showed in Proposition 3.2.5 that πS is prime if and only if $\pi \notin \bigcup\{q \mid q \in \mathrm{Ass}_R(R/I)\}$, so here we assume that $\pi \in \bigcup\{q \mid q \in \hat{A}^*(I)\}$, which is naturally contained in the union of the associated prime divisors of R/I. Furthermore, by Proposition 3.3.1, we know that $a_i S$ is primary if and only if $(a_i, J)R$ is primary, so we may also assume that $\pi R \notin \{a_1 R, \ldots, a_d R\}$.

Proposition 3.3.9 *Let R be a locally quasi-unmixed unique factorization domain, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $\hat{A}^*(I) = \{q_1, \ldots, q_m\}$ and let l be the non-negative integer such that $(a_i, J)R$ is a primary ideal for $i = 1, \ldots, l$ but not for $i = l+1, \ldots, d$. Define the set $\boldsymbol{W}_0$ to be $\boldsymbol{W}_0 = \left\{q \in \hat{A}^*(I) \mid q = \mathrm{Rad}((a_i, J)R) \text{ for some } i = 1, \ldots, l\right\} = \{q_1, \ldots, q_r\}$ (so $0 \leq r \leq l$, since $\mathrm{Rad}((a_i, J)R) = \mathrm{Rad}((a_j, J)R)$ may hold for some $i \neq j$ in $\{1, \ldots, l\}$). Let $\pi \in \bigcup\left\{q \mid q \in \hat{A}^*(I)\right\}$ be a prime element such that $\pi R \notin \{a_1 R, \ldots, a_l R\}$. Assume* *that π is in exactly s $(0 \leq s \leq r)$ of the elements of $\boldsymbol{W}_0$ and exactly k of the elements of $\hat{A}^*(I) \setminus \boldsymbol{W}_0$. Then*

1. *$\pi R \in \{a_{l+1}R, \ldots, a_d R\}$ if and only if πS has exactly $s + k$ asymptotic prime divisors. At least s of them have a principal primary ideal. (If we also have that I is height unmixed and $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J, then exactly s of the asymptotic primes of πS have a principal primary ideal.)*

2. *$\pi R \notin \{a_{l+1}R, \ldots, a_d R\}$ if and only if πS has exactly $s + k + 1$ asymptotic prime divisors. At least s of these have a principal primary ideal, and at least $s + 1$ of them have a principal primary ideal if I is height unmixed and there exist h and x as in Proposition 3.2.9(3) for $p = \pi R$.*

Proof. For each of the s elements $q \in \mathbf{W}_0$ and the k elements $q \in \hat{A}^*(I) \setminus \mathbf{W}_0$ that contain π, qS is a height-one prime ideal containing πS, and therefore a minimal (and hence asymptotic) prime divisor of πS. Thus πS has at least $s + k$ asymptotic primes. For each of the s elements $\mathbf{W}_0$ such that $\pi \in q$, we have by hypothesis that q contains a primary ideal $(a_i, J)R$ for some $i = 1, \ldots l$. Then by Proposition 3.3.1, qS contains the principal primary ideal $a_i S$, so at least s of the asymptotic primes of πS have a principal primary ideal. (If $\pi R \in \{a_{l+1}R, \ldots, a_d R\}$, I is height unmixed, and $a_1, \ldots, a_d$ satisfy the Radical Property with respect to J, then by Proposition 3.3.6, these are the only prime divisors of πS that have principal primary ideals.)

Let $p = P \cap R$, where P is any asymptotic prime of πS. Then we have that either $\mathrm{ht}(p) = 1$ or $\mathrm{ht}(p) = n + 1$ by Lemma 3.1.15(3). Also by Lemma 3.1.15, we have that $\mathrm{ht}(p) = n + 1$ if and only if $a \in p$ if and only if $p \in \hat{A}^*(I)$, or equivalently, that $\mathrm{ht}(p) = 1$ if and only if $a \notin p$. Note that if $\mathrm{ht}(p) = 1$, $\pi R = p$, so $a \notin p = \pi R$ and $\pi R \notin \{a_{l+1}R \ldots, a_d R\}$. Further, $P = \pi R[\frac{1}{a}] \cap S$, and by Proposition 3.2.9 $\pi R[\frac{1}{a}] \cap S$ has a principal primary ideal if and only if there exist x and h as in Proposition 3.2.9(3). Thus if there exist such x and h, πS has at least $s+1$ asymptotic primes with principal primary ideals.

If πS has exactly $s+k+1$ asymptotic prime divisors, we must have $\pi R[\frac{1}{a}] \cap S$ is an asymptotic prime of πS in addition to the $s+k$ elements $qS \in \hat{A}^*(aS)$ which contain π. By Lemma 3.1.15, as we observed above, this means that $\pi R \notin \{a_{l+1}R, \ldots, a_d R\}$. The reverse implication is clear.

Then if πS has exactly $s+k$ asymptotic primes, by the above paragraph we must have that $\pi R \in \{a_{l+1}R, \ldots, a_d R\}$. ∎

Chapter 4

Special Cases

For this chapter we will assume that R is a locally quasi-unmixed unique factorization domain, $a, b_1, \ldots, b_n$ is an R-sequence, $I = (a, b_1, \ldots, b_n)R$ is height unmixed, and $S = R[I/a]$. The results of sections 1 and 2 are extensions of the results in sections 5 and 6 of [9] repsectively.

4.1 When a is a primary element

In this section we consider the case where a is a power of a single prime element, $a = a_1^{c_1}$. If this is the case, we say that a is a *primary element*. This allows us to obtain some additional results. In particular, we have a necessary and sufficient condition for S to be a Krull domain with finite cyclic class group.

Our first result shows that if a is a primary element and $\mathrm{Rad}(I)$ is not prime, then S is not a UFD.

Theorem 4.1.1 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Assume $a = a_1^{c_1}$ is a power of a prime element a_1 in R. If P is a height-one prime ideal in S*

that is the radical of a principal ideal, if $\mathrm{ht}(P \cap R) = 1$*, and if* $P \cap R \subset q \in \hat{A}^*(I)$*, then* $P \cap R \subseteq \mathrm{Rad}(I)$*. Therefore, if* $\mathrm{Rad}(I)$ *is not prime, then for each height-one prime ideal* p *in* R *that is contained in at least one, but not all, asymptotic prime divisors of* I *it holds that* $pR[\frac{1}{a}] \cap S$ *is not the radical of any principal ideal (and hence has no principal primary ideals).*

Proof. The second statement follows from the first, so it suffices to prove the first.

Suppose that $q \neq q'$ are asymptotic primes of I such that $P \cap R \subset q$ and $P \cap R \not\subseteq q'$. (Note that $a \notin P \cap R$ since $\mathrm{ht}(P \cap R) = 1$ by Lemma 3.1.15.) Let $p = P \cap R = \pi R$, so $a \notin \pi R$, and $P = \pi R[\frac{1}{a}] \cap S$, so P is the only prime ideal of S lying over p by Lemma 3.1.15. We will show that P is not the radical of a principal ideal, so this contradiction to the hypothesis shows that $\mathrm{Rad}(I)$ is prime.

Assume that $P = \mathrm{Rad}(\beta S)$ for some $\beta \in S$. Since $\mathrm{ht}(qS) = 1$ by Lemma 3.1.5, for any nonzero $r \in p$ we have $rR \subseteq \pi R \subset q$, so $rS \subseteq qS$, thus qS is a minimal prime divisor of rS. Therefore P is the only height-one prime divisor of βS, so $\beta \in P \setminus R$. Now let $\beta = \frac{x}{a^h}$, where $x \in I^h \setminus (I^{h+1} \cup aR)$ and $h > 0$, as in Lemma 3.2.3. Then $\pi^m \in \beta S$, since $P = \mathrm{Rad}(\beta S)$ and $\pi \in P \cap R$, so $\pi^m = \beta\gamma$ for some $\gamma \in S$. There are then two cases.

Case (1): If $\gamma \in R$, $\pi^m = (x\gamma)/a^h$ becomes $a^h \pi^m = x\gamma$. Since $x \notin aR$ and a is a_1-primary, unique factorization of R gives us $x = \omega a_1^f \pi^e$ for some unit ω of R, some non-negative integer $f < c_1$ and some $e > 0$. (Observe that $e > 0$ because $x = \beta a^h \in \beta S \cap R \subseteq \mathrm{Rad}(\beta S) \cap R = P \cap R = \pi R$.) If $f = 0$, $\pi^e R = xR \subseteq I$, hence $\pi R = p \subset \mathrm{Rad}(I) \subseteq q'$, a contradiction. Therefore $f > 0$, so $\beta = x/a^h = (\omega a_1^f \pi^e)/a_1^{hc_1} = (\omega \pi^e)/a_1^{hc_1 - f}$, so $\pi^e R \subseteq a_1^{hc_1-f} S \cap R \subseteq \mathrm{Rad}(a_1^{hc_1-f} S) \cap R \subseteq \mathrm{Rad}(aS) \cap R$, and $\operatorname{Rad}(aS) \cap R = \operatorname{Rad}(I)$ by Lemma 3.1.5. Then $\pi^e \in \operatorname{Rad}(I)$ and $\pi \in q'$, which is a contradiction, so $\gamma \notin R$.

Case (2): Since $\gamma \in S \backslash R$, let $\gamma = \frac{y}{a^k}$, where $y \in I^k \backslash (I^{k+1} \cup aR)$ and $k > 0$, as in Lemma 3.2.3. We have $\pi^m = \frac{x}{a^h}\frac{y}{a^k}$, or $\pi^m a^{h+k} = \pi^m a_1^{c_1(h+k)} = xy$, with $h+k \geq 2$. Then by unique factorization in R, $xy \in a_1^{c_1}R$, which contradicts the fact that $x \notin a_1^{c_1}R = aR$ and $y \notin a_1^{c_1}R = aR$. So the assumption that I has two distinct asymptotic prime divisors gives the contradiction that P is not the radical of a principal ideal. ■

The next theorem characterizes when S is a Krull domain with torsion class group, under the running hypotheses of this section.

Theorem 4.1.2 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Assume $a = a_1^{c_1}$ is the power of a prime element $a_1 \in R$. Then the following are equivalent.*

1. *S is a Krull domain with finite cyclic class group.*

2. *S is a Krull domain with torsion class group.*

3. $\operatorname{Rad}(I)$ *is prime and I is integrally closed.*

Proof. It is clear that (1) $\Rightarrow$ (2). By Theorem 3.1.10(2), (3) $\Rightarrow$ (1).

For (2) $\Rightarrow$ (3): By Remark 3.1.8, S a Noetherian Krull domain implies that I is integrally closed. If $\operatorname{Rad}(I)$ is not prime, then there is an element $r \in I$ that is in some $q \in \hat{A}^*(I)$ and not in some other $q' \in \hat{A}^*(I)$. Then rR has a height-one prime divisor $p \subset q$ (and $p \not\subseteq q'$), so by Theorem 4.1.1, $pR[\frac{1}{a}] \cap S = P$ (which has $\operatorname{ht}(P) = 1$ by Lemma 3.1.15(2)) has no principal primary ideal. But S has torsion class group, and by [7, Proposition 6.8], this is equivalent to every height-one prime ideal of S having a principal primary ideal, so we have a contradiction. Thus $\operatorname{Rad}(I)$ is prime. ■

The following is a corollary to Proposition 3.3.6.

Corollary 4.1.3 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \dots, b_n$ be an R-sequence, let $I = (a, b_1, \dots, b_n)$ be height unmixed, and let $S = R[I/a]$. Assume that $a = a_1^{c_1}$ for some prime element a_1 of R, and that P is a height-one prime ideal in S such that* $\mathrm{ht}(P \cap R) > 1$. *If P is the radical of a principal ideal, then $P = (P \cap R)S =$* $\mathrm{Rad}(a_1 S)$ *and* $\mathrm{Rad}(I) = P \cap R$. *Additionally, if I is height unmixed, then $a_1 S$ is P-primary and I is $P \cap R$-primary.*

Proof. It follows from Proposition 3.3.6, since a_1 satisfies the Radical Property with respect to J. ■

The next remark restates Proposition 3.2.9 under the additional hypothesis that a is a primary element, and again when a is a prime element.

Remark 4.1.4 Let R be a locally quasi-unmixed unique factorization domain, let the elements $a, b_1, \dots, b_n$ be an R-sequence, let $I = (a, b_1, \dots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $a = a_1^{c_1}$ be a power of a prime element a_1 in R, let $\hat{A}^*(I) = \{q_1, \dots, q_m\}$, let $a_1 \notin p = \pi R \subseteq q_1 \cup \cdots \cup q_m$, and let $P = pR[\frac{1}{a}] \cap S$. Then it follows from Proposition 3.2.9 that $p = \pi R$ is such that P has a principal primary ideal if and only if there exist positive integers e, h and a nonnegative integer k such that $\pi^e b_1^k \in I^h \backslash (aR \cup q_1 I \cup \cdots \cup q_m I)$ (so $k < c_1$), and then $((\pi^e a_1^k)/a^h)S$ is P-primary. If $a = a_1$ is a prime element in R, then $p = \pi R$ ($\neq aR$) is such that P has a principal primary ideal if and only if there exist positive integers e, h such that $\pi^e \in I^h \setminus (aR \cup q_1 I \cup \cdots \cup q_m I)$, and then $(\pi^e/a^h)S$ is P-primary.

Remark 4.1.5 Let R be a locally quasi-unmixed unique factorization domain, let the elements $a, b_1, \dots, b_n$ be an R-sequence, let $I = (a, b_1, \dots, b_n)R$ be height unmixed, and $S = R[I/a]$.

1. If $a_iS = S$ for all but one i (say $a_1S \neq S$), then the results in this section hold concerning S. Since $1/(a_2 \cdots a_d)$ is a unit in S and $C = R[1/(a_2 \cdots a_d)]$ is a UFD such that $aC = a_1^{c_1}C$, $a, b_1, \ldots, b_n$ is a C-sequence, and $S = C[I/b] = C[I/a_1^{c_1}]$.

2. The results in this section hold for the Rees ring $\mathfrak{R}(R, I)$, as is shown in the next section.

4.2 Application to the Rees Ring

In this section we apply the previous results to the Rees ring $\mathfrak{R}(R, I)$, where R is a locally quasi-unmixed UFD, $a, b_1, \ldots, b_n$ are an R-sequence, $I = (a, b_1, \ldots, b_n)R$ and $S = R[I/a]$. (Recall that $\mathfrak{R}(R, I) = R[u, ta, tb_1, \ldots, b_n]$, where $u = 1/t$, is a monoidal transform over $R[u]$.)

Remark 4.2.1 Let R be a locally quasi-unmixed unique factorization domain, let I be generated by the R-sequence $a, b_1, \ldots, b_n$, and let $A = R[u]$, where u is an indeterminate. Then A is a locally quasi-unmixed UFD, $u, a, b_1, \ldots, b_n$ is an A-sequence, and $\mathfrak{R} = \mathfrak{R}(R, I) = R[u, ta, tb_1, \ldots, tb_n] = A[\frac{a}{u}, \frac{b_1}{u}, \ldots, \frac{b_n}{u}]$. Therefore the results in the previous sections apply with A and $u, a, b_1, \ldots, b_n$ in place of R and $a, b_1, \ldots, b_n$. Also, u is a prime element in A, so by Lemma 3.1.5 there is a one-to-one correspondence between the elements of $\mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}/u\mathfrak{R})$ and the elements of $\mathrm{Ass}_A(A/(u, I)A)$, which have a natural one-to-one correpsondence with $\mathrm{Ass}_R(R/I)$. (The respective sets of asymptotic primes $\hat{A}^*(u\mathfrak{R})$, $\hat{A}^*((u, I)A)$, and $\hat{A}^*(I)$ are analogously in one-to-one correspondence.) Also, if I is height unmixed,then the results of section 4.1 apply to $\mathfrak{R} = \mathfrak{R}(R, I)$.

For the next proposition we temporarily lift the restrictions on $a, b_1, \ldots, b_n$ and R for the sake of generality. The proposition is essentially a restatement of Theorem 3.1.10 in terms of $\mathfrak{R}$ as a monoidal transform over $R[u]$.

Proposition 4.2.2 *Let R be an integrally closed Noetherian domain, let $a, b_1, \ldots, b_n$ be an asymptotic sequence, and $I = (a, b_1, \ldots, b_n)R$. Let $\mathfrak{R} = \mathfrak{R}(R, I)$ and let $A = R[u]$. Then:*

1. *If I is integrally closed, we have that $\mathfrak{R}$ is integrally closed and there is a surjective homomorphism $\phi : \mathrm{Cl}(\mathfrak{R}) \to \mathrm{Cl}(\mathfrak{R}[\frac{1}{u}])$ whose kernel is generated by the classes of elements of $\hat{A}^*(u\mathfrak{R})$.*

2. *If R is locally quasi-unmixed, I is integrally closed, $\mathrm{Rad}(I)$ is prime (in particular, if I is primary), and if $\mathrm{Cl}(R)$ is torsion (resp. finite, resp. trivial), then $\mathrm{Cl}(\mathfrak{R})$ is torsion (resp. finite, resp. finite cyclic).*

3. *If I is prime, then $u\mathfrak{R} \in \mathrm{Spec}(\mathfrak{R})$ and the divisor class groups $\mathrm{Cl}(R)$ and $\mathrm{Cl}(\mathfrak{R})$ are isomorphic.*

Proof. Note that $\mathrm{Cl}(\mathfrak{R}[\frac{1}{u}]) = \mathrm{Cl}(R[u,t])$, and by [7, Theorem 8.1] $\mathrm{Cl}(R[u,t]) = \mathrm{Cl}(R[u]) = \mathrm{Cl}(R)$.

For (1), if I is integrally closed and generated by an asymptotic sequence, then so is $(u, I)A$. (That $(u, I)A$ is integrally closed if I is follows from [32, Proposition 1.3.5].) Thus (1) follows from Theorem 3.1.10(1) with $\mathfrak{R}$ in place of S and u in place of a.

For (2), if I is integrally closed and $\mathrm{Rad}(I)$ is prime, then so is $(u, I)A$, so (2) follows from Theorem 3.1.10(2).

For (3), uA is prime, and if I is prime, then so is $(u, I)A$, so (3) follows from Theorem 3.1.10(3). ■

For the next proposition, observe that if R is a locally quasi-unmixed domain, P is a height-one prime ideal of $\mathfrak{R}(R, I)$ such that $\mathrm{ht}(P \cap R[u]) = 1$, then by Lemma 3.1.15, $u \notin P$. In particular, if $P \cap R[u] \subseteq (u, q)R[u]$ for some $q \in \hat{A}^*(I)$, then in fact

$P \cap R[u] \subseteq qR[u]$. This proposition is essentially a restatement of Proposition 4.1.1 in terms of the Rees ring.

Proposition 4.2.3 *Let R be a locally quasi-unmixed unique factorization domain, let $a, b_1, \ldots, b_n$ be an R-sequence, and let $I = (a, b_1, \ldots, b_n)R$ be height unmixed. Let $A = R[u]$, where u is an indeterminate and $\mathfrak{R}(R, I) = \mathfrak{R}$. If P is a height-one prime ideal in $\mathfrak{R}$ that is the radical of a principal ideal, if $\mathrm{ht}(P \cap A) = 1$, and if $P \cap A \subseteq qA$ for some $q \in \hat{A}^*(I)$, then $P \cap A \subseteq (\mathrm{Rad}(I))A$. Therefore, if $\mathrm{Rad}(I)$ is not prime, then for each height-one prime ideal p in A that is contained in at least one, but not all, asymptotic prime divisors of $(u, I)A$ it holds that $pR[u, t] \cap \mathfrak{R}$ is not the radical of any principal ideal (and hence has no principal primary ideals).*

Proof. Note that u is a prime element in A, $\hat{A}^*(IA) = \left\{qA \mid q \in \hat{A}^*(I)\right\}$, and $\hat{A}^*((u, I)A) = \left\{(u, q)A \mid q \in \hat{A}^*(I)\right\}$. By the remarks preceding the proposition, if $P \cap A \subset (u, q)A$ for some $q \in \hat{A}^*(I)$, then $P \cap A \subseteq qA$. In particular, if $P \cap A \subset \bigcap\left\{(u, q)A \mid q \in \hat{A}^*(I)\right\} = \mathrm{Rad}((u, I)A)$, then $P \cap A \subseteq \bigcap\left\{qA \mid q \in \hat{A}^*(I)\right\} = (\mathrm{Rad}(I))A$. Thus the rest follows from Theorem 4.1.1. ■

It follows immediately from Proposition 4.2.3 that $\mathfrak{R}$ is not a UFD if $\mathrm{Rad}(I)$ is not prime. The next result gives a characterization of when $\mathfrak{R}$ is a locally quasi-unmixed UFD.

Theorem 4.2.4 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, and let $I = (a, b_1, \ldots, b_n)R$ be height unmixed. Then $\mathfrak{R}(R, I)$ is a locally quasi-unmixed UFD if and only if I is prime, and then $u\mathfrak{R}(R, I)$ is a prime ideal.*

Proof. If I is prime, $\mathfrak{R}(R, I)$ is a UFD and $u\mathfrak{R}(R, I)$ is prime by Proposition 4.2.2(3).

For the converse, assume I is not prime. By Proposition 4.2.3, if I is not primary (and hence by Remark 3.1.9 $\text{Rad}(I)$ is not prime), then $\mathfrak{R}(R, I)$ is not a UFD, so we may assume that I is primary, say $\text{Rad}(I) = q$. Then $u\mathfrak{R}(R, I)$ is primary for $(u, q)\mathfrak{R}(R, I)$, by Lemma 3.1.5, and u is part of a minimal basis for $(u, q)\mathfrak{R}(R, I)$ since all elements of negative degree are a multiple of u, hence $(u, q)\mathfrak{R}(R, I)$ has more than one generator and is not a principal prime ideal. Therefore $\mathfrak{R}(R, I)$ is not a UFD. ■

The following corollary strengthens Corollary 3.1.12.

Corollary 4.2.5 *Let R be a locally quasi-unmixed unique factorization domain and let $a, b_1, \ldots, b_n$ be an asymptotic sequence. If I is prime, then each of the rings $S_j = R[I/b_j]$ is a locally quasi-unmixed UFD and $b_j S_j \in \text{Spec}(S_J)$.*

Proof. Let $\mathfrak{R}(R, I) = \mathfrak{R}$. If I is prime, then by Proposition 4.2.2(3) $\mathfrak{R}$ is a UFD, so each $\mathfrak{S}_j = \mathfrak{R}[1/(tb_j)]$ is a UFD. However, $\mathfrak{S}_j = S_j[tb_j, 1/(tb_j)]$, and tb_j is transcendental over S_j and therefore prime in $S_j[tb_j]$. By Nagata's Theorem, $S_j[tb_j]$ is a UFD, so S_j is also a UFD. ■

The next theorem is a restatement of Theorem 4.1.2 for Rees rings.

Theorem 4.2.6 *Let R be a locally quasi-unmixed unique factorization domain, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, and let $I = (a, b_1, \ldots, b_n)R$ be height unmixed. Let $\mathfrak{R}(R, I) = \mathfrak{R}$. The following are equivalent:*

1. *$\mathfrak{R}$ is a Krull domain with finite cyclic class group.*

2. *$\mathfrak{R}$ is a Krull domain with torsion class group.*

3. $\text{Rad}(I)$ *is prime and I is integrally closed.*

Proof. I is primary and integrally closed if and only if $(u, I)A$ is primary and integrally closed. Furthermore, $(u, I)A$ is primary if and only if $\text{Rad}((u, I)A)$ is prime, by Remark 3.1.9, so I is primary if and only if $\mathrm{Rad}(I)$ is prime. Also, u is prime in A, so the result follows from Theorem 4.1.2. ■

Corollary 4.2.7 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $\mathfrak{R}(R, I) = \mathfrak{R}$. If the equivalent conditions in Theorem 4.2.6 hold, then for $j = 1, \ldots, n$ each $S_j = R[I/b_j]$ is a Krull domain with finite cyclic class group.*

Proof. If $\mathfrak{R}$ is a Krull domain with finite cyclic class group, then so is $\mathfrak{S}_j = \mathfrak{R}[1/(tb_j)] = S_j[tb_j, 1/(tb_j)]$ (by [7, Corollary 7.2] $\mathrm{Cl}(\mathfrak{R}) \to \mathrm{Cl}(\mathfrak{R}[1/(tb_j)])$ is a surjection, and since the homomorphic image of a finite cyclic group is finite and cyclic). The element tb_j is transcendental over B_j, thus prime in $S_j[tb_j]$, so $\mathfrak{S}_j$ is a localization of $S_j[tb_j]$ at a prime element, so by [7, Corollary 7.3], $\mathrm{Cl}(S_j[tb_j]) \cong \mathrm{Cl}(\mathfrak{S}_j)$. Finally, using [7, Theorem 8.1] we have that $\mathrm{Cl}(S_j[tb_j]) \cong \mathrm{Cl}(S_j)$. ■

Proposition 4.2.8 *Let R be a locally quasi-unmixed UFD, let $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $\mathfrak{R}(R, I) = \mathfrak{R}$. Assume that $\mathrm{Rad}((x/a^h)S) = qS$ for some $q \in \hat{A}^*(I)$. If $t^h x, ta$ is an $\mathfrak{R}$-sequence, then $\mathrm{Rad}(I) = q$ and $\mathrm{Rad}(aS) = qS$. Additionally, if I is height unmixed, then I is q-primary and aS is qS-primary.*

Proof. Let $\mathfrak{S} = S[ta, 1/(ta)] = \mathfrak{R}[1/(ta)]$. Then $u\mathfrak{S} = a\mathfrak{S}$.

By hypothesis, $\mathrm{Rad}((x/a^h)S) = qS$, so $\mathrm{Rad}(t^h x\mathfrak{S}) = q\mathfrak{S}$, so $t^h x\mathfrak{S} \cap \mathfrak{R} = (t^h x\mathfrak{R} :_{\mathfrak{R}} (ta)^k\mathfrak{R})$ and $\mathrm{Rad}(t^h x\mathfrak{S} \cap \mathfrak{R}) = q\mathfrak{S} \cap \mathfrak{R} = (u, q)\mathfrak{R}$ for all large integers k. (Note that $(u, q)\mathfrak{S} = (a, q)\mathfrak{S} = q\mathfrak{S}$.) Therefore, if $t^h x, ta$ is an $\mathfrak{R}$-sequence, then $\mathrm{Rad}(t^h x\mathfrak{R}) = (u, q)\mathfrak{R}$. Since u is a prime element in $R[u]$, it follows from Corollary 4.1.3 that $\mathrm{Rad}((u, I)R[u]) = (u, q)R[u]$ and that $\mathrm{Rad}(u\mathfrak{R}) = (u, q)\mathfrak{R}$. Therefore $\mathrm{Rad}(I) = q$, and by Lemma 3.1.5, $\mathrm{Rad}(aS) = qS$. The final statement follows from [5, Theorem 2]. ■

The last result characterizes when $pR[u,t] \cap \mathfrak{R}(R,I)$ has a principal primary ideal and when it is a principal prime ideal, where $p \subseteq \bigcup\{(u,q)A \mid q \in \hat{A}^*(I)\}$.

Proposition 4.2.9 *Let R be a locally quasi-unmixed UFD, let the elements $a, b_1, \ldots, b_n$ be an R-sequence, let $I = (a, b_1, \ldots, b_n)R$ be height unmixed, and $S = R[I/a]$. Let $\mathfrak{R}(R,I) = \mathfrak{R}$ and $R[u] = A$. Let $p = \pi A$ be a height-one prime ideal in A such that $u \notin p \subseteq \bigcup\left\{(u,q)A \mid q \in \hat{A}^*(I)\right\}$, let $P = pR[u,t] \cap \mathfrak{R}$, and let $\hat{A}^*(I) = \{q_1, \ldots, q_m\}$. Then the following hold:*

1. *P has a principal primary ideal if and only if there exist positive integers e, h such that $\pi^e \in (u,I)^h A \setminus (uA \cup (u,q_1(u,I)^h A \cup \cdots \cup (u,q_m)(u,I)^h A)$, and then $(\pi^e t^h)\mathfrak{R}$ is P-primary.*

2. *P is a principal ideal if and only if e in (1) can be chosen to be 1.*

Proof. (1) follows from Remark 4.1.4, and (2) follows from (1) and Corollary 3.2.10. ■

Chapter 5

Future Directions for Research

The Rees ring $\mathfrak{R} = R[It, t^{-1}]$ is often referred to as the *extended Rees algebra*, while the R-algebra $R[It]$ is often referred to as the *Rees algebra*. As we have seen, the extended Rees algebra may be viewed as a monoidal transform over $R[t^{-1}]$, and this allowed us to easily apply our results to the extended Rees algebra case. We will investigate which of the results hold for the (unextended) Rees algebra, $R[It]$. In particular, there is a notion of a Rees algebra for modules (which generalizes $R[It]$), but nothing analogous to the extended Rees algebra [6]. Thus to generalize our results to the module case, we will first see how they extend to the Rees algebra $R[It]$.

There is also a notion of the extended Rees algebra and asymptotic primes for multiplicative Noetherian lattices, where asymptotic primes of lattice elements show up as the centers of Rees valuations [30]. Therefore we will investigate which of our results transfer to the Noetherian multiplicative lattice case.

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