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Time-Varying Preferences, Risk Premia, and Tobin Constraints

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Time-Varying Preferences, Risk Premia, and Tobin Constraints

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

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by

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Dissertation Committee:
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ABSTRACT OF THE DISSERTATION

Time-Varying Preferences, Risk Premia, and Tobin Constraints

By

David Licata

Doctor of Philosophy in Economics

University of California, Irvine, 2015

William Branch, Chair

The first chapter of my thesis explores monetary policy in a New Keynesian model with Markov-switching risk aversion. The second considers the implications for the macroeconomic and financial properties of an RBC model of the presence of habit formation. The third examines the result of adding the "Tobin constraint" that shares equal the capital stock to a benchmark RBC mdoel. The underlying theme of these endeavors is rendering macroeconomic models more realistic via the introduction of time-varying preferences, non-linear modelling, and financial frictions.

Chapter 1

Monetary Policy in a New Keynesian model with Markov-Switching Risk Aversion

1.1 Introduction

How should monetary policy respond to changes in risk aversion? Motivated by empirical evidence of significant time-variation in risk premia, this paper employs a Markov-switching model (MSM) – a dynamic model in which the parameters are random and follow a Markov chain – in order to represent time-varying risk aversion. In particular, I replace the standard preference shock typically introduced in the general equilibrium environment underlying the "benchmark" New Keynesian model[1] with a Markov-switching "shock" to the representative agent's CRRA risk aversion coefficient. This more generalized assumption then leads to a Markov-switching linearized model that nests the benchmark linearized New Keyne-

[1]See Woodford [66] Chapters 2, 3, and 4 for a detailed discussion of this model.

sian model as a special case. Given the random variation in risk aversion induced by this specification and the resulting nonlinearity of the model, it is not obvious how monetary policy should be conducted. I consider two different specifications. First, I consider optimal Taylor-type rules in which the reaction coefficients are potentially regime-dependent. These rules are analyzed using the methodology introduced by Davig and Leeper [21].

Second, I consider optimal monetary policy with commitment in this model. In this context, the multiplicative nature of the shock to risk aversion has the property of breaking the well-known certainty equivalence principle of optimal control – namely, that in a standard linear-quadratic framework, the optimal policy function in the presence of mean zero additive shocks is identical to the optimal policy function in the absence of such shocks. Therefore, in my model the optimal policy function depends on the state of the Markov chain. Following Svensson and Williams [56], I solve for the regime-dependent optimal policy function in this model. Finally, in order to quantify the gains from commitment, I explore the optimal discretionary equilibrium, and the expected loss under this equilibrium is compared with that under the optimal policy.

1.1.1 Related Literature

Outside the context of optimal policy studies, Markov-switching macroeconomic models have been investigated by many authors in recent years. Of special importance for my analysis of simple rules below is the contribution o Davig and Leeper [21], who develop a tractable method for solving and analyzing determinacy of forward-looking rational expectations MSMs (see Section 3 below). [2]

Optimal control of MSMs has been a subject of intensive research in the control theory literature, where these models are known as Markov jump linear systems.[3] In particular, the

[2] Other notable contributions to this literature include [12] and [26].

[3] Prominent early contributions to this literature include [41] (for the continuous-time case) and [17] (for

solution of Markov jump linear-quadratic (MJLQ) control problems via a system of coupled algebraic Riccati equations (CARE) has become a standard approach. One reason for the popularity of such systems is their ability to introduce nonlinearity into an otherwise linear model in a manageable way. Indeed, it's well-established in this literature that certain classes of nonlinear systems are very well approximated by Markov jump linear systems.[4]

Aoki [5] and Chow [18] were pioneers in the attempt to apply these ideas to optimal policy problems in economic models. More recently Zampolli [67] formulated an MJLQ problem in an open-economy MSM with two regimes representing "bubble" or "no bubble" in the exchange rate. Since Zampolli's model was backward-looking, application of recursive control-theoretic algorithms to solve for the optimal policy was relatively straightforward. However, most macroeconomic models currently used for policy analysis, notably New Keynesian/DSGE models, employ forward-looking variables. This "gap" between recursive control-theoretic models and forward-looking macroeconomic models was bridged by Svensson and Williams [56], who demonstrate how to re-write a general forward-looking MSM as a recursive MSM, and then solve for the regime-dependent optimal policy function. The present study combines the optimal policy framework developed by Svensson and Williams with an analysis of simple rules in order to investigate monetary policy in a New Keynesian MSM featuring time-varying risk aversion.

The calibration employed in this paper is motivated by empirical evidence of large time-variation in risk premia.This has been abundantly documented for a wide range of asset markets and by numerous authors, among them [11], [3], [24], [42], and [48]. Such time-variation in risk premia is difficult to account for in a model with constant risk aversion .[5]

the discrete-time case).

[4]See e.g. [9].

[5]Of course, endogenous time-variation in local relative risk aversion can introduced into a model via the external habit preferences employed e.g. by Campbell and Cochrane [15], even when the model's CRRA risk aversion coefficient is constant. However, I maintain that exogenous time-variation in the CRRA coefficient itself is necessary to explain why, for example, risk premia often rise precipitously even when consumption is relatively high (as in the bursting of the tech bubble in 1999), which would not be predicted by an external habit specification.

The model which I present below instead switches stochastically between a high risk aversion regime and a low risk aversion regime, in order to approximate the exogenous component of the time-variation in risk premia observed in the data.

1.2 The Model

1.2.1 Microfoundation

Underlying the Markov-switching New Keynesian model summarized by equations (1.26)-(1.27) below is a general equilibrium problem featuring a representative household and a continuum of firms, each of which produces a differentiated good (indexed by $i \in [0, 1]$), in an environment characterized by sluggish price adjustment and monopolistic competition. The representative household maximizes the following infinite-horizon objective:

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[U_{j_T}(C_T) - \int_0^1 \nu\left(h_T(i)\right) di \right] \tag{1.1}$$

where C_T is the household's aggregate consumption in period T, $h_T(i)$ is labor supplied by the household for the production of good i in period T, and $\beta \in (0, 1)$ is the household's subjective discount factor. In order to capture time-variation in risk aversion, the household's utility function $U_{j_t}(\cdot)$ depends on the regime j_t of the Markov chain in the following way:

$$U_{j_t}(C_t) = \frac{C_t^{1-\sigma_{j_t}}}{1 - \sigma_{j_t}}$$

Thus, the household's CRRA risk aversion coefficient takes one of two values, σ_1 or σ_2, depending on the current regime $j_t \in \{1, 2\}$. The conditional expectation E_t in the household's objective (1.1) thus represents the expectation over the Markov chain given the information set at time t, as well as the expectation over the productivity shock from firms' production functions (see expression (3.13) below). In addition, I choose the following CRRA functional form for the disutility of labor supply function $\nu(\cdot)$:

$$\nu\left(h_t(i)\right) = \frac{h_t(i)^{1+\phi}}{1+\phi}$$

where $\phi > 0$ indexes the convexity of the household's disutility of labor supply, just as $\sigma_{j_t} > 0$ indexes the concavity of the household's utility of consumption.

Aggregate consumption across goods i follows the Dixit-Stiglitz CES form:

$$C_t = \left[\int_0^1 c_t^i(i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}} \tag{1.2}$$

where the constant elasticity of substitution $\theta \in (1, \infty)$ indexes the degree of substitutability between differentiated goods, and thus the degree of market power possessed by each firm i.

The associated aggregate price index is:

$$P_t = \left[\int_0^1 p_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}} \tag{1.3}$$

The household chooses the sequences $\{C_t\}$ and $\{h_t(i)\}$, $i \in [0, 1]$, to maximize (1.1) subject to a complete markets intertemporal budget constraint:

$$E_t \sum_{T=t}^{\infty} \frac{P_T C_T}{(1+i_T)^T} = E_t \sum_{T=t}^{\infty} \frac{\int_0^1 w_T(i) h_T(i) di + \int_0^1 \Pi_T(i) di}{(1+i_T)^T} \tag{1.4}$$

where $w_t(i)$ is the household's wage for producing good i in period t, and $\Pi_t(i) = p_t(i)y_t(i) - w_t(i)h_t(i)$ is the household's profit from producing good i (the firm's profit is allocated to the representative household), and the risk-free nominal interest rate i_t is determined by the monetary authority.

And market clearing in this economy implies:

$$Y_t = C_t \tag{1.5}$$

Household optimization then leads to the well-known Euler condition:

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{U_{C,j_{t+1}}(Y_{t+1})}{U_{C,j_t}(Y_t)} \frac{P_t}{P_{t+1}} \right] \right\}^{-1} \tag{1.6}$$

Note, however, that given time t information, we can integrate over the Markov chain explicitly. For example, if $j_t = 1$, we can write:

$$1 + i_t = \beta^{-1} \left\{ P_{11} E_t \left[\frac{U_{C,1}(Y_{t+1})}{U_{C,1}(Y_t)} \frac{P_t}{P_{t+1}} \right] + P_{12} E_t \left[\frac{U_{C,2}(Y_{t+1})}{U_{C,1}(Y_t)} \frac{P_t}{P_{t+1}} \right] \right\}^{-1} \tag{1.7}$$

Log-linearization of expression (1.7) then leads to the following New Keynesian "IS" curve:

$$y_t = \left[P_{11} + P_{12} \frac{\sigma_2}{\sigma_1} \right] E_t y_{t+1} - \sigma_1^{-1} \left(i_t - E_t \pi_{t+1} \right) \tag{1.8}$$

and, conversely, if $j_t = 2$ we obtain:

$$y_t = \left[P_{22} + P_{21} \frac{\sigma_1}{\sigma_2} \right] E_t y_{t+1} - \sigma_2^{-1} \left(i_t - E_t \pi_{t+1} \right) \tag{1.9}$$

Thus, the linearized IS curve takes a different form in our model than under the standard formulation without regime-switching.

Returning to the supply side of the model, each firm i has a production function given by:

$$y_t(i) = A_t f\left(h_t(i)\right) \tag{1.10}$$

where in general A_t is a stationary exogenous technology process.

And for simplicity I choose the Cobb-Douglas form for $y_t(i)$:

$$y_t(i) = A_t h_t(i) \tag{1.11}$$

Further, each firm i faces a demand curve:

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta} \tag{1.12}$$

where

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{1.13}$$

gives aggregate output and P_t is the aggregate price index given by (1.3) above.

Each period, only a measure $1-\alpha$ ($\alpha \in [0,1]$) of firms are able to set their prices optimally, while the remaining α must hold their prices fixed, following the convention of [14]. This introduces nominal rigidity in the form of sluggish price adjustment into the model. Since all firms face the same demand curve (1.12) and there are no idiosyncratic shocks, all firms that are allowed to set their price optimally in period t will choose the same profit-maximizing price p_t^*. In particular, such firms will choose their price to maximize the expected present value of their future profits:

$$E_t^i \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_T^i \left(p_t(i) \right) \right] \right\} \tag{1.14}$$

where $Q_{t,T} = \beta^{T-t} \frac{P_t}{P_T} \frac{U_{C,j_T}(Y_T)}{U_{C,j_t}(Y_t)}$ is the relevant stochastic discount factor, $\Pi_T^i(\cdot)$ is firm i's profit function, and firms further discount profits at rate α, since at time t they can expect the optimal price chosen at time t to still be in effect at time T with probability α^{T-t}. Thus the firm chooses $\{p_t(i)\}$ to maximize the expected sequence of profits (1.14) given $\{Y_T, P_T, w_T(j), A_T, Q_{t,T}\}$for $T \geq t$ and $i \in [0,1]$. Optimization then generates the following first-order condition:

$$E_t^i \left\{ \begin{array}{c} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} U_{C,j_T}(Y_T) Y_T P_T^{\theta} \\ \left[p_t^*(i) - \mu P_T s_{j_T} \left(Y_T \left(\frac{p_t^*(i)}{P_T} \right)^{-\theta}, Y_T; \tilde{\xi}_T \right) \right] \end{array} \right\} = 0 \tag{1.15}$$

where $\mu = \frac{\theta}{\theta - 1}$ is firms' (common) profit-maximizing markup and $s_{j_T}(\cdot)$ is firm i's real marginal cost function in period $T \geq t$, given the optimal price $p_t^*(i)$, set in period t.

To understand the form of the firm's real marginal cost function, first note that, in addition to the Euler condition (1.6) above, household optimization implies the following intratemporal condition:

$$\frac{\nu_h \left(h_t(i) \right)}{U_{C,j_t} \left(C_t \right)} = \frac{w_t(i)}{P_t} \tag{1.16}$$

that is, the real wage for producing good i must equal the household's marginal rate of substitution between consumption and labor for good i at each time t.

This condition, combined with the production function (3.13) above, implies that firm i's real marginal cost in period t is given by:

$$s_{j_t}\left(y_t(i), Y_t; \tilde{\xi}_t\right) = \frac{\nu_h\left(f^{-1}\left(\frac{y_t(i)}{A_t}\right)\right)}{U_{C,j_t}\left(Y_t\right)A_t}\Psi\left(\frac{y_t(i)}{A_t}\right) \tag{1.17}$$

where

$$\Psi(x) \equiv \frac{1}{f'\left(f^{-1}(x)\right)}$$

and $\tilde{\xi}_t$ is the vector of exogenous disturbances, which in our model is simply the technology disturbance $(A_t - E(A_t))$.

Substituting from (1.12) above, we can then write firm i's real marginal cost as a function of the price chosen by firm i:

$$s_{j_t}\left(Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\theta}, Y_t; \tilde{\xi}_t\right) = \frac{\nu_h\left(f^{-1}\left(\frac{Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\theta}}{A}\right)\right)}{U_{C,j_t}\left(Y_t\right)A_t}\Psi\left(\frac{Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\theta}}{A_t}\right)$$

Since each period $1-\alpha$ of firms choose the profit-maximizing price p_t^* and the remaining α must keep their prices fixed, the Dixit-Stiglitz aggregate price index evolves according to the law of motion:

$$P_t = \left[\alpha\left(P_{t-1}\right)^{1-\theta} + (1-\alpha)\left(p_t^*\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{1.18}$$

From a log-linear approximation of the FOC (1.15) above, one then obtains:

$$\hat{p}_t^* = (1-\alpha\beta)E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\hat{s}_T + p_T\right] \tag{1.19}$$

where the circumflex denotes log deviations from the steady state, and p_t is the log-linearized aggregated price index P_t. Note that the stochastic discount factors present in expression (1.15) have been differenced out following the linearization.

Quasi-differencing of (1.19) then yields:

$$\hat{p}_t^* = (1-\alpha\beta)\left[\hat{s}_t + p_t\right] + (\alpha\beta)E_t\hat{p}_{t+1}^*$$

Furthermore, from a log-linearization of the law of motion (1.18) we have:

$$\hat{p}_t^* = \frac{1}{1-\alpha}p_t - \frac{\alpha}{1-\alpha}p_{t-1}$$

Substituting this definition for $\hat{p}_t^*$ and defining $\pi_t = p_t - p_{t-1}$ we obtain:

$$\pi_t = \beta E_t \pi_{t+1} + \zeta(\hat{s}_t) \tag{1.20}$$

where

$$\zeta = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} > 0$$

This expression gives us the New Keynesian Phillips curve in terms of the real marginal cost "gap" $\hat{s}_t$. Note that since the linearized pricing condition (1.19) does not depend explicitly on σ_{j_t}, the marginal cost Phillips curve takes the same form as in the standard model without regime-switching.

I further assume that the technology process A_t is such that log-linearization leads to the simple AR(1) form:

$$a_t = \rho_a a_{t-1} + \varepsilon_t \tag{1.21}$$

with ε_t an i.i.d. mean zero random variable.

Given our parametric assumption for the utility and disutility functions, the production function, and the technology process, we can use the pricing condition (1.15) above to obtain the following log-linearized expression for the flexible price – or "natural" – level of output y_t^n:

$$y_t^n = \left(\frac{1+\phi}{\sigma_{j_t} + \phi} \right) a_t \tag{1.22}$$

Now we can define the output gap:

$$x_t = y_t - y_t^n \tag{1.23}$$

Log-linearization of the real marginal cost function (1.17) then yields the expression:

$$\hat{s}_t = (\sigma_{j_t} + \phi)\, x_t$$

Now we can rewrite the Phillips curve (1.20) above in terms of the output gap:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa_{j_t}(x_t) + e_t \tag{1.24}$$

where

$$\kappa_{j_t} = (\sigma_{j_t} + \phi)\,\frac{(1-\alpha)(1-\alpha\beta)}{\alpha} > 0$$

and an AR(1) markup shock e_t is added following standard practice. Note that once the Phillips curve is expressed in terms of the output gap, the slope κ_{j_t} does now depend on σ_{j_t} and thus on the current regime j_t.

Furthermore, using expressions (1.21), (1.22), and (1.23) above we can now rewrite the IS curve (1.8)-(1.9) in terms of the output gap as well:

$$x_t = \gamma_{j_t} E_t x_{t+1} - \sigma_{j_t}^{-1}(i_t - E_t \pi_{t+1}) + \chi_{j_t} a_t \tag{1.25}$$

where

$$\gamma_1 = \left[P_{11} + P_{12}\frac{\sigma_2}{\sigma_1}\right], \qquad \gamma_2 = \left[P_{22} + P_{21}\frac{\sigma_1}{\sigma_2}\right]$$

and

$$\chi_{j_t} = (\rho_a \gamma_{j_t} - 1)\left(\frac{1+\phi}{\sigma_{j_t} + \phi}\right)$$

Equations (1.24) and (1.25) thus define a regime-switching New Keynesian model with two endogenous state variable, π_t and x_t, two exogenous AR(1) variables, e_t and a_t, and the policymaker's control variable, i_t.

1.2.2 Statement and Calibration of the Model

Thus, we have the following linearized New Keynesian MSM:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa_{j_t}(x_t) + e_t \tag{1.26}$$

$$x_t = \gamma_{j_t} E_t x_{t+1} - \sigma_{j_t}^{-1}(i_t - E_t \pi_{t+1}) + \chi_{j_t} a_t \tag{1.27}$$

$j_t \in \{1, 2\}$ indexes the regime at time t, σ_{j_t} is the household's CRRA risk aversion coefficient, ϕ is the disutility of labor supply coefficient, β is the household's discount factor, and α is the probability that a given firm will not be able to set its price optimally in a given period, and κ_{j_t}, γ_{j_t}, and χ_{j_t} are as defined just above. The productivity shock a_t and the markup shock e_t follow AR(1) processes with autoregressive parameters ρ_a and ρ_e, respectively.

The model's only truly Markov-switching "deep parameter" is the risk aversion coefficient σ_{j_t}. However, since the Phillips curve slope parameter κ_{j_t} is a function of σ_{j_t}, γ_{j_t} is a function of σ_{j_t} as well as the elements of the Markov matrix defined by equation (1.28) below, and χ_{j_t} is a function of σ_{j_t} and γ_{j_t}, these three model parameters are also Markov-switching in practice. See Appendix A for further details.

Following [44], I choose the following values for the model's constant parameters: $\alpha = 2/3$, $\beta = 0.99$, $\phi = 1.7$, $\rho_a = 0.7$, $\rho_e = 0.35$. For the CRRA risk aversion coefficient, I choose $\sigma_1 = 1$ and $\sigma_2 = 3$, in order to produce significant time-variation in risk aversion without straying too far from the steady state. A very similar Markov-switching New Keynesian model has been considered by [57].

The risk aversion coefficient follows a two-regime Markov chain with the following transition matrix:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.95 & 0.05 \\ 0.22 & 0.78 \end{bmatrix} \tag{1.28}$$

where the transition probabilities for the low (regime 1) and high (regime 2) risk aversion regimes are matched, respectively, to the quarterly expansion and recession transition probabilities for U.S. data estimated by [16]. This calibration leads to time-variation in risk aversion at a frequency that on average matches the frequency of the U.S. business cycle. This results in asymmetric regime-switching, with periods of low risk aversion lasting longer on average than periods of high risk aversion.

1.3 Simple Rules

Taylor [60] found that historical U.S. monetary policy could be discribed by a simple reaction function of the form:

$$i_t = \bar{i} + \alpha_\pi (\pi_t - \pi^*) + \alpha_x x_t + \varepsilon_t \tag{1.29}$$

where i_t is the central bank's policy rate, $\bar{i}$ is the long-run policy rate, π_t is inflation, π^* is the central bank's inflation target, x is output, and ε_t is a random variable.[6] Taylor used 1.29 with values $\alpha_\pi = 1.5$ and $\alpha_x = 0.5$ or 1 to analyze Federal Reserve behavior over several eras since 1960.

In this spirit, in my setting I introduce a simple Markov-switching monetary policy rule in

[6] i_t,π_t, and x_t are in absolute values in Taylor's analysis, but represent log-deviations from the steady state in the rest of this paper.

my setting, following [21]

$$i_t = \alpha_{j_t,\pi}\pi_t + \alpha_{j_t,x}x_t \tag{1.30}$$

where $j_t \in \{1,2\}$ indexes the regime at time t. Though strictly sub-optimal due to their lack of history-dependence, such rules have been widely studied owing to their simplicity and capacity to approximate the behavior of actual central banks, the most notable such case being the Taylor rule for US data: $\alpha_{1,\pi} = \alpha_{2,\pi} = 1.5$, $\alpha_{1,x} = \alpha_{2,x} = 0.125$. In Section 5.4 below I consider two special cases of simple rules and then conduct a grid search for $\alpha_{i,y} \in [0,3]$ in order to determine the optimal such rule in this set.

In order to analyze the Markov-switching NK model under these two simple rules, I integrate over the distribution of the Markov chain according to the formulation of [21]:

$$\begin{aligned} x_{it} &= \gamma_i P_{i1} E_t x_{1t+1} + \gamma_i P_{i2} E_t x_{2t+1} && (1.31)\\ &\quad -\sigma_i^{-1}((\alpha_{i,\pi}\pi_{it} + \alpha_{i,x}x_{it}) - (P_{i1}E_t\pi_{1t+1} + P_{i2}E_t\pi_{2t+1})) + \chi_i a_t \\ \pi_{it} &= \beta(P_{i1}E_t\pi_{1t+1} + P_{i2}E_t\pi_{2t+1}) + \kappa_i(x_{it}) + e_t && (1.32) \end{aligned}$$

where π_{it} and x_{it} represent inflation and the output gap when the regime $j_t = i \in \{1,2\}$ and I substitute the monetary policy rule (1.30) into the New Keynesian IS curve (1.27). One can then define the expectational errors:

$$\eta^x_{1t+1} = x_{1t+1} - E_t x_{1t+1} \qquad \eta^x_{2t+1} = x_{2t+1} - E_t x_{2t+1}$$

$$\eta^\pi_{1t+1} = \pi_{1t+1} - E_t \pi_{1t+1} \qquad \eta^\pi_{2t+1} = \pi_{2t+1} - E_t \pi_{2t+1}$$

Substituting these four expressions into the four equations defined by (1.31) and (1.32) for $i \in \{1, 2\}$, one can write the resulting 4x1 system in the form:

$$AY_t = BY_{t-1} + A\eta_t + Cu_t$$

where

$$Y_t = \begin{bmatrix} x_{1t} \\ x_{2t} \\ \pi_{1t} \\ \pi_{2t} \end{bmatrix}, \quad \eta_t = \begin{bmatrix} \eta^x_{1t} \\ \eta^x_{2t} \\ \eta^\pi_{1t} \\ \eta^\pi_{2t} \end{bmatrix}, \quad u_t = \begin{bmatrix} a_t \\ e_t \end{bmatrix}$$

Having converted the Markov-switching system to a linear forward-looking rational expectations model, standard solution methods for such systems can then be employed.[7] In particular, if all of the generalized eigenvalues of (B, A) lie outside the unit circle, then a bounded solution to the system (1.31)-(1.32) exists that is unique within the class of solutions taking the minimum state variable (MSV) form:[8]

$$Y_t = Mu_t \tag{1.33}$$

Having solved the model for given simple rule coefficients, one can then obtain the expected loss under that solution using the loss function (1.38) with $\lambda_i = 0$. First note that in order for the timelessly optimal policy to be preferred, I need a loss criterion that is independent of the state of the economy at the initial period in which policy is chosen – period t in expression (1.37) above. To this end, I require that the period t state vector be set to the

[7] See e.g. [25] Chapter 10, Appendix 2
[8] See [50].

value that would be expected at an arbitrary period prior to period t, namely, zero – see [66] Chapter 7, Section 3.[9] With this constraint, the period $t+\tau$ conditional expectation of any function of $\pi_{it+\tau}$ and $x_{it+\tau}$, $i \in \{1,2\}$, is equal to the unconditional expectation for all $\tau \geq 0$. For example:

$$
\begin{aligned}
E_t(\pi_{it+\tau}) &= E(\pi_{it+\tau}) = 0 \\
E_t(x_{it+\tau}) &= E(x_{it+\tau}) = 0 \\
\forall \tau \geq 0, &\qquad i \in \{1,2\}
\end{aligned}
$$

In this case, the discounted infinite-horizon loss function converges to:

$$
E_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau} = E \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau} = \sum_{\tau=0}^{\infty} \delta^\tau E(L_{t+\tau}) = \frac{1}{1-\delta} E(L_t)
$$

where the unconditional expected period loss $E(L_t)$ is given by:

$$
\begin{aligned}
E(L_t) &= \bar{P}_1 \left[E(\pi_{1t}^2) + \lambda_x E(x_{1t}^2) \right] + \bar{P}_2 \left[E(\pi_{2t}^2) + \lambda_x E(x_{2t}^2) \right] \\
&= \left[\bar{P}_1 \sigma_{\pi 1}^2 + \bar{P}_2 \sigma_{\pi 2}^2 \right] + \lambda_x \left[\bar{P}_1 \sigma_{x1}^2 + \bar{P}_2 \sigma_{x2}^2 \right] \qquad (1.34)
\end{aligned}
$$

where $\bar{P}_1$ and $\bar{P}_2$ are the unconditional (stationary) probabilities of regimes 1 and 2 under the Markov chain. Thus the unconditional expected period loss (1.34) is equivalent (up to proportionality constant $\frac{1}{1-\delta}$) to the period t conditional expectation of the infinite horizon loss function (1.37), so long as the period t state vector equals zero. Since $\sigma_{\pi 1}^2$, $\sigma_{\pi 2}^2$, σ_{x1}^2, and

[9] In the Markov-switching case, timeless policy also requires that the Markov chain follow its stationary distribution in period t, since this is the regime that is expected at an arbitrary period prior to period t.

σ_{x2}^2 can be obtained analytically from the solution (1.33) above, (1.34) can readily be used as a criterion for ranking alternative simple rules in terms of expected loss.

1.4 Optimal Policy

Following [56], I first note that the two equation Markov-switching model (1.26)-(1.27) can be represented in matrix form as follows:

$$
\begin{aligned}
X_{t+1} &= A_{11j_{t+1}} X_t + A_{12j_{t+1}} x_t + B_{1j_{t+1}} i_t + C_{j_{t+1}} \varepsilon_{t+1} && (1.35)\\
E_t H_{j_{t+1}} x_t &= A_{21j_t} X_t + A_{22j_t} x_t + B_{2j_t} i_t && (1.36)
\end{aligned}
$$

where $j_t \in \{1, 2\}$ indexes the regime at time t, x_t is a vector of forward-looking variables (π_t and x_t in our case), i_t is a vector of control variables (i_t in our case), and X_t is a vector of predetermined variables (e_t and a_t in our case). The regime-dependent matrices A_{11j_t}, A_{12j_t}, B_{1j_t}, C_{j_t}, H_{j_t}, A_{21j_t}, A_{22j_t}, and B_{2j_t} contain the model parameters, with the regimes switching according to the transition matrix (1.28).

The policymaker's infinite-horizon problem is:

$$
\min_{\{i_{t+\tau}\}_{\tau \geq 0}} E_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau} \tag{1.37}
$$

where

$$
L_t = \pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2 \tag{1.38}
$$

subject to (1.35) and (1.36). This definition nests both a standard loss function with no interest rate penalty ($\lambda_i = 0$), as well as a loss function that penalizes interest rate variation ($\lambda_i > 0$). And, following Woodford, I employ a weight on the output gap that is welfare-based in a constant-coefficient version of our model with $\sigma_1 = \sigma_2 = E(\sigma)$:

$$\lambda_x = \frac{E(\kappa)}{\theta}$$

with the Dixit-Stiglitz CES parameter $\theta = 7.88$ and κ as defined above.

"Microfounded" reasoning also suggests that the policymaker should discount losses at the same rate as the representative household discounts utility, thus I use $\delta = 0.99$ (although the qualitative result isn't sensitive to this choice).

This forward-looking system can then be reformulated using the saddlepoint recursive method of [47]. In this approach, the policymaker's "dual" saddlepoint problem becomes:

$$\max_{\{\gamma_{t+\tau}\}_{\tau\geq 0}} \min_{\{x_{t+\tau}, i_{t+\tau}\}_{\tau\geq 0}} E_t \sum_{\tau=0}^{\infty} \delta^{\tau} \widetilde{L}_{t+\tau}$$

where

$$\widetilde{L}_{t+\tau} \equiv \begin{bmatrix} \widetilde{X}_{t+\tau} \\ \widetilde{i}_{t+\tau} \end{bmatrix}' \widetilde{W}_{j_{t+\tau}} \begin{bmatrix} \widetilde{X}_{t+\tau} \\ \widetilde{i}_{t+\tau} \end{bmatrix} \tag{1.39}$$

with $\widetilde{W}_{j_t}$ appropriately defined, and subject to the fully recursive system:

$$\widetilde{X}_{t+\tau+1} = \widetilde{A}_{j_{t+\tau+1}} \widetilde{X}_{t+\tau} + \widetilde{B}_{j_{t+\tau+1}} \widetilde{i}_{t+\tau} + \widetilde{C}_{j_{t+\tau+1}} + \varepsilon_{t+\tau+1} \tag{1.40}$$

with $\widetilde{A}_{j_t}$, $\widetilde{B}_{j_t}$, and $\widetilde{C}_{j_t}$ appropriately defined and with the expanded state and control vectors,

$$\widetilde{X}_t = \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \widetilde{i}_t \equiv \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix}$$

where Ξ_{t-1} and γ_t are the Lagrange multipliers for the forward looking variables in periods $t-1$ and t interpreted as state and control variables, respectively. The state variable Ξ_{t-1}recursively encodes the history dependence of optimal policy. For details see [56].

Now that the system is in recursive form, I can employ dynamic programming methods to obtain the optimal policy. Briefly, I can write the Bellman equation:

$$\widetilde{X}_t'\widetilde{V}_t\widetilde{X}_t + \widetilde{w}_t = \max_{\gamma_t} \min_{x_t, i_t} \left\{ \widetilde{X}_t' Q_t \widetilde{X}_t + 2\widetilde{X}_t' N_t \widetilde{i}_t + \widetilde{i}_t' R_t \widetilde{i}_t + \delta E_t \left(\widetilde{X}_{t+1}' \widetilde{V}_{t+1} \widetilde{X}_{t+1} + \widetilde{w}_{t+1} \right) \right\}$$

where the matrix $\widetilde{V}_t$ and the scalar $\widetilde{w}_t$ are components of the value function, and the matrices Q_t, N_t, and R_t result from the partition:

$$\widetilde{W}_t = \begin{bmatrix} Q_t & N_t \\ N_t' & R_t \end{bmatrix}$$

of the weighting matrix $\widetilde{W}_t$ from the loss function (1.39) of the saddlepoint problem.

Optimization then leads to a system of coupled algebraic Riccati equations (CARE):

$$\begin{aligned}
F_j &\equiv -J_j^{-1} K_j \\
J_j &\equiv R_j + \delta \sum_{k=1}^{n} P_{jk} \widetilde{B}_k' \widetilde{V}_k \widetilde{B}_k \\
K_j &\equiv N_j' + \delta \sum_{k=1}^{n} P_{jk} \widetilde{B}_k' \widetilde{V}_k \widetilde{A}_k \\
\widetilde{V}_j &= Q_j + \delta \sum_{k=1}^{n} P_{jk} \widetilde{A}_k' \widetilde{V}_k \widetilde{A}_k - K_j' J_j^{-1} K_j
\end{aligned} \tag{1.41}$$

where j and $k \in \{1, 2\}$ index regimes as above, n is the number of regimes (2 in our case) , P_{jk} is the probability of shifting from regime j to regime k (an element of the transition matrix (1.28) above), and the matrices J_t and K_t are given by:

$$\begin{aligned}
J_t &\equiv R_t + \delta E_t \widetilde{B}_{t+1}' \widetilde{V}_{t+1} \widetilde{B}_{t+1} \\
K_t &\equiv N_t' + \delta E_t \widetilde{B}_{t+1}' \widetilde{V}_{t+1} \widetilde{A}_{t+1}
\end{aligned}$$

The CARE system (1.41) can then be uncoupled and a forward iteration procedure employed which converges asymptotically to the stationary values F_j, K_j, J_j, and $\widetilde{V}_j$ which solve the CARE system. This yields the optimal policy function F_j for each of the n regimes. For details of this method see [22].

1.5 Results

1.5.1 Simple Rules

Using the welfare-based criterion (1.34) derived in Section 3 above, the expected loss for the constant coefficient Taylor rule is 0.92, while the expected loss under the optimal equilibrium is 0.23, indicating that the Taylor rule leaves significant room for improvement. In order to find the optimal simple rule within a constrained region, I conduct a grid search over $\alpha_{i,y} \in [0, 3]$. The resulting constrained optimal coefficients take the boundary values:

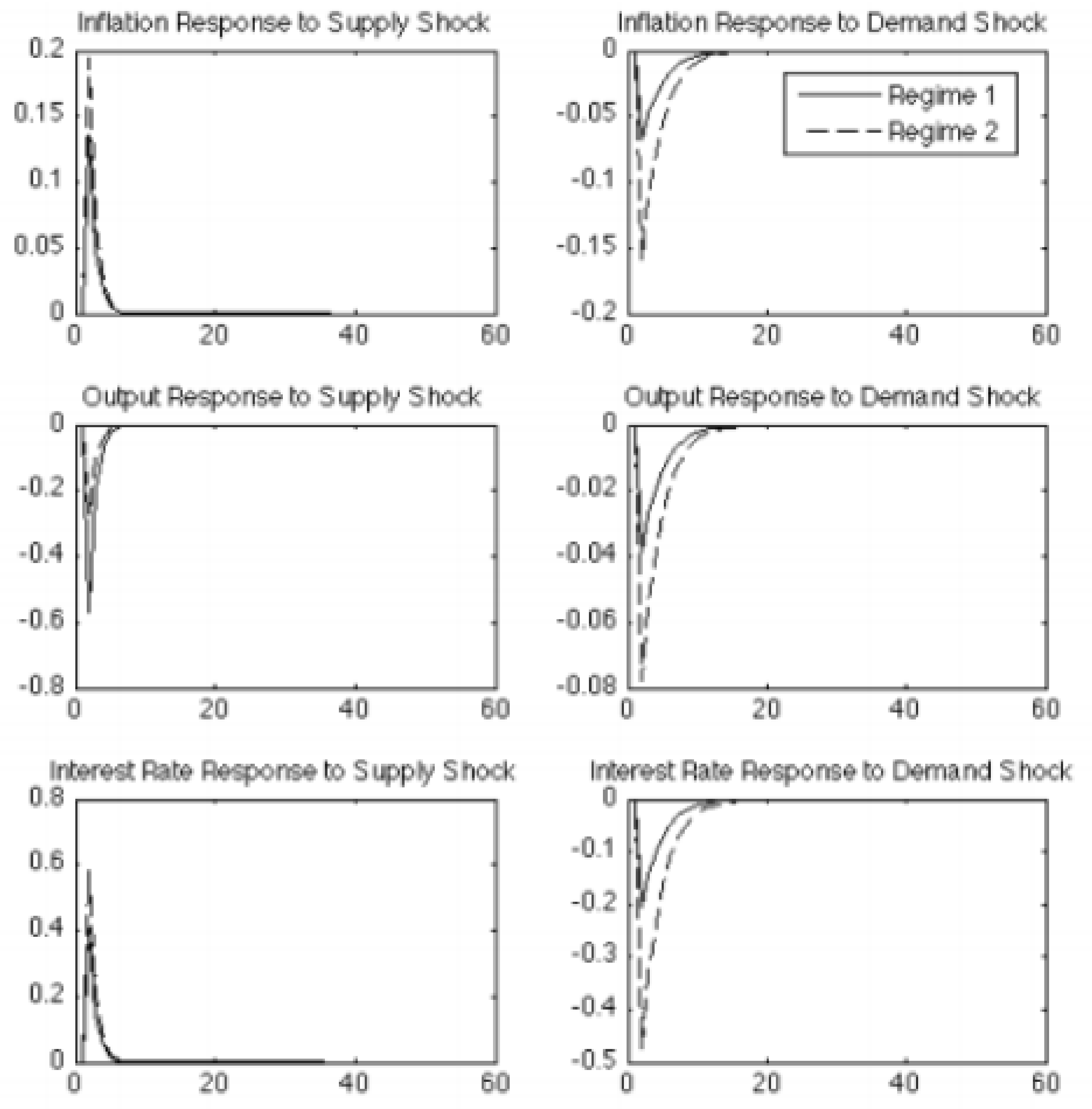
$$\begin{aligned} &\alpha_{1,\pi} = 3.00 \qquad \alpha_{1,x} = 0.00 \\ &\alpha_{2,\pi} = 3.00 \qquad \alpha_{2,x} = 0.00 \end{aligned} \tag{1.42}$$

Although the responses are constant across regimes to within the precision of the grid (0.01), in the case of inflation this is likely due to the upper bound constraint binding rather than to global optimality of a constant coefficient simple rule.

1.5.2 Simple Rules: Impulse-Response Plots

Figure 1 depicts the regime-dependent impulse-responses of inflation, the output gap, and the interest rate to one standard deviation innovations in the supply and demand shocks (separately) for the optimal simple rule (1.42). This plot makes clear that, unlike for the timelessly optimal policy under $\lambda_i = 0$ depicted in Figure 2, the constrained-optimal simple rule is unable to perfectly offset the demand shock. This observation confirms the intuition that a simple Taylor-type rule responding only to current inflation and the current output gap is not able to implement the timelessly optimal equilibrium.

Figure 1.1: Impulse-Response, Constrained-Optimal Simple Rule



1.5.3 Optimal Policy: Loss Function without Interest Rate Penalty

For the case $\lambda_i = 0$, I find that with no interest rate penalty the optimal response of the interest rate to the Lagrange multipliers associated with the inflation and the output gap constraints is larger in the high risk aversion regime (regime 2) than in the low risk aversion regime (regime 1). Using $\Phi_{j,\alpha}$ to denote the optimal response of the interest rate to the period $t-1$ Lagrange multiplier $\Xi_{t-1,\alpha}$ (discussed in Section 4 above) associated with the forward-looking constraint for state variable α in regime j, I find the following values:

$$\Phi_{1,\pi} = 0.468 \qquad \Phi_{1,x} = 5.259$$

$$\Phi_{2,\pi} = 1.383 \qquad \Phi_{2,x} = 5.610$$

Intuitively, a larger change in the interest rate is required to persuade agents to deviate from intertemporal smoothing behavior in the high risk-aversion regime, just as a higher risk premium is required to persuade agents to deviate from smoothing across states of the world. In order to gain a clearer picture of the effect of σ_{j_t} switching from σ_1 to σ_2 on the interaction between interest rates and the representative household's optimal intertemporal consumption decision, we can write the household's Euler condition explicitly by substituting the parametric form of the household's utility function into condition (1.6) from Appendix A below. First, for simplicity consider the constant-coefficient case $\sigma_1 = \sigma_2 = \bar{\sigma}$, then the Euler condition can be written:

$$\beta \left(1 + i_t\right) E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\bar{\sigma}} \left(\frac{P_t}{P_{t+1}} \right) \right] = 1 \tag{1.43}$$

It's clear from this expression that, by increasing the sensitivity of marginal utility to changes in consumption, increased $\bar{\sigma}$ reduces the sensitivity of the optimal consumption path to interest rate changes. For example, if $C_t > C_{t+1}$, $\bar{\sigma} > 1$ amplifies the "effective consumption change" $\left(\frac{C_t}{C_{t+1}}\right)^{\bar{\sigma}}$, requiring a greater reduction in i_t in order for the condition to hold relative to $\bar{\sigma} = 1$ (and conversely, $\bar{\sigma} < 1$ damps the effective consumption change). In the limit $\bar{\sigma} \to \infty$, perfect consumption smoothing regardless of interest rate movements obtains.

In the Markov-switching case, the form of the Euler condition is analogous, but we must now integrate over the Markov chain. If $j_t = 1$, we have:

$$\beta\left(1+i_t\right)E_t\left\{\left[P_{11}\left(\frac{C_t}{C_{t+1}}\right)^{\sigma_1}+P_{12}\frac{\left(C_t\right)^{\sigma_1}}{\left(C_{t+1}\right)^{\sigma_2}}\right]\left(\frac{P_t}{P_{t+1}}\right)\right\}=1 \tag{1.44}$$

Period t marginal utility depends only on σ_1. However, period $t+1$ marginal utility depends on both σ_1 and σ_2 with weights P_{11} and P_{12}. Nevertheless, since $P_{11} >> P_{12}$, the overall result is an Euler condition very similar to a constant-coefficient Euler condition with $\bar{\sigma} = \sigma_1$, but with the period $t+1$ marginal utility slightly moderated in the direction of σ_2. Again, since $P_{22} >> P_{21}$, the reverse obtains for $j_t = 2$. So the intuition from the constant-coefficient condition (1.43) that when $\bar{\sigma}$ increases from σ_1 to σ_2, greater interest rate movements are required to justify deviations from consumption smoothing broadly carries over to the case when σ_{j_t} switches from σ_1 to σ_2 in the Markov-switching condition (1.44), (and it's mirror image for the case $j_t = 2$) . This intuition is also captured in the linearized form (1.27) above. Thus, in the MSM defined by (1.26)-(1.27)-(1.28), and with $\lambda_i = 0$, the monetary authority must respond more aggressively to the Lagrange multipliers associated with both the inflation and output gap constrains in regime 2, in order to give more risk averse agents sufficient incentive to substitute consumption from the future to the present (or vice versa).

Table 1.1: Optimal Policy Coefficients for Various λ_i Values

λ_i	$\Phi_{1,\pi}$	$\Phi_{2,\pi}$	$\Phi_{1,x}$	$\Phi_{2,x}$
0	0.468	1.383	5.259	5.610
0.02	0.350	0.577	3.728	2.144
0.04	0.286	0.385	2.953	1.384
0.077	0.219	0.246	2.176	0.857
0.1	0.192	0.203	1.882	0.698
0.2	0.129	0.118	1.208	0.393

1.5.4 Optimal Policy: Loss Function with Interest Rate Penalty

Inclusion of a non-zero interest rate penalty λ_i in the loss function (1.38) may be justified in various ways. [66] Chapter 6, Section 4 shows that inclusion of such a term can be optimal if deviations from the Friedman rule ($i_t = i_t^M = 0$, where i_t^M is the nominal interest rate paid by money) are costly in terms of welfare, or in the presence of a zero lower-bound constraint.[10] Such a penalty may also be justified by the empirical fact that in practice central banker's adjust their target rates only slowly, due perhaps to the difficulty of achieving consensus for a large interest rate movement and also out of a desire to avoid "frightening" market participants. Interestingly, if such a loss function includes a sufficiently large interest rate penalty, the qualitative result reported above for the welfare-based loss function is reversed: the optimal response to the inflation and output gap multipliers is lower during the high risk aversion regime compared to the low risk aversion regime. This result obtains because the extra interest rate movement required to respond to inflation or the output gap when agents are more risk averse becomes too "expensive" in terms of interest rate variation.

Table 1 presents the optimal regime-dependent policy response for various values of λ_i. The clear trend is that as λ_i increases, the optimal policy response to both the inflation and output gap multipliers becomes less aggressive in regime 2 relative to regime 1 (for the reason just discussed). Note also that since λ_x is much smaller than one, the optimal response to the output gap multiplier switches to a stronger response in regime 1 starting with the second

[10] I also adopt Woodford's calibration $\lambda_i = 0.077$ as my baseline value for this parameter.

row of Table 1 ($\lambda_i = 0.02$), while the response to the inflation multiplier (which implicitly has weight one in the loss function) doesn't switch until the sixth row ($\lambda_i = 0.2$).

1.5.5 Optimal Policy: Impulse-Response Plots

Figures 2 and 3 depict the regime-dependent impulse-responses of inflation, the output gap, and the interest rate to one standard deviation innovations in the supply and demand shocks (separately), both for the loss function with no interest rate penalty and for the case $\lambda_i = 0.077$. Two broad trends emerge from these plots. First, with no interest rate penalty, the demand shock is perfectly offset to within machine precision. In contrast, for the case $\lambda_i = 0.077$, the the demand shock is not perfectly offset, since the interest rate penalty constrains policy from responding sufficiently aggressively to do so. Second, the magnitude of deviations of both inflation and output are significantly larger when the interest rate penalty is added to the loss function – especially for the demand shock – since the interest rate response to both of these variables is smaller in this case.

1.5.6 Gains from Commitment

In order to quantify the gains from commitment, I first consider the "implicit simple rule" under the timelessly optimal policy reported in Section 5.1 above (with $\lambda_i = 0$). That is, after simulating a sample under the optimal equilibrium, I run an OLS regression of the form (1.30). The average estimated parameter values over a number of such Monte Carlo simulations provides the response of the interest rate to current inflation and the current output gap implicit in the conduct of the optimal policy with commitment over time. Thus this rule, though misspecified, is what an econometrician would infer about the response of the interest rate to inflation and the output gap from the data generated by the model. The

Figure 1.2: Impulse-Response, No Interest Rate Penalty

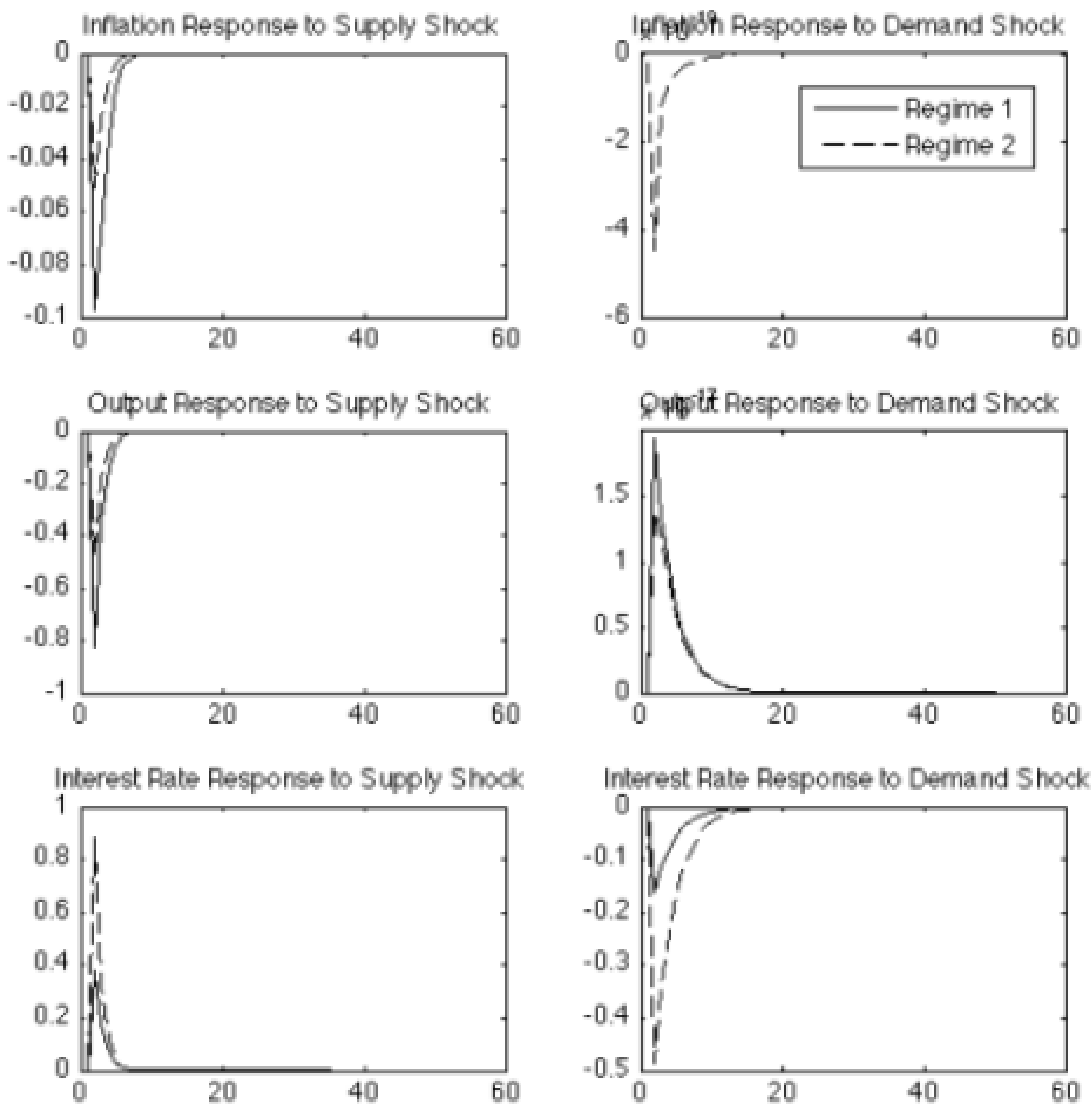
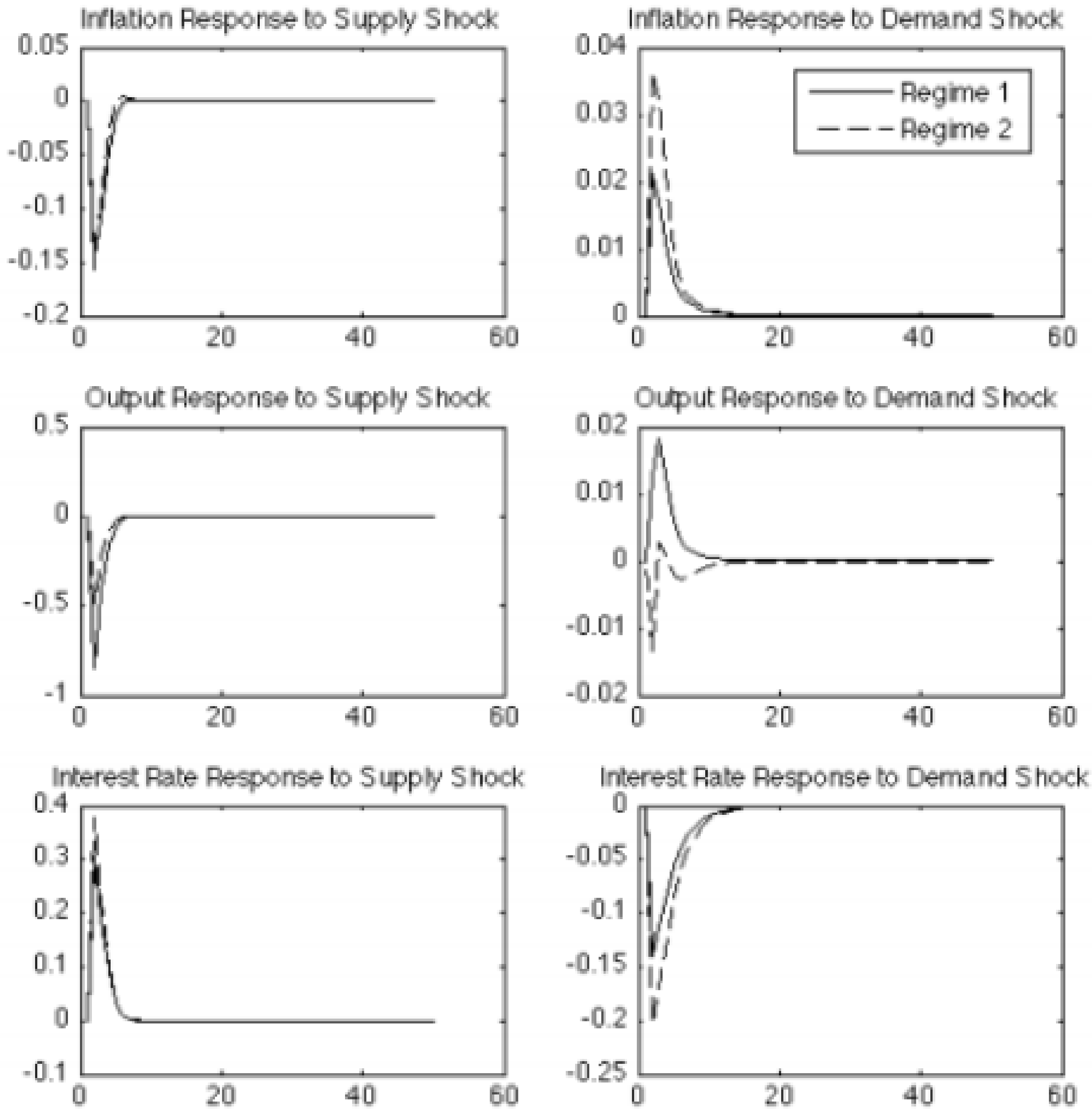


Figure 1.3: Impulse-Response, Interest Rate Penalty ($\lambda_i = 0.077$)



parameter values that result from this procedure are:

$$\begin{aligned} &\alpha_{1,\pi} = 0.576 \qquad \alpha_{1,x} = 0.453 \\ &\alpha_{2,\pi} = 1.838 \qquad \alpha_{2,x} = 1.321 \end{aligned} \tag{1.45}$$

Note that the "implicit" interest rate response to both inflation and the output gap is more aggressive in the high risk aversion regime compared to the low risk aversion regime, consistent with the qualitative result found under optimal policy with small or no interest rate penalty. However, these parameters result in indeterminacy of the resulting equilibrium, and therefore do not result in an expected loss that can be compared with that under optimal policy.

Therefore, I instead compare the expected loss under optimal policy with commitment with that under the optimal discretionary equilibrium. Specifically, if I minimize the objective (1.37) above with$\lambda_i = 0$ subject to the Phillips curve constraint (1.26), but then reoptimize each period (or equivalently, take future expectations as given), I obtain the following well-known first-order condition for each time t:

$$\pi_t = -\frac{\lambda_x}{\kappa_{j_t}} x_t \tag{1.46}$$

Combining this condition with the Phillips curve (1.26) results in a two-equation Markov-switching system that defines the optimal discretionary equilibrium. This is the optimal equilibrium that a discretionary policymaker – one who reoptimizes every period – can implement. Using the methodology described in Section 3 above, this system can then be solved and the expected loss can be calculated. The resulting expected loss under discretion is 0.28. Compared to the expected loss of 0.23 under the optimal policy with commitment,

Figure 1.4: Impulse-Responses under Optimal Discretionary Equilibrium

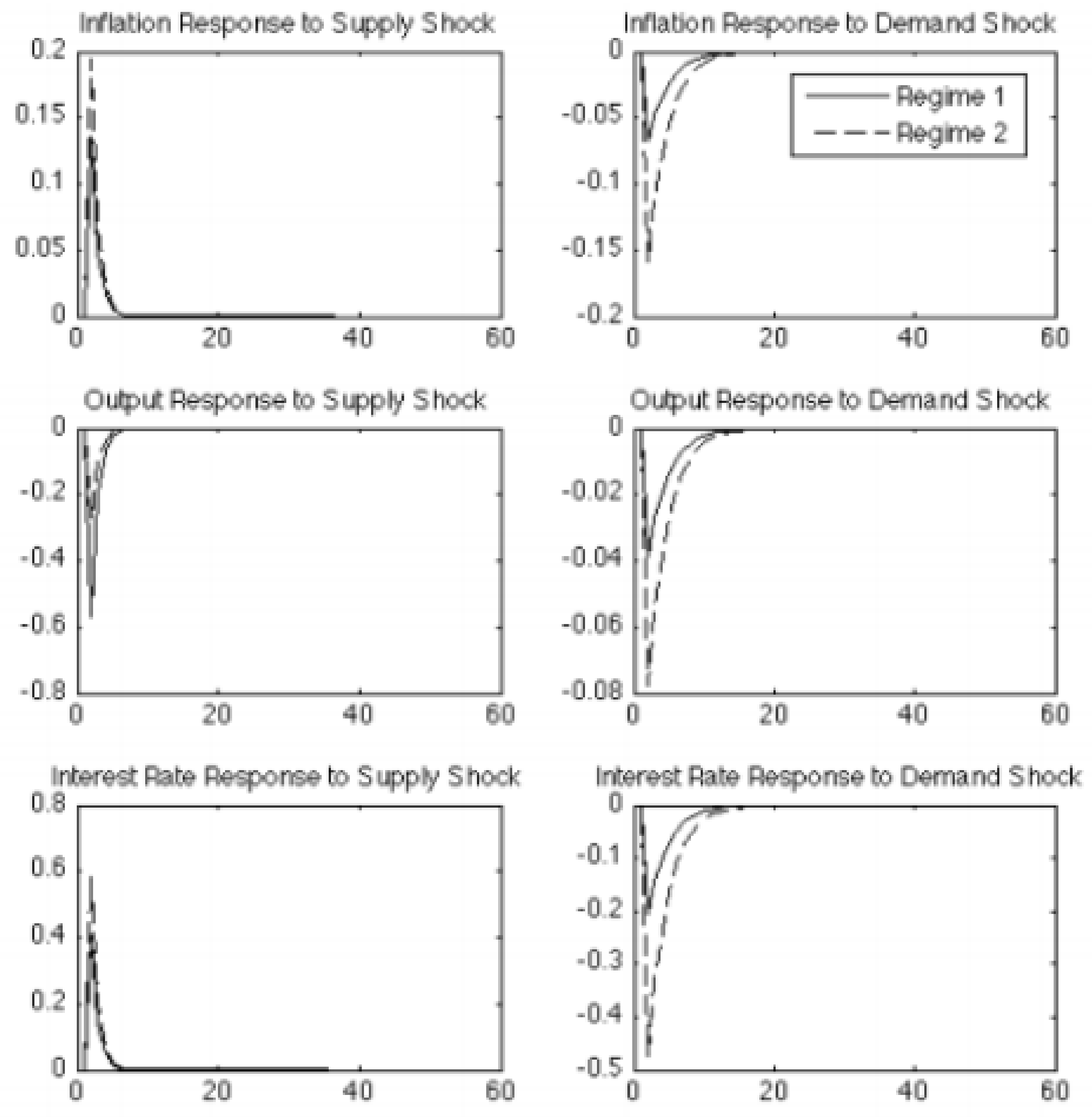


Table 1.2: Expected Loss Under the Optimal Policy Function, Discretion, and Selected Simple Rules

$\alpha_{1,\pi}$	$\alpha_{1,x}$	$\alpha_{2,\pi}$	$\alpha_{2,x}$	$E(L_t)$
-	-	-	Opt.	0.23
-	-	-	Discr.	0.28
3.00	0.00	3.00	0.00	0.41
1.5	0.125	1.5	0.125	0.92
0.576	0.453	1.838	1.321	Ind.

the optimal discretionary equilibrium thus represents a 22% increase in expected loss.

Table 2 summarizes the results discussed in Sections 5.1 and 5.6.

1.6 Conclusion

I have investigated optimal monetary policy in a New Keynesian model that switches stochastically between a high risk aversion regime and a low risk aversion regime – motivated by empirical evidence of substantial time-variation in risk premia. In particular, I replaced the preference shock typically introduced in the general equilibrium environment underlying the benchmark New Keynesian model with a Markov-switching CRRA risk aversion coefficient. I first adopted the framework of Davig and Leeper [21] in order to compute the expected loss under various regime-dependent Taylor-type rules, and found that the optimal such rule within a grid involves an aggressive inflation response in both regimes. I then employed the methodology of Svensson and Williams [56] in order to compute the optimal regime-dependent policy with commitment. Under a loss function with no interest rate penalty, I found that optimal policy is characterized by a more aggressive response to both inflation and the output gap in the high risk aversion regime. However, under a loss function that includes a sufficiently large interest rate penalty, the reverse result may obtain. The intuition for both results lies in the fact that, when agents are more risk averse, a larger change in the interest rate is required to persuade them to deviate from intertemporal smoothing behavior. Finally, I compared the optimal policy with commitment to the optimal discretionary equilibrium, and find that the gains from commitment in this model are substantial. Markov-switching provides a tractable yet powerful methodology for introducing multiplicative uncertainty and nonlinear dynamics into the models currently used for monetary policy analysis, and there is great future potential in advancing this framework.

Chapter 2

Macroeconomic Implications of the Risk Premium in a Real Business Cycle Model with Habit Formation

2.1 Introduction

The importance of risk premia – and their variation over time – for the dynamics of the business cycle seems difficult to dispute.[1] The countercyclical variation in risk premia that has frequently been documented for U.S. data[2] clearly has implications for macroeconomic agents. For example, if a negative event causes agents to become more "fearful," firms may discount risky returns to investment more strongly, and households may discount returns to risky asset holdings more strongly, resulting in reduced investment and consumption (respectively), as in fact one observes during recessions. If in turn the resulting drop in

[1] One prominent theoretical motivator of this claim is [10], in which the higher risk premium implied by the presence of intersectoral rigidity in a real business cycle model with habit formation leads to higher consumption variabilty. In turn, higher consumption variability implies a higher risk premium.

[2] See e.g. [11], [3], [24], [42], and [48].

output renders agents still more fearful, risk premia may rise yet again, causing output to fall still further. The presence of such a "negative feedback loop" between risk premia and output may at least partly explain why the U.S. recession of 2007-2009 – which was triggered by unprecedented (in the post-war record) panic in financial markets, and a precipitous rise in risk premia – also featured the largest drop in output in the post-war record.[3] The present study finds that a "risk aversion accelerator" effect of this type may indeed be present in the real business cycle (RBC) model with habit formation.

2.1.1 Agenda and Related Literature

The goal (and contribution) of the present study is to examine the implications – for both macroeconomic variables and the risk premium – of including external habit formation in a basic stochastic growth model, using a third-order perturbation solution in order to respect the nonlinearity of the risk premium. Habit formation refers to a preference specification in which the standard argument of the CRRA utility function – current consumption – is instead replaced with the difference between current consumption and an exogenous "habit stock":[4]

$$U\left(C_t\right) = \frac{\left(C_t - H_t\right)^{1-\gamma}}{1-\gamma} \tag{2.1}$$

Following standard practice in the DSGE literature, in the model I present in Section 2 below I choose $H_t = bC_{t-1}$, for $b \in [0, 1]$, so that the external habit stock is proportional

[3] Surveying an international dataset, Reinhart and Rogoff [53] report that the mean real output contraction following a major financial crisis is 9.3 percent, much deeper than that for a typical recession. Similarly, in Chapter 3 of the IMF's April 2009 World Economic Outlook, Kannan et al. [36] find that "recessions associated with financial crises tend to be unusually severe and their recoveries typically slow." While there are many factors that may contribute to this stylized fact, the increase in risk premia almost universally associated with financial crises seems likely to play at least some role.

[4] If agents don't take the habit stock to be exogenous (as in Boldrin et al. [10]), the preference is referred to as internal habit.

to to past consumption. This seemingly minor change results in agents' local relative risk aversion, which I denote R_t, varying over time – specifically, local relative risk aversion is a decreasing function of the deviation of current consumption from its habit level:

$$R_t = \frac{-C_t U_t^{CC}}{U_t^C} = \frac{\gamma C_t}{C_t - bC_{t-1}} = \frac{\gamma}{1 - \frac{bC_{t-1}}{C_t}} \tag{2.2}$$

Campbell and Cochrane [15] first demonstrated that, in a partial equilibrium consumption-based asset pricing model, an external habit specification is sufficient to produce large and time-varying risk premia broadly consistent with U.S. data without assuming an unreasonably large CRRA risk aversion coefficient, making it a candidate solution of Mehra and Prescott's [51] famous equity premium puzzle. Since then, numerous studies have included habit formation preferences in New Keynesian DSGE models, including Smets and Wouters [55] and its many successors. However, such investigations have typically linearized the model, and ignored its asset pricing implications, leaving unanswered the question of whether Campbell and Cochrane's findings carry through in a general equilibrium environment.

Jermann [32] investigated the asset-pricing implications of an RBC model with habit formation and found that, far from explaining the equity premium as it does in partial equilibrium, in general equilibrium habit formation (in the absence of capital adjustment costs) makes agents more desirous to smooth consumption, resulting in a lower risk premium. Lettau and Uhlig [43] also find that consumption is excessively smooth in an RBC model with habit formation, while Lettau [43] finds a Sharpe ratio consistent with U.S. data in an RBC model with habit formation and wage rigidity. However, all three of these papers log-linearize the model in order to generate consumption data, and those that compute asset prices condition on this data to do so. This approach is suspect since, as discussed above, it renders the model effectively risk-neutral. The present study will thus ask whether these results carry through in a third-order perturbation solution.

One widely-remarked contributor to the lack of a risk premium often found in general equilibrium models is the ability of agents to offset negative shocks by adjusting on the labor margin, as documented by Swanson [59]. Thus, in order to grant the RBC model the best chance of generating a risk premium, in the sequel I consider a calibration with low Frisch labor-supply elasticity, in addition to a benchmark calibration with higher elasticity. However, even under perfectly inelastic labor supply, capital remains as a margin on which agents can adjust to offset shocks, and Jermann introduces convex adjust costs to capital for this reason. For simplicity, I rely on the standard timing assumption that capital is fixed a period in advance, so that agents can't immediately respond to a negative shock with a change in the capital stock, as would be the case for the labor margin.

An important contribution relative to the present study is that of Boldrin et al. [10]. Using a nonlinear parametrized expectations solution method, these authors find an empirically plausible equity premium in a two-sector RBC model with intersectoral rigidity and internal habit formation. In a reference one-sector model, they do not find that habit formation increases excess returns or the Sharpe ratio, both of which I find below. However, their excess returns are derived from sector-specific claims to the marginal return to capital, whereas I define excess returns in terms of perpetual claims to the aggregate profitability of a capitalist firm. Further, the macroeconomic moments reported for their one-sector model with and without habit formation show that output variability decreases with habit formation. This is likely due to the fact they only report HP-filtered simulations, since I find the same result in the HP-filtered simulations for the my baseline RBC model. Their findings therefore need not be inconsistent with the "risk aversion accelerator" that I discuss in Section 3. Another possibility is that their internal habit specification may have meaningfully different implications – for excess returns or for the amplification mechanism just mentioned, – relative to the external habit specification I employ.

Rouwenhorst, in chapter10 of [20], explores the asset-pricing implications of a standard

RBC model and finds that the mean return of stocks relative to bonds is too low to be consistent with the historical equity premium. Rouwenhorst solves the model via value function iteration, which is a far preferable solution method compared to linearization. Still, he does not consider habit formation, and it is interesting to explore whether a perturbation approach gives a similar result. Rudebusch and Swanson [54] find a negligible term premium in a Smets-Wouters-type DSGE model with habit formation, using a high-order perturbation solution. However, in an alternative calibration with a higher CRRA coefficient – similar to the value I adopt below – they a significantly increased (though still too low) term premium. Swanson et al. [58] examine the first- through sixth-order perturbation solutions of the stochastic growth model without leisure, and note that the linear policy (consumption) function is not too bad of an approximation. However, these authors don't consider habit formation, and use relatively low curvature.

2.2 The Model

2.2.1 An RBC Model with Habit Formation

I present a decentralized, two-firm description of a basic RBC model[5] – as in Chapter 12 of [45] – with the addition of external habit formation. This model is isomorphic to the more-typically presented planner's problem , as well as to a one-firm decentralization. I also deviate from Ljungqvist and Sargent (henceforth L-S) in transferring the return to capital to the household via equity shares rather than having the capitalist firm write-over its profits to the household via debt securities.

[5]The basic RBC model is equivalent to the stochastic growth model with leisure.

2.2.2 The Household

The household chooses consumption, $\{C_t\}_{t=0}^{\infty}$, labor supply, $\{L_t\}_{t=0}^{\infty}$, and shareholdings in the capitalist firm, $\{N_t^K\}_{t=0}^{\infty}$ to maximize the infinite-horizon objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t - bC_{t-1})^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\phi}}{1+\phi} \right] \tag{2.3}$$

subject to the flow budget constraint:

$$C_t + P_t^K N_t^K = W_t L_t + N_{t-1}^K (D_t^K + P_t^K) \tag{2.4}$$

where W_t is the time-t real wage, P_t^K is the time-t price of a perpetual claim to the profitability of the capitalist firm, and D_t^K is the time-t dividend issued by the capitalist firm, and N_{t-1}^K represents the fraction of the capitalist firm's profitability that the household holds a claim to in period t. The household exhibits CRRA utility with respect to the deviation of consumption from the habit stock, with CRRA coefficient γ, and additively-separable convex CRRA disutility of labor supply, with CRRA coefficient ϕ – also equal to the inverse of elasticity of labor supply with respect to the real wage. $b \in [0,1]$ indexes the degree to which lagged consumption enters the external habit stock.

Combining the period-t first-order conditions with respect to C_t, C_{t+1}, and N_t^K results in the intertemporal pricing condition:

$$P_t^K = E_t \left[M_{t,t+1} \left(D_{t+1}^K + P_{t+1}^K \right) \right] \tag{2.5}$$

where $M_{t,t+j} = \left[\beta^j \left(\frac{C_t - bC_{t-1}}{C_{t+j} - bC_{t+j-1}} \right)^{\gamma} \right]$ is the household's stochastic discount factor (SDF) from time t to time $t+j$.

And combining the period-t first-order conditions with respect to C_t and L_t results in an intratemporal condition equating the real wage with the marginal rate of substitution between consumption and labor:

$$W_t = \frac{L_t^{\phi}}{\left(C_t - bC_{t-1}\right)^{-\gamma}} \tag{2.6}$$

2.2.3 The Capitalist Firm

The capitalist firm (or firm of type 2 in the terminology of L-S) undertakes investment and rents the resulting capital to the production firm (type 1 firm). It chooses $\{K_t\}_{t=0}^{\infty}$ to maximize the objective:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[r_t K_t - I_t\right] \tag{2.7}$$

subject to the law of motion for capital:

$$K_{t+1} = (1-\delta) K_t + I_t \tag{2.8}$$

where δ is the rate of depreciation, and r_t is the period-t rental rate earned on the capital stock.

Substituting the the capitalist firm's constraint into its objective and taking the period-t first order condition with respect to K_{t+1}, one obtains the intertemporal condition:

$$1 = E_t \left[M_{t,t+1} \left(r_{t+1} + 1 - \delta\right)\right] \tag{2.9}$$

This well-known optimality condition sets the marginal cost of capital creation equal to the expected, discounted (by the SDF) next-period marginal benefit of capital, next-period's rental rate as well as the "liquidation value" $1-\delta$, which is the marginal amount of "free capital" that will carry over into next period as a result of time-t capital creation.

2.2.4 The Production Firm

The production firm, or firm of type 1, rents capital from the type 2 firm and labor from the household, and converts them into output. It chooses $\{K_t\}_{t=0}^{\infty}$ and $\{L_t\}_{t=0}^{\infty}$ to maximize the objective:[6]

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[Y_t - r_t K_t - W_t L_t\right] \tag{2.10}$$

subject to the Cobb-Douglas production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{2.11}$$

Substituting the constraint (3.13) into the objective and taking the period-t first-order condition with respect to L_t one obtains:

$$W_t = (1-\alpha)\, A_t K_t^{\alpha} L_t^{-\alpha} \tag{2.12}$$

This condition sets the real wage equal to the marginal product of labor.

Similarly, taking the period-t first-order condition with respect to K_t one obtains a condition

[6] I could use lower-case $\{k_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$, to underscore that the production firm chooses how much capital and labor to rent independent of any supply considerations, and then – after solving the production firm's problem – impose clearing of the rental markets: $k_t = K_t$, $l_t = L_t$ for all t. However, since these subtleties are well-known, for simplicity I'll use K_t and L_t on both sides of the market.

equating the rental rate with the marginal product of capital:

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{2.13}$$

Log productivity is exogenous and follows an AR(1) process:

$$\log A_t = \rho \log A_{t-1} + \sigma \epsilon_t \tag{2.14}$$

with ϵ_t a standard normal productivity shock.

2.2.5 General Equilibrium

Market clearing implies:

$$Y_t = C_t + I_t \tag{2.15}$$

Although conditions (3.5), (3.19), (3.22), (3.13), (3.14), (3.15), and (3.17) above determine the path of $\{C_t\}_{t=0}^{\infty}$, $\{I_t\}_{t=0}^{\infty}$, $\{K_t\}_{t=0}^{\infty}$, $\{L_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, $\{W_t\}_{t=0}^{\infty}$, and$\{Y_t\}_{t=0}^{\infty}$ – given $\{A_t\}_{t=0}^{\infty}$ – independently of $\left\{N_t^K\right\}_{t=0}^{\infty}$, the same is not true of $\left\{P_t^K\right\}_{t=0}^{\infty}$, since the dividend paid by the capitalist firm includes the share-sale component of the firm's profit. However, as was observed in Section 2.3 above, only $N_t^K = 1$ for all t – which transfers the full profitability of the capitalist firm to the household – is consistent with general equilibrium, implying that the share-sale term in the capitalist firm's dividend equals 0 for all t. From the household's budget constraint, note that in equilibrium $\forall t, N_t^K = 1$ implies:

$$
\begin{aligned}
C_t &= -P_t^K \left(N_t^K - N_{t-1}^K\right) + D_t^K N_{t-1}^K + W_t L_t \\
\leftrightarrow C_t &= 0 + \left(\alpha A_t K_t^\alpha L_t^{1-\alpha} - I_t\right) + \left((1-\alpha) A_t K_t^\alpha L_t^{1-\alpha}\right) \\
\leftrightarrow C_t &= \alpha Y_t - I_t + (1-\alpha) Y_t \\
\leftrightarrow C_t &= Y_t - I_t
\end{aligned}
$$

Thus, my assumption that N_t^K is constant and equal to 1 for all t ensures that general equilibrium holds for each time t. Conversely, any other path for $\left\{N_t^K\right\}_{t=0}^{\infty}$ would imply a violation of general equilibrium.

2.2.6 Calibration

My baseline calibration is $\alpha = 0.3$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.979$, and $\sigma = 0.0072$, all of which are quite standard and are taken from King and Rebelo [39]. For the habit stock parameter I choose $b = 0.66$, which is representative of values typically found in estimations of DSGE models, following Rudebusch and Swanson [54]. For the CRRA coefficient I choose $\gamma = 5$. This moderately-high degree of curvature amplifies the importance of time-variation in risk premia for macroeconomic dynamics, and increases the importance of the higher-order terms in my third-order perturbation solution – while remaining small enough to maintain some degree of microeconomic plausibility.

Following Levine and Pearlman [44], for the CRRA "convexity of labor-disutility" measure ϕ, which is the inverse of the Frisch elasticity of labor supply, I choose 1.7 – close to but a bit higher than typical macro estimates, though still low compared to micro estimates – as my baseline calibration. However, given the interest of this parameter, in Section 3 below I will explore other values.

I take the empirical excess returns in Table 2.3 below from Kochelakota [40], and the empirical range of Sharpe ratios from Hodges [29].

2.2.7 Defining The Risk Premium

Although the asset price $\left\{P_t^K\right\}_{t=0}^{\infty}$ introduced above is "neutral" in the sense that its presence doesn't alter the equilibrium path of the model's macroeconomic variables – e.g. relative to a model in which the capitalist firm simply hands over its profit[7] to the household – it does facilitate the construction of a natural measure of the risk premium.

I further include a (neutral) risk-free one-period claim:

$$P_t^B = E_t\left[M_{t,t+1}(1)\right] \tag{2.16}$$

I will then be able to explore the risk premium in the model by constructing the excess return of the risky capital share over the risk-free rate:

$$R_{t+1}^e = \frac{r_{t+1}K_{t+1} - I_{t+1} + P_{t+1}^K}{P_t^K} - \frac{1}{P_t^B} \tag{2.17}$$

And the unconditional Sharpe ratio is given by

$$USR = \frac{E\left[R^e\right]}{\sigma(R^e)} \tag{2.18}$$

[7]Since the production firm earns zero profit, the profit of the capitalist firm in each period t is identical to that of the single firm in a one-firm decentralization of the model just presented. As a result, the dvidend issued by the single firm – and thus also the resulting excess returns and Sharpe ratios – are identical to those obtained for the capitalist firm reported here.

The USR thus "adjusts" the mean excess return for its variation over time. Although not as widely studied as the equity premium, the Sharpe ratio is a more attractive measure of the risk premium since the mean excess return is normalized by the riskiness of the return, thereby making it comparable across risky assets of various kinds, whether constructed in a model or observed in actual financial markets. However, even under efficient markets, there need not be a unique Sharpe ratio across all assets since, as noted in the introduction, the magnitude – and even the sign – of the correct risk premium depends crucially on the covariance of the risky return with the SDF. Since the dividend returned by the one-period consumption claims covaries negatively with the SDF, one expects that the Sharpe ratio for this asset will be positive.

2.2.8 Solution Method

I adopt a perturbation approach to solving the stochastic growth model. Specifically, I employ the PerturbationAIM algorithm introduced by Swanson et al. [58]. PerturbationAIM first solves for a model's deterministic steady state, and then computes the first-order solution using the AIM algorithm of Anderson and Moore [4]. Judd and Guu [35] pointed out that, once one conditions on these coefficients, determining each successively higher order of approximation reduces to a solving a linear system in the unknown coefficients.[8] Using this method, PerturbationAIM solves for higher-order coefficients of arbitrarily specified order. Although high-order perturbation solutions have the weakness of being local (and to a steady state that, as discussed above, is somewhat problematic), they are significantly more accurate than linearization,[9] and – unlike grid-based value function iteration, which suffers from a well-known curse of dimensionality – can readily be applied to models with many state variables.

[8]See Judd [34] Chapters 13-15 for a detailed discussion of perturbation methods.

[9]Precisely how much more accurate will depend on the degree of curvature and noise present in the model – see Arouba et al. [6].

2.3 Results

2.3.1 Macroeconomic Variables

Tables 1-2 report the standard deviations of macroeconomic variables for the models with and without habit formation, for the baseline calibration as well as three successively higher ϕ values. I report the raw (unfiltered) simulations since – as long as there is no long run trend present – the raw moments yield more insight into the true dynamics of the model, and in particular into the implications of the risk premium (and its time-variation) for macroeconomic variabilities, which is of primary interest in the present study.[10] Thus, while the quantitative values of the macroeconomic variabilities reported in Tables 1-2 are far too high relative to U.S. data due to the absence of filtering,[11] they are a superior source of information regarding the qualitative distinctions amongst the various models reported, and it is in this sense that Tables 1-2 are to be read.

A number of qualitative trends emerge from comparing the moments of the various models presented in Tables 1-2. First, the variability of all macroeconomic variables – except labor-supply – increases with the inverse labor-supply elasticity ϕ. This is unsurprising since, as discussed in Section 1.3 above, the presence of a labor margin allows risk-averse agents to smooth consumption more effectively by offsetting negative shocks with increased labor supply, thus reducing overall variability.

Another intriguing trend is that, for each ϕ values considered, the variability of all macroe-

[10] It's well-known in the literature that, compared to simple detrending, HP-filtering can meaningfully alter the dynamics of a time series. For the RBC model specifically, Cogley and Nason [19] demonstrate that HP-filtering can introduce spurious business-cycle dynamics.

[11] The HP-filtered counterparts of Tables 2.1 and 2.2 in fact reveal that the macroeconomic moments for my baseline model are too low compared to U.S. data or to those typically reported for the basic RBC model. This latter discrepancy is primarily due to the higher-than-usual curvature which I impose, as well as the third-order perturbation solution which "prices in" the risk premium – both of which induce excess smoothing relative to more standard results. For example, if I set $\gamma = 1$, $\phi = 1$ in my model without habit formation, and analyze the first-order solution, I find macroeconomic moments similar to those reported by King and Rebelo [39] for their basline calibration.

conomic variables – except consumption – is higher in the models with habit formation. I propose that this is the result of a "risk aversion accelerator" whereby, after a positive shock, agents in the models with habit formation become locally less risk averse (as discussed in Section 1.2 above), and thus more willing to take advantage of the temporarily higher productivity of (risky) capital on offer by investing. The simultaneous increase in intertemporal elasticity of substitution adds to this effect. The overall result is greater investment, capital and output variability, as in fact seen in Tables 2.1 and 2.2.

Ceteris paribus, this amplification mechanism might be expected to lead to higher consumption variability as well. However, there is another property of habit formation that works in the opposite direction. Note that the argument of the utility function under habit formation – the deviation of consumption from the habit stock – will generally be smaller in magnitude than consumption alone, resulting in higher local curvature. Relatedly, in a model with habit formation agents' constant-consumption relative risk aversion (equal to $\frac{\gamma}{1-b}$, see equation (2.2) in Section 1.2 above) – which obtains on average in a simulation – is greater than that in its non-habit counterpart, making habit-formation agents more desirous to smooth consumption on average. Indeed, under my calibration, constant-consumption relative risk aversion is 3 times higher under the model with habit formation. Thus, there is a tension between the time-variation in relative risk aversion implied by habit formation, which tends to amplify consumption variability, and the higher average relative risk aversion also implied by habit formation, which tends to attenuate it.

This tension is evident in the simulations reported in Tables 1 and 2. For my baseline inverse Frisch elasticity, $\phi = 1.7$, the smoothing property of habit formation dominates, and the model with habit formation has lower consumption variability. With labor-supply elasticity this high, it's simply too easy for more-risk-averse-on-average agents under habit formation to take advantage of the labor margin to smooth consumption. However, for $\phi = 3$, one arrives at a threshold for consumption variability – the two effects perfectly offset, leading

Table 2.1: Macroeconomic Statistics 1

Model	(1.7, h)	(1.7, nh)	(3, h)	(3, nh)
$\sigma(Y)$	2.744	2.722	3.112	3.076
$\sigma(C)$	1.662	1.666	1.970	1.970
$\sigma(I)$	7.348	6.964	7.914	7.465
$\sigma(K)$	4.323	4.202	4.857	4.708
$\sigma(L)$	2.147	2.124	1.742	1.720

Model (ϕ, h/nh) is the third-order perturbation solution of the model with (h) or without (nh) habit, and with inverse labor-supply elasticity ϕ. All model moments based on 100,000 Monte Carlo simulations of 500 periods each; all variables are in logs.

Table 2.2: Macroeconomic Statistics 2

Model	(6, h)	(6, nh)	(9, h)	(9, nh)
$\sigma(Y)$	3.589	3.532	3.850	3.781
$\sigma(C)$	2.374	2.366	2.596	2.584
$\sigma(I)$	8.648	8.108	9.042	8.456
$\sigma(K)$	5.516	5.328	5.867	5.656
$\sigma(L)$	1.216	1.197	0.936	0.921

to constant consumption variability across both models. For the $\phi = 6$ and $\phi = 9$ cases reported in Table 2, the amplification mechanism wins out, and consumption variability increases with habit formation. Thus, for $\phi > 3$, all macroeconomic variabilities increase with habit formation, improving the fit of the model relative to U.S. data. Thus, while many estimations of DSGE models have required high labor-supply elasticities in order to fit the data, my model fits best with a relatively low elasticity. A Frisch elasticity < 0.33 is much more consistent with microeconomic estimates. For example, Altonji [2] reports elasticities as low 0.02 ($\phi = 50$) in a range of estimates based on different samples and controls.

Table 2.3: Financial Statistics 1

Model	(1.7, h)	(1.7, nh)	(3, h)	(3, nh)	U.S. Data
r^B	0.0100	0.0100	0.0100	0.0100	0.0025
r^K	0.0100	0.0100	0.0100	0.0100	0.0175
R^e	2.22e-06	2.17e-06	2.73e-06	2.67e-06	0.0150
USR	0.0118	0.0113	0.0139	0.0134	0.3-0.9

Empirical excess returns are from Kocherlakota (1996). The range of plausible Sharpe ratios for U.S. equities (associated with different horizons and equity types) is drawn from Hodges (1997).

Table 2.4: Financial Statistics 2

Model	(6, h)	(6, nh)	(9, h)	(9, nh)
r^B	0.0100	0.0100	0.0100	0.0100
r^K	0.0100	0.0100	0.0100	0.0100
R^e	3.36e-06	3.25e-06	3.81e-06	3.62e-06
USR	0.0161	0.0154	0.0176	0.0166

2.3.2 Financial Variables

I now consider the financial statistics in Tables 2.3 and 2.4. Quantitatively, both excess returns and the USR are far too small to be consistent with U.S. data. Thus, the equity premium puzzle stands in all of the models reported. However, the USR – while off by a power of 10 – is still much closer to U.S. data than excess returns, implying that once one corrects the excess return for its variation over time, the resulting measure of the risk premium is much more empirically plausible. In summary, while a "Sharpe ratio puzzle" is present in these models, it is much less puzzling than the equity premium puzzle.

Furthermore, trends again emerge from comparing the different models to one another, and these may confirm/inform the trends that were seen for the macroeconomic moments. One clear trend is that both excess returns and the USR are increasing with ϕ. This is unsurprising since the higher consumption variability observed for increasing ϕ in Tables 1 and 2 implies that consumption risk also increases with ϕ. Thus, we expect the excess returns and Sharpe ratio demanded by agents for holding the risky capital share to increase as well.

Another important trend is that, for all ϕ values conidered, both excess returns and the USR are higher in the models with habit formation. This result is fairly intuitive for the $\phi = 6$ and $\phi = 9$ calibrations since as was just discussed, in these models habit formation implies higher consumption variability, and given this higher consumption risk one would expect higher excess returns and Sharpe ratios. Thus, it may at first seem counterintuitive that even under the baseline calibration in which habit formation implies lower consumption variability, excess returns and the USR nevertheless increase with habit formation. The solution to this puzzle lies in the (much) greater average relative risk aversion implied by habit formation – already remarked on in the discussion of the macroeconomic moments above. The decrease in consumption variability observed for $\phi = 1.7$ – reported in Table 1 – is quite small, and it is clearly outweighed by the higher average risk aversion in the habit formation model, leading to higher mean excess return and a higher USR even under this calibration. This is indicative of the type of rich insight that can be obtained by jointly modeling the dynamics of macroeconomic and financial variables in a single general equilibrium setting.

2.4 Conclusion

Unifying models of macroeconomic and financial variables is an important research agenda which has the potential to improve the fidelity and theoretical consistency of both. In particular, from a macroeconomic perspective, understanding the implications of risk premia – and their time-variation – for business cycle dynamics is of great interest. With this in mind, I have investigated the joint macroeconomic and financial implications of including external habit formation in a moderately-high-curvature calibration of the real business cycle (RBC) model. Qualitative comparison of the third-order perturbation solutions for the the models with and without habit formation yielded intriguing evidence of a "risk aversion accelerator" mechanism – whereby decreased local risk aversion following a positive productivity shock results in a greater increase in investment than would otherwise obtain – present in the models featuring habit formation. For consumption variability, a tension was evident between this amplification mechanism and the more widely cited tendency of habit formation to induce consumption smoothing, and a threshold Frisch labor-supply elasticity was identified beyond which the smoothing effect predominates. Below this empirically-plausible threshold, all macroeconomic variabilities increase with habit formation. On the financial side, I find that excess returns and the unconditional Sharpe ratio were found to be far too low to be consistent with U.S. data in all models considered. However, both are consistently higher in the models with habit formation. Thus, while including habit formation in the RBC model I presented doesn't solve the equity premium puzzle, it does increase the equity premium, as well as macroeconomic variability – improving the empirical fit of both. Given the tendency of variables in higher-order perturbation solutions of "macro-finance" models to deviate persistently from the steady state, it may be of interest in future work to pursue a more global nonlinear solution method, such as the spectral projection approach outlined by Judd [33]. Another interesting future direction would be to include Epstein-Zin preferences, in order to break the tight link between risk aversion and intertemporal substitution that is central to the "risk premium problem" in general equilibrium economies.

Chapter 3

A Real Business Cycle Model with Capital-Constrained Equity Shares

3.1 Introduction

Models that jointly describe the dynamics of both macroeconomic and financial variables in a single general equilibrium setting have been the subject of increasing interest in recent years. A central attraction of such "macro-finance" models is their potential to markedly increase the fidelity and theoretical consistency of both asset pricing models and macroeconomic models. The financial crisis and ensuing deep recession of 2007-2009 has made understanding the interactions between macroeconomic and financial variables an even more important research agenda. One important objective of the macro-finance literature is to model asset prices that are "non-neutral" in the sense that the equilibrium path of the asset price is *necessary* to specify the equilibrium path of the macroeconomic variables.[1] An

[1] Two notable recent macro-finance models featuring non-neutral asset prices are those of Jermann and Quadrini [31] and Gomes [27]. Furthermore, the family of DSGE models featuring credit frictions of various sorts (e.g. Bernanke et al. [?] and Motto et al. [52]) often implicitly, if not explicitly, assume the presence of non-neutral asset prices. Certainly, these models share with non-neutral asset pricing models the theme

attractive feature of non-neutral asset prices is that they induce two-way causality between macroeconomic and financial variables (in line with intuition) rather than one-way causality from macroeconomic to financial variables as in the case of neutral asset prices. Another fascinating property of non-neutral asset prices is that because they "affect" the macroeconomic equilibrium, policy actions that alter these asset prices can in turn have a (potentially stabilizing) effect on macroeconomic variables. This makes non-neutral asset pricing models a natural context in which to consider balance-sheet monetary policy, i.e., "quantitative easing."[2]

One simple method for specifying a non-neutral asset price is to introduce capital-constrained equity shares, in which shareholdings are allowed to vary over time provided that number of shares outstanding in each period equals the outstanding capital stock. Thus, if (say) the firm wishes to take advantage of a relatively high share price by selling shares, it may do so *only* if it increases its capital stock by the same amount. Conversely, if the firm wishes to reduce its capital stock, it may do so provided that it repurchases the shares that were backed by that capital. Thus, the constraint operates rather like a gold standard for equity shares, enforcing a fixed ratio of "paper" shares to "hard" capital. This specification takes its inspiration from Tobin's (originally in [62], and even more explicitly in [63]) assumption of capital-constrained equity share in his celebrated q-theory.[3] As such, in the sequel I will refer to the constraint requiring that the quantity of shares outstanding equal the quantity of capital outstanding as the "Tobin constraint" for brevity.

I find that introducing this constraint into an otherwise-standard decentralized real business cycle (RBC) model in which a capitalist firm adopts the "correct" objective (profit per-share)

that the financial side of the economy "matters."

[2]This is especially so in a New Keynesian setting, in which there exists a cyclical output gap that can potentially be stabilized by a monetary authority implementing the optimal path for a non-neutral asset price. (If, for example, the asset is risky, this may involve buying shares during recessions, and selling them during expansions.) This is a direction that will be pursued in the future.

[3]I explore the relation of Tobin's work and the subsequent neoclassical investment literature to the present study in Section 1.1 below.

results in an equilibrium which is identical to that chosen by the planner. I then consider a natural generalization of this model in which the capitalist firm adopts an objective that assumes a dilution cost of issuing shares – i.e. a "cost of equity" – greater than or lower than the efficient value, leading to under- and over-investment (respectively) in the steady state. This generalized model induces non-neutrality of the firm's share price. I will discuss the relation of this specification to the theoretical and empirical literature on managerial preferences when I introduce the model in Section 2.2 below. In Section 3, I explore the implications of this generalized model for the dynamics of macroeconomic variables under a variety of calibrations.

3.1.1 Related Literature

In addition to the work of Tobin himself, one strand of the macroeconomics literature that relates to the present study is the neoclassical optimal investment literature. No attempt will be made here to provide a detailed review of this literature. Rather, I shall endeavor to concisely outline how previous work compares and/or contrasts with the work to be undertaken here.

In the dynamic q-theory presented in his 1982 Nobel lecture, Tobin defines q_t (now known as average q or Tobin's q) to be "the ratio of market valuation of capital goods to normal replacement cost at time t." He further imposed the following assumption on shareholdings:

> Private capital investment is the source of new claims to physical capital, modeled as equity shares, one share for each unit of capital.

With this assumption, if $q_t > 1$, firms have an arbitrage opportunity whereby they can generate capital and sell shares to the market until $q_t \leq 1$. If $q_t < 1$, firms have no incentive to add to the capital stock and will allow capital to depreciate until $q_t \geq 1$ (negative investment is not allowed by Tobin's model). Therefore, Tobin specified that time-t investment is an increasing function of q_t, and that $q_t = 1$ on average (although his model allowed for temporary deviations from this value).

The subsequent neoclassical investment literature,[4] from the pioneering partial equilibrium work of Abel [1] and Hayashi [28] to the more recent general equilibrium models ([64], [61], [38], and [38]) of lumpy microeconomic investment, has provided increasingly sophisticated microfoundations for the *internal,* optimal-capital-choice dimension of q-theory (with average q now replaced by its marginal counterpart). However, the *external* dimension of optimal share issue and capital-constrained shares was abandoned, first by the partial equilibrium literature – in which there's no other agent to whom to sell shares – and then by the general equilibrium literature – in which either the planner's perspective is taken, so that once again there's no one to whom to sell shares, or (in the Kahn and Thomas models) shares are sold each period but are unconstrained, rendering the associated share price neutral.[5] The present study aims to provide a microfoundation for this external dimension of Tobin's theory as well, in a benchmark general equilibrium setting.

[4]See Caballero [13] for an excellent review of this literature up to the date of its publication.

[5]Specifically, in the Kahn and Thomas (2003, 2008) models, general equilibrium forces the household to purchase the entire distribution of shareholdings in the firms each period. Since these shares are sold automatically regardless of the capital stock chosen by any firm each period, the share-sale revenue is irrelevant to the firms at the margin. Furthermore, since the household is both the purchaser of shares and (via its ownership of the firms) the seller in equilibrium, it will be indifferent to these purchases in equilibrium. As a result, the share price in this model is rendered neutral.

3.2 Tobin-Constrained Models

3.2.1 First-Best Case

The Tobin constraint I consider in the present study is:

$$\forall t, N_t^K = K_t \tag{3.1}$$

i.e., the number of equity shares outstanding must equal the outstanding stock of physical capital. Below I present a standard two-firm decentralization of the real business cycle model with elastic labor supply and external habit formation, with the additional assumption that the capitalist firm can sell or buy back equity shares in each period, but must abide by the constraint (3.1) at all times – and in which the capitalist firm maximizes profit per-share. In the sequel, I will refer to this first-best case model for brevity as the FBC model.

In a standard decentralization of the RBC model, in which the number of equity shares outstanding, if modeled at all, is typically fixed, whether the firm maximizes aggregate or per-share profit is irrelevant. However, in models featuring the Tobin constraint – in which the number of shares outstanding is endogenous and time-varying – the distinction between aggregate and per-share profit max becomes significant. In particular, if the firm adopts aggregate profit as its objective (as in the $\eta = 0$ case under the more general model of Section 2.2 below), it treats share issue as "free money," and so fails to internalize the dilution cost imposed on the current shareholders. To be sure, in the Tobin-constrained models presented here there is no explicit dilution cost, since share purchases cancel out in equilibrium for the household, and it simply receives the total return to capital at no net cost each period. Still, requiring the capitalist firm to take the perspective of a hypothetical holder of one share forces it to choose its capital holding efficiently, which in turn benefits the household in equilibrium.

The Household

The household chooses consumption, $\{C_t\}_{t=0}^{\infty}$, labor supply, $\{L_t\}_{t=0}^{\infty}$, and shareholdings in the capitalist firm, $\left\{N_t^K\right\}_{t=0}^{\infty}$ to maximize the infinite-horizon objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(C_t - bC_{t-1}\right)^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\phi}}{1+\phi} \right] \tag{3.2}$$

subject to the flow budget constraint:

$$C_t + P_t^K N_{t+1}^K = W_t L_t + N_t^K (D_t^K + P_t^K) \tag{3.3}$$

where W_t is the time-t real wage, P_t^K is the period-t price of a perpetual claim to the per-share profitability of the capitalist firm, D_t^K is the per-share period-t dividend issued by the capitalist firm, and N_t^K represents the number of shares in the capitalist firm that the household holds a claim to in period t. The household exhibits CRRA utility with respect to the deviation of consumption from the habit stock, with CRRA coefficient γ, and additively-separable convex CRRA disutility of labor supply, with CRRA coefficient ϕ – also equal to the inverse of elasticity of labor supply with respect to the real wage. $b \in [0, 1]$ indexes the degree to which lagged consumption enters the external habit stock.

Combining the period-t first-order conditions with respect to C_t, C_{t+1}, and N_t^K results in the intertemporal pricing condition:

$$P_t^K = E_t \left[M_{t,t+1} \left(D_{t+1}^K + P_{t+1}^K \right) \right] \tag{3.4}$$

where $M_{t,t+j} = \left[\beta^j \left(\frac{C_t - bC_{t-1}}{C_{t+j} - bC_{t+j-1}}\right)^{\gamma}\right]$ is the household's stochastic discount factor (SDF) from time t to time $t + j$.

And combining the period-t first-order conditions with respect to C_t and L_t results in an intratemporal condition equating the real wage with the marginal rate of substitution between consumption and labor:

$$W_t = \frac{L_t^{\phi}}{(C_t - bC_{t-1})^{-\gamma}} \tag{3.5}$$

The Capitalist Firm

The capitalist firm (or firm of type 2 in the terminology of Ljungqvist and Sargent [45], Chapter 12) undertakes investment and rents the resulting capital to the production firm (type 1 firm). It chooses $\{K_t\}_{t=0}^{\infty}$, $\{I_t\}_{t=0}^{\infty}$, and $\left\{N_t^K\right\}_{t=0}^{\infty}$ to maximize the objective:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[\frac{r_t K_t - I_t + P_t^K \left(N_{t+1}^K - N_t^K\right)}{N_t^K}\right] \tag{3.6}$$

subject to the law of motion for capital:

$$K_{t+1} = (1 - \delta) K_t + I_t \tag{3.7}$$

and the Tobin constraint:

$$\forall t, N_t^K = K_t \tag{3.8}$$

where δ is the rate of depreciation, and r_t is the period-t rental rate earned on the capital stock. Substituting the the capital law of motion and the Tobin Constraint into the objective, the firm's problem is reduced to choosing $\{K_t\}_{t=0}^{\infty}$ to maximize:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[\left(r_t + 1 - \delta - P_t^K \right) + \left(P_t^K - 1 \right) \frac{K_{t+1}}{K_t} \right] \tag{3.9}$$

Taking the period-t first order condition with respect to K_{t+1}, one obtains the intertemporal condition:

$$\frac{1}{K_t} \left(1 - P_t^K \right) = E_t \left[M_{t,t+1} \left(\frac{K_{t+2}}{K_{t+1}^2} \left(1 - P_{t+1}^K \right) \right) \right] \tag{3.10}$$

The term $\left(r_t + 1 - \delta - P_t^K \right)$ in the firm's objective (3.9) is a per-share return that the capitalist firm receives regardless of its capital choice. Thus, only the cost of investment, and the share-sale (or buy-back) revenue matter to the firm at the margin. The term on term on the RHS of (3.10) represents a base effect, since choosing more capital today increases the denominator of (and thus reduces) next period's per-share profit.

As a result of the Tobin constraint, the timing of shareholdings is identical to that for capital: in period t the capitalist firm chooses next periods shareholdings, N_{t+1}^K, *at the same time* that it chooses next period's capital stock, K_{t+1}. At the end of period t, period t profit per share is handed back to the owner's of period t shareholdings, N_t^K, in the form of a dividend:

$$D_t^K = \frac{\left(r_t + 1 - \delta - P_t^K \right) K_t + \left(P_t^K - 1 \right) K_{t+1}}{K_t} \tag{3.11}$$

Next period, the purchasers of N_{t+1}^K (again, the household), receive D_{t+1}^K.

The Production Firm

The production firm, or firm of type 1, rents capital from the type 2 firm and labor from the household, and converts them into output. It chooses $\{K_t\}_{t=0}^{\infty}$ and $\{L_t\}_{t=0}^{\infty}$ to maximize the objective:[6]

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[Y_t - r_t K_t - W_t L_t \right] \tag{3.12}$$

subject to the Cobb-Douglas production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{3.13}$$

Substituting the constraint (3.13) into the objective (3.12) and taking the period-t first-order condition with respect to L_t one obtains:

$$W_t = (1 - \alpha) A_t K_t^{\alpha} L_t^{-\alpha} \tag{3.14}$$

This condition sets the real wage equal to the marginal product of labor.

Similarly, taking the period-t first-order condition with respect to K_t one obtains a condition equating the rental rate with the marginal product of capital:

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{3.15}$$

[6] I could use lower-case $\{k_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$, to underscore that the production firm chooses how much capital and labor to rent independent of any supply considerations, and then – after solving the production firm's problem – impose clearing of the rental markets: $k_t = K_t$, $l_t = L_t$ for all t. However, since these subtleties are well-known, for simplicity I'll use K_t and L_t on both sides of the market.

Log productivity is exogenous and follows an AR(1) process:

$$\log A_t = \rho \log A_{t-1} + \sigma \epsilon_t \tag{3.16}$$

with ϵ_t a standard normal productivity shock.

General Equilibrium

Market clearing implies:

$$Y_t = C_t + I_t \tag{3.17}$$

From the household's budget constraint, note that in equilibrium $\forall t, N_t^K = K_t$ implies:

$$\begin{aligned}
C_t &= -P_t^K (K_{t+1} - K_t) + D_t^K K_t + W_t L_t \\
\leftrightarrow C_t &= -P_t^K (K_{t+1} - K_t) + P_t^K (K_{t+1} - K_t) + \left(\alpha A_t K_t^{\alpha} L_t^{1-\alpha} - I_t\right) + \left((1-\alpha) A_t K_t^{\alpha} L_t^{1-\alpha}\right) \\
\leftrightarrow C_t &= \alpha Y_t - I_t + (1-\alpha) Y_t \\
\leftrightarrow C_t &= Y_t - I_t
\end{aligned}$$

Thus, the Tobin constraint ensures that general equilibrium holds in each period t.

Equivalence to the Planner

The FBC model just presented has the following remarkable property, proved in Appendix A1:

Proposition 1: *The FBC model results in an equilibrium identical to that under the planner's problem, as well as to that under a standard two-firm decentralization of the RBC model.*

Intuitively, the capitalist firm adjusts the capital stock so that the share price always equals one. This in turn causes the pricing condition for the equity share to become identical to the intertemporal condition for the capitalist firm under a standard two-firm decentralization of the RBC model. Thus, despite the apparent dissimilarities between the PSPM model and a standard decentralization – the presence of the LTC, per-share rather than aggregate profit max, shares changing hands between the household and the capitalist firm – they are in fact two ways of deriving the same model.

3.2.2 Generalized Tobin-Constrained Model

I now consider a case in which, due to imperfect monitoring, the capitalist firm may adopt an objective that deviates from that under the first-best case. In particular, the firm may assume a dilution cost of issuing shares greater than or less than that under FBC model, resulting (respectively) in under- and over-investment. This generalized model differs from the FBC model only in the capitalist firm's problem, which is presented below. In the sequel I will refer to this model as the GTC model.

The Capitalist Firm

The capitalist firm chooses $\{K_t\}_{t=0}^{\infty}$, $\{I_t\}_{t=0}^{\infty}$, and $\left\{N_t^K\right\}_{t=0}^{\infty}$ to maximize the objective:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[\frac{r_t K_t - I_t + P_t^K \left(N_{t+1}^K - N_t^K\right)}{\left(N_t^K\right)^{\eta}} \right] \tag{3.18}$$

subject to the law of motion for capital:

$$K_{t+1} = (1-\delta) K_t + I_t \tag{3.19}$$

and the Tobin constraint:

$$\forall t, N_t^K = K_t \tag{3.20}$$

where δ is the rate of depreciation, and r_t is the period-t rental rate earned on the capital stock. Thus $\eta = 1$ captures the efficient FBC model considered in Section 2.1 above. However, if $0 \leq \eta < 1$, the firm assumes a cost of equity – the dilution cost of issuing shares, represented by the denominator in (3.18) – lower than that for the efficient case (ignoring it completely at $\eta = 0$), resulting in excess capital accumulation (i.e., over-investment). Conversely, for $\eta > 1$, the firm assumes a cost of equity greater than that for the efficient case, resulting in under-investment.

Substituting the constraints (3.19) and (3.20) into the objective, the firm's problem is reduced to choosing $\{K_t\}_{t=0}^{\infty}$ to maximize:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[\left(r_t + 1 - \delta - P_t^K\right) (K_t)^{1-\eta} + \left(P_t^K - 1\right) K_{t+1} (K_t)^{-\eta} \right] \tag{3.21}$$

Taking the period-t first order condition with respect to K_{t+1}, one obtains the intertemporal condition:

$$\left(1-P_t^K\right)\left(K_t\right)^{-\eta}= \tag{3.22}$$
$$E_t\left[M_{t,t+1}\left(\eta\left(1-P_{t+1}^K\right)K_{t+2}\left(K_{t+1}\right)^{-\eta-1}+(1-\eta)\left(r_{t+1}+1-\delta-P_{t+1}^K\right)\left(K_{t+1}^{-\eta}\right)\right)\right]$$

In condition (3.10) of Section 2.1.2 above, the presence of the firm's share price in the intertemporal condition renders the share price non-neutral in the strict sense, since it is necessary to specify the macroeconomic equilbrium. However, since, as demonstrated in Appendix A1, the share price is constant in the FBC model, it's non-neutrality is not of the most interesting variety (it serves a technical role in defining the equilibrium, but doesn't behave anything like a real-world equity share price). By contrast, the presence of the firm's share price in condition (3.22) does indeed induce non-neutrality of the more meaningful variety for all cases ther than $\eta = 1$ (which latter case is equivalent to the FBC model), since for $\eta \neq 1$ the share price is *both* necessary to specify the macroeconomic equilibrium *and* dynamically varying over time.

The per-share dividend issued to shareholders each period is:

$$D_t^K=\frac{\left(r_t+1-\delta-P_t^K\right)K_t+\left(P_t^K-1\right)K_{t+1}}{K_t} \tag{3.23}$$

Given the contrasting implications of $\eta < 1$ vs. $\eta > 1$ for aggregate investment, it's natural to ask which case better represents managerial preferences. That is, do imperfectly-monitored managers prefer to over- or under-invest relative to the level that maximizes shareholder value? Interestingly, the theoretical and empirical literature on managerial preferences is distinctly undecided on this point. The theoretical literature has traditionally favored "empire-building" theories of managerial preferences, which hold that manager's prefer to over-invest since this increases their relative power and compensation.[7] However, a recent empirical investigation by Bertrand and Mullainathan [8] finds instead that managers appear to exhibit inertial "quiet life" preferences, shunning profitable new investment opportunities while also maintaining old capital that should be retired, with no effect on net investment.

In another recent study, Malmendier and Tate [46] find that over-confident CEO's – as measured by failure to divest firm-specific risk in their own portfolio's – over-invest when internal funding is abundant, but under-invest when external financing is required. Given that about 80% of investment is financed internally, this finding would seem to imply net over-investment, at least among over-confident CEO's. Yet Elsas et al. [23] find that the shares of firms undertaking major internally-financed investments outperform relative to those of firms financing such investments externally – which hardly seems consistent with the idea that firms over-invest with internal funds.

Given this lack of unanimity, the present study will attempt to shed some (modest) light on the question of whether imperfect monitoring and managerial preferences lead (in the aggregate) to over- or under-investment in Section 3 below.

Calibration

My baseline calibration is $\alpha = 0.3$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.979$, and $\sigma = 0.0072$, all of which are quite standard and are taken from King and Rebelo [39]. For the CRRA coefficient I choose $\gamma = 2$, increasing the degree of curvature a bit relative to King and Rebelo while still remaining microeconomically plausible. For the habit stock parameter I choose $b = 0.67$, which is representative of values typically found in estimations of DSGE models, following

[7] See [7], [49], [65] [30], and the more recent formal work of Kanniainen [37].

Rudebusch and Swanson [54]. I also consider the case of no habit formation: $b = 0$.

Following Levine and Pearlman [44], for the CRRA "convexity of labor-disutility" measure ϕ, which is the inverse of the Frisch elasticity of labor supply, I choose 1.7 – close to but a bit higher than typical macro estimates, though still low compared to micro estimates – as my baseline calibration. I further consider the case of perfectly inelastic labor supply, $\phi = \infty$, which corresponds to a Frisch elasticity of 0.

I consider four benchmark values for the η parameters introduced above: 0, 0.5, 1, and 1.5. The first two imply over-investment, the third replicates the first-best case, and the fourth implies under-investment.

Steady State Analysis

The nonstochastic steady-state values for the macroeconomic variables – and P^K – for the various calibrations considered here are reported in Tables 1-3. For the inelastic labor supply case, the steady states are independent of the habit formation parameter b, since this parameter only appears in the stochastic discount factors, which simply reduce to β in the steady state.Therefore, the common steady states for $\phi = \infty$, $b = 0$ and $\phi = \infty$, $b = 0.67$ are reported in Table 1.

However, once elastic labor supply is considered, b enters the steady state via the marginal rate of substitution in the intratemporal condition (3.5) above. Thus, the steady states for $\phi = 1.7$, $b = 0$ and $\phi = 1.7$, $b = 0.67$ are reported separately in Figures 2 and 3.

Notice that in all three tables, P^K's steady state value is the same increasing function of η. However, it's important to note that in the efficient case, P^K is constant and equal to one, not just in the steady state but also dynamically (see Appendix A1 below). The same does not hold for the other η values, for which P^K varies dynamically, rendering its non-neutrality

Table 3.1: Steady States for Various η Values, $b = 0$ and $b = 0.67$, $\phi = \infty$

$\eta =$	0	0.5	1	1.5
C	0.699	0.693	0.679	0.630
I	0.669	0.619	0.536	0.373
K	3.287	3.209	3.065	2.704
P^K	0.500	0.666	1.000	2.000
Y	0.986	0.963	0.920	0.811

All variables (except P^K) are in logs.

Table 3.2: Steady States for Various η Values, $b = 0$, $\phi = 1.7$

$\eta =$	0	0.5	1	1.5
C	0.491	0.482	0.464	0.412
I	0.544	0.501	0.432	0.300
K	3.079	2.998	2.850	2.486
L	-0.208	-0.211	-0.215	-0.218
P^K	0.500	0.666	1.000	2.000
Y	0.778	0.752	0.705	0.594

All variables (except P^K) are in logs.

Table 3.3: Steady States for Various η Values, $b = 0.67$, $\phi = 1.7$

$\eta =$	0	0.5	1	1.5
C	1.091	1.082	1.063	1.011
I	0.990	0.913	0.787	0.547
K	3.679	3.598	3.449	3.085
L	0.392	0.388	0.384	0.382
P^K	0.500	0.666	1.000	2.000
Y	1.378	1.351	1.304	1.193

All variables (except P^K) are in logs.

more meaningful, in the sense discussed in Section 2.2.1 above.

For the macroeconomic variables represented in Table 3, it's quite clear that steady state consumption, investment, capital, labor (when elastic), and output are all monotonically decreasing with η. This is due to the increase in capital accumulation produced by the lower assumed cost of equity as η is reduced.

The fact that steady-state consumption is higher in the $\eta < 1$ models than in the efficient $\eta = 1$ models does *not*, of course, imply that welfare is higher in the $\eta < 1$ case. Rather, the welfare cost of the excess saving (=investment) required to *reach* the higher steady-state capital of the APM model outweighs – in discounted terms – the welfare benefit of increased consumption once that higher steady state is reached.

3.3 Dynamics

What are the implications of the models presented in Sections 2 above for the dynamics of macroeconomic variables?[8] Here I consider the GTC model of Section 2.2 above under the baseline calibration and benchmark range of η values introduced in Section 2.2.2. [9] I obtain the third-order perturbation solution[10] for each calibration considered using the PerturbationAIM algorithm outlined by Swanson et al. [58], and then run Monte Carlo simulations from this solution.[11]

In Tables 4-11 I report the standard deviations of macroeconomic variables for both HP-

[8]Due to the absence of capital-adjustment costs (or a similar rigidity), the mean excess returns implied by the model introduced above are extremely small. For this reason, I omit them and focus on macroeconomic dynamics in this section. These dynamics are, of course, still influenced by the presence of the non-neutral share price.

[9]All of these (except for γ) are taken from King and Rebelo [39].

[10]For a detailed discussion of perturbation methods see Judd [34].

[11]The Mathematica code for PerturbationAIM is available on Eric Swanson's website (http://www.ericswanson.us/perturbation.html), and your author has also written a suite of supporting functions for running and analyzing simulations, which is available upon request.

Table 3.4: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0$, $\phi = \infty$, Filtered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	0.53	0.55	0.52	0.34
$\sigma(I)$	2.17	2.21	2.50	4.69
$\sigma(K)$	0.20	0.20	0.23	0.41
$\sigma(Y)$	0.94	0.94	0.94	0.95

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.5: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0$, $\phi = \infty$, Unfiltered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	4.16	4.15	4.26	4.66
$\sigma(I)$	7.54	7.89	8.77	11.00
$\sigma(K)$	5.36	5.68	6.27	6.23
$\sigma(Y)$	4.87	4.92	5.09	5.21

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.6: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0.67$, $\phi = \infty$, Filtered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	0.42	0.42	0.41	0.33
$\sigma(I)$	2.90	3.02	3.39	5.48
$\sigma(K)$	0.24	0.24	0.28	0.49
$\sigma(Y)$	0.95	0.95	0.95	0.96

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.7: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0.67$, $\phi = \infty$, Unfiltered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	4.17	4.19	4.33	4.79
$\sigma(I)$	8.12	8.57	9.56	12.21
$\sigma(K)$	5.55	5.92	6.56	6.55
$\sigma(Y)$	4.93	5.02	5.20	5.34

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.8: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0$, $\phi = 1.7$, Filtered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	0.48	0.49	0.46	0.38
$\sigma(I)$	2.33	2.37	2.78	6.21
$\sigma(K)$	0.21	0.21	0.25	0.51
$\sigma(L)$	0.05	0.05	0.05	0.36
$\sigma(Y)$	0.94	0.93	0.95	1.16

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.9: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0$, $\phi = 1.7$, Unfiltered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	3.29	3.33	3.43	3.63
$\sigma(I)$	6.84	7.20	8.17	11.01
$\sigma(K)$	4.45	4.80	5.33	4.97
$\sigma(L)$	1.04	1.03	1.05	1.37
$\sigma(Y)$	4.03	4.11	4.27	4.22

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.10: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0.67$, $\phi = 1.7$, Filtered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	0.37	0.37	0.36	0.36
$\sigma(I)$	2.91	3.01	3.49	6.97
$\sigma(K)$	0.24	0.25	0.29	0.59
$\sigma(L)$	0.05	0.06	0.05	0.37
$\sigma(Y)$	0.92	0.91	0.94	1.15

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

Table 3.11: Standard Deviations of Macroeconomic Variables for Various η Values, $b = 0.67$, $\phi = 1.7$, Unfiltered

$\eta =$	0	0.5	1	1.5
$\sigma(C)$	3.31	3.35	3.45	3.68
$\sigma(I)$	7.32	7.75	8.80	12.05
$\sigma(K)$	4.62	4.98	5.53	5.17
$\sigma(L)$	1.04	1.02	1.05	1.38
$\sigma(Y)$	4.07	4.16	4.32	4.25

Based on 100 Monte Carlo simulations of $N = 100,000$ observations each for the third-order perturbation solution of the specified model.

filtered and raw (unfiltered) simulations, since the former follow standard practice in the literature, while the latter yield more insight into the true underlying dynamics of the model. The precision of the standard deviations reported in Tables 4-11 is such that values reported should be considered to be accurate to within ± 0.01 (as determined by duplicating two of the simulations). Therefore, a difference of 0.01 between two calibrations should be disregarded, while a difference of 0.02 or more is likely to be significant.

The impact of varying η on the various terms of condition (3.22), and by extension on macroeconomic dynamics, is complex indeed. However, some clear trends emerge upon inspection of Tables 4-11. I will focus on the unfiltered simulations in Tables 5, 7, 9, and 11, since, as just stated, these more accurately represent the true underlying dynamics of the model. In all four of these tables, consumption variability increases monotonically with η, as does investment variability. However, capital variability increases as η increases from 0 to 1, and then decreases as η increases from 1 to 1.5. In Tables 9 and 11 (the elastic labor supply models), output variability follows this same trend. Labor supply variability decreases slightly as η increases from 0 to 0.5, increases slightly as η increases from 0.5 to 1, and then increases markedly as η increases from 1 to 1.5.

Comparing the no-habit models in Tables 5 and 9 to their habit formation counterparts in Tables 7 and 11, it's clear that the presence of habit formation increases the variability of all macroeconomic variables, while leaving the qualitative trends with increasing η mostly intact.

In Chapter 2 of this Dissertation, I postulate that this amplification of second moments under habit formation is due to a "risk aversion accelerator," whereby variation in local relative risk aversion – and simultaneous inverse variation in local intertemporal elasticity of substitution – leads to greater ouput and investment variability than would otherwise obtain.

Which of these calibrations best matches U.S. data? To address this question in a simple way, I construct the 2-norm (i.e., the square root of the sum of squares) of the difference between $\{\sigma(C)/\sigma(Y), \sigma(I)/\sigma(Y)\}$ for a given calibration and the empirical value of $\{0.4, 2.39\}$,[12] and then investigate which of the 16 calibrations considered here delivers the minimum 2-norm so-constructed.[13] In this case I focus on the HP-filtered simulations of Tables 4, 6, 8, and 10, since they are more directly comparable to U.S. data. Using this method, the best fitting calibration is $\eta = 0$, $b = 0$, $\phi = 1.7$, (represented by column 1 of Table 6) with a 2-norm of 0.142. The next-best-fitting calibration is $\eta = 0$, $b = 0$, $\phi = \infty$, with a 2-norm of 0.183. Therefore, the $\eta = 0$ case, implying aggregate over-investment, seems to be preferred by the data. This result can be read as lending modest support to "empire-building" theories of managerial preferences

3.4 Conclusion

This investigation has sought to understand the consequences of introducing capital-constrained equity shares into an otherwise-standard decentralized real business cycle model. In particular, the equity shares outstanding of a capitalist firm were allowed to vary over time subject to the constraint that shares outstanding equal the outstanding capital stock. If the firm adopts per-share profit as its objective, this model was found to result in an equilibrium identical to that chosen by the planner. However, if the firm disregards the shareholder's perspective and adopts an objective that deviates from per-share profit, the resulting equilibrium features a

[12] This empirical value is taken from [10].

[13] [32] conducts a calibration along similar macroeconomic dimensions – and via a comparable methodology.

share price for the firm which is "non-neutral" in the sense that it is necessary to specify the macroeconomic equilibrium. I explored a generalized model in which the firm can adopt an objective that assumes a cost of equity higher or lower than that under the efficient case, resulting in under- or over-investment (respectively) – motivated by the large theoretical and empirical literature on managerial preferences. The resulting equilibria naturally incorporate a non-neutral share price (except for the single point-calibration corresponding to the efficient case). I found a calibration that assumes a cost of equity lower than that under the efficient case – resulting in aggregate over-investment – to provide the best fit to empirical data in terms of the standard deviations of consumption and investment relative to that of output, lending tentative support to "empire-building" theories of managerial preferences. In future work it will be of interest to explore the implications of the capital-constrained equity shares for excess returns in the context of a model featuring a more quantitatively meaningful risk premium. This can be achieved through a capital rigidity such as capital adjustment costs, or by introducing Epstein-Zin preferences. The presence of a non-neutral share price also makes it straightforward (in a future model) introduce a monetary authority that uses its balance sheet to affect the share price – thereby altering (and potentially stabilizing) the macroeconomic equilibrium – in a form of quantitative easing.

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Appendix A

Proof of Proposition 1

A1: Proposition 1

Recall the capitalist firm's intertemporal condition for the First-Best Case model in Section 2.2.1:

$$\frac{1}{K_t}\left(1 - P_t^K\right) = E_t\left[M_{t,t+1}\left(\frac{K_{t+2}}{K_{t+1}^2}\left(1 - P_{t+1}^K\right)\right)\right]$$

Since K_{t+1} is a choice variable being determined in period t without knowledge of period $t+1$ shocks, we can rearrange this as:

$$1 - P_t^K = \frac{K_t}{K_{t+1}} E_t\left[M_{t,t+1}\left(\frac{K_{t+2}}{K_{t+1}}\left(1 - P_{t+1}^K\right)\right)\right] \tag{A.1}$$

Now consider the expression:

$$\frac{K_{t+1}}{K_t}\left(1 - P_t^K\right) \tag{A.2}$$

Substituting (A.1) into (A.2) we obtain:

$$E_t\left[M_{t,t+1}\left(\frac{K_{t+2}}{K_{t+1}}\left(1-P^K_{t+1}\right)\right)\right] \tag{A.3}$$

Advancing (A.1) a period and substituting into (A.3) we get:

$$E_t\left[M_{t,t+1}\left(E_{t+1}\left[M_{t+1,t+2}\left(\frac{K_{t+3}}{K_{t+2}}\left(1-P^K_{t+2}\right)\right)\right]\right)\right]$$

Now we can certainly pull the β terms of the stochastic discount factors out of the integrals represented by the conditional expectations. Doing so, we find:

$$\beta^2 E_t\left[\left(\frac{U_{C_{t+1}}}{U_{C_t}}\right)\left(E_{t+1}\left[\left(\frac{U_{C_{t+2}}}{U_{C_{t+1}}}\right)\left(\frac{K_{t+3}}{K_{t+2}}\left(1-P^K_{t+2}\right)\right)\right]\right)\right] \tag{A.4}$$

where $U_{C_{t+J}}$ denotes period $t+j$ marginal utility. Now, as we continue to iteratively advance (A.1) a period, substitute into the current version of (A.4), and pull the additional β term out of the expectations, the term in the power of β in front of the expression clearly goes to $\beta^\infty = 0$. Therefore we have:

$$\frac{K_{t+1}}{K_t}\left(1-P^K_t\right) = 0$$

Now since $\frac{K_{t+1}}{K_t} > 0$,[1] and since similar reasoning can be applied to (A.2) advanced forward

[1] Although there's no explicit constraint preventing the capital stock from going to zero, in practice as long as $\beta > 0$ the capitalist firm (or planner) will never have an incentive to choose a next-period capital stock of zero, since this would mean zero output and (because negative investment, even if allowed, is not possible with a current capital stock of zero) zero consumption next period. If the standard Inada conditions are satisfied by the utility function (as they are in our case with respect to consumption for $b = 0$ and with respect to the deviation of consumption from the external habit stock for $b > 0$), then zero consumption (or zero deviation of consumption from the habit stock) will never be allowed. Refer to Section 2.1.1 to review our assumptions regarding household preferences.

or backward to any other period, we obtain:

$$\forall t, P_t^K = 1 \tag{A.5}$$

The capitalist firm will adjust the capital stock so that marginal cost of generating capital is equal to the marginal benefit of selling shares to the household in each period. Thus, this is a "q-condition" that captures Tobin's no-arbitrage logic in a microfounded way.

Now recall that the pricing condition for the equity share is:

$$P_t^K = E_t \left[M_{t,t+1} \left(\frac{\left(r_{t+1} + 1 - \delta - P_{t+1}^K\right) K_{t+1} + \left(P_{t+1}^K - 1\right) K_{t+2}}{K_{t+1}} + P_{t+1}^K \right) \right] \tag{A.6}$$

Substituting (A.5) into (A.6) we get:

$$1 = E_t \left[M_{t,t+1} \left(r_{t+1} + 1 - \delta \right) \right]$$

But this is simply the capitalist firm's intertemporal condition under the standard two-firm decentralization of the RBC model, which in turn is well-known to be isomorphic to the model under the planner. Thus, there exists an isomorphism from the FBC model to the standard two-firm decentralization, and another from that model to the planner's model. Of necessity, therefore, these three models result in identical equilibria, yielding the desired result.