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Authors

Ayyer, Arvind
Klee, Steven
Schilling, Anne

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MARKOV CHAINS FOR PROMOTION OPERATORS

ARVIND AYYER, STEVEN KLEE, AND ANNE SCHILLING

Dedicated to Mohan Putcha and Lex Renner on the occasion of their 60th birthdays.

ABSTRACT. We consider generalizations of Schützenberger’s promotion operator on the set \mathcal{L} of linear extensions of a finite poset. This gives rise to a strongly connected graph on \mathcal{L} . In earlier work [AKS12], we studied promotion-based Markov chains on these linear extensions which generalizes results on the Tsetlin library. We used the theory of \mathcal{R} -trivial monoids in an essential way to obtain explicitly the eigenvalues of the transition matrix in general when the poset is a rooted forest. We first survey these results and then present explicit bounds on the mixing time and conjecture eigenvalue formulas for more general posets. We also present a generalization of promotion to arbitrary subsets of the symmetric group.

1. INTRODUCTION

Schützenberger [Sch72] introduced the notion of evacuation and promotion on the set of linear extensions of a finite poset P of size n . This generalizes promotion on standard Young tableaux defined in terms of jeu-de-taquin moves. Haiman [Hai92] as well as Malvenuto and Reutenauer [MR94] simplified Schützenberger’s approach by expressing the promotion operator ∂ in terms of more fundamental operators τ_i ($1 \leq i < n$), which either act as the identity or as a simple transposition. A beautiful survey on this subject was written by Stanley [Sta09].

In earlier work, we considered a slight generalization of the promotion operator [AKS12] defined as $\partial_i = \tau_i \tau_{i+1} \cdots \tau_{n-1}$ for $1 \leq i \leq n$ with $\partial_1 = \partial$ being the original promotion operator. In Section 2 we define the extended promotion operator, give examples and state some of its properties. We survey our results on Markov chains based on the operators ∂_i , which act on the set of all linear extensions of P (denoted $\mathcal{L}(P)$) in Section 3.

Our results [AKS12] can be viewed as a natural generalization of the results of Hendricks [Hen72, Hen73] on the Tsetlin library [Tse63], which is a model for the way an arrangement of books in a library shelf evolves over time. It is a Markov chain on permutations, where the entry in the i th position is moved to the front with probability p_i . From our viewpoint, Hendricks’ results correspond to the case when P is an anti-chain and hence $\mathcal{L}(P) = S_n$ is the full symmetric group. Many variants of the Tsetlin library have been studied and there is a wealth of literature on the subject. We refer the interested reader to the monographs by Letac [Let78] and by Dies [Die83], as well as the comprehensive bibliographies in [Fil96] and [BHR99].

One of the most interesting properties of the Tsetlin library Markov chain is that the eigenvalues of the transition matrix can be computed exactly. The exact form of the eigenvalues

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was independently investigated by several groups. Notably Donnelly [Don91], Kapoor and Reingold [KR91], and Phatarfod [Pha91] studied the approach to stationarity in great detail. There has been some interest in finding exact formulas for the eigenvalues for generalizations of the Tsetlin library. The first major achievement in this direction was to interpret these results in the context of hyperplane arrangements [Bid97, BHR99, BD98]. This was further generalized to a class of monoids called left regular bands [Bro00] and subsequently to all bands [Bro04] by Brown. This theory has been used effectively by Björner [Bjö08, Bjö09] to extend eigenvalue formulas on the Tsetlin library from a single shelf to hierarchies of libraries.

We present without proof our explicit combinatorial formulas [AKS12] for the eigenvalues and multiplicities for the transition matrix of the promotion Markov chain when the underlying poset is a rooted forest in Section 4 (see Theorem 4.2). The proof of eigenvalues and their multiplicities follows from the \mathcal{R} -triviality of the underlying monoid using results by Steinberg [Ste06, Ste08]. Intuition on why the promotion monoid is \mathcal{R} -trivial is stated in Section 5.

The remainder of the paper contains new results and is outlined as follows. In Section 6, we prove a formula for the mixing time of the promotion Markov chain. This improves the result stated without proof in the Outlook section of [AKS12]. In Section 7, we present a partial conjecture for the eigenvalues of the transition matrix of posets which are not rooted forests. We give supporting data for our conjectures with formulas for all posets of size 4. Lastly, Section 8 defines a generalization of promotion on arbitrary subsets of S_n and gives a formula for its stationary distribution.

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The Markov chains presented in this paper are implemented in a `Maple` package by the first author (AA) available from his home page and in `Sage` [S⁺12, SCc12] by the third author (AS). Many of the pictures presented here were created with `Sage`.

2. EXTENDED PROMOTION ON LINEAR EXTENSIONS

Let P be an arbitrary poset of size n , with partial order denoted by \preceq . We assume that the elements of P are labeled by integers in $[n] := \{1, 2, \dots, n\}$. In addition, we assume that the poset is naturally labeled, that is if $i, j \in P$ with $i \preceq j$ in P then $i \leq j$ as integers. Let $\mathcal{L} := \mathcal{L}(P)$ be the set of its **linear extensions**,

$$(2.1) \quad \mathcal{L}(P) = \{\pi \in S_n \mid i \prec j \text{ in } P \implies \pi_i^{-1} < \pi_j^{-1} \text{ as integers}\},$$

which is naturally interpreted as a subset of the symmetric group S_n . Note that the identity permutation e always belongs to \mathcal{L} . Let P_j be the natural (induced) subposet of P consisting of elements k such that $j \preceq k$ [Sta97].

We now briefly recall the idea of **promotion** of a linear extension of a poset P . Start with a linear extension $\pi \in \mathcal{L}(P)$ and imagine placing the label π_i^{-1} in P at the location i . By the definition of the linear extension, the labels will be well-ordered. The action of promotion of π will give another linear extension of P as follows:

- (1) The process starts with a seed, the label 1. First remove it and replace it by the minimum of all the labels covering it, i , say.
- (2) Now look for the minimum of all labels covering i in the original poset, and replace it, and continue in this way.
- (3) This process ends when a label is a “local maximum.” Place the label $n + 1$ at that point.
- (4) Decrease all the labels by 1.

This new linear extension is denoted $\pi\partial$ [Sta09].

Example 2.1. Figure 1 shows a poset (left) to which we assign the identity linear extension $\pi = 123456789$. The linear extension $\pi' = \pi\partial = 214537869$ obtained by applying the promotion operator is depicted on the right. Note that indeed we place $\pi_i'^{-1}$ in position i , namely 3 is in position 5 in π' , so that 5 in $\pi\partial$ is where 3 was originally.

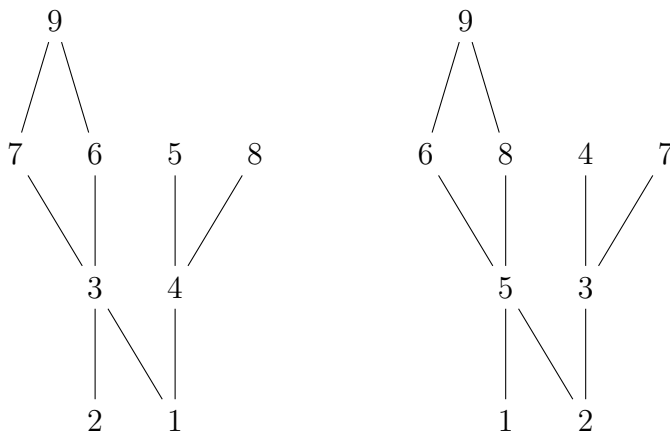


FIGURE 1. A linear extension π (left) and $\pi\partial$ (right).

Figure 2 illustrates the steps used to construct the linear extension $\pi\partial$ from the linear extension π from Figure 1.

The definition of promotion was originally motivated by the following construction. The Young diagram of a partition λ (with English notation) can naturally be viewed as a poset on the boxes of the diagram ordered according to the rule that a box is covered by any boxes immediately below it or to its right. The linear extensions of this poset are standard Young tableaux of shape λ . In this context, the definition of promotion is a natural generalization of the standard promotion operator used in the RSK algorithm. On semistandard tableaux, promotion is also used to define affine crystal structures in type A [Sh02] and it has applications to the cyclic sieving phenomenon [Rh10]. The above definition of promotion for arbitrary posets is originally due to Schützenberger [Sch72].

We now generalize the above construction to **extended promotion**, whose seed is any of the numbers $1, 2, \dots, n$. The algorithm is similar to the original one, and we describe it for seed j . Start with the subposet P_j and perform steps 1–3 in a completely analogous fashion. Now decrease all the labels strictly larger than j by 1 in P (not only P_j). Clearly this gives a new linear extension, which we denote $\pi\partial_j$. Note that ∂_n is always the identity.

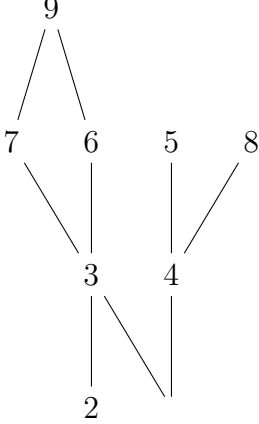
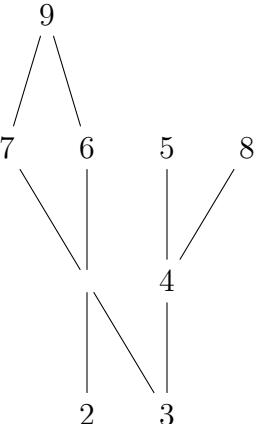
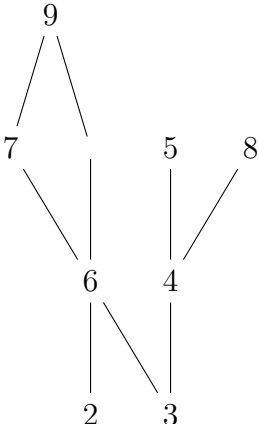
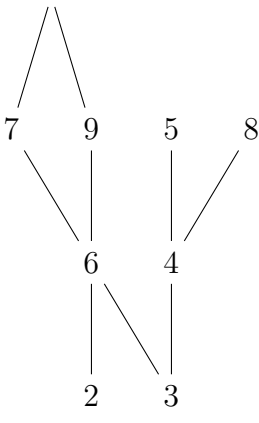
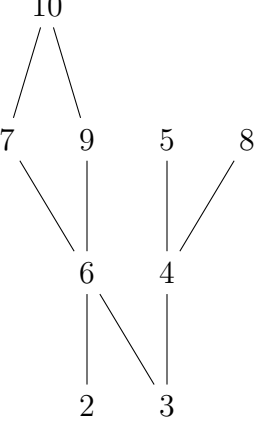
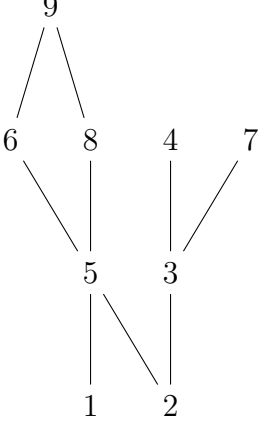
<p>Step 1: Remove the minimal element 1.</p>	<p>Step 2: The minimal element that covered 1 was 3, so replace 1 with 3.</p>	<p>Step 2 (continued): The minimal element that covered 3 was 6, so replace 3 with 6.</p>
		
<p>Step 2 (continued): The minimal element that covered 6 was 9, so replace 6 with 9.</p>	<p>Step 3: Since 9 was a local maximum, replace 9 with 10.</p>	<p>Step 4: Decrease all labels by 1. The resulting linear extension is $\partial\pi$.</p>
		

FIGURE 2. Constructing $\pi\partial$ from π .

The extended promotion operator can be expressed in terms of more elementary operators τ_i ($1 \leq i < n$) as shown in [Hai92, MR94, Sta09] and has explicitly been used to count linear extensions in [EHS89]. Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{L}(P)$ be a linear extension of a finite poset P in

one-line notation. Then

$$(2.2) \quad \pi\tau_i = \begin{cases} \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are not} \\ & \text{comparable in } P, \\ \pi_1 \cdots \pi_n & \text{otherwise.} \end{cases}$$

Alternatively, τ_i acts non-trivially on a linear extension if interchanging entries π_i and π_{i+1} yields another linear extension. Then as an operator on $\mathcal{L}(P)$,

$$(2.3) \quad \partial_j = \tau_j \tau_{j+1} \cdots \tau_{n-1}.$$

3. PROMOTION MARKOV CHAINS

We now consider two discrete-time Markov chains related to the extended promotion operator. For completeness, we briefly review the part of the theory relevant to us.

Fix a finite poset P of size n . The operators $\{\partial_i \mid 1 \leq i \leq n\}$ define a directed graph on the set of linear extensions $\mathcal{L}(P)$. The vertices of the graph are the elements in $\mathcal{L}(P)$ and there is an edge from π to π' if $\pi' = \pi\partial_i$. We can now consider random walks on this graph with probabilities given formally by x_1, \dots, x_n which sum to 1. We give two ways to assign the edge weights, see Sections 3.1 and 3.2. An edge with weight x_i is traversed with that rate. A priori, the x_i 's must be positive real numbers for this to make sense according to the standard techniques of Markov chains. However, the ideas work in much greater generality and one can think of this as an “analytic continuation.”

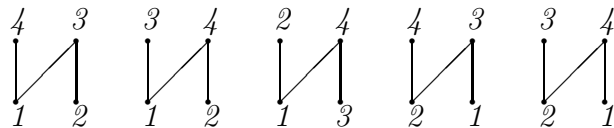
A discrete-time Markov chain is defined by the **transition matrix** M , whose entries are indexed by elements of the state space. In our case, they are labeled by elements of $\mathcal{L}(P)$. We take the convention that the (π', π) entry gives the probability of going from $\pi \rightarrow \pi'$. The special case of the diagonal entry at (π, π) gives the probability of a loop at the π . This ensures that column sums of M are one and consequently, one is an eigenvalue with row (left-) eigenvector being the all-ones vector. A Markov chain is said to be **irreducible** if the associated digraph is strongly connected. In addition, it is said to be **aperiodic** if the greatest common divisor of the lengths of all possible loops from any state to itself is one. For irreducible aperiodic chains, the Perron-Frobenius theorem guarantees that there is a unique **stationary distribution**. This is given by the entries of the column (right-) eigenvector of M with eigenvalue 1. Equivalently, the stationary distribution $w(\pi)$ is the solution of the **master equation**, given by

$$(3.1) \quad \sum_{\pi' \in \mathcal{L}(P)} M_{\pi, \pi'} w(\pi') = \sum_{\pi' \in \mathcal{L}(P)} M_{\pi', \pi} w(\pi).$$

Edges which are loops contribute to both sides equally and thus cancel out. For more on the theory of finite state Markov chains, see [LPW09].

We set up a running example that will be used for each case.

Example 3.1. Define P by its covering relations $\{(1, 3), (1, 4), (2, 3)\}$, so that its Hasse diagram is the first diagram in the list below:



The remaining graphs correspond to the linear extensions $\mathcal{L}(P) = \{1234, 1243, 1423, 2134, 2143\}$.

3.1. Uniform promotion graph. The vertices of the **uniform promotion graph** are labeled by elements of $\mathcal{L}(P)$ and there is an edge between π and π' if and only if $\pi' = \pi\partial_j$ for some $j \in [n]$. In this case, the edge is given the symbolic weight x_j .

Example 3.2. *The uniform promotion graph for the poset in Example 3.1 is illustrated in Figure 3. The transition matrix, with the lexicographically ordered basis, is given by*

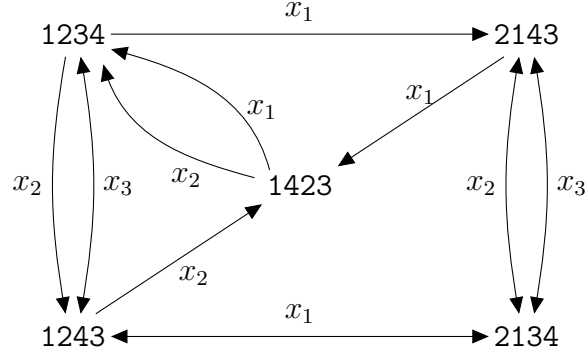


FIGURE 3. Uniform promotion graph for Example 3.1. Every vertex has four outgoing edges labeled x_1 to x_4 and self-loops are not drawn.

$$\begin{pmatrix} x_4 & x_3 & x_1 + x_2 & 0 & 0 \\ x_2 + x_3 & x_4 & 0 & x_1 & 0 \\ 0 & x_2 & x_3 + x_4 & 0 & x_1 \\ 0 & x_1 & 0 & x_4 & x_2 + x_3 \\ x_1 & 0 & 0 & x_2 + x_3 & x_4 \end{pmatrix}.$$

Note that the row sums are one although the matrix is not symmetric, so that the stationary state of this Markov chain is uniform. We state this for general finite posets in Theorem 3.6.

The variable x_4 occurs only on the diagonal in the above transition matrix. This is because the action of ∂_4 (or in general ∂_n) maps every linear extension to itself resulting in a loop.

3.2. Promotion graph. The **promotion graph** is defined in the same fashion as the uniform promotion graph with the exception that the edge between π and π' when $\pi' = \pi\partial_j$ is given the weight x_{π_j} .

Example 3.3. *The promotion graph for the poset of Example 3.1 is illustrated in Figure 4. Although it might appear that there are many more edges here than in Figure 3, this is not the case. The transition matrix this time is given by*

$$\begin{pmatrix} x_4 & x_4 & x_1 + x_4 & 0 & 0 \\ x_2 + x_3 & x_3 & 0 & x_2 & 0 \\ 0 & x_2 & x_2 + x_3 & 0 & x_2 \\ 0 & x_1 & 0 & x_4 & x_1 + x_4 \\ x_1 & 0 & 0 & x_1 + x_3 & x_3 \end{pmatrix}.$$

Notice that row sums are no longer one. The stationary distribution, as a vector written in row notation is

$$\left(1, \frac{x_1 + x_2 + x_3}{x_1 + x_2 + x_4}, \frac{(x_1 + x_2)(x_1 + x_2 + x_3)}{(x_1 + x_2)(x_1 + x_2 + x_4)}, \frac{x_1}{x_2}, \frac{x_1(x_1 + x_2 + x_3)}{x_2(x_1 + x_2 + x_4)} \right)^T.$$

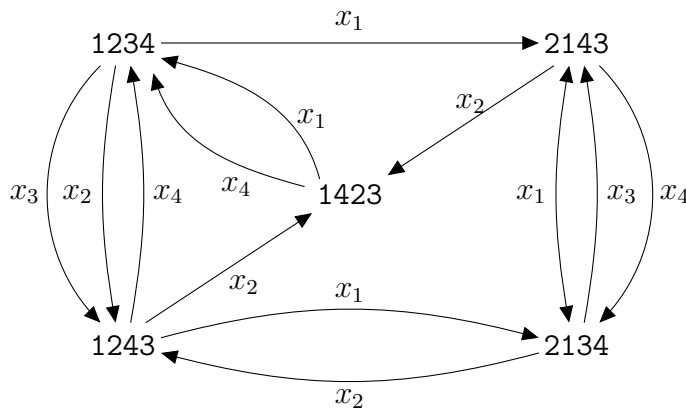


FIGURE 4. Promotion graph for Example 3.1. Every vertex has four outgoing edges labeled x_1 to x_4 and self-loops are not drawn.

Again, we will give a general such result in Theorem 3.7.

3.3. Irreducibility and stationary states. In this section we summarize some properties of the promotion Markov chains of Sections 3.1 and 3.2 and state their stationary distributions. Proofs of these statements can be found in [AKS12].

Proposition 3.4. *Consider the digraph G whose vertices are labeled by elements of \mathcal{L} and whose edges are given as follows: for $\pi, \pi' \in \mathcal{L}$, there is an edge between π and π' in G if and only if $\pi' = \pi \partial_j$ for some $j \in [n]$. Then G is strongly connected.*

Corollary 3.5. *Assuming that the edge weights are strictly positive, the two Markov chains of Sections 3.1 and 3.2 are irreducible and ergodic. Hence their stationary states are unique.*

Next we state properties of the stationary state of the two discrete time Markov chains, assuming that all x_i 's are strictly positive.

Theorem 3.6. *The discrete time Markov chain according to the uniform promotion graph has the uniform stationary distribution, that is, each linear extension is equally likely to occur.*

We now turn to the promotion graphs. In this case we find nice product formulas for the stationary weights.

Theorem 3.7. *The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathcal{L}(P)$ for the discrete time Markov chain for the promotion graph is given by*

$$(3.2) \quad w(\pi) = \prod_{i=1}^n \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}},$$

assuming $w(e) = 1$.

Remark 3.8. *The entries of w do not, in general, sum to one. Therefore this is not a true probability distribution, but this is easily remedied by a multiplicative constant Z_P depending only on the poset P .*

When P is the n -antichain, then $\mathcal{L} = S_n$. In this case, the probability distribution of Theorem 3.7 has been studied in the past by Hendricks [Hen72, Hen73] and is known as the

Tsetlin library [Tse63], which we now describe. Suppose that a library consists of n books b_1, \dots, b_n on a single shelf. Assume that only one book is picked at a time and is returned before the next book is picked up. The book b_i is picked with probability x_i and placed at the end of the shelf.

We now explain why promotion on the n -antichain is the Tsetlin library. A given ordering of the books can be identified with a permutation π . The action of ∂_k on π gives $\pi\tau_k \cdots \tau_{n-1}$ by (2.3), where now all the τ_i 's satisfy the braid relation since none of the π_j 's are comparable. Thus the k -th element in π is moved all the way to the end. The probability with which this happens is x_{π_k} , which makes this process identical to the action of the Tsetlin library.

The stationary distribution of the Tsetlin library is a special case of Theorem 3.7. In this case, Z_P of Remark 3.8 also has a nice product formula, leading to the probability distribution,

$$(3.3) \quad w(\pi) = \prod_{i=1}^n \frac{x_{\pi_i}}{x_{\pi_1} + \cdots + x_{\pi_i}}.$$

Letac [Let78] considered generalizations of the Tsetlin library to rooted trees (meaning that each element in P besides the root has precisely one successor). Our results hold for any finite poset P .

4. PARTITION FUNCTIONS AND EIGENVALUES FOR ROOTED FORESTS

For a certain class of posets, we are able to give an explicit formula for the probability distribution for the promotion graph. Note that this involves computing the partition function Z_P (see Remark 3.8). We can also specify all eigenvalues and their multiplicities of the transition matrix explicitly. Proofs of these statements can be found in [AKS12].

Before we can state the main theorems of this section, we need to make a couple of definitions. A **rooted tree** is a connected poset, where each node has at most one successor. Note that a rooted tree has a unique largest element. A **rooted forest** is a union of rooted trees. A **lower set** (resp. **upper set**) S in a poset is a subset of the nodes such that if $x \in S$ and $y \preceq x$ (resp. $y \succeq x$), then also $y \in S$. We first give the formula for the partition function.

Theorem 4.1. *Let P be a rooted forest of size n and let $x_{\preceq i} = \sum_{j \preceq i} x_j$. The partition function for the promotion graph is given by*

$$(4.1) \quad Z_P = \prod_{i=1}^n \frac{x_{\preceq i}}{x_1 + \cdots + x_i}.$$

Let L be a finite poset with smallest element $\hat{0}$ and largest element $\hat{1}$. Following [Bro00, Appendix C], one may associate to each element $x \in L$ a **derangement number** d_x defined as

$$(4.2) \quad d_x = \sum_{y \succeq x} \mu(x, y) f([y, \hat{1}]),$$

where $\mu(x, y)$ is the Möbius function for the interval $[x, y] := \{z \in L \mid x \preceq z \preceq y\}$ [Sta97, Section 3.7] and $f([y, \hat{1}])$ is the number of maximal chains in the interval $[y, \hat{1}]$.

A permutation is a **derangement** if it does not have any fixed points. A linear extension π is called a **poset derangement** if it is a derangement when considered as a permutation. Let \mathfrak{d}_P be the number of poset derangements of the poset P .

A **lattice** L is a poset in which any two elements have a unique supremum (also called join) and a unique infimum (also called meet). For $x, y \in L$ the join is denoted by $x \vee y$, whereas the meet is $x \wedge y$. For an **upper semi-lattice** we only require the existence of a unique supremum of any two elements.

Theorem 4.2. *Let P be a rooted forest of size n and M the transition matrix of the promotion graph of Section 3.2. Then*

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{d_S},$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upper sets in P . In other words, for each subset $S \subseteq [n]$, which is an upper set in P , there is an eigenvalue x_S with multiplicity d_S .

The proof of Theorem 4.2 follows from the fact that the monoid corresponding to the transition matrix M is \mathcal{R} -trivial. When P is a union of chains, which is a special case of rooted forests, we can express the eigenvalue multiplicities directly in terms of the number of poset derangements.

Theorem 4.3. *Let $P = [n_1] + [n_2] + \cdots + [n_k]$ be a union of chains of size n whose elements are labeled consecutively within chains. Then*

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{\mathfrak{d}_{P \setminus S}},$$

where $\mathfrak{d}_\emptyset = 1$.

Note that the antichain is a special case of a rooted forest and in particular a union of chains. In this case the Markov chain is the Tsetlin library and all subsets of $[n]$ are upper (and lower) sets. Hence Theorem 4.2 specializes to the results of Donnelly [Don91], Kapoor and Reingold [KR91], and Phatarford [Pha91] for the Tsetlin library.

The case of unions of chains, which are consecutively labeled, can be interpreted as looking at a parabolic subgroup of S_n . If there are k chains of lengths n_i for $1 \leq i \leq k$, then the parabolic subgroup is $S_{n_1} \times \cdots \times S_{n_k}$. In the realm of the Tsetlin library, there are n_i books of the same color. The Markov chain consists of taking a book at random and placing it at the end of the stack.

5. \mathcal{R} -TRIVIAL MONOIDS

In this section we briefly outline the proof of Theorem 4.2. More details can be found in [AKS12].

A finite **monoid** \mathcal{M} is a finite set with an associative multiplication and an identity element. Green [Gre51] defined several preorders on \mathcal{M} . In particular for $x, y \in \mathcal{M}$ the \mathcal{R} - and \mathcal{L} -order is defined as

$$(5.1) \quad \begin{aligned} x \geq_{\mathcal{R}} y & \quad \text{if } y = xu \text{ for some } u \in \mathcal{M}, \\ x \geq_{\mathcal{L}} y & \quad \text{if } y = ux \text{ for some } u \in \mathcal{M}. \end{aligned}$$

This ordering gives rise to equivalence classes (\mathcal{R} -classes or \mathcal{L} -classes)

$$\begin{aligned} x \mathcal{R} y & \quad \text{if and only if } x\mathcal{M} = y\mathcal{M}, \\ x \mathcal{L} y & \quad \text{if and only if } \mathcal{M}x = \mathcal{M}y. \end{aligned}$$

The monoid \mathcal{M} is said to be \mathcal{R} -trivial (resp. \mathcal{L} -trivial) if all \mathcal{R} -classes (resp. \mathcal{L} -classes) have cardinality one.

Now let P be a rooted forest of size n and $\hat{\partial}_i$ for $1 \leq i \leq n$ the operators on $\mathcal{L}(P)$ defined by the promotion graph of Section 3.2. That is, for $\pi, \pi' \in \mathcal{L}(P)$, the operator $\hat{\partial}_i$ maps π to π' if $\pi' = \pi \partial_{\pi_i^{-1}}$. We are interested in the monoid $\mathcal{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$. The next lemma shows that the action of the generators $\hat{\partial}_i$ for rooted forests is very similar to the action of the operators of the Tsetlin library by moving the letter i to the end; the difference in this case is that letters above i need to be reordered according to the poset.

Lemma 5.1. *Let P and $\hat{\partial}_i$ be as above, and $\pi \in \mathcal{L}(P)$. Then $\pi \hat{\partial}_i$ is the linear extension in $\mathcal{L}(P)$ obtained from π by moving the letter i to position n and reordering all letters $j \succeq i$.*

Example 5.2. *Let P be the union of a chain of length 3 and a chain of length 2, where the first chain is labeled by the elements $\{1, 2, 3\}$ and the second chain by $\{4, 5\}$. Then $41235 \hat{\partial}_1 = 41253$, which is obtained by moving the letter 1 to the end of the word and then reordering the letters $\{1, 2, 3\}$, so that the result is again a linear extension of P .*

Let M be the transition matrix of the promotion graph of Section 3.2. Define \mathcal{M} to be the monoid generated by $\{G_i \mid 1 \leq i \leq n\}$, where G_i is the matrix M evaluated at $x_i = 1$ and all other $x_j = 0$. We are now ready to state the main result of this section.

Theorem 5.3. *\mathcal{M} is \mathcal{R} -trivial.*

Remark 5.4. *Considering the matrix monoid \mathcal{M} is equivalent to considering the abstract monoid $\mathcal{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$. Since the operators $\hat{\partial}_i$ act on the right on linear extensions, the monoid $\mathcal{M}^{\hat{\partial}}$ is \mathcal{L} -trivial instead of \mathcal{R} -trivial.*

The proof of Theorem 5.3 exploits Lemma 5.1 by proving that there is an order on idempotents using right factors. For $x \in \mathcal{M}^{\hat{\partial}}$, let $\text{rfactor}(x)$ be the maximal common right factor of all elements in the image of x , that is, all elements $\pi \in \text{im}(x)$ can be written as $\pi = \pi_1 \cdots \pi_m \text{rfactor}(x)$ and there is no bigger right factor for which this is true. Let us also define the set of entries in the right factor $\text{Rfactor}(x) = \{i \mid i \in \text{rfactor}(x)\}$. Note that since all elements in the image set of x are linear extensions of P , $\text{Rfactor}(x)$ is an upper set of P . Theorem 5.3 is then established by showing that for idempotents x , the set $\text{Rfactor}(x)$ is the same as the left stabilizer $\{i \mid \hat{\partial}_i x = x\}$ which imposes a partial order.

Example 5.5. *Let P be the poset on three elements $\{1, 2, 3\}$, where 2 covers 1 and there are no further relations. The linear extensions of P are $\{123, 132, 312\}$. The monoid \mathcal{M} with \mathcal{R} -order, where an edge labeled i means right multiplication by G_i , is depicted in Figure 5. From the picture it is clear that the elements in the monoid are partially ordered.*

This confirms Theorem 5.3 that the monoid is \mathcal{R} -trivial. The proof of Theorem 4.2 now follows from [Ste06, Theorems 6.3 and 6.4] and some further considerations regarding the lattice L . For more details see [AKS12, Section 6].

6. MIXING TIMES

For random walks on hyperplane arrangements, Brown and Diaconis [BD98] (see also [AD10]) give explicit bounds for the rates of convergence to stationarity. These bounds still hold for Markov chains related to left-regular bands [Bro00]. Here we present analogous results for

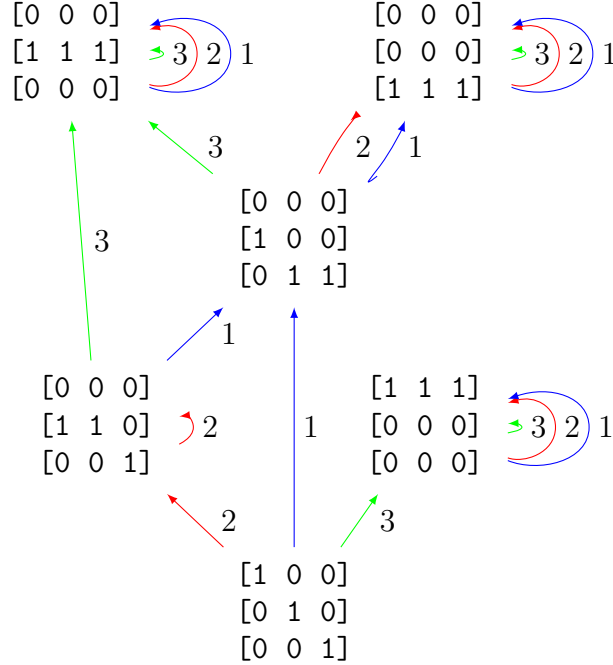


FIGURE 5. Monoid \mathcal{M} in right order for the poset of Example 5.5. With the conventions in (5.1), the identity is the biggest element in \mathcal{R} -order.

the Markov chains corresponding to the \mathcal{R} -trivial monoids of Section 4. The methods are very similar to the ones we used for Markov chains related to nonabelian sandpile models [ASST13], which also turn out to yield \mathcal{R} -trivial monoids.

The **rate of convergence** is the total variation distance from stationarity after k steps, that is,

$$\|\mathbb{P}^k - w\| = \frac{1}{2} \sum_{\pi \in \mathcal{L}(P)} |\mathbb{P}^k(t) - w(\pi)|,$$

where \mathbb{P}^k is the distribution after k steps and w is the stationary distribution.

Theorem 6.1. *Let P be a rooted forest with $n := |P|$ and $p_x := \min\{x_i \mid 1 \leq i \leq n\}$. Then, as soon as $k \geq (n^2 - 1)/p_x$, the distance to stationarity of the promotion Markov chain satisfies*

$$\|\mathbb{P}^k - w\| \leq \exp\left(-\frac{(kp_x - (n^2 - 1))^2}{2kp_x}\right).$$

The **mixing time** [LPW09] is the number of steps k until $\|\mathbb{P}^k - w\| \leq e^{-c}$ (where different authors use different conventions for the value of c). Using Theorem 6.1 we require

$$(kp_x - (n^2 - 1))^2 \geq 2kp_x c,$$

which shows that the mixing time is at most $\frac{2(n^2 + c - 1)}{p_x}$. If the probability distribution $\{x_i \mid 1 \leq i \leq n\}$ is uniform, then p_x is of order $1/n$ and the mixing time is of order at most n^3 .

The proof of Theorem 6.1 follows the same outline as the proof in [ASST13, Section 5.3]. We need to define a statistic $u(x)$ for $x \in \mathcal{M}$ such that $u(x)$ is minimal if and only if x is the constant map and furthermore

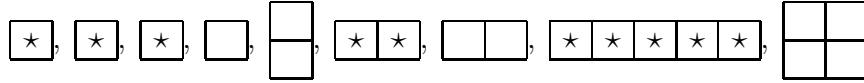


FIGURE 6. Egg-box picture for the monoid associated to the promotion Markov chain for the poset in Example 3.1.

- (1) u decreases along \mathcal{R} -order: $u(xx') \leq u(x)$ for any $x, x' \in \mathcal{M}$.
- (2) **Existence of generator with strict decrease:** There exists a generator G_i such that $u(xG_i) < u(x)$.

Unlike in [ASST13], we take $u(x) \in \mathbb{Z}_{\geq 0}^2$ with lexicographic ordering on $\mathbb{Z}_{\geq 0}^2$, that is $(x, y) < (x', y')$ if either $x < x'$, or $x = x'$ and $y < y'$. Set $u(x) := (n - |\text{Rfactor}(x)|, |\text{des}(x)|)$, where $\text{des}(x) = \{i \mid xG_i = x\}$. It is clear that $u(x) = (0, n)$ if and only if x is a constant map, which is the minimal value u can achieve. The maximal value of u is achieved by the identity $u(e) = (n, 0)$. The two conditions follow from [AKS12, Section 6]: either the right factor $\text{rfactor}(x)$ increases by right multiplication by a generator G_i ; if not, then $\{i\} \cup \text{Rfactor}(x)$ must be an upper set again and $\text{des}(xG_i) = \text{des}(x) \setminus \{j \mid j \text{ covers } i \text{ in } P\}$.

Therefore, the probability that $(n, 0) \geq u(x) > (0, n)$ after k steps of the right random walk on \mathcal{M} is bounded above by the probability of having at most $(n+1)(n-1) = n^2 - 1$ successes in k Bernoulli trials with success probability p_x . A successful step is one that decreases the statistic u . Using Chernoff's inequality for the cumulative distribution function of a binomial random variable as in [ASST13] we obtain Theorem 6.1.

7. OTHER POSETS

So far [AKS12], we have characterized posets, where the Markov chains for the promotion graph yield certain simple formulas for their eigenvalues and multiplicities. The eigenvalues have explicit expressions for rooted forests and there is an explicit combinatorial interpretation for the multiplicities as derangement numbers of permutations for unions of chains by Theorem 4.3.

However, we have not classified all possible posets, whose promotion graphs have nice properties. For example, the non-zero eigenvalues of the transition matrix of the promotion graph of the poset in Example 3.1 are given by

$$x_3 + x_4, \quad x_3, \quad 0 \quad \text{and} \quad -x_1,$$

even though the corresponding monoid is not \mathcal{R} -trivial (in fact, it is not even aperiodic). The egg-box picture of the monoid is given in Figure 6. Notice that one of the eigenvalues is negative.

On the other hand, not all posets have this property. In particular, the poset with covering relations $1 < 2, 1 < 3$ and $1 < 4$ has six linear extensions, but the characteristic polynomial of its transition matrix does not factorize at all. It would be interesting to classify all posets with the property that all the eigenvalues of the transition matrices of the promotion Markov chain are linear in the probability distribution x_i . In such cases, one would also like an explicit formula for the multiplicity of these eigenvalues.

We list all posets of size 4, which are not down forests and which nonetheless have simple linear expressions for their eigenvalues in Table 1 along with the eigenvalues. For all such posets, there is at least one eigenvalue which contains a negative term. The posets, which are not down forests and the eigenvalues of whose promotion transition matrices have nonlinear

Poset	Eigenvalues (other than 1)
	$0, 0, 0, x_2, x_3, x_2 + x_3, x_4 - x_1$
	$-x_1 - x_2$
	$0, x_3, -x_1, x_3 + x_4$
	$x_4 - x_1$
	$0, x_3 + x_4, -x_1 - x_2$

TABLE 1. All inequivalent posets of size 4 whose promotion transition matrices have simple expressions for their eigenvalues.

expressions, are given in Table 2. Comparing the two tables, it is not obvious how to characterize those posets where the eigenvalues are simple. It would be interesting to classify posets where all eigenvalues are linear in the parameters and understand the eigenvalues and their multiplicities completely. For comparison, the egg-box picture of the second poset in Table 2 is presented in Figure 7.

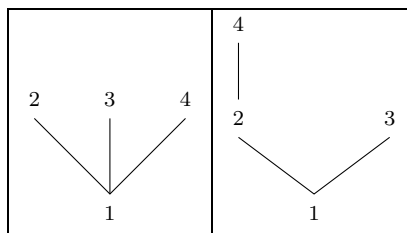


TABLE 2. All inequivalent posets of size 4 whose promotion transition matrices do not have simple expressions for their eigenvalues.

Using data from all posets which are not down forests of sizes up to 7, we have the following necessary (but not sufficient) conjecture.

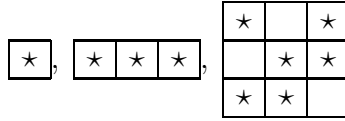


FIGURE 7. Egg-box picture for the monoid associated to the promotion Markov chain for the second poset in Table 2.

Conjecture 7.1. *Let P be a poset of size n which is not a down forest and M be its promotion transition matrix. If M has eigenvalues which are linear in the parameters x_1, \dots, x_n , then the following hold*

- (1) *the coefficients of the parameters in the eigenvalues are only one of ± 1 ,*
- (2) *each element of P has at most two successors,*
- (3) *the only parameters whose coefficients in the eigenvalues are -1 are those which either have two successors or one of whose successors have two successors.*

8. SUBSETS OF S_n

We define a generalization of the action of promotion on an arbitrary nonempty subset of S_n inspired by the ideas in [Hai92, MR94, Sta09]. Let A be such a subset and suppose $\pi = \pi_1 \cdots \pi_n \in A$ in one-line notation. Then we define the operator σ_i for $i \in \{1, \dots, n\}$ as

$$(8.1) \quad \pi \sigma_i = \begin{cases} \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n & \text{if } \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n \in A \\ \pi & \text{otherwise.} \end{cases}$$

In other words, σ_i acts non-trivially on a permutation in A if interchanging entries π_i and π_{i+1} yields another permutation in A , and otherwise acts as the identity. Then the generalized promotion operator, also denoted ∂_j , is an operator on A defined by

$$(8.2) \quad \partial_j = \sigma_j \sigma_{j+1} \cdots \sigma_{n-1}.$$

As in Sections 3.1 and 3.2, we can define a promotion graph whose vertices are the elements of the set A and where there is an edge between permutations π and π' if and only if $\pi' = \pi \partial_j$. In the uniform promotion case, such an edge has weight x_j and in the promotion case, the edge has weight x_{π_j} . In both cases, we have analogous Markov chains. We describe the stationary distribution of these chains below.

Theorem 8.1. *Assuming the promotion graph for A is strongly connected, the unique stationary state weight $w(\pi)$ of the permutation $\pi \in A$ for the corresponding discrete time Markov chain is*

- (1) *in the uniform promotion case*

$$(8.3) \quad w(\pi) = \frac{1}{|A|},$$

- (2) *in the promotion case*

$$(8.4) \quad w(\pi) = \prod_{i=1}^n \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}}.$$

The proofs are essentially identical to the proofs of Theorem 3.6 and Theorem 3.7 given in [AKS12] and are skipped.

Remark 8.2.

- (1) The entries of w do not, in general, sum to one. Therefore this is not a true probability distribution, but this is easily remedied by a multiplicative constant Z_A depending only on the subset A .
- (2) Even if the set A is such that the promotion graph is not strongly connected, (8.3) and (8.4) hold. However, the formula need not be unique. The proofs of Theorem 8.1 still go through because all we need to do is to verify that the master equation (3.1) holds.

There is a natural way to build subsets A which cannot be the set of linear extensions $\mathcal{L}(P)$ for any poset P , and whose promotion graphs are yet strongly connected. The idea is to consider a union of sorting networks. A **sorting network** from the identity permutation e to any permutation π is a shortest path from one to the other by a series of nearest-neighbor transpositions. In other words, these are maximal chains in right weak order starting at the identity. For example, one sorting network to the permutation 24153 is

$$12345 \rightarrow 12435 \rightarrow 21435 \rightarrow 24135 \rightarrow 24153.$$

Proposition 8.3. *Let $A \subset S_n$ be a union of sorting networks. Consider the digraph G_A whose vertices are labeled by the elements of A and whose edges are given as follows: for $\pi, \pi' \in A$, there is an edge between π and π' in G_A if and only if $\pi' = \pi \partial_j$ for some $j \in [n]$. Then G_A is strongly connected.*

Proof. The operators ∂_i are each invertible, which means that each vertex of G_A has exactly one edge pointing in and one pointing out for each i . Therefore, it suffices to show that there is a directed path from e to π for every π in A .

By definition of a sorting network, π can be written as $e\sigma_{i_k} \dots \sigma_{i_1}$. Although the action of each σ_{i_j} depends crucially on the set A , they satisfy $\sigma_{i_j}^2 = 1$. Using the fact that $\sigma_{n-1} = \partial_{n-1}$ and (8.2), one can recursively express each σ_{i_j} as a product of ∂_ℓ 's analogous to the proof of Lemma 2.3 in [AKS12]. \square

As a consequence of Proposition 8.3, the unique stationary distribution of a subset which is a union of sorting networks is given by (8.4). One is naturally led to ask whether the eigenvalues of these transition matrices are also linear in the parameters. This does not seem to be true in any general sense.

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(Arvind Ayyer) DEPARTMENT OF MATHEMATICS, UC DAVIS, ONE SHIELDS AVE., DAVIS, CA 95616-8633, U.S.A.

NEW ADDRESS: DEPARTMENT OF MATHEMATICS, DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE - 560012, INDIA.

E-mail address: `arvind@math.iisc.ernet.in`

(Steven Klee) SEATTLE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 901 12TH AVENUE E, SEATTLE, WA 98122, U.S.A

E-mail address: `klee@math.ucdavis.edu`

(Anne Schilling) DEPARTMENT OF MATHEMATICS, UC DAVIS, ONE SHIELDS AVE., DAVIS, CA 95616-8633, U.S.A.

E-mail address: `anne@math.ucdavis.edu`