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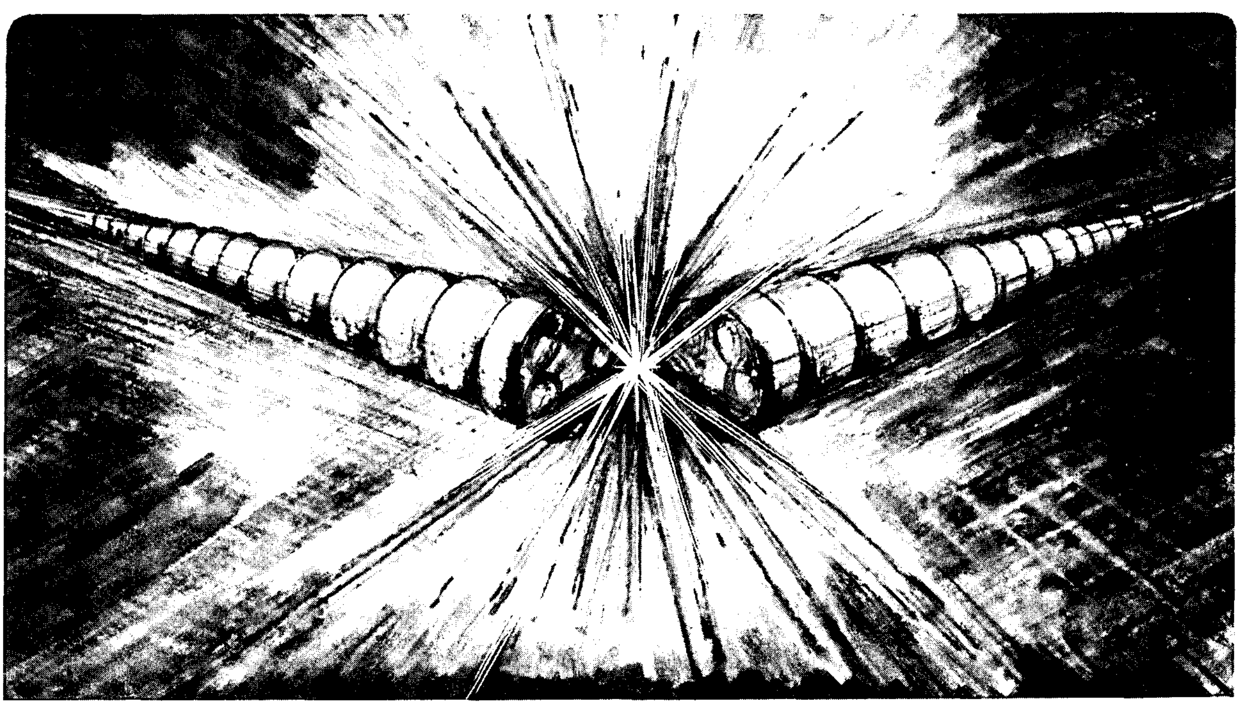



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# The Phase-Space-Lagrangian Action Principle and The Generalized K-χ Theorem*

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# THE PHASE-SPACE-LAGRANGIAN ACTION PRINCIPLE AND THE GENERALIZED K-$\chi$ THEOREM

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## Abstract

The covariant coupled equations for plasma dynamics and the Maxwell field are expressed as a phase-space-Lagrangian action principle. The linear interaction is transformed to the bilinear beat Hamiltonian by a gauge-invariant Lagrangian Lie transform. The result yields the generalized linear susceptibility directly.

The fundamental relation between *nonlinear* ponderomotive effects and *linear* plasma response has come to be known as the K-$\chi$ theorem. Ponderomotive effects are embodied in the oscillation-center Hamiltonian K(z), introduced by Dewar.[1] It describes the (oscillation-*averaged*) orbit of a single particle in an oscillatory field, with the dominant effects *quadratic* in the wave amplitude. On the other hand, the linear susceptibility $\chi$, a functional of the unperturbed particle distribution, describes the *oscillatory* current density *linear* in the wave amplitude.

This surprising relation between the quadratic Hamiltonian $K_2$ and the susceptibility $\chi$, for the case of a *single* wave, was observed[2] some years ago, and then proved by Johnston[3] and by Cary.[4] The underlying reason for the relation, however, became clear only with the recent development[5] of phase-space Lagrangian action principles, and the realization that the plasma action term quadratic in the wave amplitude $[-\int f(z)K_2(z)]$ was *simultaneously* both the oscillation-center energy and the plasma part of the wave Lagrangian. (To be sure, this fact was at least implicit in the earlier work of Dewar[1] and of Johnston.[3]) The importance of this realization, with its embodiment in the action principle, is best exemplified in Similon's recent study[6] of self-consistency in the stabilization of a confined plasma by the ponderomotive effects of an electromagnetic wave.

The ponderomotive *beat* Hamiltonian, introduced by Johnston[7] for the scattering of *two* waves, and now of especial use for the theory of free-electron lasers and beat-wave accelerators, is a conceptually simple extension of oscillation-center ideas to particles that resonant with the beat of two primary waves. Its utility led Grebogi[8] to the conjecture that it too is related to the linear susceptibility. This paper presents a simple proof of that desired relation, and then illustrates it by an explicit calculation.

That calculation, in turn, is based on the use of a powerful new perturbation technique, invented by Littlejohn[9] for a system governed by a *phase-space* Lagrangian. Whereas the standard Hamiltonian perturbation theories (such as the Hamiltonian Lie transform[10]) preserve the Poisson structure, the new method enables one to perform the desired averaging directly on the Poisson (or symplectic) structure. As a result, the generator of the transform can be made gauge invariant and physically meaningful.

The calculation is here outlined for a field-free background. The extension to the case of a strong background field is conceptually easy, but of course algebraically complex, and will be published later. We begin with the definition of the two-point linear susceptibility tensor,[11,5] as a functional derivative:

$$\chi^{\mu\nu}(x_1,x_2) = \delta j^{\mu}(x_1)/\delta A_{\nu}(x_2) \, . \tag{1a}$$

It is convenient to use covariant notation, with metric (1, 1, 1, -1) and c = 1. Thus ($x = \mathbf{x},t$), $j^{\mu} = (\mathbf{j}, \rho)$, $A_{\nu} = (\mathbf{A}, -\phi)$. In terms of the Fourier transforms [e.g., $j^{\mu}(k) = \int d^4x \, j^{\mu}(x) \exp(-ik\cdot x)$; $k_{\mu} = (\mathbf{k},-\omega)$], the susceptibility reads

$$\chi^{\mu\nu}(k_1,k_2) = \delta j^{\mu}(k_1)/\delta A_{\nu}(k_2). \tag{1b}$$

In Eq. (1), $j$ is the linear current response to a perturbing electromagnetic potential $A$. Since $j$ must be invariant under gauge transformations of $A$, the susceptibility must satisfy $\chi^{\mu\nu}(k, k') \, k'_{\nu} = 0$. In addition, charge conservation ($\partial j^{\mu}/\partial x^{\mu} = 0$) implies that $k_{\mu}\chi^{\mu\nu}(k,k') = 0$. Because each particle responds to the perturbing field independently, the current density is additive in the particles; hence the susceptibility is a *linear* functional of the unperturbed distribution.

The ponderomotive Hamiltonian $K_2$ (z) is (by definition) that term, of the oscillation-center Hamiltonian K(z), which is quadratic in the perturbing potential. Its most general form is thus

$$K_2(z) = \tfrac{1}{2}\int d^4x_1 \int d^4x_2\, A_\mu(x_1)\, A_\nu(x_2)\; K^{\mu\nu}(z;x_1,x_2) \qquad (2a)$$

$$= \tfrac{1}{2}\int d^4k_1 \int d^4k_2\, A^*_\mu(k_1)\, A_\nu(k_2)\, K^{\mu\nu}(z;k_1,k_2)\,, \qquad (2b)$$

(We absorb $(2\pi)^{-4}$ into the element $d^4k$.)

We may interpret the integrand of (2b) as the contribution, to the oscillation-center Hamiltonian, of the nonlinear beat between two plane waves with wave-vectors $k_1$ and $k_2$. The relation we wish to prove, the "generalized K-$\chi$ theorem," is

$$K^{\mu\nu}(z;k_1,k_2) = -\,\delta\chi^{\mu\nu}(k_1,k_2)/\delta f(z)\,. \qquad (3)$$

That this relation has not heretofore been observed is probably due to the fact that almost all calculations of $\chi$ make specific assumptions on the form of *f*. However, the functional derivative in (3) requires the susceptibility for completely general *f*.

One restriction which we do make is that f include only those particles which are *non-resonant* with the *primary* waves $k_1$, $k_2$. Hence $\chi$ is Hermitian, and $K_2$ is real. The proper treatment of primary resonances is a large subject in itself, with important contributions especially by Dewar and his co-workers.[12]

The system action S is a functional of the potential field A(x), and of the particle orbits in 8-dimensional phase space, denoted $z^\alpha(\tau) \equiv [r^\mu(\tau), \pi_\mu(\tau)]$, with $\pi_\mu = (\pi, -h)$ the *kinetic* 4-momentum, h the kinetic energy, and $\tau$ an arbitrary orbit parameter. The *single*-particle action is $S_1 = \int(\pi\cdot dr + eA(r)\cdot dr)$. We demand that $\delta S_1 = 0$ for variation of orbits constrained to the 7-dimensional mass surface $0 = H(z) = (\pi^2 + m^2)/2m$. With a Lagrangian multiplier $\lambda(\tau)$, we have

$$0 = \delta\int[\pi\cdot dr + eA(r)\cdot dr - \lambda(\tau)d\tau\, H(z)]\,. \qquad (4)$$

Variation with respect to (wrt) $r(\tau)$ yields $d\pi_\mu = eF_{\mu\nu}\, dr^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, while variation wrt $\pi(\tau)$ yields $dr^\mu = \lambda(\tau)\, d\tau\, \pi^\mu/m$. The mass constraint determines $\lambda^2(\tau) = -(dr(\tau)/d\tau)^2$; if one wishes $d\tau$ to represent the particle's proper-time interval, then $\lambda = 1$.

The total action is $S = \Sigma_i S_i + S_m$, where $S_i$ is the action of particle i, and $S_m = \int d^4x\, F_{\mu\nu}F^{\nu\mu}/16\pi$ is the Maxwell action. The interaction part of S can be expressed as $\int d^4x\, j^\mu(x)\, A_\mu(x)$, with $[u^\mu = dr^\mu/d\tau = \pi^\mu/m]$

$$j^\mu(x) = \Sigma_i\, e \int d\tau\, u^\mu(\tau)\, \delta^4(x-r(\tau)) = e\int d^8z\, f(z)\, u^\mu \delta^4(x-r).$$

We have introduced the particle phase-space density (for each species)

$$f(z) = \Sigma_i \int d\tau_i\, \delta^4(r - r_i(\tau_i))\, \delta^4(\pi - \pi_i(\tau_i)). \qquad (5)$$

Variation of S wrt A(x) yields the Maxwell equation $\partial_\mu F^{\mu\nu}(x) = -4\pi\, j^\nu(x)$.

The distribution f satisfies the Vlasov equation[5] $\{f,H\} = 0$, in terms of the *noncanonical* Poisson bracket (PB) $\{g_1, g_2\} = J^{\alpha\beta}(z)(\partial g_1/\partial z^\alpha)(\partial g_2/\partial z^\beta)$. The Poisson tensor J(z) is the reciprocal of the Lagrange tensor (or symplectic 2-form),[9] $\omega_{rr} = eF(r)$, $\omega_{\pi r} = -\omega_{r\pi} = I$, $\omega_{\pi\pi} = 0$. Thus $J^{rr} = 0$, $J^{r\pi} = -J^{\pi r} = I$, $J^{\pi\pi} = eF(r)$, and the PB is expressed in the physical variables r, $\pi$, F:

$$\{g_1,g_2\} = (\partial g_1/\partial r)\cdot(\partial g_2/\partial \pi) - (\partial g_1/\partial \pi)\cdot(\partial g_2/\partial r)$$
$$+ e(\partial g_1/\partial \pi)\cdot F(r)\cdot(\partial g_2/\partial \pi). \qquad (6)$$

For a wave field $F_{\mu\nu}(x)$, oscillations occur in the PB (6) for nonresonant particles. Our aim is to transform away this term, linear in F, by a change of variables from particle coordinates $z^\alpha$ to oscillation-center (OC) coordinates $\dot{z}^\alpha(z;F)$. The *linear oscillation* induced by F is denoted $\tilde{z} \equiv z - \dot{z}$. We see that $\tilde{z}^\alpha(z)$ is a physically meaningful *vector field*; it is the generator of the Lagrangian Lie transform.

In terms of the Fourier transform $F_{\mu\nu}(k)$, the linearized particle equations yield the oscillation:

$$\begin{cases} \tilde{\pi}(\dot{z};F) = e \int d^4k\, F(k)\cdot \bar{u}\ (ik\cdot\bar{u})^{-1} \exp ik\cdot\bar{r}\,, \\ \tilde{r}(\dot{z};F) = -(e/m) \int d^4k\, F(k)\cdot \bar{u}\ (k\cdot\bar{u})^{-2} \exp ik\cdot\bar{r}\,, \end{cases} \tag{7}$$

as a *vector field* on OC phase space. In order that (7) be well-defined, we consider only that portion of phase space which has no primary resonances; i.e., $k\cdot\bar{u} \neq 0$ for all k such that $F(k) \neq 0$.

Our aim is to make the PB *canonical*, when expressed in OC variables:

$$\{g_1,g_2\} = (\partial g_1/\partial\bar{r})\cdot(\partial g_2/\partial\bar{\pi}) - (\partial g_1/\partial\bar{\pi})\cdot(\partial g_2/\partial\bar{r})\,. \tag{8}$$

Space limitations permit us only to quote the result of using the Lagrangian Lie transform, which is based on differential-geometric methods.[13] We obtain the OC Hamiltonian $K(\bar{z}) = H(\bar{z}) + K_2(\bar{z})$, with the ponderomotive term given by the *virial:*[14]

$$K_2(\dot{z};F) = -\tfrac{1}{2}\tilde{r}(\dot{z};F)\cdot[eF(\bar{r})\cdot\bar{u}]. \tag{9}$$

The canonical Hamiltonian equations then yield

$$d\bar{\pi}/d\tau = -\partial K/\partial\bar{r} = -\partial K_2/\partial\bar{r} \tag{10a}$$

for the ponderomotive force, and

$$d\bar{r}/d\tau = \partial K/\partial\bar{\pi} = \bar{\pi}/m + \partial K_2/\partial\bar{\pi}\,, \tag{10b}$$

a *gauge - invariant* expression for the *canonical* OC momentum $\bar{\pi}$, in terms to the OC velocity $d\bar{r}/d\tau$ and the quadratic term (related to wave monentum). (The mass constraint now reads $0 = H(z) = K(\bar{z})$; i.e., the Hamiltonian transforms as a scalar under the coordinate change.)

The one-particle action is now, in the OC representation, including the Hamiltonian constraint:

$$S_1 = \int(\vec{\pi}\cdot d\vec{r} - K(\bar{z};F)\, d\tau). \tag{11}$$

The terms of $\sum_i S_i$ quadratic in F are thus

$$S^{(2)} = -\sum_i \int d\tau_i\, K_2(z_i(\tau_i);F) = -\int d^8z\, f(z)\, K_2(\bar{z};F)\,. \tag{12}$$

Noting from (9) and (7) that $K_2$ and $\bar{z}$ are manifestly gauge - invariant, we proceed to express $K_2$ in the desired form (2b), using $F_{\mu\nu}(k) = i(k_\mu A_\nu - k_\nu A_\mu)$; we obtain

$$K^{\mu\nu}(z;k_1,k_2) = (e^2/m)\, [(k_1\cdot u)^{-2} + (k_2\cdot u)^{-2}]\, \mathcal{K}^{\mu\nu}(u;k_1,k_2)\, \exp[i\, (k_2-k_1)\cdot r]\ ; \tag{13}$$

with $\mathcal{K}^{\mu\nu}(u;k_1,k_2) = k_1\cdot u\ \ k_2^\mu u^\nu + k_2\cdot u\ \ u^\mu k_1^\nu - k_1\cdot k_2\ u^\mu u^\nu - k_1\cdot u\ \ k_2\cdot u\ g^{\mu\nu}$.

On substituting (2b) into (12), we obtain

$$S^{(2)} = -\int d^8z\, f(z)\, \tfrac{1}{2} \int d^4k_1 \int d^4k_2\, A_\mu(k_1)\, A_\nu(k_2)\, K^{\mu\nu}(\bar{z};k_1,k_2) \tag{14}$$

Recalling that $j^\mu(x) = \delta\Sigma_i S_i/\delta A_\mu(x)$, we see from (1a) that

$$\chi^{\mu\nu}(x_1,x_2) = \delta^2 S/\delta A_\mu(x_1)\delta A_\nu(x_2)\,,$$

$$\text{or} \qquad \chi^{\mu\nu}(k_1,k_2) = \delta^2 S/\delta A^*_\mu(k_1)\delta A_\nu(k_2)\,. \tag{15}$$

Applying this to (14), we obtain

$$\chi^{\mu\nu}(k_1,k_2) = -\int d^8z\, f(z)\, K^{\mu\nu}(\bar{z};k_1,k_2), \tag{16}$$

which is equivalent to the desired theorem (3).

If we set $k_2 = k_1$ in (13) and (16), we obtain the covariant form of the single-wave K-$\chi$ theorem.[5,15]

In summary, we have indicated that a phase-space transformation from particle to oscillation-center coordinates, using the oscillation vector field as the generator of a Lagrangian Lie transform, converts the Poisson Bracket to canonical but gauge-invariant form, and converts the linear interaction to a bilinear form, which simultaneously is the beat Hamiltonian and expresses the generalized linear susceptibility.

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