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THE GELFAND–NAIMARK THEOREM FOR JB^* -TRIPLES

YAAKOV FRIEDMAN AND BERNARD RUSSO

JB^* -triples (which will be defined later) occur in the study of bounded symmetric domains in finite and infinite dimensions. Kaup [21] showed the equivalence of the two categories: bounded symmetric domains in complex Banach spaces; and JB^* -triples. This result extended the work of Koecher [24], who introduced Jordan triple systems as a vehicle for classifying finite dimensional bounded symmetric domains. Koecher also showed their connection to symmetric Lie algebras, cf. [20].

A concrete example of a JB^* -triple, called a J^* -algebra, was introduced by Harris [15]. This class, which includes all C^* -algebras, all Jordan operator algebras, some Lie algebras and all Hilbert spaces, was shown to have the following property: the open unit ball is a homogeneous domain, i.e., the group of biholomorphic automorphisms acts transitively on it.

In [10] the authors showed that JB^* -triples occur naturally in the solution of the contractive projection problem for C^* -algebras. Since the morphism in this problem may not preserve order, its image takes us out of the category of C^* -algebras (whose morphisms are completely positive) and even out of the category of Jordan algebras (whose morphisms are positive) to a category based only on geometry and not depending on order structure, cf. [11]. Actually the above mentioned problem was solved by the authors in the wider category of J^* -algebras, and by Kaup [22] and Stachó [28] for JB^* -triples.

By combining the contractive projection theorem with the ultrafilter version of the principle of local reflexivity, Dineen [8] showed that the second dual of a JB^* -triple is itself a JB^* -triple. By refining the ultrafilter used in Dineen's proof, Barton and Timoney [5] showed that the extended triple product is separately weak*-continuous. Thus, the second dual of a JB^* -triple is a JBW^* -triple, i.e., a dual JB^* -triple with a separately weak*-continuous triple product.

The structures of a JBW^* -triple and its predual were studied by the authors in [12]. For example it was proved that each JBW^* -triple decomposes into a direct sum of atomic and purely nonatomic JBW^* -subtriples. Also, a study of JBW^* -triples was undertaken by Horn [18], who obtained a classification of type I JBW^* -triples. In particular Horn proved that the JBW^* -factors of type I are precisely the Cartan factors (these will be defined below).

Using these results, we are now able to prove the following Gelfand–Naimark type theorem.

THEOREM 1. *Every JB^* -triple is isometrically isomorphic to a subtriple of a product of Cartan factors.*

This theorem answers many questions about JB^* -triples since it reduces their study to that of J^* -algebras and the two exceptional Cartan factors.

In [3], Alfsen–Shultz–Størmer proved a representation theorem of Gelfand–Naimark type for a class of Jordan algebras called JB -algebras. Their result, which generalizes the classification of formally real Jordan algebras [19], has renewed the interest in applying Jordan algebras to the quantum mechanical formalism.

Our proof of Theorem 1, not using order, when specialized to a JB^* -algebra, i.e., complexification of a JB -algebra, yields a shorter proof of this representation theorem, reducing it to the classification of type I Jordan factors [29], the second dual [27], and the atomic decomposition [2: Lemma 5.2].

Theorem 1, and the representation theorem of Alfsen–Shultz–Størmer are generalizations of the original Gelfand–Naimark theorem for C^* -algebras [13].

The proof of Theorem 1 is based on several recent developments in the subject. In order to make this paper self-contained, these preliminary results will be formulated in detail in §1. Some of the ideas involved in their proofs will be discussed so that the reader can obtain a general view of the field. The main results and their consequences will be stated and proved in §2.

The authors wish to thank Tom Barton and G. Horn for some helpful suggestions.

§1. Preliminaries. In C^n with $n > 1$ a Riemann mapping theorem is false and a classification of simply connected domains is intractable. However, a classification of bounded symmetric domains in any C^n was obtained using the tools of Lie algebras and holomorphy [7], and later using Jordan triple systems [24], [25]. A domain \mathcal{D} in a complex Banach space is *symmetric* if for each a in \mathcal{D} there is a biholomorphic map s_a of \mathcal{D} onto \mathcal{D} with $s_a = s_a^{-1}$, such that a is an isolated fixed point of s_a (cf. [30], [31], [32]).

Every such domain in C^n is a finite product of irreducible domains and each irreducible domain is holomorphically equivalent to a domain belonging to one of six classes of domains, called classical Cartan domains, which can be realized as matrices.

The (classical) Cartan domains are the unit balls of the (finite dimensional) Cartan factors of types 1 to 6. *Cartan factors* are defined as follows (cf. [15], [18], [33]): $B(H, K)$, $\{x \in B(H) : x^t = x\}$ and $\{x \in B(H) : x^t = -x\}$, where j is a conjugation on H and $x^t = jx^*j$, are Cartan factors of types 1, 2, 3 respectively. Any self-adjoint closed subspace of $B(H)$, of dimension ≥ 3 , for which x^2 is a scalar for each x , is a Cartan factor of type 4. The Cartan factors of types 1 to 4 are J^* -algebras, i.e., norm closed subspaces of $B(H, K)$ which are closed under the symmetrized triple product $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$. Here, H and K denote arbitrary Hilbert spaces.

The Cartan factors of types 5 and 6 consist of matrices over the eight dimensional complex algebra known as the Cayley numbers. C^6 consists of all 3

by 3 self-adjoint matrices and has a natural Jordan algebra structure. C^5 consists of all 1 by 2 matrices and can be embedded in C^6 as a subtriple.

Note that each of the Cartan factors of type 1–6 can be embedded as a subtriple of a JB^* -algebra. Moreover, in this embedding the Cartan factor is the range of some contractive projection on the JB^* -algebra.

In order to motivate the precise definition of JB^* -triple it will be convenient to sketch the construction of the triple product, starting from a bounded symmetric domain \mathcal{D} in a complex Banach space V .

For a bounded circled symmetric domain $\mathcal{D} \subset V$, let G_0 denote the component of the identity of the group $G = \text{Aut}(\mathcal{D})$ of all biholomorphic automorphisms of \mathcal{D} , and let $K \subseteq G_0$ be the isotropy group at 0, i.e., the set of automorphisms of \mathcal{D} which fix 0. The adjoint action of the symmetry $s(x) = -x$ at 0 gives rise to a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra \mathfrak{g}_0 of G_0 , which consists of complete holomorphic vector fields on \mathcal{D} . Here \mathfrak{k} is the Lie subalgebra corresponding to K and the subspace \mathfrak{p} satisfies $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ and $\text{Ad } K(\mathfrak{p}) \subset \mathfrak{p}$. Evaluation at 0 gives rise to an isomorphism of real vector spaces $\mathfrak{p} \simeq V$. For $v \in V$ we denote the unique vector field ξ in \mathfrak{p} with $\xi(0) = v$ by ξ_v . The key to defining a triple product structure on V lies in the following fact (cf. Loos [25: Lemma 2.3]), which involves simple calculations.

PROPOSITION. *For a bounded symmetric domain \mathcal{D} , the expression $v - \xi_v(z)$ is a homogeneous quadratic polynomial in z which is complex anti-linear in v .*

Armed with this basic fact we set $Q_z v = v - \xi_v(z)$ and linearize in z to obtain a bilinear form $Q_{z,w} = \frac{1}{2}(Q_{z+w} - Q_z - Q_w)$, a triple product $\{xyz\} = Q_{x,z}y$ and a linear transformation $D(x, y)$ given by $D(x, y)z = \{xyz\}$. Note that

$$\begin{aligned} \{xyz\} &\text{ is linear and symmetric in } x \text{ and } z \\ &\text{ and complex anti-linear in } y. \end{aligned} \tag{1}$$

We shall now derive the main identity for a Jordan triple system.

Recall that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Thus for $u, v \in V$, $[\xi_u, \xi_v]$ is a linear transformation. This linear transformation is equal to $2(D(u, v) - D(v, u))$, as a short calculation shows.

Since D is sesquilinear, we get

$$[\xi_{iu}, \xi_u] = 4iD(u, u).$$

This means that $iD(u, u) \in \mathfrak{k}$, so in particular, it is a complete vector field on the domain \mathcal{D} , i.e., $\exp itD(u, u)$ is a one parameter group of biholomorphic automorphisms of \mathcal{D} . Moreover $S_t := \exp itD(u, u) \in K$, $t \in \mathbb{R}$.

But, as shown in Loos [25: Lemma 2.11] elements of K act by automorphisms of the triple structure, i.e., for $g \in K$,

$$g(Q_x y) = Q_{gx} \cdot gy \quad \text{for } x, y \in V.$$

We now have $S_t(\{xyz\}) = \{S_t x, S_t y, S_t z\}$ and differentiating with respect to t

at $t = 0$ shows that $\delta := iD(u, u)$ is a derivation in the sense that

$$\delta \{xyz\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\}. \quad (2)$$

Now, for $u, v \in V$, we have $D(u + v, u + v) = D(u, u) + D(u, v) + D(v, u) + D(v, v)$, which shows that $i(D(u, v) + D(v, u))$ is also a derivation.

Using (2) with $\delta = i(D(u, v) + D(v, u))$ and retaining all terms linear in u we obtain the identity

$$\{iuv \{xyz\}\} = \{\{iuvx\}yz\} + \{x \{iuvy\}z\} + \{xy \{iuvw\}\},$$

i.e.,

$$\{uv \{xyz\}\} - \{xy \{vz\}\} = \{\{uvx\}yz\} - \{x \{vuy\}z\}. \quad (3)$$

Identity (3) was introduced in algebra as the defining identity for a Jordan triple system. As we have seen, (2) is equivalent to (3) and thus can be used to define a Jordan triple system in a more natural way.

A complex vector space V endowed with a triple product $\{xyz\}$ satisfying (1) and (2) is called a *Jordan triple system*. A *JB*-triple* is a Banach space V equipped with a triple product satisfying (1), (2) for which $D : U \times U \rightarrow B(U)$ is continuous,

$$\begin{aligned} D(u, u) \text{ is hermitian (in the numerical range sense)} \\ \text{with spectrum in } [0, \infty); \text{ and} \end{aligned} \quad (4)$$

$$\|D(u, u)\| = \|u\|^2, \quad \text{for all } u \text{ in } V. \quad (5)$$

An important link between infinite dimensional holomorphy and functional analysis was the observation by Harris [15], that the open unit ball of a C^* -algebra is a bounded symmetric domain. He showed that the Möbius transformations

$$A \rightarrow (I - BB^*)^{-1/2}(A + B)(I + B^*A)^{-1}(I - B^*B)^{1/2}$$

act transitively on the unit ball. A close examination of this formula shows that it involves only the symmetrized triple product $\frac{1}{2}(AB^*C + CB^*A)$ and is therefore valid for J^* -algebras. Analogues of the Möbius transformations are given by Kaup [20, 21] for arbitrary JB^* -triples, thus showing that the unit ball of a JB^* -triple is a bounded symmetric domain.

Equivalence of the two categories: bounded symmetric domains, JB^* -triples, can be used to show that the category of JB^* -triples is stable under contractive projections, as in:

THEOREM A (Kaup). *Let P be a contractive projection on a JB^* -triple U . Then*

- (a) $P(U)$ is a JB^* -triple in the triple product (a, b, c)
 $= P(\{abc\})$, for $a, b, c \in P(U)$;
- (b) $P\{PaPbPc\} = P\{PabPc\}$, for $a, b, c \in U$.

Sketch of proof. Let B denote the open ball of U and let D denote the open unit ball of $P(U)$. We know that B is a bounded symmetric domain and it suffices to show that D is a bounded symmetric domain with $v - \xi_v(z) = P\{zvz\}$ for $v \in P(U)$, and $z \in D$.

For $u \in U$, the vector field $w \rightarrow u - \{wuw\}$ is complete on B , since it equals ξ_u . Since P is contractive a short functional analysis argument shows that the vector field $Y: z \rightarrow P(u - \{zuz\})$ is complete on D . The same argument applies to the vector field $\tilde{Y}: z \rightarrow P(u - \{zPuz\})$ on D .

The complete vector fields Y and \tilde{Y} on D have the same value (namely Pu) and the same derivative (namely 0) at $z = 0$. By the infinite dimensional version of Cartan's uniqueness theorem [23], $Y \equiv \tilde{Y}$, which implies the conditional expectation formula (b). The implicit function theorem now implies that the group $\text{Aut}(D)$ acts transitively on D . Therefore D is a bounded symmetric domain and the triple product on $P(U)$ is given as in (a).

A geometric version of Theorem A was already contained in a theorem of Stachó [28] involving nonlinear projections.

A striking application of Theorem A has been given in August of 1984 by S. Dineen [8]. It is based on the following result from the theory of ultraproducts of Banach spaces [17: Prop. 6.7], called the principle of local reflexivity.

PROPOSITION. *For any Banach space E , there is a set I , an ultrafilter \mathcal{Q} on I , an isometry $J: E'' \rightarrow (E)_{\mathcal{Q}}$ from the bidual of E into the ultrapower, such that the range of J is the image of the contractive projection P on $(E)_{\mathcal{Q}}$ defined by $P(x_i)_{\mathcal{Q}} = J(w^* - \lim_{\mathcal{Q}} x_i)$.*

It is easy to show that if E is a JB^* -triple then so is any ultrapower $(E)_{\mathcal{Q}}$. Therefore Theorem A implies:

THEOREM B (Dineen). *The bidual of JB^* -triple U is a JB^* -triple containing U as a JB^* -subtriple. The triple product on U'' is given by $\{xyz\} = w^* - \lim_{\mathcal{Q}} \{Jx, Jy, Jz\}$.*

Note that if $Jx = (x_i)_{\mathcal{Q}}$ then $w^* - \lim_{\mathcal{Q}} x_i = x$ and thus $\{xyz\} = w^* - \lim_{\mathcal{Q}} \{x_i y_i z_i\}$. Barton and Timoney [5] were able to obtain the separate w^* -continuity of the triple product on U'' by refining, in a nontrivial way, the ultrafilter \mathcal{Q} . They also made use of the conditional expectation formula (b) of Theorem A. Thus we have:

THEOREM C (Barton-Timoney). *The bidual U'' of a JB^* -triple U is a JBW^* -triple, i.e., a dual Banach space which is a JB^* -triple for which the triple product is w^* -continuous in each variable.*

Horn [18: Satz (3.22)] showed that a JBW^* -triple possesses a unique predual. Hence we may speak of *the* w^* -topology without ambiguity.

The authors studied properties of JBW^* -triples in [12]. The result from [12] which we shall use here is the following theorem. A *tripotent* in a Jordan triple system is an element e with $e = \{eee\}$. The tripotent e is *minimal* if $\{eUe\} = Ce$. Elements a, b are *orthogonal* if $D(a, b) = 0$. An *ideal* in a Jordan triple system is a linear subspace which contains $\{abc\}$ whenever it contains either a or b .

THEOREM D (Friedman–Russo). *Every JBW^* -triple U with predual U_* decomposes into an orthogonal ℓ^∞ -direct sum of w^* -closed ideals A and N , where A is the w^* -closure of the linear span of its minimal tripotents, and N has no minimal tripotents. Moreover $N = \{x \in U : g(x) = 0 \text{ for all extreme points } g \text{ of the unit ball of } U_*\}$.*

Recall that there is no order structure on a JB^* -triple. Therefore a decomposition into atomic and nonatomic parts requires new techniques in this nonordered setting. The main tools needed are: symmetry of transition probabilities, [12: Lemma 2.2], Hilbert ball property [12: Prop. 5], extreme ray property [12: Prop. 7]. These properties are analogues of properties established for the state space of a Jordan algebra by Alfsen–Shultz [2]. By contrast the corresponding result for JBW -algebras is elementary once you know that the set of projections forms a complete lattice.

The final recent result that we shall use in our proof of Theorem 1 is the following corollary to Horn’s classification theorem [18: Kor. 9.1.2].

A JBW^* -triple U is said to be a *factor* if it cannot be written as an ℓ^∞ -direct sum of two nonzero ideals, or equivalently [18: Kor. 5.3] if 0 and U are the only w^* -closed ideals in U . If furthermore U contains a minimal tripotent, then U is a JBW^* -triple factor of type I (cf. [18: Satz 5.9]).

THEOREM E (Horn). *Every JBW^* -triple factor of type I is isomorphic to a Cartan factor of type k , $1 \leq k \leq 6$.*

§2. Main results. The following two propositions, together with Theorem E will yield a proof of Theorem 1.

A JBW^* -triple is *atomic* if it equals the w^* -closure of the linear span of its minimal tripotents.

PROPOSITION 1. *Every JB^* -triple has an isometric representation into an atomic JBW^* -triple.*

Proof. Let U be a JB^* -triple and let $U'' = A \oplus^\infty N$ be the decomposition of the JBW^* -triple U'' into atomic and nonatomic ideals. Let $\pi : U \rightarrow U''$ be the canonical injection and let $\sigma : U'' \rightarrow A$ be the canonical projection. The $\sigma \circ \pi$ is a norm decreasing homomorphism of U into A . To show that $\sigma \circ \pi$ is norm preserving let $x \in U$, $\|x\| = 1$, and let g be an extreme point of the nonempty convex w^* -compact set $\{f \in U' : \|f\| = 1 = f(x)\}$. It follows that g is an extreme point of the unit ball of U' and therefore $g(N) = 0$. Thus $1 = \|x\| = \|\pi(x)\| \geq \|\sigma(\pi(x))\| \geq |g(\sigma(\pi(x)))| = |g(\pi(x))| = |g(x)| = 1$. Q.E.D.

PROPOSITION 2. *Every atomic JBW^* -triple U is an ℓ^∞ -direct sum of JBW^* -triple factors of type I.*

Proof. Let e be a minimal tripotent of U . Then the intersection J_e of all w^* -closed ideals which contain e is a w^* -closed ideal. If J_e is not a factor it decomposes into 2 nonzero orthogonal ideals, resulting in a decomposition of e

into two orthogonal elements, which must be tripotents: $e = e_1 + e_2$. By minimality of e , $e = e_1$ or $e = e_2$, contradiction. Thus each minimal tripotent belongs to a factor subtriple of U .

Since each w^* -closed ideal in U is an M -summand [18: Satz 5.2], for any two minimal tripotents e_1, e_2 either $J_{e_1} = J_{e_2}$ or J_{e_1} is orthogonal to J_{e_2} .

By Zorn's lemma $U = \bigoplus_{\alpha}^{\infty} J_{\alpha}$ where $\{J_{\alpha}\}$ is an orthogonal family of factor subtriples of U . Since U is atomic, each J_{α} is of type I. Q.E.D.

By combining Propositions 1 and 2 and Theorem E we have:

THEOREM 1. *Every JB^* -triple is isometrically isomorphic to a subtriple of an ℓ^{∞} -direct sum of Cartan factors.*

Our first application of Theorem 1 is an alternative form of the embedding, which is analogous to [6: (1.11a)]. It follows since the Cartan factors of types 1–4 can be embedded into some $B(H)$ and the Cartan factors of type 5–6 can be embedded into $\mathcal{C}(S, C^6)$ where S is the Stone–Cech compactification of a discrete set.

COROLLARY 1. *Every JB^* -triple is isometrically isomorphic to a subtriple of*

$$B(H) \oplus \mathcal{C}(S, C^6)$$

As noted in §1, every Cartan factor can be considered as a subtriple of a JB^* -algebra. This fact together with Theorem 1 gives the following corollary which answers a question posed by Kaup [21: (5.10)].

COROLLARY 2. *Every JB^* -triple is isomorphic to a subtriple of a JB^* -algebra.*

In the terminology of Loos–McCrimmon, [26], Corollary 2 says that every JB^* -triple is j^* -special. Note that, on the algebra side, it is not known if every Jordan triple system is j^* -special.

The following Corollary also answers a question of Kaup [21: (5.9)]. It follows from Theorem 1 since its conclusion is known to be valid in any Cartan factor.

COROLLARY 3. *In a JB^* -triple,*

$$\|\{xyz\}\| \leq \|x\| \|y\| \|z\|.$$

In [4] Araki–Elliott proved that the axiom $\|xy\| \leq \|x\| \|y\|$ in a C^* -algebra is redundant, and it was noted in [1] that the proof of Araki–Elliott is valid in a JB -algebra. Corollary 3 is the analogue of these results for JB^* -triples.

The next applications of Theorem 1 deal with questions of speciality, i.e., of representing JB^* -triples as J^* -algebras. We shall first give a proof of a recent result of Barton–Timoney [5: Th. 3.6].

A *representation* of a JB^* -triple is a homomorphism into another JB^* -triple. A *factor representation* is a homomorphism with w^* -dense range on a JB^* -triple into a JBW^* -triple factor.

COROLLARY 4. *Every Cartan factor occurring in the proof of Theorem 1 is the w^* -closure of the range of a type I factor representation. Hence every JB^* -triple has a separating family of factor representations.*

Proof. Write $U'' = (\bigoplus_{\alpha}^{\infty} C_{\alpha}) \oplus^{\infty} N$ and let P_{α} be the natural projection of U'' onto C_{α} . Since U is w^* -dense in U'' , $\pi_{\alpha} := P_{\alpha} \circ \pi$ is a representation of U with $\pi_{\alpha}(U)$ w^* -dense in C_{α} . Q.E.D.

In Loos–McCrimmon [26], it is shown that the Cartan factors of types 5 and 6 are not isomorphic to J^* -algebras. They also introduce an analogue of Glennie's identity, namely

$$f(x, y, z, w) = \{ Q_x z, w, Q_{x,y} Q_z Q_y w \} - Q_x Q_z Q_{x,y} Q_w Q_y z \\ - \{ Q_y z, w, Q_{y,x} Q_z Q_x w \} + Q_y Q_z Q_{y,x} Q_w Q_x z. \quad (6)$$

and they showed that $f(x, y, z, w) = 0$ for all x, y, z, w in a J^* -algebra, but $f(x, y, z, w)$ can be nonzero for some choice of elements x, y, z, w in the Cartan factor C^5 .

From this it follows that there is no homomorphism of a J^* -algebra onto C^5 or C^6 . Thus we have from Corollary 4:

COROLLARY 5. *A JB^* -triple U is isomorphic to a J^* -algebra if and only if f is identically zero in U , where f is the analogue of Glennie's identity defined by (6).*

In [16], Harris asked whether the quotient of a J^* -algebra M by a closed ideal J is a J^* -algebra. By Corollary 5 the answer is yes, since M/J is a JB^* -triple [21: p. 523], for which $f = 0$.

Recall that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB^* -triple. It is known (cf. [20]) that if U is a JB^* -subtriple of a JB^* -triple V , then the symmetry at a point a of the unit ball U_0 is the restriction of the corresponding symmetry at a with respect to V_0 . Thus Theorem 1 implies:

COROLLARY 6. *Every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to a subdomain of an ℓ^{∞} -direct sum of Cartan domains of types 1–6 in such a way that the symmetry s_a at each point is the restriction of a direct sum of symmetries at points of Cartan domains.*

We now give an alternative form of Theorem 1 analogous to [3: Th. 9.5].

THEOREM 2. *Every JB^* -triple U contains a unique norm closed ideal J such that U/J is isomorphic to a J^* -algebra and J is purely exceptional in the sense that every homomorphism of J into a J^* -algebra is zero.*

Proof. Let f denote the generalized Glennie identity (6) and let J be the closed ideal generated by $\{f(a, b, c, d) : a, b, c, d \in U\}$. By Corollary 5 U/J is isomorphic to a J^* -algebra.

To show that J is purely exceptional let $\varphi : J \rightarrow M$ be a homomorphism into a J^* -algebra M . Then φ'' is a homomorphism of J'' into M'' . By Horn [18: Satz 5.2] there is a norm one projection p of U'' onto J'' which is a homomorphism. Therefore $\varphi := \varphi'' \circ p \circ \pi$, where $\pi : U \rightarrow U''$ is the canonical map, is a homomorphism of U into M'' which extends φ . Since M'' is a J^* -algebra [9: Prop. 2.1], $0 = f(\overline{\varphi}(a), \overline{\varphi}(b), \overline{\varphi}(c), \overline{\varphi}(d)) = \overline{\varphi}(f(a, b, c, d)) = \varphi(f(a, b, c, d))$ i.e., φ is zero on J , as required.

Suppose K is another purely exceptional ideal with U/K a J^* -algebra. The homomorphism $U \rightarrow U/K$ has kernel K and since U/K is a J^* -algebra, this homomorphism must annihilate J . Therefore $J \subset K$.

Since U/J is a J^* -algebra, so is $K/J \subset U/J$. Because K is purely exceptional the homomorphism $K \rightarrow K/J$ must be zero, i.e., $K \cap J = K$ or $K \subset J$. Q.E.D.

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