Title
ULTRAHOMOGENEOUS AND EXISTENTIALLY CLOSED HEYTING ALGEBRAS

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Abstract. Aspects of ultrahomogeneous and existentially closed Heyting algebras are studied. The isomorphism type, non-simplicity, and non-amenability of the automorphism group of the Fraïssé limit of finite Heyting algebras are examined among others.

In these notes, we examine countable ultrahomogeneous existentially closed (e.c.) Heyting algebras. The existence of model-completion T^* of the theory T of Heyting algebra [7] is of interest in its relation to second-order intuitionistic propositional logic. Countable ultrahomogeneous Heyting algebras are paradigmatic models of T^*, one of them being its prime model.

In the first section, we see that there are uncountably many countable ultrahomogeneous e.c. Heyting algebras. The remainder of the article concerns the prime model L of T^*. In the second section, we study the countable atomless Boolean algebra definable in L. In the last section, we look at the automorphism group of L with the Kechris-Pestov-Todorcević correspondence in mind, where it will be proved *inter alia* that $\mathrm{Aut}(L)$ is not amenable. In the Appendix, we study issues related to the axiomatization of T^*.

It is an important future task to investigate the combinatorics of the age $\mathrm{Age}(L)$ of L, in particular about the existence of order expansion of $\mathrm{Age}(L)$ with the Ramsey property and the ordering property, and the metrizability of $\mathrm{Aut}(L)$.

Preliminaries. Let T be the theory of Heyting algebras. The model completion T^* of T exists. It is axiomatized by

$$T \cup \{U(\theta' \to \theta) \mid \theta \text{ existential}\} \tag{1}$$

where U denotes universal closure, θ' is a quantifier-free formula such that $T \models U(\theta \to \theta')$, and $T + U(\theta' \to \theta)$ is a conservative extension with respect to the universal formulas (θ' is the result of applying the QE algorithm in [7] to θ).

We will study an ultrahomogeneous model L of T^* later in this article. Neither T nor T^* is locally finite, but L is. However, L is not uniformly locally finite, so T^* is not $\aleph_0$-categorical. T^* is not uncountably categorical either because of its instability. The Fraïssé limit L is the prime model of T^*. In many cases, the Fraïssé limit of a class of finite structures is pseudofinite. However, this is not the case for the complete theory $T^* + (0 \neq 1)$; there is a sentence ϕ implied by the theory that is not satisfied by any finite structure of the same signature. Indeed, take ϕ to be the conjunction of the density of the partial order (see Ghilardi and Zawadowski [7, Proposition 4.28]), $0 \neq 1$, and $\bigwedge T$.

We review an important construction of Heyting algebras (this material appears in, e.g., Chagrov and Zakharyaschev [3]). For an arbitrary poset $\mathbb{P}$, the poset of upward closed sets, or *up-sets*, of $\mathbb{P}$ ordered by inclusion has a Heyting algebra structure. We call this Heyting algebra is *dual* of $\mathbb{P}$. Conversely, if L is a *finite*

Heyting algebra, then one can associate with L the poset $\mathbb{P}$ of join-prime elements of L with the reversed order. One can show that the dual of $\mathbb{P}$ is isomorphic to L.

Suppose that L and L' are the duals of $\mathbb{P}$ and $\mathbb{P}'$, respectively, and that $f : \mathbb{P} \to \mathbb{P}'$ is *p-morphic*, i.e., f is monotonic with

$$\forall u \in \mathbb{P}\, \forall v \geq f(u)\, \exists w \geq u\, f(w) = u,$$

then the function f^* defined on L' that maps each up-set with its inverse image under f is a Heyting algebra homomorphism $L' \to L$. We call f^* the *dual* of f as well. If f is injective, then f^* is surjective; if f is surjective, then f^* is a Heyting algebra embedding.

1. Countable Ultrahomogeneous Heyting algebras

The model completion T^* is the theory of the Fraïssé limit L of finite nontrivial Heyting algebras, which exists [7]. The strong amalgamation property of T was proved by Maksimova [13]; in fact, her construction establishes the *superamalgamation property* for the class of finite Heyting algebras. Recall that a Fraïssé class $\mathcal{K}$ of poset expansions has the superamalgamation property if for every diagram $A_1 \hookleftarrow A_0 \hookrightarrow A_2$ of inclusion maps in $\mathcal{K}$, the amalgamation property of $\mathcal{K}$ is witnessed by a diagram $A_1 \hookrightarrow A \hookleftarrow A_2$ of inclusion maps in such a way that $A_1 \downarrow_{A_0} A_2$, where $\downarrow$ is the ternary relations for subsets of A defined as:

$$S \underset{T}{\downarrow} U \iff \forall a \in S\, \forall b \in T \begin{Bmatrix} a \leq b & \Longrightarrow & \exists c \in U\, a \leq c \leq b \\ b \leq a & \Longrightarrow & \exists c \in U\, b \leq c \leq a \end{Bmatrix}.$$

For the sake of completeness, we directly show the superamalgamation property for the class $\mathcal{K}$ of finite Heyting algebras. Let $A_1 \hookleftarrow A_0 \hookrightarrow A_2$ be a diagram of inclusions in $\mathcal{K}$. Consider the dual $\mathbb{P}_i$ of A_i $(i = 0, 1, 2)$. Consider $\mathbb{P} := \{(p_1, p_2) \in \mathbb{P}_1 \times \mathbb{P}_2 \mid p_1 \sim p_2\}$, where $p \sim p'$ if and only if $p \leq c \iff p' \leq c$ for every $c \in \mathbb{P}_0$. One can show that (the restriction of) the projection $\pi_i : \mathbb{P} \twoheadrightarrow \mathbb{P}_i$ is a p-morphism for $i = 1, 2$ and that π_i induces an embedding of A_i into a finite Heyting algebra A. Choose a copy of A so that $A_i \subseteq A$. It suffices to show that if $a_1 \leq^A a_2$ for $a_i \in A_i$, there exists $a_0 \in A_0$ such that $a_1 \leq^A a_0 \leq^A a_2$ (We shall omit superscripts from $\leq^A$ henceforth). Let $a_0 = \bigwedge\{a \in A_0 \mid a_1 \leq a_0\} \in A_0$. We claim that $a_1 \leq a_0 \leq a_2$. It suffices to show that for every $p_2 \leq a_0$ prime in A_2, we have that $p_2 \leq a_2$. We prove this by showing the existence of $p_1 \leq a_1$ prime in A_1 such that $p_1 \sim p_2$; then, since $(p_1, p_2) \in \mathbb{P}$, and $(p_1, p_2) \in \pi_1^*(a_1) \subseteq \pi_2^*(a_2)$, we would conclude that $p_2 \leq a_2$. Let $c := \bigwedge\{a \in A_0 \mid p_1 \leq a\}$, and consider the elements below $a_1 \wedge c$ prime in A_1. Assume by way of contradiction that all of them are bounded from above by some element in $\{a \in A_0 \mid p_1 \not\leq a\}$. Then, so is $a_1 \wedge c$, i.e., $a_1 \wedge c \leq d$ for some $d \in A_0$ with $p_1 \not\leq d$. Now, we have $a_1 \leq c \to d \in A_0$. By the definition of a_0, this means that $a_0 \leq c \to d$. Since $p_1 \leq c \to d$, and $p_1 \leq c$, we have $p_1 \leq c \wedge (c \to d) \leq d$, which is a contradiction. Hence, one can find a $p_1 \leq a_1$ prime in A_1 such that $p_1 \sim p_2$.

We introduce notation naming structures obtained by the superamalgamation property: Let D be the diagram $B \hookleftarrow A \hookrightarrow C$ in $\mathrm{Age}(L)$, where $\mathrm{Age}(L)$ the age of L is regarded as a category whose morphisms are the embeddings. The superamalgamation property for $\mathrm{Age}(L)$ gives rise to a subalgebra $\bigsqcup D$ of L such that there are embeddings $\iota^D_{\hookleftarrow} : B \hookrightarrow \bigsqcup D$ and $\iota^D_{\hookrightarrow} : C \hookrightarrow \bigsqcup D$ with $\iota^D_{\hookleftarrow}(B) \downarrow_{\iota^D_{\hookrightarrow}(A)} \iota^D_{\hookrightarrow}(C)$. One can show that $\iota^D_{\hookleftarrow}(B) \setminus \iota^D_{\hookleftarrow}(A)$ and $\iota^D_{\hookrightarrow}(C) \setminus \iota^D_{\hookrightarrow}(A)$ are disjoint.

The following is a model-theoretic argument that L is e.c.:

Proof. Consider a quantifier-free formula $\phi_0(x, y)$ and a tuple $\overline{a} \in L$. Note that $\langle \overline{a} \rangle^L$ is finite by the construction of L, so there is a quantifier-free formula $\psi(\overline{y})$ such that for any Heyting algebra L'' and $b \in L$, we have

$$L'' \models \psi(\overline{b}) \iff \langle \overline{b} \rangle^{L''} \cong \langle \overline{a} \rangle^L.$$

Now suppose that there is $L' \supset L$ such that $L' \models \exists x\, \phi_0(x, \overline{a})$. This implies the formula

$$\exists x\, \exists \overline{y} [\phi_0(x, y) \wedge \psi(\overline{y})] \tag{2}$$

is satisfiable over T. By the extended form of the finite model property for T that works for equations as well as inequations [4], there is a finite Heyting algebra L_0 satisfying (2). By construction, without loss of generality $L_0 \subseteq L$. Let $\xi, \overline{b} \in L_0$ be the witness to $\exists x$, $\exists \overline{y}$, resp. The isomorphism $\langle \overline{b} \rangle^L \to \langle \overline{a} \rangle^L$ induces another $i : L \to L$ by ultrahomogeneity. It follows that $i(\xi)$ solves the formula $\phi_0(x, \overline{a})$ in L. □

Fact 1.1. There are continuum many finitely generated Heyting algebras up to isomorphism.

This fact is probably well known, but a proof is included for the sake of completeness.

Proof. Regard the chain $\omega+1$ as a Heyting algebra. Note that every order-preserving injection $\omega + 1 \to \omega + 1$ is a Heyting algebra embedding. Since such functions are in bijective correspondence with strings in ω^ω, there are continuum many of them. Now consider the free Heyting algebra F with one generator. Construct an embedding $\iota : (\omega + 1) \to F$ recursively as follows: let $\iota(0) = 0$ and $\iota(\omega) = 1$; having defined $\iota(n)$ for $n < \omega$, define $\iota(n + 1)$ to be the join of the two successors of $\iota(n)$. It can be checked directly that ι is a Heyting algebra embedding. By Abogatma and Truss [1, Theorem 2.4], we conclude that there are continuum many finitely generated Heyting algebras up to isomorphism. □

Proposition 1.2. Let $\mathcal{K}$ be an inductive class of finitely generated structures with the amalgamation property, and let $A \in \mathcal{K}$. There exists an ultrahomogeneous structure $A^\sharp \in \mathcal{K}$ that is existentially closed in $\mathcal{K}$ and extends A.

Proof. We construct $A^\sharp$ as the union of an ω-chain $A_0 \subseteq A_1 \subseteq \cdots$ of structures in $\mathcal{K}$. Let $A_0 = A$. Fix a bijection $\pi : \omega \times \omega \to \omega$ such that $\pi(i, k) < i$ for $i, k < \omega$.

Having A_i constructed, we extend A_i to A_{i+1} as follows:

- **Case $i = 2i'$:** Apply the well-known construction to A_i to obtain A_{i+1} so that $A_{i+1} \models \phi(\bar{a})$ whenever $\phi(\bar{x})$ is an existential formula, $\bar{a}$ is in A_i, and there exists $C \in \mathcal{K}$ such that $A_i \subseteq C$ and that $C \models \phi(\bar{a})$.
- **Case $i = 2i' + 1$:** We do the construction in the proof of Abogatma and Truss [1, Lemma 2.3], which is included for the sake of completeness. There are at most countably many partial isomorphisms of A_i, i.e., isomorphisms between substructures of A_i; enumerate them as $(\varphi_{ik})_{k<\omega}$. Take (j, k) such that $\pi(j, k) = i'$. Let A'_{i+1} be the structure in $\mathcal{K}$ witnessing the amalgamation property for the diagram

$$A_i \overset{\iota_1}{\hookleftarrow} \operatorname{dom} \varphi_{jk} \overset{\iota_2 \circ \varphi_{jk}}{\hookrightarrow} A_i,$$

where ι_1, ι_2 are the inclusion maps of the correct types. Replace A_{i+1} with an isomorphic copy if need be so that $A_i \subseteq A_{i+1}$. Note that φ_{jk} is extended to a partial isomorphism $\tilde{\varphi}_{jk}$ of A'_{i+1}, where $\operatorname{dom} \tilde{\varphi}'_{jk} = A_i$. One can use a similar construction to obtain A_{i+1} with a partial isomorphism $\tilde{\varphi}_{jk}$ extending $\tilde{\varphi}'_{jk}$ such that $A_i \subseteq \operatorname{ran} \tilde{\varphi}_{jk}$.

That $A^\sharp$ is e.c. in $\mathcal{K}$ can be proved as usual. Let $\varphi : B \to C$ be an isomorphism where B, C are finitely generated substructures of $A^\sharp$. Let $j < \omega$ be such that the finitely many generators of B and C are contained in A_j; in fact, we have $B, C \subseteq A_j$. Take $k < \omega$ so that $\varphi = \varphi_{jk}$. By the construction of A_{i+1} from A_i, where $i = 2\pi(j,k) + 1$, φ is extended by a partial automorphism $\tilde{\varphi}_{jk}$ of $A^\sharp$, where $\operatorname{dom} \tilde{\varphi}_{jk} \cap \operatorname{ran} \tilde{\varphi}_{jk} \supseteq A_i$. Note that $i > j$. By repeating this, one obtains a chain $\varphi = \varphi_0 \subsetneq \varphi_1 \subsetneq \cdots$, where $\operatorname{dom} \varphi_m < \operatorname{dom} \varphi_n$ whenever $m < n$, so the union $\bigcup_{n<\omega} \varphi$ has the domain $A^\sharp$, which is evidently an isomorphism $A^\sharp \to A^\sharp$. We have seen that $A^\sharp$ is ultrahomogeneous. □

Corollary 1.3. There are continuum many countable ultrahomogeneous e.c. Heyting algebras.

Proof. This follows immediately from the preceding propositions as a single countable ultrahomogeneous e.c. Heyting algebra has at most countable substructures up to isomorphism. □

2. Definable Countable Atomless Boolean Algebras

In the next section where we study the topological group of automorphisms of L, first-order interpretations of B in L would be useful. Of course, the countable atomless Boolean algebra embeds in L by the weak homogeneity of L. However:

Proposition 2.1. No substructure of L that is a countable atomless Boolean algebra is a relativized reduct.

Proof. We show that for any countable atomless Boolean algebra $B \subseteq L$ there are an automorphism σ of L over $\overline{a}$ and a distinct countable atomless Boolean algebra $B' \subseteq L$ such that $\sigma(B) = B'$ setwise. (Then the domain of B will be seen to be undefinable.)

Recall that B is the union of an ω-chain $B_0 \subseteq B_1 \subseteq \dots$ of finite Boolean algebras. We construct an ω-sequence $A_0, A_1, \dots$ of finite Boolean algebras that are subalgebras of L and an ω-chain $B'_0 \subseteq B'_1 \subseteq \dots$ such that $B_k, B'_k \subseteq A_k$, that $B_k \cong B'_k$, and that $B_k \neq B'_k$.

Let D_0 be the diagram $B_0 \hookleftarrow \mathbf{2} \hookrightarrow B_0$. Let $A'_0 = \bigsqcup D$. By the weak homogeneity of L, there is an embedding $i_0 : A'_0 \to L$ such that $(i_0 \circ \iota^D_{\hookleftarrow}) \restriction B_0$ is the identity. Let A_0 be $\operatorname{ran} i_0$ and B'_0 be $\operatorname{ran}(i_0 \circ \iota^D_{\hookrightarrow})$.

Having A_k and B'_k defined, we define A_{k+1} and B'_{k+1} as follows. the diagram $A_k \hookleftarrow B'_k \hookrightarrow B_{i+1}$. Let $\tilde{D}_{k+1}$ be the diagram $A_k \hookleftarrow B_i \hookrightarrow B_{i+1}$, and D_{k+1} be $B_{i+1} \hookleftarrow B_i \hookrightarrow \bigsqcup \tilde{D}_{k+1}$. By appealing to the weak homogeneity of L as before, take an embedding $i_{i+1} : \bigsqcup D_{k+1} \hookrightarrow L$ so that $B'_{i+1} := \operatorname{ran}(i_{i+1} \circ \iota^{D_{k+1}}_{\hookrightarrow} \circ \iota^{\tilde{D}_{k+1}}_{\hookrightarrow})$ extends B'_i and that $B_{i+1} = \operatorname{ran}(i_{i+1} \circ \iota^{D_{k+1}}_{\hookleftarrow})$. Finally, let $A_{k+1} = \operatorname{ran} i_{i+1}$.

By construction and by the ultrahomogeneity of L, the two substructures B_k and B'_k are conjugate under an automorphism of L. Let B and B' be the unions of B_k's and B'_k's, respectively. Then B and B' are conjugate under an automorphism of L, and $B \neq B'$. □

If we drop the requirement that a copy of B in L be a subalgebra of L, we do obtain a natural interpretation as follows:

Proposition 2.2. There is an atomless Boolean algebra which is a relativized reduct of L.

Proof. The set B of fixed points of $1-(1-\cdot)$ in L is a Boolean algebra by setting $a \wedge^B b = \neg\neg(a \wedge^L b)$ and the remaining operations of B the restrictions of the corresponding operations of L. (Note that B is not a substructure of L.)

Suppose that $a \in B$ is an atom of B. We show that a is also an atom of L. To see this, assume the contrary, and let b be such that $0 < b < a$, where $b \notin B$. Since $b \notin B$, we have $1-(1-b) \neq b$; since $1-(1-c) \leq c$ for all $c \in B$, we have $1-(1-b) < b$. Now $1-(1-b) \in B$ and $0 < 1-(1-b)$ (since $1-b < 1$), so we have $0 < 1-(1-b) < a$, contradicting the assumption that a is an atom of B.

We have seen that any atom in B is an atom of L. Since there is no join-irreducible elements (let alone atoms) in L [7, Proposition 4.28.(iii)], B is atomless. □

3. Automorphism Group

In this last section, we look at the automorphism group of L with the Kechris-Pestov-Todorcević correspondence in mind.

Recall that the extreme amenability of topological groups are of interest only if they are not locally compact [9]. It is well known that $\mathrm{Aut}(M)$ for a countable ω-categorical M is not locally compact [12]. Even though L is not ω-categorical, we can show the following.

Proposition 3.1. The topological group $\mathrm{Aut}(L)$ is not locally compact.

Proof. It suffices to show that for every finite subset $S \subseteq L$ there is an infinite orbit in $\mathrm{Aut}(L)_{(S)} \curvearrowright L$. Note that for every finite subalgebra $A \subseteq L$, there exists $a \in L \setminus A$ such that a is join-prime in $\langle Aa \rangle^L$. By repeatedly using this, take an ω-sequence $(a_i)_{i<\omega}$ of elements of L such that $a_i \in L \setminus \langle Sa_0a_1 \dots a_{i-1} \rangle^L$ is join-prime in $\langle Sa_0a_1 \dots a_i \rangle^L$ for $i < \omega$. By construction, there exists an automorphism $\phi_i : L \to L$ fixing S pointwise such that $\phi_i(a_i) = a_{i+1}$ for $i < \omega$. Hence, the orbit of a_0 under $\mathrm{Aut}(L)_{(S)}$ is infinite. □

An obvious strategy to study $\mathrm{Aut}(L)$ is to relate it to $\mathrm{Aut}(B)$, where B is the countable atomless Boolean algebra. The following lemma gives rise to a topological embedding of the former into the latter.

Lemma 3.2.

(1) Let $f : H \to H_1$ be a Heyting algebra homomorphism between finite algebras. There are finite Boolean algebras $B(H)$ and $B(H_1)$ and a Boolean algebra homomorphism $B(f) : B(H) \to B(H_1)$. There are interior operators $^\circ, ^{\circ_1}$ on $B(H), B(H_1)$ such that $B(H)^\circ \cong H$ and $B(H_1)^{\circ_1} \cong H_1$. If f is injective, so is $B(f)$; if f is surjective, so is $B(f)$.

(2) There is an interior operator $^\circ$ on the countable atomless Boolean algebra B such that B° is isomorphic to the universal ultrahomogeneous countable Heyting algebra L.

(3) An automorphism $L \to L$ can be extended (as a function between pure sets) to another $B \to B$. Moreover, there is an embedding $\mathrm{Aut}(L) \hookrightarrow \mathrm{Aut}(B)$ that is a homeomorphism onto its image.

Proof.

(1) Let P and P' be the dual posets of H and H_1, respectively. There is a p-morphism $D(f) : P_1 \to P$ that is the dual of f. $D(f)$ is surjective if f is injective. Let $B(H) = \mathscr{P}(P)$ and $B(H_1) = \mathscr{P}(P_1)$. $D(f)$ induces a Boolean algebra homomorphism $B(f) : B(H) \to B(H_1)$. $B(f)$ is injective if $D(f)$ is surjective. Likewise, $B(f)$ is surjective if f is. Let $^\circ, ^{\circ_1}$ be the operations that take a subset to the maximal up-set contained by that set.

(2) Let $(L_i)_{i<\omega}$ be a chain of finite Heyting algebras used in the construction of L; so $\bigcup_i L_i = L$. Let $B_i = B(L_i)$ and $^{\circ_i}$ be an interior operator such that ${B_i}^{\circ_i} \cong L_i$. We may take $B_i \subseteq B_{i+1}$ for $i < \omega$. Then $^{\circ_{i+1}}$ extends $^{\circ_i}$. Let $B = \bigcup_i B_i$ and $^\circ = \bigcup {}^{\circ_i}$. Then $B^\circ = (\bigcup_i B_i)^\circ = \bigcup_i {B_i}^{\circ_i} = \bigcup_i H_i = H$. It remains to show that B is atomless. Take an arbitrary $a \in B$ that is nonzero. Take $i < \omega$ such that $a \in B_i$. Let P_i be the poset dual to L_i; then a is a nonempty subset of P_i. Take some $w \in a$. Let P' be the poset obtained from P_i by replacing w with the 2-chain $\{w_1 < w_2\}$. Let $\pi : P' \twoheadrightarrow P_i$ be the surjection that maps the chain to $\{w\}$ and is the identity elsewhere. This is a p-morphism, and it induces $\iota : L_i \hookrightarrow L'$, where L' is the dual of P'. Take $k < \omega$ such that there is an embedding $\iota' : L' \hookrightarrow L_k$ such that $\iota' \circ \iota$ is the identity on L_i. Write L' for that image of L'. Let $b = (a \setminus \{w\}) \cup \{w_1\}$. Then $b \in B_k = B(L_k) \subseteq B$ and $0 < b < a$.

(3) Let $f : L \to L$ be an automorphism. Let $f_k : L_k \to L'_k$ be the restriction of f to L_k where $L'_k = f(L_k)$. Each f_k is an automorphism. By the fact above, f_k induces a Boolean algebra automorphism $B(f_k) : B(L_k) \to B(L'_k)$ for each $k < \omega$; and by construction $B(f_j)$ extends $B(f_k)$ for each $k < j < \omega$. Let $\hat{f} = \bigcup_k B(f_k)$. Then $\hat{f}$ is an isomorphism $B \to B$.

Let $g : L \to L$ be another automorphism. We need to show $\hat{f} \circ \hat{g} = (f \circ g)\hat{}$. Let $a \in B$ be arbitrary. It suffices to show that $\hat{f}(\hat{g}(a)) = (f \circ g)\hat{}(a)$. Take $i < \omega$ such that $g(a), a \in B_i = B(L_i)$. Then $(f \circ g)\hat{}(a) = B((f \circ g)|_{L_k})(a) = B(f_k)(B(g_k)(a)) = \hat{f}(\hat{g}(a))$.

Let $\iota : \mathrm{Aut}(L) \to \mathrm{Aut}(B)$ be the map $f \mapsto \hat{f}$. The map ι is a group homomorphism as seen above, and it is clearly injective.

Next, we show that ι is continuous. Let $\bar{b}$ be a tuple in B. It suffices to show that for an automorphism $f : L \to L$ the value of $\hat{f}(\bar{b})$ is determined by the value of $f(\bar{a})$ for a tuple $\bar{a}$ in L. There exists $k < \omega$ such that $\bar{b}$ is in $B_k = B(L_k)$. Let $f_k : L_k \to L'_k$ be an automorphism that is a restriction of f. Then $\hat{f}(\bar{b}) = B(f_k)(\bar{b})$. Let $\bar{a}$ be an enumeration of the finite algebra L_k; then $\bar{a}$ is what we needed.

Finally, we show that the image $\iota(U)$ is open in $\operatorname{ran}\iota \subseteq \mathrm{Aut}(B)$ for an arbitrary basic open set U of $\mathrm{Aut}(L)$. Indeed, let U be the set of $f : L \to L$ fixing the values of f at $\bar{a} \in L$; then $\hat{g} \in \iota(U)$ in and only if $\hat{g} \restriction B_0 = \hat{f} \restriction B_0$ for $g : L \to L$, where B_0 is the Boolean subalgebra of B generated by $\bar{a}$. □

Note that despite $L \subseteq B$, the structure L is not interpretable in B because the latter is $\aleph_0$-categorical whereas the former is not.

There is another way $\mathrm{Aut}(B)$ and $\mathrm{Aut}(L)$ can be related. Recall the interpretation of B in L from Proposition 2.2, and let $h_{\neg\neg} : \mathrm{Aut}(L) \to \mathrm{Aut}(B)$ be the continuous homomorphism that it induces.

Lemma 3.3. Consider the copy of B as a relativized reduct of L as before. Every element L is a finite join of elements of B.

Proof. Let $a \in L$ be arbitrary. Take a finite subalgebra $H \subseteq L$ so $a \in H$, and let $\mathbb{P}$ be the dual poset of H so we may identify an element of H with an up-set of $\mathbb{P}$. Possibly by replacing L by another finite Heyting algebra into which L embeds, we may assume that $\mathbb{P}$ is a forest. Furthermore, without loss of generality, we may assume that a is principal. Suppose that a is generated by $x \in \mathbb{P}$. If x is a root, then a itself is regular, so there remains nothing to show. Suppose not, and let x^- be the predecessor of x. Let $\mathbb{P}_1, \mathbb{P}_2$ be disjoint posets isomorphic to that induced by $a \subseteq \mathbb{P}$. Let $\mathbb{P}' := (\mathbb{P} \setminus a) \sqcup \mathbb{P}_1 \sqcup \mathbb{P}_2$ whose partial order is the least containing those of the summands and $x^- \leq \mathbb{P}_1$, $x^- \leq \mathbb{P}_2$. Consider the surjective p-morphism $\mathbb{P}' \twoheadrightarrow \mathbb{P}$ that collapses $\{\min \mathbb{P}_1, \min \mathbb{P}_2\}$ into x, and let $i : H \hookrightarrow H'$ be the Heyting algebra embedding it induces. Note that $\mathbb{P}_i \in H'$ is regular for $i = 1, 2$ and that $i(a) = \mathbb{P}_1 \vee \mathbb{P}_2$. Let $\phi : H' \to H_r(a)$ be an isomorphism such that $H_r(a)$ is a subalgebra of L and $\phi \restriction H$ is the identity. Let $r_1(a) := \phi(\mathbb{P}_1)$ and $r_2(a) := \phi(\mathbb{P}_2)$. We have $a = r_1(a) \vee r_2(a)$ and $r_i(a) \in B$ $(i = 1, 2)$ as promised. □

Proposition 3.4. The continuous homomorphism $h_{\neg\neg}$ is injective and is a homeomorphism onto its image. However, $h_{\neg\neg}$ is not surjective, and its image is a non-dense non-open set.

Proof. The first claim is immediate. We show that $h_{\neg\neg}$ is not surjective.

Consider the 3-element chain C_3, which can be regarded as a Heyting algebra, and let $a \in C_3$ be such that $0 < a < 1$. Note that a is irregular and a principal up-set in the dual finite poset of C_3. Let D be the diagram $C_3 \hookleftarrow \mathbf{2} \hookrightarrow C_3$, where $\mathbf{2}$ is the 2-element Heyting algebra. Let $a_0 = \iota^D_{\hookleftarrow}(a)$, $a_{1.5} = \iota^D_{\hookrightarrow}(a)$, and $H = H_r(a_{1.5})$. Next, let D' be the diagram $H \hookleftarrow \iota^D_{\hookleftarrow}(C_3) \hookrightarrow H$. Let $a_{1i} = \iota^{D'}_{\hookleftarrow}(r_i(a_{1.5}))$, $a_{2i} = \iota^{D'}_{\hookrightarrow}(r_i(a_{1.5}))$ and $a_{0i} = r_i(a_0)$ for $i = 1, 2$.

The Boolean subalgebra B_6 generated by a_{ji} $(0 \leq j \leq 2, 1 \leq i \leq 2)$ in B has six atoms, each permutation of which extends to an automorphism of B. Consider the permutation $a_{ji} \mapsto a_{(j+1 \bmod 3)i}$, which extends to an automorphism of B_6, which in turn extends to $\phi \in \mathrm{Aut}(B)$ by ultrahomogeneity of B. By construction,

$$\bigvee_L \phi(\{a_{11}, a_{12}\}) \neq \bigvee_L \phi(\{a_{21}, a_{22}\})$$

showing that ϕ is not in the range of $h_{\neg\neg}$.

The last paragraph also shows that the image of $h_{\neg\neg}$ is not dense. To see that $\operatorname{ran} h_{\neg\neg}$ is not open, let $\bar{b}$ be an arbitrary tuple in B, and we show that $\mathrm{Aut}(B)_{(\bar{b})} \setminus \operatorname{ran} h_{\neg\neg} \neq 0$. Take a finite subalgebra H of L such that H generates $\langle \overline{a} \rangle^B$ as a Boolean algebra. Let D'' be the diagram[1] $H \hookleftarrow \mathbf{2} \hookrightarrow \bigsqcup D$. The image $\operatorname{ran} \iota^{D''}_{\hookrightarrow}$ generates a copy B_6' of B_6. Take an automorphism ψ_0 on $\bigsqcup D''$ $\psi_0 \restriction B_6'$ is as constructed in the preceding paragraph and that $\psi_0 \restriction \operatorname{ran} \iota^{D''}_{\hookleftarrow}$ is the identity.[2] The automorphism ψ_0 extends to another $\phi \in \mathrm{Aut}(B)$, which is in $\mathrm{Aut}(B)_{(\bar{b})} \setminus \operatorname{ran} h_{\neg\neg}$. □

[1]To be more precise, one can replace $\bigsqcup D$ by an appropriate copy by the weak homogeneity of L.

[2]The existence of such an automorphism can be proved in terms of the concrete representation of the $\bigsqcup D''$.

We will show the non-amenability of $\mathrm{Aut}(L)$ later in this section. Before doing so, we find it interesting to see that $\mathrm{Aut}(L)$ is distinct from the automorphism groups of better-known ultrahomogeneous structures.

Lemma 3.5. Let N be a countable strongly 2-homogeneous structure, $p \in S_1(N)$, and M be an ω-categorical structure in a possibly different countable language. Let $f_M : \omega \to \omega$ be defined by $f_M(n) = |S_n^M(0)|$. Suppose that for every $n_0 < \omega$ there exist $m < \omega$ and a set X of m-types realized in N such that for every $q(x_1, \ldots, x_m)$ and $i < m$ we have $p(x_i) \subseteq q(x_1, \ldots, x_m)$ and that $f_M(n_0 m) < |X|$. Then:

(1) The topological group $\mathrm{Aut}(N)$ is not topologically isomorphic to $\mathrm{Aut}(M)$.
(2) The abstract group $\mathrm{Aut}(N)$ is not isomorphic to $\mathrm{Aut}(M)$ if $\mathrm{Aut}(M)$ has the small index property.

More generally, an analogous statement about a strongly $(\kappa+1)$-homogeneous N, a κ-type p, and sets X of $\kappa \cdot m$-types of N holds true.

Proof. The second claim is a corollary of the first (see, e.g., Hodges [8, Lemma 4.2.6]).

By way of contradiction, assume that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are topologically isomorphic. First, we see:

Claim. There exists $n_0 = n_0(p) < \omega$ and a function $c : p(N) \to M^{n_0}$ such that for any formula $\phi(x_1, \ldots, x_m)$ in the language of N there is a formula $\phi^*(\overline{x_1}, \ldots, \overline{x_m})$ in the language of M with

$$N \models \phi(b_1, \ldots, b_m) \iff M \models \phi^*(c(b_1), \ldots, c(b_m))$$

for every $b_1, \ldots, b_m \in N$ with $b_i \in p(N)$ $(1 \le i \le m)$.

In other words, N is Ind-interpretable in M. Before proving this claim, we note that if $(a_1, \ldots, a_m) \in \phi(N^m) \triangle \psi(N^m)$, then $c(a_1) \ldots c(a_m) \in \phi^*(M^{n_0 m}) \triangle \psi^*(M^{n_0 m})$.

We adapt the proof of a well-known fact [8, Lemma 7.3.7] combined with the strong 2-homogeneity of N to prove this claim. Let $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ be a topological isomorphism. By the strong 2-homogeneity, the realizers of p in N form an orbit of $\mathrm{Aut}(N) \curvearrowright N$. Fix $b \models p$ in N. Since h is continuous, $h^{-1}(\mathrm{Aut}(N)_{(b)})$ is open and hence contains $\mathrm{Aut}(M)_{(\overline{a})}$ for some $n_0 < \omega$ and $\overline{a} \in M^{n_0}$. Define $c : p(N) \to M^{n_0}$ so that for $b' \in p(N)$ we have $c(b') = h^{-1}(\alpha) \cdot \overline{a}$, where α is the unique element of $\mathrm{Aut}(N)$ such that $\alpha(b) = b'$. Now let $\phi(x_1, \ldots, x_m)$ be as in the assumption of the claim, and consider $X := \{c(\overline{d}) \in M^{n_0 m} \mid \overline{d} \in N^m, N \models \phi(\overline{d})\}$. Since h is a group homomorphism, we can easily show that X is an orbit of $\mathrm{Aut}(M) \curvearrowright M^{n_0 m}$. By the $\aleph_0$-homogeneity of M, X is 0-definable, say, by ϕ^*.

Take $m < \omega$ and X as in the assumption. For $q \in X$ let p^* be the possibly partial $n_0 m$-type $\{\phi^* \mid \phi \in q^*\}$ over 0 of M. By construction, if a tuple $\overline{a} \in N^m$ realizes q, we have $c(\overline{a}) \models q^*$. We conclude that $|\{q^* \mid q \in X\}| > f_M(n_0 m)$, a contradiction. $\square$

Corollary 3.6. The topological group $\mathrm{Aut}(L)$ is not realized as the automorphism group of any of the following structures:

- the countable atomless Boolean algebra B,
- the Fraïssé limit D of finite distributive lattices, or
- countable ultrahomogeneous structures in finite relational languages.

Moreover, $\mathrm{Aut}(L)$ is not isomorphic to $\mathrm{Aut}(B)$ or $\mathrm{Aut}(D)$ as abstract groups.

Proof. We will handle the cases of B and D first. Recall that $\mathrm{Aut}(B)$ and $\mathrm{Aut}(D)$ have the small index property [17, 6]. Since $\mathrm{Th}(L)$, $\mathrm{Th}(B)$, and $\mathrm{Th}(D)$ eliminate quantifiers, we may replace "types" with "quantifier-free types" in applying the preceding lemma to these structures. Since f_D grows asymptotically faster than f_B, it suffices to prove the conclusion for D. Let $\mathbf{V}$ be the variety of Gödel algebras, i.e., Heyting algebras satisfying the equation $(x \to y) \vee (y \to x) = 1$. This is a locally finite variety. For a tuple of variables $\overline{x}$, write $F^{\mathbf{V}}_{\overline{x}}$ for the free $\mathbf{V}$-algebra generated by $\overline{x}$, and let p be $\mathrm{qftp}^{F^{\mathbf{V}}_{\overline{x}}}(x/0)$. Let $m < \omega$ be arbitrary and $\overline{x} = x_1 \dots x_m$. Consider $H_a := F^{\mathbf{V}}_{\overline{x}} \times (F^{\mathbf{V}}_{\overline{x}}/\theta_a)$, where θ_a is the principal filter generated by $a \in F^{\mathbf{V}}_{\overline{x}}$. This is a $\mathbf{V}$-algebra. Now, let $X_m = \{\mathrm{qftp}^{H_a}(\overline{x'}/0) \mid a \in F^{\mathbf{V}}_m\}$, where $\overline{x'} := (x_1, x_1) \dots (x_m, x_m)$. By construction, we have $p((x_i, x_i)) \subseteq q(\overline{x'})$ whenever $q(\overline{x'}) \in X_m$ and $1 \le i \le m$. Moreover, as H_a is finite for every $a \in F^{\mathbf{V}}_m$, every type in X_m is realized in L.

Let $n_0 < \omega$ be given. We have

$$f_D(n) = \sum_{i=1}^{n} S(n,i) i! M(i) \le n n! M(n) \max_i S(n,i),$$

where $n = n_0 m$, $S(\cdot,\cdot)$ are Stirling numbers of the second kind, and $M(i)$ is the i-th Dedekind number. Furthermore, by $\log \max_i S(n,i) = O(n \log n)$ [14] and $\log_2 M(n) = O\left(\binom{n}{n/2}\right)$ [11], we have

$$\log f_D(n) = O(n^2) + O\left(\binom{n}{n/2}\right) + O(n \log n) = O\left(\binom{n}{n/2}\right),$$

where we assumed n_0 is even without loss of generality. On the other hand, Valota [18] showed that $|X_m| = |F^{\mathbf{V}}_m| = (d(m))^2 + d(m)$, where $d(0) = 1$, and

$$d(k) = \prod_{i=0}^{k-1} (d(i) + 1)^{\binom{n}{i}}.$$

Therefore, $\log |X_m| = O(d(m))$, and

$$\log d(m) \ge \sum_{i=0}^{m-1} \binom{n}{i} \log d(i).$$

One can show by induction that $\log d(m)$ is at least the m-th Fubini number, which is strictly greater than $m!$ asymptotically [15]. Therefore, there exists m such that $|X_m| > f_D(n_0 m)$ as $\binom{n_0 m}{n_0 m/2} \sim 4^{n_0 m}/\sqrt{\pi n_0 m}$.

Finally, it is known that for every countable ultrahomogeneous structure M in a finite relational language, f_M is bounded from above by the exponential of a polynomial [2], so the claim follows from the argument above. $\square$

We now proceed to showing the non-amenability of $\mathrm{Aut}(L)$.

Definition 3.7. Let H be a finite nondegenerate Heyting algebra. We write $I(b)$ for the set of join-prime elements below or equal to b for $b \in H$. Let $\prec$ be an arbitrary linear extension of the partial order on $I(1)$ induced from H. We define a total order $\prec^{\mathrm{alex}}$ on H extending $\prec$ by the following:

$$a \prec^{\mathrm{alex}} a' \iff \max_{\prec}(I(a) \triangle I(a')) \in I(a').$$

This is clearly a total order, which is known as the anti-lexicographic order. We call this a *natural ordering* on H.

An expansion of a finite nondegenerate Heyting algebra H by a natural total order is called a *finite Heyting algebra with a natural ordering.*

It is easy to check that if $(H, \prec)$ is a finite Heyting algebra with a natural ordering, and H happens to be a Boolean algebra, then $(H, \prec)$ is a finite Boolean algebra with a natural ordering in the sense of Kechris, Pestov, and Todorcević [9].

Proposition 3.8. The class $\mathcal{K}^*$ of finite Heyting algebras with a natural ordering is a reasonable Fraïssé expansion of $\mathrm{Age}(L)$.

Proof. We show that $\mathcal{K}^*$ is reasonable and that $\mathcal{K}^*$ has the amalgamation property. (Other claims are clear.) In what follows, for a totally ordered set $(X, <)$ and $Y, Z \subseteq X$, we write $Y < Z$ to mean that $y < z$ whenever $y \in Y$ and $z \in Z$.

Let $H_1 \subseteq H_2$ be finite Heyting algebra, and let $\prec_1^{\mathrm{alex}}$ be an arbitrary admissible total order on H_1. We show that there exists an admissible order on H_2 extending $\prec_1^{\mathrm{alex}}$. Let $\pi : \mathbb{P}_2 \twoheadrightarrow \mathbb{P}_1$ be the surjective p-morphism dual to the inclusion map $H_1 \hookrightarrow H_2$. Note that identifying $I(1_{H_i})$ with $\mathbb{P}_i$ as pure sets, an admissible total order of H_i extends the dual of the order of $\mathbb{P}_i$ for $i = 1, 2$.

Suppose that for $p, q \in \mathbb{P}_1$ we have $p \prec_1 q$. Since $\prec_1^{\mathrm{alex}}$ is admissible, $p \not\leq q$. Take arbitrary $p', q' \in \mathbb{P}_2$ such that $\pi(p') = p$ and that $\pi(q') = q$. Since π is order-preserving *a fortiori*, we have $p' \not\leq q'$.

Let $R = (\leq \setminus \Delta) \cup \{(p', q') \mid \pi(p') \prec_2 \pi(q')\}$ be a binary relation on $\mathbb{P}_2 = I(1_{H_2})$, where Δ is the diagonal relation. It can be shown by induction from the fact in the preceding paragraph that R contains no cycle. Therefore, R can be extended to a total order $\prec_2$. Furthermore, for $p, q \in \mathbb{P}_1$, we have $\pi^{-1}(p) \prec_2 \pi^{-1}(q)$; *a fortiori*, $\pi^{-1}(p) \prec_2^{\mathrm{alex}} \pi^{-1}(q)$. This shows that $\prec_2^{\mathrm{alex}}$ extends $\prec_1^{\mathrm{alex}}$.

Next, we show the amalgamation property for $\mathcal{K}^*$. Let D be the diagram $H_1 \hookleftarrow H_0 \hookrightarrow H_2$ in $\mathrm{Age}(L)$ and let $\prec_i^{\mathrm{alex}}$ be an arbitrary admissible ordering on H_i for $i = 1, 2$. Recall the dual poset $\mathbb{P}$ of $\bigsqcup D$ is a sub-poset of the product order $\mathbb{P}_1 \times \mathbb{P}_2$, where $\mathbb{P}_i$ is the dual of H_i $(i = 1, 2)$ [13]. Define a total order $\prec$ on $\mathbb{P}$ so it extends the product order of $\prec_1$ and $\prec_2$.

We first show that $\prec$ extends the dual of the order of $\mathbb{P}$. Assume that $(p_1, p_2) \leq (q_1, q_2)$ for $(p_i, q_j) \in \mathbb{P}$ and $1 \leq i, j \leq 2$. (Recall that $p_i, q_i \in \mathbb{P}_i$.) Since the order of $\mathbb{P}$ is induced by the product of those of $\mathbb{P}_1$ and $\mathbb{P}_2$, we have $p_i \leq q_i$ for $i = 1, 2$. Because $\prec_i$ extends the dual of the order of $\mathbb{P}_i$, we have $p_i \succ_i q_i$ $(i = 1, 2)$. By the construction of $\prec$, we have $(p_1, p_2) \succ (q_1, q_2)$ as desired.

We then show that $(\bigsqcup D, \prec^{\mathrm{alex}})$ witnesses the amalgamation property. Because of the strong amalgamation property of $\mathrm{Age}(L)$, it suffices to show that $\prec^{\mathrm{alex}}$ extends $\iota^D_{\hookleftarrow}(\prec_1^{\mathrm{alex}})$ and $\iota^D_{\hookrightarrow}(\prec_2^{\mathrm{alex}})$. Take $p, p' \in \mathbb{P}_1$, and assume that $p \prec p'$ (the other case can be handled in a similar manner). Since $\iota^D_{\hookleftarrow}$ is induced by the projection $\pi_1 : \mathbb{P} \twoheadrightarrow \mathbb{P}_1$, it suffices to show that $\pi^{-1}(p) \prec^{\mathrm{alex}} \pi^{-1}(p')$. Now, it is easy to see that, in fact, $\pi^{-1}(p) \prec \pi^{-1}(p')$ by the construction of $\prec$. □

Corollary 3.9. $\mathrm{Aut}(L)$ is not amenable.

Proof. Consider the Boolean algebras that witness the conditions (i) and (ii) of [10, Proposition 2.2] for the class of finite Boolean algebras with natural orderings [9, Remark 3.1]; call them A_1 and A_2. Since $A_1, A_2 \in \mathcal{K}$, and the Heyting algebra embeddings $A_1 \to A_2$ are exactly the Boolean algebra embeddings $A_1 \to A_2$, the pair A_1, A_2 witness the conditions (i) and (ii) of the same propositions for $\mathcal{K}^*$. □

Finally, we study the aspects of the combinatorics of Age(L) pertaining to the extreme amenability of Aut(L). The Kechris-Pestov-Todorcević correspondence concerns order expansions of the ages of ultrahomogeneous structures with the ordering property [9]. One can make an empirical observation that the ordering property of an order expansion of a Fraïssé class have been proved by two classes of arguments, one of which is based on a lower-dimensional Ramsey property, with the other argument rather trivially following from the order-forgetfulness of the expansion. The former is applied to many classes of relational structures such as graphs, whereas the latter is used with the countable atomless Boolean algebras and the infinite-dimensional vector space over a finite field. Our structure L is similar to the latter classes of structures. However, we see the following.

Proposition 3.10. There is no Fraïssé order class of isomorphism types that expands the class of finite Heyting algebras and is order-forgetful.

Proof. Suppose that such a class $\mathcal{K}^*$ exists. Let H be an arbitrary finite Heyting algebra, and consider the action of $\mathrm{Aut}(H)$ on the set of binary relations on H. Since $\mathcal{K}^*$ is closed under isomorphs, the set of admissible orderings A_L on H is a union of orbits. Since $\mathcal{K}^*$ is order-forgetful, A_L consists of a single orbit.

Now, consider the poset $\mathbb{P}'$ that is the disjoint union of two 2-chains, with its quotient $\mathbb{P}$ obtained by collapsing one of the 2-chains into a point. The canonical surjection $\mathbb{P}' \twoheadrightarrow \mathbb{P}$ is p-morphic, which induces a Heyting algebra embedding $H \hookrightarrow H'$. Let $a, b \in H'$ correspond to the two 2-chains. Clearly, H is rigid whereas there is an automorphism $\phi : H' \to H'$ under which a and b are conjugates. Consider an admissible ordering $\prec$ on H'; without loss of generality, we may assume $a \prec b$. Writing the action of $\mathrm{Aut}(H')$ by superscripts, we have $b \prec^\phi a$. Since $\mathcal{K}^*$ is a Fraïssé class, the restrictions of $\prec$ and $\prec^\phi$ to H, respectively, are admissible orderings on H. Now, we have $\prec \cap H^2 \neq \prec^\phi \cap H^2$, as witnessed by $(a, b) \in H^2$. These cannot belong to the same orbit of A_H as H is rigid. □

From this point on, we study Aut(L) as an abstract group. In fact, we will show the normality of Aut(L) by an argument applicable to many other ultrahomogeneous lattices.

Lemma 3.11. If M is a countable ultrahomogeneous structure with Age(M) having the superamalgamation property, then M has an automorphism $g : M \to M$ that moves almost maximally with respect to $\mathop{\smile\!\!\!\!\!\!|}$ in the sense of Tent and Ziegler [16, Lemma 5.3].

Proof. A back-and-forth construction. Enumerate M as $(a_i)_{i<\omega}$ and all the realized 1-types over all finite subsets of M as $(p_i)_{i<\omega}$. We construct g as the union of the chain $0 = g_0 \subseteq g_1 \subseteq \cdots$, each of which is a partial isomorphism with a finite domain. Along the way, we construct a chain $0 = S_0 \subseteq S_1 \subseteq \cdots$ of realized 1-types. Suppose that g_j has been constructed. To construct g_{j+1}, one does the following:
If $j = 3i$. If a_i is in $\mathrm{dom}\, g_j$, then $g_{j+1} := g_j$. Otherwise, let g_{j+1} extend g_j so $g_{j+1}(a_i)$ may be a realization of $g_j(p)$ outside $\mathrm{ran}\, g_j$, which exists due to the strong amalgamation, where p is the type of a_i over $\mathrm{dom}\, g_j$.
If $j = 3i + 1$. Similar as above, but use range instead of domain.
If $j = 3i + 2$. Let k be the least such that p_k is over X, that $X \subseteq \mathrm{dom}\, g_j$, and that $p_k \notin S_i$. (There may not be such k, in which case $g_{j+1} := g_j$ and $S_{i+1} := S_i$, but there will be such k for infinitely many i because of the other two kinds of

stages.) Let $S_{i+1} := S_i \cup \{p_k\}$. If all realizers of p_k is in $\operatorname{dom} g_j$, then $g_{j+1} := g_j$. If not, apply the strong amalgamation to obtain infinitely many realizers of p_k. Since $\operatorname{dom} g_j$ is finite, there exists $a \models p_k$ outside $\operatorname{dom} g_j$. Let D be the diagram $\langle aX\rangle \hookleftarrow \langle X\rangle \hookrightarrow \langle aX\rangle$, and let the diagram $\langle aX\rangle \overset{\text{incl.}}{\hookrightarrow} A \overset{\iota}{\hookleftarrow} \langle aX\rangle$, where $A \subseteq M$, witness the superamalgamation property for D. Now let $g_{j+1} := g_j \cup \{(a, \iota(a))\}$. (Replace $\iota(a')$ by something else if need be so $\iota(a') \notin \operatorname{ran} g_j$ by replacing the amalgam by one with more copies of $\langle aX\rangle$.) By the superamalgamation property, we have $a \mathop{\smile\!\!\!\!\!|}_X g_{j+1}(a)$. □

Theorem 3.12. Let M be a countable ultrahomogeneous structure with $\operatorname{Age}(M)$ having the superamalgamation property. Moreover, assume that the amalgamation property of $\operatorname{Age}(M)$ is witnessed canonically and functorially by $\otimes$ in the sense of Tent and Ziegler [16, Example 2.2.1] (the superamalgamation property need not be witnessed in this manner). Then, the abstract group $\operatorname{Aut}(M)$ is normal.

Proof. First, observe that the proof of [16, Lemma 2.8] depend only on the stationarity and existence properties of an independence relation on M. Hence, apply the proof of the lemma to the independence relation induced by $\otimes$ as in [16, Example 2.2.1], which has the stationarity and existence properties, to conclude that the topological group $\operatorname{Aut}(M)$ has a dense conjugacy class. By the superamalgamation property of $\operatorname{Age}(M)$, one can show that the relation $\mathop{\smile\!\!\!\!\!|}$ satisfies all defining properties of a stationary independence relation but the stationarity. In fact, the invariance of $\mathop{\smile\!\!\!\!\!|}$ follows from the ultrahomogeneity of M. The monotonicity and the symmetry of $\mathop{\smile\!\!\!\!\!|}$ are obvious by the shape of the definition of $\mathop{\smile\!\!\!\!\!|}$. To show transitivity, assume that $A \mathop{\smile\!\!\!\!\!|}_{BC} D$ and that $A \mathop{\smile\!\!\!\!\!|}_B C$. To show $A \mathop{\smile\!\!\!\!\!|}_B D$, take an arbitrary $a \in A$ and $d \in D$. Suppose $a \le d$. (The case of $d \le a$ can be handled in a similar way.) Since $A \mathop{\smile\!\!\!\!\!|}_{BC} D$, there is $b \in BC$ such that $a \le b \le d$. If $b \in B$, we are done. Otherwise, $b \in C$, so by $A \mathop{\smile\!\!\!\!\!|}_B C$, there is $b' \in B$ such that $a \le b' \le b$. Now we have $a \le b' \le d$. Finally, to show the existence property of $\mathop{\smile\!\!\!\!\!|}$, let p be a realized type over a finite set B and C a finite set. Let $\overline{a}$ be a tuple realizing p. Now consider the diagram D:

$$\langle \overline{a}B\rangle \hookleftarrow \langle B\rangle \hookrightarrow \langle BC\rangle$$

Let the diagram $\langle \overline{a}B\rangle \overset{\iota}{\hookrightarrow} A \overset{\text{incl.}}{\hookleftarrow} \langle BC\rangle$, where $A \subseteq M$, witness the superamalgamation propriety for D. It is clear that $\iota(\overline{a}) \mathop{\smile\!\!\!\!\!|}_B C$. By examining the proofs of [16, Theorem 2.7 and Lemma 5.3], one can see that they do not depend on the stationarity of $\mathop{\smile\!\!\!\!\!|}$. Therefore, for g constructed in the preceding lemma, every element of $\operatorname{Aut}(M)$ is the product of 16 conjugates of g. □

Corollary 3.13. $\operatorname{Aut}(L)$ is normal.

By the result by Maksimova, our argument shows the normality of the automorphism group of the Fraïssé limit of finite members of each of the 7 nontrivial subvariety of Heyting algebras with the (super-)amalgamation property. Moreover, our argument seems to be applicable to other Fraïssé classes of lattice expansions with the superamalgamation property.

Appendix A. Axiomatization

Following Darnière and Junker [5], we follow the formalism of co-Heyting algebras, or cHAs for short. They are exactly the order-theoretic dual of Heyting algebras. Let T be the theory of co-Heyting algebras. This is a theory in the language of lattices expanded by a binary function symbol $-$, where $x - y$ is the supremum of elements z for which $y \vee z \geq x$, which always exists in a co-Heyting algebra. As before, we write T^* for the model-completion of T.

We write $y \ll x$ iff $y \leq x$ and $x - y = 0$. Darnière and Junker [5, Section 4] lists two axioms D1 and S1 that are satisfied by e.c. co-Heyting algebras:

D1: For every a, c such that $c \ll a \neq 0$ there exists a nonzero element b such that:
$$c \ll b \ll a.$$

S1: For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq 0$ there exists nonzero elements a_1 and a_2 such that:
$$a - a_2 = a_1 \geq b_1$$
$$a - a_1 = a_2 \geq b_2$$
$$a_1 \wedge a_2 = b_1 \wedge b_2.$$

D1 is of the form (1), but S1 is not; in particular, the consequent of D1 does not imply the antecedent over T. However, consider the following condition:
(AS1′)
$(b_1 = a$ and $b_2 = 0)$ or $(b_2 = a$ and $b_1 = 0)$ or $(b_1 < a$ and $b_2 < a$ and $b_1 \wedge b_2 \ll a)$.

The same construction as in [5, Lemma 4.2] shows that AS1′ implies the consequent of S1 in T^*. It can also be seen that the consquent of S1 implies AS1′ over T. I refer to the conditional obtained from S1 by replacing the antecedent with AS1′ as S1′.

Proposition A.1. D1 does not imply S1′; a fortiori, it does not axiomatize T^*.

Proof. It suffices to show that, given a finite cHA L with $x, y \in L$ such that $x \ll y$ and $a, b_1, b_2 \in L$ witnessing the failure of S1′, there is a finite $L' \supset L$ such that $L' \models \exists z(x \ll z \ll y)$, and that a, b_1, b_2 still witness the failure of S1′. For let L_0 be a cHA as in the hypothesis of the claim; the usual argument gives rise to a chain $L_0 \subset L_1 \subset \dots$, where L_{n+1} is constructed by applying the claim to L_n, the union $\bigcup_n L_n$ of which will satisfy D1 and the negation of S1′.

In fact, the following construction in [5, Lemma 4.1] works. Let $y_1, \dots, y_r$ be the join-irreducible components of y in L. Let $\mathcal{I}_0$ be the poset of the join-irreducible elements of I; let $\mathcal{I}$ be the poset obtained from $\mathcal{I}_0$ by replacing each y_i by the chain $\{\eta_i < y_i\}$. The p-morphism $\mathcal{I} \twoheadrightarrow \mathcal{I}_0$ that collapses each chain $\{\eta_i < y_i\}$ to y_i induces a cHA embedding $L \hookrightarrow L'$, where L' is the cHA of downsets of $\mathcal{I}$. An element $z \in L'$ is in (the image of) L if and only if there is $1 \leq i \leq r$ such that $\eta_i \in z$ and that $y_i \notin z$. Suppose that there are $a_1, a_2 \in L'$ witnessing the consequent of S1′. By hypothesis, one of them is in $L' \setminus L$; without loss of generality, assume a_1 is. There is $1 \leq i \leq r$ such that $\eta_i \in a_1$ and that $y_i \notin a_1$. By the consequent of S1′, $a = a_1 \vee a_2 \in L$. Since $\eta_i \in a_1 \cup a_2$, we have that $y_i \in a_1 \cup a_2$. Hence, $y_i \in a_2$, and thus $\eta_i \in a_2$. Therefore, $\eta_i \in a_1 \cap a_2$, and $y_i \notin a_1 \cap a_2$. However, $a_1 \wedge a_2 = b_1 \wedge b_2 \in L$, which leads to a contradiction. □

Lemma A.2. For a finite cHA L and $a, b \in L$, we have $a \ll b$ if and only if for every join-irreducible component b' of b we have $a \wedge b' < b'$.

Proof. Note that to prove quantifier-free formulas one may just treat elements of a cHA as closed sets in a space. If concepts of higher quantifier complexity (e.g., irreducibility) are involved, care must be taken.

Let $(b_i)_{i<k}$ be the join-irreducible components of b. Then

$$\begin{aligned}
b - a = b &\iff \bigvee_i b_i - a = \bigvee_i b_i \\
&\iff \bigvee_i (b_i - a) = \bigvee_i b_i && \text{identity in cHAs} \\
&\iff \bigvee_i (b_i - a) \geq \bigvee_i b_i \\
&\iff \forall i \ \bigvee_j (b_j - a) \geq b_i && \text{definition of } \bigvee \\
&\iff \forall i \exists j \ b_j - a \geq b_i && \text{join-primality of } b_j \\
&\iff \forall i \ b_i - a \geq b_i && \text{no other } j \text{ than } i \text{ can satisfy that} \\
&\iff \forall i \ b_i - (a \wedge b_i) \geq b_i \\
&\iff \forall i \ (a \wedge b_i) < b_i && \text{by join-primality of } b_i\text{; see [5].}
\end{aligned}$$

□

Proposition A.3. S1′ does not imply D1.

Proof. We use a similar argument as before. We let L_0 be the minimal nontrivial cHA, and we apply to L_n the construction in [5, Lemma 4.2] to obtain L_{n+1}. Note that for $n < \omega$ there is no chain consisting of more than one element in the poset of join-irreducible elements of L_n with the induced order.

We claim that for $n < \omega$ there is no nonzero $z \in L_n$ such that $0 \ll z \ll 1$—that is, 0 and 1 witness the failure of D1. Indeed, suppose that there is such a $z \neq 0$. There exists a join-irreducible component u' of 1 such that $u' \wedge z \neq 0$ since $z \neq 0$ and by distributivity. Take a join-irreducible component z' of $z \wedge u'$. We now have a nontrivial chain $\{z' < u'\}$ of join-irreducible elements. □

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