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Quasi-Galois theory in tensor-triangulated categories

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Quasi-Galois Theory in Tensor-Triangulated Categories

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by

Bregje Ellen Pauwels

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Abstract of the Dissertation

Quasi-Galois Theory in Tensor-Triangulated Categories

by

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Professor Paul Balmer, Chair

We consider separable ring objects in symmetric monoidal categories and investigate what it means for an extension of ring objects to be (quasi)-Galois. Reminiscent of field theory, we define splitting ring extensions and examine how they occur. We also establish a version of quasi-Galois-descent for ring objects.

Specializing to tensor-triangulated categories, we study how extension-of-scalars along a quasi-Galois ring object affects the Balmer spectrum. We define what it means for a separable ring to have constant degree, which turns out to be a necessary and sufficient condition for the existence of a quasi-Galois closure. Finally, we illustrate the above for separable rings occurring in modular representation theory.

The dissertation of Bregje Ellen Pauwels is approved.

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Vita

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Introduction

Classical Galois theory studies field extensions L/K through the *Galois group* Γ, that is the group of automorphisms of L that fix K. Writing L^Γ for the subfield of elements in L fixed by Γ, we call a field extension *Galois with group* Γ if $L^\Gamma = K$. For a polynomial $f \in K[x]$, the *splitting field* of f over K is the smallest extension over which f decomposes into linear factors. A field extension L/K is sometimes called *quasi-Galois*[1] when L is the splitting field for some polynomial in $K[x]$. We highlight some facts from Galois theory for fields (see [Bou81] or [Kap72], for instance) to consult later. Let L/K be a finite field extension and f an irreducible separable polynomial with coefficients in K. Then,

(a) The field L is a splitting field of f over K if and only if L is the smallest extension of K such that $L \otimes_K K[x]/(f) \cong L^{\times \deg(f)}$.

(b) There exists a unique (up to isomorphism) splitting field of f over K.

(c) The field extension L/K is quasi-Galois if and only if any irreducible polynomial with a root in L factors completely in L.

(d) There exists a field extension N/L such that N is quasi-Galois over K and no other field between L and N is quasi-Galois over K. This field is unique up to isomorphism and we call N the *quasi-Galois closure* of L over K.

In this dissertation, we adapt the above ideas to the context of (commutative, separable) ring objects in symmetric monoidal categories. The generalisation of Galois theory from fields to rings originated with Auslander and Goldman in [AG60, App.]. They considered commutative separable algebras S that are projective over the base ring R. For a finite group Γ of ring automorphisms of S

[1] see Bourbaki [Bou81, §9]. In the literature, a quasi-Galois extension is sometimes called normal or Galois, probably due to the fact that those notions coincide when L/K is separable.

fixing R, the extension S/R is called *Galois* with group Γ if the maps $R \hookrightarrow S^\Gamma$ and

$$S \otimes_R S \longrightarrow \prod_{\gamma \in \Gamma} S : x \otimes y \longmapsto (x \cdot \gamma(y))_{\gamma\in\Gamma} \tag{0.0.1}$$

are isomorphisms. Five years later, Chase, Harrison and Rosenberg gave six characterisations of Galois extensions of commutative rings [CHR65]. Further generalisations are aplenty [Kre67, CS69, KT81, Hes09]. In particular, Rognes [Rog08] introduced a Galois theory up-to-homotopy. The objects of study are brave new rings: commutative monoids in categories of structured spectra. The maps in the definition of Auslander and Goldman are now required to be isomorphisms in the stable homotopy category.

In our version, the analogue of a separable field extension will be a commutative separable ring A in a symmetric monoidal idempotent-complete category $\mathcal{K}$, with special emphasis on tensor-triangulated categories. Here, A is called *separable* if the multiplication map $A \otimes A \to A$ has a right inverse $A \to A \otimes A$ which is an A, A-bimodule morphism. Throughout the rest of the introduction, we assume all ring objects are commutative. Consider the following two examples:

Algebraic Geometry If $V \to X$ is an étale morphisms of schemes, we can understand the derived category of V as the category of A-modules for some separable ring A in the derived category of X (see [Bal14a, Th.3.5, Rem.3.8]).

Modular Representation theory Let $H < G$ be finite groups and $\Bbbk$ a field. Consider the ring object $A^G_H := \Bbbk(G/H)$ in $\mathcal{K} := \Bbbk G-\mathrm{mod}$ with all $[g] \in G/H$ being orthogonal idempotents. Then, Balmer ([Bal15]) shows that A^G_H is separable and there is an equivalence $A^G_H-\mathrm{Mod}_{\mathcal{K}} \cong \Bbbk H-\mathrm{mod}$ such that extension-of-scalars coincides with the restriction Res^G_H. We can consider the ring A^G_H in any category that receives $\Bbbk G-\mathrm{mod}$, say the derived category $\mathcal{K} = \mathrm{D}^b(\Bbbk G-\mathrm{mod})$ or stable category $\mathcal{K} = \Bbbk G-\mathrm{stab}$, and the equivalence will still hold. Likewise, extension-of-scalars along a suitable separable ring recovers restriction to a subgroup in equivariant stable homotopy theory, in equivariant KK-theory and in equivariant derived categories ([BDS14]). In particular, sometimes results obtained for an opportune subgroup H can be extended to the whole group G by performing descent along the ring A^G_H. This technique was most notably used in [Bal15] to describe the kernel and image of the restriction homomorphism $T(G) \to T(H)$, where $T(G)$ denotes the group of endotrivial $\Bbbk G$-modules and $[G : H]$ is invertible in $\Bbbk$.

Separable ring objects allow a sound notion of degree [Bal14b], and our first Galois-flavoured result shows that the degree of A provides a bound for the number of ring endomorphisms of A in $\mathcal{K}$ (Theorem 2.2.4). Another reason we turn to separable rings is that the category of A-modules in $\mathcal{K}$ remains symmetric monoidal. We can therefore consider *algebras* over a separable ring, meaning ring objects in the module category.

The condition $R \xrightarrow{\cong} S^\Gamma$ in Auslander and Goldman's definition is delicate in a category without equalizers and is the topic of further work. As it turns out, the second condition 0.0.1 is interesting in its own right. Let $(\mathcal{K}, \otimes, \mathbb{1})$ be an idempotent-complete symmetric monoidal category and A a ring object in $\mathcal{K}$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$. We think of A as a ring extension of $\mathbb{1}$ and consider a group Γ of ring automorphisms of A in $\mathcal{K}$. We then define the ring homomorphism

$$\lambda_\Gamma : A \otimes A \longrightarrow \prod_{\gamma \in \Gamma} A_\gamma$$

by $\mathrm{pr}_\gamma \lambda_\Gamma = \mu(1 \otimes \gamma)$.

Definition. We call (A, Γ) *quasi-Galois* in $\mathcal{K}$ if $\lambda_\Gamma : A \otimes A \to \prod_{\gamma\in\Gamma} A$ is an isomorphism.

To illustrate, let R be a commutative ring and S a commutative R-algebra. Suppose (S, Γ) is a Galois extension of R in the sense of Auslander and Goldman, where Γ is some finite group of ring automorphisms of S over R. In particular, S is projective and separable as an R-module. Then, (S, Γ) is quasi-Galois in the symmetric monoidal categories $R-\mathrm{Mod}$ and $\mathrm{D}^{\mathrm{perf}}(R)$. If S is an indecomposable ring, it moreover follows that Γ contains all ring endomorphisms of S over R:

Theorem. (3.1.2). *Let A be a nonzero separable indecomposable ring object of finite degree in $\mathcal{K}$. Let Γ be the set of ring endomorphisms of A in $\mathcal{K}$. The following are equivalent:*

(i) $|\Gamma| = \deg(A)$.

(ii) $A \otimes A \cong A^{\times t}$ *as left A-modules for some $t \in \mathbb{N}$.*

(iii) Γ *is a group and* (A, Γ) *is quasi-Galois in* $\mathcal{K}$.

Following classical field theory, we introduce splitting rings (compare to (a)):

Definition. Let A and B be ring objects in $\mathcal{K}$ and suppose B is indecomposable. We say B *splits* A if $B \otimes A \cong B^{\times \deg(A)}$ as (left) B-algebras. We call B a *splitting ring* of A if B splits A and any ring morphism $C \to B$, where C is an indecomposable ring object splitting A, is an isomorphism.

Under mild conditions on $\mathcal{K}$, Corollary 3.2.7 shows B is quasi-Galois if and only if B is a splitting ring of some nonzero ring object A in $\mathcal{K}$; our terminology matches up with classical field theory. Moreover, Proposition 3.2.6 shows that every separable ring object in $\mathcal{K}$ has (possibly multiple) splitting rings.

If in addition, we assume that $\mathcal{K}$ is tensor-triangulated, we can say more about the way splitting rings arise. Examples of tensor-triangulated categories appear in many different shapes, be it in algebraic geometry, homological algebra, stable homotopy theory or modular representation theory. Paul Balmer [Bal05] has introduced the *spectrum* of an (essentially small) tensor-triangulated category $\mathcal{K}$, providing an algebro-geometric approach to the study of triangulated categories.

In short, the spectrum $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$ is the set of all prime thick $\otimes$-ideals $\mathcal{P} \subsetneq \mathcal{K}$. The *support* of an object x in $\mathcal{K}$ is the subset $\mathrm{supp}(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{K})$. The complements of these supports form a basis for a topology on $\mathrm{Spc}(\mathcal{K})$. A complete description of the spectrum can come from a classification of the thick $\otimes$-ideals in the category, as in modular representation theory, algebraic geometry and stable homotopy theory. Still, classifying the thick $\otimes$-ideals remains an open challenge for many tensor-triangulated categories. When such a classification is unknown, for instance in the derived category of G-equivariant vector bundles over a scheme, new information about the spectrum could mean progress in the classification problem. Tensor-triangular geometry seeks to deliver techniques for the study of the spectrum, which in turn could mean progress in the classification problem.

Extension-of-scalars along separable ring objects can play a central role in this endeavor. When A is a separable ring in a tensor-triangulated category, the category of A-modules in $\mathcal{K}$ remains tensor-triangulated [Bal11, Cor.4.3]. We can thus consider algebras over a separable ring without leaving the tensor-triangulated world or descending to a model category. In particular, we can study the continuous map

$$\mathrm{Spc}(F_A) : \mathrm{Spc}(A\!-\!\mathrm{Mod}_{\mathcal{K}}) \longrightarrow \mathrm{Spc}(\mathcal{K}) \tag{0.0.2}$$

induced by the extension-of-scalars functor $F_A : \mathcal{K} \to A\!-\!\mathrm{Mod}_{\mathcal{K}}$.

Thus motivated, we translate our quasi-Galois theory to the tensor-triangular setting. We assume $\mathcal{K}$ is nice (say, $\mathrm{Spc}(\mathcal{K})$ is Noetherian or $\mathcal{K}$ satisfies Krull-Schmidt). Proposition 5.2.8 provides an analogue to (c):

Proposition. *Let A be a separable ring in $\mathcal{K}$ such that the spectrum* $\mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})$ *is connected, and suppose B is an A-algebra with* $\mathrm{supp}(A) = \mathrm{supp}(B)$*. If B is quasi-Galois in $\mathcal{K}$, then B splits A.*

With one eye on future applications of the theory, we are on the lookout for separable rings whose Galois theory and geometry interact well. That is, we would like to control the support of the splitting rings. Recall that the local category $\mathcal{K}_{\mathcal{P}}$ at the prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ is the idempotent-completion of the Verdier quotient $\mathcal{K}/\mathcal{P}$. We say a separable ring A has *constant degree* if its degree as a ring in $\mathcal{K}_{\mathcal{P}}$ is the same for every prime $\mathcal{P} \in \mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{K})$. Finally, Proposition 5.2.6 and Theorem 5.2.9 provide a version of (b) and (d):

Theorem. *If A has connected support and constant degree, there exists a unique splitting ring A^* of A. Furthermore,* $\mathrm{supp}(A) = \mathrm{supp}(A^*)$ *and A^* is the unique quasi-Galois closure of A in $\mathcal{K}$. That is, any A-algebra morphism $B \to A^*$ with B quasi-Galois and indecomposable in $\mathcal{K}$, is an isomorphism.*

Initially, we were motivated to consider rings that behave "Galois"as a means to study the map 0.0.2. We note that any group Γ of ring automorphisms of A acts on $A-\mathrm{Mod}_{\mathcal{K}}$ and on its spectrum $\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$. Then,

Theorem. (5.2.2). *If (A, Γ) is quasi-Galois in $\mathcal{K}$,*

$$\mathrm{supp}(A) \cong \mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})/\Gamma.$$

In particular, we recover $\mathrm{Spc}(\mathcal{K})$ from $\mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})$ when $\mathrm{supp}(A) = \mathrm{Spc}(\mathcal{K})$. This happens exactly when A is *nil-faithful*, that is when $A \otimes f = 0$ implies f is $\otimes$-nilpotent. In fact, when the functor $A \otimes -$ is faithful and (A, Γ) is quasi-Galois, we find that $\mathcal{K} \cong (A-\mathrm{Mod}_{\mathcal{K}})^{\Gamma}$. Here, the category $(A-\mathrm{Mod}_{\mathcal{K}})^{\Gamma}$ has objects $(x, (\delta_{\gamma} : x \to x^{\gamma})_{\gamma \in \Gamma})$, where x is an A-module in $\mathcal{K}$, we write x^{γ} for x with γ-twisted A-action, and $(\delta_{\gamma})_{\gamma \in \Gamma}$ is a family of A-linear isomorphisms satisfying some cocycle condition. More generally, if (A, Γ) is quasi-Galois, Corollary 6.3.3 shows

$$\mathrm{Desc}_{\mathcal{K}}(A) \cong (A-\mathrm{Mod}_{\mathcal{K}})^{\Gamma},$$

where $\mathrm{Desc}_{\mathcal{K}}(A)$ is the descent category of A in the sense of [Mes06].

In the last chapter of the dissertation, we compute degrees and splitting rings for the separable rings $A^G_H := \Bbbk(G/H)$ from above. The degree of A^G_H in $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$ is simply $[G : H]$, and A^G_H is quasi-Galois if and only if H is normal in G. Accordingly, the quasi-Galois closure of A^G_H in the derived category $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$ is the ring A^G_N, where N is the normal core of H in G (Cor. 7.0.13). On the other hand, Theorem 7.0.15 shows that the degree of A^G_H in $\Bbbk G-\mathrm{stab}$ is the greatest $0 \leq n \leq [G : H]$ such that there exist distinct $[g_1], \ldots, [g_n]$ in $H\backslash G$ with p dividing $|H^{g_1} \cap \ldots \cap H^{g_n}|$. In that case, the splitting rings of A_H are exactly the $A^G_{H^{g_1} \cap \ldots \cap H^{g_n}}$ with $g_1, \ldots, g_n$ as above. Finally, A^G_H is quasi-Galois in $\Bbbk G-\mathrm{stab}$ if and only if p does not divide $|H \cap H^g \cap H^{gh}|$ for $g \in G - H$ and $h \in H - H^g$.

CHAPTER 1

Preliminaries: Rings in monoidal categories

In this dissertation, we interpret the notion of Galois extensions of rings in a broader context. Our playing field will be symmetric monoidal idempotent-complete categories and the main characters are objects that behave like rings. In this first chapter, we give a short overview of these concepts. For the definition of a symmetric monoidal category, we refer to [Mac98, Section XI.1].

Throughout this chapter, $(\mathcal{K}, \otimes, \mathbb{1})$ **will denote a symmetric monoidal additive category.**

Notation 1.0.1. For objects $x_1, \ldots, x_n$ in $\mathcal{K}$ and a permutation $\tau \in S_n$, we will write $\tau : x_1 \otimes \ldots \otimes x_n \to x_{\tau(1)} \otimes \ldots \otimes x_{\tau(n)}$ to denote the isomorphism that permutes the tensor factors.

Definition 1.0.2. A *ring object* $A \in \mathcal{K}$ is a triple (A, μ, η) with associative multiplication $\mu : A \otimes A \to A$ and two-sided unit $\eta : \mathbb{1} \to A$. That is,

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes \mathbb{1} \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & \mathbb{1} \otimes A \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

commute. We call a ring *commutative* when $\mu(12) = \mu$. If (A, μ_A, η_A) and (B, μ_B, η_B) are rings, we call a morphism $\alpha : A \to B$ in $\mathcal{K}$ a *ring morphism* if the diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "\alpha\otimes\alpha"] & A \arrow[d, "\alpha"] \\ B \otimes B \arrow[r, "\mu_B"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{1} \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "\alpha"] \\ & B \end{tikzcd}$$

commute.

Convention 1.0.3. We will often call A a *ring in* $\mathcal{K}$ instead of a ring object. **All rings are assumed commutative.**

Definition 1.0.4. Let (A, μ, η) be a ring object in $\mathcal{K}$. A *left A-module* is a pair $(x \in \mathcal{K}, \varrho : A \otimes x \to x)$ such that the action ϱ is compatible with the ring structure in the usual way. That is, the diagrams

$$\begin{tikzcd} A \otimes A \otimes x \arrow[r, "1\otimes\varrho"] \arrow[d, "\mu\otimes 1"] & A \otimes x \arrow[d, "\varrho"] \\ A \otimes x \arrow[r, "\varrho"] & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{1} \otimes x \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes x \arrow[d, "\varrho"] \\ & x \end{tikzcd}$$

commute. *Right modules* are defined analogously. Every object $x \in \mathcal{K}$ gives rise to a *free* A-module $A \otimes x$ with action given by $\varrho : A \otimes A \otimes x \xrightarrow{\mu\otimes 1} A \otimes x$. If (x, ϱ_1) and (y, ϱ_2) are left A-modules, a morphism $\alpha : x \to y$ is said to be *A-linear* if the diagram

$$\begin{tikzcd} A \otimes x \arrow[r, "\varrho_1"] \arrow[d, "1\otimes\alpha"] & x \arrow[d, "\alpha"] \\ A \otimes y \arrow[r, "\varrho_2"] & y \end{tikzcd}$$

commutes.

Definition 1.0.5. A ring A in $\mathcal{K}$ is called *separable* if the multiplication map $\mu : A \otimes A \to A$ has an A, A-bilinear section. That is, there exists a morphism $\sigma : A \to A \otimes A$ such that $\mu\sigma = 1_A$ and the diagram

$$\begin{tikzcd} & A \otimes A \arrow[dl, "\sigma\otimes 1"'] \arrow[d, "\mu"] \arrow[dr, "1\otimes\sigma"] & \\ A \otimes A \otimes A \arrow[dr, "1\otimes\mu"'] & A \arrow[d, "\sigma"] & A \otimes A \otimes A \arrow[dl, "\mu\otimes 1"] \\ & A \otimes A & \end{tikzcd}$$

commutes.

Notation 1.0.6. Let A and B be ring objects in $\mathcal{K}$. We write $A \times B$ for the ring $A \oplus B$ with component-wise multiplication. We will also consider the ring structure on $A \otimes B$ given by $(\mu_A \otimes \mu_B)(23) : (A \otimes B)^{\otimes 2} \to (A \otimes B)$. We write A^e for the enveloping ring $A \otimes A^{\mathrm{op}}$. If A and B are separable, then so are A^e, $A \otimes B$ and $A \times B$. Conversely, A and B are separable whenever $A \times B$ is separable.

Remark 1.0.7. Of course, left A^e-modules are just A, A-bimodules. Furthermore, any A-linear morphism $A \to A$ is A^e-linear because A is assumed commutative. Finally, any two A-linear morphisms $A \to A$ commute.

Example 1.0.8. Let R be a commutative ring. The category of R-modules forms a symmetric monoidal category $(R\text{−Mod}, \otimes_R, R)$. The (commutative, separable) ring objects in this category are just (commutative, separable) R-algebras.

Example 1.0.9. Let G be a finite group and let $\Bbbk$ be a field. We write $\Bbbk G - \mathrm{Mod}$ for the category of left $\Bbbk G$-modules. For any $\Bbbk G$-modules M and N, the tensor product $M \otimes_{\Bbbk} N$ inherits the structure of a $\Bbbk G$-module by letting G act diagonally. Then, $(\Bbbk G - \mathrm{Mod}, \otimes_{\Bbbk}, \Bbbk)$ is symmetric monoidal, and the ring objects are $\Bbbk$-algebras equipped with an action of G via algebra automorphisms.

1.1 The Eilenberg-Moore category of modules

We study the category of A-modules for a ring (A, μ, η) in $\mathcal{K}$. The results in this section all appear in [Bal14b, §1].

Definition 1.1.1. The *Eilenberg-Moore category* $A - \mathrm{Mod}_{\mathcal{K}}$ has A-modules as objects and A-linear morphisms. We will write $F_A : \mathcal{K} \to A - \mathrm{Mod}_{\mathcal{K}}$ for the extension-of-scalars, given by $F_A(x) = (A \otimes x, \mu \otimes 1)$, and write $U_A : A\text{–}\mathrm{Mod}_{\mathcal{K}} \to \mathcal{K}$

for the forgetful functor $U_A(x,\varrho) = x$. The adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[d, shift right, "F_A"'] \arrow[d, phantom, "\dashv"] \\ A\!-\!\mathrm{Mod}_{\mathcal{K}} \arrow[u, shift right, "U_A"'] \end{tikzcd}$$

is also called the Eilenberg-Moore adjunction, see [EM65].

The *Kleisli category* A–$\mathrm{Free}_{\mathcal{K}}$ is the full subcategory of A–$\mathrm{Mod}_{\mathcal{K}}$ on free A-modules. In other words, the objects are the same as $\mathcal{K}$, writing $F_A(x) \in A$–$\mathrm{Free}_{\mathcal{K}}$ for $x \in \mathcal{K}$, with morphisms $\mathrm{Hom}_A(F_A(x), F_A(y)) := \mathrm{Hom}_{\mathcal{K}}(x, A \otimes y)$. Let's write $\overline{f} : F_A(x) \to F_A(y)$ for the morphism corresponding to $f : x \to A \otimes y$ in $\mathcal{K}$. The Eilenberg-Moore adjunction restricts to $F_A : \mathcal{K} \underset{\longrightarrow}{\overset{\longleftarrow}{}} A\!-\!\mathrm{Free}_{\mathcal{K}} : U_A$, see [Kle65].

Example 1.1.2. Let R be a commutative ring and A a commutative finite étale (flat and separable) R-algebra. Then $\mathcal{K} := \mathrm{D}^{\mathrm{perf}}(R)$, the homotopy category of bounded complexes of finitely generated projective R-modules, is a symmetric monoidal category. Since A is R-flat, the object $A = A[0]$ in $\mathcal{K}$ keeps its ring structure. Then A is a separable ring object in $\mathcal{K}$ and the category of A-modules $A\!-\!\mathrm{Mod}_{\mathcal{K}}$ is equivalent to $\mathrm{D}^{\mathrm{perf}}(A)$ [Bal11, Th.6.5].

Definition 1.1.3. We say $\mathcal{K}$ is *idempotent-complete* if every idempotent morphism splits. That is, for all $x \in \mathcal{K}$, any morphism $e : x \to x$ with $e^2 = e$ yields a decomposition $x \cong x_1 \oplus x_2$ under which e becomes $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Remark 1.1.4. Every additive category $\mathcal{K}$ can be embedded in an idempotent-complete category $\mathcal{K}^\natural$ in such a way that $\mathcal{K} \hookrightarrow \mathcal{K}^\natural$ is fully faithful and every object in $\mathcal{K}^\natural$ is a direct summand of some object in $\mathcal{K}$. We call $\mathcal{K}^\natural$ the *idempotent-completion* of $\mathcal{K}$.

Remark 1.1.5. If $\mathcal{K}$ is idempotent-complete, the module category $A\!-\!\mathrm{Mod}_{\mathcal{K}}$ is idempotent-complete too. When A is moreover separable, $A\!-\!\mathrm{Mod}_{\mathcal{K}}$ is equivalent to the idempotent-completion of $A\!-\!\mathrm{Free}_{\mathcal{K}}$, see [Bal11]. In particular, any A-module x is a direct summand of the free module $F_A(U_A(x))$.

We can define a tensor product $\otimes_A$ on the Kleisli category $A-\mathrm{Free}_{\mathcal{K}}$ by

$$F_A(x) \otimes_A F_A(y) := F_A(x \otimes y)$$

on objects and $\overline{f} \otimes_A \overline{g} = \overline{(\mu \otimes 1 \otimes 1)(23)(f \otimes g)}$ for morphisms $f : x \to A \otimes x'$ and $g : y \to A \otimes y'$ in $\mathcal{K}$:

$$x \otimes y \xrightarrow{f \otimes g} A \otimes x' \otimes A \otimes y' \xrightarrow{(23)} A \otimes A \otimes x' \otimes y' \xrightarrow{\mu \otimes 1 \otimes 1} A \otimes x' \otimes y'.$$

The tensor $\otimes_A$ yields a symmetric monoidal structure on $A-\mathrm{Free}_{\mathcal{K}}$ with unit $A = F_A(\mathbb{1})$. If we moreover assume the ring A is separable and $\mathcal{K}$ is idempotent-complete, idempotent-completion conveys the tensor $\otimes_A$ from $A-\mathrm{Free}_{\mathcal{K}}$ to $A-\mathrm{Mod}_{\mathcal{K}}$.

We could also define a tensor product $\otimes'$ directly on objects (x, ϱ) in $A-\mathrm{Mod}_{\mathcal{K}}$; the following lemma is key. As A is commutative, we let ϱ denote both the left and right action of A on x.

Lemma 1.1.6. *Let (A, μ, η, σ) be a separable ring in $\mathcal{K}$ and suppose (x, ϱ_1) and (y, ϱ_2) are A-modules. Consider the endomorphism*

$$v_{x,y} = v : \quad x \otimes y \xrightarrow{1 \otimes \eta \otimes 1} x \otimes A \otimes y \xrightarrow{1 \otimes \sigma \otimes 1} x \otimes A \otimes A \otimes y \xrightarrow{\varrho_1 \otimes \varrho_2} x \otimes y.$$

Then, $v(\varrho_1 \otimes 1) = v(1 \otimes \varrho_2) = (\varrho_1 \otimes \varrho_2)(1 \otimes \sigma \otimes 1) : x \otimes A \otimes y \longrightarrow x \otimes y$, and any morphism $f : x \otimes y \to z$ with $z \in \mathcal{K}$ such that $f(\varrho_1 \otimes 1) = f(1 \otimes \varrho_2) : x \otimes A \otimes y \to z$, satisfies $fv = f$. In particular, v is idempotent.

Proof. First, note that $v(\varrho_1 \otimes 1) = (\varrho_1 \otimes \varrho_2)(1 \otimes \sigma \otimes 1)$ follows from

$$\begin{tikzcd}
x \otimes A \otimes y \arrow[r, "1 \otimes \sigma \otimes 1"] & x \otimes A^{\otimes 2} \otimes y \arrow[r, "\varrho_1 \otimes \varrho_2"] & x \otimes y \\
x \otimes y \arrow[u, "1 \otimes \eta \otimes 1"] & x \otimes A^{\otimes 3} \otimes y \arrow[u, "\varrho_1 \otimes 1 \otimes 1 \otimes 1"] \arrow[r, "1 \otimes \mu \otimes 1 \otimes 1"] & x \otimes A^{\otimes 2} \otimes y \arrow[u, "\varrho_1 \otimes \varrho_2"'] \\
x \otimes A \otimes y \arrow[u, "\varrho_1 \otimes 1"] \arrow[r, "1 \otimes 1 \otimes \eta \otimes 1"] \arrow[rr, bend right, "1"'] & x \otimes A^{\otimes 2} \otimes y \arrow[u, "1 \otimes 1 \otimes \sigma \otimes 1"] \arrow[r, "1 \otimes \mu \otimes 1"] & x \otimes A \otimes y, \arrow[u, "1 \otimes \sigma \otimes 1"']
\end{tikzcd}$$

in which the lower right square commutes because σ is A, A-bilinear. A similar diagram shows $v(1 \otimes \varrho_2) = (\varrho_1 \otimes \varrho_2)(1 \otimes \sigma \otimes 1)$. Finally, for morphisms f as in the lemma, we see that the diagram

$$\begin{tikzcd}
x \otimes A \otimes y \arrow[r, "1\otimes\sigma\otimes 1"] \arrow[dr, equal] & x \otimes A^{\otimes 2} \otimes y \arrow[r, "\varrho_1\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu\otimes 1"] & x \otimes A \otimes y \arrow[r, "1\otimes\varrho_2"] \arrow[d, "\varrho_1\otimes 1"] & x \otimes y \arrow[d, "f"] \\
x \otimes y \arrow[u, "1\otimes\eta\otimes 1"] \arrow[r, "1\otimes\eta\otimes 1"] \arrow[rr, bend right, "1"'] & x \otimes A \otimes y \arrow[r, "\varrho_1\otimes 1"] & x \otimes y \arrow[r, "f"] & z
\end{tikzcd}$$

commutes. $\square$

Seeing how $v : x \otimes y \to x \otimes y$ is idempotent and $\mathcal{K}$ is idempotent-complete, we can define $x \otimes' y$ as the direct summand $\operatorname{im}(v)$ of $x \otimes y$, with projection $p_{x,y} = p$ and inclusion $j_{x,y} = j$:

$$v : x \otimes y \xrightarrow{p} x \otimes' y \xrightarrow{j} x \otimes y. \tag{1.1.7}$$

By Lemma 1.1.6, we get a split coequaliser in $\mathcal{K}$,

$$x \otimes A \otimes y \underset{1\otimes\varrho_2}{\overset{\varrho_1\otimes 1}{\rightrightarrows}} x \otimes y \xrightarrow{p} x \otimes' y, \tag{1.1.8}$$

and A acts on $x \otimes' y$ by

$$A \otimes x \otimes' y \xrightarrow{1\otimes j} A \otimes x \otimes y \xrightarrow{\varrho_1\otimes 1} x \otimes y \xrightarrow{p} x \otimes' y.$$

For $f : x \to x'$ and $g : y \to y'$ in $A-\mathrm{Mod}_{\mathcal{K}}$, we consider the commutative diagram

$$\begin{tikzcd}
v: & x \otimes y \arrow[r, "1\otimes\eta\otimes 1"] \arrow[d, "f\otimes g"] & x \otimes A \otimes y \arrow[r, "1\otimes\sigma\otimes 1"] \arrow[d, "f\otimes 1\otimes g"] & x \otimes A \otimes A \otimes y \arrow[r, "\varrho_1\otimes\varrho_2"] \arrow[d, "f\otimes 1\otimes 1\otimes g"] & x \otimes y \arrow[d, "f\otimes g"] \\
v': & x' \otimes y' \arrow[r, "1\otimes\eta\otimes 1"] & x' \otimes A \otimes y' \arrow[r, "1\otimes\sigma\otimes 1"] & x' \otimes A \otimes A \otimes y' \arrow[r, "\varrho_1'\otimes\varrho_2'"] & x' \otimes y'
\end{tikzcd}$$

and get a map

$$f \otimes' g : \operatorname{im}(v) = x \otimes' y \longrightarrow \operatorname{im}(v') = x' \otimes' y'.$$

Remark 1.1.9. The diagram

$$\begin{tikzcd}
A \otimes x \otimes y \arrow[r, "\varrho_1 \otimes 1"] \arrow[d, "1\otimes 1\otimes \eta \otimes 1"] & x \otimes y \arrow[r, "1\otimes \eta \otimes 1"] & x \otimes A \otimes y \arrow[r, "1\otimes \sigma \otimes 1"] & x \otimes A^{\otimes 2} \otimes y \arrow[d, "\varrho_1 \otimes \varrho_2"] \\
A \otimes x \otimes A \otimes y \arrow[r, "1\otimes 1\otimes \sigma \otimes 1"] & A \otimes x \otimes A^{\otimes 2} \otimes y \arrow[r, "1\otimes \varrho_1 \otimes \varrho_2"] & A \otimes x \otimes y \arrow[r, "\varrho_1 \otimes 1"] & x \otimes y
\end{tikzcd}$$

commutes. In fact, v is A^e-linear and $x \otimes' y$ is a direct summand of $x \otimes y$ as A^e-modules.

Proposition 1.1.10. *Suppose $\mathcal{K}$ is idempotent-complete and (A, μ, η, σ) is a separable ring in $\mathcal{K}$. The tensor products $\otimes'$ and $\otimes_A$ on $A-\mathrm{Mod}_{\mathcal{K}}$ are naturally isomorphic. They yield a symmetric monoidal structure $\otimes_A : A\text{–}\mathrm{Mod}_{\mathcal{K}} \times A\text{–}\mathrm{Mod}_{\mathcal{K}} \to A\text{–}\mathrm{Mod}_{\mathcal{K}}$ on the Eilenberg-Moore category under which F_A becomes monoidal. We will write $\mathbb{1}_A := A$ for the unit object in $A-\mathrm{Mod}_{\mathcal{K}}$.*

Proof. It is not hard to see that $\otimes_A$ defines a symmetric monoidal structure on the Kleisli category $A\text{–}\mathrm{Free}_{\mathcal{K}}$, with unit $A = F_A(\mathbb{1})$. What is more, extension-of-scalars $F_A : \mathcal{K} \to A\text{–}\mathrm{Free}_{\mathcal{K}}$ is monoidal by construction. After idempotent-completion, the symmetric monoidal structure carries over to $A-\mathrm{Mod}_{\mathcal{K}}$. Thus, the proposition follows if we show that $\otimes'$ and $\otimes_A$ agree on $A-\mathrm{Free}_{\mathcal{K}}$. Given free A-modules $F_A(x) = A \otimes x$ and $F_A(y) = A \otimes y$, we can identify $F_A(x) \otimes F_A(y) \cong x \otimes A^{\otimes 2} \otimes y$ and note that the endomorphism from Lemma 1.1.6,

$$v : F_A(x) \otimes F_A(y) \longrightarrow F_A(x) \otimes F_A(y),$$

is given by the first row of the commuting diagram

$$\begin{tikzcd}
x \otimes A^{\otimes 2} \otimes y \arrow[r, "1\otimes 1\otimes \eta \otimes 1\otimes 1"] \arrow[dr, equal] & x \otimes A^{\otimes 3} \otimes y \arrow[r, "1\otimes 1\otimes \sigma \otimes 1\otimes 1"] \arrow[d, "1\otimes \mu \otimes 1\otimes 1"] & x \otimes A^{\otimes 4} \otimes y \arrow[r, "1\otimes \mu \otimes \mu \otimes 1"] \arrow[d, "1\otimes \mu \otimes 1\otimes 1\otimes 1"] & x \otimes A^{\otimes 2} \otimes y \arrow[d, equal] \\
& x \otimes A^{\otimes 2} \otimes y \arrow[r, "1\otimes \sigma \otimes 1\otimes 1"] \arrow[dr, "1\otimes \mu \otimes 1"'] & x \otimes A^{\otimes 3} \otimes y \arrow[r, "1\otimes 1\otimes \mu \otimes 1"] & x \otimes A^{\otimes 2} \otimes y \\
& & x \otimes A \otimes y. \arrow[ur, "1\otimes \sigma \otimes 1"'] &
\end{tikzcd}$$

In other words, the following diagram commutes

$$\begin{tikzcd}
F_A(x) \otimes F_A(y) \arrow[r, two heads, "p"] \arrow[d, "\cong"] \arrow[rr, bend left, "v"] & F_A(x) \otimes' F_A(y) \arrow[r, hook, "j"] & F_A(x) \otimes F_A(y) \arrow[d, "\cong"] \\
A \otimes A \otimes x \otimes y \arrow[r, two heads, "\mu\otimes 1\otimes 1"] & A \otimes x \otimes y \arrow[r, hook, "\sigma\otimes 1\otimes 1"] & A \otimes A \otimes x \otimes y
\end{tikzcd}$$

and $F_A(x) \otimes' F_A(y) = \mathrm{im}(v)$ is isomorphic to $A \otimes x \otimes y$ in $\mathcal{K}$.

Recall that the A-action on $F_A(x) \otimes' F_A(y)$ is given by the first row of

$$\begin{tikzcd}
A \otimes F_A(x) \otimes' F_A(y) \arrow[r, hook, "1\otimes j"] \arrow[d, "\cong"] & A \otimes A \otimes x \otimes A \otimes y \arrow[r, "\mu\otimes 1\otimes 1\otimes 1"] \arrow[d, "(34)"] & F_A(x) \otimes F_A(y) \arrow[r, two heads, "p"] & F_A(x) \otimes' F_A(y) \arrow[d, "\cong"] \\
A \otimes A \otimes x \otimes y \arrow[r, "1\otimes\sigma\otimes 1\otimes 1"] \arrow[rr, bend right, "1"'] & A \otimes A \otimes A \otimes x \otimes y \arrow[r, "1\otimes\mu\otimes 1\otimes 1"] & A \otimes A \otimes x \otimes y \arrow[r, "\mu\otimes 1\otimes 1"] & A \otimes x \otimes y,
\end{tikzcd}$$

so $F_A(x) \otimes' F_A(y)$ is isomorphic to $F_A(x \otimes y) = F_A(x) \otimes_A F_A(y)$ as A-modules.

For morphisms $\overline{f} : F_A(x) \to F_A(x')$ and $\overline{g} : F_A(y) \to F_A(y')$ in $A-\mathrm{Free}_{\mathcal{K}}$ corresponding to $f : x \to A \otimes x'$ and $g : y \to A \otimes y'$ in $\mathcal{K}$, the tensor product $\overline{f} \otimes_A \overline{g}$ is the bar of

$$x \otimes y \xrightarrow{f\otimes g} A \otimes x' \otimes A \otimes y' \xrightarrow{(23)} A \otimes A \otimes x' \otimes y' \xrightarrow{\mu\otimes 1\otimes 1} A \otimes x' \otimes y'.$$

So, $\overline{f} \otimes_A \overline{g}$ is the top row of

$$A \otimes x \otimes y \xrightarrow{1\otimes f\otimes g} A^{\otimes 2} \otimes x' \otimes A \otimes y' \xrightarrow{(34)} A^{\otimes 3} \otimes x' \otimes y' \xrightarrow{\mu\otimes 1\otimes 1\otimes 1} A^{\otimes 2} \otimes x' \otimes y' \xrightarrow{\mu\otimes 1\otimes 1} A \otimes x' \otimes y'$$

$\overline{f} \otimes' \overline{g}$ is given by

$$(A \otimes x) \otimes' (A \otimes y) \xrightarrow{j} A \otimes x \otimes A \otimes y \xrightarrow{1\otimes f\otimes g} A^{\otimes 2} \otimes x' \otimes A^{\otimes 2} \otimes y' \xrightarrow{\mu\otimes 1\mu\otimes 1} A \otimes x \otimes A \otimes y \xrightarrow{p} (A \otimes x) \otimes' (A$$

$$F_A(x) \otimes' F_A(y) \xrightarrow{j} F_A(x) \otimes F_A(y) \xrightarrow{\overline{f}\otimes\overline{g}} F_A(x') \otimes F_A(y') \xrightarrow{p} F_A(x') \otimes' F_A(y'),$$

where $\overline{f} \otimes \overline{g}$ is the map

$$A \otimes x \otimes A \otimes y \xrightarrow{1\otimes f\otimes 1\otimes g} A^{\otimes 2} \otimes x' \otimes A^{\otimes 2} \otimes y' \xrightarrow{\mu\otimes 1\otimes\mu\otimes 1} A \otimes x' \otimes A \otimes y'.$$

Under the correspondence $F_A(x) \otimes' F_A(y) \cong A \otimes x \otimes y$, this becomes

$$\begin{tikzcd}
A \otimes x \otimes y \arrow[r, "\sigma\otimes1\otimes1"] & A^{\otimes2} \otimes x \otimes y \arrow[r, "1\otimes1\otimes f\otimes g"] \arrow[d, "\mu\otimes1\otimes1"] & A^{\otimes3} \otimes x' \otimes A \otimes y' \arrow[r, "(34)"] & A^{\otimes2} \otimes x' \otimes A^{\otimes2} \otimes y' \arrow[d, "\mu\otimes1\otimes\mu\otimes1"] \\
& A \otimes x \otimes y \arrow[r, "1\otimes f\otimes g"] & A^{\otimes2} \otimes x' \otimes A \otimes y' \arrow[r, "\mu\otimes1\otimes1\otimes1"] & A \otimes x' \otimes A \otimes y'
\end{tikzcd}$$

Hence $\overline{f} \otimes' \overline{g}$ is given by

$$\begin{tikzcd}
A \otimes x \otimes y \arrow[r, "\sigma\otimes1\otimes1"] \arrow[dr, equal] & A^{\otimes2} \otimes x \otimes y \arrow[d, "\mu\otimes1\otimes1"] & A^{\otimes3} \otimes x' \otimes y' \arrow[r, "(\mu\otimes1\otimes1)(\mu\otimes1\otimes1\otimes1)"] & A \otimes x' \otimes y', \\
& A \otimes x \otimes y \arrow[r, "1\otimes f\otimes g"] & A^{\otimes2} \otimes x' \otimes A \otimes y' \arrow[u, "(34)"] &
\end{tikzcd}$$

which is the bar of

$$x \otimes y \xrightarrow{f\otimes g} A \otimes x' \otimes A \otimes y' \xrightarrow{(23)} A \otimes A \otimes x' \otimes y' \xrightarrow{\mu\otimes1\otimes1} A \otimes x' \otimes y'.$$

We conclude that

$$\begin{tikzcd}
F_A(x) \otimes' F_A(y) \arrow[r, "\cong"] \arrow[d, "\overline{f}\otimes'\overline{g}"] & A \otimes x \otimes y \arrow[d, "\overline{(\mu\otimes1\otimes1)(23)(f\otimes g)}=\overline{f}\otimes_A\overline{g}"] \\
F_A(x') \otimes' F_A(y') \arrow[r, "\cong"] & A \otimes x' \otimes y'
\end{tikzcd}$$

commutes and the tensor products $F_A(x) \otimes' F_A(y)$ and $F_A(x) \otimes_A F_A(y)$ are naturally isomorphic. $\square$

Proof. It is not hard to see that $\otimes_A$ defines a symmetric monoidal structure on the Kleisli category A–Free$_{\mathcal{K}}$, with unit $A = F_A(\mathbb{1})$. What is more, extension-of-scalars $F_A : \mathcal{K} \to A$–Free$_{\mathcal{K}}$ is monoidal by construction. After idempotent-completion, the symmetric monoidal structure carries over to $A-\mathrm{Mod}_{\mathcal{K}}$. Thus, the proposition follows if we show that $\otimes'$ and $\otimes_A$ agree on $A-\mathrm{Free}_{\mathcal{K}}$. Given free A-modules $F_A(x) = A \otimes x$ and $F_A(y) = A \otimes y$, we can identify $F_A(x) \otimes F_A(y) \cong x \otimes A^{\otimes2} \otimes y$ and note that the endomorphism from Lemma 1.1.6,

$$v : F_A(x) \otimes F_A(y) \longrightarrow F_A(x) \otimes F_A(y),$$

is given by the first row of the commuting diagram

$$\begin{tikzcd}
x \otimes A^{\otimes 2} \otimes y \arrow[r, "1\otimes 1\otimes \eta\otimes 1\otimes 1"] \arrow[dr, equal] & x \otimes A^{\otimes 3} \otimes y \arrow[r, "1\otimes 1\otimes \sigma\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu\otimes 1\otimes 1"] & x \otimes A^{\otimes 4} \otimes y \arrow[r, "1\otimes\mu\otimes\mu\otimes 1"] \arrow[d, "1\otimes\mu\otimes 1\otimes 1\otimes 1"] & x \otimes A^{\otimes 2} \otimes y \arrow[d, equal] \\
 & x \otimes A^{\otimes 2} \otimes y \arrow[r, "1\otimes\sigma\otimes 1\otimes 1"] \arrow[dr, "1\otimes\mu\otimes 1"'] & x \otimes A^{\otimes 3} \otimes y \arrow[r, "1\otimes 1\otimes\mu\otimes 1"] & x \otimes A^{\otimes 2} \otimes y \\
 & & x \otimes A \otimes y. \arrow[ur, "1\otimes\sigma\otimes 1"'] &
\end{tikzcd}$$

In other words, the following diagram commutes

$$\begin{tikzcd}
F_A(x) \otimes F_A(y) \arrow[r, two heads, "p"] \arrow[d, "\cong"] \arrow[rr, bend left, "v"] & F_A(x) \otimes' F_A(y) \arrow[r, hook, "j"] & F_A(x) \otimes F_A(y) \arrow[d, "\cong"] \\
A \otimes A \otimes x \otimes y \arrow[r, two heads, "\mu\otimes 1\otimes 1"] & A \otimes x \otimes y \arrow[r, hook, "\sigma\otimes 1\otimes 1"] & A \otimes A \otimes x \otimes y
\end{tikzcd}$$

and $F_A(x) \otimes' F_A(y) = \mathrm{im}(v)$ is isomorphic to $A \otimes x \otimes y$ in $\mathcal{K}$.

Recall that the A-action on $F_A(x) \otimes' F_A(y)$ is given by the first row of

$$\begin{tikzcd}
A \otimes F_A(x) \otimes' F_A(y) \arrow[r, hook, "1\otimes j"] \arrow[d, "\cong"] & A \otimes A \otimes x \otimes A \otimes y \arrow[r, "\mu\otimes 1\otimes 1\otimes 1"] \arrow[d, "(34)"] & F_A(x) \otimes F_A(y) \arrow[r, two heads, "p"] & F_A(x) \otimes' F_A(y) \arrow[d, "\cong"] \\
A \otimes A \otimes x \otimes y \arrow[r, "1\otimes\sigma\otimes 1\otimes 1"] \arrow[rr, bend right, "1"'] & A \otimes A \otimes A \otimes x \otimes y \arrow[r, "1\otimes\mu\otimes 1\otimes 1"] & A \otimes A \otimes x \otimes y \arrow[r, "\mu\otimes 1\otimes 1"] & A \otimes x \otimes y,
\end{tikzcd}$$

so the corresponding action on $A \otimes x \otimes y$

$$\begin{tikzcd}
A \otimes F_A(x) \otimes' F_A(y) \arrow[r, hook, "1\otimes j"] & A \otimes A \otimes x \otimes A \otimes y \arrow[r, "\mu\otimes 1\otimes 1\otimes 1"] & F_A(x) \otimes F_A(y) \arrow[d, two heads, "p"] \\
 & & F_A(x) \otimes' F_A(y),
\end{tikzcd}$$

which corresponds to

$$\begin{tikzcd}
A \otimes A \otimes x \otimes y \arrow[r, "1\otimes\sigma\otimes 1\otimes 1"] \arrow[dr, equal] & A \otimes A \otimes A \otimes x \otimes y \arrow[r, "\mu\otimes 1\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu\otimes 1\otimes 1"] & A \otimes A \otimes x \otimes y \arrow[d, "\mu\otimes 1\otimes 1"] \\
 & A \otimes A \otimes x \otimes y \arrow[r, "\mu\otimes 1\otimes 1"] & A \otimes x \otimes y,
\end{tikzcd}$$

so $F_A(x) \otimes' F_A(y)$ is isomorphic to $F_A(x \otimes y) = F_A(x) \otimes_A F_A(y)$ as A-modules. For morphisms $\overline{f} : F_A(x) \to F_A(x')$ and $\overline{g} : F_A(y) \to F_A(y')$ in $A-\mathrm{Free}_{\mathcal{K}}$ corresponding to $f : x \to A \otimes x'$ and $g : y \to A \otimes y'$ in $\mathcal{K}$, the tensor product $\overline{f} \otimes' \overline{g}$ is given by

$$F_A(x) \otimes' F_A(y) \xrightarrow{j} F_A(x) \otimes F_A(y) \xrightarrow{\overline{f}\otimes\overline{g}} F_A(x') \otimes F_A(y') \xrightarrow{p} F_A(x') \otimes' F_A(y').$$

Under the correspondence $F_A(x) \otimes' F_A(y) \cong A \otimes x \otimes y$, this becomes

$$\begin{tikzcd}
A \otimes x \otimes y \arrow[r, "\sigma\otimes1\otimes1"] & A^{\otimes 2} \otimes x \otimes y \arrow[d, "(23)"] & & A^{\otimes 2} \otimes x' \otimes y' \arrow[r, "\mu\otimes1\otimes1"] & A \otimes x' \otimes y', \\
& A \otimes x \otimes A \otimes y \arrow[rr, "\overline{f}\otimes\overline{g}"] & & A \otimes x' \otimes A \otimes y' \arrow[u, "(23)"'] &
\end{tikzcd}$$

where $\overline{f} \otimes \overline{g}$ is the map

$$A \otimes x \otimes A \otimes y \xrightarrow{1\otimes f\otimes 1\otimes g} A^{\otimes 2} \otimes x' \otimes A^{\otimes 2} \otimes y' \xrightarrow{\mu\otimes1\otimes\mu\otimes1} A \otimes x' \otimes A \otimes y'.$$

Hence $\overline{f} \otimes' \overline{g}$ is given by

$$\begin{tikzcd}
A \otimes x \otimes y \arrow[r, "\sigma\otimes1\otimes1"] \arrow[dr, equal] & A^{\otimes 2} \otimes x \otimes y \arrow[d, "\mu\otimes1\otimes1"] & & A^{\otimes 3} \otimes x' \otimes y' \arrow[r, "(\mu\otimes1\otimes1)(\mu\otimes1\otimes1\otimes1)"] & A \otimes x' \otimes y', \\
& A \otimes x \otimes y \arrow[rr, "1\otimes f\otimes g"] & & A^{\otimes 2} \otimes x' \otimes A \otimes y' \arrow[u, "(34)"'] &
\end{tikzcd}$$

which is the bar of

$$x \otimes y \xrightarrow{f\otimes g} A \otimes x' \otimes A \otimes y' \xrightarrow{(23)} A \otimes A \otimes x' \otimes y' \xrightarrow{\mu\otimes1\otimes1} A \otimes x' \otimes y'.$$

We conclude that

$$\begin{tikzcd}
F_A(x) \otimes' F_A(y) \arrow[r, "\cong"] \arrow[d, "\overline{f}\otimes'\overline{g}"] & A \otimes x \otimes y \arrow[d, "\overline{(\mu\otimes1\otimes1)(23)(f\otimes g)}=\overline{f}\otimes_A\overline{g}"] \\
F_A(x') \otimes' F_A(y') \arrow[r, "\cong"] & A \otimes x' \otimes y'
\end{tikzcd}$$

commutes and the tensor products $F_A(x) \otimes' F_A(y)$ and $F_A(x) \otimes_A F_A(y)$ are naturally isomorphic. $\square$

Remark 1.1.11. The canonical A-linear isomorphism $A \otimes_A x \xrightarrow{\cong} x$ is given by the map

$$A \otimes_A x \overset{j}{\hookrightarrow} A \otimes x \xrightarrow{\varrho} x,$$

with inverse

$$x \xrightarrow{\eta \otimes 1} A \otimes x \overset{p}{\twoheadrightarrow} A \otimes_A x.$$

Indeed, $\varrho j p(\eta \otimes 1_x) = \varrho v(\eta \otimes 1_x) = \varrho(\eta \otimes 1_x) = 1_x$ by Lemma 1.1.6 and the diagram

$$\begin{tikzcd}
A \otimes_A x \arrow[r, hook, "j"] \arrow[d, "\eta\otimes 1 \otimes_A 1"] \arrow[dd, bend right, "1"'] & A \otimes x \arrow[r, "\varrho"] & x \arrow[d, "\eta\otimes 1"] \\
A \otimes A \otimes_A x \arrow[r, hook, "1\otimes j"] \arrow[d, "\mu\otimes_A 1"] & A \otimes A \otimes x \arrow[r, "1\otimes\varrho"] \arrow[d, "\mu\otimes 1"] & A \otimes x \arrow[d, two heads, "p"] \\
A \otimes_A x \arrow[r, hook, "j"] \arrow[rr, bend right, "1"'] & A \otimes x \arrow[r, two heads, "p"] & A \otimes_A x
\end{tikzcd}$$

commutes because p is an A, A-bimodule morphism.

Remark 1.1.12. Recall that the endomorphism ring of the unit object in a symmetric monoidal category is commutative, and composition coincides with the tensor product. See [Bal10a], for instance. In particular, any two A-linear endomorphisms $A \to A$ commute.

Proposition 1.1.13. (Projection Formula). *Suppose $\mathcal{K}$ is idempotent-complete and let (A, μ, η) be a separable ring in $\mathcal{K}$. For all $x \in A-\mathrm{Mod}_{\mathcal{K}}$ and $y \in \mathcal{K}$, there is a natural isomorphism $U_A(x \otimes_A F_A(y)) \cong U_A(x) \otimes y$ in $\mathcal{K}$.*

Proof. Proving this for free modules $x \in A-\mathrm{Free}_{\mathcal{K}}$ is sufficient, so let $x = F_A(z)$ with $z \in \mathcal{K}$. We show that

$$U_A(x \otimes_A F_A(y)) = U_A(F_A(z) \otimes_A F_A(y)) = U_A(F_A(z \otimes y)) \cong A \otimes z \otimes y \cong U_A(x) \otimes y$$

naturally in x and y. For $x' = F(z') \in A-\mathrm{Free}_{\mathcal{K}}$ and $\overline{f} : x \to x'$ in $A-\mathrm{Free}_{\mathcal{K}}$ corresponding to $f : z \to A \otimes z'$ in $\mathcal{K}$, we note that $\overline{f} \otimes_A 1_{F_A(y)} = \overline{f} \otimes_A \overline{\eta \otimes 1_y}$ is the bar of

$$z \otimes y \xrightarrow{f\otimes\eta\otimes 1} A \otimes z' \otimes A \otimes y \xrightarrow{(23)} A \otimes A \otimes z' \otimes y \xrightarrow{\mu\otimes 1\otimes 1} A \otimes z' \otimes y,$$

so $\overline{f} \otimes_A 1_{F_A(y)} = \overline{f \otimes 1}$. Therefore, the required diagram

$$\begin{tikzcd}
U_A(F_A(z) \otimes_A F_A(y)) \arrow[r, phantom, "="] \arrow[dd, "U_A(\overline{f}\otimes_A 1)"] & U_A(F_A(z \otimes y)) \arrow[r, phantom, "\cong"] \arrow[dd, "U_A(\overline{f\otimes 1})"] & A \otimes z \otimes y \arrow[r, phantom, "\cong"] \arrow[d, "1\otimes f\otimes 1"] & U_A(x) \otimes y \arrow[dd, "U_A(\overline{f})\otimes 1"] \\
 & & A \otimes A \otimes z \otimes y \arrow[d, "\mu\otimes 1\otimes 1"] & \\
U_A(F_A(z') \otimes_A F_A(y)) \arrow[r, phantom, "="] & U_A(F_A(z' \otimes y)) \arrow[r, phantom, "\cong"] & A \otimes z' \otimes y \arrow[r, phantom, "\cong"] & U_A(x') \otimes y
\end{tikzcd}$$

commutes. Naturality in y follows easily. $\square$

Remark 1.1.14. Explicitly, the isomorphism $U_A(x \otimes_A F_A(y)) \cong U_A(x) \otimes y$ from Proposition 1.1.13 is given by

$$\phi_{x,y} : \quad x \otimes_A (A \otimes y) \xhookrightarrow{j} x \otimes (A \otimes y) \xrightarrow{(12)} A \otimes x \otimes y \xrightarrow{\varrho} x \otimes y,$$

where $\varrho : A \otimes x \to x$ is the action of A on x. In particular, $\phi_{x,y}$ is left A-linear.

1.2 Rings in the Eilenberg-Moore category

This section contains a haphazardous collection of results on algebras over a ring object. Again, all of the results can be found in [Bal14b, §1].

Let (A, μ_A, η_A) and (B, μ_B, η_B) be rings in $\mathcal{K}$. We say that B is an A-*algebra* if there is a ring morphism $h : A \to B$ in $\mathcal{K}$. In that case, we can consider the A-module structure on B given by

$$A \otimes B \xrightarrow{h\otimes 1} B \otimes B \xrightarrow{\mu_B} B, \tag{1.2.1}$$

making μ_B into an A^e-linear morphism. We define a functor $F_h : A\!-\!\mathrm{Free}_{\mathcal{K}} \to B\!-\!\mathrm{Free}_{\mathcal{K}}$ on the Kleisli category by setting

$$F_h(F_A(x)) = F_B(x) \qquad \text{and} \qquad F_h(\overline{f}) = \overline{(h \otimes 1_x) f}$$

for $F_A(x)$ in $A-\mathrm{Free}_{\mathcal{K}}$ and $f : x \to A \otimes x'$ in $\mathcal{K}$. It is not hard to check that F_h is monoidal, with

$$F_h(F_A(x) \otimes_A F_A(y)) = F_h(F_A(x \otimes y)) = F_B(x \otimes y) = F_B(x) \otimes_B F_B(y).$$

Indeed, for morphisms $f : x \to A \otimes x'$ and $g : y \to A \otimes y'$ in $\mathcal{K}$ we know that $\overline{f} \otimes_A \overline{g} = \overline{(\mu_A \otimes 1 \otimes 1)(23)(f \otimes g)}$ and

$$F_h(\overline{f}) \otimes_B F_h(\overline{g}) = \overline{(\mu_B \otimes 1 \otimes 1)(23)(h \otimes 1 \otimes h \otimes 1)(f \otimes g)},$$

and the following diagram commutes:

$$\begin{tikzcd}
x \otimes y \arrow[r, "f\otimes g"] \arrow[d, "f\otimes g"] & A \otimes x' \otimes A \otimes y' \arrow[r, "(23)"] \arrow[d, "h\otimes 1\otimes h\otimes 1"] & A \otimes A \otimes x' \otimes y' \arrow[r, "\mu_A\otimes 1\otimes 1"] \arrow[d, "h\otimes h\otimes 1\otimes 1"] & A \otimes x' \otimes y' \arrow[d, "h\otimes 1\otimes 1"] \\
A \otimes x' \otimes A \otimes y' \arrow[r, "h\otimes 1\otimes h\otimes 1"] & B \otimes x' \otimes B \otimes y' \arrow[r, "(23)"] & B \otimes B \otimes x' \otimes y' \arrow[r, "\mu_B\otimes 1\otimes 1"] & B \otimes x' \otimes y'.
\end{tikzcd}$$

If $\mathcal{K}$ is moreover idempotent-complete and the rings A and B are separable, idempotent-completion yields a monoidal functor $F_h : A-\mathrm{Mod}_{\mathcal{K}} \longrightarrow B-\mathrm{Mod}_{\mathcal{K}}$. Alternatively, if x is an A-module, we can consider the B-module structure on $B \otimes_A x$ given by

$$B \otimes B \otimes_A x \xrightarrow{\mu_B \otimes_A 1} B \otimes_A x$$

and define a functor $F'_h : A-\mathrm{Mod}_{\mathcal{K}} \longrightarrow B-\mathrm{Mod}_{\mathcal{K}}$ by setting

$$F'_h(x) = B \otimes_A x \qquad \text{and} \qquad F'_h(f) = 1_B \otimes_A f$$

for A-linear morphisms f.

Proposition 1.2.2. *Suppose that $\mathcal{K}$ is idempotent-complete and (A, μ_A, η_A) and (B, μ_B, η_B) are separable rings in $\mathcal{K}$. If there is a ring morphism $h : A \to B$ in $\mathcal{K}$, then the functors F_h and F'_h defined above are naturally isomorphic.*

Proof. It suffices to show that F_h and F'_h agree on $A-\mathrm{Free}_{\mathcal{K}}$, so let $F_A(x)$ be the free A-module on $x \in \mathcal{K}$. By the Projection Formula 1.1.13, we have a natural isomorphism

$$F'_h(F_A(x)) = B \otimes_A F_A(x) \xrightarrow[\cong]{\phi_{B,x}} B \otimes x = F_B(x) = F_h(F_A(x)),$$

in $\mathcal{K}$, which is B-linear seeing how

$$\begin{tikzcd}
(B\otimes B)\otimes_A F_A(x) \arrow[r, "\phi_{B\otimes B,x}"] \arrow[d, "\mu_B\otimes_A 1_{F_A(x)}"] & B\otimes B\otimes x \arrow[d, "\mu_B\otimes 1_x"] \\
B\otimes_A F_A(x) \arrow[r, "\phi_{B,x}"] & B\otimes x.
\end{tikzcd}$$

commutes by naturality of ϕ. In fact, Remark 1.1.14 shows that $\phi_{B,x}$ is given by

$$\phi_{B,x}: \quad B\otimes_A F_A(x) \xhookrightarrow{j} B\otimes A\otimes x \xrightarrow{1\otimes h\otimes 1} B\otimes B\otimes x \xrightarrow{\mu_B\otimes 1} B\otimes x,$$

with j defined as in 1.1.7. Let $\overline{f}: F_A(x)\to F_A(y)$ be a morphism in $A-\mathrm{Free}_{\mathcal{K}}$, corresponding to $f: x\to A\otimes y$ in $\mathcal{K}$. Then, $F'_h(\overline{f}) = 1\otimes_A \overline{f}$ is given by

$$F'_h(\overline{f}): \quad B\otimes_A(A\otimes x) \xrightarrow{1\otimes_A 1\otimes f} B\otimes_A(A\otimes A\otimes y) \xrightarrow{1\otimes_A\mu_A\otimes 1} B\otimes_A(A\otimes y)$$

and F_h maps $\overline{f}$ to $F_h(\overline{f}) = \overline{(h\otimes 1)f}$:

$$F_h(\overline{f}): \quad B\otimes x \xrightarrow{1\otimes f} B\otimes A\otimes y \xrightarrow{1\otimes h\otimes 1} B\otimes B\otimes y \xrightarrow{\mu_B\otimes 1} B\otimes y.$$

So, it suffices to show

$$\begin{tikzcd}
B\otimes_A(A\otimes A\otimes y) \arrow[rr, "1\otimes_A\mu_A\otimes 1"] \arrow[d, "\phi_{B,A\otimes y}"] & & B\otimes_A(A\otimes y) \arrow[d, "\phi_{B,y}"] \\
B\otimes A\otimes y \arrow[r, "1\otimes h\otimes 1"] & B\otimes B\otimes y \arrow[r, "\mu_B\otimes 1"] & B\otimes y
\end{tikzcd}$$

commutes, which follows because ϕ is left A-linear (Remark 1.1.14). $\square$

Remark 1.2.3. Suppose (A,μ_A,η_A) and (B,μ_B,η_B) are separable rings in $\mathcal{K}$ and $h: A\to B$ is a ring morphism.

1. The following diagram commutes up to isomorphism:

$$\begin{tikzcd}
& \mathcal{K} \arrow[dl, "F_A"'] \arrow[dr, "F_B"] & \\
A-\mathrm{Mod}_{\mathcal{K}} \arrow[rr, "F_h"] & & B-\mathrm{Mod}_{\mathcal{K}}.
\end{tikzcd}$$

Indeed, $F_h(F_A(x)) \cong F_B(x)$ for every $x \in \mathcal{K}$, and for morphisms $f : x \to y$ in $\mathcal{K}$ we see that

$$F_h(F_A(f)) = F_h(\overline{\eta_A \otimes f}) \cong \overline{(h \otimes 1)(\eta_A \otimes f)} = \overline{\eta_B \otimes f} = F_B(f)$$

in $B-\mathrm{Mod}_{\mathcal{K}}$.

2. If $k : B \to C$ is another ring morphism then $F_{kh} \cong F_k F_h$. As before, it suffices to check this on the Kleisli category. For $F_A(x) \in A-\mathrm{Free}_{\mathcal{K}}$, clearly $F_k(F_h(F_A(x))) \cong F_k(F_B(x)) \cong F_C(x)$, and

$$F_k(F_h(\overline{f})) \cong F_k(\overline{(h \otimes 1)f}) \cong \overline{(k \otimes 1)(h \otimes 1)f} \cong F_{kh}(f)$$

for every morphism $f : x \to A \otimes y$ in $\mathcal{K}$.

Proposition 1.2.4. *Suppose $\mathcal{K}$ is idempotent-complete and let A be a separable ring in $\mathcal{K}$. There is a one-to-one correspondence between A-algebras B in $\mathcal{K}$ and rings $\overline{B}$ in $A-\mathrm{Mod}_{\mathcal{K}}$. Under this correspondence, B is separable if and only if $\overline{B}$ is.*

Proof. If B is a ring in $\mathcal{K}$ and $h : A \to B$ a ring morphism, we can equip B with an A^e-module structure as in 1.2.1. We will write ϱ for both the left and right action of A on B, and $(\overline{B}, \varrho)$ for the corresponding object in $A-\mathrm{Mod}_{\mathcal{K}}$. As before, we will write j and p for the inclusion and projection maps of the direct summand $B \otimes_A B$ in $B \otimes B$,

$$v : \quad B \otimes_A B \overset{j}{\hookrightarrow} B \otimes B \overset{p}{\twoheadrightarrow} B \otimes_A B.$$

Since $\mu_B(\varrho \otimes 1) = \mu_B(1 \otimes \varrho) : B \otimes A \otimes B \to B$, the coequaliser 1.1.8 gives a map $\overline{\mu} : B \otimes_A B \longrightarrow B$ such that $\overline{\mu} p = \mu_B$:

$$\begin{tikzcd} B \otimes B \arrow[r, two heads, "p"] \arrow[dr, "\mu_B"'] & B \otimes_A B \arrow[d, "\overline{\mu}"] \\ & B. \end{tikzcd}$$

In fact, we have that $\overline{\mu} = \mu_B j$, seeing how $\mu_B = \mu_B v = \mu_B jp$ by Lemma 1.1.6. This shows that $\overline{\mu}$ is A-linear, and we can define the commutative ring $(\overline{B}, \overline{\mu}, \overline{\eta} := h)$ in $A-\mathrm{Mod}_{\mathcal{K}}$. To show that $\overline{\mu}(1_B \otimes_A \overline{\eta}) = 1_B$, for instance, note that

$$\begin{tikzcd}
B \arrow[r, "1\otimes\eta_A"] \arrow[rr, bend left, "1\otimes\eta_B"] \arrow[d, equal] & B\otimes A \arrow[r, "1\otimes h"] \arrow[d, two heads, "p"] & B\otimes B \arrow[r, "\mu_B"] \arrow[d, two heads, "p"] & B \arrow[d, equal] \\
B \arrow[r, "\cong"] & B\otimes_A A \arrow[r, "1\otimes_A\overline{\eta}"] & B\otimes_A B \arrow[r, "\overline{\mu}"] & B
\end{tikzcd} \tag{1.2.5}$$

commutes. Now, suppose B is separable. The section $\sigma_B : B \to B\otimes B$ is A-linear, seeing how the diagram

$$\begin{tikzcd}
A\otimes B \arrow[r, "h\otimes 1"] \arrow[d, "1\otimes\sigma_B"] & B\otimes B \arrow[r, "\mu_B"] \arrow[d, "1\otimes\sigma_B"] & B \arrow[d, "\sigma_B"] \\
A\otimes B\otimes B \arrow[r, "h\otimes 1\otimes 1"] & B\otimes B\otimes B \arrow[r, "\mu_B\otimes 1"] & B\otimes B
\end{tikzcd}$$

commutes. Then, the ring $\overline{B}$ in $A-\mathrm{Mod}_{\mathcal{K}}$ is separable with section $p\sigma_B : \overline{B} \to \overline{B}\otimes_A \overline{B}$ for $\overline{\mu}$. Indeed, $\overline{\mu} p\sigma_B = \mu_B jp\sigma_B = \mu_B v\sigma_B = \mu_B\sigma_B = 1_B$ and $p\sigma_B$ is (left) $\overline{B}$-linear because the following diagram commutes:

$$\begin{tikzcd}
B\otimes_A B \arrow[r, "1\otimes_A\sigma_B"] \arrow[d, "j"] \arrow[dd, bend right=60, "\overline{\mu}"'] & B\otimes_A B\otimes B \arrow[r, "1\otimes_A p"] \arrow[d, "j\otimes 1"] & B\otimes_A B\otimes_A B \arrow[d, "j\otimes_A 1"] \arrow[dd, bend left=60, "\overline{\mu}\otimes_A 1"] \\
B\otimes B \arrow[r, "1\otimes\sigma_B"] \arrow[d, "\mu_B"] & B\otimes B\otimes B \arrow[r, "1\otimes p"] \arrow[d, "\mu_B\otimes 1"] & B\otimes B\otimes_A B \arrow[d, "\mu_B\otimes_A 1"] \\
B \arrow[r, "\sigma_B"] & B\otimes B \arrow[r, "p"] & B\otimes_A B.
\end{tikzcd}$$

Right $\overline{B}$-linearity follows similarly.

Conversely, for any ring $(\overline{B}, \overline{\mu}, \overline{\eta})$ in $A-\mathrm{Mod}_{\mathcal{K}}$, write $h := U_A(\overline{\eta}) : A \to B$. One easily verifies that $B := U_A(\overline{B})$ is a commutative ring in $\mathcal{K}$ with unit $\eta_B : \mathbb{1} \xrightarrow{\eta_A} A \xrightarrow{h} B$ and multiplication

$$\mu_B : \quad B\otimes B \overset{p}{\twoheadrightarrow} B\otimes_A B \xrightarrow{\overline{\mu}} B.$$

In particular, the commuting diagram 1.2.5 shows that $\mu_B(1_B \otimes \eta_B) = 1_B$. If we moreover assume that $\overline{B}$ is separable in $A-\mathrm{Mod}_{\mathcal{K}}$ with section $\overline{\sigma}$ for $\overline{\mu}$, it follows that B is separable in $\mathcal{K}$ with section $j\overline{\sigma} : B \to B \otimes B$. That is, $\mu_B j\overline{\sigma} = \overline{\mu} p j \overline{\sigma} = \overline{\mu}\,\overline{\sigma} = 1_B$ and $j\overline{\sigma}$ is (left) B-linear because the following diagram commutes:

$$\begin{tikzcd}
B \otimes B \arrow[r, "1\otimes\overline{\sigma}"] \arrow[d, "p"] \arrow[dd, bend right, "\mu_B"'] & B \otimes B \otimes_A B \arrow[r, "1\otimes j"] \arrow[d, "p\otimes_A 1"] & B \otimes B \otimes B \arrow[d, "p\otimes 1"] \arrow[dd, bend left, "\mu_B\otimes 1"] \\
B \otimes_A B \arrow[r, "1\otimes_A\overline{\sigma}"] \arrow[d, "\overline{\mu}"] & B \otimes_A B \otimes_A B \arrow[r, "1\otimes_A j"] \arrow[d, "\overline{\mu}\otimes_A 1"] & B \otimes_A B \otimes B \arrow[d, "\overline{\mu}\otimes 1"] \\
B \arrow[r, "\overline{\sigma}"] & B \otimes_A B \arrow[r, "j"] & B \otimes B.
\end{tikzcd}$$

Right B-linearity follows similarly. Finally,

$$\begin{tikzcd}
A \otimes A \arrow[r, "h\otimes 1"] \arrow[d, "p"] \arrow[dd, bend right, "\mu"'] & B \otimes A \arrow[r, "1\otimes h"] \arrow[d, "p"] & B \otimes B \arrow[r, "\mu_B"] \arrow[d, "p"] & B \arrow[d, equal] \\
A \otimes_A A \arrow[r, "\overline{\eta}\otimes_A 1"] \arrow[d, "\cong"] & B \otimes_A A \arrow[r, "1\otimes_A\overline{\eta}"] \arrow[d, "\cong"] & B \otimes_A B \arrow[r, "\overline{\mu}"] & B \arrow[d, equal] \\
A \arrow[r, "h"] & B \arrow[rr, equal] & & B
\end{tikzcd}$$

commutes so $h = U_A(\overline{\eta})$ is a ring morphism. $\square$

Proposition 1.2.6. *Suppose $\mathcal{K}$ is idempotent-complete, A is a separable ring in $\mathcal{K}$ and B is a separable A-algebra, say $\overline{B} \in \mathcal{L} := A$–$\mathrm{Mod}_{\mathcal{K}}$. There is an equivalence $B-\mathrm{Mod}_{\mathcal{K}} \simeq \overline{B}-\mathrm{Mod}_{\mathcal{L}}$ such that*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F_A"] \arrow[d, "F_B"] & \mathcal{L} \arrow[d, "F_{\overline{B}}"] \\
B-\mathrm{Mod}_{\mathcal{K}} \arrow[r, "\simeq"] & \overline{B}-\mathrm{Mod}_{\mathcal{L}},
\end{tikzcd}$$

commutes up to isomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "F_A"] \arrow[d, "F_B"] & \mathcal{L} \arrow[d, "F_{\overline{B}}"] \\
B-\mathrm{Free}_{\mathcal{K}} \arrow[r] & \overline{B}-\mathrm{Free}_{\mathcal{L}},
\end{tikzcd}$$

where the bottom row is defined on objects by sending $F_B(x)$ to $F_{\overline{B}}(F_A(x))$. On morphisms, it is defined by the sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_B(F_B(x), F_B(y)) &= \mathrm{Hom}_{\mathcal{K}}(x, B \otimes y) = \mathrm{Hom}_{\mathcal{K}}(x, U_A(\overline{B}) \otimes y) \cong \mathrm{Hom}_{\mathcal{K}}(x, U_A(\overline{B} \otimes_A F_A(y)) \\ &\cong \mathrm{Hom}_A(F_A(x), \overline{B} \otimes_A F_A(y)) = \mathrm{Hom}_{\overline{B}}(F_{\overline{B}}(F_A(x)), F_{\overline{B}}(F_A(y))),\end{aligned}$$

where the first isomorphism is given by the Projection Formula 1.1.13. Seeing how $F_{\overline{B}}(x)$ is a direct summand of $F_{\overline{B}}(F_A(x))$ for every $x \in \mathcal{L}$, it follows that the bottom row is an equivalence up to direct summands. After idempotent-completion, the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F_A"] \arrow[d, "F_B"] & \mathcal{L} \arrow[d, "F_{\overline{B}}"] \\ B\text{-}\mathrm{Mod}_{\mathcal{K}} \arrow[r, "\simeq"] & \overline{B}\text{-}\mathrm{Mod}_{\mathcal{L}}, \end{tikzcd}$$

commutes up to isomorphism and the bottom row is an equivalence. □

Proposition 1.2.7. *Suppose $\mathcal{K}$ and $\mathcal{L}$ are symmetric monoidal idempotent-complete categories, $F : \mathcal{K} \to \mathcal{L}$ is a monoidal functor and A is a separable ring in $\mathcal{K}$. Then, $B := F(A)$ is a separable ring in $\mathcal{L}$ and there exists a monoidal functor $\overline{F} : A\text{-}\mathrm{Mod}_{\mathcal{K}} \to B\text{-}\mathrm{Mod}_{\mathcal{L}}$ such that $\overline{F}F_A \cong F_B F$ and $U_B\overline{F} = FU_A$,*

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "F_A"] \arrow[d, "F"] & A\text{-}\mathrm{Mod}_{\mathcal{K}} \arrow[l, shift left, "U_A"] \arrow[d, "\overline{F}"] \\ \mathcal{L} \arrow[r, shift left, "F_B"] & B\text{-}\mathrm{Mod}_{\mathcal{L}} \arrow[l, shift left, "U_B"]. \end{tikzcd}$$

Proof. If (A, μ_A, η_A) is a separable ring in $\mathcal{K}$, it is not hard to see that $B := F(A)$ is a separable ring in $\mathcal{L}$ with multiplication $\mu_B : B \otimes B \cong F(A \otimes A) \xrightarrow{F(\mu_A)} F(A) = B$ and unit $\eta_B : \mathbb{1}_{\mathcal{L}} \cong F(\mathbb{1}_{\mathcal{K}}) \xrightarrow{F(\eta_A)} F(A) = B$. We define the monoidal functor $\overline{F}$ on objects $(x, \varrho) \in A\text{-}\mathrm{Mod}_{\mathcal{K}}$ by $\overline{F}(x) = F(x)$ with B-module structure given by $B \otimes F(x) \cong F(A \otimes x) \xrightarrow{F(\varrho)} F(x)$. On morphisms, $\overline{F}$ is given by $\overline{F}(f) = F(f)$. Now, $U_B\overline{F} = FU_A$ follows immediately and we can check that

$$\overline{F}(F_A(x)) = F(A \otimes x) \cong B \otimes F(x) = F_B(F(x)).$$

The restricted functor $\overline{F} : A\text{–Free}_{\mathcal{K}} \to B\text{–Free}_{\mathcal{L}}$ is clearly monoidal, and therefore so is $\overline{F} : A-\mathrm{Mod}_{\mathcal{K}} \to B-\mathrm{Mod}_{\mathcal{L}}$. $\square$

CHAPTER 2

Separable rings

In this chapter, we collect some concepts and results on separable rings. In particular, we define the degree $\deg(A)$ of a separable ring A in $\mathcal{K}$ and study its relation to the number of ring morphisms $A \to B$.

Throughout this chapter, $(\mathcal{K}, \otimes, \mathbb{1})$ **will denote an idempotent-complete symmetric monoidal category.**

Definition 2.0.1. Let us call a ring A in $\mathcal{K}$ *indecomposable* when A is nonzero and the only idempotent A-linear morphisms $A \to A$ in $\mathcal{K}$ are the identity 1_A and 0. In other words, A is an indecomposable ring if it doesn't decompose as a direct sum of nonzero A^e-modules, or as a product of nonzero rings (see Lemma 2.1.2).

Remark 2.0.2. Let A be a separable ring in $\mathcal{K}$. Recalling the one-to-one correspondence between A-algebras B in $\mathcal{K}$ and rings $\overline{B}$ in A–$\mathrm{Mod}_{\mathcal{K}}$, from Proposition 1.2.4, the ring B is indecomposable if and only if $\overline{B}$ is.

Lemma 2.0.3. *Let (A, μ, η) be a separable ring in $\mathcal{K}$.*

(a) For every ring morphism $\alpha : A \to \mathbb{1}$, there exists a unique idempotent A-linear morphism $e : A \to A$ such that $\alpha e = \alpha$ and $e\eta\alpha = e$.

(b) If $\mathbb{1}$ is indecomposable and $\alpha_i : A \to \mathbb{1}$ with $1 \leq i \leq n$ are distinct ring morphisms with corresponding idempotent morphisms $e_i : A \to A$ as above, then $e_i e_j = \delta_{i,j} e_i$ and $\alpha_i e_j = \delta_{i,j} \alpha_i$.

Proof. For (a), consider the right A-linear morphism $e := (\alpha \otimes 1)\sigma : A \to A$, where σ is a separability morphism. Since A is commutative, it is also a left A-linear morphism. Idempotence follows from the commutativity of

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[ddr, "\sigma"'] & A \otimes A \arrow[r, "\alpha\otimes 1"] \arrow[d, "1\otimes\sigma"] & A \arrow[d, "\sigma"] \\
& A \otimes A \otimes A \arrow[r, "\alpha\otimes 1\otimes 1"] \arrow[d, "\mu\otimes 1"] & A \otimes A \arrow[d, "\alpha\otimes 1"] \\
& A \otimes A \arrow[r, "\alpha\otimes 1"] & A
\end{tikzcd}$$

and $\alpha e = \alpha(\alpha \otimes 1)\sigma = \alpha\mu\sigma = \alpha$ is clear. Finally, $e\eta\alpha$ is the top row of the commuting diagram

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "\alpha\otimes 1_{\mathbb{1}}"] \arrow[d, equal] & \mathbb{1} \otimes \mathbb{1} \arrow[r, "1_{\mathbb{1}}\otimes\eta"] & \mathbb{1} \otimes A \arrow[r, "1_{\mathbb{1}}\otimes\sigma"] & \mathbb{1} \otimes A \otimes A \arrow[r, "1_{\mathbb{1}}\otimes\alpha\otimes 1_A"] & \mathbb{1} \otimes A \arrow[d, equal] \\
A \otimes \mathbb{1} \arrow[r, "1_A\otimes\eta"] \arrow[dr, equal] & A \otimes A \arrow[r, "1_A\otimes\sigma"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[r, "1_A\otimes\alpha\otimes 1_A"] \arrow[d, "\mu\otimes 1_A"] & A \otimes A \arrow[r, "\alpha\otimes 1_A"] & \mathbb{1} \otimes A \\
& A \arrow[r, "\sigma"] & A \otimes A \arrow[urr, "\alpha\otimes 1_A"'] & &
\end{tikzcd}$$

so $e\eta\alpha = e$ indeed. Now, for any other A-linear morphism e' such that $\alpha e' = \alpha$ and $e'\eta\alpha = e'$, we see that $e = e\eta\alpha = e\eta\alpha e' = ee' = e'e = e'\eta\alpha e = e'\eta\alpha = e'$ by Remark 1.1.12.

For (b), let $1 \leq i, j \leq n$ and consider the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha_i"] \arrow[d, "1\otimes\eta"] & \mathbb{1} \arrow[r, "\eta"] & A \arrow[d, "e_j"] \\
A \otimes A \arrow[r, "1\otimes e_j"] \arrow[d, "\mu"] & A \otimes A \arrow[r, "\alpha_i\otimes 1"] \arrow[d, "\mu"] & A \arrow[d, "\alpha_i"] \\
A \arrow[r, "e_j"] & A \arrow[r, "\alpha_i"] & \mathbb{1},
\end{tikzcd}$$

which shows that $\alpha_i e_j \eta\alpha_i = \alpha_i e_j$. Then, $(\alpha_i e_j \eta)(\alpha_i e_j \eta) = \alpha_i e_j e_j \eta = \alpha_i e_j \eta$ so $\alpha_i e_j \eta : \mathbb{1} \to \mathbb{1}$ is idempotent and therefore 0 or $1_{\mathbb{1}}$. In the first case, $\alpha_i e_j = \alpha_i e_j \eta \alpha_j = 0$. If $\alpha_i e_j \eta = 1_{\mathbb{1}}$, we get $\alpha_i e_j = \alpha_i e_j \eta \alpha_i = \alpha_i$ by the above diagram and also $\alpha_i e_j = \alpha_i e_j \eta \alpha_j = \alpha_j$, thus $i = j$. Hence, $\alpha_i e_j = \delta_{i,j}\alpha_i$ and $e_i e_j = e_i \eta \alpha_i e_j = e_i \eta \alpha_i \delta_{i,j} = e_i \delta_{i,j}$. $\square$

Proposition 2.0.4. *Let A be a separable ring in $\mathcal{K}$ and suppose A decomposes as a product of indecomposable A-algebras $A \cong A_1 \times \ldots \times A_n$. This decomposition is unique up to (possibly non-unique) isomorphism.*

Proof. Let A be as above, with decomposition $A \cong A_1 \times \ldots \times A_n$ for some indecomposable A-algebras $A_1, \ldots, A_n$. We show that for any two rings B, C with $A \cong B \times C$, we can find $0 \leq k \leq n$ such that $B \cong A_1 \times \ldots \times A_k$ and $C \cong A_{k+1} \times \ldots \times A_n$ as A-algebras, possibly after reordering the A_i. The proposition then follows immediately. The category $A-\mathrm{Mod}_{\mathcal{K}}$ decomposes as

$$A-\mathrm{Mod}_{\mathcal{K}} \cong A_1-\mathrm{Mod}_{\mathcal{K}} \times \ldots \times A_n-\mathrm{Mod}_{\mathcal{K}},$$

under which $\mathbb{1}_A$ corresponds to $(\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n})$. Accordingly, the A-algebras B and C can be written as $(B_1, \ldots, B_n)$ and $(C_1, \ldots, C_n)$ respectively, with $B_i, C_i \in A_i-\mathrm{Mod}_{\mathcal{K}}$ for every i. Given $\mathbb{1}_A \cong B \times C$, we see $\mathbb{1}_{A_i} \cong B_i \times C_i$ for every i. The indecomposability of $\mathbb{1}_{A_i}$ then gives $B_i = 0$ or $B_i = \mathbb{1}_{A_i}$. Without loss of generality, we can assume $(B_1, \ldots, B_n) \cong (\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_k}, 0, \ldots, 0)$ in $A-\mathrm{Mod}_{\mathcal{K}}$ for some $0 \leq k \leq n$. We conclude $B \cong A_1 \times \ldots \times A_k$ and $C \cong A_{k+1} \times \ldots \times A_n$ as A-algebras. $\square$

Remark 2.0.5. The above argument moreover shows that if a ring $A \cong B \times C$ in $\mathcal{K}$ has an indecomposable ring factor A_1, then A_1 is a ring factor of B or C.

2.1 Degree of a separable ring

In this section, we recall Balmer's definition of the degree of a separable ring in a tensor-triangulated category, see [Bal14b], and show the definition still holds in a non-triangulated setting. Unless stated otherwise, we only assume $\mathcal{K}$ is an idempotent-complete symmetric monoidal category. Whenever A is a ring and B is an A-algebra in $\mathcal{K}$, we will write $\overline{B}$ for the corresponding ring object in A–$\mathrm{Mod}_{\mathcal{K}}$ (as in Proposition 1.2.4).

Lemma 2.1.1. *Let (A, μ_A, η_A) and (B, μ_B, η_B) be separable rings in $\mathcal{K}$ and suppose $f : A \to B$ and $g : B \to A$ are ring morphisms such that $g \circ f = 1_A$. Equipping A with the structure of B^e-module via the morphism g, there exists a B^e-linear morphism $\widetilde{f} : A \to B$ such that $g \circ \widetilde{f} = 1_A$. In particular, A is a direct summand of B as a B^e-module.*

Proof. First, consider the A-module structure on B given by f and note that $g : \; B \to A$ is A-linear:

$$\begin{tikzcd}
A \otimes B \arrow[r, "f\otimes 1"] \arrow[d, equal] & B \otimes B \arrow[r, "\mu_B"] \arrow[d, "g\otimes g"] & B \arrow[d, "g"] \\
A \otimes B \arrow[r, "1\otimes g"] & A \otimes A \arrow[r, "\mu_A"] & A.
\end{tikzcd}$$

We can then apply Lemma 2.0.3 to the ring morphism $\bar{g} : \overline{B} \to \mathbb{1}_A$ in $A-\mathrm{Mod}_{\mathcal{K}}$ to find an idempotent $\overline{B^e}$-linear morphism $\bar{e} : \overline{B} \to \overline{B}$ such that $\bar{g}\bar{e} = \bar{g}$ and $\bar{e}\eta_{\bar{B}}\bar{g} = \bar{e}$. Forgetting the A-action, we are left with an idempotent B^e-linear morphism e such that $ge = g$ and $efg = e$. Let $\widetilde{f} := ef$. Now we equip A with a B^e-module structure via the morphism g and show that $\widetilde{f}$ is indeed B^e-linear. It is left B-linear because the diagram

$$\begin{tikzcd}
B \otimes A \arrow[r, "g\otimes 1"] \arrow[dd, "1\otimes f"] & A \otimes A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"] & A \arrow[d, "f"] \\
 & B \otimes B \arrow[r, "\mu_B"] \arrow[d, "e\otimes 1"] & B \arrow[d, "e"] \\
B \otimes B \arrow[r, shift left, "efg\otimes 1 = e\otimes 1"] \arrow[r, shift right, "1\otimes e"'] & B \otimes B \arrow[r, "\mu_B"] & B
\end{tikzcd}$$

commutes, where $\mu_B(1 \otimes e) = e\mu_B = \mu_B(e \otimes 1)$ in the last row since e is B^e-linear. By a similar argument, $\widetilde{f}$ is also right B-linear. Finally, $g\widetilde{f} = gef = gf = 1_A$. $\square$

Lemma 2.1.2. ([Bal14b, Lem.2.2]). *Let A be a ring in $\mathcal{K}$, A_1 and A_2 two A^e-modules and $h : A \xrightarrow{\sim} A_1 \oplus A_2$ an A^e-linear isomorphism. Then A_1 and A_2 admit unique ring structures under which h becomes a ring isomorphism $h : A \xrightarrow{\sim} A_1 \times A_2$.*

Theorem 2.1.3. *Let A and B separable rings in $\mathcal{K}$ and suppose $f : A \to B$ and $g : B \to A$ are ring morphisms such that $g \circ f = 1_A$. Then there exists a separable ring C and a ring isomorphism $h : B \xrightarrow{\sim} A \times C$ such that $\mathrm{pr}_1 h = g$. Equipping C with an A-module structure via the morphism $\mathrm{pr}_2 hf$, it is unique up to isomorphism of A-algebras.*

Proof. Most of the proof in [Bal14b, Th.2.2] still holds, adjusting to the non-triangulated case by way of Lemma 2.1.1. $\square$

For a separable ring (A, μ, η) in $\mathcal{K}$, we can apply Theorem 2.1.3 to the morphisms $f = 1_A \otimes \eta : A \to A \otimes A$ and $g = \mu : A \otimes A \to A$. Thus we find a separable ring A' and a ring isomorphism $h : A \otimes A \xrightarrow{\sim} A \times A'$ such that $\mathrm{pr}_1 h = \mu$. The resulting A-algebra A' is unique up to isomorphism.

Definition 2.1.4. ([Bal14b]). The *splitting tower* of a separable ring A

$$\mathbb{1} = A^{[0]} \xrightarrow{\eta} A = A^{[1]} \to A^{[2]} \to \ldots \to A^{[n]} \to A^{[n+1]} \to \ldots$$

is defined inductively by $A^{[n+1]} = (A^{[n]})'$, where we consider $A^{[n]}$ as a ring in $A^{[n-1]} - \mathrm{Mod}_{\mathcal{K}}$. If $A^{[d]} \neq 0$ and $A^{[d+1]} = 0$, we say the *degree* of A is d and write $\deg(A) = d$. We say A has *infinite degree* if $A^{[d]} \neq 0$ for all $d \geq 0$.

Remark 2.1.5. Regarding $A^{[n]}$ as a ring in $A^{[n-1]} - \mathrm{Mod}_{\mathcal{K}}$, we have $(A^{[n]})^{[m]} \cong A^{[n+m-1]}$ for all $m \geq 1$ by construction. Thus, $\deg(A^{[n]}) = \deg(A) - n + 1$ when $A^{[n]} \neq 0$.

Example 2.1.6. Let R be a commutative ring with no idempotents but 0 and 1. Suppose A is a commutative projective separable R-algebra. By [DI71, Prop.2.2.1], A is finitely generated as an R-module. The degree of A as a separable ring in $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(R)$ (see Example 1.1.2) recovers its rank as an R-module.

Proposition 2.1.7. *Let A and B be separable rings in $\mathcal{K}$.*

(a) *Let $F : \mathcal{K} \to \mathcal{L}$ be an additive monoidal functor. For every $n \geq 0$, the rings $F(A^{[n]})$ and $F(A)^{[n]}$ are isomorphic. In particular,* $\deg_{\mathcal{L}}(F(A)) \leq \deg_{\mathcal{K}}(A)$.

(b) *Suppose A is a B-algebra. Then* $\deg_{B\text{–}\mathrm{Mod}_{\mathcal{K}}}(F_B(A)) = \deg_{\mathcal{K}}(A)$.

(c) *For $n \geq 1$, we have $F_{A^{[n]}}(A) \cong \mathbb{1}_{A^{[n]}}^{\times n} \times A^{[n+1]}$ as $A^{[n]}$-algebras.*

Proof. (a) and (c) are proved in [Bal14b]. To prove (b), observe that $A^{[n]}$ is a B-algebra and therefore a direct summand of $F_B(A^{[n]}) \cong F_B(A)^{[n]}$. Hence, $F_B(A)^{[n]} \neq 0$ when $A^{[n]} \neq 0$ and $\deg_{B\text{–}\mathrm{Mod}_{\mathcal{K}}}(F_B(A)) \geq \deg_{\mathcal{K}}(A)$. □

Lemma 2.1.8. ([Bal14b, Lem.3.11]). *Let $n \geq 1$. For $A = \mathbb{1}^{\times n} \in \mathcal{K}$, we have $A^{[2]} \cong A^{\times(n-1)}$ as A-algebras.*

Proof. We prove there is an A-algebra isomorphism $\lambda : A \otimes A \xrightarrow{\sim} A \times A^{\times n-1}$ with $\mathrm{pr}_1 \lambda = \mu$. Let's write $A = \prod_{i=0}^{n-1} \mathbb{1}_i$, $A \otimes A = \prod_{0 \leq i,j \leq n-1} \mathbb{1}_i \otimes \mathbb{1}_j$ and $A^{\times n} = \prod_{k=0}^{n-1} \prod_{i=0}^{n-1} \mathbb{1}_{ik}$ with $\mathbb{1} = \mathbb{1}_i = \mathbb{1}_{ik}$ for all i, k. Define $\lambda : A \otimes A \to A^{\times n}$ by mapping the factor $\mathbb{1}_i \otimes \mathbb{1}_j$ identically to $\mathbb{1}_{i(i-j)}$, where we take the indices to be in $\mathbb{Z}_n$. It is not hard to see that λ is an A-algebra isomorphism and $\mathrm{pr}_{k=0} \lambda = \mu_A$. □

Corollary 2.1.9. *Let $n \geq 1$. The degree of $\mathbb{1}^{\times n} \in \mathcal{K}$ is n and $(\mathbb{1}^{\times n})^{[n]} \cong \mathbb{1}^{\times n!}$ in $\mathcal{K}$.*

Proof. Write $A := \mathbb{1}^{\times n}$. The result is clear for $n = 1$, and we proceed by induction on n, using Lemma 2.1.8. Applying the induction hypothesis to $A^{[2]} \cong \mathbb{1}_A^{\times(n-1)}$ in $A-\mathrm{Mod}_{\mathcal{K}}$, we get $\deg_{A\text{–}\mathrm{Mod}_{\mathcal{K}}}(A^{[2]}) = n - 1$ and $A^{[n]} \cong (A^{[2]})^{[n-1]} \cong \mathbb{1}_A^{\times(n-1)!} \cong (\mathbb{1}^{\times n})^{\times(n-1)!} \cong \mathbb{1}^{\times n!}$. □

Corollary 2.1.10. *Let A be a separable ring of finite degree in $\mathcal{K}$. Then* $\deg(A \times \mathbb{1}^{\times n}) \geq \deg(A) + n$.

Proof. Let $B := A^{[\deg(A)]}$. By Proposition 2.1.7(c), we know $F_B(A \times \mathbb{1}^{\times n}) \cong F_B(A) \times F_B(\mathbb{1}^{\times n}) \cong \mathbb{1}_B^{\times \deg(A)} \times \mathbb{1}_B^{\times n}$. So, $\deg(F_B(A \times \mathbb{1}^{\times n})) = \deg(A) + n$ and the result follows from Proposition 2.1.7(a). □

Definition 2.1.11. We call $\mathcal{K}$ *nice* if every separable ring A of finite degree has a decomposition $A \cong A_1 \times \ldots \times A_n$ for some indecomposable rings $A_1, \ldots, A_n$ in $\mathcal{K}$.

Example 2.1.12. The categories $\Bbbk G - \mathrm{mod}$, $\mathrm{D}^b(\Bbbk G - \mathrm{mod})$ and $\Bbbk G - \mathrm{stab}$ (see Section 7) are nice categories. More generally, every essentially small idempotent-complete symmetric monoidal category that satisfies Krull-Schmidt is nice.

Example 2.1.13. Let X be a Noetherian scheme. Then $\mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes over X, is nice (see Lemma 5.0.1).

2.2 Counting ring morphisms

Lemma 2.2.1. *Let A be a separable ring in $\mathcal{K}$. If $\mathbb{1}$ is indecomposable and there are n distinct ring morphisms $A \to \mathbb{1}$, then A has $\mathbb{1}^{\times n}$ as a ring factor. In particular, there are at most* $\deg A$ *distinct ring morphisms $A \to \mathbb{1}$.*

Proof. Let $\alpha_i : A \to \mathbb{1}$, $i = 1, \ldots, n$ be distinct ring morphisms with corresponding idempotent A-linear morphisms $e_i : A \to A$ as in Lemma 2.0.3. Since $A^e\text{–}\mathrm{Mod}_{\mathcal{K}}$ is idempotent-complete, e_1 yields a decomposition $A \cong A_1 \oplus A_1'$ of A^e-modules under which $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In fact, since the e_i are orthogonal idempotents, they yield a decomposition $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus A'$ of A^e-modules. By Lemma 2.1.2, the A_i and A' admit unique ring structures such that $A \cong A_1 \times A_2 \times \cdots \times A_n \times A'$. The first claim follows because every factor A_i is ring isomorphic to $\mathbb{1}$ via $\alpha_i \, \mathrm{incl}_{A_i}$. The statement about the degree is now an immediate consequence of Corollary 2.1.10. $\square$

Let A be a separable ring in $\mathcal{K}$ and $n \geq 1$. The property $F_{A^{[n]}}(A) \cong \mathbb{1}_{A^{[n]}}^{\times n} \times A^{[n+1]}$ from Proposition 2.1.7(c) characterises the ring $A^{[n]}$ in the following way:

Proposition 2.2.2. *Let B be an indecomposable separable ring in $\mathcal{K}$. The following are equivalent:*

(i) $\mathbb{1}_B^{\times n}$ *is a ring factor of* $F_B(A)$ *in* $B-\mathrm{Mod}_{\mathcal{K}}$*.*

(ii) There exists a ring morphism $A^{[n]} \to B$ *in* $\mathcal{K}$*.*

(iii) There exist (at least) n *distinct ring morphisms* $A \to B$ *in* $\mathcal{K}$*.*

Remark 2.2.3. When $\mathcal{K}$ is nice (Def. 2.1.11), (i) and (ii) remain equivalent even for decomposable rings B.

Proof. (i)⇒(iii) Suppose $\mathbb{1}_B^{\times n}$ is a ring factor of $F_B(A)$ in $B-\mathrm{Mod}_{\mathcal{K}}$ and write $\mathrm{pr}_i : B \otimes A \to B$ with $i = 1, \ldots, n$ for the corresponding projections in $\mathcal{K}$. The ring morphisms

$$\alpha_i : A \xrightarrow{\eta_B \otimes 1_A} B \otimes A \xrightarrow{\mathrm{pr}_i} B,$$

satisfy $\mu_B(1_B \otimes \alpha_i) = \mathrm{pr}_i(\mu_B \otimes 1_A)(1_B \otimes \eta_B \otimes 1_A) = \mathrm{pr}_i$ as pr_i is a B-algebra morphism for $i = 1, \ldots, n$. Hence, the α_i are all distinct. For (iii)⇒(i), let $\alpha_i : A \to B$, $i = 1, \ldots, n$ be distinct ring morphisms. Seeing how $\alpha_i = \mu_B(1_B \otimes \alpha_i)(\eta_B \otimes 1_A)$ for every i, the B-algebra morphisms $\mu_B(1_B \otimes \alpha_i) : B \otimes A \to B$ are also distinct. Having found n distinct ring morphisms $F_B(A) \to \mathbb{1}_B$ in $B-\mathrm{Mod}_{\mathcal{K}}$, Lemma 2.2.1 shows $\mathbb{1}_B^{\times n}$ is a ring factor of $F_B(A)$.

We show (i)⇒(ii) by induction on n. The case $n = 1$ is just (iii) and has already been proven. Now suppose $n > 1$ and $\mathbb{1}_B^{\times n}$ is a ring factor of $F_B(A)$. By the induction hypothesis, there exists a ring morphism $A^{[n-1]} \to B$. Thus B is an $A^{[n-1]}$-algebra, let us write $\overline{B}$ for the corresponding separable ring in $A^{[n-1]}$–$\mathrm{Mod}_{\mathcal{K}}$. By Proposition 2.1.7(c), we know

$$F_{\overline{B}}(F_{A^{[n-1]}}(A)) \cong F_{\overline{B}}(\mathbb{1}_{A^{[n-1]}}^{\times(n-1)} \times A^{[n]}) \cong \mathbb{1}_{\overline{B}}^{\times(n-1)} \times F_{\overline{B}}(A^{[n]}).$$

On the other hand, the commuting diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{F_{A^{[n-1]}}} & A^{[n-1]}-\mathrm{Mod}_{\mathcal{K}} \\ \big\downarrow{\scriptstyle F_B} & & \big\downarrow{\scriptstyle F_{\overline{B}}} \\ B-\mathrm{Mod}_{\mathcal{K}} & \xrightarrow{\simeq} & \overline{B}-\mathrm{Mod}_{A^{[n-1]}\text{–}\mathrm{Mod}_{\mathcal{K}}} \end{array}$$

from Proposition 1.2.6 shows that $F_B(A)$ is mapped to $F_{\overline{B}}(F_{A^{[n-1]}}(A))$ under the equivalence $B\text{–Mod}_{\mathcal{K}} \simeq \overline{B}\text{–Mod}_{A^{[n-1]}\text{–Mod}_{\mathcal{K}}}$. So, $F_{\overline{B}}(F_{A^{[n-1]}}(A))$ has $\mathbb{1}_{\overline{B}}^{\times n}$ as a ring factor. Remark 2.0.5 shows that we can compare indecomposable factors and it follows that $\mathbb{1}_{\overline{B}}$ is a ring factor of $F_{\overline{B}}(A^{[n]})$. By the induction hypothesis, there exists a ring morphism $A^{[n]} \to \overline{B}$ in $A^{[n-1]}\text{–Mod}_{\mathcal{K}}$ and therefore in $\mathcal{K}$. To show (ii)⇒(i), we suppose B is an $A^{[n]}$-algebra and write $\overline{B}$ for the corresponding separable ring in $A^{[n]}\text{–Mod}_{\mathcal{K}}$. Using Proposition 1.2.6 again, $F_B(A)$ is mapped to $F_{\overline{B}}(F_{A^{[n]}}(A))$ under the equivalence $B\text{–Mod}_{\mathcal{K}} \simeq \overline{B}\text{–Mod}_{A^{[n]}\text{–Mod}_{\mathcal{K}}}$. By Proposition 2.1.7(c), $F_{\overline{B}}(F_{A^{[n]}}(A)) \cong F_{\overline{B}}(\mathbb{1}_{A^{[n]}}^{\times n} \times A^{[n+1]}) \cong \mathbb{1}_{\overline{B}}^{\times n} \times F_{\overline{B}}(A^{[n+1]})$, so $\mathbb{1}_B^{\times n}$ is a ring factor of $F_B(A)$ in $B\text{–Mod}_{\mathcal{K}}$. □

Theorem 2.2.4. *Let A and B be separable rings in $\mathcal{K}$ and suppose A has finite degree and B is indecomposable. There are at most* $\deg(A)$ *distinct ring morphisms from A to B.*

Proof. Suppose there are n distinct ring morphisms from A to B. By Proposition 2.2.2, we know $\mathbb{1}_B^{\times n}$ is a ring factor of $F_B(A)$. Then, $n \le \deg_{B\text{–Mod}_{\mathcal{K}}}(F_B(A)) \le \deg_{\mathcal{K}}(A)$ by Corollary 2.1.10 and Proposition 2.1.7(a). □

Remark 2.2.5. Let A and B be separable rings in $\mathcal{K}$.

- Theorem 2.2.4 is evidently false when B is not indecomposable, say $A = B = \mathbb{1}^{\times n}$. Indeed, $\deg(A) = n$ but A has at least $n!$ ring endomorphisms.
- When $\mathcal{K}$ is nice and A, B are separable rings of finite degree in $\mathcal{K}$, the number of ring morphisms from A to B is finite even when B is decomposable, since B can be written as a finite product of indecomposable rings.

CHAPTER 3

Finite quasi-Galois theory

As before, $(\mathcal{K}, \otimes, \mathbb{1})$ denotes an idempotent-complete symmetric monoidal category. For now, we only assume (A, μ, η) is a nonzero ring in $\mathcal{K}$ and Γ is a finite set of ring endomorphisms of A containing 1_A. Consider the ring $\prod_{\gamma\in\Gamma} A$, writing $\prod_{\gamma\in\Gamma} A_\gamma$ to keep track of the different copies of A, and define $\varphi_1 : A \to \prod_{\gamma\in\Gamma} A_\gamma$ by $\mathrm{pr}_\gamma \varphi_1 = 1$ and $\varphi_2 : A \to \prod_{\gamma\in\Gamma} A_\gamma$ by $\mathrm{pr}_\gamma \varphi_2 = \gamma$ for all $\gamma \in \Gamma$. Thus, φ_1 renders the (standard) left A-algebra structure on $\prod_{\gamma\in\Gamma} A_\gamma$ and we can introduce a right A-algebra structure via φ_2.

Definition 3.0.1. We define the ring morphism

$$\lambda_\Gamma = \lambda : \quad A \otimes A \longrightarrow \textstyle\prod_{\gamma\in\Gamma} A_\gamma$$

by $\mathrm{pr}_\gamma \lambda = \mu(1 \otimes \gamma)$. Note that $\lambda(1 \otimes \eta) = \varphi_1$ and $\lambda(\eta \otimes 1) = \varphi_2$,

$$\begin{tikzcd} & A \arrow[dl, shift right, "1\otimes\eta"'] \arrow[dl, shift left, "\eta\otimes 1"] \arrow[dr, shift left, "\varphi_1"] \arrow[dr, shift right, "\varphi_2"'] & \\ A \otimes A \arrow[rr, "\lambda"] & & \prod_{\gamma\in\Gamma} A_\gamma, \end{tikzcd} \tag{3.0.2}$$

so λ is an A^e-algebra morphism.

Lemma 3.0.3. *Suppose $\lambda_\Gamma : A \otimes A \to \prod_{\gamma\in\Gamma} A_\gamma$ is an isomorphism.*

(a) There is an A^e-linear morphism $\sigma : A \to A \otimes A$ with $\mu(1 \otimes \gamma)\sigma = \delta_{1,\gamma}$ for every $\gamma \in \Gamma$. In particular, A is separable.

(b) Let $\gamma \in \Gamma$. If there exists a nonzero A-linear morphism $\alpha : A \to A$ with $\alpha\gamma = \alpha$, then $\gamma = 1$.

(c) Let $\gamma \in \Gamma$. If there exists a nonzero ring B and ring morphism $\alpha : A \to B$ with $\alpha\gamma = \alpha$, then $\gamma = 1$.

(d) The ring A has degree $|\Gamma|$ in $\mathcal{K}$.

Proof. To prove (a), define the A^e-linear morphism $\sigma := \lambda^{-1} \operatorname{incl}_1 : A \to A \otimes A$. The following diagram shows that $\mu(1 \otimes \gamma)\sigma = \delta_{1,\gamma}$:

$$\begin{tikzcd} A \arrow[rr, "\sigma"] \arrow[dr, hook, "\operatorname{incl}_1"'] & & A \otimes A \arrow[r, "1\otimes\gamma"] \arrow[dr, "\lambda"] & A \otimes A \arrow[rr, "\mu"] & & A. \\ & \prod_{\gamma\in\Gamma} A_\gamma \arrow[ur, "\lambda^{-1}"'] & & \prod_{\gamma\in\Gamma} A_\gamma \arrow[urr, "\operatorname{pr}_\gamma"'] & & \end{tikzcd}$$

For (b), suppose $\alpha\gamma = \alpha$ and $\sigma : A \to A \otimes A$ as in (a). Then $\alpha = \alpha\mu\sigma = \mu(1 \otimes \alpha)\sigma = \mu(1 \otimes \alpha)(1 \otimes \gamma)\sigma = \alpha\mu(1 \otimes \gamma)\sigma = \alpha\delta_{\gamma,1}$. Hence, $\alpha = 0$ or $\gamma = 1$. For (c), suppose $\alpha\gamma = \alpha$ and $\sigma : A \to A \otimes A$ as in (a). Then $\alpha = \alpha\mu\sigma = \mu(\alpha \otimes \alpha)\sigma = \mu(\alpha \otimes \alpha)(1 \otimes \gamma)\sigma = \alpha\mu(1 \otimes \gamma)\sigma = \alpha\delta_{\gamma,1}$. Again, $\alpha = 0$ or $\gamma = 1$. Finally, let $d = |\Gamma|$ be the order of the set. Given that $F_A(A) \cong \mathbb{1}_A^{\times d}$ in $A\text{–Mod}_{\mathcal{K}}$, Proposition 2.1.7(b) shows $\deg(A) = d$. □

Definition 3.0.4. We say (A, Γ) is *quasi-Galois* in $\mathcal{K}$ when A is a nonzero ring, Γ is a finite group of ring automorphisms of A and $\lambda_\Gamma : A \otimes A \to \prod_{\gamma\in\Gamma} A_\gamma$ is an isomorphism. We also call $F_A : \mathcal{K} \longrightarrow A\text{–Mod}_{\mathcal{K}}$ a *quasi-Galois extension* with group Γ.

Example 3.0.5. Let $A := \mathbb{1}^{\times n}$ and $\Gamma = \{\gamma_i \mid 0 \le i \le n-1\} \cong \mathbb{Z}_n$ where γ_1 is the permutation matrix corresponding to $(12\cdots n)$ and $\gamma_i := \gamma_1^i$. Then (A, Γ) is quasi-Galois. Indeed, in the notation of the proof of Lemma 2.1.8, γ_i sends the summand $\mathbb{1}_j$ identically to $\mathbb{1}_{j+i} \hookrightarrow A$ and the isomorphism λ is precisely the λ_Γ from above.

More generally, let Γ be any finite group and let $A := \prod_{\gamma\in\Gamma} \mathbb{1}_\gamma$, with $\mathbb{1} = \mathbb{1}_\gamma$ for every $\gamma \in \Gamma$ and component-wise structure. Then (A, Γ) is quasi-Galois, where $\gamma \in \Gamma$ acts on A by sending $\mathbb{1}_{\gamma'} \hookrightarrow A$ identically to $\mathbb{1}_{\gamma\gamma'} \hookrightarrow A$. In particular, this example shows that Γ does not always contain all ring automorphisms of A.

Example 3.0.6. Let R be a commutative ring, A a commutative R-algebra and Γ a finite group of ring automorphisms of A over R. Suppose A is a Galois extension of R with Galois group Γ in the sense of Auslander and Goldman ([AG60]). In particular, A is a finitely generated projective R-module and A is separable as an R-algebra by [DI71, Prop. 3.1.2]. Then, A is quasi-Galois with group Γ as a ring object in the symmetric monoidal categories $R-\mathrm{Mod}$ and $\mathrm{D}^{\mathrm{perf}}(R)$ (see Example 1.1.2).

Lemma 3.0.7. *Suppose* (A, Γ) *is quasi-Galois of degree* d *in* $\mathcal{K}$ *and* $F : \mathcal{K} \to \mathcal{L}$ *is an additive monoidal functor with* $F(A) \neq 0$. *Then* $F(A)$ *is quasi-Galois of degree* d *in* $\mathcal{L}$ *with group* $F(\Gamma) = \{F(\gamma) \mid \gamma \in \Gamma\}$. *In particular, being quasi-Galois is stable under extension-of-scalars.*

Proof. Seeing how

$$F(\lambda_\Gamma) : F(A) \otimes F(A) \cong F(A \otimes A) \longrightarrow \prod_{\gamma \in \Gamma} F(A)$$

is an isomorphism in L, it is enough to show $\lambda_{F(\Gamma)} = F(\lambda_\Gamma)$. Recall that λ_Γ is defined by $\mathrm{pr}_\gamma \lambda_\Gamma = \mu_A(1_A \otimes \gamma)$, hence $\mathrm{pr}_\gamma F(\lambda_\Gamma) = \mu_{F(A)}(1_{F(A)} \otimes_A F(\gamma))$. In particular, the morphisms $\mu_{F(A)}(1_{F(A)} \otimes_A F(\gamma))$ with $\gamma \in \Gamma$ are all distinct, therefore so are the $F(\gamma)$. This shows that $F(\prod_\Gamma A) \cong \prod_{F(\Gamma)} F(A)$ and $\lambda_{F(\Gamma)} = F(\lambda_\Gamma)$. $\square$

3.1 Quasi-Galois theory for indecomposable rings

Proposition 3.1.1. *Suppose* (A, Γ) *is quasi-Galois in* $\mathcal{K}$.

(a) If B *is a separable indecomposable* A*-algebra,* Γ *acts faithfully and transitively on the set of ring morphisms from* A *to* B. *In particular, there are exactly* $\deg(A)$ *distinct ring morphisms from* A *to* B.

(b) If A is indecomposable, any ring endomorphism of A is an automorphism and belongs to Γ.

Proof. The set of ring morphisms from A to B is non-empty, as B is an A-algebra, and Γ acts on it by precomposition. Faithfulness follows from Lemma 3.0.3(c). The action is transitive because the set of ring morphisms from A to B has no more than $\deg A = |\Gamma|$ elements by Theorem 2.2.4. This proves (a). For (b), note that the ring A has at most $\deg A = |\Gamma|$ ring endomorphisms by Theorem 2.2.4, so Γ must provide all of them. $\square$

So, for an indecomposable ring A of finite degree, we can simply say A is quasi-Galois, with the understanding that the Galois group Γ contains all ring endomorphisms of A.

Theorem 3.1.2. *let A be a nonzero separable indecomposable ring of finite degree. Let Γ be the set of ring endomorphisms of A. The following are equivalent:*

(i) $|\Gamma| = \deg(A)$.

(ii) $F_A(A) \cong \mathbb{1}_A^{\times t}$ *as rings in* $A{-}\mathrm{Mod}_{\mathcal{K}}$*, for some* $t > 0$.

(iii) λ_Γ *is an isomorphism.*

(iv) Γ *is a group and* (A, Γ) *is quasi-Galois in* $\mathcal{K}$.

Proof. Let $d := \deg(A)$. (i)⇒(ii) Lemma 2.2.2 shows that $|\Gamma| = \deg(A)$ implies $\mathbb{1}_A^{\times d}$ is a ring factor of $F_A(A)$. Since $\deg(F_A(A)) = d$, Corollary 2.1.10 shows $F_A(A) \cong \mathbb{1}_A^{\times d}$. To prove (ii)⇒(iii), first note that $t = d$ follows from Proposition 2.1.7(b). Let $l : A \otimes A \xrightarrow{\cong} A^{\times d}$ be an A-algebra isomorphism. Consider the ring morphisms

$$\alpha_i : A \xrightarrow{\eta\otimes 1} A \otimes A \xrightarrow{l} A^{\times d} \xrightarrow{\mathrm{pr}_i} A, \qquad i = 1, \ldots, d,$$

and note that $\mu(1_A \otimes \alpha_i) = \mathrm{pr}_i\, l(\mu \otimes 1_A)(1_A \otimes \eta \otimes 1_A) = \mathrm{pr}_i\, l$ for every i, so the α_i are all distinct. Now, $\Gamma = \{\alpha_i \mid 1 \leq i \leq d\}$ by Theorem 2.2.4 and $l = \lambda_\Gamma$ in the notation of Definition 3.0.1. For (iii)⇒(iv), it is enough to show that every $\gamma \in \Gamma$ is an automorphism. By Lemma 3.0.3 (a), we can find an A^e-linear morphism $\sigma : A \to A \otimes A$ such that $\mu(1 \otimes \gamma)\sigma = \delta_{1,\gamma}$ for every $\gamma \in \Gamma$. Let $\gamma \in \Gamma$ and note that $(1 \otimes \gamma)\sigma : A \to A \otimes A$ is nonzero since $\gamma = \mu(\gamma \otimes 1)(1 \otimes \gamma)\sigma$ is nonzero. Hence there exists $\gamma' \in \Gamma$ such that $\mathrm{pr}_{\gamma'}\, \lambda_\Gamma(1 \otimes \gamma)\sigma = \mu(1 \otimes \gamma')(1 \otimes \gamma)\sigma = \delta_{1,\gamma'\gamma}$ is nonzero. This means $\gamma'\gamma = 1$ and $\gamma'(\gamma\gamma') = \gamma'$ so $\gamma\gamma' = 1$ by Lemma 3.0.3(c). (iv)⇒(i) is the last part of Lemma 3.0.3. □

Corollary 3.1.3. *Quasi-Galoisness is stable under passing to indecomposable factors. De facto, if A, B and C are separable rings in $\mathcal{K}$ with $A \cong B \times C$ as rings, $F_A(A) \cong \mathbb{1}_A^{\times d}$ as A-algebras and B is indecomposable, then B is quasi-Galois.*

Proof. Under the decomposition $A-\mathrm{Mod}_\mathcal{K} \cong B-\mathrm{Mod}_\mathcal{K} \times C-\mathrm{Mod}_\mathcal{K}$, the isomorphism $F_A(A) \cong \mathbb{1}_A^{\times d}$ corresponds to $(F_B(B \times C), F_C(B \times C)) \cong (\mathbb{1}_B^{\times d}, \mathbb{1}_C^{\times d})$. Given that $\mathbb{1}_B$ is indecomposable and that $\mathbb{1}_B^{\times d}$ has $F_B(B)$ as a ring factor, we see $F_B(B) \cong \mathbb{1}_B^{\times t}$ for some $1 \leq t \leq d$ by Proposition 2.0.4 and Remark 2.0.5. The result now follows from Theorem 3.1.2. □

Corollary 3.1.4. *Let A be a separable ring in $\mathcal{K}$ and B an indecomposable A-algebra. If B is quasi-Galois in $\mathcal{K}$, then the ring $\overline{B}$ is quasi-Galois in $A-\mathrm{Mod}_\mathcal{K}$.*

Proof. This follows immediately from Lemma 3.0.7 and Corollary 3.1.3, seeing how $\overline{B}$ is an indecomposable ring factor of the quasi-Galois ring $F_A(B)$ in A–$\mathrm{Mod}_\mathcal{K}$. □

3.2 Splitting rings

Definition 3.2.1. Let A and B be separable rings in $\mathcal{K}$. We say B *splits* A if $F_B(A) \cong \mathbb{1}_B^{\times \deg(A)}$ in $B-\mathrm{Mod}_\mathcal{K}$. We call an indecomposable ring B a *splitting ring* of A if B splits A and any ring morphism $C \to B$, where C is an indecomposable ring splitting A, is an isomorphism.

Remark 3.2.2. Let A be a separable ring in $\mathcal{K}$ with $\deg(A) = d$. The ring $A^{[d]}$ in $\mathcal{K}$ splits itself by Proposition 2.1.7(a),(c) and Corollary 2.1.9:

$$F_{A^{[d]}}(A^{[d]}) \cong (F_{A^{[d]}}(A))^{[d]} \cong (\mathbb{1}_{A^{[d]}}^{\times d})^{[d]} \cong \mathbb{1}_{A^{[d]}}^{\times d!}.$$

In particular, $A^{[d]}$ has constant degree $d!$ in $\mathcal{K}$ by Lemma 5.1.6.

Lemma 3.2.3. *Let A be a separable ring in $\mathcal{K}$ that splits itself. If A_1 and A_2 are indecomposable ring factors of A, then any ring morphism $A_1 \to A_2$ is an isomorphism.*

Proof. Suppose A splits itself and let A_1, A_2 be indecomposable ring factors of A. Suppose there exists a ring morphism $f : A_1 \to A_2$ and write $\overline{A_2}$ for the corresponding ring in A_1–$\mathrm{Mod}_{\mathcal{K}}$. Then, $\overline{A_2}$ is an indecomposable ring factor of $F_{A_1}(A_2)$ and hence of $F_{A_1}(A)$. On the other hand, $F_{A_1}(A) \cong \mathbb{1}_{A_1}^{\times \deg(A)}$ because A splits itself. By Proposition 2.0.4, this means we have an isomorphism of rings $\overline{A_2} \cong \mathbb{1}_{A_1}$ or, forgetting the A_1-action, a ring isomorphism $g : A_2 \xrightarrow{\cong} A_1$ in $\mathcal{K}$. Note that A_1 is quasi-Galois by Corollary 3.1.3, so the ring morphism $gf : A_1 \to A_1$ is an isomorphism by Proposition 3.1.1(b). The lemma now follows. □

Lemma 3.2.4. *Suppose $\mathcal{K}$ is nice (see Def. 2.1.11) and let A, B be separable rings in $\mathcal{K}$. If B is indecomposable and there exists a ring morphism $A \to B$ in $\mathcal{K}$, then there exists a ring morphism $C \to B$ for some indecomposable ring factor C of A.*

Proof. Since $\mathcal{K}$ is nice, we can write $A \cong A_1 \times \ldots \times A_n$ with A_i indecomposable for $1 \leq i \leq n$. If there exists a ring morphism $A \to B$ in $\mathcal{K}$, Proposition 2.2.2 shows that $\mathbb{1}_B$ is a ring factor of $F_B(A) \cong F_B(A_1) \times \cdots \times F_B(A_n)$. Since $\mathbb{1}_B$ is indecomposable, it is a ring factor of some $F_B(A_i)$ with $1 \leq i \leq n$ by Proposition 2.0.4. The lemma now follows from Proposition 2.2.2. □

The following lemma is an immediate consequence of Proposition 2.2.2:

Lemma 3.2.5. *Let A and B be separable rings in $\mathcal{K}$ and suppose B is indecomposable. Then B splits A if and only if B is an $A^{[\deg(A)]}$-algebra.*

Proposition 3.2.6. *Suppose $\mathcal{K}$ is nice and let A be a separable ring in $\mathcal{K}$. An indecomposable ring B is a splitting ring of A if and only if B is a ring factor of $A^{[\deg(A)]}$. In particular, any separable ring in $\mathcal{K}$ has a splitting ring and at most finitely many.*

Proof. Let $d := \deg(A)$ and suppose B is a splitting ring of A. By the above lemma, B is an $A^{[d]}$-algebra and there exists a ring morphism $C \to B$ for some indecomposable ring factor C of $A^{[d]}$ by Lemma 3.2.4. Since C splits A, the ring morphism $C \to B$ is an isomorphism.
On the other hand, suppose B is a ring factor of $A^{[d]}$. Then B splits A. Let C be an indecomposable separable ring splitting A and suppose there is a ring morphism $C \to B$. As before, C is an $A^{[d]}$-algebra and there exists a ring morphism $C' \to C$ for some indecomposable ring factor C' of $A^{[d]}$. The composition $C' \to C \to B$ is an isomorphism by Remark 3.2.2 and Lemma 3.2.3. This means B is a ring factor of the indecomposable ring C, so $C \cong B$. $\square$

Corollary 3.2.7. *Suppose $\mathcal{K}$ is nice. An indecomposable separable ring B in $\mathcal{K}$ is quasi-Galois if and only if there exists a nonzero ring A in $\mathcal{K}$ such that B is a splitting ring of A.*

Proof. If B is indecomposable and quasi-Galois of degree t, then $B^{[2]} \cong \mathbb{1}_B^{\times(t-1)}$. Hence B is the unique splitting ring of B:

$$B^{[t]} \cong (B^{[2]})^{[t-1]} \cong (\mathbb{1}_B^{\times(t-1)})^{[t-1]} \cong B^{\times(t-1)!}.$$

Now suppose B is a splitting ring for some A in $\mathcal{K}$, say with $\deg(A) = d > 0$. By Proposition 3.2.6, $F_B(B)$ is a ring factor of $F_B(A^{[d]}) \cong F_B(A)^{[d]} \cong (\mathbb{1}_B^{\times d})^{[d]} = \mathbb{1}_B^{\times d!}$, so $F_B(B) \cong \mathbb{1}_B^{\times t}$ for some $t > 0$. Hence, B is quasi-Galois by Theorem 3.1.2. $\square$

CHAPTER 4

Tensor-triangulated categories

Many everyday triangulated categories come equipped with a symmetric monoidal structure $\otimes$, from algebraic geometry and stable homotopy theory to modular representation theory. Using the $\otimes$-structure, Paul Balmer [Bal05] has introduced the spectrum of a tensor-triangulated category, providing an algebro-geometric approach to the study of triangulated categories. In this chapter, we review some of the main concepts from tensor-triangular geometry, and recall some bits and pieces on separable rings that we will need later. In particular, we by no means aim to provide a complete overview of the theory. We refer to [Nee01] for an extensive account of triangulated categories, to [Kra10] for a summary on localizations and to [Bal10b] for a great introduction to tensor-triangular geometry.

Definition 4.0.1. Let $\mathcal{K}$ be a triangulated category. We call a subcategory *replete* if it is closed under isomorphisms. A *triangulated subcategory* of $\mathcal{K}$ is a full replete additive subcategory $\mathcal{L} \subset \mathcal{K}$ which is closed under (de)suspension and taking cones. That is, whenever $x \to y \to z \to \Sigma x$ is an exact triangle of $\mathcal{K}$ and $x, y \in \mathcal{L}$, then also $z \in \mathcal{L}$. A *thick subcategory* is a triangulated subcategory that is closed under direct summands.

Proposition 4.0.2. ([BS01, Prop. 3.2]). *Let $\mathcal{K}$ be a triangulated category. The idempotent-completion $\mathcal{K}^\natural$ (see Remark 1.1.4) admits a unique triangulated category structure such the embedding $\mathcal{K} \hookrightarrow \mathcal{K}^\natural$ is exact.*

Definition 4.0.3. A *tensor-triangulated category* is a triangulated category $\mathcal{K}$, equipped with a symmetric monoidal structure $(\otimes, \mathbb{1})$ that is compatible with the triangulation. In particular, the bifunctor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is exact in each variable. For a more precise account, we refer to [San14]. We call a functor *tensor-triangulated* when it is exact and monoidal.

Remark 4.0.4. If $\mathcal{K}$ is a tensor-triangulated category, then the idempotent-completion $\mathcal{K}^\natural$ remains tensor-triangulated and the embedding $\mathcal{K} \hookrightarrow \mathcal{K}^\natural$ is a tensor-triangulated functor.

Definition 4.0.5. Let $\mathcal{K}$ be a tensor-triangulated category. A triangulated subcategory $\mathcal{J} \subset \mathcal{K}$ of $\mathcal{K}$ is said to be $\otimes$-*ideal* if $x \in \mathcal{K}$ and $y \in \mathcal{J}$ implies that $x \otimes y \in \mathcal{J}$.

Remark 4.0.6. Let $\mathcal{K}$ be a tensor-triangulated category and $\mathcal{J} \subset \mathcal{K}$ a thick $\otimes$-ideal. Recall that the Verdier quotient $\mathcal{K}/\mathcal{J}$ has the same objects as $\mathcal{K}$ and morphisms obtained by calculus of fractions, inverting those morphisms whose cone is in $\mathcal{J}$. Then, $\mathcal{K}/\mathcal{J}$ inherits a canonical tensor-triangulated structure making the localization functor $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$ tensor-triangulated.

Definition 4.0.7. We call a tensor-triangulated category $\mathcal{K}$ *strongly closed* if there exists a bi-exact functor $\hom : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{K}$ with a natural isomorphism $\mathrm{Hom}_{\mathcal{K}}(x \otimes y, z) \cong \mathrm{Hom}_{\mathcal{K}}(x, \hom(y, z))$ and such that every object in $\mathcal{K}$ is *strongly dualizable*, that is the natural morphism $\hom(x, 1) \otimes y \xrightarrow{\sim} \hom(x, y)$ is an isomorphism for all x, y in $\mathcal{K}$.

Example 4.0.8. [Bal07, Prop. 4.1] Let X be a Noetherian scheme. Then $\mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes over X, is a strongly closed tensor-triangulated category, with derived tensor product $- \otimes^{L}_{\mathcal{O}_X}$. The internal hom is given by the derived Hom sheaf $R\mathcal{H}om_{\mathcal{O}_X}$.

4.1 Tensor-triangular geometry

In this section, we briefly recall some tensor-triangular geometry and refer the reader to [Bal05] for precise statements and motivation.

Throughout the rest of this section, $(\mathcal{K}, \otimes, \mathbb{1})$ will denote an essentially small tensor-triangulated category.

Definition 4.1.1. A *prime ideal* of $\mathcal{K}$ is a thick $\otimes$-ideal $\mathcal{P}$ with the property that $x \otimes y \in \mathcal{P}$ implies that either $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The *spectrum* $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$ is the set of all prime ideals $\mathcal{P} \subsetneq \mathcal{K}$.

Remark 4.1.2. The spectrum $\mathrm{Spc}(\mathcal{K})$ is indeed a set because $\mathcal{K}$ is assumed essentially small.

Definition 4.1.3. The *support* of an object x in $\mathcal{K}$ is

$$\mathrm{supp}(x) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{K}).$$

The complements $\mathcal{U}(x) := \mathrm{Spc}(\mathcal{K}) - \mathrm{supp}(x)$ of these supports form an open basis for what we call the *Balmer topology* on $\mathrm{Spc}(\mathcal{K})$. The study of tensor-triangulated categories and their spectrum is called *tensor-triangular geometry*.

Remark 4.1.4. The Balmer topology on $\mathrm{Spc}(\mathcal{K})$ appears to be a "reverse" version of the familiar Zariski topology on the prime spectrum of a commutative ring. Indeed, the closed sets in $\mathrm{Spc}(\mathcal{K})$ have the form

$$Z(\mathcal{E}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | \mathcal{E} \cap \mathcal{P} \neq \emptyset\},$$

for some family of objects $\mathcal{E} \subset \mathcal{K}$. This is not what one would expect in algebraic geometry, and familiar notions from algebraic geometry may take on surprising new meanings in tensor-triangular geometry. For instance, the closed points in $\mathrm{Spc}(\mathcal{K})$ are the minimal primes, and $\mathcal{K}$ is local if $\mathrm{Spc}(\mathcal{K})$ has a unique minimal prime.

Theorem 4.1.5. ([Bal05, Prop. 3.2]). *The support* $\mathrm{supp}(-)$ *assigns a closed subset* $\mathrm{supp}(x) \subset \mathrm{Spc}(\mathcal{K})$ *to any object* $x \in \mathcal{K}$ *and satisfies the following properties:*

1. $\mathrm{supp}(0) = \emptyset$ *and* $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$

2. $\mathrm{supp}(x \oplus y) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$

3. $\mathrm{supp}(\Sigma x) = \mathrm{supp}(x)$*, where* $\Sigma : \mathcal{K} \to \mathcal{K}$ *denotes the suspension.*

4. $\mathrm{supp}(z) \subset \mathrm{supp}(x) \cup \mathrm{supp}(y)$ *if there is an exact triangle* $x \to y \to z \to \Sigma x$.

5. $\mathrm{supp}(x \otimes y) = \mathrm{supp}(x) \cap \mathrm{supp}(y)$

The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is in some sense the universal such assignment satisfying the above properties. For a more precise statement, see [Bal05, Prop. 3.2].*

Remark 4.1.6. ([Bal05, Cor. 2.4]). An object $x \in \mathcal{K}$ is called $\otimes$-nilpotent if $x^{\otimes n} = 0$ for some $n \geq 1$. Then, x is $\otimes$-nilpotent if and only if $\mathrm{supp}(x) = \emptyset$. What is more, a separable ring A in $\mathcal{K}$ is nilpotent if and only if $A = 0$, seeing how A is a direct summand of $A \otimes A$.

Definition 4.1.7. Every tensor-triangulated functor $F : \mathcal{K} \to \mathcal{L}$ induces a continuous map

$$\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \longrightarrow \mathrm{Spc}(\mathcal{K}),$$

defined by $\mathrm{Spc}(F)(Q) := F^{-1}(Q)$.

Most of the results in this dissertation only hold for idempotent-complete categories. The following proposition, together with Remark 4.0.4, shows this is a mild condition. Indeed, we can idempotent-complete categories when needed, without affecting their spectrum.

Proposition 4.1.8. ([Bal05, Cor. 3.14]). *The embedding* $i : \mathcal{K} \to \mathcal{K}^\natural$ *of* $\mathcal{K}$ *into its idempotent-completion induces a homeomorphism* $\mathrm{Spc}(i) : \mathrm{Spc}(\mathcal{K}^\natural) \xrightarrow{\cong} \mathrm{Spc}(\mathcal{K})$.

Proposition 4.1.9. ([Bal05, Prop. 3.11]). *Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal and let $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$ denote the Verdier quotient functor. The map* $\mathrm{Spc}(q)$ *induces a homeomorphism* $\mathrm{Spc}(\mathcal{K}/\mathcal{J}) \xrightarrow{\cong} V(\mathcal{J}) \subset \mathrm{Spc}(\mathcal{K})$ *where* $V(\mathcal{J}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | \mathcal{P} \supset \mathcal{J}\}$.

Definition 4.1.10. ([Bal10a]). We call a tensor-triangulated category $\mathcal{K}$ *local* if any of the following equivalent conditions is satisfied:

1. $\mathrm{Spc}(\mathcal{K})$ is a local topological space: for every open cover $\mathrm{Spc}(\mathcal{K}) = \bigcup_{i \in I} U_i$, there exists $i \in I$ with $U_i = \mathrm{Spc}(\mathcal{K})$.
2. $\mathrm{Spc}(\mathcal{K})$ has a unique closed point.
3. $\mathcal{K}$ has a unique minimal prime.
4. For all $x, y \in \mathcal{K}$, if $x \otimes y = 0$ then x or y is $\otimes$-nilpotent.

Remark 4.1.11. For any prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, the Verdier quotient $\mathcal{K}/\mathcal{P}$ is local because the ideal $(0) = \mathcal{P}/\mathcal{P}$ is prime in $\mathcal{K}/\mathcal{P}$.

Definition 4.1.12. The local category $\mathcal{K}_{\mathcal{P}}$ at the prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ is the idempotent-completion of the Verdier quotient $\mathcal{K}/\mathcal{P}$. We write $q_{\mathcal{P}}$ for the canonical tt-functor $\mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{P} \hookrightarrow \mathcal{K}_{\mathcal{P}}$.

4.2 Separable rings in tensor-triangulated categories

Definition 4.2.1. A *tt-category* $\mathcal{K}$ is an essentially small, idempotent-complete tensor-triangulated category. A *tt-functor* $\mathcal{K} \to \mathcal{L}$ is an exact monoidal functor. We call a commutative, separable ring in $\mathcal{K}$ a *tt-ring*, after [Bal13, Bal14b].

Throughout the rest of this chapter, $\mathcal{K}$ will denote a tt-category.

Theorem 4.2.2. ([Bal11, Cor.5.18]). *Suppose A is a (not necessarily commutative) separable ring in $\mathcal{K}$. Then the category of A-modules $A-\mathrm{Mod}_{\mathcal{K}}$ has a* *unique triangulation such that the extension-of-scalars* $F_A : \mathcal{K} \to A\text{–}\mathrm{Mod}_{\mathcal{K}}$ *and the forgetful functor* $U_A : A\text{–}\mathrm{Mod}_{\mathcal{K}} \to \mathcal{K}$ *are exact.*

The above theorem shows that tt-rings preserve tt-categories:

Remark 4.2.3. If A is a tt-ring in $\mathcal{K}$, then $(A\text{–}\mathrm{Mod}_{\mathcal{K}}, \otimes_A, \mathbb{1}_A)$ is a tt-category, extension-of-scalars F_A becomes a tt-functor and U_A is exact.

Example 4.2.4. Let R be a commutative ring and A a commutative finite étale (flat and separable) R-algebra. Then $\mathcal{K} := \mathrm{D}^{\mathrm{perf}}(R)$, the homotopy category of bounded complexes of finitely generated projective R-modules, is a tt-category. We already saw in Example 1.1.2, that the category of A-modules $A\text{–}\mathrm{Mod}_{\mathcal{K}}$ is equivalent to the tt-category $\mathrm{D}^{\mathrm{perf}}(A)$. What is more, $\mathrm{Spc}(\mathcal{K})$ and $\mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})$ recover $\mathrm{Spec}(R)$ and $\mathrm{Spec}(A)$, respectively.

Proposition 4.2.5. ([Bal14b, Th.3.8]). *Suppose A is a tt-ring in $\mathcal{K}$. If the tt-ring $q_{\mathcal{P}}(A)$ has finite degree in $\mathcal{K}_{\mathcal{P}}$ for every $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, then A has finite degree and* $\deg(A) = \max_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})} \deg(q_{\mathcal{P}}(A))$.

Proposition 4.2.6. ([Bal14b, Cor. 3.12]). *Let $\mathcal{K}$ be a local tt-category and A, B tt-rings of finite degree in $\mathcal{K}$. Then the rings $A \otimes B$ and $A \times B$ have finite degree with* $\deg(A \times B) = \deg(A) + \deg(B)$ *and* $\deg(A \otimes B) = \deg(A) \cdot \deg(B)$.

Remark 4.2.7. Suppose A is a tt-ring of finite degree d in $\mathcal{K}$. Propositions 4.2.5 and 4.2.6 show that $A^{\times t}$ has degree dt.

Lemma 4.2.8. ([Bal14b, Th. 3.7]). *Suppose B is a tt-ring in $\mathcal{K}$ with* $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$. *Then* $\deg_{B\text{–}\mathrm{Mod}_{\mathcal{K}}}(F_B(A)) = \deg_{\mathcal{K}}(A)$.

Definition 4.2.9. For any tt-ring A in $\mathcal{K}$, we can consider the continuous map

$$f_A := \mathrm{Spc}(F_A) : \mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}}) \longrightarrow \mathrm{Spc}(\mathcal{K})$$

induced by the extension-of-scalars $F_A : \mathcal{K} \to A\text{–}\mathrm{Mod}_{\mathcal{K}}$.

The study of f_A is the topic of [Bal13]. We recall some of its properties here.

Lemma 4.2.10. *For any tt-ring A in $\mathcal{K}$,*

(a) $f_A^{-1}(\mathrm{supp}(x)) = \mathrm{supp}(F_A(x)) \subset \mathrm{Spc}(A\!-\!\mathrm{Mod}_{\mathcal{K}})$ *for every* $x \in \mathcal{K}$

(b) $f_A(\mathrm{supp}(y)) = \mathrm{supp}(U_A(y)) \subset \mathrm{Spc}(\mathcal{K})$ *for every* $y \in A\!-\!\mathrm{Mod}_{\mathcal{K}}$

(c) The image of f_A *is* $\mathrm{supp}(A)$.

Theorem 4.2.11. ([Bal13, Th.2.14]). *Let A be a tt-ring of finite degree in $\mathcal{K}$. Then*

$$\mathrm{Spc}((A \otimes A)\!-\!\mathrm{Mod}_{\mathcal{K}}) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathrm{Spc}(A\!-\!\mathrm{Mod}_{\mathcal{K}}) \xrightarrow{\quad f_A \quad} \mathrm{supp}(A) \tag{4.2.12}$$

is a coequaliser, where f_1, f_2 are the maps induced by extension-of-scalars along the morphisms $1 \otimes \eta$ and $\eta \otimes 1 : A \to A \otimes A$ respectively.

CHAPTER 5

Quasi-Galois theory for tt-categories

In this chapter, we consider Galois extensions in a tt-category $\mathcal{K}$, and study how our results from Chapter 3 interact with the geometry of $\mathrm{Spc}(\mathcal{K})$. Recall that a topological space is called *Noetherian* if its closed subsets satisfy the descending chain condition.

Lemma 5.0.1. *Any tt-category $\mathcal{K}$ with Noetherian spectrum $\mathrm{Spc}(\mathcal{K})$ is nice (see Def. 2.1.11).*

Proof. Let A be a separable ring of finite degree in $\mathcal{K}$. If A is not indecomposable, we can find nonzero A^e-modules $A_1, A_2 \in \mathcal{K}$ with $A \cong A_1 \oplus A_2$ as A^e-modules. By Lemma 2.1.2, A_1 and A_2 admit ring structures so that $A \cong A_1 \times A_2$ as rings. We prove that any ring decomposition of A in $\mathcal{K}$ has at most finitely many nonzero factors. Suppose there is a sequence of nontrivial decompositions $A = A_1 \times B_1$, $B_1 = A_2 \times B_2, \ldots$, with $B_n = A_{n+1} \times B_{n+1}$ for $n \geq 1$. Seeing how $\mathrm{supp}(B_n) \supseteq \mathrm{supp}(B_{n+1})$ and $\mathrm{Spc}(\mathcal{K})$ is Noetherian, there exists $k_0 \geq 1$ with $\mathrm{supp}(B_n) = \mathrm{supp}(B_{n+1})$ whenever $n \geq k_0$. By Proposition 4.2.6, we moreover know that $\deg(q_{\mathcal{P}}(B_n)) \geq \deg(q_{\mathcal{P}}(B_{n+1}))$ for every $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$. In other words, $\mathrm{supp}(B_n^{[i]}) \supseteq \mathrm{supp}(B_{n+1}^{[i]})$ for every $i \geq 0$. So, there exists $k \geq 1$ with $\mathrm{supp}(B_n^{[i]}) = \mathrm{supp}(B_{n+1}^{[i]})$ for every $i \geq 0$ and $n \geq k$. In particular, this means $\deg(q_{\mathcal{P}}(B_k)) = \deg(q_{\mathcal{P}}(B_{k+1}))$ for every $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$. Proposition 4.2.6 shows that $q_{\mathcal{P}}(A_{k+1}) = 0$ for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$. In other words, $\mathrm{supp}(A_{k+1}) = \emptyset$ and A_{k+1} is $\otimes$-nilpotent. In fact, $A_{k+1} = 0$ seeing how every ring is a direct summand of its $\otimes$-powers, a contradiction. $\square$

Example 5.0.2. Let X be a Noetherian scheme. Then $\mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes over X, is nice.

5.1 Rings of constant degree

In what follows, $\mathcal{K}$ denotes a tt-category. We will only consider tt-rings of finite degree. As of now, we have not met a tt-ring of infinite degree and it is unclear if they actually exist.

Definition 5.1.1. We say a tt-ring A in $\mathcal{K}$ has *constant degree* $d \in \mathbb{N}$ if the degree $\deg_{\mathcal{K}_{\mathcal{P}}} q_{\mathcal{P}}(A)$ equals d for every $\mathcal{P} \in \mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{K})$.

Remark 5.1.2. For any tt-ring A, we know $\mathrm{supp}(A^{[2]}) \subseteq \mathrm{supp}(A)$ because $A^{[2]} \cong A \otimes \mathrm{cone}(\eta_A)$ in $\mathcal{K}$. Now, a tt-ring A of degree d has constant degree if and only if $\mathrm{supp}(A^{[d]}) = \mathrm{supp}(A)$, seeing how $\mathcal{P} \in \mathrm{supp}(A^{[d]})$ if and only if $q_{\mathcal{P}}(A)$ has degree d.

Example 5.1.3. Let R be a commutative ring with no idempotents but 0 and 1. If A is a commutative projective separable R-algebra (see Example 2.1.6), then A has constant degree in $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(R)$ by [DI71, Th.1.4.12].

Lemma 5.1.4. *Let A be a tt-ring in $\mathcal{K}$ and $F : \mathcal{K} \to \mathcal{L}$ a tt-functor with $F(A) \neq 0$. If A has constant degree d, then $F(A)$ has constant degree d. Conversely, if $F(A)$ has constant degree d and* $\mathrm{supp}(A) \subset \mathrm{im}(\mathrm{Spc}(F))$, *then A has constant degree d.*

Proof. By Proposition 2.1.7(a), we know that $\deg(F(A)) \leq \deg(A)$. Now, if A has constant degree d,

$$\begin{aligned}\mathrm{supp}(F(A)^{[d]}) &= \mathrm{supp}(F(A^{[d]})) = \mathrm{Spc}(F)^{-1}(\mathrm{supp}(A^{[d]})) = \mathrm{Spc}(F)^{-1}(\mathrm{supp}(A)) \\ &= \mathrm{supp}(F(A)) \neq \emptyset\end{aligned}$$

shows that $F(A)$ has constant degree d. On the other hand, suppose $\mathrm{supp}(A) \subset \mathrm{im}(\mathrm{Spc}(F))$ and $F(A)$ has constant degree d. Seeing how $\mathrm{supp}(A^{[d+1]}) \subset \mathrm{im}(\mathrm{Spc}(F))$,

$\emptyset = \operatorname{supp}(F(A^{[d+1]})) = \operatorname{Spc}(F)^{-1}(\operatorname{supp}(A^{[d+1]}))$ implies $\operatorname{supp}(A^{[d+1]}) = \emptyset$, so A has degree d. Moreover,

$$\operatorname{Spc}(F)^{-1}(\operatorname{supp}(A^{[d]})) = \operatorname{supp}(F(A^{[d]})) = \operatorname{supp}(F(A)^{[d]}) = \operatorname{supp}(F(A)) = \operatorname{Spc}(F)^{-1}(\operatorname{supp}(A))$$

together with $\operatorname{supp}(A) \subset \operatorname{im}(\operatorname{Spc}(F))$ shows $\operatorname{supp}(A^{[d]}) = \operatorname{supp}(A)$, so A has constant degree d. □

Remark 5.1.5. Let A and B be tt-rings in $\mathcal{K}$ with $\operatorname{supp}(A) \cap \operatorname{supp}(B) \neq \emptyset$. The above lemma shows that if A has constant degree d, then $F_B(A)$ has constant degree d. Conversely, if $F_B(A)$ has constant degree d and $\operatorname{supp}(A) \subset \operatorname{supp}(B)$, then A has constant degree d by Lemma 4.2.10(c).

Lemma 5.1.6. *Let A be a tt-ring in $\mathcal{K}$. Then A has constant degree d if and only if there exists a tt-ring B in $\mathcal{K}$ with* $\operatorname{supp}(A) \subset \operatorname{supp}(B)$ *and such that $F_B(A) \cong \mathbb{1}_B^{\times d}$.*

Proof. If A has constant degree d, we can let $B := A^{[d]}$ and use Proposition 2.1.7(c). The other direction follows from Remark 5.1.5. □

Proposition 5.1.7. *Suppose the tt-ring A has constant degree,* $\operatorname{supp}(A)$ *is connected and there are nonzero rings B and C such that $A = B \times C$. Then B and C have constant degree too and* $\operatorname{supp}(A) = \operatorname{supp}(B) = \operatorname{supp}(C)$.

Proof. Assuming A has constant degree d, we claim that for every $1 \leq n \leq d$,

$$\operatorname{supp}(A) = \operatorname{supp}(B^{[n]}) \bigsqcup \operatorname{supp}(C^{[d-n+1]}).$$

Fix $1 \leq n \leq d$ and suppose $\mathcal{P} \in \operatorname{supp}(B^{[n]}) \subset \operatorname{supp}(A)$. This means that $\deg(q_{\mathcal{P}}(B)) \geq n$. By Proposition 4.2.6, $\deg(q_{\mathcal{P}}(C)) \leq d - n$ and hence $\mathcal{P} \notin \operatorname{supp}(C^{[d-n+1]})$. On the other hand, if $\mathcal{P} \in \operatorname{supp}(A) - \operatorname{supp}(B^{[n]})$, we get $\deg(q_{\mathcal{P}}(B)) \leq n - 1$ and $\deg(q_{\mathcal{P}}(C)) \geq d - n + 1$. So, $\mathcal{P} \in \operatorname{supp}(C^{[d-n+1]})$ and the claim follows.

Now, if A has connected support, the case $n = 1$ and the case $n = d$ give $\operatorname{supp}(A) = \operatorname{supp}(B) = \operatorname{supp}(C)$. The case $n = \deg(B)$ gives $\operatorname{supp}(B) =$

$\mathrm{supp}(A) = \mathrm{supp}(B^{[n]})$ so B and C have constant degree n and $d-n$ respectively. $\square$

Recall that for a tt-ring A and an A-algebra B in $\mathcal{K}$, we write $\overline{B}$ for the corresponding tt-ring in $A-\mathrm{Mod}_{\mathcal{K}}$. In other words, we have $B = U_A(\overline{B})$.

Proposition 5.1.8. *Let A be a tt-ring and B an A-algebra with* $\mathrm{supp}(\overline{B}) = \mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})$. *If A and $\overline{B}$ have constant degree, then B has constant degree and* $\deg_{\mathcal{K}}(B) = \deg_{A\text{–}\mathrm{Mod}_{\mathcal{K}}}(\overline{B}) \cdot \deg_{\mathcal{K}}(A)$.

Proof. We first prove the case $A = \mathbb{1}^{\times d}$. Then, $\overline{B} \in A-\mathrm{Mod}_{\mathcal{K}} \cong \mathcal{K} \times \ldots \times \mathcal{K}$ has the form $(B_1, \ldots, B_d)$ for some B_i in $\mathcal{K}$ and $B \cong B_1 \times \ldots \times B_d$. Suppose $\overline{B}$ has constant degree t on its support $\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}}) \cong \bigsqcup \mathrm{Spc}(\mathcal{K})$. In other words, B_i has support $\mathrm{Spc}(\mathcal{K})$ and constant degree t for every $1 \leq i \leq d$. For every $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, we then see that $\deg(q_{\mathcal{P}}(B)) = \sum_{i=1}^{d} \deg(q_{\mathcal{P}}(B_i)) = dt$ by Proposition 4.2.6. Hence B has constant degree dt.

Now, let A be any tt-ring of constant degree d. For $C := A^{[d]}$, we know $\widehat{A} := F_C(A) \cong \mathbb{1}_C^{\times d}$ by Proposition 2.1.7 (c). We also note that $\mathrm{supp}(B) = f_A(\mathrm{supp}(\overline{B})) = f_A(\mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})) = \mathrm{supp}(A) = \mathrm{supp}(C)$ by Lemma 4.2.10 (b), (c). Remark 5.1.5 then shows that it is enough to show that $\widehat{B} := F_C(B) \in \widehat{\mathcal{K}} := C-\mathrm{Mod}_{\mathcal{K}}$ has constant degree dt when $\overline{B}$ has constant degree t.

By Proposition 1.2.7, there exists a tt-functor $\overline{F_C} : A-\mathrm{Mod}_{\mathcal{K}} \to \widehat{A}-\mathrm{Mod}_{\widehat{\mathcal{K}}}$ such that $U_{\widehat{A}}\overline{F_C} \cong F_C U_A$ in the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "F_A"] \arrow[d, "F_C"'] & A-\mathrm{Mod}_{\mathcal{K}} \arrow[l, shift left, "U_A"] \arrow[d, "\overline{F_C}"] \\ \widehat{\mathcal{K}} \arrow[r, shift left, "F_{\widehat{A}}"] & \widehat{A}-\mathrm{Mod}_{\widehat{\mathcal{K}}} \arrow[l, shift left, "U_{\widehat{A}}"] \end{tikzcd}.$$

Writing $\widehat{\overline{B}} := \overline{F_C}(\overline{B})$, we see that $U_{\widehat{A}}(\widehat{\overline{B}}) \cong F_C\left(U_A\left(\overline{B}\right)\right) = F_C(B) = \widehat{B}$. Lemma 5.1.4 shows that $\widehat{\overline{B}}$ has constant degree t when $\overline{B}$ does. Finally, $\mathrm{supp}(\widehat{\overline{B}}) = (\mathrm{Spc}\,\overline{F_C})^{-1}(\mathrm{supp}(\overline{B})) =$

$(\operatorname{Spc}\overline{F_C})^{-1}(\operatorname{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})) = \operatorname{Spc}(\widehat{A}\text{–}\mathrm{Mod}_{\widehat{\mathcal{K}}})$. By the special case, $\widehat{B}$ has constant degree dt indeed. □

5.2 Quasi-Galois theory and tensor-triangular geometry

Lemma 5.2.1. *Let A be a tt-ring in $\mathcal{K}$ and suppose (A, Γ) is quasi-Galois. Then A has constant degree $|\Gamma|$ in $\mathcal{K}$.*

Proof. Given that $F_A(A) \cong \mathbb{1}_A^{\times|\Gamma|}$, the lemma follows from Lemma 5.1.6. □

Let A be a tt-ring in $\mathcal{K}$ and Γ a finite group of ring morphisms of A. Remark 1.2.3 shows Γ acts on $A-\mathrm{Mod}_{\mathcal{K}}$ and on its spectrum.

Theorem 5.2.2. *Suppose (A, Γ) is quasi-Galois in $\mathcal{K}$. Then,*

$$\operatorname{supp}(A) \cong \operatorname{Spc}(A-\mathrm{Mod}_{\mathcal{K}})/\Gamma.$$

Proof. Let (A, Γ) be quasi-Galois in $\mathcal{K}$. Diagram 3.0.2 yields a diagram of spectra

$$\begin{tikzcd}
 & \operatorname{Spc}(A-\mathrm{Mod}_{\mathcal{K}}) & \\
\operatorname{Spc}((A\otimes A)-\mathrm{Mod}_{\mathcal{K}}) \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] \arrow[rr, "\cong", "l"'] & & \operatorname{Spc}(\prod_{\gamma\in\Gamma} A_\gamma-\mathrm{Mod}_{\mathcal{K}}) \arrow[ul, shift right, "g_1"'] \arrow[ul, shift left, "g_2"]
\end{tikzcd}$$

where f_1, f_2, g_1, g_2, l are the maps induced by extension-of-scalars along the morphisms $1\otimes\eta$, $\eta\otimes 1$, φ_1, φ_2 and λ respectively (in the notation of Definition 3.0.1). So, the coequaliser 4.2.12 becomes

$$\textstyle\coprod_{\gamma\in\Gamma} \operatorname{Spc}(A_\gamma-\mathrm{Mod}_{\mathcal{K}}) \underset{g_2}{\overset{g_1}{\rightrightarrows}} \operatorname{Spc}(A-\mathrm{Mod}_{\mathcal{K}}) \xrightarrow{f_A} \!\!\!\!\rightarrow \operatorname{supp}(A),$$

where $g_1\,\mathrm{incl}_\gamma$ is the identity and $g_2\,\mathrm{incl}_\gamma$ is the action of γ on $\operatorname{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$. □

Remark 5.2.3. We call a tt-ring A in $\mathcal{K}$ *nil-faithful* when any morphism f in $\mathcal{K}$ with $F_A(f) = 0$ is $\otimes$-nilpotent. This is equivalent to saying $\operatorname{supp}(A) = \operatorname{Spc}(\mathcal{K})$, see [Bal13, Prop.2.15]. The above proposition thus recovers $\operatorname{Spc}(\mathcal{K})$ as the Γ-orbits of $\operatorname{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$ when (A, Γ) is nil-faithful and quasi-Galois in $\mathcal{K}$.

Remark 5.2.4. If the tt-ring A in $\mathcal{K}$ is *faithful*, that is when F_A is a faithful functor, being quasi-Galois really is being *Galois* over $\mathbb{1}$ in the sense of Auslander and Goldman (see Introduction). Indeed, [Bal12, Prop.2.12], implies that

$$\mathbb{1} \xhookrightarrow{\eta} A \underset{\eta\otimes 1}{\overset{1\otimes\eta}{\rightrightarrows}} A \otimes A$$

is an equaliser. Under the correspondence in diagram 3.0.2, this becomes

$$\mathbb{1} \xhookrightarrow{\eta} A \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \bigoplus_{\gamma\in\Gamma} A_\gamma,$$

where φ_1 and φ_2 where defined by $\mathrm{pr}_\gamma\,\varphi_1 = 1_A$ and $\mathrm{pr}_\gamma\,\varphi_2 = \gamma$ for all $\gamma \in \Gamma$.

The following lemma is a tensor-triangular version of Lemma 3.2.3.

Lemma 5.2.5. *Let A be a separable ring in $\mathcal{K}$ that splits itself. If A_1 and A_2 are indecomposable ring factors of A, then* $\mathrm{supp}(A_1) \cap \mathrm{supp}(A_2) = \emptyset$ *or* $A_1 \cong A_2$.

Proof. Suppose A splits itself and let A_1, A_2 be indecomposable ring factors of A with $\mathrm{supp}(A_1) \cap \mathrm{supp}(A_2) \neq \emptyset$. We know that $F_{A_1}(A) \cong \mathbb{1}_{A_1}^{\times \deg(A)}$ because A splits itself, and thus $F_{A_1}(A_2) \cong \mathbb{1}_{A_1}^{\times t}$ for some $t \geq 0$, seeing how $F_{A_1}(A_2)$ is a ring factor of $F_{A_1}(A)$. In fact $t > 0$, because $\mathrm{supp}(A_1 \otimes A_2) = \mathrm{supp}(A_1) \cap \mathrm{supp}(A_2) \neq \emptyset$. By Proposition 2.2.2, this means there exists a ring morphism $f : A_2 \to A_1$. Lemma 3.2.3 shows that f is an isomorphism. □

Proposition 5.2.6. *Suppose $\mathcal{K}$ is nice. When a tt-ring A in $\mathcal{K}$ has connected support and constant degree, the splitting ring A^* of A is unique up to isomorphism (see Def. 3.2.1). What is more,* $\mathrm{supp}(A) = \mathrm{supp}(A^*)$ *and A^* is quasi-Galois in $\mathcal{K}$.*

Proof. Let $d := \deg(A)$ and write $A^{[d]}$ as a product of indecomposable rings $A^{[d]} = A_1 \times \ldots \times A_n$. Note that $\mathrm{supp}(A) = \mathrm{supp}(A^{[d]})$ is connected and $A^{[d]}$ has constant degree $d!$ by Remark 3.2.2. Hence, Proposition 5.1.7 shows that $\mathrm{supp}(A) = \mathrm{supp}(A_i)$ for all $1 \leq i \leq n$. By Lemma 5.2.5, it follows that $A_i \cong A_j$ for all $1 \leq i, j \leq n$, so the splitting ring A_1 is unique (up to isomorphism) and $\mathrm{supp}(A) = \mathrm{supp}(A_1)$. Corollary 3.2.7 shows A^* is quasi-Galois in $\mathcal{K}$. □

Remark 5.2.7. In the following proposition and theorem, we will consider a tt-ring A with connected spectrum $\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$, implying A is indecomposable. When the tt-category $A-\mathrm{Mod}_{\mathcal{K}}$ is strongly closed (see Def. 4.0.7), the spectrum $\mathrm{Spc}(A\text{–}\mathrm{Mod}_{\mathcal{K}})$ is connected if and only if A is indecomposable, see [Bal07, Th.2.11].

Proposition 5.2.8. *Suppose $\mathcal{K}$ is nice and let A be a tt-ring in $\mathcal{K}$ with connected spectrum* $\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$. *Suppose B is an A-algebra with* $\mathrm{supp}(A) = \mathrm{supp}(B) \subset \mathrm{Spc}(\mathcal{K})$. *If (B, Γ) is quasi-Galois in $\mathcal{K}$, then B splits A. In particular, the degree of A in $\mathcal{K}$ is constant.*

Proof. Since B is quasi-Galois, all of its indecomposable components are quasi-Galois by Corollary 3.1.3. What is more, $\mathrm{supp}(B) = f_A(\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}}))$ is connected, so the indecomposable components of B have support equal to $\mathrm{supp}(B)$ by Proposition 5.1.7. Thus it suffices to prove the proposition with B indecomposable. Now, $F_A(B)$ is quasi-Galois by Lemma 3.0.7 and $\mathrm{supp}(F_A(B)) = f_A^{-1}(\mathrm{supp}(B)) = f_A^{-1}(\mathrm{supp}(A)) = \mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$ is connected. Let $d := \deg(B)$ and write $\overline{B}$ for the tt-ring in $A-\mathrm{Mod}_{\mathcal{K}}$ that corresponds to the A-algebra B in $\mathcal{K}$. Since $\overline{B}$ is an indecomposable ring factor of $F_A(B)$, Proposition 5.1.7 and Lemma 3.2.3 show that $F_A(B) \cong \overline{B}^{\times t}$ in $A-\mathrm{Mod}_{\mathcal{K}}$ for some $t \geq 1$. Forgetting the A-action, we get $A \otimes B \cong B^{\times t}$ in $\mathcal{K}$ and $F_B(A \otimes B) \cong F_B(B^{\times t}) \cong \mathbb{1}_B^{\times dt}$ in $B-\mathrm{Mod}_{\mathcal{K}}$. On the other hand, $F_B(A \otimes B) \cong F_B(A) \otimes_B \mathbb{1}_B^{\times d} \cong (F_B(A))^{\times d}$. It follows that $F_B(A) \cong \mathbb{1}_B^{\times t}$, with $t = \deg(A)$ by Lemma 4.2.8. Hence, B splits A and Lemma 5.1.6 shows that the degree of A is constant. $\square$

Theorem 5.2.9. (Quasi-Galois Closure). *Suppose $\mathcal{K}$ is nice and let A be a tt-ring of constant degree in $\mathcal{K}$ with connected spectrum* $\mathrm{Spc}(A-\mathrm{Mod}_{\mathcal{K}})$. *The splitting ring A^* (see Prop. 5.2.6) is the quasi-Galois closure of A. That is, A^* is quasi-Galois in $\mathcal{K}$,* $\mathrm{supp}(A) = \mathrm{supp}(A^*)$ *and any A-algebra morphism $B \to A^*$ with B quasi-Galois and indecomposable in $\mathcal{K}$, is an isomorphism.*

Proof. Proposition 5.2.6 shows that A^* is quasi-Galois in $\mathcal{K}$ and $\mathrm{supp}(A) =$

$\mathrm{supp}(A^*)$. Suppose there is an A-algebra morphism $B \to A^*$ with B quasi-Galois in $\mathcal{K}$ and indecomposable. Seeing how $\mathrm{supp}(A^*) \subset \mathrm{supp}(B) \subset \mathrm{supp}(A)$, Proposition 5.2.8 shows that B splits A. Thus $B \to A^*$ is an isomorphism by definition (3.2.1) of the splitting ring A^*. □

Remark 5.2.10. Proposition 5.2.8 shows that the assumption A has constant degree is necessary for A to have a quasi-Galois closure A^* as in Theorem 5.2.9.

CHAPTER 6

Quasi-Galois theory and descent

In this chapter, we consider the theory of descent for ring objects, and look at what happens when the ring is a quasi-Galois extension. We do not mean to give an overview of the theory of descent and refer to [Mes06] (or [Bal12] in the triangular setting) for some examples and a concise history. We will briefly recount the terminology of monads, and refer to [Mac98] for more explanations and related ideas. **We assume $(\mathcal{K}, \otimes, \mathbb{1})$ is an idempotent-complete symmetric monoidal category.**

6.1 Monads, comonads and descent

Definition 6.1.1. A *monad* (M, μ, η) on $\mathcal{K}$ is an endofunctor $M : \mathcal{K} \to \mathcal{K}$ together with natural transformations $\mu : M^2 \to M$ (multiplication) and $\eta : \mathrm{id}_{\mathcal{K}} \to M$ (two-sided unit) such that the diagrams

$$\begin{tikzcd} M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"] & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M \arrow[r, "M\eta"] \arrow[d, "\eta M"'] \arrow[dr, equal] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"] & M \end{tikzcd}$$

commute. An *M-module* (x, ϱ) in $\mathcal{K}$ is an object x in $\mathcal{K}$ together with a morphism $\varrho : M(x) \to x$ (the *M-action*) in $\mathcal{K}$ such that the following diagrams commute:

$$\begin{tikzcd} M^2(x) \arrow[r, "M(\varrho)"] \arrow[d, "\mu_x"'] & M(x) \arrow[d, "\varrho"] \\ M(x) \arrow[r, "\varrho"] & x \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, equal] & M(x) \arrow[d, "\varrho"] \\ & x. \end{tikzcd}$$

If (x, ϱ_1) and (y, ϱ_2) are M-modules, a morphism $f : x \to y$ is said to be *M-linear* if the diagram

$$\begin{tikzcd} M(x) \arrow[r, "\varrho_1"] \arrow[d, "M(f)"] & x \arrow[d, "f"] \\ M(y) \arrow[r, "\varrho_2"] & y \end{tikzcd}$$

commutes. We denote the category of M-modules and M-linear morphisms in $\mathcal{K}$ by $M-\mathrm{Mod}_{\mathcal{K}}$. Every object $x \in \mathcal{K}$ gives rise to a *free* M-module $F_M(x) = (M(x), \varrho)$ with action given by $\varrho : M^2(x) \xrightarrow{\mu_x} M(x)$. The *extension-of-scalars* $F_M : \mathcal{K} \to M-\mathrm{Mod}_{\mathcal{K}}$ is left adjoint to the forgetful functor $U_M : M-\mathrm{Mod}_{\mathcal{K}} \to \mathcal{K}$ which forgets the M-action. This adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[d, shift right=2, "F_M"'] \arrow[d, phantom, "\dashv"] \\ M-\mathrm{Mod}_{\mathcal{K}} \arrow[u, shift right=2, "U_M"'] \end{tikzcd}.$$

is called the *Eilenberg-Moore adjunction*.

Remark 6.1.2. If (A, μ_A, η_A) is a ring object in $\mathcal{K}$, then $(M, \mu, \eta) = (A \otimes -, \mu_A \otimes -, \eta_A \otimes -)$ defines a monad on $\mathcal{K}$, with $M-\mathrm{Mod}_{\mathcal{K}} = A-\mathrm{Mod}_{\mathcal{K}}$.

Definition 6.1.3. A *comonad* (N, κ, ϵ) on $\mathcal{K}$ is an endofunctor $N : \mathcal{K} \to \mathcal{K}$ together with comultiplication $\kappa : N \to N^2$ and counit $\epsilon : N \to \mathrm{id}_{\mathcal{K}}$ such that the diagrams

$$\begin{tikzcd} N \arrow[r, "\kappa"] \arrow[d, "\kappa"] & N^2 \arrow[d, "\kappa N"] \\ N^2 \arrow[r, "N\kappa"] & N^3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N \arrow[r, "\kappa"] \arrow[d, "\kappa"] \arrow[dr, equal] & N^2 \arrow[d, "N\epsilon"] \\ N^2 \arrow[r, "\epsilon N"] & N \end{tikzcd}$$

commute. An *N-comodule* (x, δ) is an object x in $\mathcal{K}$ together with a morphism $\delta : x \to N(x)$ (the *N-coaction*) in $\mathcal{K}$ such that the following diagrams commute:

$$\begin{tikzcd} x \arrow[r, "\delta"] \arrow[d, "\delta"] & N(x) \arrow[d, "N(\delta)"] \\ N(x) \arrow[r, "\kappa_x"] & N^2(x) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, "\delta"] \arrow[dr, equal] & N(x) \arrow[d, "\epsilon_x"] \\ & x. \end{tikzcd}$$

A morphism of N-comodules $f : (x, \delta_1) \to (y, \delta_2)$ is a morphism $f : x \to y$ in $\mathcal{K}$ such that the diagram

$$\begin{tikzcd} x \arrow[r, "\delta_1"] \arrow[d, "f"] & N(x) \arrow[d, "N(f)"] \\ y \arrow[r, "\delta_2"] & N(y) \end{tikzcd}$$

commutes. We write $N-\mathrm{Comod}_{\mathcal{K}}$ for the category of N-comodules in $\mathcal{K}$. The free-comodule functor $F^N : \mathcal{K} \to N-\mathrm{Comod}_{A\text{–}\mathrm{Mod}_{\mathcal{K}}}$ sends x to $(N(x), \kappa_x)$. The functor $U^N : N\text{–}\mathrm{Comod}_{\mathcal{K}} \to \mathcal{K}$ which forgets the N-coaction is left-adjoint to F^N. We call this the *co-Eilenberg-Moore adjunction*:

$$\begin{tikzcd} N-\mathrm{Comod}_{\mathcal{K}} \arrow[d, shift right, "U^N"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{K}. \arrow[u, shift right, "F^N"'] \end{tikzcd}$$

Remark 6.1.4. ([Mac98, Thm.VI.3.1]). Every adjunction $L : \mathcal{K} \leftrightarrows \mathcal{L} : R$ with unit $\eta : \mathrm{id}_{\mathcal{K}} \to RL$ and counit $\epsilon : LR \to \mathrm{id}_{\mathcal{L}}$ induces a monad $(M := RL, \mu := R\epsilon L, \eta)$ on $\mathcal{K}$ and a comonad $(N := LR, \kappa := L\eta R, \epsilon)$ on $\mathcal{L}$. We say the adjunction $L \dashv R$ *realizes* the monad M and comonad N. The (co-)Eilenberg-Moore adjunctions in Definitions 6.1.1 and 6.1.3 show that any (co)monad can be realized by an adjunction. Actually, $F_M \vdash U_M$ and $U^N \vdash F^N$ are the final adjunctions that realize the monad M and comonad N, respectively. That is, given an adjunction $L : \mathcal{K} \leftrightarrows \mathcal{L} : R$ realizing the monad $M = RL$ on $\mathcal{K}$ and comonad $N = LR$ on $\mathcal{L}$, there exist unique functors $P : \mathcal{L} \to M-\mathrm{Mod}_{\mathcal{K}}$ and $Q : \mathcal{K} \to N-\mathrm{Comod}_{\mathcal{L}}$ such that $PL = F_M$, $U_M P = R$, $QR = F^N$ and $U^N Q = L$:

$$\begin{tikzcd} & \mathcal{K} \arrow[loop above, "M"] \arrow[dl, shift right, "L"'] \arrow[dr, shift right, "F_M"'] & \\ \mathcal{L} \arrow[ur, shift right, "R"'] \arrow[rr, "P"'] & & M-\mathrm{Mod}_{\mathcal{K}} \arrow[ul, shift right, "U_M"'] \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} \mathcal{K} \arrow[rr, "Q"] \arrow[dr, shift right, "L"'] & & N-\mathrm{Comod}_{\mathcal{L}} \arrow[dl, shift right, "U^N"'] . \\ & \mathcal{L} \arrow[loop below, "N"] \arrow[ul, shift right, "R"'] \arrow[ur, shift right, "F^N"'] & \end{tikzcd}$$

The functor P is given by $P(x) = (R(x), R(\epsilon_x))$ and $P(f) = R(f)$ for objects x and morphisms f in $\mathcal{L}$. Similarly, $Q(x) = (L(x), L(\eta_x))$ and $Q(f) = L(f)$ if x is an object and f is a morphism in $\mathcal{K}$.

Definition 6.1.5. Let (M, μ, η) be a monad on $\mathcal{K}$. The *descent category* $\mathrm{Desc}_{\mathcal{K}}(M)$ for M in $\mathcal{K}$ is the category of comodules over the comonad L^M on $M-\mathrm{Mod}_{\mathcal{K}}$, where L^M is realized by the adjunction $F_M \dashv U_M$:

$$\begin{tikzcd}
M \circlearrowright \mathcal{K} \arrow[rr, "Q"] \arrow[dr, shift right, "F_M"'] & & \mathrm{Desc}_{\mathcal{K}}(M) := L^M - \mathrm{Comod}_{M-\mathrm{Mod}_{\mathcal{K}}} \arrow[dl, shift left, "U^{L^M}"'] \\
& M-\mathrm{Mod}_{\mathcal{K}} \arrow[ul, shift right, "U_M"'] \arrow[ur, shift left, "F^{L^M}"'] \arrow[loop below, "L^M := F_M U_M"] &
\end{tikzcd}$$

In the above picture, $U^{L^M} \vdash F^{L^M}$ is the co-Eilenberg-Moore adjunction for the comonad L^M on $M-\mathrm{Mod}_{\mathcal{K}}$, and Q is the comparison functor from Remark 6.1.4. In particular, $U^{L^M} Q = F_M$ and $QU_M = F^{L^M}$.

Definition 6.1.6. [Mes06] Let M be a monad on $\mathcal{K}$. We say M *satisfies effective descent* when the comparison functor $Q : \mathcal{K} \to \mathrm{Desc}_{\mathcal{K}}(M)$ is an equivalence.

We refer to [Mes06, Section 3] for necessary and sufficient conditions for $\mathcal{K}$ to satisfy effective descent. We simply herald the easy-to-state

Theorem 6.1.7. [Mes06, Cor. 3.17] *Let (M, μ, η) be a monad on $\mathcal{K}$. If the natural transformation $\eta : \mathrm{id}_{\mathcal{K}} \to M$ is a split monomorphism, then M satisfies effective descent.*

Recall that we call a ring A in $\mathcal{K}$ *faithful* when F_A is a faithful functor.

Theorem 6.1.8. [Bal12, Cor. 3.1] *Suppose $\mathcal{K}$ is an idempotent-complete tensor-triangulated category and (A, μ, η) a ring object in $\mathcal{K}$. Then A satisfies effective descent if and only if A is faithful.*

Remark 6.1.9. Let (A, μ, η) be a ring object in $\mathcal{K}$, and consider the monad $M := A \otimes - : \mathcal{K} \to \mathcal{K}$. The comonad $L^A := L^M : A-\mathrm{Mod}_{\mathcal{K}} \to A-\mathrm{Mod}_{\mathcal{K}}$ has comultiplication given by $A \otimes x \xrightarrow{1_A \otimes \eta_A \otimes 1_x} A \otimes A \otimes x$ and counit given by $A \otimes x \xrightarrow{\varrho} x$ for every $(x, \varrho) \in A-\mathrm{Mod}_{\mathcal{K}}$. We can describe the descent category $\mathrm{Desc}_{\mathcal{K}}(A) := \mathrm{Desc}_{\mathcal{K}}(M)$ explicitly. An object (x, ϱ, δ) in $\mathrm{Desc}_{\mathcal{K}}(A)$ is an object x in $\mathcal{K}$, together with an A-module structure $\varrho : A \otimes x \to x$ and *descent datum* $\delta : x \to A \otimes x$, a comodule structure on x, compatible with the A-module structure in the following way:

$$\begin{tikzcd}
A \otimes A \otimes x \arrow[r, "1\otimes\varrho"] \arrow[d, "\mu\otimes 1"] \arrow[dr, phantom, "(a)"] & A \otimes x \arrow[r, "1\otimes\delta"] \arrow[d, "\varrho"] \arrow[dr, phantom, "(b)"] & A \otimes A \otimes x \arrow[d, "\mu\otimes 1"] \\
A \otimes x \arrow[r, "\varrho"] & x \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dl, equal] \arrow[dr, phantom, "(c)"] & A \otimes x \arrow[d, "1\otimes\eta\otimes 1"] \\
x \arrow[u, "\eta\otimes 1"] & A \otimes x \arrow[l, "\varrho"] \arrow[r, "1\otimes\delta"] & A \otimes A \otimes x
\end{tikzcd} \tag{6.1.10}$$

commutes (see [Bal12, Rem. 1.4]). A morphism $f : (x, \varrho_1, \delta_1) \to (y, \varrho_2, \delta_2)$ in $\mathrm{Desc}_{\mathcal{K}}(A)$ is an A-linear morphism in $\mathcal{K}$ that is compatible with the descent datum:

$$\begin{tikzcd}
x \arrow[r, "\delta_1"] \arrow[d, "f"] & A \otimes x \arrow[d, "1_A\otimes f"] \\
y \arrow[r, "\delta_2"] & A \otimes y.
\end{tikzcd}$$

The comparison functor $Q : \mathcal{K} \longrightarrow \mathrm{Desc}_{\mathcal{K}}(A)$ maps objects $x \in \mathcal{K}$ to $(A \otimes x, \mu \otimes 1_x, 1_A \otimes \eta \otimes 1_x)$ and maps morphisms f to $1_A \otimes f$.

6.2 A comonad induced by ring automorphisms

Notation 6.2.1. Let (A, μ, η) be a ring in $\mathcal{K}$ and suppose Γ is a group of ring automorphisms of A. Seeing how A is commutative, any left A-module (x, ϱ) has a right A-module structure given by $x \otimes A \xrightarrow{(12)} A \otimes x \xrightarrow{\varrho} x$. Let $\gamma \in \Gamma$. Recall from Definition 3.0.1 that we write A_γ for the A, A-bimodule A with (standard) left A-action $A \otimes A \xrightarrow{\mu} A$ and right A-action $A \otimes A \xrightarrow{1\otimes\gamma} A \otimes A \xrightarrow{\mu} A$. On the other hand, for any left A-module (x, ϱ), we can twist the left A-action on x as follows:

$$A \otimes x \xrightarrow{\gamma\otimes 1} A \otimes x \xrightarrow{\varrho} x$$

and keep the (standard) right A-action $x \otimes A \xrightarrow{(12)} A \otimes x \xrightarrow{\varrho} x$. We will denote the resulting A, A-bimodule by x^γ. In particular, we have an equality of A, A-bimodules $(x^{\gamma_1})^{\gamma_2} = x^{\gamma_1\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. Finally, for any A, A-bilinear morphism $f : x \to y$, the morphism $f^\gamma := f : x^\gamma \to y^\gamma$ is still A, A-bilinear.

Remark 6.2.2. Let (x, ϱ) be an A-module in $\mathcal{K}$. For every $\gamma \in \Gamma$, we note that the maps $\gamma : x \to x^\gamma$ and $\varrho : A^\gamma \otimes x \to x^\gamma$ are left A-linear.

Proposition 6.2.3. *Let (A, μ, η) be a ring in $\mathcal{K}$ and suppose Γ is a group of ring automorphisms of A. The endofunctor*

$$N = N^\Gamma := \ (- \otimes_A \prod_{\gamma\in\Gamma} A^\gamma) : A{-}\mathrm{Mod}_{\mathcal{K}} \longrightarrow A{-}\mathrm{Mod}_{\mathcal{K}} : x \mapsto \prod_{\gamma\in\Gamma} x^\gamma$$

defines a comonad (N, κ, ϵ) on $A{-}\mathrm{Mod}_{\mathcal{K}}$, with comultiplication

$$\kappa_x : \ \prod_{\gamma\in\Gamma} x^\gamma \longrightarrow \prod_{\gamma_1,\gamma_2\in\Gamma} (x^{\gamma_1})^{\gamma_2} \cong \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2}$$

given by $\mathrm{pr}_{\gamma_1,\gamma_2} \kappa_x = \mathrm{pr}_{\gamma_1\gamma_2}$ and with counit $\epsilon_x := \mathrm{pr}_1 : \prod_{\gamma\in\Gamma} x^\gamma \to x$.

Proof. Let $x \in A{-}\mathrm{Mod}_{\mathcal{K}}$. To check that comultiplication is coassociative, we consider

$$\begin{tikzcd}
\prod\limits_{\gamma\in\Gamma} x^\gamma \arrow[r, "\kappa_x"] \arrow[d, "\kappa_x"] & \prod\limits_{\gamma_2',\gamma_2\in\Gamma} (x^{\gamma_2'})^{\gamma_2} \arrow[d, "\prod_{\gamma_2\in\Gamma}(\kappa_x)^{\gamma_2}"] \arrow[ddr, bend left, "\mathrm{pr}_{\gamma_1\gamma_3,\gamma_2}"] & \\
\prod\limits_{\gamma_1,\gamma_1'\in\Gamma} (x^{\gamma_1})^{\gamma_1'} \arrow[r, "\kappa_{(\prod_{\gamma_1\in\Gamma} x^{\gamma_1})}"'] \arrow[ddrr, phantom] \arrow[drr, bend right, "\mathrm{pr}_{\gamma_1,\gamma_3\gamma_2}"'] & \prod\limits_{\gamma_1,\gamma_3,\gamma_2\in\Gamma} x^{\gamma_1\gamma_3\gamma_2} \arrow[dr, "\mathrm{pr}_{\gamma_1,\gamma_3,\gamma_2}"'] & \\
 & & x^{\gamma_1\gamma_3\gamma_2},
\end{tikzcd}$$

which shows that

$$\mathrm{pr}_{\gamma_1,\gamma_3,\gamma_2}(\kappa_{N(x)})\kappa_x = \mathrm{pr}_{(\gamma_1\gamma_3)\gamma_2} = \mathrm{pr}_{\gamma_1(\gamma_3\gamma_2)} = \mathrm{pr}_{\gamma_1,\gamma_3,\gamma_2}(N(\kappa_x))\kappa_x$$

for every $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. Furthermore, ϵ is a two-sided counit, because

$$\begin{tikzcd}
\prod_{\gamma\in\Gamma} x^\gamma \arrow[r, "\kappa_x"] \arrow[d, "\kappa_x"] \arrow[dr, equal] & \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2} \arrow[d, "\prod_{\gamma_2\in\Gamma}(\epsilon_x)^{\gamma_2}"] \arrow[ddr, bend left, "\mathrm{pr}_{1,\gamma}"] & \\
\prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2} \arrow[r, "\epsilon_{(\prod_{\gamma_1\in\Gamma} x^{\gamma_1})}"'] \arrow[drr, bend right, "\mathrm{pr}_{\gamma,1}"'] & \prod_{\gamma\in\Gamma} x^\gamma \arrow[dr, "\mathrm{pr}_\gamma"] & \\
& & x^\gamma
\end{tikzcd}$$

shows that $\mathrm{pr}_\gamma(\epsilon_{N(x)})\kappa_x = \mathrm{pr}_{\gamma 1} = \mathrm{pr}_{1\gamma} = \mathrm{pr}_\gamma(N(\epsilon_x))\kappa_x$ for every $\gamma \in \Gamma$. $\square$

Definition 6.2.4. The category $(A-\mathrm{Mod}_{\mathcal{K}})^\Gamma$ has objects $(x, \varrho, (\delta_\gamma : x \to x^\gamma)_{\gamma\in\Gamma})$, where (x, ϱ) is an A-module in $\mathcal{K}$ and $(\delta_\gamma)_{\gamma\in\Gamma}$ is a family of left A-linear isomorphisms $\delta_\gamma : x \to x^\gamma$, with $\delta_1 = 1_x$ and satisfying the cocycle condition

$$\begin{tikzcd}
x \arrow[r, "\delta_{\gamma_1}"] \arrow[dr, "\delta_{\gamma_2\gamma_1}"'] & x^{\gamma_1} \arrow[d, "(\delta_{\gamma_2})^{\gamma_1}"] \\
& x^{\gamma_2\gamma_1} = (x^{\gamma_2})^{\gamma_1}
\end{tikzcd}$$

for any $\gamma_1, \gamma_2 \in \Gamma$. Morphisms $f : (x, \varrho_1, (\delta_\gamma)_{\gamma\in\Gamma}) \to (y, \varrho_2, (\beta_\gamma)_{\gamma\in\Gamma})$ in $(A\text{–}\mathrm{Mod}_{\mathcal{K}})^\Gamma$ are A-linear morphisms $f : (x, \varrho_1) \to (y, \varrho_2)$ such that $\beta_\gamma f = f^\gamma \delta_\gamma$ for every $\gamma \in \Gamma$.

Proposition 6.2.5. *Let (A, μ, η) be a ring in $\mathcal{K}$ and suppose Γ is a group of ring automorphisms of A. The category $(A-\mathrm{Mod}_{\mathcal{K}})^\Gamma$ is isomorphic to the category of N^Γ-comodules.*

Proof. We define a functor $N-\mathrm{Comod}_{A\text{–}\mathrm{Mod}_{\mathcal{K}}} \longrightarrow (A\text{–}\mathrm{Mod}_{\mathcal{K}})^\Gamma$ by sending objects $((x, \varrho), \delta : x \to N(x)) \in N-\mathrm{Comod}_{A\text{–}\mathrm{Mod}_{\mathcal{K}}}$ to $(x, \varrho, (\delta_\gamma)_{\gamma\in\Gamma}) \in (A-\mathrm{Mod}_{\mathcal{K}})^\Gamma$, with

$$(\delta : x \to \textstyle\prod_{\gamma\in\Gamma} x^\gamma) \longmapsto (\delta_\gamma : x \xrightarrow{\mathrm{pr}_\gamma\,\delta} x^\gamma)_{\gamma\in\Gamma}.$$

Suppose $((x, \varrho), \delta) \in N-\mathrm{Comod}_{A\text{–}\mathrm{Mod}_{\mathcal{K}}}$. The following diagram

$$\begin{tikzcd}
x \arrow[r, "\delta"] \arrow[d, "\delta"] & \prod_{\gamma\in\Gamma} x^{\gamma} \arrow[r, "\mathrm{pr}_{\gamma_2}"] \arrow[d, "\prod_{\gamma\in\Gamma}(\delta)^{\gamma}"] & x^{\gamma_2} \arrow[d, "\delta^{\gamma_2}"] \\
\prod_{\gamma\in\Gamma} x^{\gamma} \arrow[r, "\kappa_x"] \arrow[d, "\mathrm{pr}_{\gamma_1\gamma_2}"] & \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2} \arrow[r, "\mathrm{pr}_{\gamma_2}"] \arrow[d, "\mathrm{pr}_{\gamma_1,\gamma_2}"] & \prod_{\gamma_1\in\Gamma} x^{\gamma_1\gamma_2} \arrow[d, "\mathrm{pr}_{\gamma_1}"] \\
x^{\gamma_1\gamma_2} \arrow[r, equal] & x^{\gamma_1\gamma_2} \arrow[r, equal] & x^{\gamma_1\gamma_2}
\end{tikzcd}$$

shows that $(\delta_\gamma)_{\gamma\in\Gamma}$ satisfies the cocycle condition $(\delta_{\gamma_1})^{\gamma_2}\delta_{\gamma_2} = \delta_{\gamma_1\gamma_2}$. What is more, we know

$$\begin{tikzcd}
x \arrow[r, "\delta"] \arrow[dr, equal] & \prod_{\gamma\in\Gamma} x^{\gamma} \arrow[d, "\epsilon_x=\mathrm{pr}_1"] \\
& x
\end{tikzcd}$$

and hence $\delta_1 = 1_x$ as desired. Now, suppose (x, ϱ, δ) and (y, ϱ', δ') are N-comodules in $A-\mathrm{Mod}_{\mathcal{K}}$. An A-linear morphism $f : x \longrightarrow y$ is a morphism of N-comodules if

$$\begin{tikzcd}
x \arrow[r, "\delta"] \arrow[d, "f"] & \prod_{\gamma\in\Gamma} x^{\gamma} \arrow[d, "\prod_{\gamma\in\Gamma} f^{\gamma}"] \\
y \arrow[r, "\delta'"] & \prod_{\gamma\in\Gamma} y^{\gamma}
\end{tikzcd}$$

commutes. This happens precisely when

$$\begin{tikzcd}
x \arrow[r, "\delta"] \arrow[d, "f"] & \prod_{\gamma\in\Gamma} x^{\gamma} \arrow[r, "\mathrm{pr}_{\gamma}"] \arrow[d, "\prod_{\gamma\in\Gamma} f^{\gamma}"] & x^{\gamma} \arrow[d, "f^{\gamma}"] \\
y \arrow[r, "\delta'"] & \prod_{\gamma\in\Gamma} y^{\gamma} \arrow[r, "\mathrm{pr}_{\gamma}"] & y^{\gamma}
\end{tikzcd}$$

commutes for every $\gamma \in \Gamma$, so $f : x \to y$ is a morphism in $(A-\mathrm{Mod}_{\mathcal{K}})^{\Gamma}$. $\square$

6.3 Descent and quasi-Galois theory

Recall from Definition 3.0.4 that we have an A, A-bilinear map $\lambda : A \otimes A \to \prod_{\gamma\in\Gamma} A_\gamma$ given by $\mathrm{pr}_\gamma \lambda = \mu(1 \otimes \gamma)$. Furthermore, we will write $\phi = \phi_{A,x}$:

$A \otimes x \to F_A(A) \otimes_A x$ for the Projection Formula isomorphism in $\mathcal{K}$ (Prop. 1.1.13).

Lemma 6.3.1. *Let (A, μ, η) be a ring in $\mathcal{K}$ and suppose Γ is a group of ring automorphisms of A. The map $\lambda_x : A \otimes x \to \prod_{\gamma \in \Gamma} x^\gamma$ defined by*

$$\lambda_x : \quad A \otimes x \xrightarrow{\phi} F_A(A) \otimes_A x \xrightarrow{\lambda \otimes_A 1_x} \textstyle\prod_{\gamma \in \Gamma} A_\gamma \otimes_A x \cong \prod_{\gamma \in \Gamma} x$$

is left A-linear for every $x \in A\!-\!\mathrm{Mod}_{\mathcal{K}}$.

Proof. Let $(x, \varrho) \in A\!-\!\mathrm{Mod}_{\mathcal{K}}$. We note that λ_x is given by $\mathrm{pr}_\gamma \lambda_x = \varrho(\gamma \otimes 1)$:

$$\begin{tikzcd}
A \otimes x \arrow[r, "\phi"] \arrow[d, "\gamma \otimes 1_x"] & (A \otimes A) \otimes_A x \arrow[r, "\lambda \otimes_A 1_x"] \arrow[d, "(1 \otimes \gamma) \otimes_A 1_x"] & \prod_{\gamma \in \Gamma} A_\gamma \otimes_A x \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_\gamma"] & \prod_{\gamma \in \Gamma} x \arrow[d, "\mathrm{pr}_\gamma"] \\
A \otimes x \arrow[r, "\phi"] \arrow[rrr, bend right, "\varrho"'] & (A \otimes A) \otimes_A x \arrow[r, "\mu \otimes_A 1_x"] & A_\gamma \otimes_A x \arrow[r, "\cong"] & x,
\end{tikzcd}$$

in which the left square commutes because $\phi_{y,x}$ is natural in y. It follows that

$$\mathrm{pr}_\gamma \lambda_x : \quad A \otimes x \xrightarrow{\gamma \otimes 1_x} A^\gamma \otimes x \xrightarrow{\varrho} x^\gamma$$

is left A-linear for every $\gamma \in \Gamma$ (see Remark 6.2.2). $\square$

Let (A, μ, η) be a ring in $\mathcal{K}$. Recall that we defined the comonad $L = L^A = F_A U_A$ on $A\!-\!\mathrm{Mod}_{\mathcal{K}}$, with comultiplication given by $A \otimes x \xrightarrow{1_A \otimes \eta_A \otimes 1_x} A \otimes A \otimes x$ and counit given by $A \otimes x \xrightarrow{\varrho} x$ for every $(x, \varrho) \in A\!-\!\mathrm{Mod}_{\mathcal{K}}$ (Def. 6.1.5). On the other hand, for Γ a group of ring automorphisms of A, we defined the monad $(N = N^\Gamma, \kappa, \epsilon)$ on $A\!-\!\mathrm{Mod}_{\mathcal{K}}$ in Proposition 6.2.3.

Proposition 6.3.2. *Let (A, μ, η) be a ring in $\mathcal{K}$ and suppose Γ is a group of ring automorphisms of A. Then,*

$$\lambda_x : \quad L(x) = A \otimes x \longrightarrow \textstyle\prod_{\gamma \in \Gamma} x^\gamma = N(x)$$

defines a morphism $\lambda : L \Rightarrow N$ of comonads on $A\!-\!\mathrm{Mod}_{\mathcal{K}}$:

$$\begin{tikzcd}
A\!-\!\mathrm{Mod}_{\mathcal{K}} \arrow[r, "U_A"] \arrow[dr, "N"'] & \mathcal{K} \arrow[d, "F_A"] \\
& A\!-\!\mathrm{Mod}_{\mathcal{K}}\,.
\end{tikzcd}$$

Proof. Recall that λ_x is given by $\mathrm{pr}_\gamma \lambda_x = \varrho(\gamma \otimes 1) : A \otimes x \to x^\gamma$ for every $(x, \varrho) \in A-\mathrm{Mod}_{\mathcal{K}}$. To show that $(\lambda_x)_{x \in A\text{–}\mathrm{Mod}_{\mathcal{K}}}$ is natural, we note that the diagram

$$\begin{tikzcd}
A \otimes x \arrow[r, "\gamma\otimes 1"] \arrow[d, "1_A \otimes f"] & A \otimes x \arrow[r, "\varrho_1"] \arrow[d, "1_A \otimes f"] & x \arrow[d, "f"] \\
A \otimes y \arrow[r, "\gamma\otimes 1"] & A \otimes y \arrow[r, "\varrho_2"] & y
\end{tikzcd}$$

commutes for every A-linear map $f : (x, \varrho_1) \to (y, \varrho_2)$. We still need to check that $(\lambda_x)_{x \in A\text{–}\mathrm{Mod}_{\mathcal{K}}}$ defines an *morphism of monads*. That is, we check that the diagrams

$$\begin{tikzcd}
A \otimes x \arrow[r, "\lambda_x"] \arrow[dr, "\varrho"'] & \prod_{\gamma\in\Gamma} x^\gamma \arrow[d, "\epsilon_x = \mathrm{pr}_1"] \\
& x
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes x \arrow[r, "1_A \otimes \eta \otimes 1_x"] \arrow[d, "\lambda_x"] & A \otimes A \otimes x \arrow[d, "\lambda_{N(x)}(L(\lambda_x)) = N(\lambda_x)\lambda_{L(x)}"] \\
\prod_{\gamma\in\Gamma} x^\gamma \arrow[r, "\kappa_x"] & \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2}
\end{tikzcd}$$

commute. Commutativity of the first diagram is clear, because $\mathrm{pr}_1 \lambda_x = \varrho(1 \otimes 1)$. To show the second diagram commutes, let us compute

$$N(\lambda_x)\lambda_{L(x)}(1 \otimes \eta \otimes 1) = \prod_{\gamma\in\Gamma} (\lambda_x)^\gamma \lambda_{A\otimes x}(1 \otimes \eta \otimes 1) :$$

the diagram

$$\begin{tikzcd}
A \otimes x \arrow[r, "1\otimes\eta\otimes 1"] \arrow[d, "\gamma_2\otimes 1"] & A \otimes A \otimes x \arrow[r, "\lambda_{A\otimes x}"] \arrow[d, "\gamma_2 \otimes 1 \otimes 1"] & \prod_{\gamma_2\in\Gamma} (A \otimes x)^{\gamma_2} \arrow[r, "\prod_{\gamma_2\in\Gamma} \lambda_x^{\gamma_2}"] \arrow[d, "\mathrm{pr}_{\gamma_2}"] & \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2} \arrow[d, "\mathrm{pr}_{\gamma_2}"] \\
A \otimes x \arrow[r, "1\otimes\eta\otimes 1"] \arrow[rr, bend right, "1"'] & A \otimes A \otimes x \arrow[r, "\mu\otimes 1"] & (A \otimes x)^{\gamma_2} \arrow[r, "\lambda_x^{\gamma_2}"] \arrow[d, "(\gamma_1\otimes 1)^{\gamma_2}"] & \prod_{\gamma_1\in\Gamma} x^{\gamma_1\gamma_2} \arrow[d, "\mathrm{pr}_{\gamma_1}"] \\
& & (A \otimes x)^{\gamma_2} \arrow[r, "\varrho^{\gamma_2}"] & x^{\gamma_1\gamma_2}
\end{tikzcd}$$

commutes, which shows

$$\mathrm{pr}_{\gamma_1,\gamma_2}(\prod_{\gamma_2\in\Gamma} \lambda_x^{\gamma_2})\lambda_{A\otimes x}(1 \otimes \eta \otimes 1) = \varrho(\gamma_1 \otimes 1)(\gamma_2 \otimes 1).$$

On the other hand, we compute $\kappa_x\lambda_x$ and find

$$\begin{tikzcd}
A \otimes x \arrow[r, "\lambda_x"] \arrow[d, "\gamma_1\gamma_2\otimes 1"] & \prod_{\gamma\in\Gamma} x^{\gamma} \arrow[r, "\kappa_x"] \arrow[d, "\mathrm{pr}_{\gamma_1\gamma_2}"] & \prod_{\gamma_1,\gamma_2\in\Gamma} x^{\gamma_1\gamma_2} \arrow[d, "\mathrm{pr}_{\gamma_1,\gamma_2}"] \\
A \otimes x \arrow[r, "\varrho"] & x^{\gamma_1\gamma_2} \arrow[r, equal] & x^{\gamma_1\gamma_2}
\end{tikzcd}$$

so that $\mathrm{pr}_{\gamma_1,\gamma_2}(\kappa_x)\lambda_x = \varrho(\gamma_1\gamma_2 \otimes 1)$. □

Corollary 6.3.3. *Let* (A, Γ) *be quasi-Galois in* $\mathcal{K}$. *The comonads* L^A *and* N^Γ *on* $A{-}\mathrm{Mod}_{\mathcal{K}}$ *are isomorphic and*

$$\mathrm{Desc}_{\mathcal{K}}(A) \simeq N^\Gamma{-}\mathrm{Comod}_{A\text{–}\mathrm{Mod}_{\mathcal{K}}} \simeq (A{-}\mathrm{Mod}_{\mathcal{K}})^\Gamma.$$

Proof. This follows immediately from Proposition 6.3.2 and Proposition 6.2.5. □

Corollary 6.3.4. *Let* A *be a faithful ring in* $\mathcal{K}$ *and suppose* (A, Γ) *is quasi-Galois. Then* $\mathcal{K} \simeq (A{-}\mathrm{Mod}_{\mathcal{K}})^\Gamma$.

Proof. This follows from Theorem 6.1.8 and Corollary 6.3.3. □

CHAPTER 7

Quasi-Galois representation theory

Let G be a finite group and $\Bbbk$ a field with characteristic p dividing $|G|$. We write $\Bbbk G{-}\mathrm{mod}$ for the category of finitely generated left $\Bbbk G$-modules. Its bounded derived category $\mathrm{D}^b(\Bbbk G{-}\mathrm{mod})$ and stable category $\Bbbk G{-}\mathrm{stab}$ are nice tt-categories; both have tensor $\otimes_\Bbbk$ with diagonal G-action and the unit is the trivial representation $\mathbb{1} = \Bbbk$. Rickard [Ric89] proved there is an equivalence of tt-categories

$$\Bbbk G{-}\mathrm{stab} \cong \mathrm{D}^b(\Bbbk G{-}\mathrm{mod}) \big/ \mathrm{K}^b(\Bbbk G - \mathrm{proj}).$$

The Balmer spectrum $\mathrm{Spc}(\mathrm{D}^b(\Bbbk G{-}\mathrm{mod}))$ of the derived category is homeomorphic to the homogeneous spectrum $\mathrm{Spec}^h(H^\bullet(G, \Bbbk))$ of the graded-commutative cohomology ring $H^\bullet(G, \Bbbk)$. Accordingly, the Balmer spectrum $\mathrm{Spc}(\Bbbk G{-}\mathrm{stab})$ of the stable category is homeomorphic to the projective support variety $\mathcal{V}_G(\Bbbk) := \mathrm{Proj}(H^\bullet(G, \Bbbk))$, see [Bal05].

Notation 7.0.1. Let $H \leq G$ be a subgroup. We define the $\Bbbk G$-module $A_H = A_H^G := \Bbbk(G/H)$ to be the free $\Bbbk$-module with basis G/H and left G-action given by $g \cdot [x] = [gx]$. We also define $\Bbbk G$-linear maps $\mu : A_H \otimes_\Bbbk A_H \longrightarrow A_H$ given for every $\gamma, \gamma' \in G/H$ by

$$\gamma \otimes \gamma' \longmapsto \begin{cases} \gamma & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma' \end{cases}$$

and $\eta : \mathbb{1} \longrightarrow A_H$ by sending $1 \in \Bbbk$ to $\sum_{\gamma \in G/H} \gamma \in \Bbbk(G/H)$.
We will write $\mathcal{K}(G)$ to denote $\Bbbk G{-}\mathrm{mod}$, $\mathrm{D}^b(\Bbbk G{-}\mathrm{mod})$ or $\Bbbk G{-}\mathrm{stab}$ and consider the object A_H in each of these categories.

Theorem 7.0.2. ([Bal15, Prop.3.16, Th.4.4]). *Let $H \leq G$ be a subgroup. Then,*

(a) The triple (A_H, μ, η) is a commutative separable ring object in $\mathcal{K}(G)$.

(b) There is an equivalence of categories $\phi : \mathcal{K}(H) \xrightarrow{\simeq} A_H-\mathrm{Mod}_{\mathcal{K}(G)}$ sending $V \in \mathcal{K}(H)$ to $\Bbbk G \otimes_{\Bbbk H} V \in \mathcal{K}(G)$ with A_H-action

$$\varrho : \Bbbk(G/H) \otimes_{\Bbbk} (\Bbbk G \otimes_{\Bbbk H} V) \longrightarrow \Bbbk G \otimes_{\Bbbk H} V$$

given for $\gamma \in G/H$, $g \in G$ and $v \in V$ by $\gamma \otimes g \otimes v \longmapsto \begin{cases} g \otimes v & \text{if } g \in \gamma \\ 0 & \text{if } g \notin \gamma \end{cases}.$

(c) The following diagram commutes up to isomorphism:

$$\begin{tikzcd} & \mathcal{K}(G) \arrow[dl, "\mathrm{Res}^G_H"'] \arrow[dr, "F_{A_H}"] & \\ \mathcal{K}(H) \arrow[rr, "\phi", "\cong"'] & & A_H-\mathrm{Mod}_{\mathcal{K}(G)} . \end{tikzcd}$$

The above shows that subgroups $H \leq G$ provide indecomposable separable rings A_H in $\mathcal{K}(G)$, along which extension-of-scalars becomes restriction to the subgroup.

Proposition 7.0.3. *The ring object A_H has degree $[G : H]$ in* $\Bbbk G-\mathrm{mod}$ *and* $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$.

Proof. We refer to [Bal14b, Cor. 4.5] for $\mathcal{K}(G) = \mathrm{D}^b(\Bbbk G-\mathrm{mod})$. The case $\mathcal{K}(G) = \Bbbk G-\mathrm{mod}$ follows likewise from considering the fiber functor $\mathrm{Res}^G_{\{1\}}$. □

Lemma 7.0.4. *Let $\mathcal{K}(G)$ denote* $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$ *or* $\Bbbk G-\mathrm{stab}$ *and consider the subgroups $K \leq H \leq G$. Then* $\mathrm{supp}(A_H) = \mathrm{supp}(A_K) \subset \mathrm{Spc}(\mathcal{K}(G))$ *if and only if every elementary abelian subgroup of H is conjugate in G to a subgroup of K.*

Proof. This follows from [Eve91, Th.9.3.2], seeing how $\mathrm{supp}(A_H) = (\mathrm{Res}^G_H)^*(\mathrm{Spc}(\mathcal{K}(H)))$ can be written as a union of disjoint pieces coming from conjugacy classes in G of elementary abelian subgroups of H. □

Notation 7.0.5. For any two subgroups $H, K \leq G$, we write ${}_H[g]_K$ for the equivalence class of $g \in G$ in $H\backslash G/K$, just $[g]$ if the context is clear. We will write $H^g := g^{-1}Hg$ for the conjugate subgroups of H.

Remark 7.0.6. Let $H, K \leq G$ be subgroups and choose a complete set $T \subset G$ of representatives for $H\backslash G/K$. Consider the Mackey isomorphism

$$\coprod_{g \in T} G/(K \cap H^g) \xrightarrow{\cong} G/K \times G/H,$$

sending $[x] \in G/(K \cap H^g)$ to $([x]_K, [xg^{-1}]_H)$. The resulting ring isomorphism in $\mathcal{K}(G)$ (see [Bal13, Constr. 3.1]),

$$A_K \otimes A_H \xrightarrow{\cong} \prod_{g \in T} A_{K \cap H^g} \tag{7.0.7}$$

sends $[x]_K \otimes [y]_H$ to $[xk]_{K \cap H^g}$, with $g \in T$ such that ${}_H[g]_K = {}_H[y^{-1}x]_K$ and $k \in K$ such that $y^{-1}xkg^{-1} \in H$. This yields an A_K-algebra structure on $A_{K \cap H^y}$ for every $y \in T$, given by

$$A_K \xrightarrow{1 \otimes \eta} A_K \otimes A_H \cong \prod_{g \in T} A_{K \cap H^g} \xrightarrow{\mathrm{pr}_y} A_{K \cap H^y},$$

which sends $[x]_K \in G/K$ to $\sum_{[k] \in K/K \cap H^y} [xk]_{K \cap H^y} \in A_{K \cap H^y}$. In the notation of Theorem 7.0.2(b), this just means $A_{K \cap H^y} = \phi(A^K_{K \cap H^y})$ in $A_K\!-\!\mathrm{Mod}_{\mathcal{K}(G)}$.

Lemma 7.0.8. *For $x, y \in G$ we have ${}_H[x]_{H^y} = {}_H[y]_{H^y}$ if and only if ${}_H[x] = {}_H[y]$.*

Proof. If $[x] = [y]$ in $H\backslash G/H^y$, there are $h, h' \in H$ with $x = hy(y^{-1}h'y) = hh'y$. □

Corollary 7.0.9. *Let $x, g_1, g_2, \ldots, g_n \in G$ and $1 \leq i \leq n$. Then ${}_H[x]_{H \cap H^{g_1} \cap \ldots \cap H^{g_n}} = {}_H[g_i]_{H \cap H^{g_1} \cap \ldots \cap H^{g_n}}$ if and only if ${}_H[x] = {}_H[g_i]$.*

Notation 7.0.10. We will write $S \subset G$ to denote some complete set of representatives for $H\backslash G/H$. Likewise, for $g_1, g_2, \ldots, g_n \in G$ we fix a complete set $S_{g_1, g_2, \ldots, g_n} \subset G$ of representatives for $H\backslash G/H \cap H^{g_1} \cap \ldots \cap H^{g_n}$.

Lemma 7.0.11. *Let $1 \leq n < [G:H]$. In $\mathcal{K}(G)$, there is an isomorphism of rings*

$$A_H^{[n+1]} \cong \prod_{g_1,\,\dots,\,g_n} A_{H \cap H^{g_1} \cap \dots \cap H^{g_n}},$$

where the product runs over all $g_1 \in S$, $g_2 \in S_{g_1}$, $\dots$, $g_n \in S_{g_1, g_2, \dots, g_{n-1}}$ with ${}_H[1], {}_H[g_1], \dots, {}_H[g_n]$ distinct in $H \backslash G$.

Proof. By Remark 7.0.6,

$$A_H \otimes A_H \cong \prod_{g \in S} A_{H \cap H^g} = A_H \times \prod_{\substack{g \in S \\ {}_H[g] \neq {}_H[1]}} A_{H \cap H^g}$$

so Proposition 2.0.4 shows $A_H^{[2]} \cong \prod_{\substack{g \in S \\ {}_H[g] \neq {}_H[1]}} A_{H \cap H^g}$. Now suppose

$$A_H^{[n]} \cong \prod_{g_1\,\dots,\,g_{n-1}} A_{H \cap H^{g_1} \cap \dots \cap H^{g_{n-1}}}$$

for some $1 \leq n < [G:H]$, where the product runs over all $g_1 \in S$, $g_2 \in S_{g_1}$, $\dots$, $g_{n-1} \in S_{g_1, g_2, \dots, g_{n-2}}$ with ${}_H[1], {}_H[g_1], \dots, {}_H[g_{n-1}]$ distinct in $H \backslash G$. Then

$$A_H^{[n]} \otimes A_H \cong \prod_{g_1\,\dots,\,g_{n-1}} A_{H \cap H^{g_1} \cap \dots \cap H^{g_{n-1}}} \otimes A_H \cong \prod_{g_1\,\dots,\,g_{n-1}} \prod_{g_n \in S_{g_1, g_2, \dots, g_{n-1}}} A_{H \cap H^{g_1} \cap \dots \cap H^{g_n}}$$

by Remark 7.0.6. We note that every $g_n \in S_{g_1, g_2, \dots, g_{n-1}}$ with ${}_H[g_n] = {}_H[1]$ or ${}_H[g_n] = {}_H[g_i]$ for $1 \leq i \leq n-1$ provides a copy of $A_H^{[n]}$. By Corollary 7.0.9, this happens exactly n times. Hence,

$$A_H^{[n]} \otimes A_H \cong \left(A_H^{[n]}\right)^{\times n} \times \prod_{g_1,\,\dots,\,g_n} A_{H \cap H^{g_1} \cap \dots \cap H^{g_n}},$$

where the product runs over all $g_1 \in S$, $g_2 \in S_{g_1}$, $\dots$, $g_n \in S_{g_1, g_2, \dots, g_{n-1}}$ with ${}_H[1], {}_H[g_1], \dots, {}_H[g_n]$ distinct in $H \backslash G$, and the lemma now follows from Propositions 2.1.7(c) and 2.0.4. $\square$

Corollary 7.0.12. *Let $d := [G:H]$ and suppose $\mathcal{K}(G)$ is $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$ or $\Bbbk G-\mathrm{mod}$. There is an isomorphism of rings*

$$A_H^{[d]} \cong \left(A_{\mathrm{norm}_H^G}\right)^{\times \frac{d!}{[G:\mathrm{norm}_H^G]}},$$

where $\mathrm{norm}_H^G := \bigcap_{g \in G} g^{-1}Hg$ is the normal core of H in G.

Proof. From the above lemma we know that $A_H^{[d]} \cong \prod_{g_1,\dots,g_{d-1}} A_{H\cap H^{g_1}\cap\ldots\cap H^{g_{d-1}}}$, where the product runs over some $g_1,\ldots g_{d-1} \in G$ with $\{{}_H[1], {}_H[g_1], \ldots, {}_H[g_{d-1}]\} = H\backslash G$. So, $A_H^{[d]} \cong A_{\mathrm{norm}_H^G}^{\times t}$ for some $t \geq 1$. Furthermore, $\deg(A_{\mathrm{norm}_H^G}) = [G : \mathrm{norm}_H^G]$ and $\deg(A_H^{[d]}) = d!$ by Remark 3.2.2 and Proposition 7.0.3, so $t = \frac{d!}{[G:\mathrm{norm}_H^G]}$ by Remark 4.2.7. □

Corollary 7.0.13. *The ring A_H in $\mathrm{D}^b(\Bbbk G{-}\mathrm{mod})$ has constant degree $[G : H]$ if and only if norm_H^G contains every elementary abelian subgroup of H. In that case, its quasi-Galois closure is $A_{\mathrm{norm}_H^G}$. Furthermore, A_H is quasi-Galois in $\mathrm{D}^b(\Bbbk G-$ mod) if and only if H is normal in G.*

Proof. The first statement follows immediately from Lemma 7.0.4 and Corollary 7.0.12. By Proposition 3.2.6, the splitting ring of A_H is $A_{\mathrm{norm}_H^G}$, so the second statement is Theorem 5.2.9. Since A_H is an indecomposable ring, it is quasi-Galois if and only if it is its own splitting ring. Hence A_H is quasi-Galois if and only if $A_{\mathrm{norm}_H^G} \cong A_H$, which yields $\mathrm{norm}_H^G = H$ by comparing degrees. □

Remark 7.0.14. Let $H \leq G$ be a subgroup. Recall that $A_H \cong 0$ in $\Bbbk G$–stab if and only if p does not divide $|H|$. On the other hand, The Mackey Formula 7.0.7 shows that $\mathrm{Res}_H^G \mathrm{Ind}_H^G(\Bbbk) = \Bbbk \oplus (\mathrm{proj})$ if and only if $\mathrm{Ind}_{H\cap H^g}^H(\Bbbk)$ is projective for every $g \in G - H$. Hence, $A_H \cong \Bbbk$ in $\Bbbk G{-}\mathrm{stab}$ if and only if H is *strongly p-embedded* in G, that is p divides $|H|$ and p does not divide $|H \cap H^g|$ if $g \in G - H$.

Theorem 7.0.15. *Let $H \leq G$ and consider the ring A_H in $\Bbbk G{-}\mathrm{stab}$. Then,*

(a) The degree of A_H is the greatest $0 \leq n \leq [G : H]$ such that there exist distinct $[g_1], \ldots, [g_n]$ in $H\backslash G$ with p dividing $|H^{g_1} \cap \ldots \cap H^{g_n}|$.

(b) The ring A_H is quasi-Galois if and only if p divides $|H|$ and p does not divide $|H \cap H^g \cap H^{gh}|$ whenever $g \in G - H$ and $h \in H - H^g$.

(c) If A_H has degree n, the degree is constant if and only if there exist distinct $[g_1], \ldots, [g_n]$ in $H\backslash G$ such that $H^{g_1} \cap \ldots \cap H^{g_n}$ contains a G-conjugate of

every elementary abelian subgroup of H.

In that case, A_H has quasi-Galois closure given by $A_{H^{g_1}\cap\ldots\cap H^{g_n}}$.

Proof. For (a), recall that $\deg(A_H)$ is the greatest n such that $A_H^{[n]} \neq 0$, thus such that there exist distinct ${}_H[1], {}_H[g_1], \ldots, {}_H[g_{n-1}]$ with $|H \cap H^{g_1} \cap \ldots \cap H^{g_{n-1}}|$ divisible by p. To show (b), we note that

$$F_{A_H}(A_H) \cong \prod_{g\in S} A_{H\cap H^g} \cong \mathbb{1}_{A_H}^{\times \deg(A_H)}$$

corresponds to $\prod_{g\in S} A^H_{H\cap H^g} \cong \Bbbk^{\times \deg(A_H)}$ under the equivalence $A_H\text{–Mod}_{\Bbbk G\text{–stab}} \cong \Bbbk H-\mathrm{stab}$ (see Remark 7.0.6). So, A_H is quasi-Galois if and only if $A_H \neq 0$ and for every $g \in G$, either $A^H_{H\cap H^g} = 0$ or $A^H_{H\cap H^g} \cong \Bbbk$. By Remark 7.0.14, this means either $p \nmid |H \cap H^g|$, or $p \mid |H \cap H^g|$ but $p \nmid |H \cap H^g \cap H^{gh}|$ when $h \in H - H^g$. Equivalently, p does not divide $|H \cap H^g \cap H^{gh}|$ whenever $g \in G - H$ and $h \in H - H^g$. For (c), suppose A_H has constant degree n. By Proposition 5.2.6, any indecomposable ring factor of $A_H^{[n]}$ is isomorphic to the splitting ring A_H^*, so Lemma 7.0.11 shows that the quasi-Galois closure is given by $A_H^* \cong A_{H^{g_1}\cap\ldots\cap H^{g_n}}$ for all distinct ${}_H[g_1], \ldots, {}_H[g_n]$ with $|H^{g_1} \cap \ldots \cap H^{g_n}|$ divisible by p. Then, $\mathrm{supp}(A_H) = \mathrm{supp}(A_H^*) = \mathrm{supp}(A_{H^{g_1}\cap\ldots\cap H^{g_n}})$ so $H^{g_1} \cap \ldots \cap H^{g_n}$ contains a G-conjugate of every elementary abelian subgroup of H. On the other hand, if there exist distinct $[g_1], \ldots, [g_n]$ in $H\backslash G$ such that $H^{g_1} \cap \ldots \cap H^{g_n}$ contains a G-conjugate of every elementary abelian subgroup of H, then $\mathrm{supp}(A_H^{[n]}) = \mathrm{supp}(A_{H^{g_1}\cap\ldots\cap H^{g_n}}) = \mathrm{supp}(A_H)$, so the degree of A_H is constant. □

Example 7.0.16. Let $p = 2$ and suppose $G = S_3$ is the symmetric group on 3 elements $\{1, 2, 3\}$. Consider the subgroup $H := \{(), (12)\} \cong S_2$ of permutations fixing $\{3\}$. Its conjugate subgroups in G are the subgroups of permutations fixing $\{1\}$ and $\{2\}$ respectively, so $\mathrm{norm}_H^G = \{()\}$. Then, A_H is a faithful ring of degree 3 in $\mathrm{D}^b(\Bbbk G-\mathrm{mod})$ with $\mathrm{supp}(A_H) = \mathrm{Spc}(\mathrm{D}^b(\Bbbk G-\mathrm{mod}))$. On the other hand, $\mathrm{supp}(A_H^{[3]})$ contains only one point, so A_H does not have constant degree in $\mathrm{D}^b(\Bbbk G-$

mod). When considered in $\Bbbk G$–stab however, the ring A_H is quasi-Galois of degree 1, since H is strongly p-embedded in G.

Example 7.0.17. Let $p = 2$. Suppose $G = S_4$ is the symmetric group on 4 elements $\{1, 2, 3, 4\}$ and $H \cong S_3$ is the subgroup $\langle (12), (123) \rangle$ of permutations fixing $\{4\}$. The intersections $H \cap H^g$ with $g \in G - H$ each fix two elements of $\{1, 2, 3, 4\}$ pointwise, so $H \cap H^g \cong S_2$. Furthermore, the intersections $H \cap H^{g_1} \cap H^{g_2}$ with $[1], [g_1], [g_2]$ distinct in $H \backslash G$ are trivial. So, the ring A_H in $\Bbbk G-$stab has constant degree 2 and $A_H^{[2]}$ is a faithful A-algebra. The quasi-Galois closure of A_H in $\Bbbk G-$stab is A_{S_2}, with $S_2 = \{(), (12)\}$ embedded in H.

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