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AN APPROXIMATE THEORY FOR THE VIBRATIONS OF HOLLOW, ELASTIC RODS

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ABSTRACT

A system of approximate, one-dimensional equations is derived for axially symmetric motions of hollow, elastic rods of circular cross section. The theory is valid for a range of wall thicknesses from the very thin to thick walls and in fact is valid in the limit for the solid cylinder. The theory takes into account the coupling between the longitudinal, radial, and axial shear modes. The theory is based on expansions of the displacements in a series of orthogonal polynomials in the radial co-ordinate, retaining only the earliest terms representing the longitudinal, radial, and axial shear deformations. To offset the error introduced by omission of the terms of higher order, four adjustment factors are introduced and chosen in such a way that the behavior of the first three branches of the exact frequency spectrum is reproduced at long wave lengths.

Comparison is made between the three spectral lines developed from this theory when the propagation constant is real and the comparable spectral lines from the exact three-dimensional theory.

§ 1. INTRODUCTION

Approximate theories governing the free vibrations of infinitely long, hollow, elastic cylindrical shells are very numerous and an additional theory might seem superfluous. However, most of the theories are based on assumptions which restrict their validity to shells whose thickness is small compared to the inner radius and to low frequencies [1 - 8].[1] Those that accomodate thick walled cylinders [9, 10] only claim to predict accurate behaviour for the fundamental or longitudinal mode, thus also restricting themselves to low frequencies. Further, very few of the theories have been put to the only valid test, that is comparison with the exact three-dimensional theory.

The theory developed here is applicable to cylinders having a complete range of wall thicknesses from the thin wall to the very thick wall and in fact is valid in the limit for the solid cylinder. It reproduces motions corresponding to the lowest three modes of axisymmetric motion which admits a frequency range much greater than any previous theory. The three spectral lines representing the three modes match very closely the corresponding three lines from the exact theory for a variety of wall thicknesses and Poisson's ratios.

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[1] Numbers in brackets designate References at end of paper.

Examination of the spectral lines derived from the exact theory due to Gazis [11, 12] shows that the second mode is a radial mode and the third an axial shear mode for physically real values of Poisson's ratio and that they both have a strong influence on the fundamental mode and on each other. The fourth mode on the other hand has a cut-off frequency independent of Poisson's ratio and it is much higher than that of the third mode. One may predict therefore that the fourth and all higher modes will not have a marked influence on the lowest three modes. Accordingly, the theory includes motions associated with the lowest three modes.

The theory is developed following the method used for solid rods by Mindlin and McNiven [13]. The theory is based on expansions of the displacements in a series of orthogonal polynomials in the radial co-ordinate, retaining only the earliest terms representing the longitudinal, radial, and axial shear deformations. To offset the error introduced by omission of the terms of higher order, four adjustment factors are introduced and chosen in such a way that the behavior of the first three branches of the exact frequency spectrum is reproduced at long wave lengths. To make the paper self-contained, the derivation of the exact frequency equation is outlined in the second section.

§ 2. THREE-DIMENSIONAL THEORY

2.1 Frequency Equation.

Motions in isotropic, elastic solids are governed by the displacement equations of motion which are, in the absence of body forces,

$$\nabla^2 \mathbf{u} + [1/(1 - 2\nu)]\nabla\nabla \cdot \mathbf{u} = (\rho/\mu) \frac{\partial^2 \mathbf{u}}{\partial t^2} , \qquad (1)$$

where $\mathbf{u}$ is the displacement vector, the constants ρ, μ, and ν designate the mass density, the shear modulus, and Poisson's ratio, respectively, and ∇ is the usual del operator.

Solution of Eq. (1) is obtained by expressing $\mathbf{u}$ in terms of a dilatational scalar potential ϕ and an equivoluminal vector potential $\mathbf{H}$ according to

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{H} \qquad (2)$$

where the divergence of $\mathbf{H}$ is zero throughout the body. Eq. (1) is satisfied if ϕ and $\mathbf{H}$ satisfy the equations

$$\Box_1^2 \phi = 0, \quad \Box_2^2 \mathbf{H} = 0 \qquad (3)$$

in which

$$\Box_n^2 = \nabla^2 - \frac{1}{V_n^2} \frac{\partial^2}{\partial t^2} \qquad (n = 1, 2)$$

$$V_1^2 = k^2 \frac{\mu}{\rho} \, , \; V_2^2 = \frac{\mu}{\rho} \tag{4}$$

and

$$k^2 = \frac{2\,(1 - \nu)}{(1 - 2\nu)} \, .$$

We refer the rod to the cylindrical coordinates r, θ, and z, (see Fig. 1), and call the scalar components of $\mathbb{H}$ in this system H_r, H_θ, and H_z. The motions of the rod possess torsionless axisymmetry about the z axis if

$$u_r = u_r\,(r, z, t)\,, \quad u_\theta = 0\,, \quad u_z = u_z\,(r, z, t).$$

For this case, Sternberg [14] has shown that the vector ψ can be completely described in Eq. (2) by the potentials ϕ and H_θ. Using Sternberg's theorem and cylindrical coordinates, Eqs. (3) reduce to

$$\begin{aligned} V_1^2 \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] \phi &= \frac{\partial^2 \phi}{\partial t^2} \\ V_2^2 \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right] H_\theta &= \frac{\partial^2 H_\theta}{\partial t^2} \, . \end{aligned} \tag{5}$$

For the trial solutions of Eq. (5) we assume that ϕ and H_θ have the form of waves travelling along the z axis; that is,

$$\begin{aligned} \phi\,(r, z, t) &= f(r)\,\mathrm{expi}\,(\gamma z - \omega t) \\ H\,(r, z, t) &= h(r)\,\mathrm{expi}\,(\gamma z - \omega t) \end{aligned} \tag{6}$$

where γ and ω designate the wavenumber in the z direction, and the circular frequency, respectively. The functions f and h must now satisfy the equations

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + p^2\right] f = 0$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(q^2 - \frac{1}{r^2}\right)\right] h = 0, \qquad (7)$$

where

$$p^2 = \omega^2/V_1^2 - \gamma^2$$

$$q^2 = \omega^2/V_2^2 - \gamma^2. \qquad (8)$$

The general solutions of Eqs. (7) are given in terms of the Bessel functions J and Y with arguments pr and qr, or the modified Bessel functions I and K with arguments $pr = |pr|$ and $qr = |qr|$, depending on whether p and q, determined by Eqs. (8), are real or imaginary.

Using the notation adopted by Gazis [11], the general solutions of Eqs. (7) are

$$f(r) = A_1 Z_o (pr) + A_2 W_o (pr)$$

$$h(r) = A_3 Z_1 (qr) + A_4 W_1 (qr) \qquad (9)$$

where Z denotes a J or I function and W denotes a Y or K function, and the zero and one subscripts denote the order of the Bessel functions. In (9) the arguments of Z and W are real.

From Eqs. (2) and (6) we have

$$u_r = [\, f' - (i\gamma) h \,] \exp i (\gamma z - \omega t)$$

$$u_z = \left[(i\gamma)f + h' + h/r\right] \operatorname{expi}(\gamma z - \omega t) \tag{10}$$

where prime denotes differentiation with respect to r.

The boundary stresses are given by the stress-strain relations

$$\begin{aligned} \tau_{rr} &= \lambda \Delta + 2\mu\epsilon_{rr} \\ \tau_{rz} &= 2\mu\epsilon_{rz} \end{aligned} \tag{11}$$

where λ is the usual Lamé constant, and Δ is the dilatation given by

$$\Delta = \nabla^2\phi = -(p^2 + \gamma^2)\, f \operatorname{expi}(\gamma z - \omega t)\,. \tag{12}$$

Using the strain-displacement relations

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \end{aligned} \tag{13}$$

and Eqs. (10), the boundary stresses are given in terms of the potentials f and h according to

$$\begin{aligned} \tau_{rr} &= \mu\left\{\left[(2 - k^2)\,\gamma^2 - (pk)^2\right] f - \frac{2}{r} f' - 2(i\gamma)h'\right\} \\ \tau_{rz} &= \mu\left[2(i\gamma) f' + (\gamma^2 - q^2)\, h\right] \end{aligned} \tag{14}$$

where prime, as before, denotes differentiation with respect to r.

The boundary conditions, specifying the traction-free inner and outer surfaces, are

$$\begin{aligned} \tau_{rr}(a) &= 0 \quad , \quad \tau_{rz}(a) = 0 \\ \tau_{rr}(b) &= 0 \quad , \quad \tau_{rz}(b) = 0\,. \end{aligned} \tag{15}$$

Substitution of Eqs. (14) into Eqs. (15) and use of Eqs. (9) gives four homogeneous equations in terms of the amplitudes A_j (j = 1 - 4). The nontrivial solution of the equations is obtained by setting the determinant of the coefficients of the A_j equal to zero, giving

$$|C_{ij}| = 0 \;, \quad (i, j = 1 - 4) \tag{16}$$

where the i indicates the row and j the column. The elements of the determinant are

$$C_{11} = \delta\,(2\zeta^2 - \Omega^2)\, Z_o\,(\delta\alpha) + 2\alpha\lambda_1 Z_1\,(\delta\alpha)$$

$$C_{12} = \delta\,(2\zeta^2 - \Omega^2)\, W_o\,(\delta\alpha) + 2\alpha W_1\,(\delta\alpha)$$

$$C_{13} = 2\zeta\,[\delta\beta Z_o\,(\delta\beta) - Z_1\,(\delta\beta)]$$

$$C_{14} = 2\zeta\,[\delta\beta\lambda_2 W_o\,(\delta\beta) - W_1\,(\delta\beta)]$$

$$C_{21} = -\,2\zeta\alpha\lambda_1 Z_1\,(\delta\alpha)$$

$$C_{22} = -\,2\zeta\alpha W_1\,(\delta\alpha)$$

$$C_{23} = (2\zeta^2 - \Omega^2)\, Z_1\,(\delta\beta)$$

$$C_{24} = (2\zeta - \Omega^2)\, W_1\,(\delta\beta)$$

$$C_{31} = \delta a^*(2\zeta^2 - \Omega^2)\, Z_o\,(\delta\alpha a^*) + 2\alpha\lambda_1 Z_1\,(\delta\alpha a^*) \tag{17}$$

$$C_{32} = \delta a^*(2\zeta^2 - \Omega^2)\, W_o\,(\delta\alpha a^*) + 2\alpha W_1\,(\delta\alpha a^*)$$

$$C_{33} = 2\zeta\,[\delta\beta a^* Z_o\,(\delta\beta a^*) - Z_1\,(\delta\beta a^*)]$$

$$C_{34} = 2\zeta\,[\delta\beta a^*\lambda_2 W_o\,(\delta\beta a^*) - W_1\,(\delta\beta a^*)]$$

$$C_{41} = -2\zeta\alpha\lambda_1 Z_1 (\delta\alpha a^*)$$

$$C_{42} = -2\zeta\alpha W_1 (\delta\alpha a^*)$$

$$C_{43} = (2\zeta^2 - \Omega^2) \; Z_1 (\delta\beta a^*)$$

$$C_{44} = (2\zeta^2 - \Omega^2) \; W_1 (\delta\beta a^*).$$

The elements defined by Eqs. (17) are given in terms of dimensionless quantities. These quantities have the following relationship to quantities already defined.

$$\Omega = \omega/\omega_1^s \quad , \qquad \alpha = p \cdot a/\delta$$

$$\omega_1^s = V_2 \cdot \delta/a \; , \qquad \beta = q \cdot a/\delta \qquad (18)$$

$$\zeta = \gamma \cdot a/\delta \; , \qquad a^* = b/a \, .$$

In the above δ is the lowest root of Eq. (24) with $\Omega_1^s = 1$. The λ_i's ($i = 1, 2$) are introduced in order to account for the differences in the recursion equations involving the first derivatives of the different kinds of Bessel functions.

The appropriate choice of each λ_i depends on whether a J or Y function or an I or K function is used and the choice is made by reference to Table 1.

Table 1

$\lambda_1 =$	1 -1	$\frac{\Omega}{\zeta}$	$\geq$ $<$	$\frac{V_1}{V_2}$
$\lambda_2 =$	1 -1	$\frac{\Omega}{\zeta}$	$\geq$ $<$	1

For a given rod the geometric and physical parameters (a^* and ν, respectively) are established. When these are substituted in Eq. (16), it becomes an equation implicitly relating the normalized frequency Ω and the dimensionless propagation constant ζ. The combinations of Ω and ζ that satisfy Eq. (16) are obtained by numerical analysis, as explained by Gazis [12], and discussed later in section 4. The pairs of roots when plotted on the Ω - ζ plane, form frequency spectra. Because of the transcendental nature of the frequency equation, an infinite number of spectral lines can be formed from its roots. Only the lowest four are explored in this paper. The lowest three are shown in Figs. 3-10 by solid lines and the fourth by dash-dot lines.

The determinant $|C_{ij}|$ of Eq. (16) coincides, with the determinant D_3 shown by Gazis in Eq. (30) of his paper [11] for the case of motions having axial symmetry.

2. 2 Motions Having Infinite Wavelength.

When the wavelength is infinite (propagation constant ζ is zero), not only do the dilatational and equivoluminal modes become uncoupled, but the motions become either pure radial or pure axial depending on the frequency.

As the propagation constant is zero, these resonant frequencies are called the cutoff frequencies of the appropriate axial shear or radial modes. The equations governing these frequencies are established next and, subsequently, the equations giving the displacement distributions along the radius accompanying each frequency are developed.

When $\zeta = 0$, the frequency Eq. (16) can be factored into three equations.

The first is

$$\Omega^6 = 0. \tag{19}$$

This can be interpreted as identifying the mode whose frequency approaches zero, as the wavelength becomes infinitely long. This has been historically called the longitudinal mode. The phase velocity of this mode for infinite wavelength is obtained by taking the $\lim\limits_{\substack{\Omega \to 0 \\ \zeta \to 0}} \frac{\Omega}{\zeta}$ in Eq. (16). This process gives

$$\left(\frac{\Omega}{\zeta}\right)^2 = \left(\frac{V}{V_2}\right)^2 = 2\,(1 + \nu) \qquad \text{or} \qquad V^2 = \frac{E}{\rho}$$

which is the "bar" velocity as given by elementary theory and that obtained by Pochhammer [15] for the analogous limiting case of the solid rod.

The second equation

$$[2J_1(\bar{\delta}\Omega_j^r) - k^2(\bar{\delta}\Omega_j^r)\, J_o(\bar{\delta}\Omega_j^r)] \quad [2Y_1(a^*\bar{\delta}\Omega_j^r) - k^2(a^*\bar{\delta}\Omega_j^r)\, Y_o(a^*\bar{\delta}\Omega_j^r)]$$
$$- \; [2Y_1(\bar{\delta}\Omega_j^r) - k^2(\bar{\delta}\Omega_j^r)\, Y_o(\bar{\delta}\Omega_j^r)] \quad [2J_1(a^*\bar{\delta}\Omega_j^r) - k^2(a^*\bar{\delta}\Omega_j^r)\, J_o(a^*\bar{\delta}\Omega_j^r)] = 0 \tag{20}$$

where $\bar{\delta} = \delta/k$ (21)

establishes the cutoff frequencies of the radial modes. Omitting exponentials, the displacement distribution for the j^{th} radial mode ($j = 1, 2, \ldots$) is given by

$$u_r = (-A_1/a)(\bar{\delta}\Omega_j^r)\,[J_1(\bar{\delta}\Omega_j^r\, r/a) + (A_{2/A_1})\, Y_1(\bar{\delta}\Omega_j^r\, r/a)] \tag{22}$$

where the amplitude ratio A_{2/A_1} is obtained from

$$\frac{A_2}{A_1} = - \frac{\left[2J_1(\bar{\delta}\Omega_j^r) - k^2(\bar{\delta}\Omega_j^r)\; J_o(\bar{\delta}\Omega_j^r)\right]}{\left[2Y_1(\bar{\delta}\Omega_j^r) - k^2(\bar{\delta}\Omega_j^r)\; Y_o(\bar{\delta}\Omega_j^r)\right]} \tag{23}$$

The third equation

$$J_1(\delta\Omega_i^s)\, Y_1(a^*\delta\Omega_i^s) - Y_1(\delta\Omega_i^s)\, J_1(a^*\delta\Omega_i^s) = 0 \tag{24}$$

establishes the cutoff frequencies Ω_i^s of the axial shear modes. The displacement distribution for the i^{th} axial shear mode is given by.

$$u_z = \left(A_{3/a}\right)(\delta\Omega_i^s)\left[J_o(\delta\Omega_i^s\, r/a) + \left(A_{4/A_3}\right)Y_o(\delta\Omega_i^s\, r/a)\right] \quad (25)$$

where the amplitude ratio A_{4/A_3} is obtained from

$$A_{4/A_3} = -J_1\left(\delta\Omega_i^s\right) \Big/ Y_1\left(\delta\Omega_i^s\right). \quad (26)$$

We now restrict our attention to the lowest three modes. The lowest cutoff frequency will obviously be $\Omega = 0$ but study shows, what is not obvious, that the second mode is almost always a radial mode for real materials. For example, for a rod whose outside diameter is eight times its inner diameter the second mode will be a radial mode for all $\nu < \cdot 3364$. For thinner tubes the critical Poisson's ratio will be even higher. The third mode will be an axial shear mode, so the second and third cutoff frequencies will be denoted by Ω_1^r and Ω_1^s respectively.

The displacements for the longitudinal mode are axial and are uniform over a cross section.

The distribution of displacements along a radial line for the radial mode is obtained using the lowest root of Eq. (20) in Eqs. (22) and (23). This distribution is shown in Fig. 2.

The distribution for the axial shear mode is obtained using the lowest root of Eq. (24) in Eqs. (25) and (26) and is also shown in Fig. 2.

§ 3. APPROXIMATE THEORY

3.1 Expansion in Infinite Series

An approximate theory can be constructed that will predict a relationship between frequency and wavelength for as many modes as desired. What follows are the beginning steps in the generation of a theory accounting for "n" modes.

We start by choosing the radial dependency of the radial and axial displacements. These functions will be polynomials $f_n(r)$ which satisfy the orthogonality conditions

$$\left\langle f_m(r)\,,\; f_n(r)\right\rangle = \int_a^b f_m(r)\, f_n(r)\, r\, dr = 0 \qquad m \neq n \tag{27}$$

for reasons which will be apparent later.

The radial and axial components of displacement incorporating these polynomials will be as follows:

$$u_r = \sum_{n=o}^{\infty} \mathcal{U}_n(r)\, u_n(z,\, t) \tag{28}$$

$$u_z = \sum_{n=o}^{\infty} \mathcal{W}_n(r)\, w_n(z,\, t) \tag{29}$$

where $\mathcal{U}_o(r) = r/a\,;\; \mathcal{U}_1(r) = r/a - \frac{B_{11}}{a}\, r^3,$

$$\mathcal{U}_n(r) = r/a + \sum_{k=o}^{n} (-1)^k \frac{B_{nk}}{a}\, r^{2k+1} \tag{30}$$

$$\mathcal{W}_o(r) = 1;\; \mathcal{W}_1(r) = 1 - A_{11}r^2;\; \mathcal{W}_2(r) = 1 - A_{21}r^2 + A_{22}r^4,$$

$$\mathcal{W}_n(r) = 1 + \sum_{k=1}^{n} (-1)^k A_{nk} r^{2k}.$$

Each term in the series represents a mode of motion and the notation is such that the subscript for a mode matches the number of nodal circles existing in the cutoff frequency displacement distribution.

A few of the coefficients B_{nk} and A_{nk} are given below.

$$B_{11} = \frac{3(b^2 + a^2)}{2(b^4 + b^2a^2 + a^4)},$$

$$A_{11} = \frac{2}{b^2 + a^2}, \tag{31}$$

$$A_{21} = \frac{6(b^2 + a^2)}{(b^4 + 4b^2a^2 + a^4)}, \qquad A_{22} = \frac{6}{(b^4 + 4b^2a^2 + a^4)}.$$

3.2 Three-Mode Theory

We retain in our theory only the first three modes namely the longitudinal, first radial, and first axial shear. The displacements appropriate to this theory are derived from Eqs. (28) and (29). As there is only one mode having radial motions we retain only the first term in Eq. (28) and we retain the first two terms of Eq. (29) representing the longitudinal and first axial shear modes. The displacements are

$$u_r = (r/a)\, u_o(z, t)$$
$$u_z = w_o(z, t) + (1 - A_{11} r^2)\, w_1(z, t). \tag{32}$$

Stress-equation of motion. Eqs. (32) are substituted in the equations

$$\int_{-\ell}^{\ell}\int_{a}^{b}\left(\frac{\partial \tau_{rr}}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{rz}}{r}-\rho\frac{\partial^2 u_r}{\partial t^2}\right)\delta u_r\, r\, dr\, dz = 0$$

$$\int_{-\ell}^{\ell}\int_{a}^{b}\left(\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \tau_{zz}}{\partial z}+\frac{\tau_{rz}}{r}-\rho\frac{\partial^2 u_z}{\partial t^2}\right)\delta u_z\, r\, dr\, dz = 0 \qquad (33)$$

which are obtained from the variational equations of motion [16].

In Eqs. (33), τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, and τ_{rz} are components of stress derivable from a strain-energy-function U by differentiation with respect to components of strains ϵ_{rr}, $\epsilon_{\theta\theta}$, ϵ_{zz}, and ϵ_{rz}, respectively. Performing the integration with respect to r and setting the coefficients of δu_o, δw_o, and δw_1 equal to zero, we obtain three equations of motion involving stress-resultant forces.

$$-\frac{P_{ro}}{a}+\frac{\partial Q_o}{\partial z}+R_o = \frac{\rho(b^4-a^4)}{4a^2}\frac{\partial^2 u_o}{\partial t^2}$$

$$\frac{\partial P_{zo}}{\partial z}+X_o = \rho/2\,(b^2-a^2)\frac{\partial^2 w_o}{\partial t^2} \qquad (34)$$

$$\frac{\partial P_{z1}}{\partial z}+2A_{11}aQ_o+X_1 = \frac{\rho(b^2-a^2)^3}{6(b^2+a^2)^2}\frac{\partial^2 w_1}{\partial t^2}$$

where

$$P_{ro} = \int_a^b (\tau_{rr}+\tau_{\theta\theta})\, r\, dr$$

$$P_{zo} = \int_a^b \tau_{zz}\, r\, dr$$

$$P_{z1} = \int_a^b r\,(1 - A_{11} r^2)\,\tau_{zz}\,dr \tag{35}$$

$$Q_o = \int_a^b \frac{r^2}{a}\,\tau_{rz}\,dr$$

$$R_o = \left[\frac{r^2}{a}\,\tau_{rr}\right]_a^b$$

$$X_o = \left[r\,\tau_{rz}\right]_a^b, \quad X_1 = \left[r(1 - A_{11} r^2)\,\tau_{rz}\right]_a^b .$$

Components of strain. Upon substituting Eqs. (32) in the usual strain-displacement relations, we obtain

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} = \frac{u_o}{a} \\ \epsilon_{\theta\theta} &= \frac{u_r}{r} = \frac{u_o}{a} \\ \epsilon_{zz} &= \frac{\partial u_z}{\partial z} = w_o' + (1 - A_{11} r^2)\,w_1' \\ \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \frac{r}{2a}\,(u_o' - 2A_{11} a w_1) \end{aligned} \tag{36}$$

where prime indicates differentiation with respect to z.

Energy-densities. From the strain-energy-density of the three dimensional theory:

$$2U = (\tau_{rr}\epsilon_{rr} + \tau_{\theta\theta}\,\epsilon_{\theta\theta} + \tau_{zz}\,\epsilon_{zz} + 2\,\tau_{rz}\,\epsilon_{rz}) \tag{37}$$

we define an energy-density

$$\overline{U} = \int_a^b U\,r\,dr \tag{38}$$

Substituting Eqs. (36) in Eq. (37) and this, in turn, substituted in Eq. (38), we obtain, after performing the integrations:

$$2\overline{U} = P_{ro}\frac{u_o}{a} + P_{zo}w_o' + P_{z1}w_1' + Q_o(u_o' - 2A_{11}aw_1). \quad (39)$$

Defining modified strains as,

$$\begin{aligned} S_{ro} &= \frac{u_o}{a} \\ S_{zo} &= w_o' \\ S_{z1} &= w_1' \\ \Gamma_o &= u_o' - 2A_{11}aw_1 \end{aligned} \quad (40)$$

and substituting Eqs. (40) in Eqs. (39) and (36), we obtain

$$2\overline{U} = P_{ro}S_{ro} + P_{zo}S_{zo} + P_{z1}S_{z1} + Q_o\Gamma_o \quad (41)$$

and

$$\begin{aligned} \epsilon_{rr} &= \epsilon_{\theta\theta} = S_{ro} \\ \epsilon_{zz} &= S_{zo} + (1 - A_{11}r^2)S_{z1} \\ \epsilon_{rz} &= \frac{r}{2a}\Gamma_o . \end{aligned} \quad (42)$$

Similarly, from the kinetic -energy-density

$$K = \rho/2\left[\left(\frac{\partial u_r}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right] \quad (43)$$

we define an energy-density

$$\overline{K} = \int_a^b K\, r\, dr$$

$$= \frac{\rho}{2}\left[\frac{(b^4 - a^4)}{4a^2}\left(\frac{\partial u_o}{\partial t}\right)^2 + \frac{(b^2 - a^2)}{2}\left(\frac{\partial w_o}{\partial t}\right)^2 + \frac{(b^2 - a^2)^3}{6(b^2 + a^2)^2}\left(\frac{\partial w_1}{\partial t}\right)^2\right]. \tag{44}$$

Stress-strain relations. If Eqs. (42) are substituted in the stress-strain relations

$$\begin{aligned}
\tau_{rr} &= \partial U/\partial \epsilon_{rr} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\epsilon_{rr} \\
\tau_{\theta\theta} &= \partial U/\partial \epsilon_{\theta\theta} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\epsilon_{\theta\theta} \\
\tau_{zz} &= \partial U/\partial \epsilon_{zz} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\
\tau_{rz} &= \frac{1}{2}\ \partial U/\partial \epsilon_{rz} = 2\mu\epsilon_{rz}
\end{aligned} \tag{45}$$

we obtain, after performing the integration, the relation between the components of stress resultants and modified strain as:

$$\begin{aligned}
P_{ro} &= G_{ro}S_{ro} + G_{zo}S_{zo} \\
P_{zo} &= D_{ro}S_{ro} + D_{zo}S_{zo} \\
P_{z1} &= E_{z1}S_{z1} \\
Q_o &= B_o\Gamma_o
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
D_{zo} &= \frac{1}{2}(\lambda + 2\mu)(b^2 - a^2) \\
G_{ro} &= 2(\lambda + \mu)(b^2 - a^2) \\
E_{z1} &= \frac{(\lambda + 2\mu)(b^2 - a^2)^3}{6(b^2 + a^2)^2}
\end{aligned} \tag{47}$$

$$D_{ro} = G_{zo} = \lambda(b^2 - a^2)$$

$$B_o = \frac{\mu}{4a^2}(b^4 - a^4).$$

3.3 Introduction of Adjustment Factors η_i

The quality of such an approximate theory may be judged by how well it predicts the phase and group velocities for trains of waves having a given wavelength and how well it predicts the accompanying motions; that is, how well these quantities match those of the exact theory.

As we have omitted the higher order terms in Eqs. (28) and (29), we have omitted the higher modes each of which influences to a different degree the phase and group velocities. Further, by having to choose displacement patterns that satisfy a radial orthogonality condition the motions do not match those from the exact theory even when the wavelength is infinite. It is advisable therefore to introduce means for compensating for the omission of the higher order terms and for the prejudiced displacement patterns. Accordingly, we introduce adjustment factors $\eta_i (i = 1 - 4)$. We replace S_{ro} by $\eta_1 S_{ro}$ and Γ_o by $\eta_2 \Gamma_o$ in the strain-energy-density and $\dot{u}_o$ by $\eta_3 \dot{u}_o$ and $\dot{w}_1$ by $\eta_4 \dot{w}_1$ in the kinetic-energy-density, where the η_i are constants for a given rod, whose values are determined later on, and the dot indicates differentiation with respect to time. Then the adjusted energy densities are

$$2\overline{U} = G_{ro}\eta_1^2 S_{ro} + D_{zo}S_{zo}^2 + E_{z1}S_{z1}^2 + (G_{zo} + D_{ro})\,\eta_1 S_{ro}S_{zo}$$

$$+ B_o\eta_2^2\Gamma_o^2 \tag{48}$$

$$2\overline{K} = \rho\left[\frac{(b^4 - a^4)}{4a^2}\,\eta_3^2\dot{u}_o^2 + \frac{(b^2 - a^2)}{2}\,\dot{w}_o^2 + \frac{(b^2 - a^2)^3}{6(b^2 + a^2)^2}\,\eta_4^2\dot{w}_1^2\right]. \tag{49}$$

The adjusted stress resultant strain-displacement relations, derived from the strain-energy-density function (48), and Eqs. (40), are

$$P_{ro} = \partial\overline{U}/\partial S_{ro} = G_{ro}\eta_1^2 S_{ro} + G_{zo}\eta_1 S_{zo}$$

$$= G_{ro}\eta_1^2\,\frac{u_o}{a} + G_{zo}\eta_1 w_o'$$

$$P_{zo} = \partial\overline{U}/\partial S_{zo} = D_{zo}S_{zo} + D_{ro}\eta_1 S_{ro}$$

$$= D_{zo}w_o' + D_{ro}\eta_1\,\frac{u_o}{a} \tag{50}$$

$$P_{z1} = \partial\overline{U}/\partial S_{z1} = E_{z1}S_{z1}$$

$$= E_{z1}w_1'$$

$$Q_o = \partial\overline{U}/\partial\Gamma_o = B_o\eta_2^2\Gamma_o$$

$$= B_o\eta_2^2(u_o' - 2A_{11}aw_1)$$

where prime indicates differentiation with respect to z.

Letting the surface traction vanish, the adjusted stress-resultant equations of motion, derived from the variational equations of motion with constants η_3 and η_4 introduced into the kinetic energy are

$$Q_o' - \frac{P_{ro}}{a} = \frac{\rho(b^4 - a^4)}{4a^2}\,\eta_3^2\,\ddot{u}_o$$

$$P_{zo}' = \frac{\rho}{2}(b^2 - a^2)\,\ddot{w}_o \qquad (51)$$

$$2A_{11}aQ_o + P_{z1}' = \frac{\rho(b^2 - a^2)^3}{6(b^2 + a^2)^2}\,\eta_4^2\,\ddot{w}_1 .$$

Substituting Eqs. (47) in (50), and these, in turn, in Eqs. (51), the displacement equations of motion are

$$\mu(b^2 + a^2)\,\eta_2^2 u_o'' - 8(\lambda + \mu)\,\eta_1^2 u_o$$

$$- 4\lambda a \eta_1 w_o' - 4\mu a \eta_2^2 w_1' = \rho(b^2 + a^2)\,\eta_3^2\,\ddot{u}_o$$

$$(\lambda + 2\mu)\,a^2 w_o'' + 2\lambda a \eta_1 u_o' = \rho a^2 \ddot{w}_o \qquad (52)$$

$$(\lambda + 2\mu)(b^2 - a^2)^2 w_1'' - 24\mu\,(b^2 + a^2)\,\eta_2^2 w_1$$

$$+ 6\mu\,\frac{(b^2 + a^2)^2}{a}\,\eta_2^2 u_o' = \rho(b^2 - a^2)^2\,\eta_4^2\,\ddot{w}_1 .$$

3.4 <u>Frequency Equation</u>

For the trial solutions of the equations of motion (52), we consider again free harmonic motions, in the form

$$u_o = B_1 \cos \gamma z \exp i\omega t$$

$$w_o = B_2 \sin \gamma z \exp i\omega t$$
$$w_1 = B_3 \sin \gamma z \exp i\omega t. \tag{53}$$

Substituting Eqs. (53) in (52), we obtain the characteristic equation

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{vmatrix} = 0 \tag{54}$$

where

$$\begin{aligned} a_{11} &= B\eta_2^2\delta^2\zeta^2 + 8(k^2 - 1)\eta_1^2 - B\eta_3^2\delta^2\Omega^2 \\ a_{22} &= 2\delta^2(k^2\zeta^2 - \Omega^2) \\ a_{33} &= 6\left(k^2 \frac{A^2}{B^2}\delta^2\zeta^2 + \frac{24}{B}\eta_2^2 - \frac{A^2}{B^2}\eta_4^2\delta^2\Omega^2\right) \\ a_{12} &= 4(k^2 - 2)\eta_1\delta\zeta \\ a_{13} &= 12\eta_2^2\delta\zeta \end{aligned} \tag{55}$$

and

$$\begin{aligned} A &= a^{*2} - 1 \\ B &= a^{*2} + 1 . \end{aligned} \tag{56}$$

By formally letting $A = B = 1$ in Eqs. (55), which can be derived by normalizing appropriate quantities in Eqs. (18) and (30) with respect to "b" instead of "a" and letting "a" equal to zero, Eqs. (55) coincide with Eqs. (45) of Mindlin and McNiven [13] for the case of a solid rod.

For the evaluation of adjustment factors η_i, as well as for plotting the approximate frequency spectra, it is convenient to expand the determinantal Eq. (54) in the polynomial form

$$c_1(\delta\Omega)^6 - c_2(\delta\Omega)^4 + c_3(\delta\Omega)^2 - c_4 = 0 \tag{57}$$

where

$$c_1 = \frac{A^2}{B}\eta_3^2\eta_4^2$$

$$c_2 = \left[\frac{A^2}{B}\eta_2^2\eta_4^2 + k^2\frac{A^2}{B}\eta_3^2 + k^2\frac{A^2}{B}\eta_3^2\eta_4^2\right](\delta\zeta)^2 + \left[8(k^2-1)\frac{A^2}{B^2}\eta_1^2\eta_4^2 + 24\eta_2^2\eta_3^2\right]$$

$$c_3 = \left[k^2\frac{A^2}{B}\eta_2^2 + k^2\frac{A^2}{B}\eta_2^2\eta_4^2 + k^4\frac{A^2}{B}\eta_3^2\right](\delta\zeta)^4 \tag{58}$$

$$+ \left[8k^2(k^2-1)\frac{A^2}{B^2}\eta_1^2 + 8(3k^2-4)\frac{A^2}{B^2}\eta_1^2\eta_4^2 + 24k^2\eta_2^2\eta_3^2\right](\delta\zeta)^2$$

$$+ \left[192\frac{(k^2-1)}{B}\eta_1^2\eta_2^2\right]$$

$$c_4 = \left[k^4\frac{A^2}{B}\eta_2^2\right](\delta\zeta)^6 + \left[8k^2(3k^2-4)\frac{A^2}{B^2}\eta_1^2\right](\delta\zeta)^4 + \left[192\frac{(3k^2-4)}{B}\eta_1^2\eta_2^2\right](\delta\zeta)^2.$$

3.5 Evaluation of Adjustment Factors η_i^2

Eq. (57) is the frequency equation relating the square of the frequency to the square of the wave number, with the radii, Poisson's ratios, and the adjustment factors η_i^2 as the parameters. The three spectral branches derived from the roots of this equation should match, as closely as possible, the corresponding branches from frequency equation (16) of the exact theory. In general, the frequency Ω must be real; but the wave number ζ along the axis of the rod may be real, imaginary, or complex. In this paper, we restrict ourselves to real wave numbers.

The match between the three branches of the cubic equation and the analogous branches of Eq. (16) may be improved by means of the adjustment factors η_i; but, since the η_i are constants for a given rod, a perfect match can be made, in general, only at one value of ζ for each of them.

Now large, real ζ corresponds to frequencies high enough to enter the frequency range of modes that have not been included in the approximate equations so that in general the applicability of the approximate equations is limited to frequencies below the lowest frequency, for real wave numbers, of the lowest neglected mode and the correspondingly small wave numbers.

It is important that the approximate theory match the exact theory for long wavelengths so that ideally the matching should be made at the cutoff frequencies for which the wavelength is infinite. For the solid rod theory, developed in [13], it was possible to match four quantities at the cutoff frequencies for the evaluation of all four adjustment factors. The intercepts were matched as well as the curvatures of the second and third spectral lines at the intercepts. For the Hollow Rod theory the intercepts are likewise matched but because the frequency equations are much more complicated than for the solid rod it is not possible to match the intercept curvatures. As an alternative the spectral lines from the approximate theory are made to pass through two points on the spectral lines from the exact theory on the Ω - ζ plane apart from the Ω axis, that is at points for which the wavelength is finite.

We first match the intercepts. From the exact theory the lowest three intercepts are

$$0\ ,\ 1\ ,\ \Omega_1^r$$

where Ω_1^r is the lowest root of Eq. (20). From the approximate theory the intercepts are

$$0,\ \frac{2\sqrt{6B}}{\delta A}\cdot\frac{\eta_2}{\eta_4}\quad ,\quad \frac{2\sqrt{2(k^2-1)}}{\delta\sqrt{B}}\ \frac{\eta_1}{\eta_3} \tag{59}$$

Setting comparable intercepts equal gives the two equations

$$\left(\frac{\eta_2}{\eta_4}\right)^2 = P_2 = \frac{\delta^2 A^2}{24B}$$

$$\left(\frac{\eta_1}{\eta_3}\right)^2 = P_1 = \frac{(\delta\Omega_1^r)^2\ B}{8\,(k^2-1)}\ . \tag{60}$$

It can be observed that the intercept of the fundamental mode needs no adjustment. Neither does the slope of the fundamental mode at the intercept. The slopes from both the exact and approximate theories are

$$\frac{\Omega}{\zeta} = \sqrt{2(1+\nu)}.$$

It remains to establish two more matching points leading to two more equations.

After much exploration it was decided to match the first and third spectral lines at $\zeta = 0.6$. Matching at these two points maintained a match for longer wavelengths and gave real, positive adjustment factors which is a necessary condition to maintain real strains and positive definite energy densities. If we call the frequencies of the first and third modes for $\zeta = \bar{\zeta} = 0.6$, $\bar{\Omega}_1$ and $\bar{\Omega}_3$, we obtain two more equations in the adjustment factors by substituting the $\bar{\Omega}_1$ and $\bar{\Omega}_3$ from the exact equation (16) successively in the approximate Eq. (57).

The four equations can be reduced to the following equation in η_4

$$G_1 \eta_4^4 + G_2 \eta_4^2 + G_3 = 0 \qquad (61)$$

where

$$G_1 = D_{31} D_{13} - D_{11} D_{33}$$

$$G_2 = D_{11}D_{34} + D_{12}D_{33} - D_{31}D_{14} - D_{32}D_{13} \tag{62}$$

$$G_3 = D_{14}D_{32} - D_{12}D_{34}$$

and for i = 1 or 3

$$D_{i1} = (\delta\bar{\Omega}_i)^2 \left[(\delta\bar{\Omega}_i)^2 - k^2 (\delta\bar{\zeta})^2\right]$$

$$D_{i2} = k^2 (\delta\bar{\zeta})^2 \left[(\delta\bar{\Omega}_i)^2 - k^2 (\delta\bar{\zeta})^2\right] \tag{63}$$

$$D_{i3} = \left\{\frac{A^2}{B}\right\}(\delta\bar{\Omega}_i)^6 - \left\{\left[k^2 \frac{A^2}{B}\right](\delta\bar{\zeta})^2 + \left[8(k^2-1)\frac{A^2}{B^2}P_1 + 24P_2\right]\right\}(\delta\bar{\Omega}_i)^4$$

$$+\left\{\left[8(3k^2-4)\frac{A^2}{B^2}P_1 + 24k^2P_2\right](\delta\bar{\zeta})^2 + \left[192\frac{(k^2-1)}{B}P_1P_2\right]\right\}(\delta\bar{\Omega}_i)^2$$

$$-\left\{\left[192\frac{(3k^2-4)}{B}P_1P_2\right](\delta\bar{\zeta})^2\right\}$$

$$D_{i4} = \left\{\left[k^2 \frac{A^2}{B}\right](\delta\bar{\zeta})^2\right\}(\delta\bar{\Omega}_i)^4 - \left\{\left[k^4 \frac{A^2}{B}\right](\delta\bar{\zeta})^4 +\right.$$

$$\left.\left[8k^2(k^2-1)\frac{A^2}{B^2}P_1\right](\delta\bar{\zeta})^2\right\}(\delta\bar{\Omega}_i)^2 + \left\{\left[8k^2(3k^2-4)\frac{A^2}{B^2}P_1\right](\delta\bar{\zeta})^4\right\}.$$

Eq. (61) has two real solutions but only one is acceptable on physical grounds. Having η_4, η_3 is obtained from

$$\eta_3^2 = \frac{A^2}{B}P_2\eta_4^2(\delta\bar{\zeta})^2\frac{\left(D_{i1}\eta_4^2 - D_{i2}\right)}{\left(D_{i3}\eta_4^2 - D_{i4}\right)} \quad (i = 1 \text{ or } 3). \tag{64}$$

Finally η_1 and η_2 are obtained from Eq. (60).

Tables II - V give the values of the η_i^2 for some typical values of the parameters.

It is important to note that for the special case where the rod is solid the four adjustment factors obtained, for a variety of Poisson's ratios, are exactly the same as those derived using the equations of Mindlin and McNiven [13].

Having the η_i^2 the roots of Eq. (57) are found and plotted on the Ω - ζ plane. The spectral lines formed from these points are shown as dotted lines in Figs. 3-10.

§ 4. NUMERICAL ANALYSIS AND FREQUENCY SPECTRA

4.1 Numerical Analysis

Numerical analysis using a digital computer was required in many aspects of the solution, but only the solutions of the frequency equations are discussed, as the other analyses were carried out using similar methods.

The frequency equations (16) and (57) implicitly relate the normalized frequency Ω and the dimensionless propagation constant ζ, and are influenced by the parameters ν and a^*. The equations, therefore, can be written

$$F(\Omega, \zeta; \nu, a^*) = 0. \qquad (65)$$

They were each analysed for twenty eight sets of parameters, viz., $a^* = 1.1, 2.0, 4.0$, and 8.0, and for each a^*, $\nu = 0.20, 0.21, 0.23,$ $0.25, 0.27, 0.29$, and 0.31. After the introduction of the parameters, Eq. (65) can be solved either by adopting successive values of Ω and finding a set of roots ζ for each Ω, or by adopting values of ζ and finding the roots Ω. The latter scheme was adopted as the roots Ω can be more easily identified. A pair of roots (Ω, ζ) establishes a point on the Ω - ζ plane of the frequency spectrum. Only roots involving real ζ were established.

The method of finding the roots of Eq. (16) is as follows. An adopted value of the propagation constant $\bar{\zeta}$ is introduced into the equation which can then be satisfied by a set of roots $\bar{\Omega}_i$. A coarse mesh size $\Delta\Omega$ and a fine mesh size $\Delta^2\Omega$, giving plotting accuracy, are adopted. The frequency range $0 < \Omega < 2$ is then scanned from a low frequency $\bar{\Omega}_o$ to the upper limit. The values of Eq. (65) are found for $(\bar{\Omega}_o + n\,\Delta\Omega,\ \zeta)$ $(n = 0, 1, 2, \ldots)$ until a change of sign of the values indicate a root. The interval in which the change in sign occurs is then immediately scanned using the smaller mesh size $\Delta^2\Omega$ until a change in sign of the value of Eq. (65) establishes a root $(\bar{\Omega}_1)$ within plotting accuracy. Starting now at $\bar{\Omega}_1 + \Delta\Omega$, and moving up the spectrum, the root $\bar{\Omega}_2$ is established in the same way, and so on.

As noted in Ref. 12, spurious roots appear along those lines where α and β equal zero. These roots were distinguished easily from the true roots during the plotting of the spectral lines.

For Eq. (57), we determine the values of η_i^2, as explained earlier, and adopt a value of $\bar{\zeta}$. It then becomes a cubic equation in Ω^2. It is easy to determine the three roots Ω_i^2 of this equation using one of the polynomial subroutines.

4.2 Frequency Spectra

Frequency spectra have been formed from the roots of the frequency equations obtained numerically as described in the previous section. Figs. 3-10 show the spectra for the eight physical rods described. The range of frequencies has been restricted to $0 \leqslant \Omega \leqslant 2.0$, which admits the lowest four spectral lines from the exact theory, and all three branches from the approximate theory. The range of propagation constant extends, for each case, from $\zeta = 0$ (infinite wavelength) to $\zeta = 0.8$.

For the values of the geometric parameter (a*) considered, there is no value of ν in the range $0.20 \leq \nu \leq 0.31$, for which $\Omega_1^r = \Omega_1^s = 1.0$. For the solid rod, it was found, that for $\nu = \nu_c = 0.2833$ $\Omega_1^r = 1$. The nearest value of ν to the range, when $\Omega_1^r = 1$, is $\nu_c = .3364$ for a* = 8.0 as explained previously.

The ratio of the phase velocity (V) and group velocity (V_g) to the velocity of equivoluminal waves (V_2) can be obtained from Figs. 3-10, in as much as

$$\frac{V}{V_2} = \frac{\omega}{\gamma V_2} = \frac{\Omega}{\zeta}$$

$$\frac{V_g}{V_2} = \frac{d\omega}{V_2 d\gamma} = \frac{d\Omega}{d\zeta} \quad (66)$$

Thus the slope of the straight line from the origin to a point on a branch is proportional to the phase velocity, and the slope of the branch is proportional to the group velocity.

§ 5. UNIQUENESS AND ORTHOGONALITY

A uniqueness theorem, analogous to Neumann's [13, 16], will establish the initial and boundary conditions appropriate to three-mode theory.

Consider two systems of displacements, strains, and stresses which satisfy the strain-displacements relations (40), the associated compatibility condition.

$$\frac{1}{a}\frac{\partial \Gamma_o}{\partial z} + 2A_{11}S_{z1} - \frac{\partial^2 S_{ro}}{\partial z^2} = 0, \tag{67}$$

the stress-strain relations (50), and the stress equation of motion (34). Let the differences between corresponding components of displacement, strain, and stress constitute a "difference system" of these quantities and let K_2 and U_2 be the kinetic and strain energy densities of the difference system. Then the sum of K_2 and U_2 in a bar of length 2ℓ at time t is

$$K_t + U_t = K_o + U_o + \int_o^t dt \int_{-\ell}^{\ell} (\dot{K}_2 + \dot{U}_2)\, dz \tag{68}$$

where K_o and U_o are the values of K_t and U_t at an initial time $t = 0$.

$$\text{Now } \dot{K}_2 = \rho\left[\frac{(b^4 - a^4)}{4a^2}\eta_3^2 \ddot{u}_o \dot{u}_o + \frac{(b^2 - a^2)}{2}\ddot{w}_o \dot{w}_o + \frac{(b^2 - a^2)}{6(b^2 + a^2)}\eta_4^2 \ddot{w}_1 \dot{w}_1\right] \tag{69}$$

where u_o, w_o, and w_1 are the displacements of the difference system; and

$$\dot{U}_2 = \frac{\partial U_2}{\partial S_{ro}} \dot{S}_{ro} + \frac{\partial U_2}{\partial S_{zo}} \dot{S}_{zo} + \frac{\partial U_2}{\partial S_{z1}} \dot{S}_{z1} + \frac{\partial U_2}{\partial \Gamma_o} \dot{\Gamma}_o \tag{70}$$

where the strains are those of the difference system. From Eqs. (40) and (50), Eq. (70) becomes

$$\begin{aligned} \dot{U}_2 &= P_{ro} \frac{\dot{u}_o}{a} + P_{zo}\dot{w}_o' + P_{z1}\dot{w}_1' + Q_o (\dot{u}_o' - 2A_{11} a\dot{w}_1) \\ &= \left(\frac{P_{ro}}{a} - Q_o'\right)\dot{u}_o - P_{zo}'\dot{w}_o - (P_{z1}' + 2A_{11} aQ_o)\,\dot{w}_1 \\ &\quad + (P_{zo}\dot{w}_o + P_{z1}\dot{w}_1 + Q_o\dot{u}_o)' \end{aligned} \tag{71}$$

where prime and dot, as before, indicate differentiation with respect to z and t, respectively. Hence

$$\begin{aligned} (\dot{K}_2 + \dot{U}_2) = &- \left[Q_o' - \frac{P_{ro}}{a} - \rho\, \frac{(b^4 - a^4)}{4a^2} \eta_3^2 \ddot{u}_o\right] \dot{u}_o \\ &- \left[P_{zo}' - \rho \frac{(b^2 - a^2)}{2} \ddot{w}_o\right] \dot{w}_o \\ &- \left[P_{z1}' + 2A_{11} aQ_o - \frac{\rho (b^2 - a^2)^3}{6(b^2 + a^2)^2} \eta_4^2 \ddot{w}_1\right] \dot{w}_1 \\ &+ \left[P_{zo}\dot{w}_o + P_{z1}\dot{w}_1 + Q_o\dot{u}_o\right]' \end{aligned} \tag{72}$$

or, using the stress equations of motion (34),

$$(\dot{K}_2 + \dot{U}_2) = R_o \dot{u}_o + X_o \dot{w}_o + X_1 \dot{w}_1 + [P_{zo} \dot{w}_o + P_{z1} \dot{w}_1 + Q_o \dot{u}_o]' . \quad (73)$$

Upon substituting Eq. (73) into Eq. (68), we have, finally

$$K_t + U_t = K_o + U_o + \int_o^t dt \int_{-\ell}^{\ell} (R_o \dot{u}_o + X_o \dot{w}_o + X_1 \dot{w}_1)\, dz$$

$$+ [P_{zo} \dot{w}_o + P_{z1} \dot{w}_1 + Q_o \dot{u}_o]_{-\ell}^{\ell} . \quad (74)$$

Then, by the usual arguments based on the positive definiteness of K_2 and U_2, uniqueness of solution is insured if the following are specified,

(i) Throughout the rod, the initial values of u_o, w_o, w_1 and $\dot{u}_o$, $\dot{w}_o$, $\dot{w}_1$.

(ii) Throughout the rod, one member of each of the three products $R_o u_o$, $X_o w_o$, $X_1 w_1$.

(iii) At each end of the rod, one member of each of the three products $P_{zo} w_o$, $P_{z1} w_1$, and $Q_o u_o$.

By a closely related procedure, it may be shown that in two solutions

$$(u_o, w_o, w_1) = (u_{op}, w_{op}, w_{1p}) \exp i\omega_p t$$
$$(u_o, w_o, w_1) = (u_{oq}, w_{oq}, w_{1q}) \exp i\omega_q t \quad (75)$$

of the homogeneous ($R_o = X_o = X_1 = 0$) stress equations of motion (34), the characteristic functions satisfy the orthogonality condition

$$\int_{-\ell}^{\ell} \left(3B\eta_3^2 u_{op}u_{oq} + 6\, w_{op}w_{oq} + 2\, \frac{A^2}{B^2}\eta_4^2 w_{1p}w_{1q}\right) dz = 0 \qquad (76)$$

for $\omega_p \neq \omega_q$ and homogeneous end conditions (iii).

$a^* = 1\cdot 1$, $\delta = 31\cdot 4267$				
ν	η_1^2	η_3^2	η_2^2	η_4^2
0·20	0·401325	1·064051	2·337040	2·845992
0·21	0·398401	1·079190	2·430611	2·959940
0·23	0·395163	1·121668	2·587513	3·151012
0·25	0·381052	1·139998	2·664081	3·244254
0·27	0·368688	1·165718	2·759261	3·360162
0·29	0·366005	1·232827	2·984602	3·634577
0·31	0·353429	1·280322	1·138162	3·821578

Table II

$a^* = 2\cdot 0$, $\delta = 3\cdot 1965$				
ν	η_1^2	η_3^2	η_2^2	η_4^2
0·20	0·394295	0·881355	1·670803	2·180292
0·21	0·390406	0·889170	1·735301	2·264457
0·23	0·378151	0·896708	1·752411	2·286785
0·25	0·370561	0·919328	1·889082	2·465132
0·27	0·358553	0·935558	1·914999	2·498952
0·29	0·346977	0·958394	1·942561	2·534918
0·31	0·340080	1·002438	2·105549	2·747608

Table III

$a^* = 4 \cdot 0$, $\delta = 1 \cdot 1118$				
ν	η_1^2	η_3^2	η_2^2	η_4^2
0·20	0·493658	0·800074	1·062626	1·558852
0·21	0·487715	0·799452	1·106085	1·622606
0·23	0·476251	0·801344	1·101373	1·615692
0·25	0·454181	0·786957	1·110590	1·629213
0·27	0·437677	0·783598	1·106261	1·622863
0·29	0·431979	0·803462	1·049796	1·540030
0·31	0·410059	0·797118	1·164164	1·707806

Table IV

$a^* = 8 \cdot 0$, $\delta = 0 \cdot 4998$				
ν	η_1^2	η_3^2	η_2^2	η_4^2
0·20	0·686266	0·917775	0·855902	1·346714
0·21	0·673244	0·905798	0·894568	1·407551
0·23	0·666135	0·907477	0·878628	1·382472
0·25	0·621471	0·857672	0·972849	1·530723
0·27	0·590321	0·827546	0·974109	1·532705
0·29	0·580178	0·828065	1·048586	1·649890
0·31	0·553994	0·807003	1·054721	1·659544

Table V

CAPTIONS FOR FIGURES

Figure 1: The hollow rod showing the reference coordinates and dimensions.

Figure 2: Displacement distributions for the lowest three modes for motions having infinite wavelength. The distributions are for the first radial, longitudinal, and first axial shear modes.

Figures 3-10: Spectra of frequency vs. real propagation constant showing comparison between the exact and approximate theories.

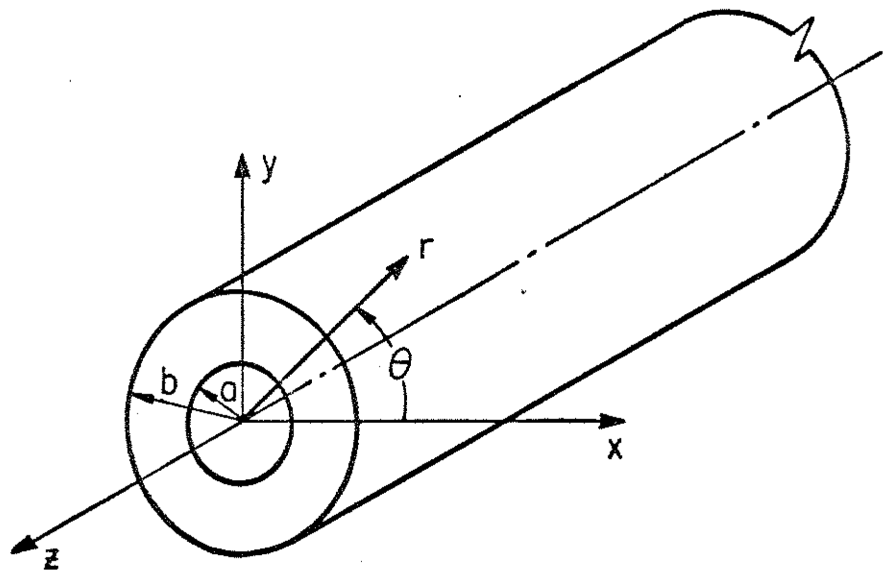
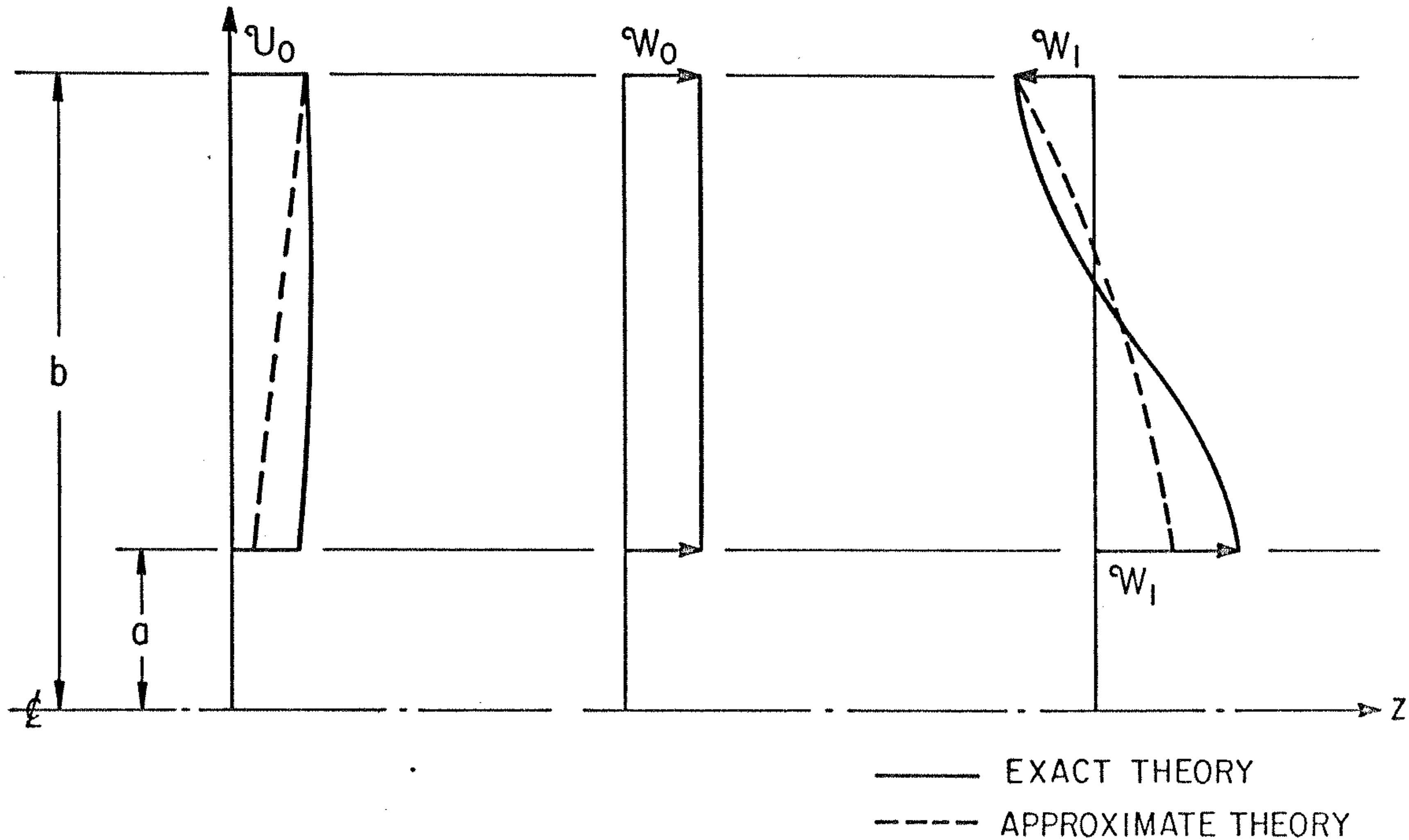
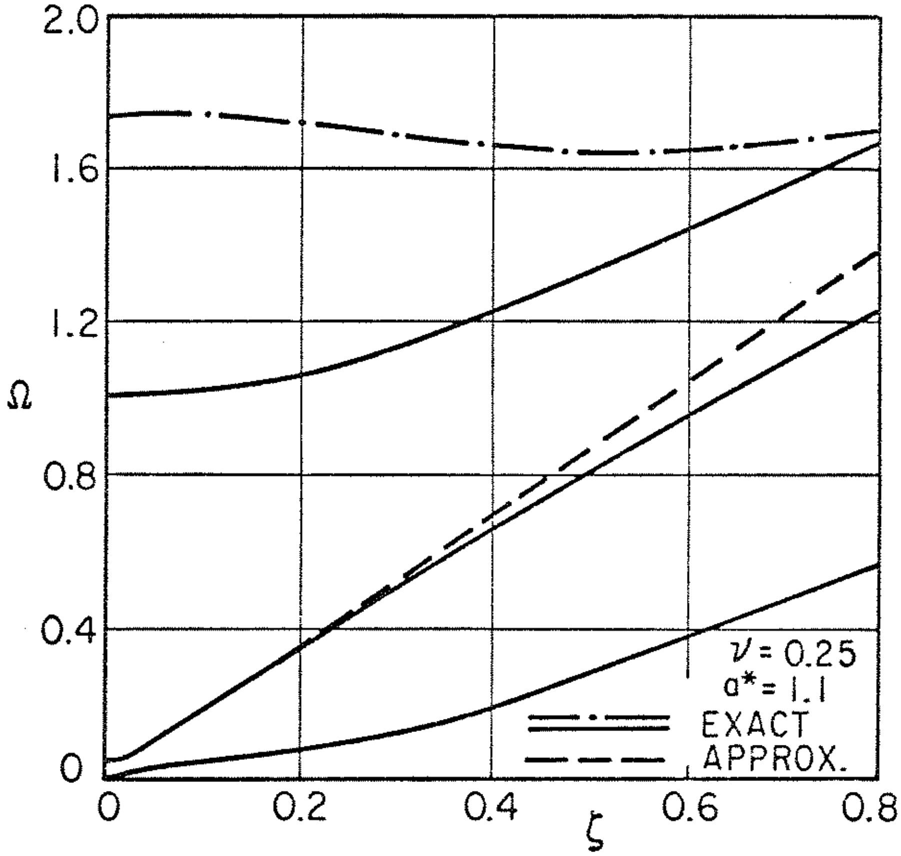


FIG. 1



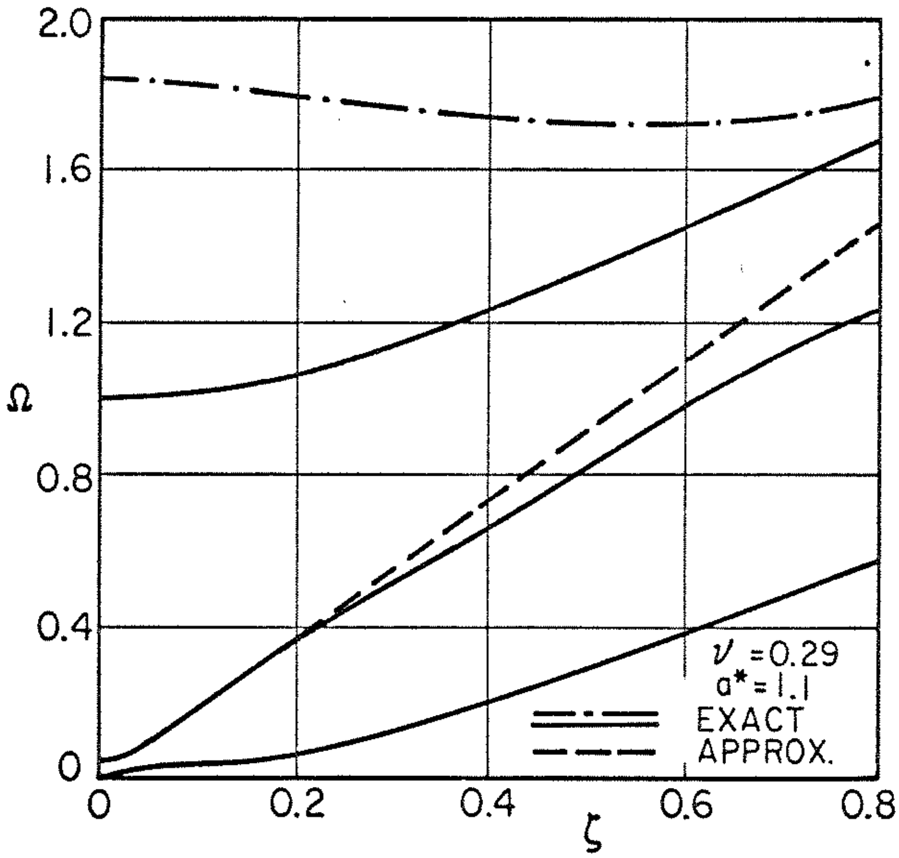
$\mathcal{U}_0$
$\mathcal{W}_0$
$\mathcal{W}_1$
$\mathcal{W}_1$
b
a
℄
z
EXACT THEORY
APPROXIMATE THEORY

FIG. 2



2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.25
a* = 1.1
EXACT
APPROX.

FIG. 3



2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.29
a* = 1.1
EXACT
APPROX.

FIG. 4

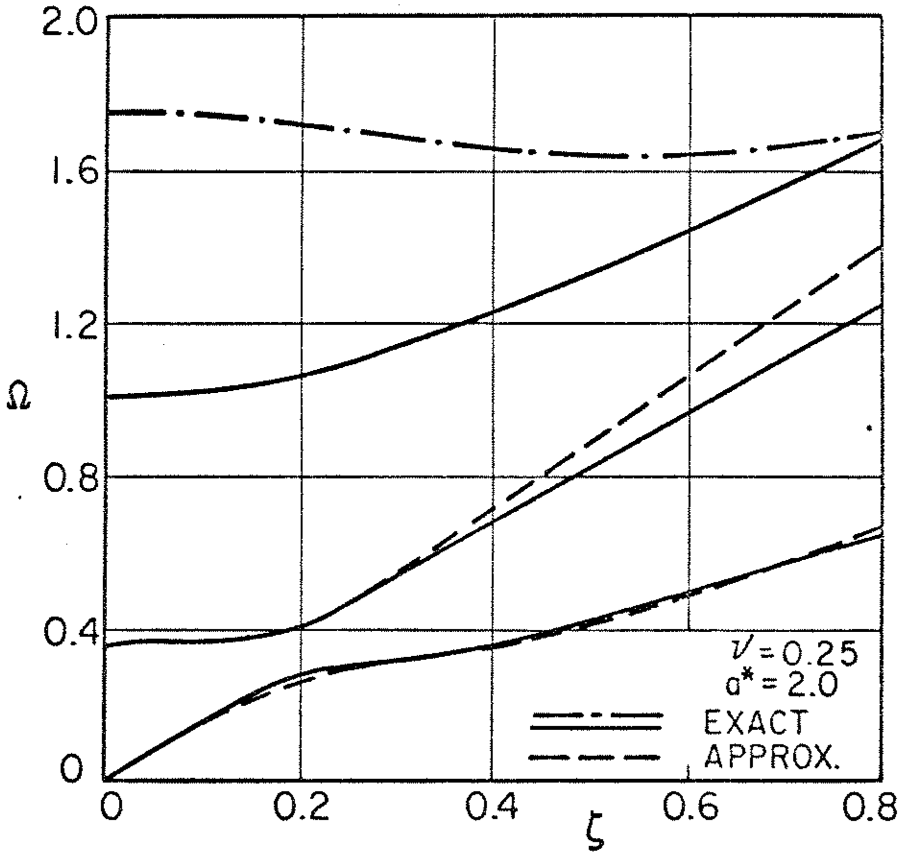

2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.25
a* = 2.0
EXACT
APPROX.

FIG. 5

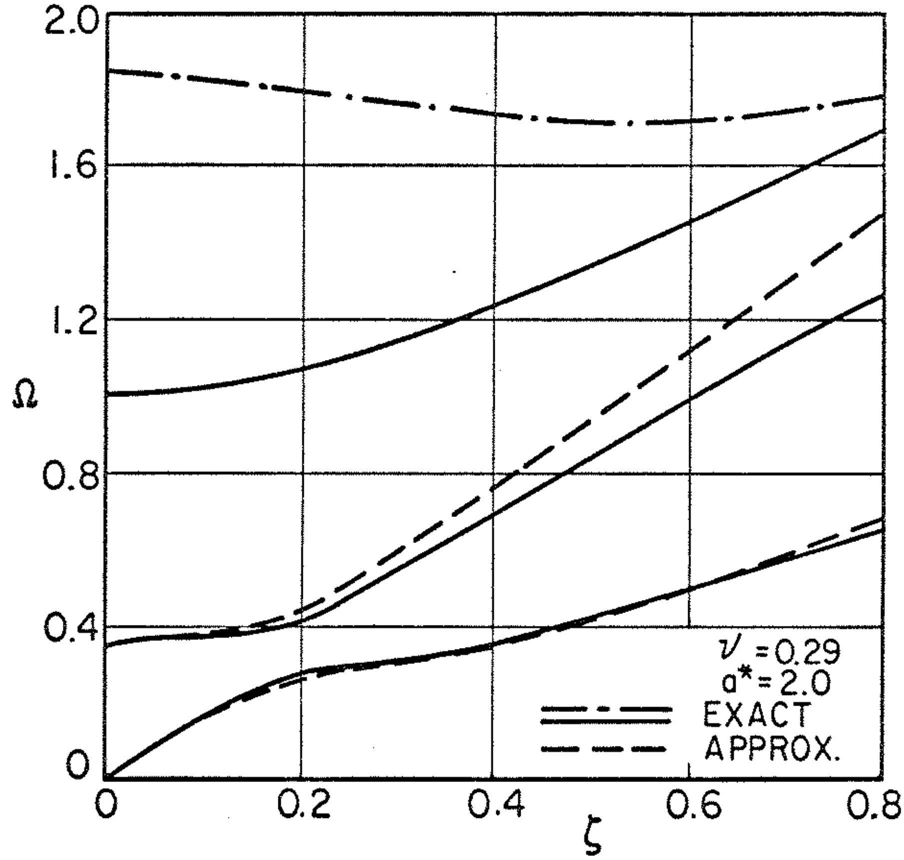

2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.29
a* = 2.0
EXACT
APPROX.

FIG. 6

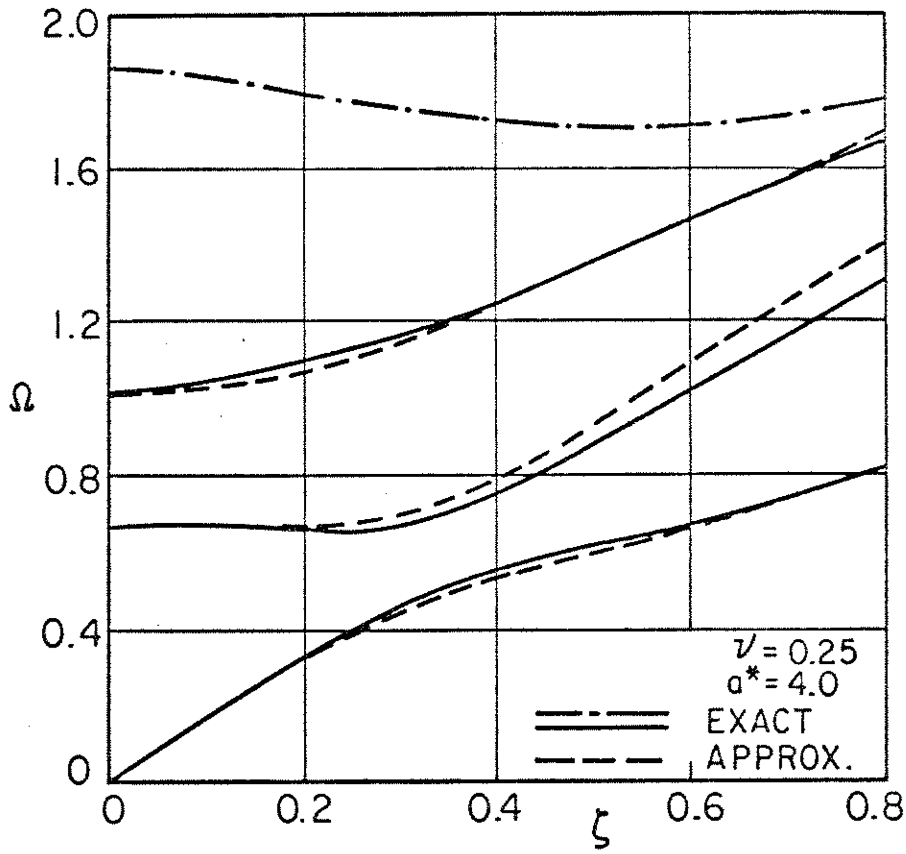

2.0
1.6
1.2
Ω
0.8
0.4
0
$\nu = 0.25$
$a^* = 4.0$
EXACT
APPROX.
0
0.2
0.4
ζ
0.6
0.8

FIG. 7

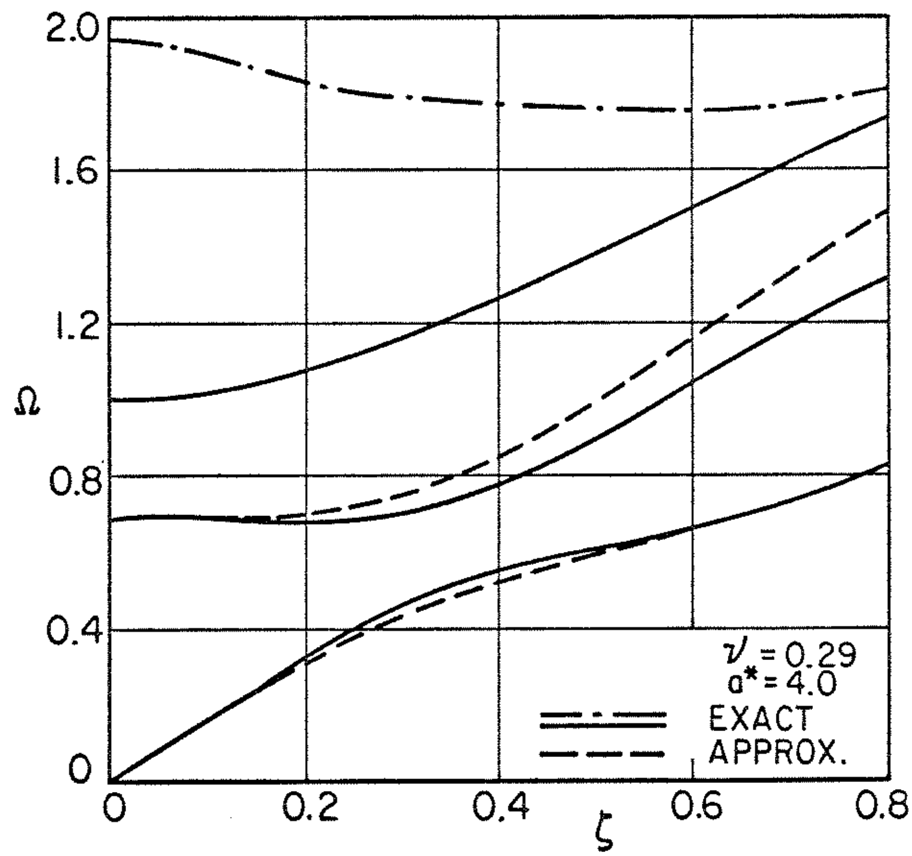

2.0
1.6
1.2
Ω
0.8
0.4
0
$\nu = 0.29$
$a^* = 4.0$
EXACT
APPROX.
0
0.2
0.4
ζ
0.6
0.8

FIG. 8

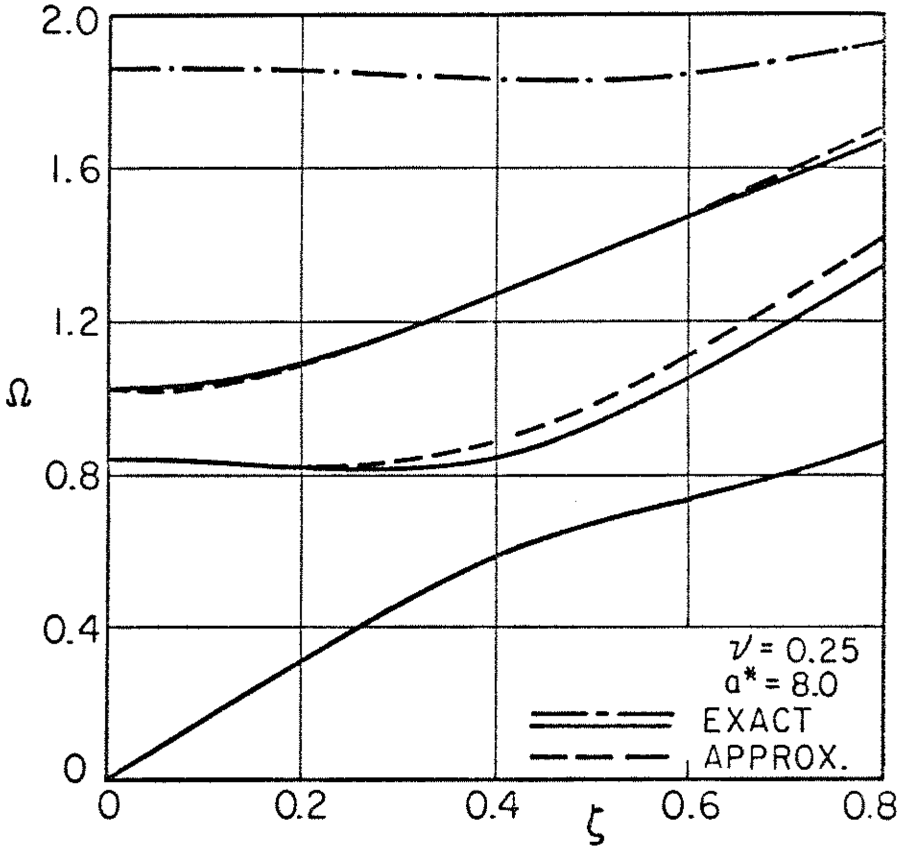

2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.25
a* = 8.0
EXACT
APPROX.

FIG. 9

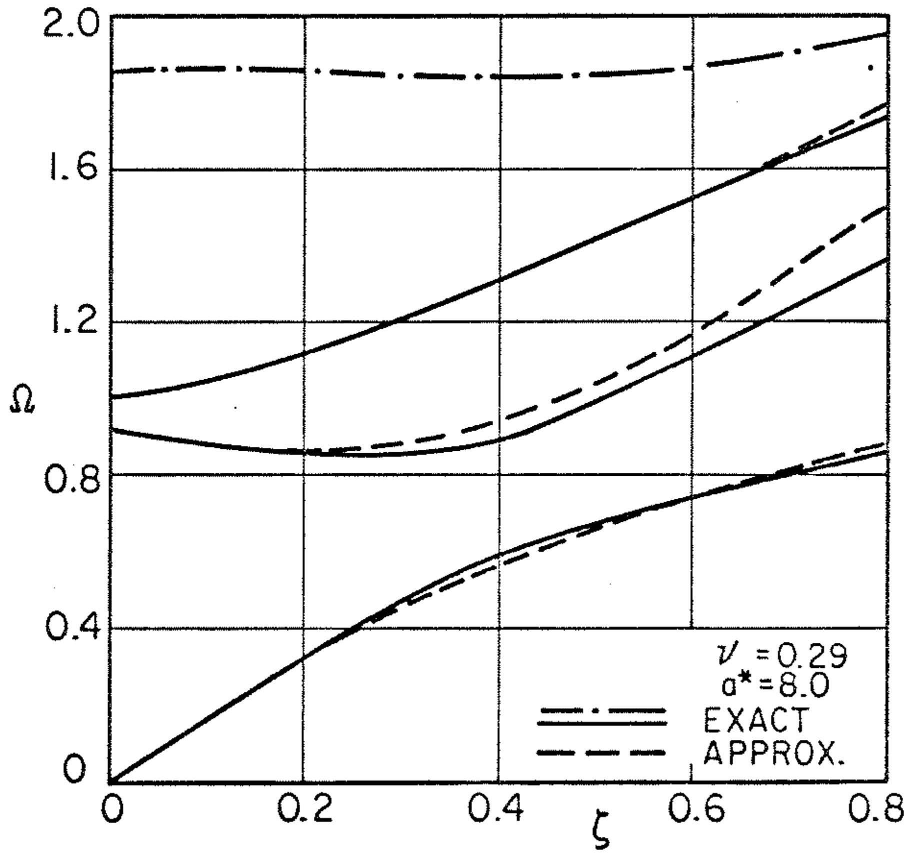

2.0
1.6
1.2
Ω
0.8
0.4
0
0
0.2
0.4
0.6
0.8
ζ
ν = 0.29
a* = 8.0
EXACT
APPROX.

FIG. 10

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