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Higher Fitting Ideals of p-adic Realizations of Abstract 1-Motives and a Special Case of the Breuil-Schneider Conjecture

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Higher Fitting Ideals of p-adic Realizations of Abstract 1-Motives and a Special Case of the Breuil-Schneider Conjecture

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Robert Allen Snellman Jr.

Committee in charge:

Professor Cristian Popescu, Chair
Professor Claus Sorensen, Co-Chair
Professor Ronald Graham
Professor Russell Impagliazzo
Professor Kiran Kedlaya

2017

The dissertation of Robert Allen Snellman Jr. is approved, and it is acceptable in quality and form for publication on microfilm:

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Co-Chair

_____
Chair

University of California, San Diego

2017

DEDICATION

To my sweetheart Amy

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The completion of this dissertation would not have been possible had it not been for the constant encouragement and support of many people. I'd like to thank my parents for greatly sacrificing on behalf of my academic pursuits, as well as supporting and encouraging me along my journey.

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VITA

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ABSTRACT OF THE DISSERTATION

Higher Fitting Ideals of p-adic Realizations of Abstract 1-Motives and a Special Case of the Breuil-Schneider Conjecture

by

Robert Allen Snellman Jr.

Doctor of Philosophy in Mathematics

University of California San Diego, 2017

Professor Cristian Popescu, Chair

Professor Claus Sorensen, Co-Chair

Part I: Starting with a rational odd prime $p > 2$ and a cyclic extension $F/\mathbb{Q}$ whose Galois group has order coprime to p, Kurihara's conjecture [24] gives an explicit description of all higher Fitting ideals of large p-power quotients of the classical Iwasawa module X, over correspondingly large p-power quotients of the classical Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$. The generators of these higher Fitting ideals are, essentially, special values of equivariant L-functions. A complete proof of Kurihara's conjecture was recently given by Popescu-Stone in full generality [41]. This dissertation conjectures a generalization of Kurihara's conjecture to so-called "semi-nice" extensions F/k where F is CM and k is totally real. In particular, this generalized conjecture specializes to Kurihara's original setting with $k = \mathbb{Q}$ and F a CM field given by the fixed field of $\overline{F}$ by the kernel of an odd Dirichlet character χ of order coprime to p, such that χ is not the Teichmüller character ω. Under certain hypotheses a proof of the generalized conjecture is given, away from the Teichmüller component. The methods of proof employed for the generalized conjecture are similar to those used by Popescu and Stone in their proof of Kurihara's conjecture.

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Part II: From a potentially semistable representation ρ of the absolute Galois group of a p-adic field $L/\mathbb{Q}_p$, Breuil and Schneider [4] construct a locally algebraic representation BS(ρ). The Breuil-Schneider conjecture asserts the equivalence between BS(ρ) carrying a GL$_n$-invariant norm, and the existence of a certain (φ, N)-module with admissible filtration. In the indecomposable case, an unconditional proof of BS(ρ) was given by Sorensen [40]. Assuming Sorensen's result for subrepresentations and quotient representations of ρ, we prove BS(ρ) is true under some additional hypotheses on ρ.

Part I

Chapter 1

Basic Notions in Number Theory

In this section we give a brief overview of certain fundamental topics in algebraic number theory. Proofs of these results can be found in [28], [22], and [46].

1.1 Number Fields

A number field k is a finite field extension of the rational number field $\mathbb{Q}$, the degree of the extension is denoted $n = [k : \mathbb{Q}]$, whereby k may be viewed as an n-dimensional $\mathbb{Q}$-algebra. The set of $\mathbb{Q}$-algebra embeddings of k into the complex numbers $\mathbb{C}$, denoted $\mathrm{Hom}_{\mathbb{Q}}(k, \mathbb{C})$, has two types of embeddings

Definition 1.1. *Let $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(k, \mathbb{C})$, then*

- *If $\sigma(k) \subset \mathbb{R} \subset \mathbb{C}$, then σ is called a real embedding of k.*

- *If $\sigma(k) \not\subset \mathbb{R}$, then σ is called a complex embedding of k.*

Post composing any complex embedding $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(k, \mathbb{C})$ with complex conjugation $\iota \in \mathrm{Aut}(\mathbb{C})$ yields another complex embedding $\iota \circ \sigma \in \mathrm{Hom}_{\mathbb{Q}}(k, \mathbb{C})$, therefore, when talking about complex embeddings of a number field one talks about pairs of complex embeddings. The standard decomposition of $n = [k : \mathbb{Q}]$ in terms of the number of real embeddings of k, denoted r_1, and the number of pairs of complex embeddings of k, denoted r_2, is $n = r_1 + 2r_2$.

The field k contains a subring called the ring of integers of k, denoted $\mathcal{O}_k$, given by

$$\mathcal{O}_k = \{a \in k : f(a) = 0 \text{ for some monic } f(x) \in \mathbb{Z}[x]\}.$$

The subring $\mathcal{O}_k$ is a Dedekind domain, hence any nonzero ideal $\mathfrak{a} \subset \mathcal{O}_k$ factors uniquely (up to units and permutation of factors) into the product of prime ideals.

The group of fractional ideals of k, denoted I_k, consists of all finitely generated $\mathcal{O}_k$-submodules of k, and is viewed as the free abelian group on the prime ideals of $\mathcal{O}_k$ so that any fractional ideal $\mathfrak{b} \in I_k$ can be written uniquely as

$$\mathfrak{b} = \prod_{i=1}^{m} \mathfrak{p}_i^{r_i},$$

where, for all $1 \le i \le m$, $\mathfrak{p_i}$ is a prime ideal of $\mathcal{O}_k$ and $0 \neq n_i \in \mathbb{Z}$. If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ then

$$\mathfrak{p}^{-1} = \{x \in k : x\mathfrak{p} \subset \mathcal{O}_k\}.$$

Considering the homomorphism

$$\phi : k^\times \to I_k$$
$$x \mapsto (x) = x\mathcal{O}_k,$$

whose image is P_k and whose kernel is $\mathcal{O}_k^\times$, we obtain the fundamental short exact sequence

$$1 \longrightarrow \mathcal{O}_k^\times \longrightarrow k^\times \overset{\phi}{\longrightarrow} I_k \longrightarrow \mathrm{Cl}_k \longrightarrow 0$$

where $\mathrm{Cl}_k := I_k/P_k$ is the ideal class group of k. One can show, wither with Minkowski's theory of numbers or an idèlic topological argument, that Cl_k is a finite abelian group. The order of Cl_k is called the class number of k and denoted h_k.

1.2 Completions

Let k be a number field

Definition 1.2. *A valuation on k is a function $v : k \to \mathbb{R} \cup \{\infty\}$ satisfying*

i.) $v(0) = \infty$

ii.) $v(xy) = v(x) + v(y)$ for all $x, y \in k$

iii.) $v(x + y) \ge \min\{v(x), v(y)\}$ for all $x, y \in k$. We call a valuation v discrete of rank one if it takes values in $\mathbb{Z} \cup \{\infty\}$.

For our purposes we only discuss valuations which are discrete of rank one, and simply call them valuations of k. If v is a valuation of k, then the valuation ring associated to v is

$$\mathcal{O}_v = \{x \in k : v(x) \geq 0\},$$

which has a unique maximal ideal

$$\mathfrak{m}_v = \{x \in: v(x) > 0\}.$$

The residue field associated to v is

$$\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v,$$

and is a finite field of cardinality Nv.

A nonarchimedean absolute value $|\cdot|$ of k is an absolute value which satisfies the strong triangle inequality

$$|x + y+ \leq \max\{|x|, |y|\}$$

for all $x, y \in k$. Given a nonarchimedean absolute value $|\cdot|$ of k, we obtain a discrete rank one valuation of k by defining

$$v(x) := -\log|x|,$$

for $x \in k$, where $\log(0) := -\infty$. Conversely, if v is a discrete rank one valuation of k, then defining

$$|x|_v := q^{-v(x)},$$

for all $x \in k$, where $0 < q < 1$, yields a nonarchimedean absolute value of k. In this way, one obtains a correspondence between nonarchimedean absolute values and discrete rank one valuations of k.

If $(0) \neq \mathfrak{p} \subset \mathcal{O}_k$ is a prime ideal we obtain a discrete rank one valuation on k by defining

$$v_{\mathfrak{p}}(x) = \begin{cases} m_{\mathfrak{p}}, & \text{if } x\mathcal{O}_k = \mathfrak{p}^{m_{\mathfrak{p}}}I \text{ where } (I, \mathfrak{p}) = 1 \\ \infty, & \text{if } x = 0. \end{cases}$$

Considering equivalence classes of absolute values and valuations on k, we obtain the following one-to-one correspondences

$$\{\text{Nonarchimedean absolute values on } k\}/\sim$$

$$\updownarrow$$

$$\{\text{Discrete rank one valuations on } k\}/\sim$$

$$\updownarrow$$

$$\{\text{Nonzero prime ideals } \mathfrak{p} \subset \mathcal{O}_k\}$$

We will call an equivalence class of nontrivial absolute values of k a prime of k, and denote primes of a field by v or w. The nonarchimedean primes are called finite primes and the archimedean primes are called infinite, and correspond to the real and complex embeddings of k. Each prime of k contains an absolute value with a standard normalization. Let $x \in k^{\times}$, then the standard normalizations are

   i.) If v corresponds to a real embedding σ of k, then $|x|_v = |\sigma(x)|_{\infty}$.

   ii.) If v corresponds to a complex embedding σ of k, then $|x|_v = |\sigma(x)|^2$.

   iii.) If v is a finite prime, then $|x|_v = Nv^{-v(x)}$.

These normalizations give the product rule

Lemma 1.1. *For all $x \in k^{\times}$ we have*

$$\prod_v |x|_v = 1,$$

where the product is taken over all normalized absolute values of k.

If v is a prime of k, then the completion of k with respect to the absolute value corresponding to v, is denoted k_v. The following is a classification of completions of number fields

Proposition 1.2. *Let k be a number field,*

  - *If v is a finite prime of k, then k_v is a locally compact field and is a finite extension of $\mathbb{Q}_p$ where $(p) = \mathfrak{m}_v \cap \mathbb{Z}$ is the prime lying below $\mathfrak{m}_v$.*

  - *If v is an infinite prime of k, then k_v is isomorphic to either $\mathbb{R}$ (if v is a real prime) or $\mathbb{C}$ (if v is a complex prime).*

1.3 Galois Extensions of Number Fields

Let F/k be a Galois extension of number fields with $G = \mathrm{Gal}(F/k)$. If v is a finite prime of k with corresponding prime ideal $\mathfrak{p}_v \subset \mathcal{O}_k$, then

$$\mathfrak{p}_\mathfrak{v} \mathcal{O}_F = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i},$$

where, for all $1 \leq i \leq g$, $\mathfrak{P}_i$ is a prime ideal of $\mathcal{O}_F$ lying above $\mathfrak{p}_v$. Let w_i be the prime of F corresponding to the prime ideal $\mathfrak{P}_i \subset \mathcal{O}_F$ and denote the residue field associated to w_i by $\kappa(w_i) = \mathcal{O}_F/\mathfrak{P}_i$, which is a finite field extension of $\kappa(v)$. We choose to denote primes by v, w in order to limit notation, although one should keep in mind the underlying prime ideals. The exponents e_i are called the ramification indices, and the integers

$$f_i = [\kappa(w_i) : \kappa(v)]$$

are called the inertial degrees. Since F/k is Galois $f := f_1 = f_2 = \ldots = f_g$ and $e := e_1 = e_2 = \ldots = e_g$. We call the prime v unramified if $e = 1$, totally ramified if $e = g$, and completely split if $e = f = 1$.

Since the Galois group G acts transitively on the primes of $\mathcal{O}_F$ lying above v, if w is a prime of F lying above v, the deomposition and inertia groups of w are $G_w := \{\sigma \in G : \sigma(w) = w\}$, and $I_w := \{\sigma \in G : \sigma(x) \equiv x \pmod{w} \text{ for all } x \in \mathcal{O}_F\}$, respectively. These subgroups give an important short exact sequence

$$1 \longrightarrow I_w \longrightarrow G_w \longrightarrow \mathrm{Gal}(\kappa(w)/\kappa(v)) \longrightarrow 1. \tag{1.1}$$

The extension $\kappa(w)/\kappa(v)$ is a finite cyclic extension of degree f whose Galois group is generated by the arithmetic Frobenius φ_v, namely

$$\mathrm{Gal}(\kappa(w)/\kappa(v)) = \langle \varphi_v : x \mapsto x^{Nv} \rangle.$$

By exactness of (1.1)

$$G_w/I_w \simeq \mathrm{Gal}(\kappa(w)/\kappa(v)).$$

Since $\mathrm{Gal}(\kappa(w)/\kappa(v))$ is cyclic of order f, there is a unique coset σI_w of G_w which lifts φ_v. We call any representative of this unique coset a Frobenius associated to w and denote it by σ_w in general, or Frob_w if we view w as a prime ideal. In general σ_w depends on the

inertia subgroup I_w, however, if the prime v is unramified i.e. $I_w = \{1\}$, then $\sigma_w \in G_w$ is the unique element whose restriction to $\kappa(w)$ is φ_v.

The properties of the Frobenius can be found in [22], but for the sake of completeness we record them here.

Lemma 1.3. *Let F/k be a Galois extension of number fields, v a prime of k, and w, w' primes of F lying over v. Then, there exists $\tau \in G$ such that*

$$G_{w'} = \tau G_w \tau^{-1} \qquad and \qquad \sigma_{w'} = \tau \sigma_w \tau^{-1}.$$

Lemma 1.4. *Let $F/E/k$ be extensions of k such that F/k and E/k are Galois. Let v be a prime of k with u a prime of E lying over v and w a prime of F lying over u. Then, via the Galois restriction map $\mathrm{res}_E : \mathrm{Gal}(F/k) \to \mathrm{Gal}(E/k)$*

$$\sigma_u = \mathrm{res}_E(\sigma_w).$$

Lemma 1.5. *Let E/k and F/k be Galois extensions of k. Let v be a prime of k with w a prime of EF lying above v. Then, via the injective map $\mathrm{Gal}(EF/k) \to \mathrm{Gal}(E/k) \times \mathrm{Gal}(F/k)$*

$$\sigma_w \mapsto (\mathrm{res}_E(\sigma_w), \mathrm{res}_F(\sigma_w)).$$

In particular, lemma 1.5 shows that a prime v splits completely in a compositum extension if and only if v splits completely in each constituent of the compositum. Furthermore, since $\#G_w = ef$, a prime v of k splits completely in an extension F, if and only if the Frobenius element σ_w (for any $w|v$) is trivial.

1.4   Group Rings and Group Characters

Let G be a finite abelian group and R a commutative ring with unit 1.

Definition 1.3. *The group ring associated to R and G is*

$$R[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma : a_\sigma \in R \right\},$$

with ring operations addition and multiplication.

Example 1. *Let $G = \langle \sigma \rangle$ be cyclic of order n and $R = \mathbb{Z}$. Consider the elements*

$$x = 1 + \sigma + \sigma^2 + \ldots + \sigma^{n-1} \qquad and \qquad y = 1 - \sigma.$$

Then,

$$x + y = (1 + \sigma + \sigma^2 + \ldots + \sigma^{n-1}) + (1 - \sigma) = 1 + \sigma^2 + \ldots + \sigma^{n-1}$$

and

$$xy = (1 + \sigma + \sigma^2 + \ldots + \sigma^{n-1})(1 - \sigma) = 1 - \sigma^n = 0.$$

In particular, the group ring $R[G]$ is not an integral domain in general.

Definition 1.4. *Let G be a finite abelian group. The character group of G is*

$$\widehat{G} = \{\chi : G \to \mathbb{C}^\times : \chi \text{ is a homomorphism}\},$$

with group operation $\chi\psi(\sigma) = \chi(\sigma)\psi(\sigma)$, for all $\sigma, \tau \in G$.

We will view characters $\chi \in \widehat{G}$ as being extend to a homomorphism of the group ring $R[G]$ via

$$\chi\left(\sum_{\sigma \in G} a_\sigma \sigma\right) = \sum_{\sigma \in G} a_\sigma \chi(\sigma).$$

The following orthogonality relations play an important role in character decompositions of modules over group rings

Lemma 1.6 (Orthogonality Relations). *Let G be a finite abelian group, then*

i.) For $\chi, \psi \in \widehat{G}$,

$$\frac{1}{|G|}\sum_{\sigma \in G} \chi(\sigma^{-1})\psi(\sigma) = \delta(\chi, \psi) = \begin{cases} 1, & if \ \chi = \psi \\ 0, & if \ \chi \neq \psi. \end{cases}$$

ii.) For $\sigma, \tau \in G$,

$$\frac{1}{|G|}\sum_{\chi \in \widehat{G}} \chi(\sigma^{-1})\chi(\tau) = \delta(\sigma, \tau) = \begin{cases} 1, & if \ \sigma = \tau \\ 0, & if \ \sigma \neq \tau. \end{cases}$$

Associated to each $\chi \in \widehat{G}$ is an element e_χ which plays the role of a projection operator.

Definition 1.5. *Attached to each $\chi \in \widehat{G}$ is the idempotent element*

$$e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma \in \frac{1}{|G|} R[G].$$

Lemma 1.7. *For R and G as above*

i.) $\sigma e_\chi = \chi(\sigma)e_\chi$ *for all $\sigma \in G$.*

ii.) $e_\chi e_\psi = \delta(\chi, \psi)e_\chi$ *for all $\chi, \psi \in \widehat{G}$.*

iii.) $\psi(e_\chi) = \delta(\chi, \psi)$ *for any $e_\chi \in R[G]$ and $\psi \in \widehat{G}$.*

iv.) $\displaystyle\sum_{\chi \in \widehat{G}} e_\chi = 1.$

1.5 Character Decomposition of Modules

Let R and G be defined as in section 1.4. We now outline the relationship between the elements e_χ, for $\chi \in \widehat{G}$, and decompositions of $R[G]$-modules M. Proofs of the below statements can be found in [32].

Definition 1.6. *Let M be an $R[G]$-module and $\chi \in \widehat{G}$. The χ-isotypic component of M, or χ-component of M is*

$$M^\chi := \{m \in M : \sigma m = \chi(\sigma)m \text{ for all } \sigma \in G\}.$$

If $x = e_\chi m$ for some $m \in M$, then

$$\sigma x = \sigma e_\chi m = \chi(\sigma)e_\chi m = \chi(\sigma)x.$$

Conversely, if $x \in M^\chi$ then

$$e_\chi x = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma x = \frac{1}{|G|} \sum_{\sigma \in G} x = x.$$

Therefore,

$$M^\chi = e_\chi M,$$

and we have the following character decomposition of M

Lemma 1.8. *If M is an $R[G]$-module, then the map $m \mapsto (e_\chi m)_{\chi \in \widehat{G}}$ gives the deocmposition*

$$M \simeq \bigoplus_{\chi \in \widehat{G}} M^\chi.$$

1.6   Zeta Functions and L-Functions

Let F/k be a finite abelian extension of number fields of Galois group G. Let S be a finite set of primes of k satisfying $S \supset S_{ram}(F/k) \cup S_\infty$, where S_∞ denotes the infinite primes of k. Since S contains the ramified primes of k, any $v \notin S$ is necessarily unramified, and therefore has a unique Frobenius automorphism $\sigma_v \in G_v$. To simplify notation we abbreviate the above information by $(F/k, S)$.

Definition 1.7. *Let $(F/k, S)$ be as above, and let $\chi \in \widehat{G}$. The S-imprimitive (or S-incomplete) L-function associated to χ is given as the Euler product*

$$L_S(\chi, s) := \prod_{v \notin S} (1 - \chi(\sigma_v) Nv^{-s})^{-1}.$$

It is well known that $L_S(\chi, s)$ converges uniformly and absolutely on compact subsets of the half plane $\mathrm{Re}(s) > 1$, and therefore defines a holomorphic function on that half-plane. Moreover, $L_S(\chi, s)$ admits meromorphic continuation to the entire complex plane with a simple pole at $s = 1$ if $\chi = \mathbf{1}_G$, where $\mathbf{1}_G$ denotes the trivial character of G.

Alternatively, one can write the S-imprimitive L-function as the following Dirichlet series

$$L_S(\chi, s) = \sum_{v \notin S} \frac{\chi(v)}{Nv^s},$$

where $\chi(v) = \chi((\mathfrak{p}_v, F/k))$ for $\mathfrak{p}_v$ the prime ideal of $\mathcal{O}_k$ corresponding to the prime v of k, and $(\cdot, F/k)$ is the Artin homomorphism associated to the extension F/k. The importance of these L-functions is illustrated in the following examples

Example 2.   1. *If $\chi = \mathbf{1}_G$ and $S = S_\infty$ then we obtain the Dedekind zeta function of k,*

$$\zeta_k(s) = \prod_v (1 - Nv^{-s})^{-1}.$$

This function admits meromorphic continuation to the whole complex plane with a

simple pole at $s = 1$, whose residue is given by the analytic class number formula

$$\lim_{s \to 1}(s-1)\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2}R_k h_k}{w_k\sqrt{|d_k|}},$$

where r_1, r_2, R_k, h_k, d_k, and w_k denote the number of real embeddings, pairs of complex embeddings, regulator, class number, discriminant, and number of roots of unity in the field k, respectively.

2. *If $K = k = \mathbb{Q}$ and $S = S_\infty = \{\sigma_\infty\}$, then we obtain the familiar Riemann zeta function*

$$\zeta_\mathbb{Q}(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p(1-p^{-s})^{-1}.$$

The G-equivariant, S-imprimitive L-function combines all of the above L-functions into one all encompassing function allowing for the consideration of all characters at once, rather than one at a time.

Definition 1.8. *Let $(F/k, S)$ be as above, then the G-equivariant, S-imprimitive L-function associated to the data $(F/k, S)$ is $\Theta_{F/k,S} : \mathbb{C} \to \mathbb{C}[G]$ where*

$$\Theta_{F/k,S}(s) := \sum_{\chi \in \widehat{G}} L_S(\chi^{-1}, s)e_\chi.$$

The function $\Theta_{F/k,S}$ is a meromorphic function on $\mathbb{C}$ with the property that for any $\chi \in \widehat{G}$

$$\chi(\Theta_{F/k,S}(s)) = \chi\left(\sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s)e_\psi\right)$$

$$= \sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s)\chi(e_\psi)$$

$$= \sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s)\delta(\chi, \psi) \qquad \text{by property } iii.)\text{ of lemma 1.7}$$

$$= L_S(\chi^{-1}, s).$$

It is often useful to express $\Theta_{F/k,S}$ in terms of partial zeta functions. In fact, a deep result of Deligne-Ribet, which was proved independently by Cassou-Noguès, gives a "rationality" statement about certain values of $\Theta_{F/k,S}$ at non-positive integers.

Definition 1.9. *Let $(F/k, S)$ be as above, then the S-incomplete partial zeta function associated to the data $(F/k, S)$ is $\zeta_S : G \times \mathbb{C} \to \mathbb{C}$ where*

$$\zeta_S(s, \sigma) := \sum_{\substack{v \notin S \\ (v, F/k) = \sigma}} \frac{1}{Nv^{-s}}.$$

One quickly verifies the following relationship between ζ_S and L_S

Lemma 1.9. *For $(F/k, S)$ as above, we have the following*

1. $L_S(\chi, s) = \sum_{\sigma \in G} \chi(\sigma) \zeta_S(s, \sigma)$

2. $\zeta_S(s, \sigma) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \chi(\sigma^{-1}) L_S(\chi, s)$.

Proof. See [32, p.265]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With the partial zeta function defined, we have an alternative definition of $\Theta_{F/k,S}$, which is equivalent to the original definition in definition 1.8

Lemma 1.10. *With $(F/k, S)$ as above, we have*

$$\Theta_{F/k,S}(s) = \sum_{\sigma \in G} \zeta_S(s, \sigma) \sigma^{-1}.$$

Proof. The proof makes use of the fact that two elements $f, g \in \mathbb{C}[G]$ are equal if and only if $\chi(f) = \chi(g)$ for all $\chi \in \widehat{G}$, which follows from the character decomposition of $\mathbb{C}[G]$. Therefore, let $\chi \in \widehat{G}$, then

$$
\begin{aligned}
\chi(\sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s) e_\psi &= \sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s) \chi(e_\psi) \\
&= \sum_{\psi \in \widehat{G}} L_S(\psi^{-1}, s) \delta(\chi, \psi) \qquad \text{by property 3 of lemma 1.7} \\
&= L_S(\chi^{-1}, s).
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\chi(\sum_{\sigma \in G} \zeta_S(s, \sigma) \sigma^{-1}) &= \sum_{\sigma \in G} \chi(\sigma^{-1}) \zeta_S(s, \sigma) \\
&= L_S(\chi^{-1}, s) \qquad \text{by property 1 of lemma 1.9.} \qquad \square
\end{aligned}
$$

A first step towards an integrality statement for special values of ζ_S was accomplished by Klingen and Siegel

Theorem 1.11 (Klingen-Siegel)**.** *Let $(F/k, S)$ be as above. For all $\sigma \in G$ and all $n \in \mathbb{Z}_{\geq 1}$*

$$\zeta_S(1 - n, \sigma) \in \mathbb{Q}.$$

The final step towards an integrality statement for special values of ζ_S was proved by Deligne-Ribet and independently by Cassou-Noguès

Theorem 1.12 (Deligne-Ribet, Cassou-Noguès)**.** *Let $(F/k, S)$ be as above. For all $\sigma \in G$ and all $n \in \mathbb{Z}_{\geq 1}$*

$$W_F^{(n)} \zeta_S(1 - n, \sigma) \in \mathbb{Z}.$$

Therefore, using the alternate definition of $\Theta_{F/k, S}$ we obtain that $W_K^{(n)} \Theta_{F/k, S}(1 - n) \in \mathbb{Z}[G]$. This integral element is called the Stickelberger element associated to $(F/k, S)$.

We will be interested in defining a T-modified version of the above equivariant L-function $\Theta_{F/k, S}$. This T-modified L-function appears in the equivariant main conjecture of Popescu and Greither and establishes a link between special values of L-functions and Fitting ideals of p-adic 1-motives. We therefore postpone the definition of the T-modified L-function until the section on p-adic 1-motives.

1.7 Main Theorems of Class Field Theory

One of the most important achievements in 20th century mathematics was the rigorous development of (abelian) class field theory for abelian extensions K/k of local and global fields. There are many approaches to the study of class field theory, namely, ideals, idèles, central simple algebras, and group cohomology (specifically Tate cohomology) to name a few, and each approach has its own advantage depending on the context. Our brief treatment will focus on the ideal and idèlic formulations, ultimately leading to the main theorems of global class field theory and the introduction of the local Artin map for local fields. We will ultimately use the appendix of [45] (written by Karl Rubin) to formulate the main theorems of class field theory, however, other excellent resources on this material are [27], [2], [22], [39], [7], [28], and [13].

1.7.1 Main theorems in terms of ideals

Let S be a finite set of finite primes of a number field k. Denote $I_{k,S}$ the free abelian group generated by prime ideals $\mathfrak{p} \notin S$, therefore, any element $\mathfrak{a} \in I_{k,S}$ can be written uniquely as

$$\mathfrak{a} = \prod_{i=1}^{t} \mathfrak{p}_i^{n_i},$$

where, for all $i = 1, 2, \ldots, t$, $\mathfrak{p}_i \notin S$ and $n_i \in \mathbb{Z}$. The S-units of k are given by

$$k_S := \{x \in k^\times : (x) \subset I_{k,S}\} = \{x \in k^\times : \operatorname{ord}_{\mathfrak{p}}(x) = 0 \text{ for all } \mathfrak{p} \in S\},$$

and we have a natural map

$$k_S \longrightarrow I_{k,S}$$
$$x \longmapsto (x)$$

which gives rise to the important exact sequence

$$1 \longrightarrow U_k \longrightarrow k_S \longrightarrow I_{k,S} \longrightarrow \operatorname{Cl}_k \longrightarrow 0$$

where Cl_k denotes the ideal class group of k.

A modulus $\mathfrak{m}$ of k (or replete divisor of k following [28]) is a formal product

$$\mathfrak{m} = \prod_v \mathfrak{p}_v^{n_v},$$

where

$$n_v = \begin{cases} 0, & \text{if } v \text{ complex} \\ 0 \text{ or } 1, & \text{if } v \text{ real} \\ > 0, & \text{if } v \text{ finite} \end{cases} .$$

Therefore, any modulus $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_f$ where $\mathfrak{m}_\infty$ is the product of corresponding infinite primes of $\mathfrak{m}$ and $\mathfrak{m}_f$ is an integral ideal of $\mathcal{O}_k$. Given a modulus $\mathfrak{m}$ of k with $\mathfrak{m}$ given as a the product above, define

$$k_{\mathfrak{m},1} := \{x \in k^\times : x_v > 0 \text{ for all } v \text{ real, and } x_v \in \mathcal{U}_v^{(n_v)} \text{ for all } v \text{ finite }\}$$

where $\mathcal{U}_v^{(n_v)} = 1 + \mathfrak{p}_v^{n_v}$. Denoting $I_{k,\mathfrak{m}} = I_{k,S}$ where $S := \{\mathfrak{p} \text{ prime } : \mathfrak{p}|\mathfrak{m}_f\}$, we have a natural map

$$k_{\mathfrak{m},1} \xrightarrow{\iota} I_{k,\mathfrak{m}}$$
$$x \longmapsto (x)$$

which gives rise to

$$C_{\mathfrak{m}} := I_{k,\mathfrak{m}}/\iota(k_{\mathfrak{m},1}),$$

the **ray class group of** k **modulo** $\mathfrak{m}$.

If K/k is a finite abelian extension of number fields, we let $N_{K/k}$ denote the norm associated to this extension, and recall that if $\mathfrak{P} \subset \mathcal{O}_K$ is a prime ideal with $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_k$, then $N_{K/k}(\mathfrak{P}) = \mathfrak{p}^{f_{K/k}}$ where $f_{K/k} := [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_k/\mathfrak{p}]$ is the residue degree. If $\Updownarrow$ is a modulus of k, we let $\mathfrak{m}'$ be the modulus of K consisting of primes lying above those in $\mathfrak{m}$. We let $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K/k)$ denote the (unique) Frobenius automorphism corresponding to an unramified finite prime $\mathfrak{p} \subset \mathcal{O}_k$. If S is a finite set of primes of k such that $S \supset S_{\mathrm{ram}}(K/k)$, then the global Artin map associated to K/k is

$$\psi_{K/k} : I_{k,S} \longrightarrow \mathrm{Gal}(K/k)$$

where, if $\mathfrak{a} = \prod_{i=1}^{t} \mathfrak{p}_i^{n_i} \in I_{k,S}$, then

$$\psi_{K/k}(\mathfrak{a}) = \prod_{i=1}^{t} \mathrm{Frob}_{\mathfrak{p}_i}^{n_i}.$$

With the above notation in prime, we are now ready to state the main theorems of (global) class field theory.

Theorem 1.13. *[45, Theorem 1, p.398] Let K/k be a finite abelian extension. Then there exists a modulus $\mathfrak{m}$ of k such that*

- *A prime $\mathfrak{p}$ ramifies in K/k if and only if $\mathfrak{p}|\mathfrak{m}$.*

- *If $\mathfrak{M}$ is another modulus of k with $\mathfrak{m}|\mathfrak{M}$, then there is a subgroup H with $\iota(k_{\mathfrak{M},1}) \subset H \subset I_{k,\mathfrak{M}}$ such that the global Artin map induces an isomorphism*

$$I_{k,\mathfrak{M}}/H \simeq \mathrm{Gal}(K/k).$$

In fact, $H = \iota(k_{\mathfrak{M},1}) N_{K/k}(I_{K,\mathfrak{M}'})$

Theorem 1.14. *[45, Theorem 2, p.398] Let $\mathfrak{m}$ be a modulus of k and H a subgroup of $I_{k,\mathfrak{m}}$ with $\iota(k_{\mathfrak{m},1}) \subset H \subset I_{k,\mathfrak{m}}$. Then there exists a unique abelian extension K/k with ramification occurring at primes $\mathfrak{p}|\mathfrak{m}$ (if ramified at all), such that $H = \iota(k_{\mathfrak{m},1}) N_{K/k}(I_{K,\mathfrak{m}'})$ and*

$$I_{k,\mathfrak{m}} \simeq \mathrm{Gal}(K/k).$$

In particular, for any modulus $\mathfrak{m}$ of k, there exists a field extension $L_{\mathfrak{m}}/k$ ($L_{\mathfrak{m}}$ is called the **ray class field modulo $\mathfrak{m}$**) such that the global Artin map gives an isomorphism $C_{\mathfrak{m}} \simeq \mathrm{Gal}(L_{\mathfrak{m}}/k)$.

The classical application of these theorems is given as follows, choose the modulus $\mathfrak{m} = \mathcal{O}_k$ and the subgroup $H = \iota(k_{\mathfrak{m},1})$, then, $I_{k,\mathfrak{m}} = I_k$ so Theorem 1.14 gives an everywhere unramified, abelian extension K/k such that $\mathrm{Gal}(K/k) \simeq I_k/\iota(k^{\times}) = \mathrm{Cl}_k$. This unique field K is called the **Hilbert class field of** k and often denoted H_k. In fact, H_k is the maximal, abelian, everywhere unramified extension of k, where maximality is a consequence of uniqueness.

1.7.2 Main theorems in terms of idèles

Let k be a number field. The group of idèles of k is the restricted product

$$J_k := \prod_v{}' (k_v^{\times}, \mathcal{U}_v) := \{(x_v)_v : x_v \in \mathcal{U}_v \text{ for almost all } v\},$$

where the product is taken over all primes v of k (both finite and infinite). If S is a finite set of primes of k, then the S-idèles of k are

$$J_{k,S} := \prod_{v \in S} k_v^{\times} \times \prod_{v \notin S} \mathcal{U}_v.$$

Is $S' \subset S$ then $J_{k,S'} \subset J_{k,S}$, and therefore, we view $J_k = \bigcup_S J_{k,S}$. We topologize J_k be letting $J_{k,S}$ have the product topology, and then defining $J_{k,S}$ to be open in J_k (it is important to note that this topology is not the topology induced from the product topology). With this topology, J_k becomes a locally compact topological group.

We embed $k^{\times}$ in J_k via the diagonal embedding $x \mapsto (x, x, \dots)$ and view $k^{\times} \subset$

J_k. With this identification, the idèle class group of k is then defined to be

$$C_k := J_k/k^\times,$$

and is a generalization of the classical ideal class group Cl_k in the following sense. We have a surjection

$$\varphi : J_k \longrightarrow \mathrm{Cl}_k$$
$$(x_v) \longmapsto \prod_{\substack{v \\ \text{finite}}} \mathfrak{p}_v^{\mathrm{ord}_v(x_v)}$$

with $\ker \varphi = k^\times \prod_v \mathcal{U}_v$, therefore,

$$\mathrm{Cl}_k \simeq J_k/k^\times \prod_v \mathcal{U}_v.$$

Since $k^\times \subset k^\times \prod_v \mathcal{U}_v$, we have a surjection

$$C_k \longrightarrow \mathrm{Cl}_k.$$

If K/k is a finite extension and w is a prime of K lying over a prime v of k, then the local norm maps $N_{K_w/k_v} : K_w \longrightarrow k_v$ allow us to define a norm map at the level of idèles $N_{K/k} : J_K \longrightarrow J_k$, namely

$$N((x_w)_w) := (\prod_{w|v} N_{K_w/k_v}(x_w))_v.$$

With the above norm map, we can now formulate the idèlic version of the main theorems of global class field theory.

Theorem 1.15. *[45, Theorem 11, p.405] Let K/k be a finite abelian extension, then there is an isomorphism*

$$J_k/N_{K/k}J_K \simeq \mathrm{Gal}(K/k).$$

A prime $\mathfrak{p}$ is unramified in K/k if and only if $\mathcal{U}_\mathfrak{p} \subset k^\times N_{K/k}J_K$.

Theorem 1.16. *[45, Theorem 12, p.405] If H is an open subgroup of finite index in J_k with $k^\times \subset H$, then there exists a unique finite abelian extension K/k such that $H = k^\times N_{K/k}J_K$.*

Chapter 2

Conjectures of Rubin, Stark, and Gross

This section outlines two conjectures, the Rubin-Stark conjecture, and a generalization of a conjecture of Gross. Gross originally formulated his conjecture in [17], however, we will follow the treatment found in [15] for the generalization of Gross's conjecture needed in this manuscript. We adopt the same notation in [15] albeit slightly modified to coincide with the notation in this thesis.

2.1 Evaluation Maps

Let R be a commutative ring with 1 and M an R-module. Denote the R-module dual of M by $M^* := \mathrm{Hom}_R(M, R)$, and let $r \in \mathbb{Z}_{\geq 0}$. For any R-algebra S denote $SM := S \otimes_R M$. For any $\phi \in M^*$ there is an R-linear homomorphism at the level of exterior powers

$$\phi^{(r)} : \bigwedge_R^r M \longrightarrow \bigwedge_R^{r-1} M,$$

defined on elementary wedges by

$$\phi^{(r)}(m_1 \wedge m_2 \wedge \ldots \wedge m_r) = \sum_{i=1}^r (-1)^{i+1} \phi(m_i) m_1 \wedge \ldots \wedge \widehat{m_i} \wedge \ldots \wedge m_r,$$

where the $\widehat{m_i}$-term is omitted. Therefore, for all $0 \le i \le r$, there is an R-linear homomorphism

$$\bigwedge_R^r M^* \otimes_R \bigwedge_R^i M \longrightarrow \bigwedge_R^{r-i} M \tag{2.1}$$

given on elementary wedges by

$$(\phi_1 \wedge \ldots \wedge \phi_r) \otimes (m_1 \wedge \ldots \wedge m_i) \mapsto \phi_r^{(r-i+1)} \circ \ldots \circ \phi_1^{(r)}(m_1 \wedge \ldots \wedge m_r).$$

In the case $i = r$, the map in (2.1) is the determinant

$$\phi_r^{(1)} \circ \ldots \circ \phi_1^{(r)}(m_1 \wedge \ldots \wedge m_r) = \det(\phi_i(m_j)_{i,j}).$$

We illustrate this in the simple case $i = r = 2$.

Example 3. *If $i = r = 2$, then the map in (2.1) is*

$$\begin{aligned}
(\phi_1 \wedge \phi_2) \otimes (m_1 \wedge m_2) &\mapsto \phi_2^{(1)}(\phi_1^{(2)}(m_1 \wedge m_2)) \\
&= \phi_2^{(1)}(\phi_1(m_1)m_2 - \phi_1(m_2)m_1) \\
&= \phi_1(m_1)\phi_2(m_2) - \phi_1(m_2)\phi_2(m_1) \\
&= \det \begin{pmatrix} \phi_1(m_1) & \phi_1(m_2) \\ \phi_2(m_1) & \phi_2(m_2) \end{pmatrix}.
\end{aligned}$$

Let K denote the total ring of fractions of R, so that K is the localization of R at the multiplicative set consisting of nonzero divisors. There is an R-module homomorphism

$$\operatorname{Hom}_R(M, R) \longrightarrow \operatorname{Hom}_K(KM, K) \tag{2.2}$$

$$\phi \mapsto \psi : \frac{m}{s} \mapsto \frac{\phi(m)}{s}.$$

Therefore, there is an R-module homomorphism

$$E_r : \bigwedge_R^r M \otimes \bigwedge_R^r KM \longrightarrow K,$$

given by

$$(\phi_1 \wedge \ldots \wedge \phi_r) \otimes (m_1 \wedge \ldots \wedge m_r) \mapsto \psi_r^{(1)} \circ \ldots \circ \psi_1^{(r)}(m_1 \wedge \ldots \wedge m_r) = \det(\psi_i(m_j)_{i,j}),$$

where ψ_i corresponds to ϕ_i under (2.2).

Definition 2.1. *For R, M, and r as above, we define*

$$\mathcal{L}_R(M, r) := \left\{ \epsilon \in \bigwedge_K^r KM : E_r((\phi_1 \wedge \ldots \wedge \phi_r) \otimes \epsilon) \in R, \text{ for all } \phi_1, \ldots, \phi_r \in M^* \right\}.$$

If $R = \mathbb{Z}[G]$, then any element $\epsilon \in \mathcal{L}_{\mathbb{Z}[G]}(M, r)$ yields a $\mathbb{Z}[G]$-linear evaluation map

$$\mathrm{ev}_{\mathbb{Z}[G],\epsilon} : \bigwedge_{\mathbb{Z}[G]}^r M^* \longrightarrow \mathbb{Z}[G],$$

given on elementary wedges by

$$\mathrm{ev}_{\mathbb{Z}[G],\epsilon}(\phi_1 \wedge \ldots \wedge \phi_r)(\epsilon) = \phi_r^{(1)} \circ \ldots \circ \phi_1^{(r)}(\epsilon).$$

Originally, Rubin [33] constructed his regulator (henceforth called the Rubin-Stark regulator) for $\mathbb{Z}[G]$ modules, and refers to $\mathrm{ev}_{\mathbb{Z}[G],\epsilon}$ as a determinant pairing associated to particular modules. In order to establish a link between the Rubin-Stark conjecture and a conjecture of Gross we will need a version of the Rubin-Stark regulator amenable to $R[G]$-modules, for arbitrary commutative $\mathbb{Z}$-algebras R.

The reader is referred to [15] for more details of the following constructions. Let R be a commutative ring with 1 and G a finite abelian group. Then there is a ring isomorphism

$$R \otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq R[G] \tag{2.3}$$

$$a \otimes x \mapsto ax.$$

For any finitely generated $\mathbb{Z}[G]$-module M there is an $R[G]$-module

$$M_R^* := \mathrm{Hom}_{\mathbb{Z}[G]}(M, R[G]),$$

which can be viewed in terms of the following $R[G]$-module isomorphism

$$M^* \otimes_{\mathbb{Z}} R \simeq M_R^* \qquad (2.4)$$

$$\phi \otimes a \mapsto \psi : m \mapsto a\phi(m).$$

Combining (2.3) and (2.4) gives an $R[G]$-evaluation map

$$\textstyle\bigwedge_{R[G]}^r M_R^* \xrightarrow{\;\sim\;} \bigwedge_{R[G]}^r (M^* \otimes_{\mathbb{Z}} R) \xrightarrow{\;\sim\;} R \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[G]}^r M^* \xrightarrow{1 \otimes \mathrm{ev}_{\mathbb{Z}[G],\epsilon}} R \otimes_{\mathbb{Z}} \mathbb{Z}[G] \xrightarrow{\;\sim\;} R[G].$$

$$\mathrm{ev}_{R[G],\epsilon}$$

The connection between the Rubin-Stark conjecture and a conjecture of Gross will involve a $\mathbb{Z}_{\geq 0}$-graded commutative ring R given by powers of augmentation ideals associated to Galois groups of prescribed field extensions. It is this connection which will allow us to prove the generalized version of Kurihara's conjecture for p-adic realizations of abstract 1-motives.

2.2 Idempotents

Let F/k be a finite abelian extension of Galois group $G = \mathrm{Gal}(F/k)$ where F is CM and k is totally real. Let S be a finite set of primes of k such that $S \supset S_{\mathrm{ram}}(F/k) \cup S_\infty$ and let S_F denote the primes of F lying above those in S. We first define the modules which play an important role in defining the Rubin-Stark regulator.

Let Y_S denote the free abelian group on the set S_F so that

$$Y_S := \bigoplus_{w \in S_F} \mathbb{Z}w.$$

Let $\mathrm{aug} : Y_S \to \mathbb{Z}$ be the augmentation map, i.e. the $\mathbb{Z}$-linear map sending any $\sigma \in G$ to 1. We have a short exact sequence of $\mathbb{Z}[G]$-modules

$$1 \longrightarrow X_S \longrightarrow Y_S \xrightarrow{\mathrm{aug}} \mathbb{Z} \longrightarrow 0$$

where

$$X_S := \left\{ \sum_{\sigma \in G} a_\sigma \sigma : \sum_{\sigma \in G} a_\sigma = 0 \right\},$$

and $\mathbb{Z}$ is endowed with the trivial G-action.

The S-units (or more appropriately the S_F-units) are denoted U_S and are defined by

$$U_S = \{x \in F^\times : \mathrm{ord}_w(x) = 0 \text{ for all } w \notin S_F\},$$

i.e. U_S consists of those nonzero elements of F whose principal ideal is only divisible by primes in S. Let T be a finite set of primes of k disjoint from S such that, if T_F denotes the set of primes of F lying above primes in T, then $F_{T_F}^\times$ is torsion-free. The set T_F gives a subgroup $U_{S,T}$ of U_S given by

$$U_{S,T} = \{x \in U_S : x \equiv 1 \pmod{w} \text{ for all } w \in S_F\}.$$

The classical S-modified Dirichlet logarithm map $\lambda_S : U_S \longrightarrow X_S$ is given by

$$\lambda_S(u) = -\sum_{w \in S_F} \log |u|_w w$$

where $u \in U_S$ is not a root of unity. Tensoring the above map with $\mathbb{Q}$ gives an isomorphism of $\mathbb{Q}[G]$-modules

$$\mathbb{Q}U_S \simeq \mathbb{Q}X_S.$$

Let $\chi \in \hat{G}$, then since the order of vanishing $r_{S,\chi}$ of $L_S(\chi, s)$ at $s = 0$ is

$$r_{S,\chi} = \begin{cases} \#\{v \in S : \chi(G_v) = \{1\}\}, & \text{if } \chi \neq \mathbf{1}_G; \\ \#S - 1, & \text{if } \chi = \mathbf{1}_G \end{cases}$$

we quickly see that for if χ and χ' are conjugate under the action of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then

$$r_{S,\chi} = r_{S,\chi'}.$$

Therefore, letting $\widehat{G_{S,r}} = \{\chi \in \hat{G} : r_{S,\chi} = r\}$ we define

$$e_{S,r} := \sum_{\chi \in \widehat{G_{S,r}}} e_\chi.$$

The equivalence of orders of vanishing between conjugate characters shows $e_{S,r} \in \mathbb{Q}[G]$

is an idempotent. We therefore have a decomposition of $1 \in \mathbb{Q}[G]$ via

$$1 = \sum_{r \in \mathbb{Z}_{\geq 0}} e_{S,r}.$$

The importance of using the idempotents $e_{S,r}$ is due to the fact that the $e_{S,r}\mathbb{Q}[G]$-modules $e_{S,r}\mathbb{Q}X_S$ and $e_{S,r}\mathbb{Q}U_S$ are free of rank r.

2.3   The Rubin-Stark Conjecture

Let $r \in \mathbb{Z}_{\geq 0}$. In addition to $S \supset S_{\mathrm{ram}}(F/k) \cup S_\infty$ we additionally assume $S \supset \{v_1, \ldots, v_r\}$ where the v_i are distinct primes of k, all of which split completely in F. Furthermore, we assume $|S| \geq r + 1$ so that there exists $v_0 \in S \smallsetminus \{v_1, \ldots, v_r\}$. Let $w_0, w_1, \ldots, w_r$ be primes of F lying above $v_0, v_1, \ldots, v_r$, respectively, and form the $(r+1)$-tuple $W = (w_0, w_1, \ldots, w_r)$.

Lemma 2.1. *For all $1 \leq i \leq r$, the $\mathbb{Z}[G]$-module generated by w_i is free of rank 1.*

Proof. Since v_i splits completely in F, its decomposition group $G_{v_i} = \{1\}$. The lemma then follows from the isomorphisms $\mathbb{Z}[G]w_i \simeq \mathbb{Z}[G/G_{v_i}] \simeq \mathbb{Z}[G]$. $\square$

The above result yields the direct sum decomposition of X_S

$$X_S \simeq X_{S \smallsetminus \{v_1, \ldots, v_r\}} \oplus \left(\bigoplus_{i=1}^r \mathbb{Z}[G](w_i - w_0) \right). \tag{2.5}$$

Tensoring (2.5) with $\mathbb{Q}[G]$ and passing to the $e_{S,R}\mathbb{Q}[G]$-component gives the $e_{S,r}\mathbb{Q}[G]$-isomorphism

$$e_{S,r}\mathbb{Q}X_S \simeq \bigoplus_{i=1}^r e_{S,r}\mathbb{Q}[G](w_i - w_0)$$

of the free $e_{S,r}\mathbb{Q}[G]$-module $e_{S,r}\mathbb{Q}X_S$. Consider now the free $e_{S,r}\mathbb{Z}[G]$-submodule of $e_{S,r}\mathbb{Q}X_S$ generated by the basis elements $x_i := e_{S,r}(w_i - w_0)$. The basis $\{x_i\}_{i=1,\ldots,r}$ gives a canonical generator $e_{S,r}x_1^* \wedge \ldots \wedge x_r^*$ of the rank 1 $e_{S,r}\mathbb{Q}[G]$-module $\bigwedge_{e_{S,r}\mathbb{Q}[G]}^r (e_{S,r}\mathbb{Q}X_S)^*$. This canonical generator is then used to define the Rubin-Stark regulator R_W.

Definition 2.2. *The Rubin-Stark regulator is*

$$R_W : e_{S,r} \bigwedge_{\mathbb{Q}[G]}^r \mathbb{Q}U_S \longrightarrow e_{S,r}\mathbb{Q}X_S,$$

given by

$$R_W(e_{S,r}u_1 \wedge \ldots \wedge u_r) = \mathrm{ev}_{e_{S,r}\mathbb{Q}X_S}(\tilde{\lambda}_S(e_{S,r}u_1 \wedge \ldots \wedge u_r) \otimes (e_{S,r}x_1^* \wedge \ldots \wedge x_r^*)).$$

More explicitly, using the relation $x_j^*(u_i) = e_{S,r}(-\sum_{\sigma \in G} \log |u_i^\sigma|_{w_j} \sigma^{-1})$, the formula for R_W is

$$R_W(e_{S,r}u_1 \wedge \ldots \wedge u_r) = e_{S,r} \det \left(-\sum_{\sigma \in G} \log |u_i^\sigma|_{w_j} \sigma^{-1} \right).$$

For our purposes, we will consider the Rubin-Stark regulator on a subgroup of the S-unit group. Let $U_{S,T} = \{x \in U_S : x \equiv 1 \pmod{w} \text{ for all } w \in T\}$, then $U_{S,T}$ is a subgroup of U_S of finite index, hence $\mathbb{Q}U_{S,T} = \mathbb{Q}U_S$, where instead of λ_S we consider the corresponding Dirichlet regulator $\lambda_{S,T}$ on $U_{S,T}$. Consequently, for any $e_{S,r}u_1 \wedge \ldots \wedge u_r \in e_{S,r} \bigwedge_{\mathbb{Q}[G]}^r \mathbb{Q}U_{S,T}$ the Rubin-Stark regulator R_W has value

$$R_W(e_{S,r}u_1 \wedge \ldots \wedge u_r) = \mathrm{ev}_{e_{S,r}\mathbb{Q}X_S}(\tilde{\lambda}_{S,T}(e_{S,r}u_1 \wedge \ldots \wedge u_r) \otimes (e_{S,r}x_1^* \wedge \ldots \wedge x_r^*)).$$

Definition 2.3. *For $(F/k, S, T, r)$ as above, Rubin's lattice is*

$$\Lambda_{S,T} := \left\{ \epsilon \in e_{S,r} \bigwedge_{\mathbb{Q}[G]}^r \mathbb{Q}U_{S,T} : (\phi_r^{(1)} \circ \ldots \circ \phi_1^{(r)})(\epsilon) \in \mathbb{Z}[G] \text{ for all } \phi_1, \ldots, \phi_r \in U_{S,T}^* \right\}.$$

Recall that $\Theta_{F/k,S,T}$ is a holomorphic function and therefore has a Taylor series expansion about $s = 0$,

$$\Theta_{F/k,S,T}(s) = \sum_{n=0}^{\infty} \frac{\Theta_{F/k,S,T}^{(n)}(0)}{n!} s^n.$$

The Rubin-Stark conjecture gives a conjectural link between values of R_W at certain elements of $\Lambda_{S,T}$, and $\Theta_{F/k,S,T}^{(r)}(0)$.

Conjecture 2.2 (Rubin-Stark)**.** *For $(F/k, S, T, r)$ as above, there exists a unique element $\epsilon_{S,T} \in \Lambda_{S,T}$ such that*

$$R_W(\epsilon_{S,T}) = \Theta_{F/k,S,T}^{(r)}(0).$$

The unique element $\epsilon_{S,T}$ is called the Rubin-Stark element associated to $(F/k, S, T, r)$.

2.4   Augmentation Ideals

Gross's conjecture applies to a triple of field extensions $L/F/k$ where F/k is a finite abelian extension satisfying the Rubin-Stark conjecture, and L/k is an abelian extension. Let $\mathcal{G} = \mathrm{Gal}(L/k)$, $\Gamma = \mathrm{Gal}(L/F)$, and $G = \mathrm{Gal}(F/k)$. Associate to Γ are two tpes of augmentation ideals.

Definition 2.4. *The augmentation ideal $I(\Gamma)$ associated to Γ is the kernel of the surjective $\mathbb{Z}$-linear augmentation map*

$$\mathbb{Z}[\Gamma] \xrightarrow{\mathrm{aug}} \mathbb{Z}$$

given by sending $\gamma \in \Gamma$ to 1 and extending $\mathbb{Z}$-linearly. Therefore, sits in the short exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow I(\Gamma) \longrightarrow \mathbb{Z}[\Gamma] \xrightarrow{\mathrm{aug}} \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}$ is endowed with the trivial Γ action.

Remark. *For our purposes the extension L/F will always be finite, however, it is important to mention that one can define augmentation ideals for infinite abelian extensions L/F simply by taking projective limits of augmentation ideals associated to the finite subextensions $F \subset F' \subset L$, where the limit is taken with respect to Galois restriction maps associated to the extensions.*

Definition 2.5. *Given $L/F/k, \mathcal{G}, \Gamma$, and G as above, the Γ-relative augmentation ideal I_Γ of $\mathbb{Z}[\mathcal{G}]$ is the kernel of the projection $\mathbb{Z}[\mathcal{G}] \twoheadrightarrow \mathbb{Z}[G]$ induced by the Galois restriction map $\mathcal{G} \longrightarrow G \simeq \mathcal{G}/\Gamma$.*

In [15] the $\mathbb{Z}$-graded commutative unital rings

$$R(\Gamma) := \bigoplus_{n \geq 0} I(\Gamma)^n / I(\Gamma)^{n+1} \qquad R_\Gamma := \bigoplus_{n \geq 0} I_\Gamma^n / I_\Gamma^{n+1},$$

are considered along with important relationships between $R(\Gamma)$ and R_Γ.

Lemma 2.3. [31, p. 92] *With notation as above the following hold*

i.) $I_\Gamma^n = \bigoplus_{\sigma \in G} \tilde{\sigma} I(\Gamma)^n$ where $\tilde{\sigma} \in \mathcal{G}$ denotes a lift of σ via Galois restriction.

ii.) For any $n \in \mathbb{Z}_{\geq 0}$, there is a $\mathbb{Z}[G]$-module isomorphism

$$I(\Gamma)^n/I(\Gamma)^{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq I_{\Gamma}^n/I_{\Gamma}^{n+1}$$

$$\hat{\iota} \otimes \sigma \mapsto \widehat{\tilde{\sigma}\iota}$$

where $\hat{\iota}$ denotes the class of $\iota \in I(\Gamma)^n$ in $I(\Gamma)^n/I(\Gamma)^{n+1}$, and $\tilde{\sigma} \in \mathcal{G}$ denotes a lift of σ via Galois restriction.

iii.) There are $\mathbb{Z}[G]$-graded isomorphisms

$$R(\Gamma)[G] \simeq R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[G] \simeq R_{\Gamma}.$$

2.5 A Generalization of Gross's Conjecture

The notation $(L/F/k, S, T, r)$ will denote the following data

i.) $r \in \mathbb{Z}_{\geq 0}$

ii.) L/k is abelian with Galois group $\mathcal{G} = \mathrm{Gal}(L/k)$

iii.) $G = \mathrm{Gal}(F/k)$

iv.) $\Gamma = \mathrm{Gal}(L/F)$

v.) S is a finite set of primes of k satisfying $S \supset S_{\mathrm{ram}(L/k)} \cup S_\infty \cup \{v_1, v_2, \ldots, v_r\}$, where, for each $1 \leq i \leq r$, v_i is a finite prime of k which splits completely in F.

vi.) For each $i \in \{1, \ldots, r\}$ we fix a prime w_i of F lying over v_i.

vii.) T is a finite set of primes of k satisfying $T \cap S = \varnothing$ and if T_L denotes the primes of L lying above those in T, then $L_{T_L}^\times$ is torsion-free.

viii.) The data $(F/k, S, T, r)$ satisfies the Rubin-Stark conjecture, in particular there is a unique element $\epsilon_{S,T} \in \Lambda_{S,T}$ satisfying

$$R_W(\epsilon_{S,T}) = \Theta_{F/k,S,T}^{(r)}(0).$$

For each $i \in \{1, \ldots, r\}$ we let k_{v_i} and F_{w_i} denote the completions of k and F with respect to the normalized absolute values associated to the primes v_i and w_i, respectively, and

consider the following extensions

Since v_i splits completely in F, the decomposition group $G_{v_i} = \{1\}$, therefore $G_{w_i}(L/F) = G_{v_i}(L/k)$. We identify $\mathrm{Gal}(Lf_{w_i}/F_{w_i}) = G_{w_i}(L/F) = G_{v_i}(L/k)$. Composing the Artin reciprocity map associated to the extension LF_{w_i}/F_{w_i} with the inclusion $G_{w_i}(L/F) \hookrightarrow \Gamma$, we get a morphism

$$\rho_{w_i} : F_{w_i}^{\times} \longrightarrow \Gamma.$$

Corresponding to ρ_{w_i}, Gross makes the following definitions

Definition 2.6. *For all $i \in \{1, \ldots, r\}$ and $w_i | v_i$ as above, define the following $\mathbb{Z}[G]$-linear homomorphisms*

$$\psi_{w_i} : U_{S,T} \xrightarrow{\phi_{w_i}} R(\Gamma)[G] \xrightarrow[\sim]{\iota} R_{\Gamma}$$

where

$$\phi_{w_i}(u) = \sum_{\sigma \in G} (\widehat{\rho_{w_i}(u^{\sigma})} - 1)\sigma^{-1}.$$

Notice that $\rho_{w_i}(u^{\sigma}) \in \Gamma$, so $(\rho_{w_i}(u^{\sigma}) - 1) \in I(\Gamma)$, and $(\widehat{\rho_{w_i}(u^{\sigma})} - 1) \in I(\Gamma)/I(\Gamma)^2 \subset R(\Gamma)$. Letting $\epsilon_{S,T} \in \Lambda_{S,T}$ be the unique Rubin-Stark element for $(F/k, S, T, r)$, we obtain an evaluation map $\mathrm{ev}_{\epsilon_{S,T}, R(\Gamma)}$ such that

$$\mathrm{ev}_{\epsilon_{S,T}, R(\Gamma)}(\phi_{w_1} \wedge \ldots \wedge \phi_{w_r}) \in \left(I(\Gamma)^r/I(\Gamma)^{r+1}\right)[G].$$

Therefore, composing $\mathrm{ev}_{\epsilon_{S,T}, R(\Gamma)}$ with the isomorphism $\iota : R(\Gamma)[G] \xrightarrow{\sim} R_{\Gamma}$ gives an evaluation map $\mathrm{ev}_{\epsilon_{S,T}, R_{\Gamma}}$ such that

$$\mathrm{ev}_{\epsilon_{S,T}, R_{\Gamma}}(\psi_{w_1} \wedge \ldots \wedge \psi_{w_r}) \in I_{\Gamma}^r/I_{\Gamma}^{r+1}.$$

Definition 2.7. *For $(L/F, k, S, T, r)$ as above and $W = (w_0, w_1, \ldots, w_r)$, the Gross*

regulator $R_{W,\mathrm{Gross}}$ *is the* $\mathbb{Z}[G]$*-regulator*

$$R_{W,\mathrm{Gross}} : \Lambda_{S,T} \longrightarrow I_\Gamma^r / I_\Gamma^{r+1},$$

given by

$$R_{W,\mathrm{Gross}}(\epsilon_{S,T}) = \mathrm{ev}_{\epsilon_{S,T}, R_\Gamma}(\psi_{w_1} \wedge \ldots \wedge \psi_{w_r}).$$

The generalization of Gross's conjecture [17] is then formulated as follows

Conjecture 2.4. *Let* $(L/F/k, S, T, r)$ *be as above, then the following hold*

i.) $\Theta_{L/k,S,T}(0) \in I_\Gamma^r$

ii.) $R_{W,\mathrm{Gross}}(\epsilon_{S,T}) = \Theta_{L/k,S,T}(0) \pmod{I_\Gamma^{r+1}}.$

Chapter 3

p-adic Realizations of Abstract 1-Motives

We introduce the notion of abstract 1-motives and their p-adic realizations as defined in [15]. These objects will play a similar role to the classical Iwasawa module considered by Kurihara in his original conjecture.

Definition 3.1. *An abstract 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ consists of the data*

  i.) A free abelian group L of finite rank

  ii.) A divisible abelian group J of finite local corank

  iii.) A group homomorphism $\delta : L \to J$.

In order to mimic the construction of p-adic Tate modules, which will turn out to be our p-adic realizations of abstract 1-motives, we need to make sense of torsion points of abstract 1-motives. Let $n \in \mathbb{Z}_{\geq 1}$ and consider the following commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J[n] & \longrightarrow & J \times_L^n L & \longrightarrow & L & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & J[n] & \longrightarrow & J & \xrightarrow{\ n\ } & J & \longrightarrow & 0
\end{array}
$$

where $J \times_L^n L = \{(j,l) \in J \times L : \delta(l) = nj\}$ is the fibre product of L and J, and the maps from $J \times_L^n L$ to J or L are the projection maps given by the definition of the fibre product of abelian groups.

Definition 3.2. *For an abstract 1-motive* $\mathcal{M} = [L \xrightarrow{\delta} J]$*, and* $n \in \mathbb{Z}_{\geq 1}$*, the* n*-torsion points of* $\mathcal{M}$ *are defined by*

$$\mathcal{M}[n] := (J \times_L^n L) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}.$$

Applying the functor $* \mapsto * \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ to the exact sequence

$$0 \longrightarrow J[n] \longrightarrow J \times_L^n L \longrightarrow L \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow J[n] \longrightarrow \mathcal{M}[n] \longrightarrow L \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

For integers $n|m$, we have the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J[m] & \longrightarrow & \mathcal{M}[m] & \longrightarrow & L \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \frac{m}{n}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi} & & \\
0 & \longrightarrow & J[n] & \longrightarrow & \mathcal{M}[n] & \longrightarrow & L \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}
\tag{3.1}
$$

where $\pi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the natural projection, $\frac{m}{n} : J[m] \to J[n]$ is the multiplication by $\frac{m}{n}$ map, and $\mathcal{M}[m] \to \mathcal{M}[n]$ is $(j,l) \otimes \hat{1} \mapsto (\frac{m}{n}j, l) \otimes \hat{1}$. Restricting to the case where m and n are powers of a prime $p \in \mathbb{Z}$ inspires the following

Definition 3.3. *For* $p \in \mathbb{Z}$ *a prime, the* p*-adic realization of the abstract 1-motive* $\mathcal{M} = [L \xrightarrow{\delta} J]$ *is*

$$T_p(\mathcal{M}) := \varprojlim_n \mathcal{M}[p^n].$$

For m, n powers of a prime $p \in \mathbb{Z}$, taking projective limits in (3.1) gives an exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow T_p(J) \longrightarrow T_p(\mathcal{M}) \longrightarrow L \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0. \tag{3.2}$$

The p-adic realizations of abstract 1-motives of interest to us are those arising from classical arithmetic data. We briefly introduce these arithmetic objects, and explain how they lead to the p-adic realizations of interest.

3.1 Generalized Class Groups

Iwasawa used the term "$\mathbb{Z}_p$-field" in reference to infinite Galois extensions K of a number field, whose Galois group is isomorphic to $(\mathbb{Z}_p, +)$. We similarly adopt this terminology in what follows. Let K be a $\mathbb{Z}_p$-field and let v be a finite prime of K. Inside the prime v is the canonically normalized valuation $\mathrm{ord}_v : K \to \Gamma_v$, where Γ_v is the value group of the valuation. If the prime v lies above the p-adic valuation of $\mathbb{Q}$, then $\Gamma_v = \mathbb{Z}[\frac{1}{p}]$, however, if $\ell \neq p$ is prime and v lies over an ℓ-adic prime of $\mathbb{Q}$, then $\Gamma_v = \mathbb{Z}$.

Our exposition will rely heavily on finite sets of primes of certain fields, therefore, to set notation we make the convention that if k is a finite extension of a number field, a finite set of primes of k will be denoted with roman S_k or T_k, where the subscript carries the obvious meaning. However, if k is an infinite extension of some number field, then we will use script notation $\mathcal{S}_k$ or $\mathcal{T}_k$ for our sets of primes. When treating definitions for fields which can either be number fields or $\mathbb{Z}_p$-fields, we will use the script notation for finite sets of primes of the field.

Let K be either a number field or a $\mathbb{Z}_p$-field. By definition, the divisor group of K is

$$\mathrm{Div}_K := \bigoplus_{\substack{v \\ \text{finite}}} \Gamma_v \cdot v.$$

Let $\mathcal{S}$ be a finite set of primes of K such that $\mathcal{S} \supset \mathcal{S}_p$, where $\mathcal{S}_p$ is the set of primes of K which extend the p-adic valuation of $\mathbb{Q}$. Let $\mathcal{T}$ be a finite set of primes of K satisfying $\mathcal{T} \cap \mathcal{S} = \varnothing$. The $\mathcal{T}$-modified divisor group of K is then

$$\mathrm{Div}_{K,\mathcal{T}} := \bigoplus_{\substack{v \notin \mathcal{T} \\ \text{finite}}} \Gamma_v \cdot v.$$

The $\mathcal{T}$-units of K are defined to be

$$K_{\mathcal{T}}^{\times} := \{ x \in K^{\times} : \mathrm{ord}_v(x) > 0 \text{ for all } v \in \mathcal{T} \}.$$

There is a divisor map $\mathrm{div}_K : K_{\mathcal{T}}^{\times} \longrightarrow \mathrm{Div}_{K,\mathcal{T}}$ given by

$$\mathrm{div}_K(x) = \sum_{v \notin \mathcal{T}} \mathrm{ord}_v(x) \cdot v,$$

whose kernel is given by

$$U_{K,\mathcal{T}} := \{x \in U_K : \mathrm{ord}_v(x-1) > 0 \text{ for all } v \in \mathcal{T}\}.$$

The generalized ideal class group associated to K and $\mathcal{T}$ is

$$C_{K,\mathcal{T}} := \frac{\mathrm{Div}_{K,\mathcal{T}}}{\mathrm{div}_K(K_{\mathcal{T}}^{\times})},$$

which behaves well under finite extensions of K, namely, if M/K is a finite extension of fields (either number fields or $\mathbb{Z}_p$-fields), and $\mathcal{S}_K$, $\mathcal{T}_K$ are finite sets of primes of K with $\mathcal{S}_M$, $\mathcal{T}_M$ the corresponding primes of M extending those in $\mathcal{S}_K$ and $\mathcal{T}_K$ respectively, then there is a natural injective homomorphism of abelian groups

$$\mathrm{Div}_{K,\mathcal{S}_K} \longrightarrow \mathrm{Div}_{M,\mathcal{S}_M}$$
$$v \mapsto \sum_{w|v} e(w|v)w$$

where $e(w|v)$ is the ramification index of $w|v$. This homomorphism descends to a map of generalized ideal class groups

$$C_{K,\mathcal{T}_K} \longrightarrow C_{M,\mathcal{T}_M}, \tag{3.3}$$

which, in general, is not injective.

Our interest will be in the p-part of $C_{K,\mathcal{T}_K}$ which we denote

$$J_{K,\mathcal{T}_K} := C_{K,\mathcal{T}_K} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

If K is a $\mathbb{Z}_p$-field, the classical μ-invariant conjecture of Iwasawa states that if $\mu_K = 0$ then

$$J_{K,\mathcal{T}_K} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_K},$$

where λ_K is the λ_K-invariant of Iwasawa theory. We will assume the vanishing of the Iwasawa μ-invariant in our construction of our p-adic realizations.

If K is of CM-type, i.e. K is a $\mathbb{Z}_p$-field extension of a CM number field, then K carries a unique complex conjugation automorphism, which we denote j. Under the

action of j we can decompose the module $J_{K,\mathcal{T}_K}$ as

$$J_{K,\mathcal{T}_K} = J^-_{K,\mathcal{T}_K} \oplus J^+_{K,\mathcal{T}_K},$$

where $J^\pm_{K,\mathcal{T}_K} = \left(\frac{1\pm j}{2}\right) J_{K,\mathcal{T}_K}$ denotes the $\pm$-eigenspace of $J_{K,\mathcal{T}_K}$ under the action of j. In particular, j acts by -1 on $J^-_{K,\mathcal{T}_K}$, and therefore we can view $J^-_{K,\mathcal{T}_K} \simeq J_{K,\mathcal{T}_K}/(1+j)$. The last fact needed before defining our p-adic realizations is the following

Lemma 3.1. *[15, Lemma 2.8] Let K be a $\mathbb{Z}_p$-field and $\mathcal{T}$ a finite, non-empty set of finite primes in K disjoint from $\mathcal{S}_p$. Then,*

1. *If $\mu_K = 0$, then $J_{K,\mathcal{T}}$ is a p-torsion, divisible, abelian group of finite local corank.*

2. *If K is of CM-type, p an odd prime, $\mathcal{T}$ j-invariant, and $\mu_K = 0$, then $J^-_{K,\mathcal{T}}$ is a p-torsion, divisible, abelian group of finite local corank.*

3.2 p-adic Realizations of Abstract 1-Motives Associated to Arithmetic Data

Let p be an odd prime, K a $\mathbb{Z}_p$-field, $\mathcal{S}$ a finite set of primes of K with $S \supset \mathcal{S}_p$, and $\mathcal{T}$ a finite set of finite primes of K satisfying $\mathcal{T} \cap \mathcal{S} = \varnothing$. Assuming $\mu_K = 0$ we associate to the data $(K, \mathcal{S}, \mathcal{T})$ the 1-motive

$$\mathcal{M}_{\mathcal{S},\mathcal{T}} := [\mathrm{Div}_K(\mathcal{S} \smallsetminus \mathcal{S}_p) \xrightarrow{\delta} J_{K,\mathcal{T}}].$$

From Lemma 3.1, $J_{K,\mathcal{T}}$ is a torsion, divisible, abelian group of finite local corank, and $\mathrm{Div}_K(\mathcal{S} \smallsetminus \mathcal{S}_p)$ is a free abelian group of finite rank $\#(\mathcal{S} \smallsetminus \mathcal{S}_p)$. The group homomorphism δ maps $D \in \mathrm{Div}_K(\mathcal{S} \smallsetminus \mathcal{S}_p)$ to $\delta(D) := \hat{D} \otimes 1 \in J_{K,\mathcal{T}} := C_{K,\mathcal{T}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\hat{D}$ denotes the class of the divisor D in $C_{K,\mathcal{T}}$. From (3.2) we have an exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow T_p(J_{K,\mathcal{T}}) \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}) \longrightarrow \mathrm{Div}_K(\mathcal{S} \smallsetminus \mathcal{S}_p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0. \qquad (3.4)$$

The above construction applies to any $\mathbb{Z}_p$-field K. We will primarily be interested in the special case when K is the cyclotomic $\mathbb{Z}_p$-extension of a CM number field, the constructions of which we briefly describe.

Let p be an odd prime and F/k a finite abelian extension of number fields of Galois group $G = \mathrm{Gal}(F/k)$ where F is CM and k is totally real. Let $\mu_{p^\infty} := \mu_{p^\infty}(F)$

denote the set of p-power roots of unity in an algebraic closure $\overline{F}$ of F, and consider the infinite extension $F(\mu_{p^\infty})/F$. This is a Galois extension with $\mathrm{Gal}(F(\mu_{p^\infty})/F) \simeq \Delta \times \Gamma_F$, where Δ is a finite group of order coprime to p, and $\Gamma_F \simeq (\mathbb{Z}_p, +)$. The cyclotomic $\mathbb{Z}_p$-field of F, denoted F_∞, is the subfield of $F(\mu_{p^\infty})$ fixed by the action of Δ, therefore, F_∞/F is a Galois extension with Galois group $\Gamma_F := \mathrm{Gal}(F_\infty/F) \simeq (\mathbb{Z}_p, +)$. Since the only closed subgroups of $\mathbb{Z}_p$ are either $\{0\}$ or $p^n \mathbb{Z}_p$ for some $n \in \mathbb{Z}_{\geq 0}$, by infinite Galois theory we have an infinite sequence of intermediate extensions of F_∞/F. More precisely, for any $n \in \mathbb{Z}_{\geq 0}$, if we let γ denote a topological generator of Γ_F, there is a unique intermediate field $F \subset F_n \subset F_\infty$ such that

i.) $\Gamma_n := \mathrm{Gal}(F_\infty/F_n) = \langle \gamma^{p^n} \rangle$

ii.) F_n/F is Galois with Galois group $\mathrm{Gal}(F_n/F) = \Gamma_F/\Gamma_n$, in particular, $[F_n : F] = p^n$.

The extension F_∞/k is Galois and we denote $\mathcal{G}_F = \mathrm{Gal}(F_\infty/k)$. The above is summarized in the following field diagram
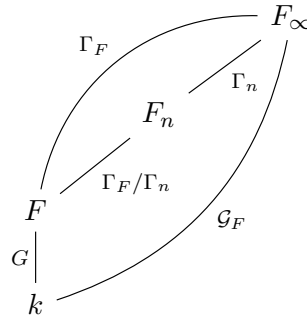
With notation as above let S and T be finite sets of primes of k such that $S \supset S_p$ and $T \cap S = \varnothing$. For $n \in \mathbb{Z}_{\geq 0}$, let S_n, T_n and $S_{n,p}$ denote the sets of places of F_n lying above those in S, T, and S_p, respectively. Let $\mathcal{S}_F$ and $\mathcal{T}_F$ denote the sets of places of F_∞ lying above those in S and T. Since S_n, T_n, and $S_{n,p}$ are all Γ_F/Γ_n-invariant we have natural $\mathbb{Z}_p[\Gamma_F/\Gamma_n]$-module structures on J_{F_n, T_n} and $\mathrm{Div}_{F_n}(S_n \smallsetminus S_{n,p})$. The classical Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma_F]] = \varprojlim_n \mathbb{Z}_p[\Gamma_F/\Gamma_n]$ maps surjectively onto $\mathbb{Z}_p[\Gamma_F/\Gamma_n]$ and therefore J_{F_n, T_n} and $\mathrm{Div}_{F_n}(S_n \smallsetminus S_{n,p})$ carry Λ-module structures. The natural maps

$$J_{F_n, T_n} \longrightarrow J_{F_{n+1}, T_{n+1}} \qquad \text{and} \qquad \mathrm{Div}_{F_n}(S_n \smallsetminus S_{n,p}) \longrightarrow \mathrm{Div}_{F_{n+1}}(S_{n+1} \smallsetminus S_{n+1,p})$$

are Λ-linear, hence we obtain natural Λ-module structures on

$$J_{F_\infty,\mathcal{T}_F} \simeq \varinjlim_n J_{F_n,T_n} \quad \text{and} \quad \mathrm{Div}_{F_\infty}(\mathcal{S} \smallsetminus \mathcal{S}_p) \simeq \varinjlim_n \mathrm{Div}_{F_n}(S_n \smallsetminus S_{n,p}).$$

In particular, the exact sequence of (3.4) is exact in the category of Λ-modules. Furthermore, the $\mathcal{G}_F$-invariance of $\mathcal{S}_F$ and $\mathcal{T}_F$ make (3.4) exact in the category of $\mathbb{Z}_p[[\mathcal{G}_F]]$-modules. Lastly, since F is CM, and therefore F_∞ is CM, and the sets $\mathcal{S}_F, \mathcal{T}_F$ are $\mathcal{G}_F$-invariant, and j-invariant (here j is the unique complex conjugation automorphism of $\mathcal{G}_F$), the exact sequence in (3.4) gives two exact sequences of $\mathbb{Z}_p[[\mathcal{G}_F]]$-modules, namely

$$0 \longrightarrow T_p(J_{K,\mathcal{T}})^\pm \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^\pm \longrightarrow \mathrm{Div}_K(\mathcal{S} \smallsetminus \mathcal{S}_p)^\pm \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0. \quad (3.5)$$

3.2.1 Equivariant Main Conjecture in Iwasawa Theory

To the data $(F/k, S)$ above we associated the G-equivariant L-function $\Theta_{F/k,S} : \mathbb{C} \longrightarrow \mathbb{C}[G]$ defined in Definition 1.8. For our purposes we introduce a modified version of $\Theta_{F/k,S}$. Letting T be a finite set of primes of k such that $T \cap S = \varnothing$ and, if T_F denotes the set of primes of F lying above those in T, then $F_{T_F}^\times$ is torsion free, the T-modified G-equivariant L-function is defined as

Definition 3.4. *Let $(F/k, S, T)$ be as above. The T-modified, S-imprimitive, G-equivariant L-function associated to the data $(F/k, S, T)$ is $\Theta_{F/k,S,T} : \mathbb{C} \to \mathbb{C}[G]$ given by*

$$\Theta_{F/k,S,T}(s) := \Theta_{F/k,S}(s)\delta_T(s)$$

where

$$\delta_T(s) := \prod_{v \in T}(1 - Nv^{1-s}\sigma_v^{-1}),$$

is a holomorphic $\mathbb{C}[G]$-valued function.

If T contains two primes of different residual characteristic then, for all $m \in \mathbb{Z}_{\geq 1}$, we have

$$\Theta_{F/k,S,T}(1 - m) \in \mathbb{Z}[G].$$

Our primary interest will be the case when $m = 1$.

If F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension of the CM field F, with $\mathcal{G}_F = \mathrm{Gal}(F_\infty/k)$,

then for all $n \in \mathbb{Z}_{\geq 0}$ we can construct the equivariant L-function

$$\Theta_{F_n/k, S_{F_n}, T_{F_n}} : \mathbb{C} \longrightarrow \mathbb{C}[\mathrm{Gal}(F_n/k)].$$

If T is chosen to contain two primes of different residual characteristic, and $m = 1$, then for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$\Theta_{F_n/k, S_{F_n}, T_{F_n}}(0) \in \mathbb{Z}[\mathrm{Gal}(F_n/k)] \subset \mathbb{Z}_p[\mathrm{Gal}(F_n/k)].$$

Via Artin inflation, the $\Theta_{F_n/k, S_{F_n}, T_{F_n}}(0)$ form a coherent sequence in the equivariant Iwasawa algebra $\mathbb{Z}_p[[\mathcal{G}_F]]$, whereby we denote the limit

$$\Theta_{\mathcal{S}_F, \mathcal{T}_F}^{(\infty)} := \Theta_{\mathcal{S}_F, \mathcal{T}_F}^{(\infty)}(0) := (\Theta_{F_n/k, S_{F_n}, T_{F_n}}(0))_n \in \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(F_n/k)] = \mathbb{Z}_p[[\mathcal{G}_F]].$$

The equivariant main conjecture in Iwasawa theory, proved by Popescu and Greither, is then given as follows

Theorem 3.2. [15, Theorem 5.6] *For $(F/k, S, T, p)$ as above, and assuming $\mu_F = 0$, the following equality of ideals in $\mathbb{Z}_p[[\mathcal{G}_F]]^-$ holds*

$$\mathrm{Fit}^0_{\mathbb{Z}_p[[\mathcal{G}_F]]^-}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}})^-) = (\Theta_{\mathcal{S}, \mathcal{T}}^{(\infty)}).$$

Chapter 4

A Generalized Conjecture for p-adic Realizations of Abstract 1-Motives

4.1 Fitting Ideals and Kurihara's Conjecture

We give a brief exposition of Fitting ideals, a comprehensive treatment of Fitting ideals can be found in [30]. Let R be a commutative Noetherian ring and M a finitely presented R-module with presentation

$$R^m \xrightarrow{\ f\ } R^n \longrightarrow M \longrightarrow 0. \tag{4.1}$$

Choose bases of R^m and R^n so that the R-module homomorphism f is given by an $n \times m$ matrix A. Associated to the presentation in (4.1) is an increasing stabilizing sequence of ideals of R

$$\mathrm{Fit}_R^0(M) \subseteq \mathrm{Fit}_R^1(M) \subseteq \ldots \subseteq \mathrm{Fit}_R^{n-1}(M) \subseteq \mathrm{Fit}_R^n(M) = R = R \ldots$$

where $\mathrm{Fit}_R^i(M)$ is the ith Fitting ideal of M, which is the ideal of R generated by determinants of all $(m-i) \times (m-i)$ minors of the matrix A. The inclusive nature of these ideals follows from elementary cofactor expansion. The definition of Fitting ideal does not depend on the chosen presentation, nor does it depend on the chosen bases. The following lemma is found in [26] and contains the properties of Fitting ideals relevant to

this manuscript.

Lemma 4.1. *Let M, M', and M'' be finitely presented R-modules, then the following hold*

i.) If $M \twoheadrightarrow M'$, then

$$\mathrm{Fit}_R^0(M) \subset \mathrm{Fit}_R^0(M').$$

ii.) If $0 \to M' \to M \to M'' \to 0$, then

$$\mathrm{Fit}_R^0(M')\mathrm{Fit}_R^0(M'') \subset \mathrm{Fit}_R^0(M).$$

iii.) If $\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$ denotes the annihilator of M, then

$$\mathrm{Fit}_R^0(M) \subset \mathrm{Ann}_R(M).$$

For rings R which are principal ideal domains (PIDs), the following example illustrates how Fitting ideals classify modules up to isomorphism.

Example 4. *Suppose R is PID and M is a finitely generated R-module. From the fundamental theorem of finitely generated modules over a PID, we have an isomorphism*

$$M \simeq R^m \oplus R/a_1 R \oplus R/a_2 R \oplus \ldots \oplus R/a_n R,$$

where $a_1 | a_2 | \ldots | a_n$. In general, if $m > 0$, i.e. M is not torsion, then

$$\mathrm{Fit}_R^i(M) = 0$$

for all $0 \le i \le m$. Therefore, in what follows we assume M is a finitely generated torsion *R-module, and therefore, by the fundamental theorem*

$$M \simeq R/a_1 R \oplus R/a_2 R \oplus \ldots \oplus R/a_n R,$$

where $a_1 | a_2 | \ldots | a_n$.

Since the definition of Fitting ideals does not depend on the chosen presentation, nor on the chosen bases, we let

$$R^n \xrightarrow{\ f\ } R^n \longrightarrow M \longrightarrow 0$$

be the presentation of M such that the matrix A, corresponding to the R-module morphism f, is diagonal

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

By definition, $\mathrm{Fit}_R^0(M)$ is the ideal of R generated by all $(n-0) \times (n-0)$ minors of A. Since there is only one such minor

$$\mathrm{Fit}_R^0(M) = (a_1 a_2 \dots a_n).$$

Remark. *Observe $\mathrm{Fit}_R^0(M) \subset (a_n) = \mathrm{Ann}_R(M)$ illustrating the third property of lemma 4.1.*

By definition, $\mathrm{Fit}_R^1(M)$ is the ideal of R generated by all $(n-1) \times (n-1)$ minors of A, consequently

$$\mathrm{Fit}_R^1(M) = \left(\frac{a_1 a_2 \dots a_n}{a_i} : 1 \leq i \leq n \right).$$

However, for any $1 \leq i \leq n$

$$a_1 a_2 \dots \hat{a}_i \dots a_n = a_1 a_2 \dots a_{n-1} \cdot \frac{a_n}{a_i},$$

where $\hat{\cdot}$ denotes the omission of the ith term, and $\frac{a_n}{a_i} \in R$ due to the divisibility relations amongst the invariant factors. Therefore,

$$\mathrm{Fit}_R^1(M) = (a_1 a_2 \dots a_{n-1}).$$

Continuing in this manner, for all $0 \leq i \leq n-1$ we obtain

$$\mathrm{Fit}_R^i(M) = (a_1 a_2 \dots a_{n-i})$$

and for all $i \geq n$

$$\mathrm{Fit}_R^i(M) = R.$$

Therefore, knowing the Fitting filtration of M over a PID R yields the invariant factors of M and therefore determines the isomorphism class of M.

In general, if R is a commutative Noetherian ring with unit and M and N are

two finitely presented R-modules such that, for all $i \geq 0$

$$\operatorname{Fit}_R^i(M) = \operatorname{Fit}_R^i(N),$$

is it necessarily true that M and N are isomorphic as R-modules? Unfortunately, the answer to this question is *no* in general. If all Fitting ideals of two R-modules M and N are equal, then the two R-modules are said to be quasi-isomorphic, or pseudo-isomorphic, denoted $M \sim N$. Unfortunately, the converse to the above is false, namely, there exist quasi-isomorphic R-modules whose Fitting ideals are not equal, as illustrated below.

Example 5. *Let $\Lambda = \mathbb{Z}_p[[T]]$ be the classical one-variable Iwasawa algebra. Let $\mathfrak{m} = (p, T)$ denote the unique maximal ideal of Λ and consider the two Λ-modules $M = \Lambda/\mathfrak{m}\Lambda$ and $N = 0$. Since $M \simeq \mathbb{F}_p$, the finite field with p-elements, M and N are quasi-isomorphic, denoted $M \sim N$. However, computing Fitting ideals,*

$$\operatorname{Fit}_\Lambda^0(M) = \mathfrak{m} \neq \Lambda,$$

whereas, for all $i \geq 0$

$$\operatorname{Fit}_\Lambda^i(N) = \Lambda.$$

Therefore, quasi-isomorphism is not sufficient to ensure equality of Fitting ideals.

For the ring Λ the notion of quasi-isomorphism can be detected locally as given by the following lemma

Lemma 4.2. *Let X and Y be two finitely generated torsion Λ-modules, then the following hold:*

i.) $X \sim Y$ if and only if $X_\mathfrak{p} \simeq Y_\mathfrak{p}$ for all height one primes $\mathfrak{p} \in \operatorname{Spec}(\Lambda)$.

ii.) If $\operatorname{ht}(\mathfrak{p}) = 1$, then $\Lambda_\mathfrak{p}$ is a PID.

iii.) If M and N are finitely generated torsion R-modules where R is a PID, and $\operatorname{Fit}_R^i(M) = \operatorname{Fit}_R^i(N)$ for all $i \geq 0$, then $M \simeq N$ as R-modules.

In particular, if M and N are two finitely generated torsion Λ-modules such that, for all $i \geq 0$

$$\operatorname{Fit}_\Lambda^i(M) = \operatorname{Fit}_\Lambda^i(N),$$

then, for all height one primes $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$

$$\mathrm{Fit}^i_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{Fit}^i_{\Lambda_\mathfrak{p}}(N_\mathfrak{p})$$

and therefore, by property three of Lemma 4.2 (since $\Lambda_\mathfrak{p}$ is a PID), we have

$$M_\mathfrak{p} \simeq N_\mathfrak{p},$$

hence by the first property of lemma 4.2

$$M \sim N$$

as Λ-modules.

With the establishment of the definitions and properties of Fitting ideals, we now formulate Kurihara's conjecture. The original statement of the conjecture may be found in [24], but for the reader's convenience we recall the statement here. Let $p > 2$ be prime and χ an odd Dirichlet character of order coprime to p such that $\chi(p) \neq 1$ and $\chi \neq \omega$ where ω is the Teichmüller character. Let $F := \overline{\mathbb{Q}}^{\ker(\chi)}$ be the fixed field determined by χ and let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of F with Galois group Γ_F. Setting $\Lambda_\chi := \Lambda^\chi_F = \mathbb{Z}_p(\chi)[[\mathrm{Gal}(F_\infty/F)]]$, the χ-component of the classical Iwasawa module $X^\chi_{F_\infty}$ over Λ_χ is given by

$$X^\chi_{F_\infty} = \varprojlim_n (\mathrm{Cl}_{F_n} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\chi,$$

where the projective limit is taken with respect to the norm maps.

Conjecture 4.3 (Kurihara). *Let $r \in \mathbb{Z}_{>0}$ and fix $N > 0$ large, then for any $i \geq 0$*

$$\mathrm{Fit}^i_{\Lambda_\chi/p^N \Lambda_\chi}(X^\chi_{F_\infty}/p^N X^\chi_{F_\infty}) = \mathscr{F}^i,$$

where $\mathscr{F}^i$ is the ideal of Λ_χ generated by $\Theta^\chi_{F_\infty/\mathbb{Q}}(0)$ and $\delta_{i_1,\ldots,i_r}(\Theta^\chi_{L_\infty}(0))$ where

(i.) $L/\mathbb{Q}$ ranges over all abelian fields satisfying $L \cap F_\infty = F$ and

$$\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(F/\mathbb{Q}) \times \mathrm{Gal}(L/F),$$

and

$$\mathrm{Gal}(L/F) \simeq \underbrace{\mathbb{Z}/p^N\mathbb{Z} \times \ldots \times \mathbb{Z}/p^N\mathbb{Z}}_{r}$$

(ii.) $(i_1, i_2, \ldots, i_r)$ *ranges over integers satisfying* $i_1 + i_2 + \ldots + i_r \leq i$.

The case $i = 1$ was resolved by Kurihara using Euler systems methods, however, a full proof of the conjecture was recently given by Popescu and Stone [41, Theorem 4.6] using different methods.

4.2   Semi-nice Extensions and Homological Algebra

For our generalization of Kurihara's conjecture to p-adic realizations of abstract 1-motives, we will be interested in certain classes of abelian extensions.

Definition 4.1. *Let F/k be a finite abelian extension of number fields of Galois group G such that F is CM and k is totally real. Let $p > 2$ be prime and $j \in G$ the unique complex conjugation automorphism of F, then the extension F/k is called semi-nice if the following conditions hold*

i.) $\mathrm{Cl}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p = \{1\}$.

ii.) $j \in G_v$ *for all* $v \in S_{\mathrm{ram}}(F/k) \cup S_{k,p}$.

We now establish some important homological algebra properties of finite $\mathbb{Z}_p[G]$-modules M, where G is a finite abelian group, and use these properties to establish relationships between our p-adic realizations of abstract 1-motives and modified Iwasawa modules in semi-nice extensions. The ideas can be traced back to work of Popescu-Greither found in [15].

Lemma 4.4. *Let G be a finite abelian group and M a finite $\mathbb{Z}_p[G]$-module satisfying* $\mathrm{pd}_{\mathbb{Z}_p[G]} M \leq 1$. *Let $M^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ denote the Pontrjagin dual of M, made into a G-module via the covariant G-action, $(\sigma \cdot f)(x) = f(\sigma x)$ for all $\sigma \in G$, $x \in M$, and $f \in M^\vee$. Then, for all $i \geq 0$*

$$\mathrm{Fit}^i_{\mathbb{Z}_p[G]}(M^\vee) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(M).$$

Proof. Consider the exact sequence of $\mathbb{Z}_p[G]$-modules with trivial G-action,

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0. \tag{4.2}$$

Applying the left exact functor $* \mapsto \text{Hom}_{\mathbb{Z}_p}(M, *)$ to (4.2), we get the long exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \longrightarrow \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p) \longrightarrow \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow$$
$$\longrightarrow \text{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \longrightarrow \text{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Q}_p) \longrightarrow 0 \longrightarrow \cdots$$

$$(4.3)$$

Since M is finite $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, and since $\mathbb{Q}_p$ is a divisible $\mathbb{Z}_p$-module, $\text{Ext}_{\mathbb{Z}_p}(M, \mathbb{Q}_p) = 0$, therefore, the long exact sequence in (4.3) becomes the short exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \text{Ext}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \longrightarrow 0 \qquad (4.4)$$

showing that we have an isomorphism

$$M^\vee \simeq \text{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p). \qquad (4.5)$$

in the category of finite $\mathbb{Z}_p[G]$-modules. Since M is finite and $\text{pd}_{\mathbb{Z}_p[G]} M \le 1$, we have a presentation

$$0 \longrightarrow \mathbb{Z}_p[G]^{\oplus n} \xrightarrow{\alpha} \mathbb{Z}_p[G]^{\oplus n} \longrightarrow M \longrightarrow 0. \qquad (4.6)$$

Letting $\{e_1, e_2, \ldots, e_n\}$ be a basis for the left-most $\mathbb{Z}_p[G]^{\oplus n}$ and $\{f_1, f_2, \ldots, f_n\}$ a basis for the middle $\mathbb{Z}_p[G]^{\oplus n}$ in (4.6), set $A = (a_{ij})$ for the $n \times n$ matrix of α with respect to these chosen bases where

$$\alpha(e_j) = \sum_{i=1}^{n} a_{ij} e_i.$$

Applying the contravariant functor $* \mapsto \text{Hom}_{\mathbb{Z}_p}(*, \mathbb{Z}_p)$ to the sequence in (4.6) gives the long exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \longrightarrow \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p) \xrightarrow{\alpha^*} \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p) \longrightarrow$$
$$\longrightarrow \text{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \longrightarrow \text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p) \longrightarrow 0$$

$$(4.7)$$

However, $\mathbb{Z}_p[G]^{\oplus n}$ is a free $\mathbb{Z}_p$-module, and M is finite, therefore

$$\text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p) = 0. \qquad \text{and} \qquad \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) = 0. \qquad (4.8)$$

Combining (4.5) and (4.8) with the long exact sequence in (4.7), and utilizing the canonical $\mathbb{Z}_p[G]$-module isomorphism

$$\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p[G])$$

$$\varphi \mapsto \tilde{\varphi} : x \mapsto \sum_{\sigma \in G} \varphi(\sigma^{-1}x)\sigma,$$

we obtain the short exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G], \mathbb{Z}_p[G])^{\oplus n} \xrightarrow{\alpha^*} \text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G], \mathbb{Z}_p[G])^{\oplus n}$$

$$\downarrow \qquad\qquad\qquad (4.9)$$

$$M^\vee \longrightarrow 0.$$

Using the notation

$$\mathbb{Z}_p[G]^* = \text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G], \mathbb{Z}_p[G]),$$

the short exact sequence in (4.9) becomes

$$0 \longrightarrow (\mathbb{Z}_p[G]^*)^{\oplus n} \xrightarrow{\alpha^*} (\mathbb{Z}_p[G]^*)^{\oplus n} \longrightarrow M^\vee \longrightarrow 0. \qquad (4.10)$$

We now compute the matrix for $\alpha^* : (\mathbb{Z}_p[G]^*)^{\oplus n} \to (\mathbb{Z}_p[G]^*)^{\oplus n}$ with respect to the dual bases $\{f_1^*, f_2^*, \ldots, f_n^*\}$ and $\{e_1^*, e_2^*, \ldots, e_n^*\}$, respectively.

By definition $\alpha^*(\varphi) = \varphi \circ \alpha$ for any $\varphi \in \text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p[G]^{\oplus n}, \mathbb{Z}_p[G])$, hence, for fixed $1 \leq i \leq n$

$$\alpha^*(f_i)(e_j) = f_i^*(\alpha(e_j))$$

$$= f_i^* \left(\sum_{i=1}^n a_{ij} f_i \right)$$

$$= a_{ij}.$$

Therefore, the ith column of the matrix associated to α^* is the ith row of the matrix

for α, consequently, the matrix for α^* is A^T. Since there is a one-to-one correspondence between determinants of minors of A and A^T, we have for all $i \geq 0$

$$\mathrm{Fit}^i_{\mathbb{Z}_p[G]}(M^\vee) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(M). \qquad \square$$

4.3 Linking $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^-$ and X_T^-

Let $(F/k, S, T, p)$ be as above with $G := \mathrm{Gal}(F/k)$. We use the notations

$$C_{F_n,T} := (\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \qquad \text{and} \qquad J_T := \varinjlim_n C_{F_n,T}.$$

From the definition of $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})$ we have the short exact sequence

$$0 \longrightarrow T_p(J_T)^- \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^- \longrightarrow (\mathrm{Div}(S \smallsetminus S_{k,p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \longrightarrow 0.$$

Since each of the above modules are $\mathbb{Z}_p$-free, applying the $\mathbb{Z}_p$-module functor $* \mapsto \mathrm{Hom}_{\mathbb{Z}_p}(*, \mathbb{Z}_p)$ gives the short exact sequence

$$0 \longrightarrow ((\mathrm{Div}(S \smallsetminus S_{k,p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-)^* \longrightarrow (T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^-)^* \longrightarrow (T_p(J_T)^-)^* \longrightarrow 0 \ .$$

From here we obtain the following two lemmas

Lemma 4.5. *If $j \in G_v$ for all $v \in S_{\mathrm{ram}} \smallsetminus S_{k,p}$ and $\mu = 0$, then $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^- \simeq T_p(J_T)^-$.*

Proof. For any $v \in S_{\mathrm{ram}} \smallsetminus S_{k,p}$ fix some $w(v)|v$ in F_∞, then

$$(\mathrm{Div}(S \smallsetminus S_{k,p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- = \bigoplus_{v \in S \smallsetminus S_{k,p}} \mathbb{Z}_p[[\mathcal{G}_F]]^- w(v)$$

$$= \bigoplus_{v \in S \smallsetminus S_{k,p}} \mathbb{Z}_p[[\mathcal{G}_F/\mathcal{G}_{F,v}]]^- w(v)$$

however, by assumption $j \in \mathcal{G}_{F,v}$ for all $v \in S \smallsetminus S_{k,p}$, and therefore j simultaneously acts via $+1$ and -1 on $(\mathrm{Div}(S \smallsetminus S_{k,p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$, therefore $(\mathrm{Div}(S \smallsetminus S_{k,p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- = \{0\}$. From the fundamental exact sequence above, we obtain the result

$$T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^- \simeq T_p(J_T)^-. \qquad \square$$

Lemma 4.6. *If $j \in G_v$ for all $v \in S_{k,p}$ and $\mu = 0$, then $T_p(J_T)^- \simeq X_T^-$.*

Proof. Let $\Gamma = \overline{\langle \gamma \rangle}$ and set $\gamma_n = \gamma^{p^n}$ which is a generator of $\mathrm{Gal}(F_\infty/F_n)$. We always have the exact sequence

$$X_T^- \xrightarrow{1-\gamma_n} X_T^- \longrightarrow C_{F_n,T} \longrightarrow 0 \tag{4.11}$$

however, since X_T^- has no nonzero finite submodules, (4.11) is actually a short exact sequence

$$0 \longrightarrow X_T^- \xrightarrow{1-\gamma_n} X_T^- \longrightarrow C_{F_n,T} \longrightarrow 0 \tag{4.12}$$

Taking $\mathbb{Z}_p$-module duals of (4.12) we obtain

$$0 \longrightarrow C_{F_n,T}^* \longrightarrow (X_T^-)^* \xrightarrow{(1-\gamma_n)^*} (X_T^-)^* \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p}(C_{F_n,T}, \mathbb{Z}_p) \longrightarrow 0 \tag{4.13}$$

where $\mathrm{Ext}^1_{\mathbb{Z}_p}(X_T^-, \mathbb{Z}_p) = 0$ since we are assuming $\mu = 0$. Moreover, $C_{F_n,T}$ is finite, therefore, $C_{F_n,T}^* = 0$, hence (4.13) gives the short exact sequence

$$0 \longrightarrow (X_T^-)^* \xrightarrow{(1-\gamma_n)^*} (X_T^-)^* \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p}(C_{F_n,T}, \mathbb{Z}_p) \longrightarrow 0. \tag{4.14}$$

In (4.5) we have the isomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(C_{F_n,T}, \mathbb{Z}_p) \simeq C_{F_n,T}^\vee, \tag{4.15}$$

therefore, combining (4.14) and (4.15)

$$(X_T^-)^*/(1-\gamma_n)^*(X_T^-)^* \simeq C_{F_n,T}^\vee,$$

and therefore,

$$\varprojlim_n (X_T^-)^*/(1-\gamma_n)^*(X_T^-)^* \simeq \varprojlim_n C_{F_n,T}^\vee. \tag{4.16}$$

However, consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (X_T^-)^* & \xrightarrow{(1-\gamma_{n+1})^*} & (X_T^-)^* & \longrightarrow & (X_T^-)^*/(1-\gamma_{n+1})^*(X_T^-)^* & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & (X_T^-)^* & \xrightarrow{(1-\gamma_n)^*} & (X_T^-)^* & \longrightarrow & (X_T^-)^*/(1-\gamma_n)^*(X_T^-)^* & \longrightarrow & 0
\end{array}
$$

where the map $(X_T^-)^* \to (X_T^-)^*$ is given by multiplication by $\frac{1-\gamma_{n+1}}{1-\gamma_n}$. Since $\frac{1-\gamma_{n+1}}{1-\gamma_n} \in \mathfrak{m}^n$

where $\mathfrak{m} = (p, \gamma) \subset \Lambda$ is the maximal ideal, taking projective limits and using $\cap_{n \geq 0} \mathfrak{m}^n = 0$ gives

$$(X_T^-)^* \simeq \varprojlim_n (X_T^-)^*/(1 - \gamma_n)^*(X_T^-)^*,$$

hence

$$(X_T^-)^* \simeq \varprojlim_n (X_T^-)^*/(1 - \gamma_n)^*(X_T^-)^* \simeq \varprojlim_n C_{F_n,T}^\vee. \tag{4.17}$$

From the definition of the p-adic realization of 1-motives we have

$$\begin{aligned}
T_p(J_T)^* &= \mathrm{Hom}_{\mathbb{Z}_p}(J_T, \mathbb{Q}_p/\mathbb{Z}_p) \\
&= \mathrm{Hom}_{\mathbb{Z}_p}(\varinjlim_n C_{F_n,T}, \mathbb{Q}_p/\mathbb{Z}_p) \\
&\simeq \varprojlim_n \mathrm{Hom}_{\mathbb{Z}_p}(C_{F_n,T}, \mathbb{Q}_p/\mathbb{Z}_p) \\
&= \varprojlim_n C_{F_n,T}^\vee, \tag{4.18}
\end{aligned}$$

therefore, combining (4.17) and (4.18) we obtain the isomorphism

$$(T_p(J_T)^-)^* \simeq (X_T^-)^*.$$

Since the $\mathbb{Z}_p$-module duals are isomorphic, taking the second $\mathbb{Z}_p$-module dual yields the desired isomorphism

$$T_p(J_T)^- \simeq X_T^-. \qquad \square$$

Corollary 4.7. *If $j \in G_v$ for all $v \in S_{\mathrm{ram}} \cup S_{k,p}$ and $\mu = 0$, then*

$$T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}})^- \simeq T_p(J_T)^- \simeq X_T^-.$$

Proof. This is simply a combination of lemmas 4.5 and 4.6 above. $\qquad \square$

Since $T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-$ is finitely presented, consider the presentation

$$0 \longrightarrow \mathbb{Z}_p[[\mathcal{G}_F]]^{\oplus n} \xrightarrow{\varphi} \mathbb{Z}_p[[\mathcal{G}_F]]^{\oplus n} \longrightarrow T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^- \longrightarrow 0$$

where bases are chosen so that the determinant of the matrix representing φ is $\Theta_{\mathcal{S},\mathcal{T}}^{(\infty)}(0)$.

Taking Γ_m-coinvariants gives the sequence

$$0 \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{pm}}^- f_i \xrightarrow{\overline{\varphi}} \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{pm}}^- v_i \longrightarrow (\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \longrightarrow 0$$

where the f_i and v_i are chosen such that the determinant of $\overline{\varphi}$ is $\Theta_{S_m,T_m}(0)$, the classes of the v_i generate $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$, and such that the inclusion

$$\bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_\mathcal{F}]]_{\Gamma^{pm}}^- f_i \to \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_\mathcal{F}]]_{\Gamma^{pm}}^- v_i,$$

is the divisor map $\mathrm{div} : (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \to \mathrm{Div}_{F_m,T}$. Alternatively, one can see the elements $f_1, \ldots, f_n$ as those elements of F_m whose divisors are supported on $v_1, \ldots, v_m$.

4.4   Totally Ramified Extensions

Let $p > 2$ be a prime, F/k an abelian extension where F is CM and k totally real, and fix some large positive integer $N > 0$. We prove the existence of finite primes λ of k and cyclic extensions of k with prescribed ramification in the following lemma.

Lemma 4.8. *Let (F, k, p) be as above and let $h_{k,p} := \#(\mathrm{Cl}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. If $h_{k,p} = 1$ then there exist infinitely many finite primes λ of k with associated cyclic extensions $k(\lambda)^*/k$ satisfying*

  i.) $\mathrm{Gal}(k(\lambda)^*/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$.

  ii.) $k(\lambda)^*/k$ is totally ramified at λ and contains no unramified subextensions.

  iii.) $k(\lambda)^*/k$ and F/k are linearly disjoint.

Proof. Consider the extensions

$$M = k(\mu_{p^N}, (\mathcal{O}_k^\times)^{1/p^N})$$
$$|$$
$$K = k(\mu_{p^N})$$
$$|$$
$$k$$

The Tchebotarev density theorem applied to the Galois extension MF/k, asserts the existence of infinitely many unramified primes of k which split completely in

MF. Let λ be a finite unramified prime of k which splits completely in MF/k, and denote λ' and λ'' primes above λ in K and M, respectively. Furthermore, denote $\mathcal{U}_\lambda$ and $\kappa(\lambda)$ the units and residue field associated to λ, respectively. Since λ splits completely in M/k the completions satisfy $M_{\lambda''} = K_{\lambda'} = k_\lambda$. Since $M_{\lambda''} = k_\lambda(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})$ and $K_{\lambda'} = k_\lambda(\mu_{p^N})$ we see $\mu_{p^N} \subset \mathcal{U}_\lambda \subset k_\lambda$ and $\mathcal{O}_k^{\times 1/p^N} \subset \mathcal{U}_\lambda \subset k_\lambda$. Letting ℓ denote the prime of $\mathbb{Q}$ lying below λ and using that λ splits completely in M/k we have $\ell \neq p$. Since $\mu_{p^N} \subset \kappa(\lambda)$ and $\mathcal{U}_\lambda^{(1)}$ is an ℓ-group, the decomposition $\mathcal{U}_\lambda = \kappa(\lambda)^\times \times \mathcal{U}_\lambda^{(1)}$ shows $\mu_{p^N} \subset \kappa(\lambda)^\times$. Since $\kappa(\lambda)^\times$ is cyclic and contains μ_{p^N}, it must contain a unique subgroup of order p^N. Since $\ell \neq p$ and $\mathcal{U}_\lambda^{(1)}$ is an ℓ-group, the multiplication by p^N map is an automorphism of $\mathcal{U}_\lambda^{(1)}$, whereby

$$\mathcal{U}_\lambda^{p^N} \simeq \kappa(\lambda)^{\times p^N} \times \mathcal{U}_\lambda^{(1)}.$$

The inclusion $\mathcal{O}_k^\times \subset \mathcal{U}_\lambda^{p^N}$ provides a natural surjection

$$\mathcal{U}_\lambda / \mathcal{U}_\lambda^{(1)} \mathcal{O}_k^\times \twoheadrightarrow \mathcal{U}_\lambda / \mathcal{U}_\lambda^{(1)} \mathcal{U}_\lambda^{p^N} = \kappa(\lambda)^\times / \kappa(\lambda)^{\times p^N}.$$

Furthermore, since $\kappa(\lambda)^\times / \kappa(\lambda)^{\times p^N}$ is a p-group, we have a surjection at the level of p-primary parts

$$(\mathcal{U}_\lambda / \mathcal{U}_\lambda^{(1)} \mathcal{O}_k^\times) \otimes_\mathbb{Z} \mathbb{Z}_p \twoheadrightarrow \kappa(\lambda)^\times / \kappa(\lambda)^{\times p^N},$$

and since $\kappa(\lambda)^\times / \kappa(\lambda)^{\times p^N}$ contains a unique subgroup of order p^N, the same holds true for $(\mathcal{U}_\lambda / \mathcal{U}_\lambda^{(1)} \mathcal{O}_k^\times) \otimes_\mathbb{Z} \mathbb{Z}_p$.

Consider the short exact sequence

$$0 \longrightarrow \frac{k^\times \prod_v \mathcal{U}_v}{k^\times \left(\prod_{v \neq \lambda} \mathcal{U}_v \times \mathcal{U}_\lambda^{(1)} \right)} \longrightarrow \frac{J_k}{k^\times \left(\prod_{v \neq \lambda} \mathcal{U}_v \times \mathcal{U}_\lambda^{(1)} \right)} \longrightarrow \frac{J_k}{k^\times \prod_v \mathcal{U}_v} \longrightarrow 0 \qquad (4.19)$$

where

$$\frac{J_k}{k^\times \prod_v \mathcal{U}_v} \simeq \mathrm{Cl}_k$$

and

$$\frac{J_k}{k^\times \left(\prod_{v \neq \lambda} \mathcal{U}_v \times \mathcal{U}_\lambda^{(1)} \right)} \simeq \mathrm{Cl}_{k,T}$$

are the ideal class group of k and ray class group of k of conductor $T := \{\lambda\}$, respectively.

We denote by $k(\lambda)$ the ray class field of k of conductor T so that

$$\mathrm{Cl}_{k,T} \simeq \frac{J_k}{k^\times(\prod_{v\neq\lambda}\mathcal{U}_v \times \mathcal{U}_\lambda^{(1)})} \simeq \mathrm{Gal}(k(\lambda)/k).$$

In (4.19) we have

$$\frac{k^\times \prod_v \mathcal{U}_v}{k^\times\left(\prod_{v\neq\lambda}\mathcal{U}_v \times \mathcal{U}_\lambda^{(1)}\right)} \simeq \frac{\mathcal{U}_\lambda}{k^\times(\mathcal{U}_\lambda^{(1)} \times \prod_{v\neq\lambda}\mathcal{U}_v) \cap \mathcal{U}_\lambda} \simeq \frac{\mathcal{U}_\lambda}{\mathcal{U}_\lambda^{(1)}\mathcal{O}_k^\times}.$$

Therefore, (4.19) becomes

$$0 \longrightarrow \frac{\mathcal{U}_\lambda}{\mathcal{U}_\lambda^{(1)}\mathcal{O}_k^\times} \longrightarrow \mathrm{Cl}_{k,T} \longrightarrow \mathrm{Cl}_k \longrightarrow 0. \qquad (4.20)$$

Applying the the functor $* \mapsto * \otimes_\mathbb{Z} \mathbb{Z}_p$ to (4.20) gives the short exact sequence

$$0 \longrightarrow \left(\frac{\mathcal{U}_\lambda}{\mathcal{U}_\lambda^{(1)}\mathcal{O}_k^\times}\right) \otimes_\mathbb{Z} \mathbb{Z}_p \longrightarrow \mathrm{Cl}_{k,T} \otimes_\mathbb{Z} \mathbb{Z}_p \longrightarrow \mathrm{Cl}_k \otimes_\mathbb{Z} \mathbb{Z}_p \longrightarrow 0, \qquad (4.21)$$

whose accompanying field diagram is

where $k(\lambda)^{(p)}$ is the fixed field of $k(\lambda)$ by the non-p-part of $\mathrm{Cl}_{k,T}$ and $H_k^{(p)}$ is the p-Hilbert class field of k. The assumption $h_{k,p} = 1$ ensures the cyclic quotient group $\kappa(\lambda)^\times/\kappa(\lambda)^{\times p^N}$ contains a subgroup of order at least p^N. Without the assumption, the global units $\mathcal{O}_k^\times$ could potentially reduce the order of our desired cyclic subgroup to a power of p which is less that N. Since $(\mathcal{U}_\lambda/\mathcal{U}_\lambda^{(1)}\mathcal{O}_k^\times) \otimes_\mathbb{Z} \mathbb{Z}_p$ contains a cyclic subgroup of order p^N, we let $k(\lambda)^*$ denote the subfield of $k(\lambda)$ such that $\mathrm{Gal}(k(\lambda)^*/k) \simeq \mathbb{Z}/p^N\mathbb{Z}$. We therefore see that the chosen λ of k satisfies the desired properties. $\qquad \square$

4.5   Tchebotarev Density Constructions

The Tchebotarev density theorem will play a key role in this section, therefore we recall its statement for a general Galois extension K/k.

Theorem 4.9 (Tchebotarev Density). *Let K/k be a finite Galois extension with $G = \mathrm{Gal}(K/k)$ and let $C \subset G$ be a conjugacy class in G. Then, the set of unramified primes $\mathfrak{p} \subset \mathcal{O}_k$ satisfying $C_\mathfrak{p} = C$ has density $\frac{|C|}{|G|}$. Here $C_\mathfrak{p} = [\mathrm{Frob}_\mathfrak{P}]$ denotes the G-conjugacy class of $\mathrm{Frob}_\mathfrak{P}$ for some (any) $\mathfrak{P}|\mathfrak{p}$.*

Lemma 4.10. *Let $A/B/C$ be a triple of Galois extension of number fields with $\mathcal{G} := \mathrm{Gal}(A/C)$ and $\mathcal{H} = \mathrm{Gal}(A/B) \trianglelefteq \mathcal{G}$. For any $h \in \mathcal{H}$, there exist infinitely many primes λ_A in A such that $\mathrm{Frob}_{\lambda_A} = h$ where $\mathrm{Frob}_{\lambda_A} \in \mathcal{G}$.*

Proof. Let $h \in \mathcal{H}$ and let C_h denote the $\mathcal{G}$-conjugacy class of h in $\mathcal{G}$. The Tchebotarev density theorem applied to the Galois extension A/C gives infinitely many primes λ'_A of A such that $[\mathrm{Frob}_{\lambda'_A}] = C_h$. Let $\sigma \in \mathcal{G}$ satisfy $\sigma \mathrm{Frob}_{\lambda'_A} \sigma^{-1} = h$, and observe that $\sigma \mathrm{Frob}_{\lambda'_A} \sigma^{-1} \in \mathcal{H}$, since $\mathcal{H} \trianglelefteq \mathcal{G}$. Applying lemma 1.3 we see $\mathrm{Frob}_{\sigma \lambda'_A} = \sigma \mathrm{Frob}_{\lambda'_A} \sigma^{-1}$, and $\sigma \lambda'_A$ is a prime of A in the same $\mathcal{G}$-orbit as λ'_A. Therefore, letting $\lambda_A := \sigma \lambda'_A$ we see

$$\mathrm{Frob}_{\lambda_A} = \mathrm{Frob}_{\sigma \lambda'_A} = \sigma \mathrm{Frob}_{\lambda_A} \sigma^{-1} = h. \qquad \square$$

We now apply lemma 4.10 to the following field diagram of Galois extensions of number fields
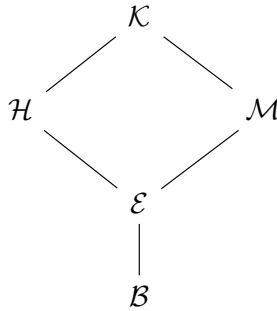
where $\mathcal{E}$ is a CM extension of the totally real $\mathcal{B}$, $\mathcal{H}$ is the Hilbert class field of $\mathcal{E}$, and $\mathcal{H}/\mathcal{E}$ and $\mathcal{M}/\mathcal{E}$ are linearly disjoint. We set $\mathcal{K} = \mathcal{H}\mathcal{M}$ and assume $\mathcal{K}/\mathcal{B}$ is Galois. Via Galois restriction we identify $\mathrm{Gal}(\mathcal{K}/\mathcal{M}) = \mathrm{Gal}(\mathcal{H}/\mathcal{E})$. If $\sigma \in \mathrm{Gal}(\mathcal{H}/\mathcal{E})$ we let $\tilde{\sigma} \in \mathrm{Gal}(\mathcal{K}/\mathcal{M})$ denote its unique lift. Let $\mathrm{Cl}_\mathcal{E}$ denote the ideal class group of $\mathcal{E}$, then the Artin map of

class field theory gives a $\mathrm{Gal}(\mathcal{E}/\mathcal{B})$-equivariant isomorphism

$$\rho : \mathrm{Cl}_{\mathcal{E}} \longrightarrow \mathrm{Gal}(\mathcal{H}/\mathcal{E}).$$

For any ideal class $\widehat{\mathfrak{c}} \in \mathrm{Cl}_{\mathcal{E}}$ let $\sigma_{\mathfrak{c}} := \rho(\mathfrak{c})$, where $\sigma_{\mathfrak{c}}$ is the product of the Frobenii associated to the fractional ideal $\mathfrak{c}$ representing the class $\widehat{\mathfrak{c}}$. We lift $\sigma_{\mathfrak{c}}$ to $\tilde{\sigma}_{\mathfrak{c}} \in \mathrm{Gal}(\mathcal{K}/\mathcal{M})$ via Galois restriction. The Tchebotarev density theorem, applied to the extension $\mathcal{K}/\mathcal{B}$, ensures the infinitude of unramified primes $\mathfrak{P}$ of $\mathcal{K}$ satisfying $[\mathrm{Frob}_{\mathfrak{P}}] = [\sigma_{\mathfrak{c}}]$, where $[\cdot]$ denotes $\mathrm{Gal}(\mathcal{K}/\mathcal{B})$-conjugacy classes. Let $\mathfrak{P}'$ be such an unramified prime. Since $\mathrm{Gal}(\mathcal{K}/\mathcal{M}) \trianglelefteq \mathrm{Gal}(\mathcal{K}/\mathcal{B})$, we can choose a prime $\mathfrak{P}$ of $\mathcal{K}$ such that $[\mathrm{Frob}_{\mathfrak{P}}] = [\mathrm{Frob}_{\mathfrak{P}'}]$ and $\mathrm{Frob}_{\mathfrak{P}} = \tilde{\sigma}_{\mathfrak{c}}$ as elements. Let $\mathfrak{p} := \mathfrak{P} \cap \mathcal{B}$ be the prime of $\mathcal{B}$ lying below $\mathfrak{P}$. Since $\sigma_{\mathfrak{c}} \in \mathrm{Gal}(\mathcal{K}/\mathcal{M})$ we have

$$\mathrm{res}_{\mathcal{M}}(\mathrm{Frob}_{\mathfrak{P}}) = \mathrm{id},$$

and therefore $\mathfrak{p}$ splits completely in $\mathcal{M}/\mathcal{B}$. Furthermore, if $\mathfrak{p}'$ is a prime of $\mathcal{H}$ lying below $\mathfrak{P}$, then

$$\mathrm{res}_{\mathcal{H}}(\mathrm{Frob}_{\mathfrak{P}}) = \mathrm{Frob}_{\mathfrak{p}'} = \sigma_{\mathfrak{c}},$$

but $\mathrm{Frob}_{\mathfrak{p}'} = \rho(\mathfrak{p})$, therefore

$$\rho(\mathfrak{c}) = \sigma_{\mathfrak{c}} = \mathrm{Frob}_{\mathfrak{p}'} = \rho(\mathfrak{p}),$$

and therefore, from bijectivity of ρ, $\widehat{\mathfrak{c}} = \widehat{\mathfrak{p}}$ as ideal classes. Consequently, the linear disjointness of the above Galois extensions facilitated the construction of a prime $\mathfrak{p}$ of $\mathcal{E}$ which splits completely in $\mathcal{M}/\mathcal{E}$ and whose ideal class is equivalent to a pre-determined class $\widehat{\mathfrak{c}}$. Let $\tilde{\mathfrak{p}} := \mathfrak{p} \cap \mathcal{B}$ be the prime of $\mathcal{B}$ lying below $\mathfrak{p}$. From section 4.4, there is a cyclic extension $\mathcal{B}_{\tilde{\mathfrak{p}}}/\mathcal{B}$ which is totally ramified at $\tilde{\mathfrak{p}}$, and which contains no unramified subextensions.

Let $\widehat{\mathfrak{c}}_1, \ldots, \widehat{\mathfrak{c}}_r$ be given ideal classes of $\mathrm{Cl}_{\mathcal{E}}$ and suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_{r-1}$ are primes of $\mathcal{B}$ such that for each $1 \leq k \leq r - 1$,

i.) $\mathfrak{p}_k$ splits completely in $\mathcal{M}\mathcal{B}_{\mathfrak{p}_1 \ldots \mathfrak{p}_{k-1}}/\mathcal{B}$

ii.) $\widehat{\mathfrak{p}}_k = \widehat{\mathfrak{c}}_k$ as ideal classes.

where the subscript $\mathfrak{p}_1 \ldots \mathfrak{p}_{k-1}$ denotes the compositum with the field
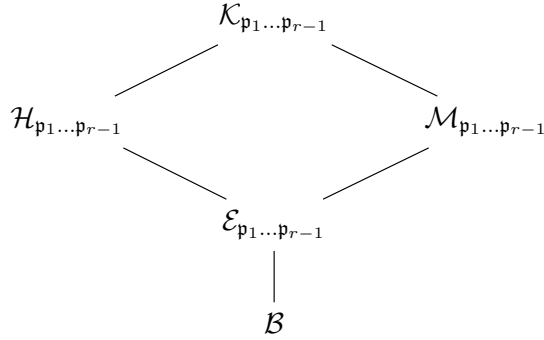
$$\mathcal{B}_{\mathfrak{p}_1 \ldots \mathfrak{p}_{k-1}} = \mathcal{B}_{\mathfrak{p}_1} \ldots \mathcal{B}_{\mathfrak{p}_{k-1}}.$$

To construct the prime $\mathfrak{p}_r$ satisfying

i.) $\mathfrak{p}_r$ splits completely in $\mathcal{MB}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}/\mathcal{B}$

ii.) $\widehat{\mathfrak{p}}_r = \widehat{\mathfrak{c}}_r$ as ideal classes,

we consider the diagram

$$
\begin{array}{ccc}
& \mathcal{K}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}} & \\
& & \\
\mathcal{H}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}} & & \mathcal{M}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}} \\
& & \\
& \mathcal{E}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}} & \\
& & \\
& \mathcal{B} &
\end{array}
$$

and apply the Tchebotarev density theorem to $\mathcal{K}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}/\mathcal{B}$ to obtain an unramified prime $\mathfrak{P}_r$ of $\mathcal{K}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}$ such that, if $\rho(\mathfrak{c}_r) = \sigma_{\mathfrak{c}_r}$ with $\tilde{\sigma}_{\mathfrak{c}_r} \in \mathrm{Gal}(K_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}/\mathcal{M}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}})$, then $\mathrm{Frob}_{\mathfrak{P}_r} = \tilde{\sigma}_{\mathfrak{c}_r}$ as elements. Letting $\mathfrak{p}_r := \mathfrak{P}_r \cap \mathcal{B}$

$$
\mathrm{res}_{\mathcal{M}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}}(\mathrm{Frob}_{\mathfrak{P}_r}) = \mathrm{id},
$$

and therefore $\mathfrak{p}_r$ splits completely in both $\mathcal{B}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}/\mathcal{B}$ and $\mathcal{M}/\mathcal{B}$. Since $\mathcal{B}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}/\mathcal{B}$ is totally ramified at $\mathfrak{p}_k$ for all $1 \leq k \leq r-1$, the complete splitting of $\mathfrak{p}_r$ in $\mathcal{B}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}$ ensures $\mathfrak{p}_r$ is distinct from $\mathfrak{p}_1, \ldots, \mathfrak{p}_{r-1}$. Furthermore, if $\mathfrak{p}'_r$ denotes a prime of $\mathcal{H}_{\mathfrak{p}_1\ldots\mathfrak{p}_{r-1}}$ lying below $\mathfrak{P}_r$, then

$$
\rho(\mathfrak{c}_r) = \sigma_{\mathfrak{c}_r} = \mathrm{Frob}_{\mathfrak{p}'_r} = \rho(\mathfrak{p}_r),
$$

hence $\widehat{\mathfrak{c}}_r = \widehat{\mathfrak{p}}_r$ as ideal classes.

The specific Frobenius conditions of splitting and equality of ideal classes associated to the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are specific to our desired situation. However, one could just as easily impose different Frobenius conditions on the $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ since the linear disjointness of the above extensions would ensure that the various conditions are simultaneously satisfied. We will impose different Frobenius conditions on primes lying in certain extensions, however, the argument that these conditions are simultaneously satisfied is the same as above.

4.5.1 Specific Tchebotarev Conditions

Let $p > 2$ be a prime and let L/k be a finite abelian extension of number fields where k is totally real, L is CM, and $G' := \mathrm{Gal}(L/k) = \Delta \times P$ where Δ has order coprime to p and P is the p-primary component of G'. We are particularly interested in the case when L is a layer in the cyclotomic $\mathbb{Z}_p$-extension of an abelian CM extension field F over k. Let $S \supset S_{\mathrm{ram}}(L/k) \cup S_{k,p} \cup S_\infty$ and T a finite set of finite primes of k such that $T \cap S = \varnothing$ and $L_{T_L}^\times$ is torsion free. We let $G_m := \mathrm{Gal}(L/k)$ where the subscript m is suggestive that $L = F_m$, the m-th layer in the cyclotomic $\mathbb{Z}_p$-extension of F.

Let $r \in \mathbb{Z}_{\geq 1}$ and fix $N > 0$, and consider ideal classes $\widehat{v}_1, \ldots, \widehat{v}_r \in (\mathrm{Cl}_{L,T} \otimes_\mathbb{Z} \mathbb{Z}_p)^-$ and elements $\widehat{f}_1, \ldots, \widehat{f}_r \in (L_T^\times / L_T^{\times p^N})^-$ as in section 4.3 above. For each $i = 1, 2, \ldots, r$, let f_i be a lift of $\widehat{f}_i$ to $(L_T^\times)^-$, and set $\mathcal{F} = \{(f_i^\sigma)^{1/p^N} : i = 1, 2, \ldots, r, \text{ for all } \sigma \in G'\}$, with $\widehat{\mathcal{F}}$ denoting reduction of $\mathcal{F}$ under the projection $(L_T^\times)^- \to (L_T^\times / L_T^{\times p^N})^-$.

Lemma 4.11. *The set $\widehat{\mathcal{F}}$ forms a $\mathbb{Z}/p^N\mathbb{Z}$-linearly independent subset of $(L_T^\times / L_T^{\times p^N})^-$.*

Proof of Lemma 4.11. Suppose $\widehat{f}_1^{\,\alpha_1} \widehat{f}_2^{\,\alpha_2} \ldots \widehat{f}_n^{\,\alpha_n} = \widehat{1}$, and lift this relation to $(L_T^\times)^-$ so that there is a $g \in (L_T^\times)^-$ with $f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_n^{\alpha_n} = g^{p^N}$. Applying the divisor map yields

$$\sum_{i=1}^n \alpha_i \mathrm{div}(f_i) = p^N \mathrm{div}(g),$$

and therefore $p^N \mathrm{div}(g) \in \bigoplus_{i=1}^n \mathbb{Z}_p[G_m]^- v_i$ so that $\mathrm{div}(g) \in \bigoplus_{i=1}^n \mathbb{Z}_p[G_m]^- v_i$, and therefore $g \in (L_T^\times \otimes_\mathbb{Z} \mathbb{Z}_p)^-$, so

$$g = f_1^{\beta_1} f_2^{\beta_2} \ldots f_n^{\beta_n}.$$

Therefore,

$$g^{p^N} = f_1^{p^N \beta_1} f_2^{p^N \beta_2} \ldots f_n^{p^N \beta_n} = f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_n^{\alpha_n},$$

and by uniqueness, $\alpha_i = p^N \beta_i$ for all $1 \leq i \leq n$, whereby $\widehat{\alpha_i} = \widehat{0}$, giving the desired linear independence. $\qquad\square$

For each $\sigma \in G'$ and $1 \leq i \leq r$ we have extensions $L(\mu_{p^N}, (f_i^\sigma)^{1/p^N})/L(\mu_{p^N})$, such that $L(\mu_{p^N}, \mathcal{F})$ is the compositum of all of these extensions. The linear independence of the set $\mathcal{F}$ ensures that these intermediate extensions are all linearly disjoint, and therefore, we can apply our general Tchebotarev argument to the extensions $L(\mu_{p^N}, \mathcal{F})/L(\mu_{p^N}, (f_i^\sigma)^{1/p^N})/L(\mu_{p^N})$ for all $\sigma \in G$ and $1 \leq i \leq r$.

We aim to show the existence of finite primes $\lambda_1, \ldots, \lambda_r$ of L, which satisfy certain Frobenius conditions. We utilize the same subscript notation as in 4.6.1 to denote composita of fields with totally ramified fields constructed from the λ_i.

Proposition 4.12. *With $(L/k, S, T, p)$ as above and $r \in \mathbb{Z}_{\geq 1}$, there exist primes $\lambda_1, \lambda_2, \ldots, \lambda_r$ of L satisfying the following*

1. *For each $1 \leq i \leq r$, if $\tilde{\lambda}_i := \lambda_i \cap k$, then $\tilde{\lambda}_i$ splits completely in $L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$.*

2. *For each $1 \leq i \leq r$, we have $\widehat{\tilde{\lambda}}_i = \widehat{v}_i$ as elements of $(\mathrm{Cl}_{L,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$.*

3. *For each $1 \leq i \leq r$, the prime $\tilde{\lambda}_i$ is inert in $L(\mu_{p^N}, (f_i^e)^{1/p^N})_{\lambda_1 \ldots \lambda_{i-1}}/L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$ and splits completely in $L(\mu_{p^N}, (f_k^\tau)^{1/p^N})_{\lambda_1 \ldots \lambda_{i-1}}/L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$ for all $1 \leq k \leq r$ with $k \neq i$ and for all $e \neq \tau \in G'$.*

The proof of the proposition will proceed by induction on the number of primes.

Proof of Proposition 4.12. To construct the prime λ_1 we consider the following field diagram

where $H_L^{p,*}$ is the subfield of the T-ray class field of L such that $\mathscr{H} := \mathrm{Gal}(H_L^{p,*}/L) = (\mathrm{Cl}_{L,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$. The superscript $\cdot^*$ denotes the need to pass to the non-Teichmüller

component of $\mathscr{H}$ in order to ensure disjointness of the extensions $H_L^{p,*}(\mu_{p^N})/L(\mu_{p^N})$ and $L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})/L(\mu_{p^N})$. If $e_\omega(\mathrm{Cl}_{L,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- = \{0\}$, then we would not need to restrict to the non-Teichmüller component.

Step 1: We first justify the labeling of Galois groups in the above diagram by showing the linearly disjointness of the associated extensions. Let $j \in G'$ denote the unique complex conjugation automorphism of L, and let $G'_N := \mathrm{Gal}(L(\mu_{p^N})/k)$. Then j acts on $\mathrm{Gal}(L(\mu_{p^N})/L)$ via lift and conjugation, i.e., considering the exact sequence

$$1 \longrightarrow \mathrm{Gal}(L(\mu_{p^N})/L) \longrightarrow G'_N \longrightarrow G' \longrightarrow 1$$

we lift $j \in G'$ to $\tilde{j} \in G'_N$, and set

$$j \cdot \sigma := \tilde{j}\sigma\tilde{j}^{-1},$$

for any $\sigma \in \mathrm{Gal}(L(\mu_{p^N})/L)$. However, since G'_N is abelian, we have $j \cdot \sigma = 1$ for all $\sigma \in \mathrm{Gal}(L(\mu_{p^N})/L)$, i.e. $\mathrm{Gal}(L(\mu_{p^N})/L)$ lives on the $+$-eigenspace for the action of j. However, by definition of $(\mathrm{Cl}_{L,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$, the action of j is by -1. Therefore, the extensions $H_L^{p,*}/L$ and $L(\mu_{p^N})/L$ are linearly disjoint. We let $H_L^{p,*}(\mu_{p^N}) := H_L^{p,*}L(\mu_{p^N})$ and identify

$$\mathrm{Gal}(H_L^{p,*}(\mu_{p^N})/L(\mu_{p^N})) \xrightarrow{\sim} \mathscr{H},$$

via Galois restriction.

The two extensions $L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})/L(\mu_{p^N})$ and $L(\mu_{p^N}, \mathcal{F})/L(\mu_{p^N})$ are Kummer extensions, and therefore come endowed with perfect G'_N-equivariant pairings, namely

$$\mathrm{Gal}(L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})/L(\mu_{p^N})) \times \mathcal{O}_k^\times/(\mathcal{O}_k^\times \cap L(\mu_{p^N})^{\times p^N}) \longrightarrow \mu_{p^N},$$

and

$$\mathrm{Gal}(L(\mu_{p^N}, \mathcal{F})/L(\mu_{p^N})) \times \langle \mathcal{F} \rangle/(\langle \mathcal{F} \rangle \cap L(\mu_{p^N})^{\times p^N}) \longrightarrow \mu_{p^N}.$$

The above pairings give isomorphisms

$$\mathrm{Gal}(L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})/L(\mu_{p^N})) \simeq \mathrm{Hom}(\mathcal{O}_k^\times/(\mathcal{O}_k^\times \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N})$$

and

$$\mathrm{Gal}(L(\mu_{p^N}, \mathcal{F})/L(\mu_{p^N})) \simeq \mathrm{Hom}(\langle \mathcal{F} \rangle/(\langle \mathcal{F} \rangle \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N}),$$

respectively. Writing $G'_N = \Delta'_N \times P'_N$, where the order of Δ'_N is coprime to $p > 2$ and P'_N is the p-primary component of $\mathrm{Gal}(L(\mu_{p^N})/k)$, we have a natural action of Δ'_N on each Hom-set above, namely

$$(\delta \cdot f)(x) := \delta f(\delta^{-1} x),$$

for all $\delta \in \Delta'_N$, and f and x in the respective Hom-sets above. However, since any $\delta \in \Delta'_N$ fixes k, and therefore $\mathcal{O}_k$, we see that Δ'_N acts on $\mathrm{Hom}(\mathcal{O}_k^\times/(\mathcal{O}_k^\times \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N})$ by

$$(\delta \cdot f)(x) = \delta f(x) = \omega(\delta) f(x),$$

where $\omega : \Delta'_N \longrightarrow \mathbb{Z}_p^\times$ is the Techmüller character associated to Δ'_N. Furthermore, the action of $j_N \in \Delta'_N$ on $\mathrm{Hom}(\langle \mathcal{F} \rangle/(\langle \mathcal{F} \rangle \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N})$ is given by

$$(j_N \cdot f)(x) = j_N f(j_N x) = \omega(j_N) f(-x) = -f(-x) = f(x),$$

and therefore, j_N acts as $+1$ on $\mathrm{Hom}(\langle \mathcal{F} \rangle/(\langle \mathcal{F} \rangle \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N})$ and acts as -1 on $\mathrm{Hom}(\mathcal{O}_k^\times/(\mathcal{O}_k^\times \cap L(\mu_{p^N})^{\times p^N}), \mu_{p^N})$, therefore, the two extensions $L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})/L(\mu_{p^N})$ and $L(\mu_{p^N}, \mathcal{F})/L(\mu_{p^N})$ are linearly disjoint. As before, we let

$$L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N}, \mathcal{F}) := L(\mu_{p^N}, \mathcal{F}) L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})$$

denote the compositum of the two extensions, and identify

$$\mathrm{Gal}(L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N}, \mathcal{F})/L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})) \xrightarrow{\sim} \mathscr{L}$$

via Galois restriction.

Let $\mathcal{G} := \mathrm{Gal}(H_L^{p,*}(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N}, \mathcal{F})/k)$ and choose a prime $\mathcal{L}_1$ in $H_L^{p,*}(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N}, \mathcal{F})$ whose Frobenius $\sigma_{\mathcal{L}_1}$ satisfies the following

- $\sigma_{\mathcal{L}_1} = \sigma_{v_1}$ where σ_{v_1} is the Frobenius associated to the ideal class $\widehat{v_1}$ via the Artin map.

- $\mathrm{res}_{L(\mu_{p^N})}(\mathcal{L}_1) = \mathrm{id}$.

- $\langle \mathrm{res}_{L(\mu_{p^N},(f_1^e)^{1/p^N})}(\mathcal{L}_1)\rangle = \mathrm{Gal}(L(\mu_{p^N},(f_1^e)^{1/p^N})/L(\mu_{p^N}))$

- $\mathrm{res}_{L(\mu_{p^N},(f_i^\sigma)^{1/p^N})}(\mathcal{L}_1) = \mathrm{id}$ for all $2 \le i \le r$ and $e \ne \sigma \in G'$.

Since the extensions in question are all linearly independent, the Frobenius conditions are satisfied simultaneously. We then let $\lambda_1 := L \cap \mathcal{L}_1$, and $\tilde{\lambda}_1 = k \cap \mathcal{L}_1$ and observe that $\tilde{\lambda}_1$ splits completely in $L(\mu_{p^N})$ since its Frobenius is trivial. Moreover, $\widehat{\lambda_1} = \widehat{v}_1$ as was shown in our general situation in 4.6.1. To show the third condition we consider the diagram

$$
\begin{array}{ccccc}
 & & L(\mu_{p^N}, \mathcal{F}) & & \\
 & \diagup & & \diagdown & \\
L(\mu_{p^N}, (f_1^e)^{1/p^N}) & & & & \mathcal{C}(e \ne \sigma \in G', 2 \le i \le r) \\
 & \diagdown & & \diagup & \\
 & & L(\mu_{p^N}) & & \\
 & & \mid & & \\
 & & k & &
\end{array}
$$

where

$$
\mathcal{C}(e \ne \sigma \in G', 2 \le i \le r) := \prod_{\substack{e \ne \sigma \in G' \\ 2 \le i \le r}} L(\mu_{p^N}, (f_i^\sigma)^{1/p^N}),
$$

denotes the compositum. We now apply the argument in 4.6.1 with the above Frobenius conditions.

Inductive Step: We now give the inductive step in the construction of the λ_i. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}$ have been constructed. We therefore have linearly independent, totally ramified cyclic extensions $k(\lambda_1)^*, \ldots, k(\lambda_{i-1})^*$ of k of order p^N. Letting

- $\mathcal{K} := H_L^{p,*}(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N}, \mathcal{F})_{\lambda_1 \ldots \lambda_{i-1}}$

- $\mathcal{H} = (H_L^{p,*})_{\lambda_1 \ldots \lambda_{i-1}}$

- $\mathcal{M} = L(\mu_{p^N}, \mathcal{O}_k^{\times 1/p^N})_{\lambda_1 \ldots \lambda_{i-1}}$

- $\mathcal{E} = L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$

- $\mathcal{B} = k$

we apply the argument in 4.6.1 to obtain the desired Frobenius conditions, namely, choose a prime $\mathcal{L}_i$ of $\mathcal{K}$ whose Frobenius $\sigma_{\mathcal{L}_i}$ satisfies

- $\sigma_{\mathcal{L}_i} = \sigma_{v_i}$ where σ_{v_i} is the Frobenius associated to the ideal class $\widehat{v_i}$ via the Artin map.

- $\mathrm{res}_{L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}}(\mathcal{L}_i) = \mathrm{id}$

- $\langle \mathrm{res}_{L(\mu_{p^N}, (f_i^e)^{1/p^N})}(\mathcal{L}_i) \rangle = \mathrm{Gal}(L(\mu_{p^N}, (f_i^e)^{1/p^N})/L(\mu_{p^N}))$

- $\mathrm{res}_{L(\mu_{p^N}, (f_j^\sigma)^{1/p^N})}(\mathcal{L}_i) = \mathrm{id}$ for all $2 \leq j \neq i \leq r$ and $e \neq \sigma \in G'$.

It is important (for linear disjointness considerations) to ensure that for all $1 \leq i \leq r$ the primes $\mathcal{L}_i$ do not divide primes in S, T, or primes dividing the f_j for all $1 \leq j \leq r$. This can be achieved since these sets are all finite and the Tchbotarev density theorem asserts the existence of infinitely many primes satisfying certain conjugacy conditions. Letting $\lambda_i := L \cap \mathcal{L}_i$ and $\tilde{\lambda}_i = k \cap \mathcal{L}_i$ we observe that $\tilde{\lambda}_i$ splits completely in $L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$ hence λ_i splits completely in $L(\mu_{p^N})$ and is distinct from $\lambda_1, \ldots, \lambda_{i-1}$. Utilizing the general situation in 4.6.1 we have $\widehat{\lambda}_i = \widehat{v}_i$. Finally, considering the diagram

$$L(\mu_{p^N}, \mathcal{F})_{\lambda_1 \ldots \lambda_{i-1}}$$

$$L(\mu_{p^N}, (f_i^e)^{1/p^N})_{\lambda_1 \ldots \lambda_{i-1}} \qquad \mathcal{C}(e \neq \sigma \in G', 2 \leq j \leq r)_{\lambda_1 \ldots \lambda_{i-1}}$$

$$L(\mu_{p^N})_{\lambda_1 \ldots \lambda_{i-1}}$$

$$k$$

and using the general argument in 4.6.1 we obtain the desired Frobenius conditions on $\mathcal{L}_i$. □

With the Tchebotarev conditions satisfied, we can now state and prove a large portion of the generalized conjecture.

4.6 The Generalized Conjecture and Proof Under Additional Conditions

Let $p > 2$ be a prime, $r \in \mathbb{Z}_{\geq 1}$, and $N > 0$ some large fixed positive integer. Let F/k be a finite abelian extension of number fields of Galois group G where F is CM and k is totally real. Let F_∞/F denote the cyclotomic $\mathbb{Z}_p$-extension of F with Galois group denoted Γ_F and let $\mathcal{G}_F = \mathrm{Gal}(F_\infty/k)$. Let L_0/k be a finite abelian extension of number fields with $G_0 := \mathrm{Gal}(L_0/k)$, such that L_0/k satisfies the following

(i.) L_0/k contains no unramified subextensions

(ii.) $G_0 = \underbrace{\mathbb{Z}/p^N\mathbb{Z} \times \ldots \times \mathbb{Z}/p^N\mathbb{Z}}_{r}$ with fixed generators $G_0 = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle \times \ldots \times \langle \sigma_r \rangle$.

(iii.) L_0/k is linearly disjoint from F_∞/k.

Let $L = L_0 F$ denote the compositum of L_0 and F, and let L_∞/L denote the cyclotomic $\mathbb{Z}_p$-extension of L with Galois group denoted Γ_L and let $\mathcal{G}_L = \mathrm{Gal}(L_\infty/k)$. Let S_k and T_k denote finite sets of primes of k such that $S_k \supset S_{\mathrm{ram}}(L_0/k) \cup S_{k,p}$ and $T_k \cap S_k = \varnothing$, where $S_{\mathrm{ram}}(L_0/k)$ denotes the set of primes of k which ramify in L_0, and $S_{k,p}$ denotes the set of primes of k lying above the p-adic valuation of $\mathbb{Q}_p$. Let $\mathcal{S}_L$ (resp. $\mathcal{T}_L$) denote the set(s) of primes of L_∞ lying above S_k (resp. T_k). Since L_0/k is linearly disjoint from F_∞/k we identify $\mathrm{Gal}(L/F)$ and $\mathrm{Gal}(L_\infty/F_\infty)$ with G_0 via Galois restriction. The underlying field diagram of interest is

$$
\begin{array}{ccc}
F_\infty & \xrightarrow{\;G_0\;} & L_\infty \\
{\scriptstyle \Gamma_F}\Big| & & \Big|{\scriptstyle \Gamma_L} \\
F & \xrightarrow{\;G_0\;} & L \\
{\scriptstyle G}\Big| & & \Big| \\
k & \xrightarrow{\;G_0\;} & L_0.
\end{array}
$$

Associated to the data $(F_\infty/k, \mathcal{S}_F, \mathcal{T}_F)$ and $(L_\infty/k, \mathcal{S}_L, \mathcal{T}_L)$ we have the p-adic realizations of abstract 1-motives $T_p(\mathcal{M}_{\mathcal{S}_L, \mathcal{T}_L})^-$ and $T_p(\mathcal{M}_{\mathcal{S}_F, \mathcal{T}_F})^-$. Since S_k, T_k consist of primes of k, and $S_k \supset S_{\mathrm{ram}}(L_0/k)$, we have $\mathcal{S}_L \supset S_{\mathrm{ram}}(L_\infty/F_\infty)$ and $\mathcal{S}_L, \mathcal{T}_L$ are G_0-invariant. The generalized conjecture is then stated as follows

Conjecture 4.13. *Let $(F/k, S, T, p, r)$ be as above with F/k a semi-nice extension, and*

fix $N > 0$ a large positive integer. If $\mu = 0$, then for any $i \geq 0$

$$\operatorname{Fit}^i_{\mathbb{Z}_p[[\mathcal{G}_F]]^-/p^N\mathbb{Z}_p[[\mathcal{G}_F]]^-} \left(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/p^N T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-\right) = \mathscr{F}^i,$$

where $\mathscr{F}^i$ is the $\mathbb{Z}_p[[\mathcal{G}_F]]^-$-ideal generated by $\Theta_{F_\infty/k}(0)$ and $\delta_{i_1,\dots,i_r}(\Theta_{L_\infty/k}(0))$ where

i.) L ranges over all abelian fields such that $L \cap F_\infty = F$ and

$$\operatorname{Gal}(L/k) \simeq \underbrace{\mathbb{Z}/p^N\mathbb{Z} \times \dots \times \mathbb{Z}/p^N\mathbb{Z}}_{r},$$

ii.) $(i_1, i_2, \dots, i_r)$ ranges over integers satisfying $i_1 + \dots + i_r \leq i$.

It should be mentioned that one inclusion of conjecture 4.13 holds only under the $\mu = 0$ assumption.

Theorem 4.14. *Let $(F/k, S, T, p)$ be as in chapter 3. If $\mu = 0$, then for all $i \geq 0$*

$$\mathscr{F}^i \subseteq \operatorname{Fit}^i_{\mathbb{Z}_p[[\mathcal{G}_F]]^-/p^N\mathbb{Z}_p[[\mathcal{G}_F]]^-} \left(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/p^N T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-\right),$$

where $\mathscr{F}^i$ is given in conjecture 4.13.

Proof of Theorem 4.14. We have a natural morphism of 1-motives, $\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F} \to \mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L}$, which induces a natural $\mathbb{Z}_p$-module homomorphism $T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F}) \longrightarrow T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^{G_0}$. Considering the minus-parts of the above p-adic realizations, and invoking [15, Prop. 4.2], we have a $\mathbb{Z}_p$-module isomorphism

$$T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^- \simeq (T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)^{G_0}. \tag{4.22}$$

Furthermore, since $T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-$ is G_0-cohomologically trivial [15, Thm 4.6]

$$T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-_{G_0} \simeq T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-, \tag{4.23}$$

and therefore, combining (4.22) and (4.23), we obtain a $\mathbb{Z}_p$-module isomorphism

$$(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)^{G_0} \simeq (T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)_{G_0}.$$

Let $\Lambda_F := \mathbb{Z}_p[[\mathcal{G}_F]] \simeq \mathbb{Z}_p[G][[\Gamma_F]]$ and $\Lambda_L := \mathbb{Z}_p[[\mathcal{G}_L]]$ so that the linear disjointness of L_0/k and F_∞/k gives $\Lambda_L = \Lambda_F[G_0]$. We let $\Lambda_F^- := \mathbb{Z}_p[G]^-[[\Gamma_F]]$ and $\Lambda_L^- := \Lambda_F^-[G_0]$.

Corresponding to our choice of generators $G_0 \simeq \langle \sigma_1 \rangle \times \ldots \times \langle \sigma_r \rangle$ is the canonical isomorphism

$$\Lambda_L \longrightarrow \frac{\Lambda_F[S_1, \ldots, S_r]}{\langle (S_i + 1)^{p^N} - 1 : i = 1, \ldots, r \rangle} \tag{4.24}$$

$$\sigma_i \mapsto \widehat{S_i + 1}.$$

Consider the augmentation exact sequence

$$0 \longrightarrow I_{G_0} \longrightarrow \Lambda_L^- \xrightarrow{\text{aug}} \Lambda_F^- \longrightarrow 0 \tag{4.25}$$

where aug is the augmentation map and $I(G_0) := \langle \sigma - 1 : \sigma \in G_0 \rangle$ is the augmentation ideal of G_0. View Λ_F^- in Λ_L^- via $\text{aug}|_{\Lambda_F} = \text{id}$ so that there is a splitting $\Lambda_L^- \simeq \Lambda_F^- \oplus I(G_0)$. The finite generatedness of the Λ_F^--module $(T_p(\mathcal{M}_{\mathcal{S}_L, \mathcal{T}_L})^-)_{G_0}$, along with sequence (4.25) combine to give the diagram

where $\pi^{-1}(Y) = Y \oplus I(G_0)^{\oplus n}$, and the sequence

$$0 \longrightarrow \pi^{-1}(Y) \longrightarrow (\Lambda_L^-)^{\oplus n} \longrightarrow (T_p(\mathcal{M}_{\mathcal{S}_L, \mathcal{T}_L})^-)_{G_0} \longrightarrow 0 \tag{4.26}$$

is exact. Applying the second property of Lemma 4.1 to (4.26)

$$\text{Fit}^0_{\Lambda_L^-}((T_p(\mathcal{M}_{\mathcal{S}_L, \mathcal{T}_L})^-)_{G_0}) \subset \sum_{i=0}^{n} \text{Fit}^i_{\Lambda_F^-}((T_p(\mathcal{M}_{\mathcal{S}_L, \mathcal{T}_L})^-)_{G_0}) I(G_0)^i. \tag{4.27}$$

The first property of Lemma 4.1 applied to the G_0-coinvariants surjection

$$T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^- \twoheadrightarrow (T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)_{G_0}$$

gives

$$\text{Fit}^0_{\Lambda_L^-}((T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)) \subset \text{Fit}^0_{\Lambda_L^-}((T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)_{G_0}), \tag{4.28}$$

and therefore, combining (4.27) and (4.28)

$$\text{Fit}^0_{\Lambda_L^-}((T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)) \subset \sum_{i=0}^n \text{Fit}^i_{\Lambda_F^-}((T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)_{G_0})I(G_0)^i. \tag{4.29}$$

The data $(L_\infty/k, \mathcal{S}_L, \mathcal{T}_L, p)$ satisfies the hypotheses of the Equivariant Main Conjecture of Popescu-Greither [15, Thm. 5.6], namely

$$\text{Fit}^0_{\Lambda_L^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-) = (\Theta^{(\infty)}_{\mathcal{S}_L,\mathcal{T}_L}(0)), \tag{4.30}$$

where $\Theta^{(\infty)}_{\mathcal{S}_L,\mathcal{T}_L}(0) := \Theta^{(\infty)}_{L_\infty/k,\mathcal{S}_L,\mathcal{T}_L}(0)$ is the equivariant p-adic L-function associated to the data $(L_\infty/k, \mathcal{S}_L, \mathcal{T}_L, p)$. Combining (4.29) and (4.30) yields

$$\Theta^{(\infty)}_{\mathcal{S}_L,\mathcal{T}_L}(0) \in \sum_{i=0}^n \text{Fit}^i_{\Lambda_F^-}((T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-)_{G_0})I(G_0)^i. \tag{4.31}$$

Using properties of Fitting ideals under surjective ring morphisms, and using the notation

$$T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N) := T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/p^N T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-$$

we obtain

$$\text{Fit}^0_{\Lambda_L^-/p^N\Lambda_L^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N)) \subset \sum_{i=0}^n \text{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N))I_G^i.$$

Using the projection form of the main conjecture with $\widehat{\Theta_L} := \Theta^{(\infty)}_{L_\infty,\mathcal{S}_L,\mathcal{T}_L}(0)$ modulo p^N,

$$\widehat{\Theta_L} \in \text{Fit}^0_{\Lambda_L^-/p^N\Lambda_L^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N)),$$

therefore, since $\widehat{\Theta_L} \in \Lambda_L^-/p^N\Lambda_L^-$, we have

$$\widehat{\Theta_L} = \sum_{i=0}^{n} \sum_{i_1+i_2+\ldots+i_r=i} \delta_{i_1,i_2,\ldots,i_r}(\widehat{\Theta_L})S_1^{i_1}\ldots S_r^{i_r},$$

but, on the other hand $\widehat{\Theta_L} \in \sum_{i=0}^{n} \mathrm{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N))I_G^i$, hence

$$\widehat{\Theta_L} = \sum_{i=0}^{n} \sum_{i_1+\ldots+i_r=i} f_{i_1,\ldots,i_r}(\widehat{\Theta_L})S_1^{i_1}\ldots S_r^{i_r}$$

with

$$f_{i_1,\ldots,i_r} \in \mathrm{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N))I_G^i$$

for all $i_1,\ldots,i_r \leq p^N$. By uniqueness of coefficients, we therefore obtain

$$\delta_{i_1,\ldots,i_r}(\widehat{\Theta_L}) \in \mathrm{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_L,\mathcal{T}_L})^-/(p^N))$$

for all i and for all $i_1+i_2+\ldots+i_r = i$. Therefore

$$\mathscr{F}^i \subseteq \mathrm{Fit}^i_{\mathbb{Z}_p[[\mathcal{G}_F]]^-/p^N\mathbb{Z}_p[[\mathcal{G}_F]]^-}\left(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/p^N T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-\right).$$

$\square$

Under additional hypotheses we can prove a large part of conjecture 4.13 namely

Theorem 4.15. *Let $(F/k, S, T, p, r)$ be as above with F/k a semi-nice extension, fix a large $N > 0$, and let ω be the Teichmüller character. Then, for any $i \geq 0$*

$$\mathrm{Fit}^i_{\mathbb{Z}_p[[\mathcal{G}_F]]^-/p^N\mathbb{Z}_p[[\mathcal{G}_F]]^-}\left((1-e_\omega)T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/p^N(1-e_\omega)T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-\right) = \mathscr{F}^i,$$

where $\mathscr{F}^i$ is the $\mathbb{Z}_p[[\mathcal{G}_F]]^-$-ideal generated by $\Theta_{F_\infty/k}(0)$ and $\delta_{i_1,\ldots,i_r}(\Theta_{L_\infty/k}(0))$ where

i.) L ranges over all abelian fields such that $L \cap F_\infty = F$ and

$$\mathrm{Gal}(L/k) \simeq \underbrace{\mathbb{Z}/p^N\mathbb{Z} \times \ldots \times \mathbb{Z}/p^N\mathbb{Z}}_{r},$$

ii.) $(i_1, i_2, \ldots, i_r)$ ranges over integers satisfying $i_1 + \ldots + i_r \leq i$.

In particular, the Teichmüller component is the only component of the conjecture left to prove. As mentioned in the Tchebotarev density section, the reason for restricting to the non-Teichmüller component was solely due to the necessity of linear disjointness of certain field extensions. Therefore, the Teichmüller condition in Theorem (4.15) is a consequence of the proof technique. Before giving the remaining inclusion of Theorem 4.15 we need a lemma from linear algebra which can be found in [41].

Lemma 4.16. *Let* $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be an* $n \times n$ *matrix with entries in a commutative ring* R, *where* A *is a* $d \times d$ *matrix. Let* $\tilde{E} := \mathrm{Adj}(E)$ *denote the adjugate (classically adjoint) of* E, *so that* $E\tilde{E} = \det(E)I_n$, *and write* $\tilde{E} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ *where* W *is a* $d \times d$ *matrix. Then,*

$$\det(W) = \det(E)^{d-1}\det(D).$$

Proof. The proof uses the following trick, let $E = (e_{i,j})$ and consider the integral domain $\mathbb{Z}[x_{i,j} : 1 \leq i, j \leq n]$ where the $x_{i,j}$ are formal variables which specialize to the $e_{i,j}$-entries via the ring homomorphism $\psi : \mathrm{M}_n(\mathbb{Z}[x_{i,j}]) \to \mathrm{M}_n(R)$ given by $\psi((x_{i,j})) = (e_{i,j})$. Then proving such a result for $\mathcal{E} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ and $\tilde{\mathcal{E}} = \begin{pmatrix} \mathcal{W} & \mathcal{X} \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix}$ in $\mathrm{M}_n(\mathbb{Z}[x_{i,j}])$, would yield the result for E and $\tilde{E}$ after applying ψ. By definition $\mathcal{E}\tilde{\mathcal{E}} = \det(E)I_n$, and therefore

$$\mathcal{A}\mathcal{W} + \mathcal{B}\mathcal{Y} = \det(E)I_d \qquad \text{and} \qquad \mathcal{C}\mathcal{W} + \mathcal{D}\mathcal{Y} = 0.$$

Setting $\mathcal{Q} = \det(\mathcal{D})\mathcal{A} - \mathcal{B}\tilde{\mathcal{D}}\mathcal{C}$ we see

$$\mathcal{Q}\mathcal{W} = \det(\mathcal{D})\det(\mathcal{E})I_d,$$

whereby, taking determinants gives

$$\det(\mathcal{Q})\det(\mathcal{W}) = (\det(\mathcal{D})\det(\mathcal{E}))^d. \tag{4.32}$$

A matrix computation shows

$$\begin{pmatrix} I_d & \mathcal{B}\tilde{\mathcal{D}} \\ 0 & I_{n-d} \end{pmatrix} \begin{pmatrix} \mathcal{Q} & 0 \\ 0 & I_{n-d} \end{pmatrix} \begin{pmatrix} I_d & 0 \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \det(\mathcal{D})\mathcal{A} & \det(\mathcal{D})\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix},$$

therefore, taking determinants yields

$$\det(\mathcal{D})\det(\mathcal{Q}) = \det(\mathcal{D})^d \det(\mathcal{E}). \tag{4.33}$$

Multiplying (4.32) by $\det(\mathcal{X})$ and combining the result with (4.33) we obtain

$$\det(\mathcal{D})^d \det(\mathcal{E})(\det(\mathcal{D})\det(\mathcal{E})^{d-1} - \det(\mathcal{W})) = 0,$$

in the integral domain $\mathbb{Z}[x_{i,j}]$. Since $\det(\mathcal{D}), \det(\mathcal{E}) \neq 0$, we conclude

$$\det(\mathcal{W}) = \det(\mathcal{E})^{d-1} \det(\mathcal{D}). \tag{4.34}$$

Applying the homomorphism ψ to (4.34) gives the desired result

$$\det(W) = \det(E)^{d-1} \det(D). \qquad \square$$

Proof of the reverse containment in Theorem 4.15. The reverse containment will be given by induction on the degree of the Fitting ideal. The finitely generated $\mathbb{Z}_p[[\mathcal{G}_F]]^-$-module $T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-$ has projective dimension $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}_F]]^-} T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^- \leq 1$. We therefore choose a presentation

$$0 \longrightarrow (\mathbb{Z}_p[[\mathcal{G}_F]]^-)^{\oplus n} \xrightarrow{\ t\ } (\mathbb{Z}_p[[\mathcal{G}_F]]^-)^{\oplus n} \longrightarrow T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^- \longrightarrow 0 \tag{4.35}$$

where bases are chosen such that the matrix representing the map t has determinant equal to $\Theta^{(\infty)}_{\mathcal{S}_F,\mathcal{T}_F}(0)$. We let $\Gamma_m := \Gamma/\Gamma^{p^m}$ denote the Galois group of the finite extension F_m/F in the cyclotomic $\mathbb{Z}_p$-extension of F. Taking Γ^{p^m}-coinvariants in (4.35) we obtain the exact sequence

$$\begin{array}{c} 0 \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]^-_{\Gamma^{p^m}} f_i \xrightarrow{\ \bar{t}\ } \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]^-_{\Gamma^{p^m}} v_i \\ \Big\downarrow \\ (\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \longrightarrow 0 \end{array} \tag{4.36}$$

where the v_i and f_i are chosen as follows. If $x_1, \ldots, x_n$ are generators for $T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-$, for each $1 \leq i \leq n$, let y_i denote the image of x_i in $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, hence the y_i are generators of $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. For each $1 \leq i \leq n$, we choose v_i a fractional ideal of $F_{m,T}$ such that v_i is completely split over k, i.e. if $u_i = v_i \cap k$, then u_i splits completely

in F_m, and such that the class of v_i in $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$ is y_i. We choose $f_1, \ldots, f_m \in (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$ such that

- The map $\bar{t} : \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^- f_i \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^- v_i$ is the $\mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^-$-linearization of the divisor map $\mathrm{div} : (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^- \longrightarrow \mathrm{Div}_{F_{m,T}}$. Therefore, we now label the map $\bar{t}$ as div.

- If $\overline{A} := (\alpha_{ij})$ is the matrix representing the map div, then $\det \overline{A} = \Theta_{F_m/k,S,T}(0)$.

- The set $\{f_i\}_{i=1,\ldots,m}$ consists of precisely those elements of $(F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$, whose images under div lie in the $\mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^-$-span of the v_i.

The idea of the proof is induction on the degree of the Fitting ideal. Since

$$\mathrm{Fit}^0_{\Lambda_F^-/p^N \Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N)) = \mathscr{F}^0,$$

the base of our induction is true. Therefore, we fix $1 \le i \le n$, and assume

$$\mathrm{Fit}^{i-1}_{\Lambda_F^-/p^N \Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N)) \subset \mathscr{F}^{i-1},$$

we then use this to show $\mathrm{Fit}^i_{\Lambda_F^-/p^N \Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N)) \subset \mathscr{F}^i$. Therefore, consider an $(n-i) \times (n-i)$ minor $\overline{A}_{n-i}$ of $\overline{A}$. After a suitable linear transformation, we may assume that our $(n-i) \times (n-i)$ minor sits in $\overline{A}$ as follows:

$$\left(\begin{array}{c|c} i \times i & * \\ \hline * & \overline{A}_{n-i \cdot} \end{array} \right)$$

The result of Popescu-Greither on the Brumer-Stark conjecture [15, Thm. 5.6] shows that $\Theta_{F_m/k,S,T}(0)$ annihilates $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$, and therefore, for each $1 \le i \le n$ we have

$$\Theta_{F_m/k,S,T}(0) v_i = \mathrm{div}(g_i)$$

for some $g_i \in (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$. Since $\mathrm{div}(g_i) \in \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^- v_i$, and by the choice of $\{f_i\}_{i=1,\ldots,n}$, we have $g_i \in \bigoplus_{i=1}^n \mathbb{Z}_p[[\mathcal{G}_F]]_{\Gamma^{p^m}}^- f_i$, and therefore, we write

$$g_i = \sum_{j=1}^n a_{ij} f_j,$$

and let

$$X := \left(\begin{array}{c|c} X_i & * \\ \hline * & (n-i) \times (n-i) \end{array} \right),$$

where X_i is the upper-left $i \times i$ minor of X. Since $\overline{A}X = \Theta_{F_m/k,S,T}(0)I_n$ and $\Theta_{F_m/k,S,T}(0)$ is a nonzero divisor, we obtain the relation

$$X = \overline{A}^*,$$

where $\overline{A}^*$ is the adjugate matrix of A. The linear algebra lemma (4.16) gives a relationship between $\det(X_i)$ and $\det(\overline{A}_i)$, namely

$$\det(X_i) = \Theta_{F_m/k,S,T}(0)^{i-1}\det(\overline{A}_i).$$

Recall in section 4.4 that in our construction of our finite abelian extensions of k, containing no unramified subextensions, we chose primes $\tilde{\lambda}_i$ of k, which split completely in a certain extension, and such that if $\lambda_i | \tilde{\lambda}_i$ is a prime of F_m lying above $\tilde{\lambda}_i$, then the classes of λ_i and v_i are the same in the class group. Since λ_i and v_i have the same ideal class, we get

$$\mathrm{div}(g(\lambda_i)) = \Theta_{F_m/k,S,T}(0)\lambda_i,$$

for a unique element $g(\lambda_i) \in (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$ and for all $1 \le i \le n$. In what follows we let $\Theta_m := \Theta_{F_m/k,S,T}(0)$. The relationship between the g_i and $g(\lambda_i)$ is given as follows:

$$\begin{aligned} \mathrm{div}(g(\lambda_i)) &= \Theta_m \lambda_i \\ &= \Theta_m v_i + \Theta_m(\lambda_i - v_i) \\ &= \mathrm{div}(g_i) + \Theta_m \mathrm{div}(\xi_i) \\ &= \mathrm{div}(g_i \xi_i^{\Theta_m}) \end{aligned}$$

where $\xi_i \in (F_{m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$ is such that $\lambda_i - v_i = \mathrm{div}(\xi_i)$, therefore

$$g(\lambda_i) = g_i \xi_i^{\Theta_m}. \tag{4.37}$$

We will need to compute wedge products of the $g(\lambda_i)$ and therefore, for ease of notation,

it will be helpful to consider (4.37) as given additively, namely

$$g(\lambda_i) = g_i + \Theta_m \xi_i.$$

The following computations are quite arduous due to the complicated formulae associated with wedge product manipulations. We first provide details for the general computation, then illustrate the general case with a simple example where the computations are easy to carry out by hand.

Computing $g(\lambda_1) \wedge \ldots \wedge g(\lambda_i)$ we obtain

$$g(\lambda_1) \wedge \ldots \wedge g(\lambda_i) = \bigwedge_{k=1}^{i} (g_k + \Theta_m \xi_k)$$

$$= \sum_{k=0}^{i} \Theta_m^k \omega_k$$

where

$$\omega_0 = g_1 \wedge \ldots \wedge g_i$$

$$\omega_1 = \sum_{k=1}^{i} g_1 \wedge \ldots \wedge \xi_k \wedge \ldots \wedge g_i$$

$$\omega_2 = \sum_{1 \le i_1 < i_2 \le i} g_1 \wedge \ldots \wedge \xi_{i_1} \wedge \ldots \wedge \xi_{i_2} \wedge \ldots \wedge g_i$$

$$\vdots$$

$$\omega_i = \xi_1 \wedge \ldots \wedge \xi_i$$

where, for all $1 \le k \le i$, the elements ξ_{i_k} replace the corresponding g_{i_k}. We illustrate this with a quick example.

Example 6. *Let $i = 3$ so we have*

$$g(\lambda_1) \wedge g(\lambda_2) \wedge g(\lambda_3) = (g_1 + \Theta_m \xi_1) \wedge (g_2 + \Theta_m \xi_2) \wedge (g_3 + \Theta_m \xi_3)$$

$$= g_1 \wedge g_2 \wedge g_3 + \Theta_m (g_1 \wedge g_2 \wedge \xi_3 + g_1 \wedge \xi_2 \wedge g_3 + \xi_1 \wedge g_2 \wedge g_3)$$

$$+ \Theta_m^2 (g_1 \wedge \xi_2 \wedge \xi_3 + \xi_1 \wedge g_2 \wedge \xi_3 + \xi_1 \wedge \xi_2 \wedge g_3)$$

$$+ \Theta_m^3 \xi_1 \wedge \xi_2 \wedge \xi_3$$

and

$$\omega_0 = g_1 \wedge g_2 \wedge g_3$$

$$\omega_1 = g_1 \wedge g_2 \wedge \xi_3 + g_1 \wedge \xi_2 \wedge g_3 + \xi_1 \wedge g_2 \wedge g_3$$

$$\omega_2 = g_1 \wedge \xi_2 \wedge \xi_3 + \xi_1 \wedge g_2 \wedge \xi_3 + \xi_1 \wedge \xi_2 \wedge g_3$$

$$\omega_3 = \xi_1 \wedge \xi_2 \wedge \xi_3$$

therefore, in the notation above, we have

$$g(\lambda_1) \wedge g(\lambda_2) \wedge g(\lambda_3) = \sum_{k=0}^{3} \Theta_m^k \omega_k.$$

Returning to our general computation of $g(\lambda_1) \wedge \ldots \wedge g(\lambda_i)$, we now expand the $g_1 \wedge \ldots \wedge g_i$ term using the relations

$$g_k = \sum_{j=1}^{n} a_{kj} f_j,$$

namely,

$$g_1 \wedge \ldots \wedge g_i = \sum_{j=1}^{n} a_{1j} f_j \wedge \ldots \wedge \sum_{j=1}^{n} a_{ij} f_j$$

$$= \det(X_i) f_1 \wedge f_2 \wedge \ldots \wedge f_i + \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} c_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k}$$

where the $c_{j_1,\ldots,j_k}$ are determinants of $i \times i$ minors of X, which, via (4.16) will be related to certain powers of Θ_m multiplied by determinants of $(n-i) \times (n-i)$ minors of $\overline{A}$. Putting all of this together, we have

$$g(\lambda_1) \wedge \ldots \wedge g(\lambda_i) = \det(X_i) f_1 \wedge \ldots \wedge f_i + \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} c_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k}$$

$$+ \Theta_m \omega_1 + \Theta_m^2 \omega_2 + \ldots + \Theta_m^i \omega_i \qquad (4.38)$$

Recall that the primes λ_i of F_m were chosen so that

- $\lambda_i \sim v_i$ in $(\mathrm{Cl}_{F_m,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^-$

- If $\tilde{\lambda}_i = \lambda_i \cap k$, then $\tilde{\lambda}_i$ splits completely in F_m.

- The maps ϕ_{λ_i} associated to λ_i satisfy

$$\phi_{\lambda_i}(f_j) \in \begin{cases} ((\mathbb{Z}/p^N\mathbb{Z})[G \times \Gamma_m]^-)^\times, & \text{if } i = j, \\ \{0\}, & \text{if } i \neq j \end{cases} .$$

Considering the extension F_m/k with $\mathcal{S}_m = S_{\mathrm{ram}}(F_m/k) \cup \{\lambda_1, \ldots, \lambda_i\}$, and observing that

$$\Theta_m g(\lambda_1) \wedge \ldots \wedge g(\lambda_i) = \Theta_m^i \epsilon_{S_m,T}, \tag{4.39}$$

where $\epsilon_{S_m,T}$ is the Rubin-Stark element associated to the data $(F_m/k, S_m, T)$, we apply the Rubin-Stark regulator to the element $\Theta_m g(\lambda_1) \wedge \ldots \wedge g(\lambda_i)$, and invoke injectivity of the regulator on the minus-side, to obtain

$$R_{S_m}(\Theta_m g(\lambda_1) \wedge \ldots \wedge g(\lambda_i)) = \Theta_m^{i+1}.$$

Multiplying equation (4.38) by Θ_m we obtain

$$\begin{aligned} \Theta_m g(\lambda_1) \wedge \ldots \wedge g(\lambda_i) = {} & \Theta_m \det(X_i) f_1 \wedge \ldots \wedge f_i \\ & + \Theta_m \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} c_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k} \\ & + \Theta_m^2 \omega_1 + \Theta_m^3 \omega_2 + \ldots + \Theta_m^{i+1} \omega_i \end{aligned} \tag{4.40}$$

and substituting (4.39) into the left-hand-side of (4.40), we have

$$\begin{aligned} \Theta_m^i \epsilon_{S_m,T} = {} & \Theta_m \det(X_i) f_1 \wedge \ldots \wedge f_i \\ & + \Theta_m \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} c_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k} \\ & + \Theta_m^2 \omega_1 + \Theta_m^3 \omega_2 + \ldots + \Theta_m^{i+1} \omega_i. \end{aligned} \tag{4.41}$$

We now utilize the following relations which are direct consequences of (4.16)

- $\det(X_i) = \Theta_m^{i-1} \det(\overline{A}_i)$

- $c_{j_1,\ldots,j_k} = \Theta_m^{i-1} d_{j_1,\ldots,j_k}$ where $d_{j_1,\ldots,j_k}$ is a determinant of an $(n-i) \times (n-i)$ minor
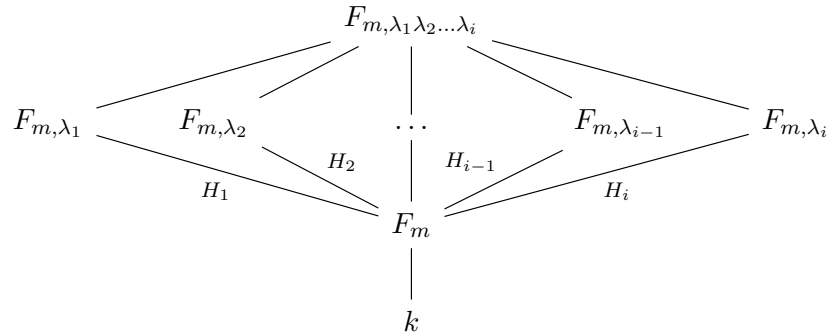
of $\overline{A}$.

Applying these relations to (4.41), we obtain

$$
\begin{aligned}
\Theta_m^i \epsilon_{S_m,T} = {}& \Theta_m^i \det(\overline{A}_i) f_1 \wedge \ldots \wedge f_i \\
& + \Theta_m^i \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} d_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k} \\
& + \Theta_m^i \omega_1 + \Theta_m^i \omega_2 + \ldots + \Theta_m^{i+1} \omega_i,
\end{aligned} \tag{4.42}
$$

and since Θ_m is a nonzero divisor, cancelling it yields the relation

$$
\begin{aligned}
\epsilon_{S_m,T} = {}& \det(\overline{A}_i) f_1 \wedge \ldots \wedge f_i \\
& + \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} d_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k} \\
& + \omega_1 + \omega_2 + \ldots + \Theta_m \omega_i.
\end{aligned} \tag{4.43}
$$

We now consider the field diagram

where we denote $H := \mathrm{Gal}(F_{m,\lambda_1 \lambda_2 \ldots, \lambda_i}/F_m)$, and aim to apply the generalization to Gross's conjecture to the triple of fields $F_{m,\lambda_1 \ldots \lambda_i}/F_m/k$. Gross's regulator is given by

$$
R_{\mathrm{Gross}} = \phi_{\lambda_1} \wedge \ldots \wedge \phi_{\lambda_i},
$$

and takes values in I_H^i/I_H^{i+1}. We let $\Theta_{\lambda_1 \ldots \lambda_i} := \Theta_{F_m/k,S_m,T}(0)$ denote the special value

associated to the extension $F_{m,\lambda_1\ldots\lambda_i}/k$ where $S_m = S \cup \{\lambda_1, \ldots, \lambda_i\}$. The generalization of Gross's conjecture then states

$$R_{\text{Gross}}(\epsilon_{S_m,T}) \equiv \Theta_{\lambda_1\ldots\lambda_i} \pmod{I_H^{i+1}}.$$

Therefore, applying Gross's regulator to (4.43) we obtain

$$\begin{aligned}
\Theta_{\lambda_1\ldots\lambda_i} \pmod{I_H^{i+1}} &\equiv R_{\text{Gross}}(\det(\overline{A}_i)f_1 \wedge \ldots \wedge f_i) \\
&+ R_{\text{Gross}}(\sum_{\substack{1 \leq j_1 < \ldots < j_k \leq i \\ (j_1,\ldots,j_k) \neq (1,2,\ldots,i)}} d_{j_1,\ldots,j_k} f_{j_1} \wedge \ldots \wedge f_{j_k}) \\
&+ R_{\text{Gross}}(\omega_1 + \omega_2 + \ldots + \Theta_m \omega_i))
\end{aligned} \tag{4.44}$$

where we have used linearity of the regulator. By definition of R_{Gross} we have

$$\begin{aligned}
R_{\text{Gross}}(\det(\overline{A}_i)f_1 \wedge \ldots \wedge f_i) &= \det(\overline{A}_i)(\phi_{\lambda_1} \wedge \ldots \wedge \phi_{\lambda_i})(f_1 \wedge \ldots \wedge f_i) \\
&= \det(\phi_{\lambda_k}(f_j))_{j,k=1,2,\ldots,i} \\
&= u_1 u_2 \ldots u_i
\end{aligned}$$

where $u_k := \phi_{\lambda_k}(f_k) \in ((\mathbb{Z}/p^N\mathbb{Z})[G \times \Gamma_m]^-)^\times$. Now, the second regulator term in (4.44) is zero due to the wedge products involved and the properties of the ϕ_{λ_k}, namely, each summand will have a determinant containing a row consisting entirely of zeros, and therefore, the determinant will be zero. The computation of the final regulator term will require some work. We illustrate the computation with ω_1. Since

$$\omega_1 = \sum_{k=1}^{i} g_1 \wedge \ldots \wedge \xi_k \wedge \ldots \wedge g_i,$$

applying $R_{\text{Gross}} = \phi_{\lambda_1} \wedge \ldots \wedge \phi_{\lambda_i}$ we get a sum of determinants of matrices all of which have $i-1$ columns involving ϕ_{λ_k} evaluated against some g_i, and precisely one column consisting of some ϕ_{λ_j} evaluated against the ξ_j, for example

$$(\phi_{\lambda_1} \wedge \ldots \wedge \phi_{\lambda_i})(g_1 \wedge \ldots \wedge g_{i-1} \wedge \xi_i) = \det \begin{pmatrix} a_{11}u_1 & a_{12}u_2 & \ldots & a_{1i}u_i \\ \vdots & \vdots & \ldots & \vdots \\ a_{i-1,1}u_1 & a_{i-1,2}u_2 & \ldots & a_{i-1,i}u_i \\ \phi_{\lambda_1}(\xi_1) & \phi_{\lambda_2}(\xi_2) & \ldots & \phi_{\lambda_i}(\xi_i) \end{pmatrix}$$

since $\phi_{\lambda_k}(g_j) = a_{jk}u_k$. Expanding this determinant along the i-th row, we see that $R_{\mathrm{Gross}}(\omega_1)$ is simply a linear combination of determinants of certain $(i-1) \times (i-1)$ minors of X, which, via (4.16) are related to determinants of $(n-(i-1)) \times (n-(i-1))$ minors of $\overline{A}$, i.e. elements in $\mathrm{Fit}^{i-1}_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$. It's important to note that the $(i-1) \times (i-1)$ minors of X are coming from the $(i-1) \times i$ block matrix of X (denoted X_{i-1} above), consisting of the first $i-1$ rows and first i columns of X. Therefore, the ω_1 computation isn't capturing the entirety of $\mathrm{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$, this is since we only computed the Gross regulator on ω_1, as we further compute the Gross regulator on the various ω_k, we obtain linear combinations of determinants of all $(i-1) \times (i-1)$ minors of X, and therefore, all $(n-(i-1)) \times (n-(i-1))$ minors of $\overline{A}$, i.e. we capture the entire Fitting ideal $\mathrm{Fit}^i_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$.

When $2 \leq k \leq i$, the value of the Gross regulator on ω_k will lie in a smaller Fitting ideal. More precisely, $R_{\mathrm{Gross}}(\omega_k)$ will be a linear combination of elements of $\mathrm{Fit}^{i-(k-1)}_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$. The reason for this is as follows, for $2 \leq k \leq i$, the determinant involved in $R_{\mathrm{Gross}}(\omega_k)$ will contain k rows given by evaluations of the ϕ_λ's against the elements ξ. When computing the determinant cofactor expansion along one of the rows given by ξ-evaluation gives us a linear combination of determinants of matrices of size $i-1$, all of which now contain at least one row consisting of ξ-evaluations. Performing a second cofactor expansion along another ξ-evaluation row, would subsequently give us another linear combination of determinants of matrices of size $i-2$. Continuing in this way, we find that once we have expanded along all ξ-evaluation rows, we end up with a linear combination of determinants of $(i-k) \times (i-k)$ minors of X, which, via (4.16) are really determinants of $(n-(i-k)) \times (n-(i-k))$ minors of $\overline{A}$, and therefore in $\mathrm{Fit}^{i-k}_{\Lambda_F^-/p^N\Lambda_F^-}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$.

We are now ready to put everything together to finish the computation in (4.44),

but first we need to apply the projection

$$\pi : I_H^i/I_H^{i+1} \longrightarrow I_H^i/(I_{H_1}^2 + \ldots + I_{H_i}^2),$$

where $\pi(\Theta_{\lambda_1 \ldots \lambda_i}) = \delta_{1,1,\ldots,1}(\Theta_{\lambda_1 \ldots \lambda_i})$.

Applying π to both sides of (4.44) and using $\pi \circ R_{\text{Gross}} = \overline{\phi}_{\lambda_1} \wedge \ldots \wedge \overline{\phi}_{\lambda_i}$, we obtain

$$
\begin{aligned}
u_1 u_2 \ldots u_i \det(\overline{A}_i) &\in \text{Fit}_{\Lambda_F^-/p^N \Lambda_F^-}^{i-1}(T_p(\mathcal{M}_{\mathcal{S}_F, \mathcal{T}_F})^-/(p^N)) + \langle \Theta_m, \delta_{1,1,\ldots,1}(\Theta_{\lambda_1 \ldots \lambda_i}) \rangle \\
&\subset \mathscr{F}^{i-1} + \langle \Theta_m, \delta_{1,1,\ldots,1}(\Theta_{\lambda_1 \ldots \lambda_i}) \rangle \\
&\subset \mathscr{F}^i
\end{aligned}
\tag{4.45}
$$

where we are using our induction hypothesis $\text{Fit}_{\Lambda_F^-/p^N \Lambda_F^-}^{i-1}(T_p(\mathcal{M}_{\mathcal{S}_F, \mathcal{T}_F})^-/(p^N)) \subset \mathscr{F}^{i-1}$. Via appropriate linear transformations, one can take any $(n-i) \times (n-i)$ minor of $\overline{A}$, and place it in the position of our $\overline{A}_i$, and therefore, applying the same computations above, the determinant of any $(n-i) \times (n-i)$ minor of $\overline{A}$ is contained in $\mathscr{F}^i$, that is to say,

$$\text{Fit}_{\Lambda_F^-/p^N \Lambda_F^-}^{i}(T_p(\mathcal{M}_{\mathcal{S}_F, \mathcal{T}_F})^-/(p^N)) \subset \mathscr{F}^i. \qquad \square$$

Using the above notation we now provide an example illustrating the above computation for the case $i = 2$.

Example 7. *Let* $\overline{A} = \left(\begin{array}{c|c} 2 \times 2 & * \\ \hline * & \overline{A}_{n-2} \end{array} \right)$ *and* $X = \left(\begin{array}{cc|c} a_{11} & a_{12} & * \\ a_{21} & a_{22} & \\ \hline * & & X_{n-2} \end{array} \right)$ *so that*

$\overline{A}X = \Theta_m I_n$ *where* I_n *is the* $n \times n$ *identity matrix. Our two primes of* F_m *are* λ_1, λ_2, *with associated elements*

$$g(\lambda_1) = g_1 + \Theta_m \xi_1 \qquad and \qquad g(\lambda_2) = g_2 + \Theta_m \xi_2,$$

where

$$g_1 = \sum_{j=1}^{n} a_{1j} f_j \qquad and \qquad g_2 = \sum_{j=1}^{n} a_{2j} f_j.$$

We therefore have

$$g(\lambda_1) \wedge g(\lambda_2) = (g_1 + \Theta_m \xi_1) \wedge (g_2 + \Theta_m \xi_2)$$

$$= g_1 \wedge g_2 + \Theta_m g_1 \wedge \xi_2 + \Theta_m \xi_1 \wedge g_2 + \Theta_m^2 \xi_1 \wedge \xi_2$$

where

$$g_1 \wedge g_2 = \sum_{j=1}^{n} a_{1j} f_j \wedge \sum_{j=1}^{n} a_{2j} f_j$$

$$= a_{11} a_{22} f_1 \wedge f_2 + a_{12} a_{21} f_2 \wedge f_1 + \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} c_{ij} f_i \wedge f_j$$

$$= (a_{11} a_{22} - a_{12} a_{21}) f_1 \wedge f_2 + \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} c_{ij} f_i \wedge f_j$$

$$= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} f_1 \wedge f_2 + \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} c_{ij} f_i \wedge f_j$$

so

$$g(\lambda_1) \wedge g(\lambda_2) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} f_1 \wedge f_2 + \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} c_{ij} f_i \wedge f_j$$

$$+ \Theta_m g_1 \wedge \xi_2 + \Theta_m \xi_1 \wedge g_2 + \Theta_m^2 \xi_1 \wedge \xi_2.$$

Multiplying by Θ_m and using the relation $\Theta_m g(\lambda_1) \wedge g(\lambda_2) = \Theta_m^2 \epsilon_{S_m,T}$, we get

$$\Theta_m^2 \epsilon_{S_m,T} = \Theta_m \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} f_1 \wedge f_2 + \Theta_m \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} c_{ij} f_i \wedge f_j$$

$$+ \Theta_m^2 g_1 \wedge \xi_2 + \Theta_m^2 \xi_1 \wedge g_2 + \Theta_m^3 \xi_1 \wedge \xi_2$$

Applying (4.16) $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \Theta_m \det(\overline{A}_{n-2})$ *and* $c_{ij} = \Theta_m d_{ij}$, *therefore*

$$\Theta_m^2 \epsilon_{S_m,T} = \Theta_m^2 \det(\overline{A}_{n-2}) f_1 \wedge f_2 + \Theta_m^2 \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} d_{ij} f_i \wedge f_j$$

$$+ \Theta_m^2 g_1 \wedge \xi_2 + \Theta_m^2 \xi_1 \wedge g_2 + \Theta_m^3 \xi_1 \wedge \xi_2$$

whereby, since Θ_m *is a nonzero divisor, we can cancel* Θ_m^2 *on both sides giving*

$$\epsilon_{S_m,T} = \det(\overline{A}_{n-2}) f_1 \wedge f_2 + \sum_{\substack{i<j \\ (i,j)\neq(1,2)}} d_{ij} f_i \wedge f_j$$

$$+ g_1 \wedge \xi_2 + \xi_1 \wedge g_2 + \Theta_m \xi_1 \wedge \xi_2. \tag{4.46}$$

We now consider the field extensions

and let $\pi : I_H^2/I_H^3 \longrightarrow I_H^2/(I_{H_1}^2 + I_{H_2}^2)$ *be the natural projection. We apply the generalized Gross conjecture to the triple* $F_{m,\lambda_1\lambda_2}/F_m/k$ *with the special value* $\Theta_{\lambda_1\lambda_2}$ *associated to* $F_{m,\lambda_1\lambda_2}/k$. *The composite* $\pi \circ R_{Gross} = \overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2}$, *applied to (4.46) gives*

$$\delta_{1,1}(\Theta_{\lambda_1\lambda_2}) = (\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(\det(\overline{A}_{n-2}) f_1 \wedge f_2) + (\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})\left(\sum_{\substack{i<j \\ (i,j)\neq(1,2)}} d_{ij} f_i \wedge f_j \right)$$

$$+ (\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(g_1 \wedge \xi_2 + \xi_1 \wedge g_2 + \Theta_m \xi_1 \wedge \xi_2). \tag{4.47}$$

We now compute each term individually,

$$(\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(\det(\overline{A}_{n-2})f_1 \wedge f_2) = \det(\overline{A}_{n-2}) \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(f_1) & \overline{\phi}_{\lambda_2}(f_1) \\ \overline{\phi}_{\lambda_1}(f_2) & \overline{\phi}_{\lambda_2}(f_2) \end{pmatrix}$$

$$= \det(\overline{A}_{n-2}) \det \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

$$= u_1 u_2 \det(\overline{A}_{n-2}).$$

$$(\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(\sum_{\substack{i<j \\ (i,j) \neq (1,2)}} d_{ij} f_i \wedge f_j) = \sum_{\substack{i<j \\ (i,j) \neq (1,2)}} d_{ij}(\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(f_i \wedge f_j)$$

$$= \sum_{\substack{i<j \\ (i,j) \neq (1,2)}} d_{ij} \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(f_i) & \overline{\phi}_{\lambda_2}(f_i) \\ \overline{\phi}_{\lambda_1}(f_j) & \overline{\phi}_{\lambda_2}(f_j) \end{pmatrix}$$

$$= 0$$

since $(i,j) \neq (1,2)$ there is at least one column in each $\begin{pmatrix} \overline{\phi}_{\lambda_1}(f_i) & \overline{\phi}_{\lambda_2}(f_i) \\ \overline{\phi}_{\lambda_1}(f_j) & \overline{\phi}_{\lambda_2}(f_j) \end{pmatrix}$ *which is zero, hence each determinant is zero.*

Finally, we have

$$(\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(g_1 \wedge \xi_2 + \xi_1 \wedge g_2 + \xi_1 \wedge \xi_2)$$

$$= (\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(g_1 \wedge \xi_2) + (\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(\xi_1 \wedge g_2) + \Theta_m(\overline{\phi}_{\lambda_1} \wedge \overline{\phi}_{\lambda_2})(\xi_1 \wedge \xi_2)$$

$$= \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(g_1) & \overline{\phi}_{\lambda_2}(g_1) \\ \overline{\phi}_{\lambda_1}(\xi_2) & \overline{\phi}_{\lambda_2}(\xi_2) \end{pmatrix} + \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(\xi_1) & \overline{\phi}_{\lambda_2}(\xi_1) \\ \overline{\phi}_{\lambda_1}(g_2) & \overline{\phi}_{\lambda_2}(g_2) \end{pmatrix} + \Theta_m \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(\xi_1) & \overline{\phi}_{\lambda_2}(\xi_1) \\ \overline{\phi}_{\lambda_1}(\xi_2) & \overline{\phi}_{\lambda_2}(\xi_2) \end{pmatrix}$$

$$= \det \begin{pmatrix} a_{11}u_1 & a_{12}u_2 \\ \overline{\phi}_{\lambda_1}(\xi_2) & \overline{\phi}_{\lambda_2}(\xi_2) \end{pmatrix} + \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(\xi_1) & \overline{\phi}_{\lambda_2}(\xi_1) \\ a_{21}u_1 & a_{22}u_2 \end{pmatrix} + \Theta_m \det \begin{pmatrix} \overline{\phi}_{\lambda_1}(\xi_1) & \overline{\phi}_{\lambda_2}(\xi_1) \\ \overline{\phi}_{\lambda_1}(\xi_2) & \overline{\phi}_{\lambda_2}(\xi_2) \end{pmatrix}$$

$$= \beta_1 a_{11} + \beta_2 a_{12} + \beta_3 a_{21} + \beta_4 a_{22} + \beta_5 \Theta_m$$

where

$$\beta_1 := u_1 \overline{\phi}_{\lambda_2}(\xi_2)$$

$$\beta_2 := -u_2\overline{\phi}_{\lambda_1}(\xi_2)$$

$$\beta_3 := -u_1\overline{\phi}_{\lambda_2}(\xi_1)$$

$$\beta_4 := u_2\overline{\phi}_{\lambda_1}(\xi_1)$$

$$\beta_5 := \overline{\phi}_{\lambda_1}(\xi_1)\overline{\phi}_{\lambda_2}(\xi_2) - \overline{\phi}_{\lambda_1}(\xi_2)\overline{\phi}_{\lambda_2}(\xi_1)$$

and the elements $a_{11}, a_{12}, a_{21}, a_{22}$ are thought of as determinants of 1×1 minors of
$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, *and therefore related to determinants of $(n-1) \times (n-1)$ minors of $\overline{A}$, i.e. elements of* $\mathrm{Fit}^1_{\Lambda^-_F/p^N\Lambda^-_F}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N))$.

Putting the above computations together and solving for $\det(\overline{A}_{n-2})$ in (4.47), we see

$$\det(\overline{A}_{n-2}) \in \mathrm{Fit}^1_{\Lambda^-_F/p^N\Lambda^-_F}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N)) + \langle \Theta_m, \delta_{1,1}(\Theta_{\lambda_1\lambda_2}) \rangle$$

$$\subset \mathscr{F}^1 + \langle \Theta_m, \delta_{1,1}(\Theta_{\lambda_1\lambda_2}) \rangle$$

$$\subset \mathscr{F}^2.$$

The above computations may seem particular to the positioning of $\overline{A}_{n-2}$, however, if B is any other $(n-2) \times (n-2)$ minor of $\overline{A}$, a suitable linear transformation (row and column swaps in this case) will place B in the position of our original $\overline{A}_{n-2}$. These row and column swaps only affect the sign of the determinant, hence they don't affect our desired inclusion result. Applying the $\overline{A}_{n-2}$ computations to the newly located minor B, we obtain

$$\det(B) \in \mathscr{F}^2.$$

Therefore, since B was an arbitrary $(n-2) \times (n-2)$ minor of $\overline{A}$, we obtain our desired inclusion, namely

$$\mathrm{Fit}^2_{\Lambda^-_F/p^N\Lambda^-_F}(T_p(\mathcal{M}_{\mathcal{S}_F,\mathcal{T}_F})^-/(p^N)) \subset \mathscr{F}^2.$$

Part II

Chapter 5

Preliminary Notions

Part two of this manuscript will have some differing notation in comparison with part one, however, all such differences will be noted to avoid confusion. This section will be devoted to establishing the tools used in the proof of our special case of the Breuil-Schneider conjecture.

5.1 Fundamental notions in p-adic Hodge theory

Let $L/\mathbb{Q}_p$ be a finite field extension and view L inside an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, so that $G_L := \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ is the absolute Galois group of L. One of the goals of p-adic Hodge theory is to study p-adic representations of G_L, which are continuous homomorphisms $\rho : G_L \longrightarrow \mathrm{GL}(V)$ where V is a finite dimensional K vector space for K a finite extension of $\mathbb{Q}_p$. We denote the category of p-adic representations of G_L by $\mathrm{Rep}_{\mathbb{Q}_p}(G_L)$.

When studying $\mathrm{Rep}_{\mathbb{Q}_p}(G_L)$ it is useful to study certain subcategories which lie in functorial equivalence with semi-linear algebraic categories. The information from the objects in the semi-linear algebraic category can then be used to understand properties of the original object in the subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_L)$. We give a brief outline of some of the subcategories of interest along with the general philosophy for constructing such functorial equivalences between said subcategories. The ensuing account follows that in [5] and should be thought of as an axiomatic approach to Fontaine's period rings.

Let F be a field, G a group, B a domain as an $F[G]$ algebra with $C = \mathrm{Frac}(B)$, and $E = B^G$.

Definition 5.1. *We say B is (F,G)-regular if $C^G = B^G$ and if $0 \neq b \in B$ is such that Fb is G-stable, then $b \in B^\times$.*

If $\mathrm{Vec}_E^{\mathrm{fin}}$ denotes the category of finite dimensional E vector spaces, then given an (F,G)-regular B, we define a functor

$$D_B : \mathrm{Rep}_F(G) \longrightarrow \mathrm{Vec}_E^{\mathrm{fin}},$$

by

$$D_B(V) := (B \otimes_F V)^G.$$

In general, it is known that $\dim_E D_B(V) \leq \dim_F V$, however, if equality holds, then the representation V is called B-*admissible*. If one restricts to the subcategory $\mathrm{Rep}_F^B(G)$ of B-admissible representations, then the functor D_B is exact and faithful.

In the notation of the above formalism, and for our study of the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_L)$, we let $G = G_L$ and $F = \mathbb{Q}_p$. Fontaine then constructs period rings B_{HT}, B_{dR}, B_{cris}, and B_{st}, which play the role of B above, and which give rise to the subcategories of $\mathrm{Rep}_{\mathbb{Q}_p}(G_L)$ consisting of Hodge-Tate $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{HT}}(G_L)$, de Rham $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_L)$, crystalline $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$, and semistable $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_L)$ representations, respectively. One can find the precise definitions of Hodge-Tate, de Rham, crystalline, and semistable representations of G_L in [5] and [10].

Each of the above subcategories comes endowed with a functor $D_{\mathrm{HT}}, D_{\mathrm{dR}}, D_{\mathrm{cris}}$, and D_{st}, which, depending on the subcategory of interest, takes values in a certain semi-linear algebraic category. The semi-linear algebraic category of interest will be the category of (ϕ, N)-modules $\mathrm{MOD}_{L'/L}$, which will be described in more detail in the section on the Breuil-Schneider conjecture. The final definition we will need in this section is that of a *potentially semistable p-adic representation of G_L.*

Definition 5.2. *A p-adic representation V of G_L is semistable if there exists a finite Galois extension L'/L such that V becomes semistable when viewed as a representation of $G_{L'}$.*

Fontaine's semistability conjecture, which was first proved by Berger and then again by Andrè-Kedlaya-Mebkhout, characterizes those potentially semistable representations of G_L.

Theorem 5.1 (Berger, Andrè-Kedlaya-Mebkhout)**.** *A p-adic representation V of G_L is*

potentially semistable if and only if it is de Rham.

5.2   The Local Langlands Correspondence

One of the main tools needed to address the smooth side of the Breuil-Schneider conjecture is the local Langlands correspondence. Before stating the correspondence, we provide descriptions of the mathematical objects involved.

Let F be a nonarchimedean local field with residue field k_F, and let $q_F := |k_F|$. Fix algebraic closures $\overline{F}$ of F and $\overline{k_F}$ of k_F, and denote the absolute Galois group of F by $G_F := \mathrm{Gal}(\overline{F}/F)$.

Definition 5.3. *The Weil group of F is the topological group*

$$W(\overline{F}/F) := \{\sigma \in G_F : \sigma \text{ induces } \Phi^n \text{ on } \overline{k_F} \text{ for any } n \in \mathbb{Z}, \},$$

where $\Phi(x) = x^{q_F}$ is the arithmetic Frobenius of $\mathrm{Gal}(\overline{k_F}/k_F)$, and the topology on $W(\overline{F}/F)$ is given by declaring the inertia subgroup $I_F \subset G_F$ to be open in $W(\overline{F}/F)$, where I_F is given the profinite topology from G_F.

The subgroup $W(\overline{F}/F)$ is dense in G_F and contains the inertia subgroup $I_F \subset G_F$. We will often view $W(\overline{F}/F)$ as sitting in the following topological short exact sequence

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & I_F & \longrightarrow & G_F & \xrightarrow{\mathrm{val}} & \widehat{\mathbb{Z}} & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & I_F & \longrightarrow & W(\overline{F}/F) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

Fixing an algebraically closed field K of characteristic zero, we have the important notion of a Weil-Deligne representation of $W(\overline{F}/F)$, namely

Definition 5.4. *A Weil-Deligne representation of $W(\overline{F}/F)$ with coefficients in K is a triple (r, N, V) consisting of*

1. *a finite dimensional K-vector space V*

2. *a continuous homomorphism $r : W(\overline{F}/F) \to \mathrm{GL}(V)$ with open kernel*

3. *an endomorphism $N : V \to V$ satisfying*

$$r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_F^{-1}(\sigma)|_F N$$

where $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W(\overline{F}/F)^{\mathrm{ab}}$ *is the reciprocity map of local class field theory, normalized by sending a uniformizer* π_F *of* F *to the geometric Frobenius* $\mathrm{Fr}_F \in \mathrm{Gal}(\overline{k_F}/k_F)$.

We call a Weil-Deligne representation (r, N, V) Frobenius semisimple, if r is semisimple as a representation of $W(\overline{F}/F)$. If (r, N, V) is not Frobenius semisimple, then its Frobenius semisimplification is $(r, N, V)^{\mathrm{F-ss}} = (r^{\mathrm{ss}}, N, V)$ where r^{ss} denotes the semisimplication of r. Recall that r^{ss} is the $W(\overline{F}/F)$-representation on V whose Jordan-Hölder components are the same as r, i.e. $r^{\mathrm{ss}} = \bigoplus_{i \geq 0} V_i/V_{i+1}$ where V_i/V_{i+1} is simple for all i. Notice that r^{ss} satisfies the same N-conjugation relation as does r. An alternate yet equivalent construction of r^{ss} comes from the Jordan decomposition of $r(\phi)$, where ϕ is a lift of the geometric Frobenius Φ. Applying Jordan decomposition to $r(\phi) \in \mathrm{GL}(V)$ gives $s, u \in \mathrm{GL}(V)$ such that $r(\phi) = su = us$, where s is semisimple and u is unipotent. Following Deligne [8, p.570], one defines r^{ss} by the formula $r^{\mathrm{ss}}(\phi^n \sigma) = s^n r(\sigma)$, for any $n \in \mathbb{Z}$ and $\sigma \in I_F$.

Our statement of the local Langlands correspondence will follow that of Harris-Taylor [18, p.2]. Following their notation, for each $n \in \mathbb{Z}_{\geq 1}$ we let $\mathrm{Irr}(\mathrm{GL}_n(F))$ denote the classes of irreducible admissible representations of $\mathrm{GL}_n(F)$ over K, and let $\mathrm{WDRep}_n(F)$ denote the classes of n-dimensional, Frobenius-semisimple Weil-Deligne representations of $W(\overline{F}/F)$ over K.

Theorem 5.2 (Local Langlands Correspondence). *Let* $\psi : F \to \mathbb{C}^\times$ *be a nontrivial additive character. For each* $n \in \mathbb{Z}_{\geq 1}$, *a local Langlands correspondence for* F *is a sequence of bijections*

$$\mathrm{rec}_n : \mathrm{Irr}(\mathrm{GL}_n(F)) \longrightarrow \mathrm{WDRep}_n(K)$$

satisfying the following

1. *If* $\pi \in \mathrm{Irr}(\mathrm{GL}_1(F))$ *then* $\mathrm{rec}_1(\pi) = \pi \circ \mathrm{Art}_F^{-1}$

2. *If* $\pi_1 \in \mathrm{Irr}(\mathrm{GL}_{n_1}(F))$ *and* $\pi_2 \in \mathrm{Irr}(\mathrm{GL}_{n_2}(F))$ *then*

$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s)$$

$$\epsilon(\pi_1 \times \pi_2, \psi, s) = \epsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), \psi, s)$$

3. *If $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$ and $\chi \in \mathrm{Irr}(\mathrm{GL}_1(F))$ then*

$$\mathrm{rec}_n(\pi \otimes (\chi \circ \det)) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi)$$

4. *If $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$ and π has central character χ, then $\det(\mathrm{rec}_n(\pi)) = \mathrm{rec}_n(\chi)$.*

5. *If $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$ then $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee$.*

5.3 Algebraic Induction

Let G be a linear algebraic group over K. Traditionally one usually works over an algebraically closed field, however in [21], Jantzen considers the more general case of K being a commutative ring with unit. By a G-module we mean a module M over the group ring $K[G]$. Given a G-module M and a closed subgroup $H < G$, we obtain a natural H-module structure on M simply by restricting the G-action to the subgroup $H < G$. In this way one obtains the restriction functor

$$\mathrm{res}_H^G : \mathrm{Mod}_G \to \mathrm{Mod}_H,$$

where Mod_G and Mod_H denote the categories of G and H-modules respectively. From this construction it is natural to ask whether one can reverse the process, i.e. given an H-module M, can one construct a functor from Mod_H to Mod_G which behaves nicely with respect to res_H^G?

Definition 5.5. *Let $H < G$ be a closed subgroup and M a finitely generated H-module. The set $\mathrm{Ind}_H^G M$, given by*

$$\mathrm{Ind}_H^G M := \{f : G \to M : f \text{ is algebraic and } f(hx) = hf(x) \text{ for all } x \in G \text{ and } h \in H\},$$

is a G-module under the left G-action of right translations, namely $(yf)(x) = f(xy)$ for all $x, y \in G$. We call $\mathrm{Ind}_H^G M$ the induction of M from H to G.

From this definition we obtain the functor

$$\mathrm{Ind}_H^G : \mathrm{Mod}_H \to \mathrm{Mod}_G$$

which is related to the restriction functor res_H^G via Frobenius reciprocity

Lemma 5.3 (Frobenius Reciprocity)**.** *Let $H < G$ be a closed subgroup of G and M an H-module, then for each G-module N we have an isomorphism*

$$\operatorname{Hom}_G(N, \operatorname{Ind}_H^G M) \xrightarrow{\sim} \operatorname{Hom}_H(\operatorname{res}_H^G N, M).$$

In particular, Ind_H^G is right adjoint to res_H^G. The following properties of the functor Ind_H^G can be found in [21] and will be used throughout our proof of a special case of the Breuil-Schneider conjecture.

Lemma 5.4. *1. If $H < H' < G$ are groups and M is an H-module, then*

$$\operatorname{Ind}_H^G M \simeq \operatorname{Ind}_{H'}^G \left(\operatorname{Ind}_H^{H'} M \right).$$

2. Let M be a G-module, then there is a canonical isomorphism

$$M \simeq \operatorname{Ind}_G^G M.$$

3. Let G, H, and G' be groups such that G' acts on G and H is stable under the action of G'. If M is a $H \rtimes G'$-module, then we have an isomorphism of $G \rtimes G'$-modules

$$\operatorname{Ind}_H^G M \simeq \operatorname{Ind}_{H \rtimes G'}^{G \rtimes G'} M.$$

4. If G, G', $H < G$, and $H' < G'$ are groups and M an H-module and M' an H'-module, then

$$(\operatorname{Ind}_H^G M) \otimes (\operatorname{Ind}_{H'}^{G'} M') \simeq \operatorname{Ind}_{H \times H'}^{G \times G'} (M \otimes M').$$

Chapter 6

A Special Case of the Breuil-Schneider Conjecture

6.1 The Breuil-Schneider Conjecture

In their paper [4], Christophe Breuil and Peter Schneider formulate a deep conjecture associated to de Rham representations. Our treatment of a special case of the Breuil-Schneider conjecture will utilize similar notations used in loc. cit. We briefly recall the statement of the general conjecture.

Let p be a rational prime number and $L/\mathbb{Q}_p$ and $K/\mathbb{Q}_p$ finite field extensions satisfying $[L : \mathbb{Q}_p] = |\operatorname{Hom}_{\mathbb{Q}_p}(L, K)|$. We call L the *base field* and K the *coefficient field*, and view both fields as living in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $q := p^f$ denote the cardinality of k_L (the residue field of L), and let L_0 denote the maximal unramified subfield of L. We let ord_p denote the additive p-adic valuation on $\mathbb{Q}_p$, normalized so that $\operatorname{ord}_p(p) = 1$. For any extension $F/\mathbb{Q}_p$ we let $W(\overline{\mathbb{Q}_p}/F)$ denote the Weil group of F.

Let $n \in \mathbb{Z}_{\geq 2}$ and set $G_n := \operatorname{GL}_n(L)$. Let L'/L be a finite Galois extension of L with maximal unramified subfield L_0'. Let $p^{f'}$ denote the cardinality $k_{L_0'}$ (the residue field of L_0') and assume $[L_0' : \mathbb{Q}_p] = |\operatorname{Hom}_{\mathbb{Q}_p}(L_0', K)|$. There is an equivalence of categories

$$\operatorname{UNRAM}_F \longleftrightarrow \operatorname{FIN}_{k_F}$$

where UNRAM_F denotes the category of finite unramified extensions of a local field F, and FIN_{k_F} is the category of finite extensions of the perfect residue field of F. The above

equivalence yields an identification of Galois groups

$$\mathrm{Gal}(F'/F) \simeq \mathrm{Gal}(k_{F'}/k_F),$$

for finite unramified extensions F'/F. In our case, we have the identification

$$\mathrm{Gal}(L_0'/\mathbb{Q}_p) \simeq \mathrm{Gal}(k_{L_0'}/\mathbb{F}_p),$$

and let $\phi_0' : L_0' \to L_0'$ denote the Frobenius of L_0' corresponding the arithmetic Frobenius $\Phi \in \mathrm{Gal}(k_{L_0'}/\mathbb{F}_p)$. We can now give a brief description of the following two important categories $\mathrm{WD}_{L'/L}$ and $\mathrm{MOD}_{L'/L}$.

We denote by $\mathrm{WD}_{L'/L}$ the category of Weil-Deligne representations over K, whose objects are triples (r, N, V) where

- $r : W(\overline{\mathbb{Q}_p}/L) \to \mathrm{GL}(V)$ is a continuous representation whose restriction to $W(\overline{\mathbb{Q}_p}/L')$ is unramified.

- V is a finite dimensional K-vector space endowed with the discrete topology. Consequently, continuity of r is equivalent to $\ker(r)$ being open.

- N is an endomorphism of V satisfying the third property of definition 5.4, implying N is nilpotent.

We denote by $\mathrm{MOD}_{L'/L}$ the category of (ϕ, N)-modules whose objects are quadruples $(\phi, N, \mathrm{Gal}(L'/L), D)$ where

- D is a free $L_0' \otimes_{\mathbb{Q}_p} K$ module of finite rank.

- $\phi : D \to D$ is a Frobenius, that is, an $L_0' \otimes_{\mathbb{Q}_p} K$-linear bijection satisfying

$$\phi((l_0' \otimes k) \cdot d) := (\phi_0'(l_0') \otimes k)\phi(d),$$

  for all $l_0' \in L_0', k \in K$, and $d \in D$.

- $N : D \to D$ is an $L_0' \otimes_{\mathbb{Q}_p} K$-linear (nilpotent) endomorphism satisfying $N\phi = p\phi N$.

- D carries an action of $\mathrm{Gal}(L'/L)$ which commutes with the actions of ϕ and N, and such that

$$\sigma((l_0' \otimes k) \cdot d) = (\sigma(l_0') \otimes k)\sigma(d)$$

for all $l_0' \in L_0', k \in K, d \in D$, and $\sigma \in \mathrm{Gal}(L'/L)$.

There are exact functors

$$\mathrm{WD} : \mathrm{MOD}_{L'/L} \to \mathrm{WD}_{L'/L}$$

and

$$\mathrm{MOD} : \mathrm{WD}_{L'/L} \to \mathrm{MOD}_{L'/L}$$

which provide an equivalence of categories. We briefly describe the construction of WD since it will be used in our proof of a special case of the Breuil-Schneider conjecture. A detailed description of WD can be found in [4] and [11].

Let $(\phi, N, \mathrm{Gal}(L'/L), D)$ be an object in $\mathrm{MOD}_{L'/L}$ and fix some embedding $\sigma_0' : L_0' \hookrightarrow K$. Corresponding to the isomorphism

$$L_0' \otimes_{\mathbb{Q}_p} K \xrightarrow{\sim} \prod_{\sigma_0':L_0'\hookrightarrow K} K \tag{6.1}$$

$$l_0' \otimes k \longmapsto (\sigma_0'(l_0')k)_{\sigma_0':L_0'\hookrightarrow K}$$

we have the decomposition

$$D \simeq \prod_{\sigma_0':L_0'\hookrightarrow K} D_{\sigma_0'} \tag{6.2}$$

where

$$D_{\sigma_0'} = (0, \ldots, 0, 1_{\sigma_0'}, 0, \ldots, 0)D$$

is a K-vector space. Setting $V := D_{\sigma_0'}$ and restricting N to the finite dimensional K-vector space V we obtain a nilpotent K-linear endomorphism of V, which, by abuse of notation, we denote by N. To construct $r : W(\overline{\mathbb{Q}_p}/L) \to \mathrm{GL}(V)$ we consider the diagram

where Galois restriction gives a surjective morphism

$$\mathrm{res}_{L'} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \to \mathrm{Gal}(L'/L).$$

For each $w \in W(\overline{\mathbb{Q}_p}/L)$ we let $\overline{w} := \mathrm{res}_{L'}(w)$ and define $r(w) := \overline{w} \circ \phi^{\alpha(w)}$, where $\alpha(w) \in f\mathbb{Z}$ is the unique integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is the $\alpha(w)$-th power of the arithmetic Frobenius $x \mapsto x^p$. Then, $r(w) \in \mathrm{End}_K(V)$, and the triple (r, N, V) is an object of $\mathrm{WD}_{L'/L}$. It is shown in [8] that the above construction does not depend on the choice of embedding $\sigma'_0 : L'_0 \hookrightarrow K$.

We now remind the reader of the statement of the Breuil-Schneider conjecture as given in [4]. Starting with the following

1. an object (r, N, V) of $\mathrm{MOD}_{L'/L}$ such that r is semisimple

2. to each embedding $\sigma : L \hookrightarrow K$, an increasing list of n integers, $i_{1,\sigma} < i_{2,\sigma} < \ldots < i_{n,\sigma}$,

a modified version [1] of the classical local Langlands correspondence applied to the object (r, N, V) gives a smooth admissible representation $\pi_{\mathrm{sm}}(r)$ of $G_n := \mathrm{GL}_n(L)$. To the strictly increasing list of n integers $i_{1,\sigma} < \ldots < i_{n,\sigma}$ we associate a new increasing sequence of integers $a_{1,\sigma} \leq a_{2,\sigma} \leq \ldots \leq a_{n,\sigma}$ where for each $1 \leq j \leq n$

$$a_{j,\sigma} := -i_{n+1-j,\sigma} - (j-1). \tag{6.3}$$

Let ρ_σ denote the irreducible algebraic representation of G_n over K of highest weight $(a_{1,\sigma}, a_{2,\sigma}, \ldots, a_{n,\sigma})$ relative to the upper triangular Borel $B_n \subset G_n$, and let $\pi_{\mathrm{alg}}(r)$ be the irreducible algebraic representation of G_n associated to $\bigotimes_{\sigma : L \hookrightarrow K} \rho_\sigma$.

Conjecture 6.1 (Breuil-Schneider). *With (r, N, V), $\pi_{\mathrm{sm}}(r)$, and $\pi_{\mathrm{alg}}(r)$ as above, the following are equivalent*

i.) The locally algebraic representation $\mathrm{BS}(r) := \pi_{\mathrm{sm}}(r) \otimes_K \pi_{\mathrm{alg}}(r)$ admits a G_n-invariant norm.

ii.) There is an object $(\phi, N, \mathrm{Gal}(L'/L), D)$ of $\mathrm{MOD}_{L'/L}$ such that

$$\mathrm{WD}(\phi, N, \mathrm{Gal}(L'/L), D)^{\mathrm{F-ss}} = (r, N, V),$$

---

[1] If r is not generic, then one replaces $\pi_{\mathrm{sm}}(r)$ with a (possibly reducible) generic representation.

and a $\mathrm{Gal}(L'/L)$-preserved admissible filtration $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ on $D_{L'}$ (i.e. for sub-objects, the Newton polygon lies above the Hodge polygon, and they have the same endpoints) such that, for all embeddings $\sigma : L \hookrightarrow K$

$$\mathrm{Fil}^i D_{L',\sigma}/\mathrm{Fil}^{i+1} D_{L',\sigma} \neq 0 \text{ if and only if } i \in \{i_{1,\sigma}, i_{2,\sigma}, \ldots, i_{n,\sigma}\}.$$

It is known that *i.)* implies *ii.)*, which was shown by Hu in his thesis [19]. In fact, Hu showed that *ii.)* is equivalent to Emerton's condition [19, p.118]. There are certain cases in which *ii.)* implies *i.)*, some of which can be found in [4, p.19-23], however, in the indecomposable case, the implication *ii.)* implies *i.)* was shown by Sorensen [40].

6.2 A Special Case of $\mathrm{BS}(\rho)$

The main result of this part of the manuscript is a proof of the following special case of the Breuil-Schneider conjecture along with a natural generalization

Theorem 6.2. *Let L, K be as above, and denote $G_L := \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$. If $\rho : G_L \to \mathrm{GL}_n(K)$ is a potentially semistable representation with subrepresentation ρ_1 and quotient representation ρ_2, then, under the following hypotheses*

i.) For each $\sigma : L \hookrightarrow K$ the Hodge-Tate weights satisfy $\mathrm{HT}_\sigma(\rho_1) < \mathrm{HT}_\sigma(\rho_2)$,

ii.) If $\mathrm{WD}(D_{\mathrm{dR}}(\rho)) = (r, N, V)$ and ϕ denotes a lift of the geometric Frobenius, and if the eigenvalues of $r(\phi)$ satisfy $\{\mu_1, \ldots, \mu_{n_1}, \nu_{n_1+1}, \ldots, \nu_{n_2}\}$ (here the $\{\mu_1, \ldots, \mu_{n_1}\}$ and $\{\nu_{n_1+1}, \ldots, \nu_{n_2}\}$ correspond to the matrix blocks for ρ_1 and ρ_2, respectively), then $\frac{\mu_k}{\nu_j} \neq p$ for all $1 \leq k \leq n_1$ and $n_1 + 1 \leq j \leq n_2$,

iii.) ρ is generic i.e. $\pi_{\mathrm{sm}}(\rho)$ is generic and irreducible,

iv.) For $i = 1, 2$ the representations $\mathrm{WD}(D_{\mathrm{dR}}(\rho_i))$ are indecomposable,

the locally algebraic representation $\mathrm{BS}(\rho)$ admits a G_n-invariant norm.

We briefly note that the condition $\mathrm{HT}_\sigma(\rho_1) < \mathrm{HT}_\sigma(\rho_2)$ simply denotes the increasing nature of the Hodge-Tate weights between the two sets, namely, if $\mathrm{HT}_\sigma(\rho_1) = \{i_{1,\sigma} < \ldots < i_{n_1,\sigma}\}$ and $\mathrm{HT}_\sigma(\rho_2) = \{j_{1,\sigma} < \ldots < j_{n_2,\sigma}\}$, then $\mathrm{HT}_\sigma(\rho_1) < \mathrm{HT}_\sigma(\rho_2)$ means $i_{n_1,\sigma} < j_{1,\sigma}$. When both ρ_1 and ρ_2 are characters, this condition is the usual ordinarity condition for Galois representations.

Proof of Theorem 6.2. We will utilize the following notations,

Notation: For $i = 1, 2$ let $n_i = \dim(\rho_i)$ so that $n = n_1 + n_2$ and set $G_n := \mathrm{GL}_n(L)$, $G_{n_1} := \mathrm{GL}_{n_1}(L)$, and $G_{n_2} := \mathrm{GL}_{n_2}(L)$, then

- T_n will denote the subgroup of diagonal matrices of G_n.

- U_n will denote the subgroup of unipotent matrices of G_n.

- B_n will denote the Borel subgroup of upper triangular matrices of G_n so that $B_n \simeq T_n \ltimes U_n$.

- Corresponding to the partition $n = n_1 + n_2$ we let P denote the unique parabolic subgroup of G_n given by

$$P := \left\{ \begin{pmatrix} g_{n_1} & * \\ 0 & g_{n_2} \end{pmatrix} : g_{n_i} \in G_{n_i} \text{ for } i = 1, 2 \right\} = MN,$$

where N is the unipotent radical and $M \simeq G_{n_1} \times G_{n_2}$.

Algebraic Side of $\mathrm{BS}(\rho)$:

In $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_L)$ we have the short exact sequence

$$1 \longrightarrow \rho_1 \longrightarrow \rho \longrightarrow \rho_2 \longrightarrow 1.$$

Applying Fontaine's faithful exact functor D_{dR} gives

$$0 \longrightarrow D_{\mathrm{dR}}(\rho_1) \longrightarrow D_{\mathrm{dR}}(\rho) \longrightarrow D_{\mathrm{dR}}(\rho_2) \longrightarrow 0$$

in $\mathrm{MOD}_{L'/L}$. Since morphisms in $\mathrm{MOD}_{L'/L}$ preserve filtration degree, the Hodge-Tate weights of ρ are sums of the Hodge-Tate weights of ρ_1 and ρ_2. Let

$$\mathrm{HT}_\sigma(\rho_1) := \{i_{1,\sigma} < \ldots < i_{n_1,\sigma}\} \qquad \text{and} \qquad \mathrm{HT}_\sigma(\rho_2) := \{j_{1,\sigma} < \ldots < j_{n_2,\sigma}\}$$

and assume $i_{n_1,\sigma} < j_{1,\sigma}$. Associated to $\mathrm{HT}_\sigma(\rho_1)$ and $\mathrm{HT}_\sigma(\rho_2)$ are sequences of increasing integers $a_{1,\sigma} \leq \ldots \leq a_{n_1,\sigma}$ and $b_{1,\sigma} \leq \ldots \leq b_{n_2,\sigma}$, respectively, which are given by (6.3). Corresponding to these sequences are characters $\lambda_\sigma : T_{n_1} \to K$ and $\mu_\sigma : T_{n_2} \to K$ defined by

$$\lambda_\sigma(t_1, \ldots, t_{n_1}) = t_1^{a_{1,\sigma}} \ldots t_{n_1}^{a_{n_1},\sigma} \qquad \text{and} \qquad \mu_\sigma(t_{n_1+1}, \ldots, t_{n_2}) = t_{n_1+1,\sigma}^{b_{1,\sigma}} \ldots t_{n_2}^{b_{n_2},\sigma}.$$

Using the decompositions $B_{n_1} = T_{n_1} \ltimes U_{n_1}$ and $B_{n_2} = T_{n_2} \ltimes U_{n_2}$, we obtain characters $\tilde{\lambda}_\sigma$ and $\tilde{\mu}_\sigma$ given by

$$\tilde{\lambda}_\sigma(b) = \lambda_\sigma(t)$$
$$\tilde{\mu}_\sigma(b') = \mu_\sigma(t')$$

where $b = tu \in B_{n_1}$ and $b' = t'u' \in B_{n_2}$. Due to our assumptions on the Hodge-Tate weights we obtain irreducible representations

$$\pi_{\mathrm{alg},\sigma}(\rho_1) = \mathrm{Ind}_{B_{n_1}}^{G_{n_1}} \tilde{\lambda}_\sigma \qquad \text{and} \qquad \pi_{\mathrm{alg},\sigma}(\rho_2) = \mathrm{Ind}_{B_{n_2}}^{G_{n_2}} \tilde{\mu}_\sigma.$$

Furthermore, the increasing nature of the Hodge-Tate weights for ρ_1 and ρ_2 along with their connection to the Hodge-Tate weights of ρ give the irreducible representation

$$\pi_{\mathrm{alg},\sigma}(\rho) := \mathrm{Ind}_{B_n}^{G_n}(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma).$$

The decompositions $P = (G_{n_1} \times G_{n_2}) \ltimes N$ and $B_n = (B_{n_1} \times B_{n_2}) \ltimes N$, combined with properties of algebraic induction give

$$\begin{aligned}
\pi_{\mathrm{alg},\sigma}(\rho) &= \mathrm{Ind}_{B_n}^{G_n}(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma) \\
&\simeq \mathrm{Ind}_P^{G_n}\left(\mathrm{Ind}_{B_n}^P(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma)\right) \\
&\simeq \mathrm{Ind}_P^{G_n}\left(\mathrm{Ind}_{(B_{n_1} \times B_{n_2}) \ltimes N}^{(G_{n_1} \times G_{n_2}) \ltimes N}(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma)\right) \\
&\simeq \mathrm{Ind}_P^{G_n}\left(\mathrm{Ind}_{B_{n_1} \times B_{n_2}}^{G_{n_1} \times G_{n_2}}(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma)\right)
\end{aligned} \tag{6.4}$$

Furthermore,

$$\begin{aligned}
\pi_{\mathrm{alg},\sigma}(\rho_1) \otimes \pi_{\mathrm{alg},\sigma}(\rho_2) &= \mathrm{Ind}_{B_{n_1}}^{G_{n_1}} \tilde{\lambda}_\sigma \otimes \mathrm{Ind}_{B_{n_2}}^{G_{n_2}} \tilde{\mu}_\sigma \\
&\simeq \mathrm{Ind}_{B_{n_1} \times B_{n_2}}^{G_{n_1} \times G_{n_2}}(\tilde{\lambda}_\sigma \otimes \tilde{\mu}_\sigma).
\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5)

$$\pi_{\mathrm{alg},\sigma}(\rho) \simeq \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{alg},\sigma}(\rho_1) \otimes \pi_{\mathrm{alg},\sigma}(\rho_2)).$$

Considering all embeddings $\sigma : L \hookrightarrow K$ gives an irreducible algebraic representation of

$\prod_{\sigma:L\hookrightarrow K}\mathrm{GL}_n(K)$, denoted

$$\bigotimes_{\sigma:L\hookrightarrow K}\pi_{\mathrm{alg},\sigma}(\rho).$$

Consider now the connected reductive algebraic group $\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n$ over the infinite field $\mathbb{Q}_p$, defined by

$$(\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n)(A) := \mathrm{GL}_n(A\otimes_{\mathbb{Q}_p}L)$$

for A any $\mathbb{Q}_p$-algebra. We therefore have

$$(\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n)(\mathbb{Q}_p) = \mathrm{GL}_n(L) \qquad \text{and} \qquad (\mathrm{Res}_{L/\mathbb{Q}_p}\mathrm{GL}_n)(K) = \prod_{\sigma:L\hookrightarrow K}\mathrm{GL}_n(K).$$

Using a result in [3, p.220], we see $\mathrm{GL}_n(L) \subset \prod_{\sigma:L\hookrightarrow K}\mathrm{GL}_n(K)$ is dense, where $\mathrm{GL}_n(L) \hookrightarrow \prod_{\sigma:L\hookrightarrow K}\mathrm{GL}_n(K)$ is embedded diagonally. We therefore let $\pi_{\mathrm{alg}}(\rho)$ be the irreducible algebraic representation of $\mathrm{GL}_n(L)$ obtained by restriction of $\bigotimes_{\sigma:L\hookrightarrow K}\pi_{\mathrm{alg},\sigma}(\rho)$ to $\mathrm{GL}_n(L)$.

Smooth Side of $\mathrm{BS}(\rho)$**:**

Since the representation ρ is potentially semistable, the subrepresentation ρ_1, and quotient ρ_2, are both potentially semistable. As above, Fontaine's faithful exact functor D_{dR} gives the short exact sequence

$$0 \longrightarrow D_{\mathrm{dR}}(\rho_2) \longrightarrow D_{\mathrm{dR}}(\rho) \longrightarrow D_{\mathrm{dR}}(\rho_2) \longrightarrow 0 \qquad (6.6)$$

in the category $\mathrm{MOD}_{L'/L}$. Since we will be working with the actual quadruples themselves, we let

$$D_{\mathrm{dR}}(\rho) := (\phi, N, \mathrm{Gal}(L'/L), D)$$
$$D_{\mathrm{dR}}(\rho_1) := (\phi_1, N_1, \mathrm{Gal}(L'/L), D_1)$$
$$D_{\mathrm{dR}}(\rho_2) := (\phi_2, N_2, \mathrm{Gal}(L'/L), D_2).$$

We must now show that the functor $\mathrm{WD}: \mathrm{MOD}_{L'/L} \to \mathrm{WD}_{L'/L}$ preserves exactness of (6.6). From (6.6) we have an exact sequence

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0 \qquad (6.7)$$

of free $L'_0 \otimes_{\mathbb{Q}_p} K$-modules of finite rank, hence $D \simeq D_1 \oplus D_2$. Corresponding to the decompositions in (6.1) and (6.2), for each $\sigma'_0 : L'_0 \hookrightarrow K$, we have the direct sum of

K-vector spaces

$$D_{\sigma_0'} \simeq D_{1,\sigma_0'} \oplus D_{2,\sigma_0'}. \tag{6.8}$$

Letting

$$\mathrm{WD}(\rho) := \mathrm{WD}(D_{\mathrm{dR}}(\rho)) := (r, N, V)$$
$$\mathrm{WD}(\rho_1) := \mathrm{WD}(D_{\mathrm{dR}}(\rho_1)) := (r_1, N_1, V_1)$$
$$\mathrm{WD}(\rho_2) := \mathrm{WD}(D_{\mathrm{dR}}(\rho_2)) := (r_2, N_2, V_2)$$

where $V := D_{\sigma_0'}$, $V_1 := D_{1,\sigma_0'}$, and $V_2 := D_{2,\sigma_0'}$ are the K-vector spaces coming from our chosen embedding σ_0' (recall that the constructions do not depend on the choice of σ_0'), we see from (6.8) that $V \simeq V_1 \oplus V_2$ and therefore we have exactness of

$$0 \longrightarrow \mathrm{WD}(\rho_1) \longrightarrow \mathrm{WD}(\rho) \longrightarrow \mathrm{WD}(\rho_2) \longrightarrow 0. \tag{6.9}$$

Since $\overline{K}$ is flat over K, tensoring (6.9) with $\overline{K}$ preserves exactness. Therefore we view the representations in (6.9) as being over $\overline{K}$.

Let $\phi \in W(\overline{\mathbb{Q}_p}/L)$ be a lift of the geometric Frobenius of $\mathrm{Gal}(\overline{k}_L/k_L)$. The action of $r(\phi)$ gives an eigenbasis for V, in which the monodromy operators N, N_1 and N_2 are related in matrix form by

$$N = \left(\begin{array}{c|c} N_1 & * \\ \hline 0 & N_2 \end{array} \right),$$

where the matrix for $r(\phi)$ is

$$r(\phi) = \left(\begin{array}{c|c} D_{n_1} & 0 \\ \hline 0 & D_{n_2} \end{array} \right),$$

where

$$D_{n_1} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{n_1} \end{pmatrix} \quad \text{and} \quad D_{n_2} = \begin{pmatrix} \nu_{n_1+1} & & \\ & \ddots & \\ & & \nu_{n_2} \end{pmatrix}.$$

Lemma 6.3. *If $\frac{\mu_i}{\nu_j} \neq p$ for all $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$, then $N = \left(\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array} \right)$,*

i.e. $N = N_1 \oplus N_2$.

Proof of Lemma 6.3. Setting $N = (c_{ij})$ we analyze the conjugation relation of the Weil-Deligne representation (r, N, V) associated to ϕ, namely $r(\phi)Nr(\phi)^{-1} = pN$. If $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$, then the (i,j)-th entry of $r(\phi)Nr(\phi)^{-1}$ is $\frac{\mu_i}{\nu_j}c_{i,j}$. Consequently, the zero matrix $r(\phi)Nr(\phi)^{-1} - pN = 0$ has (i,j)th entry $(\frac{\mu_i}{\nu_j} - p)c_{ij} = 0$. Therefore, since $\frac{\mu_i}{\nu_j} \neq p$ for all $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$, we obtain $N = \left(\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array} \right)$. $\qquad\square$

Returning to the proof of the main result, we apply Lemma 6.3 to our situation and use additivity of semisimplification of representations over an algebraically closed field of characteristic zero, i.e.

$$r^{\mathrm{ss}} \simeq r_1^{\mathrm{ss}} \oplus r_2^{\mathrm{ss}},$$

to obtain

$$\mathrm{WD}(\rho)^{\mathrm{F-ss}} \simeq \mathrm{WD}(\rho_1)^{\mathrm{F-ss}} \oplus \mathrm{WD}(\rho_2)^{\mathrm{F-ss}}. \tag{6.10}$$

Applying the local Langlands correpondence to (6.10) yields smooth irreducible admissible representations $\pi_{\mathrm{sm}}(\rho), \pi_{\mathrm{sm}}(\rho_1)$, and $\pi_{\mathrm{sm}}(\rho_2)$ such that

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) = \mathrm{WD}(\rho)^{\mathrm{F-ss}}$$
$$\mathrm{rec}_{n_1}(\pi_{\mathrm{sm}}(\rho_1)) = \mathrm{WD}(\rho_1)^{\mathrm{F-ss}}$$
$$\mathrm{rec}_{n_2}(\pi_{\mathrm{sm}}(\rho_2)) = \mathrm{WD}(\rho_2)^{\mathrm{F-ss}}$$

and

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) \simeq \mathrm{rec}_{n_1}(\pi_{\mathrm{sm}}(\rho_1)) \oplus \mathrm{rec}_{n_2}(\pi_{\mathrm{sm}}(\rho_2)).$$

Furthermore, from [18, p.252]

$$\mathrm{rec}_{n_1}(\pi_{\mathrm{sm}}(\rho_1)) \oplus \mathrm{rec}_{n_2}(\pi_{\mathrm{sm}}(\rho_2)) \simeq \mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho_1) \boxplus \pi_{\mathrm{sm}}(\rho_2)),$$

where $\boxplus$ is the Langlands sum defined in [18, p. 32]. Therefore

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) \simeq \mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho_1) \boxplus \pi_{\mathrm{sm}}(\rho_2)),$$

whereby,

$$\pi_{\mathrm{sm}}(\rho) \simeq \pi_{\mathrm{sm}}(\rho_1) \boxplus \pi_{\mathrm{sm}}(\rho_2)$$

by the injectivity of rec_n. Since ρ is generic, namely $\pi_{\mathrm{sm}}(\rho)$ is generic and irreducible, $\pi_{\mathrm{sm}}(\rho)$ is fully induced [23, p.374]

$$\pi_{\mathrm{sm}}(\rho) \simeq \pi_{\mathrm{sm}}(\rho_1) \boxplus \pi_{\mathrm{sm}}(\rho_2) \simeq \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{sm}}(\rho_1) \otimes \pi_{\mathrm{sm}}(\rho_2)).$$

Without the genericity condition on ρ, one would replace $\pi_{\mathrm{sm}}(\rho)$ by a generic principal series whose unique irreducible quotient is $\pi_{\mathrm{sm}}(\rho)$.

Existence of an Invariant Norm for $\mathrm{BS}(\rho)$: From the above computations

$$\mathrm{BS}(\rho) \simeq \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{alg}}(\rho_1) \otimes \pi_{\mathrm{alg}}(\rho_2)) \otimes_K \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{sm}}(\rho_1) \otimes \pi_{\mathrm{sm}}(\rho_2)).$$

For $i = 1, 2$ we let $\mathrm{BS}(\rho_i) := \pi_{\mathrm{alg}}(\rho_i) \otimes_K \pi_{\mathrm{sm}}(\rho_i)$ and

$$\iota : (\pi_{\mathrm{alg}}(\rho_1) \otimes \pi_{\mathrm{alg}}(\rho_2)) \otimes (\pi_{\mathrm{sm}}(\rho_1) \otimes \pi_{\mathrm{sm}}(\rho_2)) \xrightarrow{\sim} (\pi_{\mathrm{alg}}(\rho_1) \otimes \pi_{\mathrm{sm}}(\rho_1)) \otimes (\pi_{\mathrm{alg}}(\rho_2) \otimes \pi_{\mathrm{sm}}(\rho_2)),$$

the isomorphism given on elementary tensors by $\iota((f \otimes g) \otimes (f' \otimes g')) = (f \otimes f') \otimes (g \otimes g')$. Let $(f, g) \in \mathrm{BS}(\rho)$, so $f \in \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{alg}}(\rho_1) \otimes \pi_{\mathrm{alg}}(\rho_2))$ and $g \in \mathrm{Ind}_P^{G_n}(\pi_{\mathrm{sm}}(\rho_1) \otimes \pi_{\mathrm{sm}}(\rho_2))$. Then for any $p \in P$ and $x \in G_n$

$$\begin{aligned}
\iota(f \otimes g)(px) &= \iota(f(px) \otimes g(px)) \\
&= \iota(p \cdot f(x) \otimes p \cdot g(x)) \\
&= \iota(p \cdot (f(x) \otimes g(x))) \\
&= p \cdot (\iota(f(x) \otimes g(x)))
\end{aligned}$$

and therefore we obtain a G_n-equivariant map

$$\mathrm{BS}(\rho) \longrightarrow \mathrm{Ind}_P^{G_n}(\mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2)) \qquad (6.11)$$
$$(f, g) \mapsto \iota \circ (f \otimes g).$$

Since $\pi_{\mathrm{alg}}(\rho)$ is an irreducible algebraic representation and $\pi_{\mathrm{sm}}(\rho)$ is an irreducible smooth representation, a result of Dipendra Prasad [37, p.126] states that the locally algebraic representation $\mathrm{BS}(\rho)$ is irreducible, and therefore, since (6.11) is not the zero map

$$\mathrm{BS}(\rho) \hookrightarrow \mathrm{Ind}_P^{G_n}(\mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2)). \qquad (6.12)$$

To the indecomposable $\mathrm{WD}(\rho_1), \mathrm{WD}(\rho_2)$, a result of Sorensen [40] yields G_{n_i}-invariant norms $|| \cdot ||_i$ on $\mathrm{BS}(\rho_i)$ for $i = 1, 2$, respectively. Following [34] we obtain a norm [34, Proposition 17.4] $|| \cdot || = || \cdot ||_1 \otimes_K || \cdot ||_2$ on $\mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2)$ by defining

$$||u|| := \inf_u \left\{ \max_{1 \leq i \leq r} ||v_i||_1 ||w_i||_2 \right\},$$

where the infimum is taken over all representations $u = \sum_{i=1}^r v_i \otimes w_i$ where $v_i \in \mathrm{BS}(\rho_1)$ and $w_i \in \mathrm{BS}(\rho_2)$ for all $1 \leq i \leq r$. For $i = 1, 2$ the $|| \cdot ||_i$ are G_{n_i}-invariant, therefore the tensor product norm $|| \cdot ||$ is $G_{n_1} \times G_{n_2}$-invariant.

We now consider the set $C(G_n, \mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2); ||\cdot||_\infty)$ of continuous functions $f : G_n \to \mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2)$, such that $f(px) = p \cdot f(x)$ for all $p \in P$ and all $x \in G_n$, and where $|| \cdot ||_\infty$ is the sup-norm

$$||f||_\infty := \sup_{g \in G_n} ||f(g)||.$$

Let G_n act on the left of $C(G_n, \mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2); || \cdot ||_\infty)$ via right translations, namely

$$(\sigma \cdot f)(x) := f(x\sigma),$$

for all $x, \sigma \in G_n$ and all $f \in \mathrm{Ind}_P^{G_n}(\mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2))$. A priori it is unclear why $|| \cdot ||_\infty$ should be finite, however, the compactness of $P \backslash G_n$ coupled with the G_n-invariance (and therefore P-invariance) of $|| \cdot ||$ show that $|| \cdot ||_\infty$ is really defined on the compact quotient $P \backslash G_n$, namely

$$||f||_\infty = \sup_{Pg \in P \backslash G_n} ||f(Pg)|| < \infty,$$

for all $f \in C(G_n, \mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2); || \cdot ||_\infty)$. Any $x \in G_n$ gives a bijection

$$P \backslash G_n \longrightarrow P \backslash G_n$$
$$Pg \longmapsto Pgx$$

and therefore

$$||xf|| = \sup_{Pg \in P \backslash G} ||f(Pgx)|| = \sup_{Ph \in P \backslash G} ||f(Ph)|| = ||f||_\infty.$$

hence $||\cdot||_\infty$ is G_n-invariant. Viewing $\mathrm{BS}(\rho) \subset \mathrm{Ind}_P^{G_n}(\mathrm{BS}(\rho_1) \otimes_K \mathrm{BS}(\rho_2)) \subset C(G_n, \mathrm{BS}(\rho_1) \otimes_K$

BS(ρ_2); $|| \cdot ||_\infty$), restriction of the G_n-invariant norm $|| \cdot ||_\infty$ to the subspace BS(ρ) gives a G_n-invariant norm on BS(ρ). $\qquad \square$

The above proof provides the base case for an induction process used to prove a corollary to the above theorem under a certain modification of the Frobenius eigenvalues hypothesis. We briefly explain these additional Frobenius eigenvalue assumptions. Suppose ϕ is a lift of the geometric Frobenius such that (in a suitable basis) the Weil-Deligne representation (s, N, V) satisfies

$$s(\phi) = \begin{pmatrix} D_{n_1} & 0 & \ldots & 0 \\ 0 & D_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & D_{n_r} \end{pmatrix},$$

where, for each $1 \le i \le r$, the $D_{n_i} \in \mathrm{GL}_{n_i}(K)$ is the diagonal matrix

$$D_{n_i} = \begin{pmatrix} \mu_1^{(i)} & 0 & \ldots & 0 \\ 0 & \mu_2^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mu_{n_i}^{(i)} \end{pmatrix}.$$

If (in the same basis) the nilpotent endomorphism N has matrix form

$$N = \begin{pmatrix} N_{n_1} & * & \ldots & * \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & N_{n_r} \end{pmatrix},$$

then our Frobenius eigenvalue conditions are as follows

Hypothesis 6.1 (Frobenius Eigenvalue Hypothesis). *For all $1 \le i < j \le r$ if x and y are elements on the diagonals of D_{n_i} and D_{n_j}, respectively, then we assume $\frac{x}{y} \ne p$.*

We are therefore assuming that ordered quotients of eigenvalues of $s(\phi)$, occurring in distinct blocks, are not equal to p. This condition will be necessary for obtaining a direct sum decomposition of monodromy operators in the following corollary to Theorem 7.2

Corollary 6.4. *Let $L, K,$ and G_L be as above. Let $\rho : G_L \to \mathrm{GL}_n(K)$ be a potentially semistable representation of the form*

$$\rho = \left(\begin{array}{cccc|c} \rho_{n_1} & * & \cdots & * & \\ 0 & \rho_{n_2} & \ddots & \vdots & \\ \vdots & \ddots & \ddots & * & * \\ 0 & \cdots & 0 & \rho_{n_{r-1}} & \\ \hline & & 0 & & \rho_{n_r} \end{array} \right) = \left(\begin{array}{c|c} \underline{\rho} & * \\ \hline 0 & \rho_{n_r} \end{array} \right),$$

so

$$0 \longrightarrow \underline{\rho} \longrightarrow \rho \longrightarrow \rho_{n_r} \longrightarrow 0$$

is exact. If the following hypotheses hold

i.) For each $\sigma : L \hookrightarrow K$

$$\mathrm{HT}_\sigma(\rho_{n_1}) < \mathrm{HT}_\sigma(\rho_{n_2}) < \ldots < \mathrm{HT}_\sigma(\rho_{n_r}).$$

ii.) Let $\mathrm{WD}(D_{\mathrm{dR}}(\rho)) = (r, N, V)$ and denote a lift of the geometric Frobenius by ϕ. Suppose the eigenvalues of $r(\phi)$ satisfy the Frobenius eigenvalue condition as stated in Hypothesis 7.1.

iii.) ρ is generic i.e. $\pi_{\mathrm{sm}}(\rho)$ is generic and irreducible.

iv.) For $i = 1, 2, \ldots, r$ the representations $\mathrm{WD}(D_{\mathrm{dR}}(\rho_{n_i}))$ are indecomposable.

then $\mathrm{BS}(\rho)$ admits a G_n-invariant norm.

Proof. The proof will proceed by induction on r, in particular, we see that Theorem 7.2 is precisely the base case $r = 2$.

Notation: For $i = 1, 2, \ldots, r$ let $n_i := \dim(\rho_{n_i})$ so that $n = n_1 + n_2 + \ldots + n_r$ and set $G_{n_i} := \mathrm{GL}_{n_i}(L)$ for all $1 \le i \le r$, then

- T_n will denote the subgroup of diagonal matrices of G_n.

- U_n will denote the subgroup of unipotent matrices of G_n.

- B_n will denote the Borel subgroup of upper triangular matrices of G_n so that $B_n \simeq T_n \ltimes U_n$.

- Corresponding to the partition $n = n_1 + n_2 + \ldots + n_r$ we let P denote the unique parabolic subgroup of G_n given by

$$
P := \left\{ \begin{pmatrix} g_{n_1} & * & \cdots & * \\ 0 & g_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & g_{n_r} \end{pmatrix} : g_{n_i} \in G_{n_i} \text{ for all } 1 \le i \le r \right\} = MN,
$$

  where N is the unipotent radical and $M \simeq G_{n_1} \times \ldots \times G_{n_r}$.

For any $\sigma : L \hookrightarrow K$ we have characters $\underline{\lambda}_\sigma : T_{n-n_r} \to K$ and $\lambda_{n_r} : T_{n_r} \to K$ given by

$$
\underline{\lambda}_\sigma(t_1, \ldots, t_{n-n_r}) = t_1^{a_{1,\sigma}} t_2^{a_{2,\sigma}} \ldots t_{n_{r-1}}^{a_{n_{r-1},\sigma}}
$$
$$
\lambda_{n_r,\sigma}(t_{n_{r-1}+1}, \ldots, t_n) = t_{n_{r-1}+1}^{b_{n_{r-1}+1,\sigma}} t_{n_{r-1}+2}^{b_{n_{r-1}+2,\sigma}} \ldots t_n^{b_{n,\sigma}}
$$

where $a_{1,\sigma} \le a_{2,\sigma} \le \ldots \le a_{n_{r-1},\sigma}$ and $b_{n_{r-1}+1,\sigma} \le \ldots \le b_{n,\sigma}$ are the increasing sequences of integers obtained from the Hodge-Tate weights $\mathrm{HT}(\underline{\rho})$ and $\mathrm{HT}(\rho_{n_r})$, respectively, given as in (6.3). From the decompositions $B_{n-n_r} = T_{n-n_r} \ltimes U_{n-n_r}$ and $B_{n_r} = T_{n_r} \ltimes U_{n_r}$, we inflate $\underline{\lambda}_\sigma$ and $\lambda_{n_r,\sigma}$ to B_{n-n_r} and B_{n_r}, and denote the inflations by $\tilde{\underline{\lambda}}_\sigma$ and $\tilde{\lambda}_{n_r,\sigma}$, respectively. From our assumptions on the Hodge-Tate weights we obtain irreducible algebraic representations

$$
\pi_{\mathrm{alg},\sigma}(\underline{\rho}) = \mathrm{Ind}_{B_{n-n_r}}^{G_{n-n_r}} \tilde{\underline{\lambda}}_\sigma \qquad \text{and} \qquad \pi_{\mathrm{alg},\sigma}(\rho_{n_r}) = \mathrm{Ind}_{B_{n_r}}^{G_{n_r}} \tilde{\lambda}_{n_r,\sigma}.
$$

Our induction hypothesis applied to $\underline{\rho}$ gives

$$
\pi_{\mathrm{alg},\sigma}(\underline{\rho}) \simeq \mathrm{Ind}_{B_{n_1} \times \ldots \times B_{n_{r-1}}}^{G_{n_1} \times \ldots \times G_{n_{r-1}}} \left(\bigotimes_{i=1}^{r-1} \pi_{\mathrm{alg},\sigma}(\rho_{n_i}) \right),
$$

and therefore

$$
\pi_{\mathrm{alg},\sigma}(\underline{\rho}) \otimes \pi_{\mathrm{alg},\sigma}(\rho_{n_r}) \simeq \mathrm{Ind}_{B_{n_1} \times \ldots \times B_{n_r}}^{G_{n_1} \times \ldots \times G_{n_r}} \left(\bigotimes_{i=1}^{r} \pi_{\mathrm{alg},\sigma}(\rho_{n_i}) \right).
$$

Applying the same computations as in (6.4) and (6.5) to the decompositions $P = M \ltimes N$

and $B_n = (B_{n_1} \times \ldots \times B_{n_r}) \ltimes N$, we obtain the isomorphism

$$\pi_{\mathrm{alg},\sigma}(\rho) \simeq \mathrm{Ind}_P^{G_n}\left(\bigotimes_{i=1}^r \pi_{\mathrm{alg},\sigma}(\rho_{n_i})\right). \tag{6.13}$$

Taking the tensor product over all embeddings $\sigma : L \hookrightarrow K$ yields the irreducible algebraic representation of $\prod_{\sigma:L\hookrightarrow K} \mathrm{GL}_n(K)$, denoted

$$\pi_{\mathrm{alg}}(\rho) := \bigotimes_{\sigma:L\hookrightarrow K} \pi_{\mathrm{alg},\sigma}(\rho).$$

Again, using the density of the diagonal embedding $\mathrm{GL}_n(L) \subset \prod_{\sigma:L\hookrightarrow K} \mathrm{GL}_n(K)$, restriction of $\bigotimes_{\sigma:L\hookrightarrow K} \pi_{\mathrm{alg},\sigma}(\rho)$ to $\mathrm{GL}_n(L)$ gives an irreducible algebraic representation of $\mathrm{GL}_n(L)$, which we denote $\pi_{\mathrm{alg}}(\rho)$.

Smooth Side of $\mathrm{BS}(\rho)$: Applying Fontaine's functor D_{dR} to the short exact sequence

$$0 \longrightarrow \underline{\rho} \longrightarrow \rho \longrightarrow \rho_{n_r} \longrightarrow 0$$

gives the short exact sequence

$$0 \longrightarrow D_{\mathrm{dR}}(\underline{\rho}) \longrightarrow D_{\mathrm{dR}}(\rho) \longrightarrow D_{\mathrm{dR}}(\rho_{n_r}) \longrightarrow 0. \tag{6.14}$$

Modifying the standard notation slightly we let

$$\mathrm{WD}(\rho) := \mathrm{WD}(D_{\mathrm{dR}}(\rho)) := (s, N, V)$$
$$\mathrm{WD}(\underline{\rho}) := \mathrm{WD}(D_{\mathrm{dR}}(\underline{\rho})) := (\underline{s}, \underline{N}, \underline{V})$$
$$\mathrm{WD}(\rho_{n_r}) := \mathrm{WD}(D_{\mathrm{dR}}(\rho)) := (s_{n_r}, N_{n_r}, V_{n_r})$$

so that applying the exact functor WD to (6.14) gives

$$0 \longrightarrow \mathrm{WD}(\underline{\rho}) \longrightarrow \mathrm{WD}(\rho) \longrightarrow \mathrm{WD}(\rho_{n_r}) \longrightarrow 0. \tag{6.15}$$

Letting $\phi \in W(\overline{\mathbb{Q}_p}/L)$ be a lift of the geometric Frobenius of $\mathrm{Gal}(\overline{k}_L/k_L)$, we choose a basis for V given by an eigenbasis for $s(\phi)$. In this basis $N \in \mathrm{End}(V)$ has matrix form

$$N = \left(\begin{array}{c|c} \underline{N} & * \\ \hline 0 & N_{n_r} \end{array}\right).$$

However, restricting the basis of V to its subspace $\underline{V}$, the operator $\underline{N}$ has matrix form

$$\underline{N} = \begin{pmatrix} N_{n_1} & * & \cdots & * \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & N_{n_{r-1}} \end{pmatrix}$$

in which case N has matrix form

$$N = \begin{pmatrix} N_{n_1} & * & \cdots & * \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & N_{n_r} \end{pmatrix}.$$

Lemma 6.5. *Assuming the Frobenius eigenvalue hypothesis for $s(\phi)$, the matrix for the monodromy operator N is*

$$N = \begin{pmatrix} N_{n_1} & 0 & \cdots & 0 \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_{n_r} \end{pmatrix},$$

hence $N = \bigoplus_{i=1}^{r} N_{n_i}$.

Proof. Since we will need to compute with the non-diagonal upper triangular part of N, we write

$$N = \begin{pmatrix} N_{n_1} & A_{n_1,n_2} & \cdots & A_{n_1,n_r} \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n_{r-1},n_r} \\ 0 & \cdots & 0 & N_{n_r} \end{pmatrix}$$

where, for each $1 \le i \le n_{r-1}$ and $2 \le j \le n_r$, the A_{n_i,n_j} is a matrix of size $n_i \times n_j$. We let $A_{n_i,n_j} = (a_{k,l})$ where $1 \le k \le n_{r-1}$ and $1 \le l \le n_r$, so the labeling of entries of N will be written in terms of entries of each submatrix A_{n_i,n_j}, this allows for simpler notation

in the computation. As before, write $s(\phi)$ as

$$
s(\phi) = \begin{pmatrix} D_{n_1} & 0 & \cdots & 0 \\ 0 & D_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{n_r} \end{pmatrix},
$$

where, for each $1 \leq i \leq r$, the $D_{n_i} \in \mathrm{GL}_{n_i}(K)$ is the diagonal matrix

$$
D_{n_i} = \begin{pmatrix} \mu_1^{(i)} & 0 & \cdots & 0 \\ 0 & \mu_2^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_{n_i}^{(i)} \end{pmatrix}.
$$

The conjugation relation $s(\phi) N s(\phi)^{-1} = pN$ for (s, N, V) is

$$
pN = s(\phi) N s(\phi)^{-1} = \begin{pmatrix} N_{n_1} & B_{n_1,n_2} & \cdots & B_{n_1,n_r} \\ 0 & N_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{n_{r-1},n_r} \\ 0 & \cdots & 0 & N_{n_r} \end{pmatrix},
$$

where, for each $1 \leq i \leq n_{r-1}$ and $2 \leq j \leq n_r$, if $B_{n_i,n_j} = (b_{k,l})$, then $b_{k,l} = \frac{\mu_k^{(n_i)}}{\mu_l^{(n_j)}} a_{k,l}$ for all $1 \leq k \leq n_i$ and $1 \leq l \leq n_j$. Equating coefficients and using that for all i, j, k and l the quotient $\frac{\mu_k^{(n_i)}}{\mu_l^{(n_j)}} \neq p$, we find

$$
\left(\frac{\mu_k^{(n_i)}}{\mu_l^{(n_j)}} - 1 \right) a_{k,l} = 0,
$$

and therefore, $a_{k,l} = 0$ for all $1 \leq i \leq n_{r-1}$, $2 \leq j \leq n_r$, $1 \leq k \leq n_i$, and $1 \leq l \leq n_j$, hence $N = \bigoplus_{i=1}^{r} N_{n_i}$. $\qquad\square$

The short exact sequence of (6.15) along with the decompositions $V \simeq \underline{V} \oplus V_{n_r}$ and $N = \underline{N} \oplus N_{n_r}$ render an isomorphism

$$
\mathrm{WD}(\rho)^{\mathrm{F-ss}} \simeq \mathrm{WD}(\underline{\rho})^{\mathrm{F-ss}} \oplus \mathrm{WD}(\rho_{n_r})^{\mathrm{F-ss}}. \tag{6.16}
$$

By induction, we have the decompositions $\underline{V} \simeq \bigoplus_{i=1}^{r-1} V_{n_i}$ and $\underline{N} = \bigoplus_{i=1}^{r-1} N_{n_i}$, whereby

$$\mathrm{WD}(\underline{\rho})^{\mathrm{F-ss}} \simeq \bigoplus_{i=1}^{r-1} \mathrm{WD}(\rho_{n_i})^{\mathrm{F-ss}},$$

and therefore (6.16) becomes

$$\mathrm{WD}(\rho)^{\mathrm{F-ss}} \simeq \bigoplus_{i=1}^{r} \mathrm{WD}(\rho_{n_i})^{\mathrm{F-ss}}. \tag{6.17}$$

Applying the local Langlands correspondence to (6.17) we obtain smooth irreducible admissible representations $\pi_{\mathrm{sm}}(\rho), \pi_{\mathrm{sm}}(\underline{\rho})$, and $\pi_{\mathrm{sm}}(\rho_{n_r})$ such that

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) = \mathrm{WD}(\rho)^{\mathrm{F-ss}}$$

$$\mathrm{rec}_{n-n_r}(\pi_{\mathrm{sm}}(\underline{\rho})) = \mathrm{WD}(\underline{\rho})^{\mathrm{F-ss}}$$

$$\mathrm{rec}_{n_r}(\pi_{\mathrm{sm}}(\rho_{n_r})) = \mathrm{WD}(\rho_{n_r})^{\mathrm{F-ss}}$$

From (6.17)

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) \simeq \mathrm{rec}_{n-n_r}(\pi_{\mathrm{sm}}(\underline{\rho})) \oplus \mathrm{rec}_{n_r}(\pi_{\mathrm{sm}}(\rho_{n_r}))$$

$$\simeq \mathrm{rec}_n(\pi_{\mathrm{sm}}(\underline{\rho}) \boxplus \pi_{\mathrm{sm}}(\rho_{n_r}))$$

where $\boxplus$ is the Langlands sum defined in [18, p.32]. By our induction assumption

$$\mathrm{rec}_{n-n_r}(\pi_{\mathrm{sm}}(\underline{\rho})) \simeq \bigoplus_{i=1}^{r-1} \mathrm{rec}_{n_i}(\pi_{\mathrm{sm}}(\rho_{n_i})),$$

hence

$$\mathrm{rec}_n(\pi_{\mathrm{sm}}(\rho)) \simeq \mathrm{rec}_n\left(\boxplus_{i=1}^{r} \pi_{\mathrm{sm}}(\rho_{n_i})\right), \tag{6.18}$$

whereby injectivity of the local Langlands correspondence gives

$$\pi_{\mathrm{sm}}(\rho) \simeq \boxplus_{i=1}^{r} \pi_{\mathrm{sm}}(\rho_{n_i}).$$

As before, since ρ is assumed generic the Langlands sum is fully induced,

$$\pi_{\mathrm{sm}}(\rho) \simeq \boxplus_{i=1}^{r} \pi_{\mathrm{sm}}(\rho_{n_i}) \simeq \mathrm{Ind}_P^{G_n}\left(\bigotimes_{i=1}^{r} \pi_{\mathrm{sm}}(\rho_{n_i})\right),$$

and therefore the locally algebraic representation of Breuil-Schneider, $BS(\rho) = \pi_{alg}(\rho) \otimes_K \pi_{sm}(\rho)$, is

$$BS(\rho) \simeq Ind_P^{G_n}\left(\left(\bigotimes_{i=1}^r \pi_{alg}(\rho_{n_i})\right) \otimes_K \left(\bigotimes_{i=1}^r \pi_{sm}(\rho_{n_i})\right)\right).$$

Existence of an Invariant Norm for $BS(\rho)$: The proof of the existence of an invariant norm on $BS(\rho)$ follows immediately from the computations given in the proof of Theorem 7.2. In particular, letting

$$\iota : \left(\bigotimes_{i=1}^r \pi_{alg}(\rho_{n_i})\right) \otimes_K \left(\bigotimes_{i=1}^r \pi_{sm}(\rho_{n_i})\right) \xrightarrow{\sim} \bigotimes_{i=1}^r (\pi_{alg}(\rho_{n_i}) \otimes_K \pi_{sm}(\rho_{n_i}))$$

denote the isomorphism given on elementary tensors by

$$\iota((f_1 \otimes f_2 \otimes \ldots \otimes f_r) \otimes (g_1 \otimes g_2 \otimes \ldots \otimes g_r)) = (f_1 \otimes g_1) \otimes (f_2 \otimes g_2) \otimes \ldots \otimes (f_r \otimes g_r),$$

we construct a map

$$BS(\rho) \longrightarrow Ind_P^{G_n}\left(\bigotimes_{i=1}^r BS(\rho)\right) \tag{6.19}$$

$$(f_1, \ldots, f_r) \longmapsto \iota \circ (f_1 \otimes \ldots \otimes f_r)$$

which satisfies $\iota(f_1 \otimes \ldots \otimes f_r)(px) = p \cdot \iota(f_1 \otimes \ldots \otimes f_r)(x)$ for all $p \in P$ and $x \in G_n$. Since $\pi_{alg}(\rho)$ and $\pi_{sm}(\rho)$ are both irreducible and (6.19) is not the zero map, we again utilize Dipendra Prasad's result [37, p.126] to obtain the injection

$$BS(\rho) \hookrightarrow Ind_P^{G_n}\left(\bigotimes_{i=1}^r BS(\rho_{n_i})\right). \tag{6.20}$$

To the indecomposable $WD(\rho_{n_i})$ a result of Sorensen [40] applied to each $BS(\rho_{n_i})$ yields G_{n_i}-invariant norms $||\cdot||_i$ for each $1 \le i \le r$. Again utilizing the result [34, Proposition 17.4], these norms yield a $G_{n_1} \times \ldots \times G_{n_r}$-invariant norm $||\cdot|| := ||\cdot||_1 \otimes \ldots \otimes ||\cdot||_r$ on $BS(\rho)$ given by

$$||u|| := \inf_u \left\{ \max_{1 \le j \le m} ||v_j^{(1)}||_1 \ldots ||v_j^{(r)}||_r \right\}$$

where the infimum runs over all representations $u = \sum_{j=1}^m v_j^{(1)} \otimes \ldots \otimes v_j^{(r)}$ where

$v_j^{(k)} \in \mathrm{BS}(\rho_{n_k})$ for all $1 \leq j \leq m$ and $1 \leq k \leq r$.

Consider the set $C(G_n, \bigotimes_{i=1}^r \mathrm{BS}(\rho_{n_i}); ||\cdot||_\infty)$ consisting of continuous functions equipped with the sup-norm $||\cdot||_\infty$. We give $C(G_n, \bigotimes_{i=1}^r \mathrm{BS}(\rho_{n_i}); ||\cdot||_\infty)$ the left G_n-action of right translations, and observe that $P\backslash G_n$ is compact. Therefore, for any $x \in G_n$ and $f \in C(G_n, \bigotimes_{i=1}^r \mathrm{BS}(\rho_{n_i}); ||\cdot||_\infty)$

$$||xf||_\infty = \sup_{Pg \in P\backslash G_n} ||f(Pgx)|| = \sup_{Ph \in P\backslash G_n} ||f(Ph)|| = ||f||_\infty,$$

whereby $||\cdot||_\infty$ is G_n-invariant. Viewing $\mathrm{BS}(\rho) \subset C(G_n, \bigotimes_{i=1}^r \mathrm{BS}(\rho_{n_i}); ||\cdot||_\infty)$, the restriction of $||\cdot||_\infty$ to $\mathrm{BS}(\rho)$ is a G_n-invariant norm. $\qquad\square$

Chapter 7

Invariant principal ideals in a p-adic Heisenberg algebra

The work in this chapter came about from an attempt to generalize work of Peter Schneider and Jeremy Teitelbaum [38] on Schneider's 2006 ICM conjecture [35, Conjecture 2.5]. Although a proof in full generality failed, we were able to obtain a result about a certain graded p-adic Heisenberg group.

Our methods utilize the theory of p-valuable groups, which we briefly describe. A more thorough and comprehensive treatment can be found in [25] and [36].

7.1 Introduction and Notation

Let G be an abstract group whose identity we denote by 1, and let $p \in \mathbb{N}$ be a prime number with additive valuation ord_p on $\mathbb{Q}_p$ normalized so that $\mathrm{ord}_p(p) = 1$. A function $\omega : G \to (0, \infty]$ is called a p-valuation if $\omega(1) = \infty$ and, for all $g, h \in G$ the following hold

i.) $\omega(g) > \frac{1}{p-1}$

ii.) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$

iii.) $\omega([g, h]) \geq \omega(g) + \omega(h)$

iv.) $\omega(g^p) = \omega(g) + 1$

By definition, a p-valuable group is a pair (G, ω) consisting of an abstract group G along with a p-valuation ω on G satisfying the above properties. In general, the values of ω can vary widely, however, if G is a pro-p group then the values of ω are discrete in $(0, \infty]$.

Associated to any p-valuable group (G, ω) is a filtration given as follows. For any $v \in \mathbb{R}_{>0}$ define the subgroups

$$G_v = \{g \in G : \omega(g) \geq v\} \qquad \text{and} \qquad G_{v^+} = \{g \in G : \omega(g) > v\}.$$

The collection $\{G_v\}_v$ is a filtration of G with normal subgroups G_{v^+} and abelian quotients $\mathrm{gr}_v(G) := G_v/G_{v^+}$. The graded abelian group

$$\mathrm{gr}\, G = \bigoplus_{v>0} \mathrm{gr}_v G$$

is endowed with a p-power operator given on components by

$$\mathcal{P}((gG_{v^+})_v) = (g^p G_{(v+1)^+})_v.$$

The operator $\mathcal{P}$ endows $\mathrm{gr}\, G$ with the structure of an $\mathbb{F}_p[\mathcal{P}]$-module, and the rank of (G, ω) is defined to be the rank of the $\mathbb{F}_p[\mathcal{P}]$-module $\mathrm{gr}\, G$. For notational convenience we set $R = \mathbb{F}_p[\mathcal{P}]$ for the remainder of this chapter.

Definition 7.1. *Let (G, ω) be a p-valuable group. We say the elements $g_1, g_2, \ldots, g_r \in G$ are an ordered basis if both of the following conditions are satisfied*

i.) The map

$$\mathbb{Z}_p^r \to G$$
$$(x_1, \ldots, x_r) \mapsto g_1^{x_1} \ldots g_r^{x_r}$$

is a homeomorphism.

ii.) $\omega(g_1^{x_1} \ldots g_r^{x_r}) = \min_{1 \leq i \leq r} (\mathrm{ord}_p(x_i) + \omega(g_i))$

If (G, ω) is a p-valuable group of finite rank with ordered basis $\{g_1, \ldots, g_r\}$, then $\{\sigma(g_1), \ldots, \sigma(g_r)\}$ is an R-module basis of $\mathrm{gr}\, G$, where $\sigma(g) = gG_{\omega(g)^+}$ for $g \in G$.

7.2 A p-adic Heisenberg group

For $n \in \mathbb{Z}_{\geq 2}$, the $n \times n$ p-adic Heisenberg group H consists of upper triangular matrices of the form

$$
(x_{i,j}) = \begin{pmatrix}
1 & x_{1,2} & x_{1,3} & \dots & x_{1,n-1} & x_{1,n} \\
 & 1 & 0 & \dots & 0 & x_{2,n} \\
 & & \ddots & \ddots & \vdots & \vdots \\
 & & & 1 & 0 & x_{n-2,n} \\
 & & & & 1 & x_{n-1,n} \\
 & & & & & 1
\end{pmatrix}
$$

where $x_{1,i}, x_{j,n} \in \mathbb{Z}_p$ for all $2 \leq i \leq n$ and $2 \leq j \leq n-1$, with group operation matrix multiplication.

In [36, p.171] Schneider constructs a p-valuation on the open subgroup $G \subset \mathrm{GL}_n(\mathbb{Q}_p)$ where

$$
G := \{g \in \mathrm{GL}_n(\mathbb{Q}_p) : f(g-1) > \frac{1}{p-1}\}
$$

and $f : M_n(\mathbb{Q}_p) \to \mathbb{Z} \cup \{\infty\}$ is defined by

$$
f(g) = \min_{i,j}(\mathrm{ord}_p(a_{ij}))
$$

for $g = (a_{ij}) \in M_n(\mathbb{Q}_p)$. For $g \in M_n(\mathbb{Q}_p)$ define $\omega(g) = f(g-1)$, then (G, ω) is a p-valuable group. Let $\varphi : H \to G$ be defined by

$$
\varphi((x_{i,j})) = \begin{pmatrix}
1 & px_{1,2} & p^2 x_{1,3} & \dots & p^{n-2} x_{1,n-1} & p^{n-1} x_{1,n} \\
 & 1 & 0 & \dots & 0 & p^{n-2} x_{2,n} \\
 & & \ddots & \ddots & \vdots & \vdots \\
 & & & 1 & 0 & p^2 x_{n-2,n} \\
 & & & & 1 & px_{n-1,n} \\
 & & & & & 1
\end{pmatrix}.
$$

Then, for any $g, h \in H$, $\varphi(gh) = \varphi(g)\varphi(h)$ and $\ker \varphi = \{1\}$, therefore, we identify H with its image under φ and view H inside G. Let $h \in \varphi(H)$, then

$$
\omega(h) = \min_{\substack{2 \leq i \leq n \\ 2 \leq j \leq n-1}} \{\mathrm{ord}_p(x_{1,i}) + i - 1, \mathrm{ord}_p(x_{j,n}) + n - j\}.
$$

is a p-valuation on $\varphi(H)$ and therefore on H from the above identification.

Since (H, ω) is a p-valuable pro-p group, the values of ω are discrete. Therefore there exists $h_0 \in H$ such that $0 < \omega(h_0) = \min_{h \in H} \omega(h)$. By definition, $\omega(h) \geq 1$ for all $h \in H$, hence, there exists a real number C satisfying $0 < C < 1 - \frac{1}{p-1}$. Using this C we define $\omega_C := \omega - C$, which defines a new p-valuation on H with the added benefit of strict inequality on commutators, namely, for any $g, h \in H$

$$
\begin{aligned}
\omega_C([g, h]) &= \omega([g, h]) - C \\
&\geq \omega(g) + \omega(h) - C \\
&> \omega(g) - C + \omega(h) - C \\
&= \omega_C(g) + \omega_C(h).
\end{aligned}
$$

For $v \geq 0$, the filtration on (H, ω_C) is given by

$$
H_v := \{h \in H : \omega_C(h) \geq v\} \qquad \text{and} \qquad H_{v+} := \{h \in H : \omega_C(h) > v\},
$$

with associated graded abelian group

$$
\mathrm{gr}\, H = \bigoplus_{v \geq 0} \mathrm{gr}_v(H).
$$

The Lie algebra structure of $\mathrm{gr}\, H$ is given by the bracket

$$
\begin{aligned}
[\cdot, \cdot] : \mathrm{gr}_v(H) \times \mathrm{gr}_{v'}(H) &\to \mathrm{gr}_{(v+v')}(H) \\
(\xi, \eta) &\mapsto [g, h] H_{(v+v')+}
\end{aligned}
$$

where $\xi = g H_{v+}$ and $\eta = h H_{v'+}$. In [36, p.173-174] it is shown that $[\cdot, \cdot]$ is a well-defined, biadditive map, compatible with the operator $\mathcal{P}$, such that $[\xi, \xi] = 0$ and $[\xi, \eta] = -[\eta, \xi]$ for all $\xi, \eta \in \mathrm{gr}\, H$. Therefore, $\mathrm{gr}\, H$ is a graded Lie algebra over R. Furthermore, for $\xi = g H_{v+}$ and $\eta = h H_{v'+}$

$$
\omega_C([g, h]) > \omega_C(g) + \omega_C(h) \geq v + v',
$$

hence, $[\xi, \eta] = 0$ for all $\xi, \eta \in \mathrm{gr}\, H$, whereby $\mathrm{gr}\, H$ is abelian as a Lie algebra over R.

Define $E_{i,j} \in \mathrm{GL}_n(\mathbb{Q}_p)$ by $E_{i,j} = I_n + (1_{i,j})$, where I_n is the $n \times n$ identity matrix

and $(1_{i,j})$ is the $n \times n$ matrix with a one in the (i,j)th entry, and zeroes elsewhere.

Lemma 7.1. *For* $2 \leq i \leq n$ *and* $2 \leq j \leq n-1$, *the elements* $E_{1,i}$ *and* $E_{j,n}$ *form an ordered basis of* (H, ω_C).

Proof. Indeed, any $h \in H$ can be written as

$$h = \prod_{j=2}^{n-1} E_{j,n}^{x_{j,n}} \prod_{i=2}^{n} E_{1,i}^{x_{1,i}}$$

showing that the continuous map

$$\psi : (x_{1,2}, x_{1,3}, \ldots, x_{1,n}, x_{2,n}, \ldots, x_{n-1,n}) \mapsto h$$

is surjective. If $\psi(\underline{a}) = \psi(\underline{b})$ in H, then $\underline{a} = \underline{b}$ in $\mathbb{Z}_p^{2n-3}$. This follows simply from the fact that two $n \times n$ matrices are equal if and only if their corresponding coefficients are equal. Therefore, ψ is a continuous bijection from the compact space $\mathbb{Z}_p^{2n-3}$ to the Hausdorff space H, hence ψ is a homeomorphism.

Finally, we have

$$\omega_C \left(\prod_{j=2}^{n-1} E_{j,n} \prod_{i=2}^{n} E_{1,i} \right) = \min_{\substack{2 \leq i \leq n \\ 2 \leq j \leq n-1}} \{\mathrm{ord}_p(x_{1,i}) + i - 1, \mathrm{ord}_p(x_{j,n}) + n - j\} - C$$

$$= \min_{\substack{2 \leq i \leq n \\ 2 \leq j \leq n-1}} \{\mathrm{ord}_p(x_{1,i}) + i - 1 - C, \mathrm{ord}_p(x_{j,n}) + n - j - C\}$$

$$= \min_{\substack{2 \leq i \leq n \\ 2 \leq j \leq n-1}} \{\mathrm{ord}_p(x_{1,i}) + \omega_C(E_{1,i}), \mathrm{ord}_p(x_{j,n}) + \omega_C(E_{j,n})\}$$

proving the lemma. $\qquad\square$

The above lemma shows that $\mathrm{gr}\, H$ has rank $2n-3$ as a Lie algebra over R and is generated by $\{\sigma(E_{1,i}), \sigma(E_{j,n}) : i = 2, \ldots, n \text{ and } j = 2, \ldots, n-1\}$, where $\sigma(h) := hH_{\omega_C(h)^+}$ for any $h \in H$.

Now that the properties of $\mathrm{gr}\, H$ have been established, we will review similar properties associated to the graded Iwasawa algebra of H, denoted $\mathrm{gr}\, \Lambda(H)$. Proofs of the details can be found in [36, ch.VI].

7.3 The graded Iwasawa algebra $\operatorname{gr}\Lambda(H)$ and universal enveloping algebra $\mathcal{U}(\operatorname{gr}H)$

Let H be as above, then the Iwasawa algebra of H is

$$\Lambda(H) = \varprojlim_{U \triangleleft^\circ H} \mathbb{Z}_p[H/U]$$

where $U \triangleleft^\circ H$ indicates that U is an open normal subgroup of H. For $v \in \mathbb{R}_{\geq 0}$, define J_v to be the smallest closed $\mathbb{Z}_p$-submodule of $\Lambda(H)$ generated by elements of the form $p^l(h_1 - 1)\dots(h_s - 1)$ such that $l + \sum_{i=1}^{s} \omega_C(h_i) \geq v$ where $l, s \geq 0$ and $h_i \in H$ for all $1 \leq i \leq s$. Defining

$$J_{v+} = \bigcup_{v' > v} J_{v'}$$

and $\operatorname{gr}_v \Lambda(H) = J_v/J_{v+}$ we have the graded algebra

$$\operatorname{gr}\Lambda(H) = \bigoplus_{v \geq 0} \operatorname{gr}_v \Lambda(H)$$

over the graded ring

$$\operatorname{gr}\mathbb{Z}_p = \bigoplus_{n \geq 0} p^n \mathbb{Z}_p/p^{n+1}\mathbb{Z}_p.$$

We view the graded ring $\operatorname{gr}\mathbb{Z}_p$ as an R-algebra by identifying $\mathbb{F}_p \simeq \operatorname{gr}_0 \mathbb{Z}_p$ and mapping $\mathcal{P}$ to $p + p^2\mathbb{Z}_p$.

Lemma 7.2. *The map*

$$R \xrightarrow{\sim} \operatorname{gr}\mathbb{Z}_p$$
$$\sum_n \overline{a_n}\mathcal{P}^n \mapsto (a_n p^n + p^{n+1}\mathbb{Z}_p)_n$$

is an isomorphism of R-algebras.

We now discuss Schneider's exposition of the Lazard isomorphism between the graded R-algebra $\operatorname{gr}\Lambda(H)$, and the universal enveloping algebra $\mathcal{U}(\operatorname{gr}H)$ of $\operatorname{gr}H$. Let $v \in \mathbb{R}_{>0}$ and define

$$\mathcal{L}_v : \operatorname{gr}_v H \to \operatorname{gr}_v \Lambda(H)$$
$$h_v H_{v+} \mapsto h_v - 1 + J_{v+}.$$

In [36, p.197-198] it is shown that $\mathcal{L}_v$ is an R-algebra homomorphism, consequently we obtain an R-algebra homomorphism

$$\mathcal{L} := \bigoplus_{v>0} : \operatorname{gr} H \to \operatorname{gr} \Lambda(H).$$

The universal property of universal enveloping algebras gives a homomorphism of associative R-algebras $\psi : \mathcal{U}(\operatorname{gr} H) \to \operatorname{gr} \Lambda(H)$, whereby, if $\mathcal{O}$ is a finite unramified integral extension of $\mathbb{Z}_p$ with fraction field K, the Lazard isomorphism is given by extension of scalars to $\operatorname{gr} \mathcal{O}$ of the homomorphism ψ. In our case, $K = \mathbb{Q}_p$ so $\mathcal{O} = \mathbb{Z}_p$, and since $\operatorname{gr} \mathbb{Z}_p \simeq R$, extension of scalars is unnecessary, hence the Lazard isomorphism is precisely the R-algebra isomorphism

$$\psi : \mathcal{U}(\operatorname{gr} H) \simeq \operatorname{gr} \Lambda(H).$$

Since $\operatorname{gr} H$ is an abelian Lie algebra over R generated by $\{\sigma(E_{1,i}), \sigma(E_{j,n}) : 2 \leq i \leq n \text{ and } 2 \leq j \leq n - 2\}$, the universal enveloping algebra is a polynomial ring

$$\mathcal{U}(\operatorname{gr} H) = R[X_{1,2}, X_{1,3}, \ldots, X_{1,n}, X_{2,n}, \ldots, X_{n-1,n}] =: S,$$

where, for all $\leq i \leq n$ and $2 \leq j \leq n - 1$ the variables $X_{1,i}$ and $X_{j,n}$ correspond to the generators $\sigma(E_{1,i})$ and $\sigma(E_{j,,n})$, respectively.

7.4 The Contracting Monoid T^+

Define the set

$$T^+ = \{\operatorname{diag}(y_1, y_2, \ldots, y_n) \in M_n(\mathbb{Q}_p) : |y_1|_p \leq |y_2|_p \leq \ldots \leq |y_n|_p\}.$$

Under matrix multiplication, T^+ is a monoid (inverses reverse inequalities). Using the structure of $\mathbb{Q}_p^\times$, any element $t \in T^+$ is of the form $t = \operatorname{diag}(u_1 p^{m_1}, u_2 p^{m_2}, \ldots, u^n p^{m_n})$, where $u_i \in \mathbb{Z}_p^\times$ for all $1 \leq i \leq n$, and $m_1 \geq m_2 \geq \ldots \geq m_n$ are integers. For each $1 \leq i \leq n$ we define the elements

$$t_i = \operatorname{diag}(\underbrace{p, \ldots, p}_{i}, \underbrace{1, \ldots, 1}_{n-i}),$$

then any element $t = \text{diag}(u_1 p^{m_1}, \ldots, u_n p^{m_n}) \in T^+$ decomposes as

$$t = \text{diag}(u_1, \ldots, u_n) \left(\prod_{i=1}^{n} t_{i-1}^{m_{i-1} - m_i} \right) t_n^{m_n}.$$

We define an action of T^+ on $\text{gr}\, H$ by conjugation, namely,

$$t \cdot (h_v H_{v^+})_v = (t h_v t^{-1} H_{v^+})_v,$$

for any $t \in T^+$ and any $(h_v H_{v^+})_v \in \text{gr}\, H$. Although T^+ is simply a monoid, notice that the above conjugation makes sense since

$$\frac{y_i}{y_j} \in \mathbb{Z}_p$$

for all $i < j$.

Using the Lazard isomorphism

$$\mathcal{L} : \mathcal{U}(\text{gr}\, H) \simeq \text{gr}\, \Lambda(H),$$

we now write down explicitly the action of T^+ on the polynomial ring S. For each $2 \leq i \leq n$ and $2 \leq j \leq n - 2$, to explicitly write down the action of T^+ on S, it suffices to write down the action of the generators of T^+ on the generators $X_{1,i}, X_{j,n}$ of S, and extend R-linearly to all of S.

For $u \in T(\mathbb{Z}_p)$ we see

$$u \cdot X_{i,j} = \frac{u_i}{u_j} X_{i,j}$$

where $1 \leq i < j \leq n$. The action is somewhat more difficult to describe for the non-torus elements $t_i \in T^+$. The action of t_1 is given by

$$t_1 X_{1,i} = \mathcal{P} X_{1,i} \text{ for all } 2 \leq i \leq n$$
$$t_1 X_{j,n} = X_{j,n} \text{ for all } 2 \leq j \leq n - 1.$$

The element $t_n \in T^+$ acts trivially, namely $t_n X_{i,j} = X_{i,j}$ for all choices of i and j.

Finally, for $1 < m < n$, the action of t_m is given by

$$t_m X_{1,i} = \begin{cases} X_{1,i} & \text{if } 2 \leq i \leq m \\ \mathcal{P} X_{1,i} & \text{if } i > m \end{cases}$$

and

$$t_m X_{j,n} \begin{cases} \mathcal{P} X_{j,n} & \text{if } 2 \leq j \leq m \\ X_{j,n} & \text{if } j > m \end{cases}.$$

7.5   Main Result

Let p be a rational prime number and $n \in \mathbb{Z}_{\geq 2}$. Consider (H, ω), where H is the $n \times n$ Heisenberg group over the p-adic integers $\mathbb{Z}_p$, and ω is a p-valuation on H, as defined in [36, p.169]. Let $T^+ \subset \mathrm{GL}_n(\mathbb{Q}_p)$ be the set of diagonal matrices $\mathrm{diag}(y_1, y_2, \ldots, y_n) \in \mathrm{GL}_n(\mathbb{Q}_p)$ such that $|y_1|_p \leq |y_2|_p \leq \ldots \leq |y_n|_p$, where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}_p$, normalized so that $|p|_p = \frac{1}{p}$. Under matrix multiplication T^+ is a monoid, which we call a contracting monoid of $\mathrm{GL}_n(\mathbb{Q}_p)$. The Iwasawa algebra of H is

$$\Lambda(H) = \varprojlim_{U \triangleleft H} \mathbb{Z}_p[H/U],$$

where U runs over all open normal subgroups of H. Let $R = \mathbb{F}_p[\mathcal{P}]$ be a polynomial ring in the operator $\mathcal{P}$, where

$$\mathcal{P}(h) = h^p$$

for all $h \in H$. Then, utilizing Lazard's theory of p-valuable groups, we construct a finitely generated R-module $\mathrm{gr}\Lambda(H)$ which can be identified with a polynomial ring over R in $2n - 3$ variables, denoted $S = R[X_{1,2}, X_{1,3}, \ldots, X_{1,n}, X_{2,n}, \ldots, X_{n-1,n}]$, and which carries a natural action of T^+. The main result of this paper is the following

Theorem 7.3. *The T^+-invariant principal ideals $I = (f)$ of $\mathrm{gr}\Lambda(H)$, are precisely those for which the cyclic R-module Rf is T^+-invariant.*

7.6   Proof of the main result

Proof. We first show that the T^+-invariant cyclic R modules, Rf, are determined by an equality condition on certain coefficients brought about by the action of T^+ on f.

Consider the cyclic R-submodule Rf of S. By definition, Rf is T^+-invariant if and only if for all $r \in R$ and $t \in T^+$

$$t \cdot (rf) = r'f,$$

for some $r' \in R$. Since the action of T^+ on S is R-linear, the above relation reads

$$r(t \cdot f) = r'f.$$

Since the action of any $t \in T^+$ on a monomial in S is given by multiplication by some element of R, we have

$$r(t \cdot f) = \sum_{m \in \mathrm{Mon}(f)} r a_m c_m(t) m$$

where the $c_m(t) \in R$ are the coefficients determined by the action of T^+. Therefore, T^+-invariance of Rf amounts to

$$\sum_{m \in \mathrm{Mon}(f)} r a_m c_m(t) m = \sum_{m \in \mathrm{Mon}(f)} r' a_m m$$

which holds if and only if $r a_m c_m(t) - r' a_m = 0$ for all $m \in \mathrm{Mon}(f)$, and $t \in T^+$. Since R is a commutative domain, and the $a_m \neq 0$ for all $m \in \mathrm{Mon}(f)$, we see that the above equation is equivalent to $r c_m(t) = r'$ for all $m \in \mathrm{Mon}(f)$ and $t \in T^+$. Therefore, Rf is T^+-invariant, if and only if the $c_m(t)$ are equal for all $t \in T^+$.

Now, if the principal ideal $I = (f)$ is such that Rf is a T^+-invariant R-submodule of S, then for any $t \in T^+$, we have

$$t \cdot f = r_t f,$$

for some $r_t \in R$, and therefore, $t \cdot I \subset I$, showing that I is T^+-invariant. Conversely, suppose that $I = (f)$ is T^+-invariant, then for all $t \in T^+$, we have

$$t \cdot f = q_t f$$

for some $q_t \in S$. If $q_t \notin R$ for some $t \in T^+$, then q_t contains at least one nontrivial monomial term involving at least one variable of S. Therefore, the monomial terms of

$q_t f$ would differ from those monomial terms of f, hence, the equality

$$r_t f = t \cdot f = q_t f$$

would not hold. Therefore, for all $t \in T^+$, we have $q_t \in R$, and therefore f is precisely a polynomial for which the cyclic module Rf is T^+-invariant. $\qquad\square$

Originally this work grew from an attempt to generalize to $\mathrm{GL}_3(\mathbb{Z}_p)$ the work of Schneider-Teitelbaum [38, sect. 4] on $\mathrm{GL}_2(\mathbb{Z}_p)$. If K is a finite extension of $\mathbb{Q}_p$ and $G = \mathrm{GL}_2(\mathbb{Z}_p)$ with $B \subset G$ the Iwahori subgroup of matrices whose reduction modulo p is lower triangular, and if P, P^-, U, U^-, and T denote the lower triangular, upper triangular, lower unipotent, upper unipotent, and diagonal matrices, respectively, then, given a character $\chi : T \to K$, we have a finitely generated $K[[B]]$-module

$$N_\chi := k[[B]] \otimes_{K[[P]]} K^{(\chi)}.$$

The main result in section four of [38] is the following proposition

Theorem 7.4. [38, Proposition 4.1] *If $c(\chi) \notin \mathbb{N}_0$ then N_χ is a simple $K[[B]]$-module.*

Their approach is to utiilize an equivariant notion of Schikhof-duality, allowing them to translate the desired irreducibility statement into a statement about a certain algebra being simple. The algebra in question consists of formal power series in one variable with bounded coefficients, in particular, the algebra is commutative and a PID. A rough sketch of their argument is as follows:

- Choose a submodule of your principal series representation and invoke the equivariant version of Schikhof duality to consider the corresponding ideal in the algebra.

- Since the algebra is a PID, this ideal is generated by some power series.

- Using the Weierstrass preparation theorem, one can assume that the generator of this ideal is a distinguished polynomial.

- Using the actions of the torus of $\mathrm{GL}_2(\mathbb{Z}_p)$ and the upper triangular unipotent matrices, one concludes that the generating polynomial has either none or infinitely many zeroes in the p-adic unit disk. Hence, the ideal is either the whole algebra or the zero ideal.

- Invoking Schikhof duality again one concludes that the original submodule was either trivial or the whole principal series itself.

There are many problems in trying to generalize their results to the $\mathrm{GL}_3(\mathbb{Z}_p)$-setting, namely

- The algebra is a noncommutative power series algebra in three variables, no longer a PID, and therefore one cannot assume that the ideal afforded by Schikhof duality is principal.

- There is no direct analog of the Weierstrass preparation theorem for noncommutative power series rings in multiple variables, so even if the corresponding ideal were principal, one wouldn't be able to conclude that the ideal is generated by a polynomial, thereby ruling out the argument regarding zeroes of polynomials.

For these reasons, we specialized to the $n \times n$ Heisenberg group, passed to the graded Iwasawa algebra, and considered only invariant principal ideals. Understanding the behavior of ideals in noncommutative Iwasawa algebras is difficult and few results are known. For more information regarding conjectures and known results the reader is directed to the papers [20], [1].

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