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BY
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EXTERNALITIES AND HOUSING UNIT MAINTENANCE*

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ABSTRACT

This paper presents a dynamic behavioural model demonstrating the impacts of externalities generated from stocks, upon investment in these stocks. Using urban housing as an example it is shown that, dependent upon consumers' preference relations between the two goods that must be consumed jointly: housing services and neighborhood quality, there may be several stationary states for housing, and neighborhood qualities. In this case, changes in the structure of housing ownership, in demand, or in the market interest rate do not only affect stationary quality levels, but may lead to changing directions in maintenance strategies and therefore neighborhood development.

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1. INTRODUCTION

It is common knowledge that markets, even if perfectly competitive, will not achieve efficiency in resource allocation when 'external effects' are present. In particular, it is easy to show within a static model dealing with but one externality at a time, that under reasonable conditions a less than optimal quantity of such a nonmarketed good is being produced if consumption of this good is valued positively by the agents participating in the economy considered.

While there is no reason to expect this proposition to fail within a more realistic dynamic context, one might ask whether there is not more to say on the impacts of these nonmarket interactions. This we would not expect in a situation in which the externality is generated by flows. In this case the static inefficiency would simply be repeated in a sequence of inefficiently operating spot markets. However, there is more to say if the externality is generated from stocks rather than flows. We explore this case of a non-market interaction by concentrating on a specific example, the externalities generated from housing unit maintenance within an urban neighborhood.

The services from a particular dwelling unit can only be bought in a package involving among other attributes the quality of the residential neighborhood within which the dwelling unit is located. That neighborhood quality is influenced by the quality of the individual housing units located there. The revenues an individual landlord receives from renting out a dwelling unit therefore not only depend on the quality level at which he maintains the unit, but on the quality of the entire neighborhood 's housing stock, and therefore on the other landlords' maintenance strategies. Conversely, unless the entire neighborhood 's housing stock is owned by a single investor, the typical landlord is unable to reap the full benefits from his own maintenance effort, leading him to exhibit socially inefficient investment behavior. Since it is not the investment itself, but the change in the attributes of the housing stock that generates the externality, the long-run consequences of this nonmarket interaction can be appreciated fully only as we observe the evolution of housing stock characteristics over time.

In the present paper, this is done within the framework of a simple closed loop finite time differential game. The investors are assumed to own identical shares of housing stock in a neighborhood. Each one acts so as to maximize the present value over the remaining planned economic life of his housing units under rational expectations as to the development of neighborhood quality. The assumed symmetry of behavior allows us to employ phase plane analysis from which we obtain the properties of housing, and neighborhood quality trajectories. As it turns out, there will be one or more stable stationary quality levels, the number of such levels depending on whether housing and neighborhood quality attributes are substitutable for or complementary to each other, and whether these relations change in the attribute levels. If there is more than one stable stationary quality level, then intermediate unstable stationary points define bounds of initial housing, and neighborhood qualities above (below) which housing and neighborhood quality will increase over parts of the remaining economic life of the housing stock (decrease over the entire planning period). This immediately suggests two alternative policy approaches towards influencing investment decisions as affected by externalities: one of selecting the socially preferred stationary quality level and by a *one time* action pushing the system within the range of local stability of that quality level; another one of trying to shift that quality level by *permanent* policies. Thus *typical policy instruments can be applied in a twofold manner.* Observe that only the latter type of policy action can we derive from a static analysis.

The effects of changes in the ownership structure (the share of the neighborhood 's housing units owned by the typical investor) are then studied within a comparative static analysis. As one might expect, positive, stable, stationary quality levels will decrease with an increase in the number of landlords sharing housing stock in the neighborhood. Furthermore, however, levels of unstable stationary points, if they exist, will shift upwards with an increased dispersion in housing ownership, implying an increasing range of local stability of the corresponding stationary lower quality levels. Also, for sufficiently dispersed ownership, zero housing quality is always one, if not the only stationary quality level. In this situation, the quality of the entire neighborhood's housing stock will necessarily decline over its entire economic life.

Provided that several stationary quality levels continue to exist, we obtain that marginal shifts in the distribution of housing ownership may have quite nonmarginal effects on housing unit maintenance and neighborhood quality trajectories: a marginal increase in the concentration of ownership may shift the system from the range of local stability of a low quality level to that of a dramatically higher one, thus inducing housing and neighborhood quality to develop in an entirely different manner. A comparative static analysis involving exogenous changes in housing revenues, costs of maintenance, and the market interest rate reveals similar effects on stationary quality levels, with the same implications on the dynamic behavior of the system. Finally, comparative dynamics with respect to changes in the ownership structure show that housing and neighborhood quality will stay higher under more concentrated ownership as long as housing quality is nonzero.

All these conclusions are reinforced in informal extensions of the standard model that include endogenous determinations of initial housing qualities, of a final (sales) value, and of the planning period considered by the typical investor. All of this in turn suggests conjectures about possible effects of an unequal distribution of housing ownership shares among investors: first, that for any given number of landlords acting in the neighborhood, the observed inefficiency decreases with an increasing inequality in the distribution of housing ownership; second, that the inefficiencies decrease with an increase in the dispersion of initial housing qualities and/or of planned economic lives for the housing units.

In all cases the argument supporting these conjectures is that some landlords *would do better* if all owned equal shares of housing stock, and the others being initially in relatively worse position *would thereby be induced to do better*. It should be emphasized that a validation of these conjectures seems possible only in the more realistic dynamic context employed here: comparable results cannot be obtained at ease within a static model.

A validation of these conjectures would finally bring about immediate consequences for policy actions: by imposing asymmetries with respect to shares of ownership and/or initial housing qualities and/or planned economic lives via *regulatory policies* one could lead landlords to socially preferable investment behavior. Employing such a one time policy may be preferable to a policy where housing maintenance is permanently subsidized. Only the latter type of policy can typically be derived from static models.

2. THE BASIC MODEL[1]

We suppose that a dwelling unit can be described by a scalar variable q , an index aggregating the many quality dimensions incorporated in such a unit. Let $q = 0$ denote the minimum achievable quality level.

The dwelling unit will deteriorate over time if not maintained. Maintenance expenditures will slow down, if not reverse that deterioration process. This is specified in the following dynamics:

$$\dot{q} = -\delta q + f(s) \tag{1}$$

where δ refers to the deterioration rate of the typical housing unit and s to dollars spent on maintenance. These dollars are bought in a competitive capital market. f is assumed to be a twice continously differentiable, strictly concave function of s. Thus at any point in time increasing expenditures on maintenance are productive at a decreasing rate.[2]

We assume furthermore that maintenance investment is irreversible. Collecting these assumptions, we obtain

$$\begin{aligned}
&s \geqslant 0 \\
&f(0) = 0\,;\ f(s) > 0\,,\ s > 0 \\
&f'(s) > 0\,,\ f''(s) < 0\,,\ s \geqslant 0 \\
&\lim_{s \to +\infty} f'(s) = 0\,.
\end{aligned} \tag{2}$$

The revenues a landlord receives from renting out a dwelling unit are not only dependent on q , but also on neighborhood conditions. We suppose that the neighborhood can be described again by a scalar variable Q reflecting an aggregate of quality dimensions. Since our focus is on modelling the mutual interdependence of investment decisions, we let Q reflect only the quality of the neighborhood's housing stock.

The revenues net of strictly current expenditures (in particular operating costs), $R(q,Q)$ are considered twice continously differentiable and concave in both arguments. As to the sign of the cross derivative, we require it to be negative for 'large' values of q and Q .

In formal terms:

$$\left.\begin{array}{ll} R_q(q,Q) > 0 & R_{qq}(q,Q) \leqslant 0 \\ R_Q(q,Q) > 0 & R_{QQ}(q,Q) \leqslant 0 \end{array}\right\} \quad q \geqslant 0, \ Q \geqslant 0. \tag{3}$$
$$R_{Qq}(q,Q) < 0 \quad q,Q > M, \qquad 0 < M < \infty.$$

These assumptions can be derived from utility theory as follows: Consider a typical consumer endowed with an instantaneous utility funtion $u(x,q,Q)$ exhibiting the usual properties. Let x denote the numeraire good. Let finally I denote the consumer's income level.

Suppose a particular utility level $\bar{u}$ is slected (cf Fig. 1).

Fig. 1: Hypothesized relationship between x, q and Q.

For given Q' the consumer is indifferent between bundles along $x(q;Q',\bar{u})$. It follows that $I - x(q;Q',\bar{u})$ denotes his maximum willingness to pay for housing with the attributes (q,Q'). Note, that $I - x(q,Q',\bar{u})$ is concave in q for given Q' as well as in Q for given q'.

In the present context, we consider q and Q to be *complementary (substitutable)* attributes iff

$$\frac{\partial x}{\partial q}(q',Q',\bar{u}) \underset{(<)}{>} \frac{\partial x}{\partial q}(q',Q'',\bar{u}) \quad \forall Q'' > Q'.$$

Now suppose that for some version of the consumer's utility function and for large values of q and Q (and correspondingly small values of x) the marginal utility of an increase in Q declines with an additional marginal increase in q (i.e. $\frac{\partial^2}{\partial q \partial Q} u < 0$ for $q,Q \geqslant M$), while the marginal utility of an increase in Q rises with a marginal increase in x (i.e. $\frac{\partial^2}{\partial x \partial Q} u > 0$ for $q,Q \geqslant M$ and $x \leqslant x(q,Q)$). In this very plausible situation q and Q are substitutable attributes according to the above definition. A proof of this statement is relegated to the appendix (lemma 1, remark 1).

Assuming finally that all individuals in the housing market are endowed with identical indifference maps and incomes, competition in the housing market will ensure that they all end up with a consumption bundle along some indifference level $\bar{u}$. This leads us to identify

$$R(q,Q) \equiv I - x(q,Q,\bar{u}).$$

Alternatively, and much less rigidly we may consider $R(q,Q)$ to be the envelope of willingness' to pay that may be obtained in a competitive housing market by individuals endowed with different preferences and/or income.

The determination of the typical landlord's maintenance strategy for some housing unit j involves specifying the stream of maintenance investment $s_j^*(t)$ over some time interval $[0,T_j]$ that maximizes the discounted present value of net revenues to be obtained from renting out this unit, subject to (1) and his predictions as to the evolution of the components of neighborhood quality not under his control. Thus

$$\max_{s_j(t)} \int_0^{T_j} [R(q_j,Q) - s_j]\, e^{-\rho t}\, dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{q}_j = -\delta q_j + f(s_j) \qquad q_j(0) = q_{o_j} \qquad\qquad (4) \\ & Q = Q\,(q_j,\bar{q}_j) \\ & s_j \geq 0\,. \end{aligned}$$

and given $0 < T_j < \infty$, $0 < \delta < \infty$, $0 < \rho < \infty$ with $\bar{q}_j$ describing the state of all other housing units in the neighborhood.[3] All n landlords owning housing stock in the neighborhood pursue that strategy for their housing units. We call n the *ownership structure* of the neighborhood.

The solution to problem (4) is but a first step to the full specification of the behavior of the system of mutually interdependent investment trajectories. Any solution to this problem must be considered a function of the state of all other housing units $\bar{q}_j(t)$, $t \in [0,T_j]$. It follows that an incorrect prediction of the state trajectories of all other investors' housing stock would lead to a revision of investor j's optimal maintenance strategy $s_j^*(t)$. In order to obtain maintenance trajectories that indeed will be pursued (as long as the environment of our system remains stationary), we will resort to the assumption of rational expectations. In the present situation the following behavior is suggested by this assumption: the typical investor initially assumes neighborhood quality to develop in some arbitrary fashion, and determines his optimal maintenance program based on this assumption. Assuming that all other landlords will behave the same way he does, he will doubt his initial assumption and revise his policy. Only when his own maximizing plans are consistent with his assumption about the development of neighborhood quality will he be confident about his predictions and plans.

A characterization of the resulting system of mutually interdependent maintenance trajectories unfortunately is possible only by assuming that all housing units (initially) are alike, and so are investors with respect to endowments and behavior. Thus each landlord owns a share of $\frac{1}{n}$, $n \geq 1$ of the housing stock of identical initial quality, and plans are made over a common planning horizon. We finally assume Q to be simply the arithmetic mean of individual dwelling unit qualities, implying that the quality of each housing unit contributes to neighborhood quality with the same weight.

The problem of choosing n optimal maintenance policies in the general case then reduces to one of finding an optimal maintenance program for the typical investor's share of housing stock which is consistent with the assumed symmetry in endowment and behavior. The resulting set of identical "equilibrium strategies" in this externality game is possibly only one of several equilibrium strategies that may emerge, but it is, to our knowledge, the only one for which characterizations can be given.

This concludes the specification of the model. We admit that especially the latter assumptions are rather restraining, and that the solution concept employed here is delicate.[4] However, given the symmetry assumptions on investors' endowments and behavior, the behavioral assumption used here is much more reasonable than the classical Cournot assumption: here, the typical firm recognizes the other firms' reactions to its behavior, rather than being ignorant about them. Finally, the characterizations obtained below from this model are sufficiently enlightening to justify the assumptions made. They at least make way for educated guesses as to how the system will behave once one or more of the assumptions are relaxed. Such guesses will be discussed in section 6 of this paper.

3. SOLUTION OF THE BASIC MODEL

In order to solve the typical investor i's maintenance problem for his share of the housing stock, consider the current value Hamiltonian

$$H^i(q^i,s^i,\lambda^i,\bar{q}^i) = \frac{1}{n}\{ R(q^i,\frac{1}{n}q^i + \frac{n-1}{n}\ \bar{q}^i) - s^i \} + \lambda^i[-\delta q^i + f(s^i)] \qquad (5)$$

where $\lambda^i(t)$ is the costate variable (Langrangian multiplier) that reflects the contribution to the objective from increasing marginally the quality of all housing units owned by landlord i .

The Hamiltonian can be interpreted as the value of current economic activity of the landlord. More specifically, the expression in curved brackets represents current income from one dwelling unit net of maintenance expenditures for this unit. Multiplied by $\frac{1}{n}$ the first term represents the investor's total (share of) current income. The second term reflects the value of the gains or losses in housing quality as affected by current expenditures on maintenance, on future income. Recall for the latter interpretation that the bracketed expression equals $\dot{q}^i(t)$, the change in physical quality at t, which is influenced by current expenditures on maintenance. This change in physical quality is evaluated by $\lambda^i(t)$, representing the present value of the stream of future revenues from a marginal increase in the quality of (the share of) all housing units owned by investor i .

Pontryagin's maximum principle requires maximization of the Hamiltonian at any time t with respect to the control variable s^i . In the present context, maximizing the Hamiltonian intuitively means balancing present expenditure with the stream of future revenues returned from this expenditure.

Necessary conditions to be satisfied by an optimal maintenance path for an arbitrary function $\bar{q}^i(t)$ are that there exists a shadow price $\lambda^i(t)$ as a continous function of time such that

$$\dot{\lambda}^{*i} = \rho\lambda^{*i} - \frac{\partial H^i}{\partial q^i} \tag{6}$$

$$= (\delta+\rho)\,\lambda^{*i} - \frac{1}{n}[R_q(q^{*i}, \frac{1}{n}q^{*i} + \frac{n-1}{n}\bar{q}^i)$$

$$+ \frac{1}{n}R_Q(q^{*i}, \frac{1}{n}q^{*i} + \frac{n-1}{n}\bar{q}^i)], \quad \lambda^{*i}(T) = 0$$

$$\dot{q}^{*i} = \frac{\partial H^i}{\partial \lambda^i} = -\delta q^{*i} + f(s^{*i}), \qquad q^i(0) = q_0 \tag{7}$$

and the Hamiltonian is maximized with respect to $s^i(t)$

$$\frac{\partial H^i}{\partial s^i} = -\frac{1}{n} + \lambda^{*i}\cdot f'(s^{*i}) \leqslant 0 \quad \text{and} \quad \frac{\partial H^i}{\partial s^i}\cdot s^{*i} = 0\,. \tag{8}$$

Here stars denote optimal values. They are deleted below, since no confusion is possible.

The following interpretative remarks may be useful:

(i) The increase in revenues accruing from a marginal increase in the quality of the landlord's housing units is composed of two parts: R_q reflects the direct increase in revenues from increasing the quality of his housing units, and $\frac{1}{n}R_Q$ the indirect increase in revenues due to an increase in neighborhood quality that is produced by an increase in the quality of his own housing units. The term $\frac{1}{n}$ reflects the degree to which the indirect increase in revenues is internalized by the landlord himself.

(ii) Solving (6) explicitly, we obtain the interpretation of λ^i given before:

$$\lambda^i(t) = \frac{1}{n}\int_t^T [R_q(q^i(\tau), \frac{1}{n}q^i(\tau) + \frac{n-1}{n}\bar{q}^i(\tau)) \tag{9}$$

$$+ \frac{1}{n}R_Q(q^i(\tau), \frac{1}{n}q^i(\tau) + \frac{n-1}{n}\bar{q}^i(\tau))]\, e^{-(\delta+\rho)(\tau-t)}d\tau.$$

Thus, $\lambda^i(t)$ indeed represents the present value of the stream of future revenues generated from a marginal increase in the quality (the share) of all housing units owned by investor i.

(iii) If $s^i > 0$, then, from (8),

$$\lambda^i = \frac{1}{nf'(s^i)} \quad . \tag{10}$$

Note that the cost of producing q quality units is implicitly defined by $q = f(s)$. Thus, condition (10) can be interpreted as the cost of marginally increasing the quality of $\frac{1}{n}$ housing units.

From this and from (9) we conclude that at any time t , the present value of revenues from marginally increasing housing quality must equal the cost of producing that marginal quality increase. Observe finally that λ^i varies positively with s^i .

(iv) The interpretation of $\lambda^{*i}(T) = 0$ is straightforward: Since T denotes the end of the planned economic life of investor i's housing stock, any improvement at T would be wasteful.

The assumptions of rational expectations and symmetry together imply that the qualitative properties of the system can be discussed by a phase plane analysis of the pair of simultaneous autonomous differential equations (6) and (7), making use of (8) and, by assumption, $q^{*i} = \bar{q}^i$. Further on, we can also delete superscript i , by the symmetry of all trajectories.

Observe first that (8) provides a *short term condition* for the typical optimal maintenance path. It says that whenever there is positive expenditure on maintenance, i.e., $s > 0$, then the value of s is determined by the shadow value of maintenance investment only. In fact, s increases with λ , since for $s > 0$,

$$\frac{ds}{d\lambda} = - \frac{n(f'(s))^2}{f''(s)} > 0 \tag{11}$$

which follows from the concavity of f .

The *long term behavior* for the typical optimal trajectory is determined by the system of differential equations (6) and (7) together with the short term condition (8) and the boundary conditions $q(0) = q_o$ and $\lambda(T) = 0$.

We first establish the properties of the functions $\hat{\lambda}(q)$ and $\tilde{\lambda}(q)$ defined implicitly by the singular solutions to

$$\phi(\lambda,q) = (\delta+\rho)\lambda - \frac{1}{n}[R_q(q,q) + \frac{1}{n}R_Q(q,q)] \tag{12}$$

and

$$\psi(\lambda,q) = -\delta q + f(s(\lambda)) . \tag{13}$$

Differentiating the singular solution to (13) and using (11) we obtain

$$\frac{d\tilde{\lambda}}{dq} = -\delta \frac{f''(s)}{n(f'(s))^3} > 0 . \tag{14}$$

Furthermore, $\frac{1}{n\ f'(0)} > 0$ and, owing to (2), $\tilde{\lambda}(q)$ increases without bound as q does.

$\tilde{\lambda}(q)$ may be interpreted as the marginal cost of maintaining a dwelling unit at constant quality level q.[5] This cost increases with increasing dwelling unit quality. Observing that $\psi_q < 0$ and using remark (ii), we conclude that whenever $\lambda(q) \underset{(<)}{>} \tilde{\lambda}(q)$, it is (not) worthwhile to increase housing quality. In fact, if

$\lambda(q) < \frac{1}{n\, f'(0)}$, then it follows from (8) that maintenance expenditures are zero, implying in particular that at $q = 0$, $\dot{q} = 0$ whenever $\lambda(0) < \frac{1}{n\, f'(0)}$.

Let us now turn to an analysis of singular solution to (12). Observe first that

$$\hat{\lambda}(q) = \frac{1}{n(\delta+\rho)} \left(R_q + \frac{1}{n} R_Q\right)$$

is the present value of revenues over an infinite time horizon the investor obtains from his share of housing when increasing housing quality by a marginal amount, given that neighborhood quality is increased by the same amount.[6] Differentiating $\hat{\lambda}(q)$ we obtain

$$\frac{d\hat{\lambda}}{dq} \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0 \Leftrightarrow R_{qq} + \frac{n+1}{n} R_{Qq} + \frac{1}{n} R_{QQ} \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0 . \tag{15}$$

Recall the derivation of R from a representative consumer's preferences demonstrating that R_{qq} and R_{QQ} are nonpositive, and that $R_{Qq} \underset{(<)}{>} 0$ if q and Q are complementary (substitutable) attributes. Thus, if q and Q are *sufficiently strong complements* so that $\frac{n+1}{n} R_{Qq} > - R_{qq} - \frac{1}{n} R_{QQ}$, then $\hat{\lambda}(q)$ will be an increasing function; otherwise it will be decreasing. The latter will definitely hold for $q>M$. Observe finally that $\hat{\lambda}(0) > 0$ and $\phi_\lambda > 0$.

Owing to the indeterminacy of (15), we need to distinguish the following principal cases:

Case A: $\hat{\lambda}(0) > \frac{1}{n\, f'(0)}$

This reflects the situation that, at given ownership structure n ,

$$\frac{1}{\delta+\rho} \left[R_q(0,0) + \frac{1}{n} R_Q(0,0)\right] > \frac{1}{f'(s(0))}$$

Thus, under that ownership structure it is worthwhile to maintain even if the prevailing housing and neighborhood qualities are zero.

Case B: $\hat{\lambda}(0) < \frac{1}{n\ f'(0)}$

Here, we may consider two subcases, with B.1 referring to $\hat{\lambda}(q) < \tilde{\lambda}(q)$, $q > 0$ and B.2 to $\hat{\lambda}(q) > \tilde{\lambda}(q)$, some q. Thus, in the former situation it is never worthwhile to maintain at, or improve housing from a positive quality level, whereas in the latter subcase this will be profitable for some range of housing, and neighborhood quality.

Typical phase portraits corresponding to these cases are presented in figures 2, 3, and 4 respectively.

Figure 2: Phase portraits for Case A: $\hat{\lambda}(0) > \frac{1}{n\ f'(0)}$.

Figure 3: Phase portrait for Case B 1: $\hat{\lambda}(q) < \tilde{\lambda}(q) \forall q$.

Figure 4: Phase portraits for Case B2: $\hat{\lambda}(q) > \tilde{\lambda}(q)$ some q, with $\hat{\lambda}(0) < \frac{1}{n\ f'(0)}$.

Before a discussion of the optimal maintenance paths as varying with cases, the initial condition q(0) and the length of the planning period T , consider the following remarks justifying the diagrams as drawn:

(v) Case A, or case B alternatively may prevail at any given ownership structure n . Let, however, case A prevail for that ownership structure. Then as long as $\frac{1}{\delta+\rho} R_q(0,0) < \frac{1}{f'(0)}$, i.e.

as long as, at zero housing and neighborhood quality, the direct revenues generated from marginal improvement do not cover the cost, there is *always* an integer $\underline{n}$ such that $\forall\ n > \underline{n}$, $\hat{\lambda}(0) < \frac{1}{n\ f'(0)}$.

Given that eminently plausible assumption, an increasing dispersion in the ownership structure always leads us to situation B . This will be the one of major concern.

(vi) Concentrating on generic cases there is an odd number $2k+1, k=0,1,\ldots,K$ of singularities, denoted by (q^k, λ^k) in an order increasing with q , and λ . In case B, $(0, \hat{\lambda}(0))$ is always a singularity. This follows directly from the discussion of the singular solution to (13). The singularities are alternating saddle points (labelled (q_s^k, λ_s^k)) and sources (labelled (q_u^k, λ_u^k)), the one taking the lowest values of q and λ being a saddle point. Proofs of these statements again are relegated to the appendix (remarks 2, 3, lemma 2). This justifies the connection of the singularities by optimal paths as drawn in the diagrams.

We are now prepared to discuss the phase portraits in detail. Recall that any optimal maintenance path must satisfy

(a) the boundary condition for the differential equation (7), $q(0) = q_o$,

(b) the transversality condition $\lambda(T) = 0$, which may be interpreted as a boundary condition to the differential equation (6) ,

(c) the condition that travelling along the optimal trajectory takes exactly T years.

Condition (a) is self explanatory. By condition (b) we can exclude from further consideration all trajectories that do not end at the horizontal axis. Concerning condition (c), recall from the discussion of $\tilde{\lambda}(q)$ and $\hat{\lambda}(q)$ that as the vertical distance from $\tilde{\lambda}(q)$, respectively, increases, the velocity of movement in the directions prescribed by the differential equations increases. Thus as an example, travel along trajectories near the abscissa is faster than along trajectories near any of the singular points. Thus, trajectories can be ordered uniquely according to the time period that is needed for travel, subject to conditions (a) and (b).

Consider now an initial quality level $q_1(0)$ (cf figs. 2(a), 3, and 4(a)). In all cases, optimal maintenance expenditures will be nonincreasing for sufficiently small planning periods. The typical trajectory is labelled T_1. Any planning period of length $T > T_2$ results in an initially increasing, then decreasing optimal level of maintenance investment. However, optimal maintenance expenditures never lead to an increase in housing, and neighborhood quality. The typical trajectory is labelled T_3. If the planning period considered is sufficiently long, travel will be along the trajectory labelled T_* leading to the next lower saddle point. There, housing and neighborhood quality will stay constant for a nonzero period of time. Towards the end of the planning period, the optimal maintenance path will leave the saddle point towards the abscissa leading to further decreasing housing and neighborhood quality if the saddle point quality level is positive. Optimal maintenance paths will have the principal structure discussed so far whenever $q(0) \in (q_s^k, q_u^{k+1})$ $k = 0,1 \ldots, 2K+1$, or alternatively $q(0) > q_s^{2K+1}$.

Suppose now that $q(0) \in (0, q_s^0)$ or alternatively $q(0) \in (q_u^k, q_s^k)$, $k = 1,2 \ldots, 2K+1$ (cf $q_2(0)$ in figs. 2(a) and 4(a)). For $T \leqslant T_2'$, such as T_1', optimal maintenance expenditures will be nonincreasing as before, leading to nonincreasing housing, and neighborhood quality trajectories. If the planning period is of length $T > T_2'$, however, maintenance expenditures, although decreasing over time, will be heavy enough to initially *increase* housing quality. For sufficiently long planning periods, initial travel will be along the trajectory T_*' heavily drawn, up to the saddle point (q_s^k, λ_s^k). After this increase, housing quality will stay stationary for some time and then decrease, following trajectory T_*.

In conclusion, the generic cases as drawn can be distinguished be the following features:

(vii) Comparing cases A (fig. 2) and B (figs. 3 and 4) it becomes obvious that $\hat{\lambda}(0) > \frac{1}{n\, f'(0)}$ *is a necessary and sufficient condition* for $q_s^o > 0$, i.e. for housing and neighborhood quality *not* to stay stationary at a zero level.[7] This situation will, if at all, obtain *only for a sufficiently small dispersion of housing ownership in the neighborhood.*

(viii) In case B.1, $q_s^o = 0$ is the *only* stationary quality level. This case will definitely be relevant if housing ownership in the neighborhood is dispersed, and housing and neighborhood quality are considered substitutable rather than complementary $\forall\, q,\, Q \geqslant 0$.[7]

(ix) If additional positive stationary quality levels exist (such as in cases A, fig 2(b) and B.2, fig's 4), then it depends upon the initial housing and neighborhood qualities which one of these will be approached if the planning period considered is sufficiently long.

The number of stationary quality levels obviously is dependent on the shape of $\hat{\lambda}(q)$ and $\check{\lambda}(q)$; most relevant are the changes with q and Q in the preference relations between these two variables. A *necessary condition* for multiple singularities to exist is that q and Q are strongly complementary within some nonzero interval of q and Q.[7]

At this point, an empirical investigation of these cases seems to ~~be warranted~~. Without such information we cannot say much more about the general behavior of our dynamical system. Casual observation of maintenance behaviour, however, leads one to believe that there indeed are several stationary quality levels.

In any case, the existence of a multitude of such quality levels would have significant consequences for policy: In principle, the stationary points can be ordered according to welfare considerations. If the desired stationary point is not obtained by noncooperative private action, then a *one time policy intervention* may be used to shift the initial conditions of the system into the range of local stability of the desired stationary point. Conversely, were there a unique stationary quality level then a *permanent policy action* would be necessary to shift the unique stationary quality level toward more desired ones. Examples for such policies will be described in the next two sections.

4. COMPARATIVE STATICS

In this section we examine how the positive stationary quality levels, if they exist, will change with variations in some parameters. The most important result will be to establish how stationary quality levels will change with a change in the typical investor's share of housing units. In addition we will informally discuss the impacts if shifts in the demand for housing, and changes in the market interest rate.

Returning to the impacts of changes in the ownership structure, suppose there exist singularities $(q^k(n), \lambda^k(n)) \gg 0$. Consider an alternative ownership structure $n' > n$ with n' marginally greater than n. Then

$$q_u^k(n') > q_u^k(n) \quad \text{and} \quad q_s^k(n') < q_s^k(n) .$$

For a proof see again the appendix (lemma 4).

This proposition implies that positive *"stable"* stationary quality levels *decrease* with an increase in the dispersion of housing ownership, whereas *"instable"* stationary quality levels *increase*, leading to upwards shifting domains of local stability of the "stable" stationary quality levels.[8]

While the former conclusion is fully in line with the intuition developed from static models, this is not quite the case for the latter one. An illustrative example may help at this point: Suppose the initial quality of the typical housing unit is $q(0)$, with $q(0) \in (q_u^k(n), q_u^k(n'))$ for some $k > 0$. The housing and neighborhood quality will *decrease* to the next lower stable stationary quality level under dispersed ownership whereas it will *increase* under more concentrated ownership, if the typical planning period is sufficiently long. Thus, housing, and neighborhood quality trajectories may develop *in entirely different directions if the degree changes at which external effects may be internalized.*

Considering now the effects of exogenous shifts in the demand for housing, we are led to similar conclusions as long as such shifts do affect marginal revenues R_q and/or R_Q. More precisely, the effects of positive shifts in R_q and/or R_Q, respectively, do work in the same direction as a decrease in the dispersion of housing ownership. This is also the case once we consider an increase in the marginal productivity of maintenance expenditures or a decrease in the market interest rate, ρ.[9]

One consequence of all this is that *exogenous marginal shifts* in the demand for housing services provided in the neighborhood, or for that matter marginal shifts in the market interest rate may have *dramatic effects* on the further development of housing, and neighborhood quality. Again, this result is in line with observed neighborhood change. Another consequence relates to policy actions in the presence of externalities. All these exogenous shifts may be generated by the application of policy instruments such as housing allowances and construction subsidies. In the presence of multiple stationarities, however, each one of them may be employed in a one-time fashion, thus pushing, if successfully applied, the system away from a socially inferior stationarity to a socially preferred one, or in in a continuous time fashion, thus moving that socially preferred stationarity towards a socially optimal point. Note again that it is only the latter type of policy which we derive from conventional static models.

5. COMPARATIVE DYNAMICS

The result we want to establish here is that during any given planning period maintenance expenditures, as long as they are positive, will always be higher under concentrated rather than dispersed ownership of housing stock in the neighborhood. More importantly, the quality of housing stock and neighborhood quality will (as long as they are positive) permanently stay higher under concentrated ownership for sufficiently long planning periods.

In order to derive this result we employ a somewhat restrictive but very convenient assumption. It is sufficient to generate the desired result, but by no means necessary. Consider, for that matter, again ownership structures n, and n', with $n < n'$ and two optimal trajectories $(q_n(t), \lambda_n(t))$, $(q_{n'}(t), \lambda_{n'}(t))$, $t \in [0, T]$ starting at $q_n(0) = q_{n'}(0)$. Suppose that for $q > q'$

$$0 > R_q(q,q) - R_q(q',q') + \frac{1}{n} (R_q(q,q) + R_Q(q',q')) + R_Q(q',q')(\frac{1}{n} - \frac{1}{n'}) .$$

Then $q_n(t) > q_{n'}(t) \quad t \in (0, T]$ for T large enough.

The proof of this proposition is again relegated to the appendix (lemma 6 and 7, propositions 1 and 2, remarks 4 and 5).

This proposition establishes that, under the sufficient but by no means necessary condition, marginal revenues are decreasing when housing and neighborhood qualities increase, a concentration of ownership or, for that matter, central control over the allocation of maintenance expenditures leads to permanently higher housing, and neighborhood quality. While this result is intuitively obvious, it is important to keep in mind that it has been derived within a dynamic context, with externalities generated by stocks rather than flows.

We cannot conclude, as a matter of course, that exclusive control by a single firm can be favored unconditionally from a policy point of view. Such a firm may eventually gain monopoly power over segments of the urban housing market. This may have allocative and/or distributional impacts that more than outweigh the efficiency increases due to the internalization of the neighborhood effect. It therefore is important to search for alternavive remedies to the externality problem. Some will be discussed below.

6. EXTENSIONS OF THE MODEL

In this section, we will relax some of the assumptions used so far and informally discuss the implications of doing so. We will first consider the situation in which the typical investor initially constructs a housing unit, and its economic life continues beyond the end of the planning period. Then we will discuss how economies of scale in the number of units maintained, and capital market imperfections affect the results derived so far. A further comment considers the impact of changes in land use intensity. We finally speculate about the impact of relaxing the rigid symmetry assumptions underlying the present model.

It is easy to incorporate the possibility that units are constructed initially, and that their economic life continues beyond T, as long as the assumption about symmetry of stocks and behavior is maintained. We will do this by introducing explicitly a cost function relating the initial quality of housing to the construction outlay, and a function in which the terminal value of the property is determined on the basis of housing and neighborhood quality. The boundary conditions on the dynamical system (6) and (7), $q(0) = q_0$ and $\lambda^*(T) = 0$, thus are replaced by conditions of the form

$$\lambda^*(0) = \frac{1}{n} \frac{\partial C(q(0))}{\partial q(0)} \tag{16}$$

$$\lambda^*(T) = \frac{1}{n} \frac{dP(q(T),Q(T))}{dq(T)} \tag{17}$$

where $C(q(0))$ specifies the cost of constructing a unit of initial quality $q(0)$, and $P(q(T), Q(T))$ denotes the terminal value of a housing unit of quality $q(T)$, provided that neighborhood quality is $Q(T)$. Both are assumed to be given to the individual investor.

The interpretation of the new conditions is straightforward. Recalling the interpretation of λ^* given in section 3, we obtain that at initial time, the stream of revenues generated from a marginal increase in the quality of the investor's housing stock must equal the cost of producing that marginal increase. Furthermore, at terminal time the marginal value of this quality increase must be equal to the contribution to terminal value of that quality increase.

The new boundary conditions determine functions $\lambda_0(q)$ and $\lambda_T(q)$ respectively in the phase space λ/q replacing $q(0) = q_o$ and $\lambda(T) = 0$, respectively. The principal results from this extension are as follows:

(i) The initial qualities at which units are constructed do vary with the structure of ownership: Assuming $C'' > 0$, it is easily shown that the quality of initial construction increases with an increasing **concentration of ownership.** This reinforces the conclusions drawn from the analysis in sections 4 and 5.

(ii) $\lambda^*(T)$ shifts downwards with increases in n: This leads to a negative feedback on maintenance behaviour during the entire planning period, thus again reinforcing the conclusions derived earlier.

As a special case to this extension, suppose that P does not vary with the quality of the housing unit itself, but only with the quality of the neighborhood. In this form, we may interpret P as the residual value of the site upon which the housing unit is located. Suppose that the typical investor does not only determine the optimal maintenance path for a given period of length T, but together with it T itself.[10] **Then it is straightforward to show that** *T increases with a decrease in n.* This increase from T to say $T' > T$ in turn leads to an increase in housing and neighborhood quality over the entire planning period $(0,T]$, which follows directly from an inspection of the phase diagrams (figs. 2, 3, and 4).

A consideration of scale economies in the number of units maintained again reinforces the conclusions we have drawn in sections 4 and 5: Decreasing costs as the number of units maintained jointly increases do favour maintenance under concentrated ownership. A similar argument holds if we incorporate capital market imperfections. This is so because such imperfections tend to favor investors with comparatively large property holdings, thus contributing to a reduction in the capital cost of maintenance.

Consider now the effects of changes in land use intensity. Such changes typically are reflected in changing distances between housing units. However, neighborhood effects, as virtually *any* externalities, do not work pervasively in space, but are declining with increasing distance from the location at which they are generated. Within the present model, we would therefore expect that as the distance between housing units increases the relative importance of the neighborhood effect diminishes, and together with it the impact the distribution of ownership bears upon maintenance investment. Such is indeed the case as long as R_Q is decreasing with decreasing land use intensity, no matter how R_q will behave with such a change.

It is much more difficult to draw conclusions from relaxing any of the symmetry assumptions underlying the model. In principle, we should consider separately variations among the shares of investor's property holdings, variations among the initial qualities of housing units held by the different investors, and variations among the planning periods envisaged by the investors. For reasons explained in section 2, it is difficult to formally establish any results if we allow for such variations.

The following conjectures may be suggestive for further analysis:

(i) For given n, given uniform q(0), and given uniform T, increasing variations in the shares of investors' property holdings will lead to a decrease in unstable quality levels and an increase in stable ones. Furthermore, each individual housing unit will be maintained better during the entire economic life of the unit.

The reasoning behind this is that an investor with large holdings will maintain better because he is able to internalize more of the external effect. The resulting increase in neighborhood quality will induce the small property holder to maintain better as well, at least in ranges where housing and neighborhood qualities are complementary attributes.

(ii) For similar reasons, increasing variations in the initial quality of housing units cet. par. may affect positively the average quality of the housing stock if the planning period is relatively short. They most probably have no effect on stable stationary quality levels and on maintenance paths if T becomes large. The reasoning behind this is that, in general, the long run behavior of any dynamical system tends to be independent of the initial conditions imposed on it.

(iii) A similar result we would expect finally from variations in the investors' planning periods. Consider increasing variations in planning periods about some mean planning period. Since maintenance expenditures increase with increasing length of the economic life planned for a unit, an investor planning for a shorter period is again induced to maintain better at least in ranges where housing and neighborhood quality are complementary.

A verification of these conjectures would have immediate consequences for policy: By regulation one could impose asymmetries with respect to shares of ownership, or initial housing qualities, or types of housing owners (and with them in planning periods), thereby increasing long run stationary quality levels, and possibly the quality of the housing stock over the entire economic lives of the housing units involved.

7. SUMMARY

In the work reported here, we analyzed, within a dynamic behavioural model, the impacts of externalities generated from stocks upon investment in these stocks. Using urban housing as an example which can only be consumed jointly with the externality commonly called neighborhood quality, we showed that consumers' preference relations between the services consumed from the stocks themselves, and the externality variable turn out to critically influence the evolution over time of the stock in question. We demonstrated that a sufficiently strong complementarity of these service flows implies an increasing likelihood for several stationary quality levels to exist. This, in turn, leads to principal alternatives as to influencing investment behavior in the presence of such externalities: one time action to select preferred stationary quality levels, and continuous time action to shift such quality levels. The strong influence of preference relations on investment behavior and policy conclusions suggests an effort towards empirically estimating these relations.

Special further emphasis was given to the way investment behaviour changes with the degree to which the typical investor is able to internalize the benefits from his expenditures. Beyond the result that an increasing internalization increases investment efforts, it was shown that marginal modifications in this respect may dramatically affect allocation decisions. A similar conclusion applies when we consider the impacts of exogenous shifts on the demand for the services provided from the housing stock, or on the market interest rate.

The model used so far was based on rather strong assumptions as to the symmetry of endowments and in decision making. We used the intuition generated from that model in formulating conjectures about the impact of asymmetries on investment behaviour. A verification of these conjectures, which must be left for further research, could lead to rather strong conclusions on the formulation of second best policies in the presence of stock externalities.

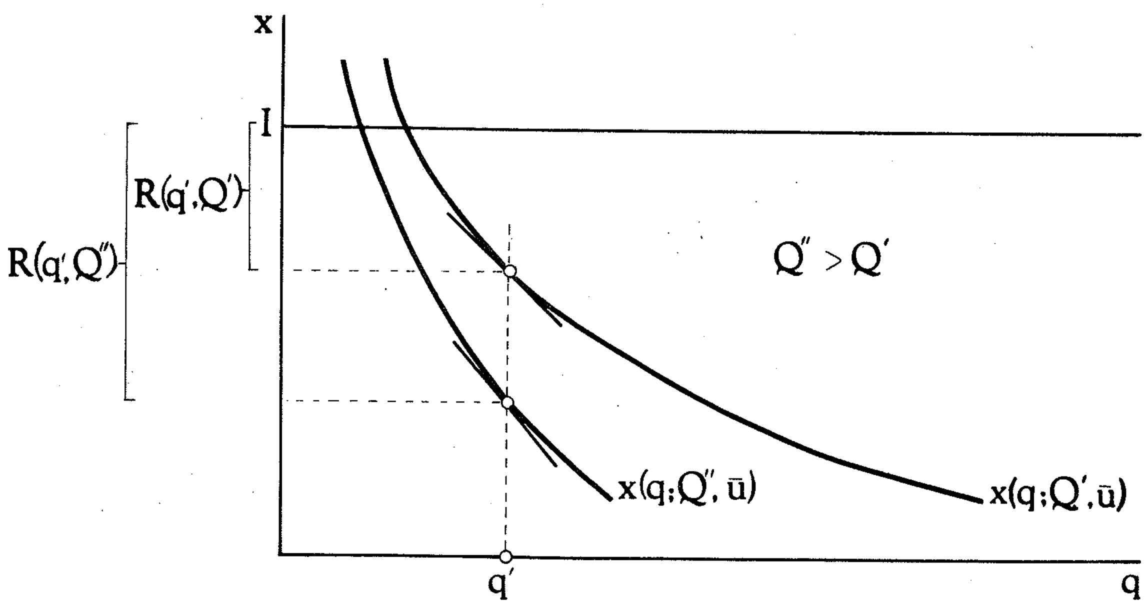
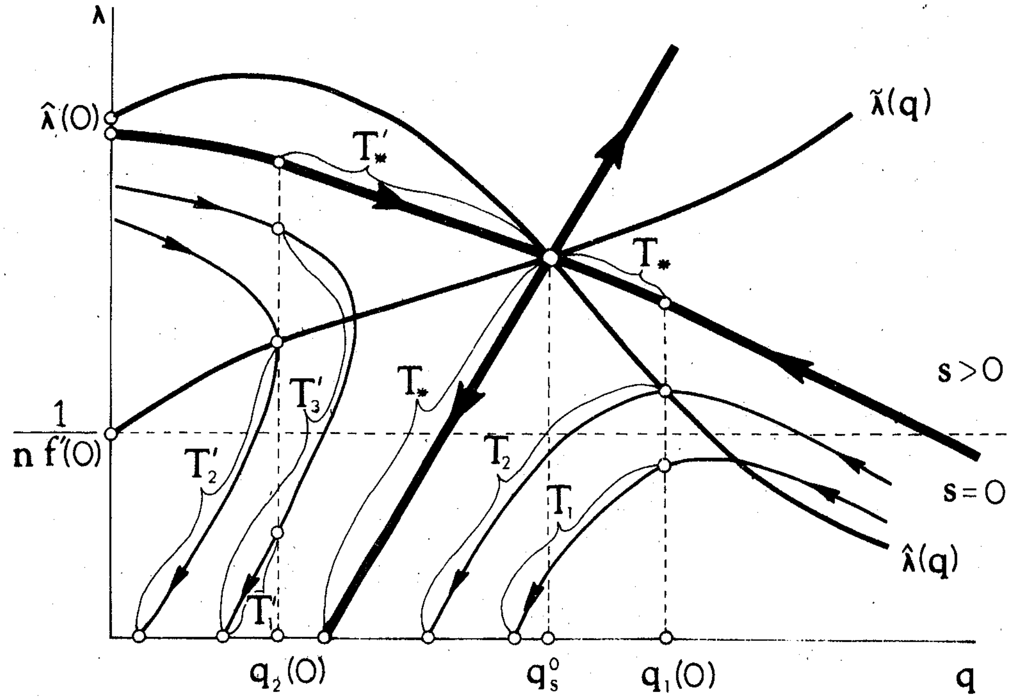
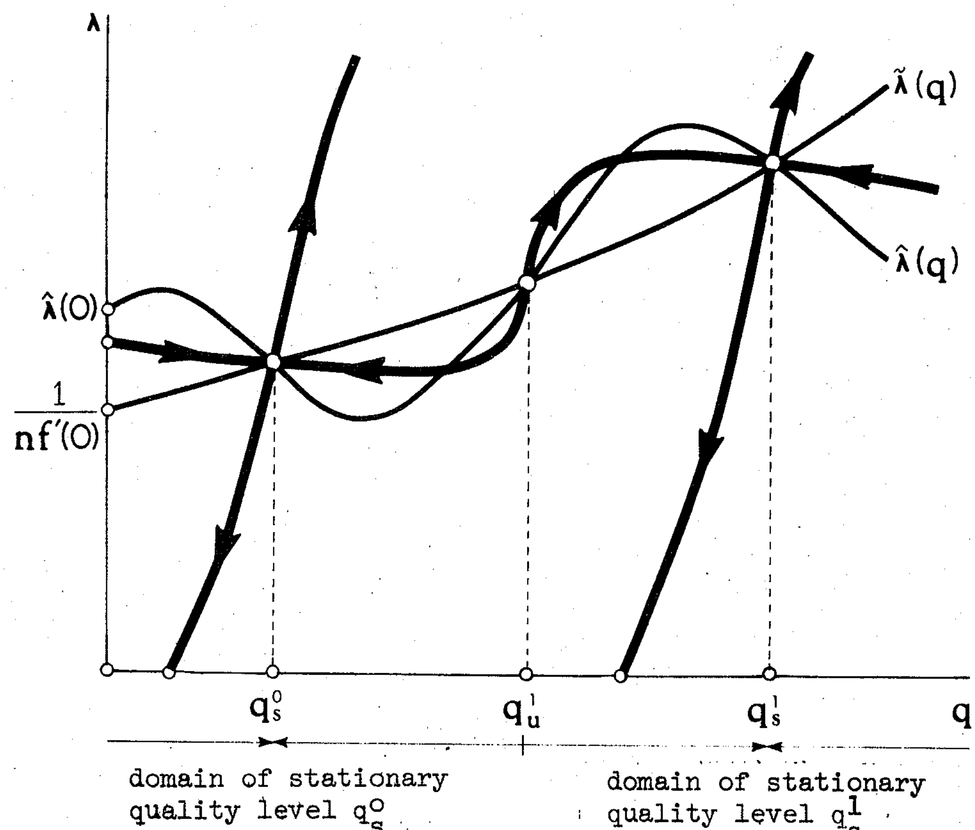


Fig. 1: Hypothesized relationship between x, q and Q.



2(a)



2(b)

Fig. 2: Phase portraits for Case A: $\hat{\lambda}(0) > \frac{1}{n\ f'(0)}$

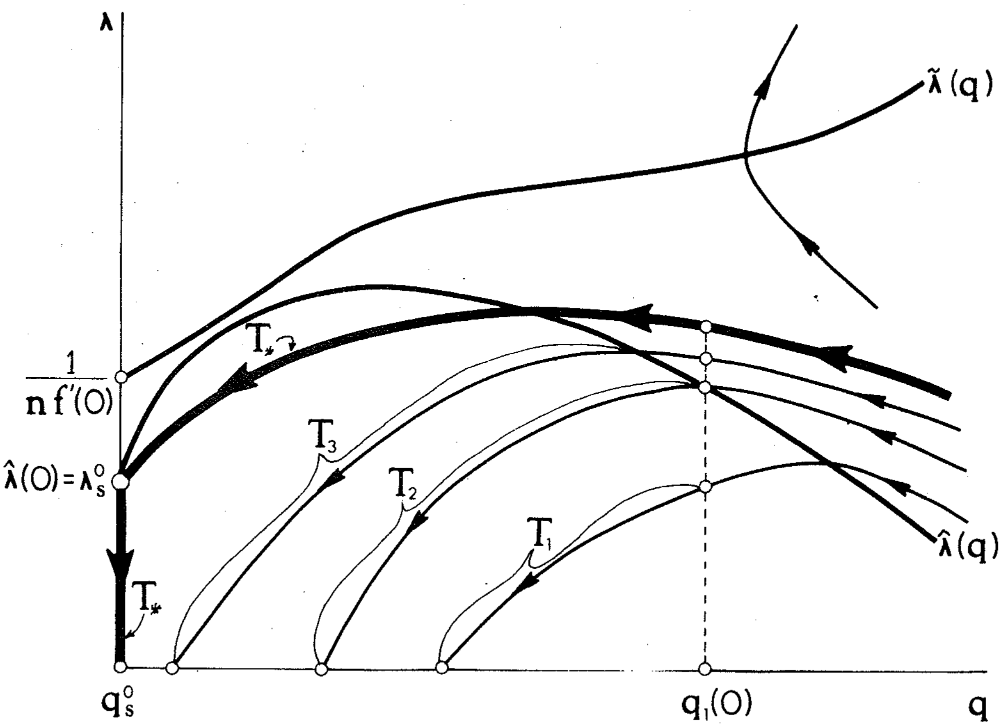
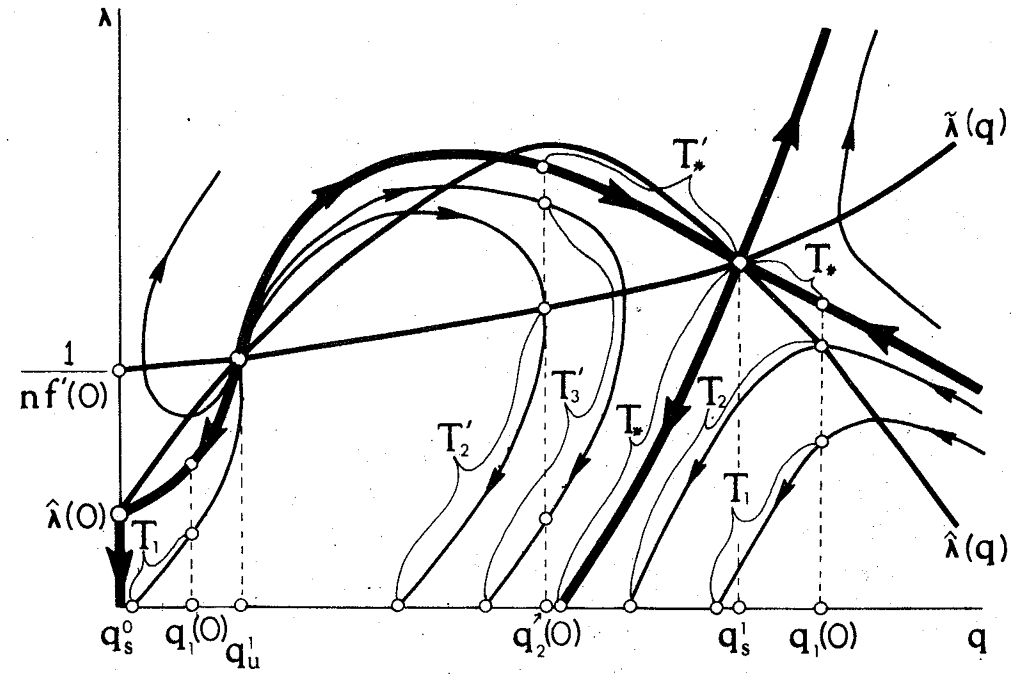
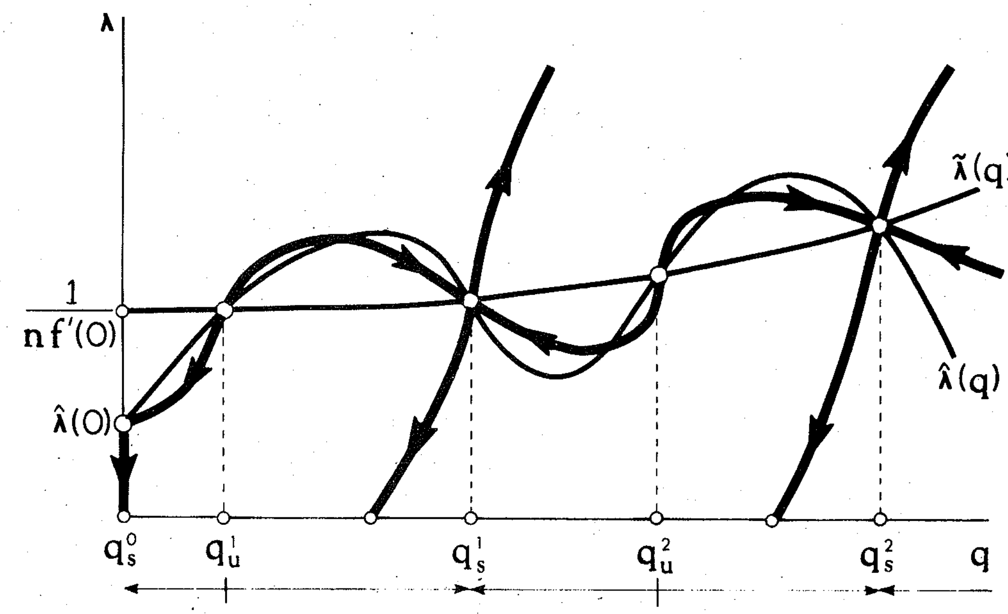


Fig. 3: Phase portrait for Case B 1: $\hat{\lambda}(q) < \tilde{\lambda}(q)\ \forall\ q$



4(a)



domain of stationary quality level q_s^o | domain of stationary quality level q_s^1 | domain of stationary quality level q_s^2

4(b)

Fig. 4: Phase portraits for Case B2: $\hat{\lambda}(q) > \tilde{\lambda}(q)$ some q, with $\hat{\lambda}(0) < \frac{1}{nf'(0)}$

APPENDIX

Lemma 1: Let $u(x, q, Q)$ be a version of the consumer's utility function exhibiting the usual properties and satisfying the conditions

$$\frac{\partial^2 u}{\partial q \partial Q}(x, q, Q) < 0 \quad \forall q, Q \geqslant M$$

$$\frac{\partial^2 u}{\partial x \partial Q}(x, q, Q) > 0 \quad \forall x < x(q, Q; \bar{u}) \text{ with } q, Q \geqslant M$$

then q and Q are substitutable attributes for sufficiently large values $(q, Q \geqslant M)$.

Proof:

$$\frac{\partial x}{\partial q}(q, Q; \bar{u}) = -\frac{\frac{\partial u}{\partial q}(x, q, Q)}{\frac{\partial u}{\partial x}(x, q, Q)}$$

$$\frac{\partial^2 x}{\partial Q \partial q}(q, Q; \bar{u}) =$$

$$= -\frac{\frac{\partial^2 u}{\partial Q \partial q}(x, q, Q)\,\frac{\partial u}{\partial x}(x, q, Q) - \frac{\partial^2 u}{\partial x \partial Q}(x, q, Q)\,\frac{\partial u}{\partial q}(x, q, Q)}{\left(\frac{\partial u}{\partial x}(x, q, Q)\right)^2}$$

and hence

$$\frac{\partial^2 x}{\partial Q \partial q}(q, Q; \bar{u}) > 0 \quad \forall q, Q \geqslant M.$$

■

Remark 1: Note that in spite of the cardinal conditions of the lemma, the relation of substitutable (complementary) attributes defined in the main body of the paper is an ordinal one.

Remark 2: From the geometry of the problem developed so far we know that

(a) in case A $\hat{\lambda}(q)$ cuts $\tilde{\lambda}(q)$ an odd number of times in R_{++}, starting with an intersection where $\hat{\lambda}(q)$ cuts $\tilde{\lambda}(q)$ from above;

(b) in Case B $\hat{\lambda}(q)$ cuts $\tilde{\lambda}(q)$ an even number of times in R_{++}, starting with an intersection where $\hat{\lambda}(q)$ cuts $\tilde{\lambda}(q)$ from below.

Remark 3: There are, of course, conceivable cases where $\hat{\lambda}(q)$ and $\tilde{\lambda}(q)$ are tangent to each other. But an inspection of (12) and (13) reveals that these are pathological cases, as marginal changes of one of the parameters, e.g. ρ or δ, would remove this irregularity.

Lemma 2: Generically (6) - (8) define a hyperbolic flow, such that

(i) $(q_s^{2k}, \lambda_s^{2k})$, $k = 0, \ldots, K$ are saddle points

(ii) $(q_u^{2k+1}, \lambda_u^{2k+1})$ $k = 0, \ldots, K$ are sources

(iii) there are no periodic solutions.

Proof: (a) Note first that the trace of the Jacobian

$$\begin{pmatrix} D_q\psi & D_\lambda\psi \\ D_q\phi & D_\lambda\phi \end{pmatrix} = \begin{pmatrix} -\delta & f' \frac{ds}{d\lambda} \\ -\frac{1}{n}\left(R_{qq} + \frac{n+1}{n} R_{qQ} + \frac{1}{n} R_{QQ}\right) & \delta+\rho \end{pmatrix}$$

is always positive and equal to ρ. Hence, the sum of the eigenvalues of the Jacobian is positive.

(b) Substituting (11) and using (12), we obtain

$$\det \begin{pmatrix} D_q\psi & D_\lambda\psi \\ D_q\phi & D_\lambda\phi \end{pmatrix} = \left(\frac{d\hat{\lambda}}{dq} - \frac{d\tilde{\lambda}}{dq}\right) (\delta+\rho)\, f' \frac{ds}{d\lambda}$$

(c) From Remark 3 and (b) we can conclude that the flow defined is generically hyperbolic.

(d) From (a) we conclude that there are no periodic solutions (see e.g. [1, p. 416]). This establishes claim (iii).

(e) The real parts of the eigenvalues of the above Jacobian have alternating signs, if its determinant is negative. Hence, a singularity is a saddle point, if

$$\frac{d\hat{\lambda}}{dq} < \frac{d\tilde{\lambda}}{dq} .$$

(f) In view of (a), the real parts of the eigenvalues of the above Jacobian are positive, if its determinant is positive. Hence, a singularity is a source iff

$$\frac{d\hat{\lambda}}{dq} > \frac{d\tilde{\lambda}}{dq} .$$

Combining (e) and (f) with Remark 2 establishes claims (i) and (ii) of the Lemma for case A. For Case B, recall that $(0, \hat{\lambda}(0))$ is always a saddle point.

■

<u>Lemma 3:</u> Consider the generic case where $\hat{\lambda}(q)$ and $\tilde{\lambda}(q)$ are nowhere tangent and $\hat{\lambda}(0) \neq \frac{1}{n\, f'(0)}$. Denote by 2K+1 again the number of singularities. Then

(i) $\hat{\lambda}(0) > \frac{1}{n\, f'(0)} \Leftrightarrow q_s^o > 0.$

(ii) $\hat{\lambda}(q) < \tilde{\lambda}(q), \ \forall q > 0, \ \hat{\lambda}(0) < \frac{1}{n\, f'(0)} \Rightarrow K = 0.$

(iii) $R_{qQ} < 0 \ \forall q, Q \geq 0 \Rightarrow K = 0.$

Proof: (i) "$\Leftarrow$" Suppose not. Then $\hat{\lambda}(0) < \frac{1}{n\, f'(0)}$. But $\tilde{\lambda}(0) \in [0, \frac{1}{n\, f'(0)}]$. Then $(0, \hat{\lambda}(0))$ is a singularity. In fact, it is a saddle point (see remark (vi)).

"$\Rightarrow$" Suppose not. Then $\hat{\lambda}(0) > \frac{1}{n\, f'(0)}$, $q_s^o = 0$. For the latter to be true, however, $\hat{\lambda}(0)$ must satisfy (8).

(ii) As singularities with $q > 0$ are excluded by $\hat{\lambda}(q) < \tilde{\lambda}(q)$, $q > 0$, the only singularity can be $(0, \hat{\lambda}(0))$. According to remark (vi), this is indeed a saddle point.

(iii) Recalling (15), we obtain the assumption made in (ii) ∎

Lemma 4: Focusing on generic cases, suppose that there exist singularities $(q^k(n), \lambda^k(n)) \gg 0$. Let n' be marginally greater than n. Then

$$q_u^k(n') > q_u^k(n) \quad \text{and} \quad q_s^k(n') < q_s^k(n).$$

Proof: Consider

$$g(q, n) \triangleq n\, [\tilde{\lambda}(q, n) - \hat{\lambda}(q, n)]$$

If $q = q_u^k(n)$ we have $g(q, n) = 0$, $\frac{\partial g}{\partial q}(q, n) < 0$.

If $q = q_s^k(n)$ we have $g(q, n) = 0$, $\frac{\partial g}{\partial q}(q, n) > 0$.

Now

$$g(q, n) = \frac{1}{f'(s(q))} - \frac{1}{\delta+\rho}\,[R_q(q, q) + \frac{1}{n} R_Q(q, q)]$$

$$< \frac{1}{f'(s(q))} - \frac{1}{\delta+\rho}\,[R_q(q, q) + \frac{1}{n'} R_Q(q, q)] = g(q, n').$$

As n' is only marginally different from n, we also have

for $q = q_u^k(n)$ $\quad \frac{\partial g}{\partial q}(q, \tilde{n}) < 0 \quad \forall\, \tilde{n} \in [n, n']$ and

for $q = q_s^k(n)$ $\quad \frac{\partial g}{\partial q}(q, \tilde{n}) > 0 \quad \forall\, \tilde{n} \in [n, n']$.

The rest follows from calculus. Let $q(n)$ be a solution to $g(q, n) = 0$:

$$0 < g(q(n), n') = g(q(n), n') = g(q(n), n') - g(q(n'), n') =$$

$$= \frac{\partial g}{\partial q}(tq(n) + (1-t)q(n'), n')(q(n) - q(n')),$$

some $t \in (0,1)$.

■

<u>Lemma 5:</u>

(i) Let R also be a function of some exogenous parameter Θ. Suppose that for $\Theta' = \Theta + \varepsilon$, ε sufficiently small and positive,

$$R_q(q, Q; \Theta') > R_q(q, Q; \Theta) \text{ and}$$

$$R_Q(q, Q; \Theta') \geqslant R_Q(q, Q; \Theta)$$

Suppose finally that there exist singularities $(q^k(\Theta), \lambda^k(\Theta)) \gg 0$. Then

$$q_u^k(\Theta) > q_u^k(\Theta'), \quad q_s^k(\Theta) < q_s^k(\Theta').$$

(ii) Let f be a function of some exogenous parameter κ, such that for $\kappa' > \kappa$, $f'(s, \kappa') > f'(s, \kappa)$. Then under analogous conditions

$$q_u^k(\kappa) > q_u^k(\kappa'), \quad q_s^k(\kappa) < q_s^k(\kappa')$$

(iii) Suppose ρ is marginally increased to ρ'. Then under analogous conditions

$$q_u^k(\rho') > q_u^k(\rho), \quad q_s^k(\rho') < q_s^k(\rho)$$

<u>Proof:</u> All the statements follow from an immediate adaption of the proof of lemma 4.

■

Lemma 6: Let $(q(T), 0)$ be the end points of two optimal trajectories $(q_n(t), \lambda_n(t))$ and $(q_{n'}(t), \lambda_{n'}(t))$, $t \in [0, T]$.

Consider assumption A: R and (n, n') are such that for $q > q'$,

$0 > R_q(q', q') - R_q(q, q) + \frac{1}{n'}(R_Q(q', q') - R_Q(q, q)) + R_Q(q, q)(\frac{1}{n'} - \frac{1}{n})$.

Then

(i) $q_n(t) \leqslant q_{n'}(t)$

(ii) $n\lambda_n(t) \geqslant n'\lambda_{n'}(t)$

and $q_n(0) < q_{n'}(0)$ for T large enough.

Proof: Let $\hat{t}_n = \sup \{ t \mid s_n(t) > 0 \}$ and $\hat{t}_{n'} = \sup \{ t \mid s_{n'}(t) > 0 \}$.

Then $s_n(t) = s_{n'}(t) = 0 \quad \forall t \in [\max(\hat{t}_n, \hat{t}_{n'}), T]$.

Hence $q_n(t) = q_{n'}(t)$ in this interval.

Moreover, using (9),

$$n\lambda_n(t) > n'\lambda_{n'}(t) \quad \forall t \in [\max(\hat{t}_n, \hat{t}_{n'}), T).$$

Now (8) implies that $\max(\hat{t}_n, \hat{t}_{n'}) = \hat{t}_n$.

Hence for some $x \geqslant \hat{t}_{n'}$,

$$s_n(t) > 0, \quad s_{n'}(t) = 0 \quad \forall t \in (x, \hat{t}_n)$$

Then (7) implies

(a) $q_n(t) < q_{n'}(t) \quad \forall t \in [x, \hat{t}_n)$

and (9) together with the assumption on R implies

(b) $n\lambda_n(t) > n'\lambda_{n'}(t) \quad \forall t \in [x, T)$.

Consider now the following two cases:

Case 1: There is an interval $[y, x]$ with $s_n(t) = s_{n'}(t) = 0$

Then the same arguments apply as above, and (i) and (ii) hold $\forall t \in [y, x]$.

Case 2: There is an interval $[y, x]$, with $s_n(t) > 0$, $s_{n'}(t) > 0$. In this case

$$n\lambda_n(t) = \frac{1}{f'(s_n(t))} > n'\lambda_{n'}(t) = \frac{1}{f'(s_{n'}(t))}$$

for $t \in [x-\varepsilon, x]$. Hence, $s_n(t) > s_{n'}(t)$ in this interval, and, again by (7), (i) and (ii) hold for $t \in [x-\varepsilon, \hat{t}_n]$. The same argument repeated gives (i) and (ii) for the whole interval $(y, \hat{t}_n)$

Because of (8), cases (c1), (c2) together with the case $s_n > 0$, $s_{n'} = 0$ are exhaustive.

As the interval $[0, T]$ can be recomposed into intervals where one of the cases holds, we have shown the claims of Lemma 6

<u>Remark 4:</u> The assumption A is always satisfied for (q, q') contained in an interval where $R_{qQ} < 0$. However, it is much stronger than needed: It in essence requires that $(q_{n'}(t) - q_n(t))$ is a nonincreasing function in t. Intuitively it is rather obvious that the desired result holds under much more general conditions.

<u>Lemma 7:</u> Let $(q_n(T), 0)$ and $(q_{n'}(T), 0)$ be the respective end points of two optimal trajectories $(q_n(t), \lambda_n(t))$ and $(q_{n'}(t), \lambda_{n'}(t))$, $t \in [0, T]$. Let $n' > n$, and suppose assumption A holds. Then

$q_n(T) < q_{n'}(T)$ implies

$q_n(t) < q_{n'}(t) \qquad \forall t \in [0, T]$

$n\lambda_n(t) > n'\lambda_{n'}(t) \qquad \forall t \in [0, T)$ for T large enough.

Proof: Slightly modified arguments of Lemma 6's proof apply. ■

Proposition 1: Consider two optimal trajectories $(q_n(t), \lambda_n(t))$, $(q_{n'}(t), \lambda_{n'}(t))$, $t \in [0, T]$ with $n' > n$, and $q_n(0) = q_{n'}(0)$. Suppose that A holds. Then

$$q_n(T) > q_{n'}(T) \quad \text{for } T \text{ large enough.}$$

Proof: Suppose not. Then $q_n(T) < q_{n'}(T)$. By Lemma 7, this implies $q_n(t) < q_{n'}(t)$, in particular $q_n(0) < q_{n'}(0)$. Contradiction. ■

Proposition 2: Consider two optimal trajectories $(q_n(t), \lambda_n(t))$, $(q_{n'}(t), \lambda_{n'}(t))$, $t \in [0, T]$, $n' > n$ starting at $q_n(0) = q_{n'}(0)$

Suppose that A': R and (n', n) are such that for $q > q'$

$$0 > R_q(q, q) - R_q(q', q') + \frac{1}{n}(R_Q(q, q) - R_Q(q', q')) + R_Q(q', q')(\frac{1}{n} - \frac{1}{n'})$$

holds.

Then $q_n(t) > q_{n'}(t)$ $t \in (0, T]$ for T large enough.

Remark 5: Whenever A' is satisfied, A is satisfied, so we can make use of Lemmas 6 and 7, as well as Proposition 1. Considering the justification of A', the argument is similar to the one made in Remark 4.

Proof: Suppose not. Then $q_n(0) = q_{n'}(0)$ and $q_n(T) > q_{n'}(T)$ from Proposition 1, but $q_n(t) < q_{n'}(t)$ some $t \in (0, T)$. It follows that there must be at least one $\tilde{t} \in (0, T)$ such that $q_n(t) = q_{n'}(t)$ and the n' trajectory cuts the n trajectory from above.

Let $\tilde{t}_m$ be the maximal element of the set of all such $\tilde{t}$.

Thus $q_n(t) > q_{n'}(t) \quad \forall\, t \in (\tilde{t}_m, T]$.

Evaluating (9) at $\tilde{t}_m$, using A', implies

$$n\,\lambda_n(\tilde{t}_m) < n'\lambda_{n'}(\tilde{t}_m)$$

Hence, using (8) either

(i) $\quad s_{n'}(\tilde{t}_m) > s_n(\tilde{t}_m) \geqslant 0$

(ii) $\quad s_{n'}(\tilde{t}_m) = s_n(\tilde{t}_m) \quad 0.$

Case (i) implies that for some $x > \tilde{t}_m$

$$q_{n'}(t) > q_n(t), \quad t \in (\tilde{t}_m, x) \qquad \text{Contradiction.}$$

If case (ii) holds in some neighborhood $[\tilde{t}_m, \tilde{t}_m+\varepsilon]$, (ii) implies that $\exists\, y > \tilde{t}_m \ni q_n(y) = q_{n'}(y)$, contradicting the choice of $\tilde{t}_m$.

If (ii) does not hold in such a neighborhood, (ii) must be a border case for (i), and the corresponding arguments apply. ■

FOOTNOTES

1 The model of individual investors' behaviour described below is in many respects similar to the ones suggested by Luenberger [2], Sweeney [6] and von Rabenau [3], with the essential difference that within our model neighborhood quality is treated explicitly and determined endogenously.

2 A case where this assumption does not hold is large scale rehabilitation of housing. It follows that rehabilitation decisions are made at discrete points in time. The impact of externalities on these decisions is discussed in Stahl [5].

3 The problem as formulated so far does not involve a terminal value of the housing stock, reflecting the idea that its planned economic life indeed ends at T_j and no scrap value will be associated to the housing stock beyond that. The case where the planned economic life of the housing stock exceeds T_j will be dealt with in section 6.

4 The principal objection one could raise here is that we cannot go beyond a characterization of that symmetric solution via necessary conditions. We cannot show that this solution exists, nor that it is stable. Indeed, due to the nonconvexity involved we cannot even show sufficiency.

5 Thus, we may call $\tilde{\lambda}(q)$ the *static supply curve for housing quality*.

6 Similarly, we may call $\hat{\lambda}$ (q) the *static demand curve for housing quality.*

7 For a proof of these statements see the appendix (lemma 3)

8 By now, it should be clear that we do *not* characterize the stability properties of the singularities in the sense of Liapunow.

9 Again, this is proven in the appendix (lemma 5).

10 In this case, T may be interpreted as the optimal date of selling the property.

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