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AH IMPROVED SOLUTION TO TEE CLASSICAL NEAR-WAKE BOUNDARY-LAYER PROBLEM

Authors
Scalise, D. Theodore
Newman, John.

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D. Theodore Scalise and John Newman

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An Improved Solution to the Classical Near-Wake Boundary-Layer Problem

by

D. Theodore Scalise and John Newman

Mechanical Engineering Department and
Inorganic Materials Research Division, Lawrence Berkeley Laboratory, and
Department of Chemical Engineering, University of California, Berkeley

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Abstract

The fluid flow in the boundary layer adjoining all solid surfaces presents a singular-perturbation problem. A change in the boundary conditions may generate additional regions in which different treatments are necessary.

This singular-perturbation property is exhibited in Goldstein's (1930) classical investigation of a fundamental problem of fluid mechanics--that of determining the fluid velocity distribution in the near-wake boundary layer of a flat plate. Using one set of coördinate variables in the series expansion, he found an approximate solution valid only for the <u>inner</u> region near the plane of the plate; with another set of coördinate variables he found an approximate solution valid only for the <u>outer</u> region, the part of the boundary layer lying farther from the plane of the plate.

In this contribution, we construct a uniformly valid expansion to the classical near-wake problem, using the method of matched asymptotic expansions. This is compared with Goldstein's inner and outer solutions at a downstream distance of half the plate length. The great improvement is evident in the fact that whereas the old solutions are discontinuous in the central region, the new solution is continuous throughout the whole flow domain, merging to the old solutions at both extremities.

Contents

1. Introduction

The study of the velocity distributions along, and in the near-wake of, an infinitely-thin flat plate in steady laminar incompressible flow has been of great interest for nearly three-quarters of a century. Goldstein's (1930) analysis, using the Blasuis (1908) solution for the trailing-edge as an initial profile for the wake flow, is a classical treatment of this subject (described by some, with whom we agree, as one of the most significant contributions of the twentieth century to the theory of fluid mechanics).

We define the "classical" near-wake boundary-layer problem for a flat plate as one which uses the Blasuis solution as the initial profile for the wake flow. Thus, the Goldstein analysis and the present study provide solutions to this classical problem. Other studies (for example, Scalise, 1971) which use a different initial profile (to account for the Kuo (1953) and Imai (1957) second-order flat plate drag) are directed toward solutions of different formulations of the near-wake problem.

In the present study the method of matched asymptotic expansions (described for example by Van Dyke, 1964) is used to construct a composite-expansion of the stream function. The analysis is presented in Section 2.; numerical and graphical results are presented in Section 3.

For convenience, a summary is included in the Appendix of the derivations of the first-order approximations to the Navier-Stokes equations including the Goldstein (1930) solution. Figure 1 shows the

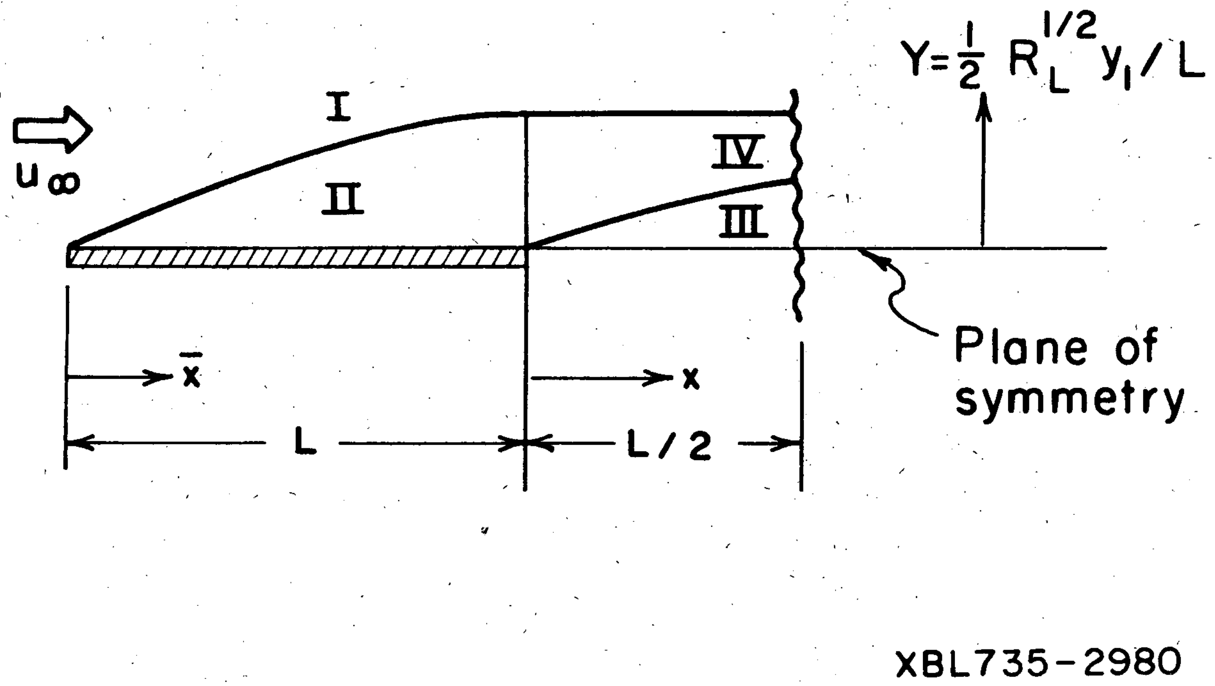

FIG. 1. FLOW REGIMES AND COORDINATE SYSTEM FOR FIRST-ORDER THEORY.
(Not to scale)

I = Inviscid region, II = Blasius boundary-layer region,
III = Goldstein inner region, IV = Goldstein outer-region.

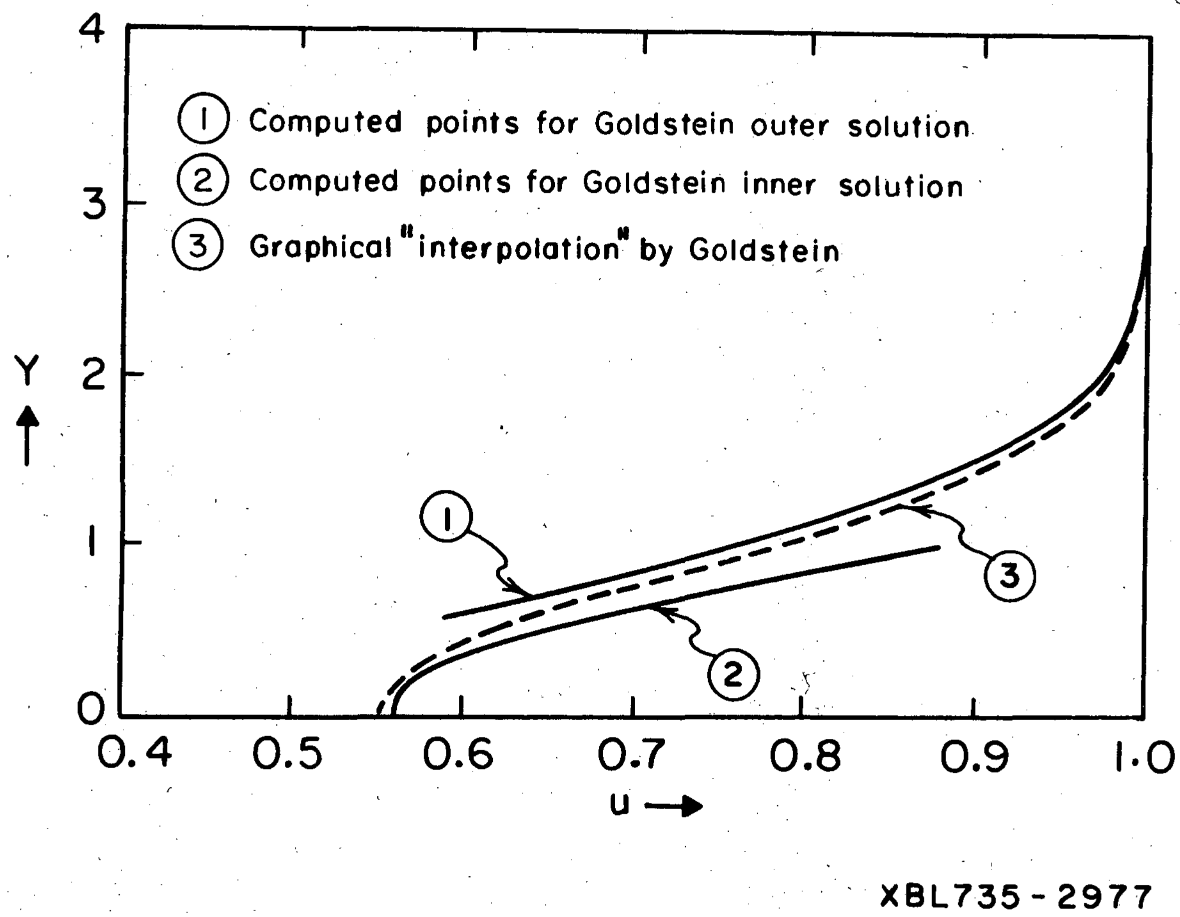

FIG. 2. GOLDSTEIN (1930) INNER AND OUTER SOLUTIONS AT $x_1 = 0.5$ L.

corresponding flow regimes together with the coördinate system.

2. Uniformly-Valid Solution by Method of Matched Asymptotic Expansions

The classical near-wake problem for a flat plate is to find the composite stream-function ψ_c such that:

$$\left\{\begin{array}{ll} 4L\left(\dfrac{\partial\psi_c}{\partial Y} * \dfrac{\partial^2\psi_c}{\partial x_1\partial Y} - \dfrac{\partial\psi_c}{\partial x_1}\dfrac{\partial^2\psi_c}{\partial Y^2}\right) = \dfrac{\partial^3\psi_c}{\partial Y^3} & \text{a} \\ \dfrac{\partial\psi_c}{\partial Y} \sim 1 \quad \text{as} \quad Y \to \infty & \text{b} \\ \psi_c \sim \dfrac{1}{2}\sqrt{\dfrac{x_1 + L}{L}}\,\zeta_{B\ell} \quad \text{as} \quad x_1 \to o & \text{c} \\ \dfrac{\partial^2\psi_c}{\partial Y^2} = 0 \ , \ \dfrac{\partial\psi_c}{\partial x_1} = 0 \quad \text{at} \quad Y = 0 \ , \ x_1 > 0 & \text{d} \end{array}\right\} \qquad (2\text{-}1)$$

To do this we construct a composite-expansion from Goldstein's inner stream function ψ_i and his outer stream function ψ_e where

$$\psi_i(\xi,\eta) = \xi^2\left[\overline{f}_o(\eta) + \overline{f}_3(\eta)\xi^3 + \overline{f}_6(\eta)\xi^6 + \ldots\right] \qquad (2\text{-}2)$$

and

$$\psi_e(\xi,Y) = \psi_o(Y) + \psi_1(Y)\xi + \frac{\psi_2(Y)}{2!}\xi^2 \ldots \qquad (2\text{-}3)$$

with ξ the streamwise coordinate, η the similarity variable for inner region and Y the outer transverse coördinate defined by:

$$\xi = (x_1/4L)^{\frac{1}{3}} \qquad (2\text{-}4)$$

$$Y = 0.5\, R_L^{\frac{1}{2}}\, y_1/L \qquad (2\text{-}5)$$

$$\eta = Y/(3\xi) \ . \qquad (2\text{-}6)$$

The asymptotic large-η form of the inner expansion is given by

$$\bar{f}_o(\infty) \sim A_o\eta^2 + B_o\eta + C_o$$

$$\bar{f}_3(\infty) \sim A_3\eta^5 + B_3\eta^4 + C_3\eta^3 + D_3\eta^2 + E_3\eta + F_3 \qquad (2\text{-}7)$$

$$\bar{f}_6(\infty) \sim A_6\eta^8 + B_6\eta^7 + C_6\eta^6 + \dots + H_6\eta + I_6$$

where the remainders are exponentially small and the coefficients A_j, B_j, C_j, ... are known.

Using additive composition (i.e. the sum of the inner and outer expansions is corrected by subtracting the part they have in common, so that it is not counted twice),

we have

$$\psi_c^{(N)} = \psi_i^{(N-1)}(\eta,\xi) + \psi_e^{(N-1)}(Y,\xi) - \psi_i^{(N-1)}(\infty,\xi) + O(\xi^N) \qquad (2\text{-}8)$$

uniformly valid to ξ^N - order. The superscripts denote the highest degree of ξ in each term; the subtracted term denotes the asymptotic large-η form of the inner expansion.

Substituting (2-2), (2-3), and (2-7) in (2-8) and writing in summation notation:

$$\psi_c^{(N)} = \sum_{o}^{N-3} \bar{f}_j(\eta)\,\xi^{j+2} + \sum_{o}^{N-1} \frac{\psi_j(Y)}{j!}\,\xi^j - \sum_{o}^{N-3} \bar{f}_j(\infty)\,\xi^{j+2} + O(\xi^N)$$

with (2-9)

$$\bar{f}_1, \bar{f}_2, \bar{f}_4, \bar{f}_5, \bar{f}_7, \bar{f}_8, \dots \equiv o.$$

For example the uniformly valid stream function to $0(\xi^3)$ is

$$\psi_c^{(3)} = \overline{f}_o(\eta)\xi^2 + \psi_o(Y) + \psi_1(Y)\xi + \psi_2(Y)\xi^2$$

$$- \left\{A_o\eta^2 + B_o\eta + C_o\right\}\xi^2 + 0(\xi^3) .$$

We now wish to obtain the uniformly valid expression for the velocity distribution. It will be seen, in the next paragraph, that taking the partial derivative of the uniformly valid stream function will not give a uniformly valid velocity distribution; instead each term must be examined to insure it proporates the same order error into the sum.

Taking the partial derivative of (2-9) and noting from (2-6) that

$$\frac{\partial\overline{f}}{\partial Y} = \frac{d\overline{f}}{d\eta}\frac{\partial\eta}{\partial Y} = \frac{1}{3\xi}\frac{d\overline{f}}{d\eta} \quad \text{we get}$$

$$\frac{\partial\psi_c^{(N)}}{\partial Y} = \sum_o^{N-3} \frac{1}{3}\frac{d\overline{f}_j}{d\eta}\xi^{j+1} + \sum_o^{N-1}\frac{\psi_j'}{j!} - \sum_o^{N-3}\frac{1}{3}\frac{d\overline{f}_j(\infty)}{d\eta}\xi^{j+1} . \qquad (2\text{-}10)$$

The highest degree of ξ in the first, second, and third sums respectively is ξ^{N-2} , ξ^{N-1} , ξ^{N-2} corresponding to ξ^{N-1} , ξ^N , ξ^{N-1} order errors. Thus we cannot say that eqn. (2-10) is characterized by a uniform error.

To insure that each sum in (2-10) propagates the same order error we change the upper limits in the first and second sums, getting

$$U_c^{(N)} \equiv \left(\frac{\partial\psi_c}{\partial Y}\right)^{(N)} = \sum_o^{N-2}\frac{1}{3}\frac{d\overline{f}_j}{d\eta}\xi^{j+1} + \sum_o^{N-1}\frac{\psi_j'(Y)}{j!}\xi^j - \sum_o^{N-2}\frac{1}{3}\frac{d\overline{f}_j(\infty)}{d\eta}\xi^{j+1} + 0(\xi^N) \qquad (2\text{-}11)$$

with $\overline{f}_1 , \overline{f}_2 , \overline{f}_4 , \overline{f}_5 , \overline{f}_7 , \overline{f}_8 \ldots \equiv 0$.

Equation (2-11) gives the uniformly valid velocity distribution to ξ^N - order. For example for $0(\xi^2)$, we have

$$U_c^{(2)} = \frac{1}{3}\frac{d\overline{f}_o}{d\eta}\xi + \psi_o(Y) + \psi_1\ (Y)\xi - \left\{\frac{2A_o}{3}\eta\xi + \frac{B_o}{3}\xi\right\} + 0(\xi^2)$$

We now wish to investigate the behavior of the uniform velocity distribution (2-11) at a downstream wake position of $x_1/L = 0.5$ for several orders of errors. Note that Goldstein's inner solution required the numerical evaluation of derivatives $\overline{f}_o{}'(\eta)$, $\overline{f}_6{}'(\eta)$, $\overline{f}_6{}'(\eta)$ for only small - η values whereas in (2-11) we need to know the value of these derivatives for all values of η in the flow domain. For example at $x_1/L = .5$ with

$$\xi = (.5/4)^{\frac{1}{3}} = .5$$

and

$$\eta = Y/1.5$$

we are interested in the range $o \leq Y \leq 3.3$ which corresponds to $o \leq \eta \leq 2.2$. Goldstein's paper tabulates $\overline{f}_j{}'(\eta)$ for $o \leq \eta \leq 1.4$. Therefore, we numerically integrated the set of ordinary differential equations (Appendix A-5.6) to extend the domain to $\eta = 2.2$ before evaluating equation (2-11). The results are discussed in the next section

3. Results

Figures 3 through 6 show graphical representations of the calculated velocity distribution at $x_1/L = 0.5$ uniformly valid to ξ^2, ξ^5, and ξ^8 - orders , compares these solutions with their inner terms only (first sum in eqn. 2-11) and their outer terms only (second sum in eqn. 2-11), and with the Goldstein inner and outer solutions. Tables I & II give the numerical results used to plot the curves in Figures 3 through 6.

Examination of these figures shows that

1. The Goldstein (1930) inner and outer solutions do not merge in the middle region (Fig. 2).
2. The improvement of the uniformly valid solutions is evident by its continuity throughout the whole flow domain and its merging to the Goldstein solutions at both extremeties (Fig. 6)
3. The behavior of the inner and outer terms of the uniformly valid solutions is graphically depicted in N = 2, 5, 8 (Fig. 5)

In summary, the method of matched asymptotic expansions was used to construct a uniformly valid expansion to the classical near-wake problem. This expansion is compared to Goldstein's inner and outer solutions at a downstream distance of half the plate length. The great improvement is evident by the fact that whereas the old solutions are discontinuous in the central region, the new solution is continuous throught the whole flow domain, merging to the old solutions at both extremities.

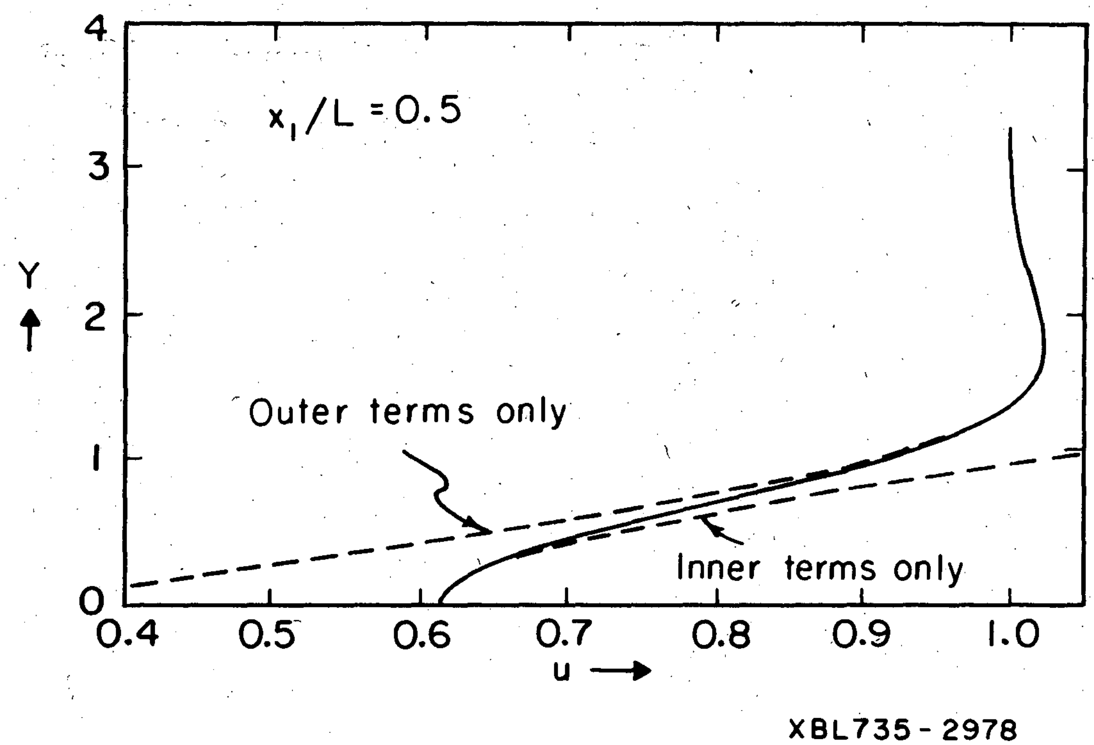

FIG. 3. COMPOSITE SOLUTION UNIFORMLY VALID TO ξ^2-ORDER.

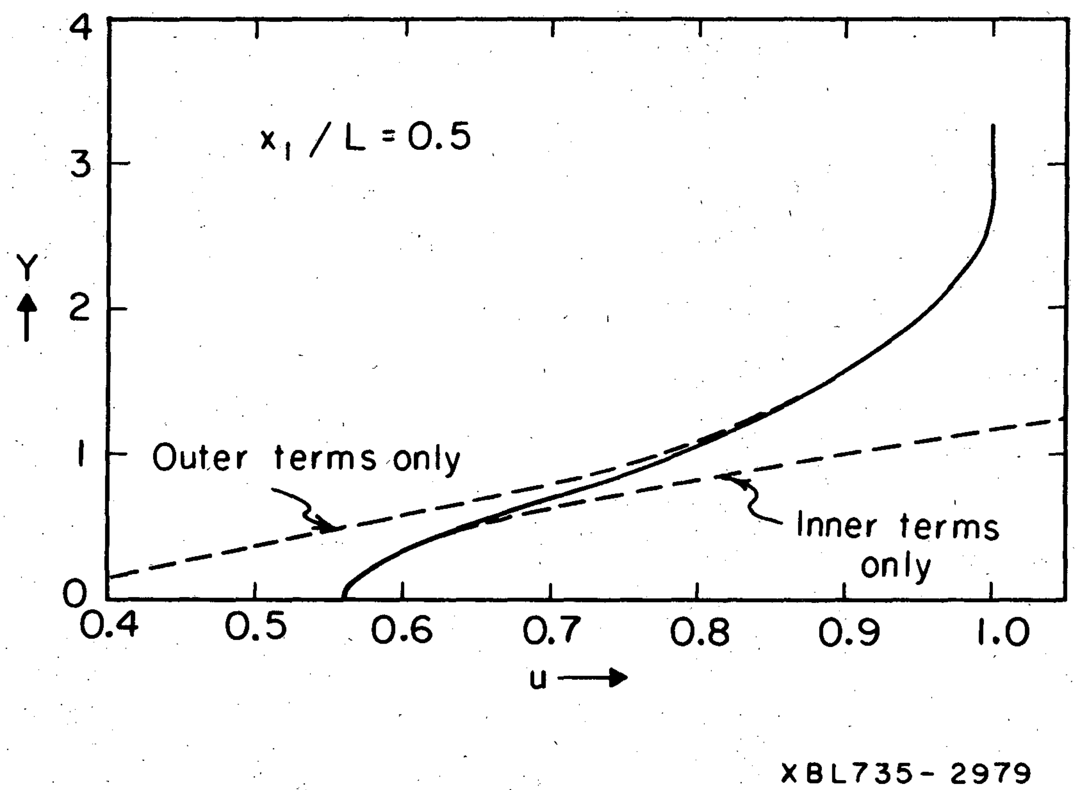

FIG. 4. COMPOSITE SOLUTION UNIFORMLY VALID TO ξ^8-ORDER.

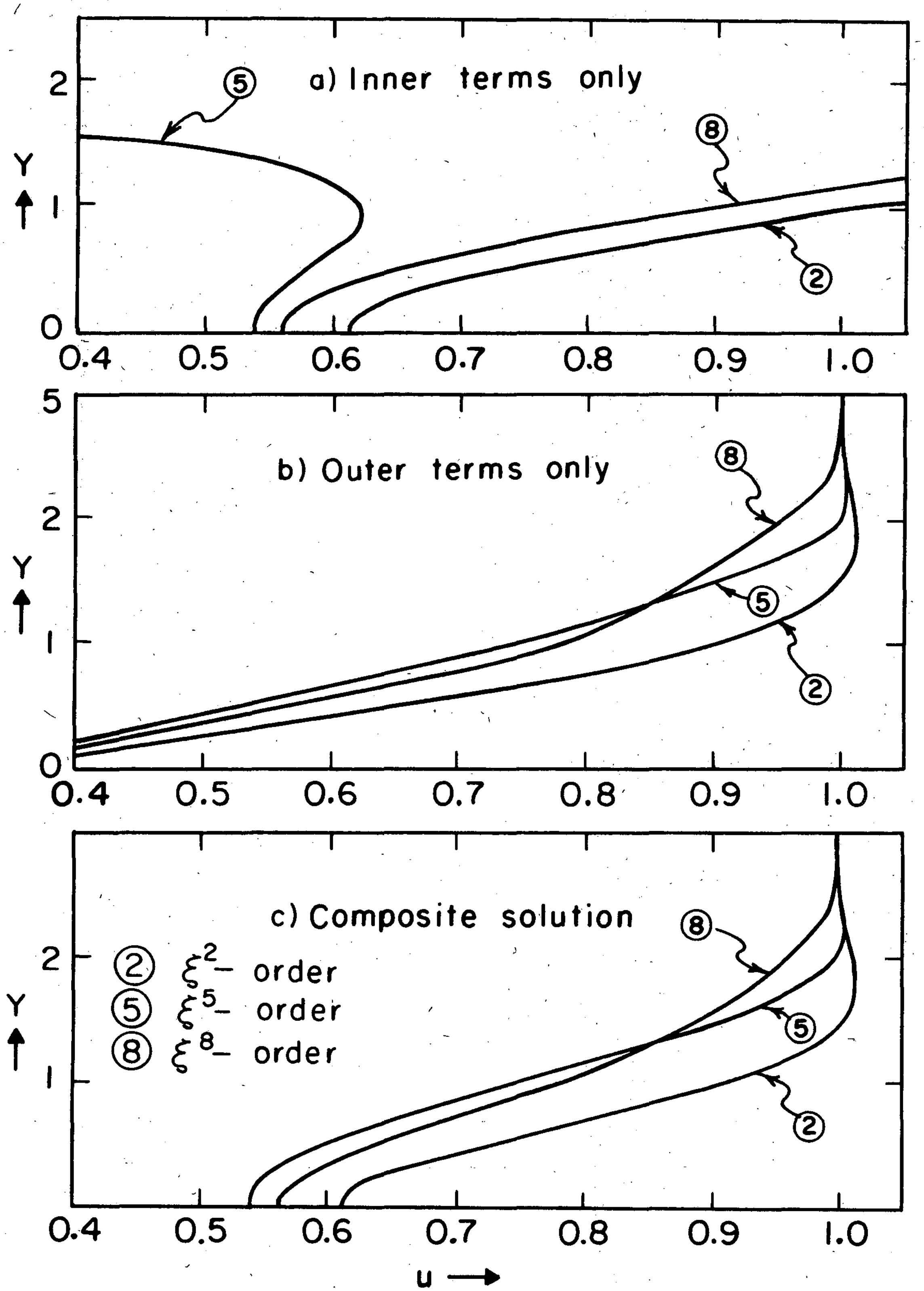

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FIG. 5. COMPARISON OF COMPOSITE SOLUTIONS UNIFORMLY VALID TO SECOND, FIFTH, AND EIGHTH ORDER
$x_1/L = 0.5$

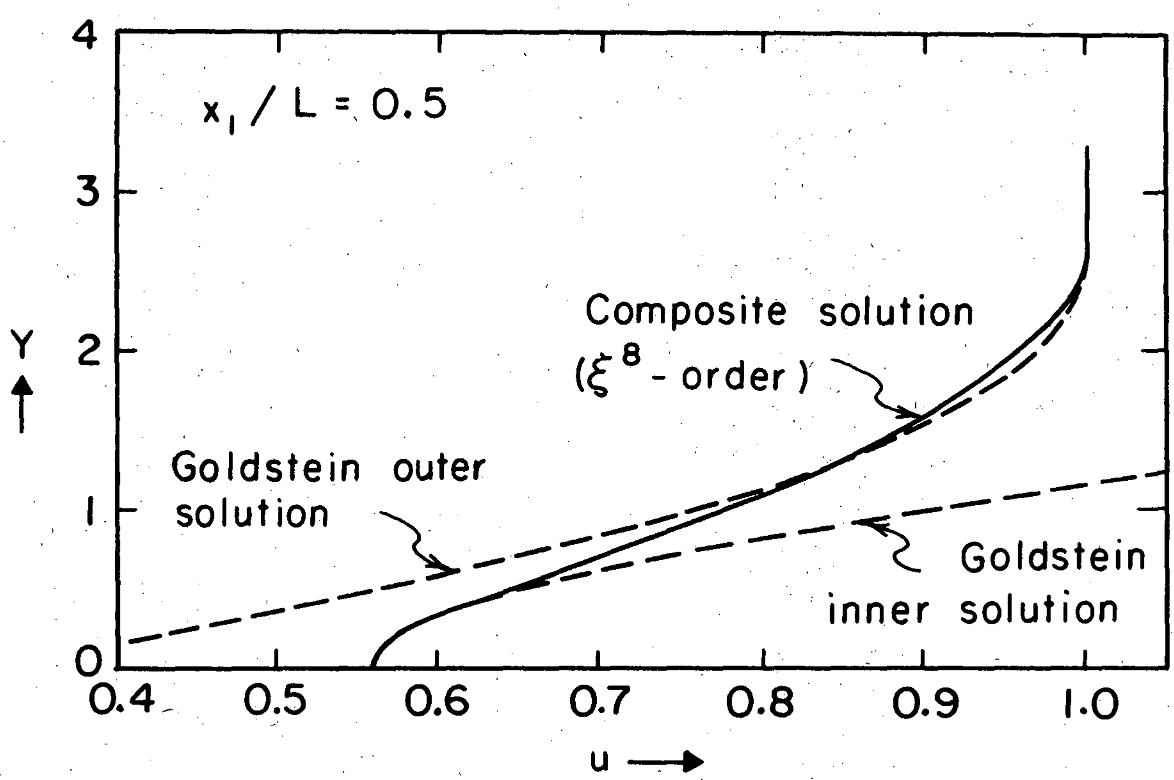

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FIG. 6. COMPARISON OF COMPOSITE SOLUTION WITH GOLDSTEIN INNER AND OUTER SOLUTIONS.

TABLE I. CALCULATED COEFFICIENTS IN THE FIRST TWO SUMS OF (2-11) FOR U_c

$X_1/L = .5$		$\overline{f}_0'$	$\overline{f}_3'$	$\overline{f}_6'$	ψ_0'	ψ_1'	$\frac{\psi_2'}{2!}$	$\frac{\psi_3'}{3!}$	$\frac{\psi_4'}{4!}$	$\frac{\psi_5'}{5!}$	$\frac{\psi_6'}{6!}$	$\frac{\psi_7'}{7!}$	$\frac{\psi_8'}{8!}$
Y	η												
0.	0.	3.6787	-3.5418	8.1174	0.	.6790	0.	0.	-.6366	0.	0.	1.4858	0.
.30	.20	3.9428	-4.8454	14.0990	.1989	.6750	-.0206	-.4425	-.6063	.1231	1.3973	1.3107	-.60
.60	.40	4.6649	-8.7917	34.7420	.3938	.6474	-.0787	-.8423	-.4045	.4469	2.4931	.1858	-2.06
.90	.60	5.6800	-15.6700	80.3770	.5748	.5785	-.1566	-1.1062	.0573	.7647	2.7024	-2.1020	-2.80
1.20	.80	6.8276	-26.3880	175.4600	.7290	.4664	-.2199	-1.1442	.6712	.7503	1.7352	-4.2663	-.98
1.50	1.00	8.0149	-42.5930	368.4500	.8461	.3300	-.2356	-.9533	1.1475	.2741	.1812	-4.3870	2.62
1.80	1.20	9.2095	-66.4720	745.5400	.9233	.2006	-.1978	-.6407	1.2448	-.3695	-.9011	-2.2293	4.58
2.10	1.40	10.4050	-100.5400	1450.0600	.9670	.1033	-.1319	-.3469	.9871	-.7419	-1.0515	.3287	3.20
2.40	1.60	11.6000	-147.6400	2715.3000	.9878	.0447	-.0705	-.1512	.6008	-.7076	-.6544	1.5698	.46
2.70	1.80	12.7960	-210.9000	4892.7000	.9962	.0162	-.0305	-.0530	.2878	-.4585	-.2522	1.4493	-1.15
3.00	2.00	13.9910	-293.8500	8506.3000	.9990	.0049	-.0107	-.0149	.1103	-.2212	-.0506	.8349	-1.25
3.30	2.20	15.1870	-400.3100	14305.0000	.9998	.0012	-.0031	-.0033	.0341	-.0827	.0052	.3477	-.74

TABLE II. CALCULATED VALUES OF $U_c^{(N)}$ FROM (2-11) WITH N = 2, 5, 8
COMPARED TO GOLDSTEIN INNER AND OUTER SOLUTIONS
AT $X_1/L = .5$

Y	Goldstein Solutions calculated herein over extended Y-domain		Uniformly Valid Solutions								
			N = 2			N = 5			N = 8		
	Inner Solution to $\theta(\xi^8)$	Outer Solution to $\theta(\xi^9)$	$U_c^{(2)}$	Inner Terms only	Outer Terms only	$U_c^{(5)}$	Inner Terms only	Outer Terms only	$U_c^{(8)}$	Inner Terms only	Outer Terms only
0.	.560	.311	.613	.613	.339	.539	.539	.300	.561	.560	.311
.300	.593	.472	.655	.657	.536	.559	.556	.438	.591	.593	.474
.600	.685	.614	.757	.777	.717	.617	.594	.567	.667	.685	.622
.900	.830	.729	.874	.947	.864	.706	.620	.690	.754	.830	.740
1.200	1.045	.820	.964	1.138	.962	.810	.588	.806	.827	1.045	.823
1.500	1.408	.892	1.011	1.336	1.011	.905	.448	.905	.882	1.408	.882
1.800	2.092	.947	1.024	1.535	1.024	.972	.150	.972	.929	2.092	.929
2.100	3.416	.979	1.019	1.734	1.019	1.004	-.360	1.004	.964	3.416	.967
2.400	5.929	.993	1.010	1.933	1.010	1.011	-1.142	1.011	.991	5.929	.991
2.700	10.480	.997	1.004	2.133	1.004	1.008	-2.261	1.008	1.001	10.480	1.001
3.000	18.362	.998	1.001	2.332	1.001	1.004	-3.790	1.004	1.002	18.362	1.003
3.300	31.444	.999	1.000	2.531	1.000	1.001	-5.809	1.001	1.000	31.444	1.002

Acknowledgement

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Nomenclature

L	plate length
p	pressure
R_L	$U_\infty L/\nu$, the Reynolds number
u,U	velocity in x direction, parallel to plate
v,V	velocity in y direction, perpendicular to plate
$\bar{x}$	distance parallel to plate measured downstream from leading edge, and divided by the plate length
x	$\bar{x} - 1$
y	distance measured perpendicularly from the plane of the plate, and divided by the plate length
Y	stretched distance from plane of plate
ζ	stream function for Blasius solution
ψ	stream function, $u_1 = \partial\psi_1/\partial y_1$
ν	kinematic viscosity
ρ	mass density

Subscripts--

c	composite solution
i	inner solution
e	external or outer solution
Bℓ	Blasius solution
1	dimensional variable (except in the expansion of ψ_e; equations 2-3 and A-5.8)
x,y,Y	partial differentiation with respect to the variable indicated
∞	freestream value

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APPENDIX

SUMMARY OF FIRST-ORDER THEORY FOR STEADY 2-DIMENSIONAL INCOMPRESSIBLE FLOW REGIONS SURROUNDING A FLAT PLATE

The derivation of the first-order approximate solutions to the exact Navier-Stokes flow in various regions is outlined below. The symbols, flow regions, and coordinate system are defined in the Nomenclature and in Fig. 1. Abbreviations used herein are as follows: fct ≡ function; ode-prob ≡ ordinary differential equation problem (governing equations with boundary conditions); pde-prob ≡ partial differential equation problem.

Description	(Eqn. No.) Eqn. Name

1. Navier-Stokes flow eqns. (1845) (Exact)

$$\left\{\begin{matrix} x = x_1/L \ , \ y = y_1/L \ , \ u = u_1/U_\infty \ , \ v = v_1/U_\infty \\ R_L = U_\infty L/\nu \ , \ p = p_1/\rho U_\infty^2 \ ; \ \psi = \psi_1/U_\infty L \end{matrix}\right\} \qquad \text{(A-1.1) Transform}$$

$$\left\{\begin{matrix} \left(u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y}\right)(u,v) = -\left(\dfrac{\partial p}{\partial x}, \dfrac{\partial p}{\partial y}\right) + \dfrac{1}{R_L}\nabla^2(u,v) & a \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 & b \end{matrix}\right\} \qquad \text{(A-1.2) Governing}$$

Integrate (1.2b) by defining ψ such that

$$\left\{u = \frac{\partial \psi}{\partial y} \ , \ v = -\frac{\partial \psi}{\partial x}\right\} \qquad \text{(A-1.3)}$$

Substitute (1.3) in (1.2a) and eliminate p to get

$$\left(\psi_y \frac{\partial}{\partial y} - \psi_x \frac{\partial}{\partial x}\right)\nabla^2\psi = \frac{1}{R_L}\nabla^2(\nabla^2\psi) \qquad \text{(A-1.4)}$$

---

2. Inviscid Euler flow eqns. (1755)

Approximation: $R_L \to \infty$ in (1.2) and (1.4) gives

$$\left\{\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(u,v) = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) \ , \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\right\} \qquad \text{(A-2.1) Governing}$$

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)\nabla^2\psi = 0 \qquad \text{(A-2.2)}$$

---

3. Prandtl boundary-layer flow eqns. (1904)

Approximations: $\left\{\begin{array}{l} R_L \to \infty \ , \ y_1 = \mathcal{O}(R_L^{-1/2} L) \ll x_1 \\ \text{within boundary-layer} \end{array}\right\}$ (A-3.1)

$$\left\{\bar{Y} = R_L^{1/2} y_1/L \ , \ \bar{V} = R_L^{1/2} v_1/L \ , \ \Psi = R_L^{1/2} \ \psi_1/(U_\infty L)\right\}$$ (A-3.2) Transform

Substitute (3.1), (3.2) in (1.2) and neglect higher order terms to get boundary-layer eqns:

$$\left\{\begin{array}{l} u_1 \dfrac{\partial u_1}{\partial x_1} + v_1 \dfrac{\partial u_1}{\partial y_1} = -\dfrac{1}{\rho}\dfrac{\partial p_1}{\partial x_1} + \nu \dfrac{\partial^2 u_1}{\partial y_1^2} \\ \dfrac{\partial p_1}{\partial y_1} = 0 \ , \ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial v_1}{\partial y_1} = 0 \end{array}\right\}$$ (A-3.3) Dimensional

$$\left\{u \frac{\partial u}{\partial x} + \bar{V}\frac{\partial u}{\partial \bar{Y}} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial \bar{Y}^2} \ , \ \frac{\partial p}{\partial \bar{Y}} = 0 \ , \ \frac{\partial u}{\partial x} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0\right\}$$ (A-3.4) Non-dim.

$$\Psi_{\bar{Y}\bar{Y}\bar{Y}} + \Psi_x \Psi_{\bar{Y}\bar{Y}} - \Psi_{\bar{Y}} \Psi_{x\bar{Y}} = -\frac{dp}{dx}$$ (A-3.5) Stream-fct

4. Blasius flow eqns. for a flat plate (1908)

Approximations: $\dfrac{dp}{dx} = 0$

$$\left\{\bar{x}_1 = x_1 + L \ , \ \bar{\eta} = \frac{1}{2} y_1\sqrt{U_\infty/\bar{x}_1\nu} \ , \ \zeta_{B\ell}(\bar{\eta}) = \psi_1/\sqrt{\bar{x}_1 U_\infty \nu}\right\}$$ (A-4.1) Transform

Substitute (4.1) in (3.3) to get

$$\left\{\zeta_{B\ell}''' + \zeta_{B\ell}\zeta_{B\ell}'' = 0 \ , \ \zeta_{B\ell}(0) = \zeta_{B\ell}'(0) = 0 \ , \ \zeta_{B\ell}'(\infty) = 2\right\}$$ (A-4.2) ode-prob

with two types of solutions as follows:

(1) numerical solution holding for all $\bar{\eta}$ giving

$\zeta_{B\ell}''(0) = 1.32824$

(2) series solution holding for small $\bar{\eta}$ giving

a_1 , a_4 , a_7 in

$$u_{B\ell}(\bar{\eta}) \equiv \frac{1}{2}\zeta_{B\ell}' \approx a_1\bar{\eta} + a_4\bar{\eta}^4 + a_7\bar{\eta}^7$$ (A-4.3) small-$\bar{\eta}$

5. Goldstein near-wake flow eqns. (1930)

$$\left\{\begin{array}{l} x = x_1/4L \ , \ Y = \frac{1}{2} R_L^{1/2} y_1/L \ , \ u = u_1/U_\infty \ , \ V = 2R_L^{1/2} v_1/U_\infty \\ \frac{dp_1}{dx_1} = 0 \ , \ \psi_G = \frac{1}{2} R_L^{1/2} \psi_1/U_\infty L \end{array}\right\} \quad \begin{array}{c}\text{(A-5.1)}\\ \text{Transform}\end{array}$$

Substitute (5.1) in (3.2) to get

$$\left\{\begin{array}{ll} u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2} & \text{a} \\ \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0 & \text{b} \\ x > 0 \ , \ Y = 0 : V = 0 \ , \ \frac{\partial u}{\partial Y} = 0 & \text{c} \\ x = 0 : u \equiv u_o \equiv \begin{cases} u_{B\ell}^N \ , \ \text{all } Y & \text{d} \\ u_{B\ell}^S \approx a_1 Y + a_4 Y^4 + a_7 Y^7 \ , \ Y << 1 & \text{e} \end{cases} \end{array}\right\} \quad \begin{array}{c}\text{(A-5.2)}\\ \text{pde-prob}\end{array}$$

where $u_{B\ell}$ ≡ Blasius solution at trailing-edge of flat plate with superscripts N , S denoting numerical and series solutions and

$$a_1 = \frac{1}{2}\alpha \ , \ a_4 = -\frac{1}{2}\alpha^2/4! \ , \ a_7 = 5.5\ \alpha^3/7! \ , \ \alpha = 1.32824$$

Integrate (5.2b) by defining ψ_G such that

$$\left\{ u = \frac{\partial \psi_G}{\partial Y} \ , \ V = -\frac{\partial \psi_G}{\partial x} \right\} \qquad \text{(A-5.3)}$$

a) Inner-solution (small Y , $0 < \xi \leq .5$)

Let

$$\left\{\begin{array}{ll} \xi = x^{1/3} \ , \ \eta = Y/3\xi & \text{a} \\ \psi_G \approx \psi_i(\xi,\eta) \equiv \xi^2\left[\bar{f}_o(\eta) + \bar{f}_3(\eta)\xi^3 + \bar{f}_6(\eta)\xi^6\right] & \text{b} \end{array}\right\} \quad \begin{array}{c}\text{(A-5.4)}\\ \text{Transform}\\ \text{Stream fct}\\ \text{expansion}\end{array}$$

Substitute (5.3), (5.4) in (5.2) and equate equal powers of ξ to zero getting

$$\left\{\begin{array}{l} \overline{f}_o''' + 2\overline{f}_o\overline{f}_o'' - \overline{f}_o'^2 = 0 \\ \overline{f}_3''' + 2\overline{f}_o\overline{f}_3'' - 5\overline{f}_o'\overline{f}_3' + 5\overline{f}_o''\overline{f}_3 = 0 \\ \overline{f}_6''' + 2\overline{f}_o\overline{f}_6'' - 8\overline{f}_o'\overline{f}_6' + 8\overline{f}_o''\overline{f}_6 = 4\overline{f}_3'^2 - 5\overline{f}_3\overline{f}_3'' \\ \text{BC:}\quad \overline{f}_o'/\eta \sim 9a_1 \,,\ \overline{f}_3'/\eta^4 \sim 3^5 a_4 \,,\ \overline{f}_6'/\eta^7 \sim 3^8 a_7 \\ \qquad\qquad\qquad \text{as}\quad \xi \to 0 \,,\ \eta \to \infty \\ \qquad \overline{f}_r(0) = \overline{f}_r''(0) = 0 \qquad r = 1,3,6 \end{array}\right\} \quad \begin{array}{c}\text{(A-5.5)}\\ \text{ode-prob}\end{array}$$

which can be solved numerically.

b) <u>Large-η solution</u>

For large-η , the numerical solution of (5.5) is of the form:

$$\left\{\begin{array}{l} \overline{f}_o(\eta) \sim A_o\eta^2 + B_o\eta + C_o \\ \overline{f}_3(\eta) \sim A_3\eta^5 + B_3\eta^4 + \ldots + F_3 \\ \overline{f}_6(\eta) \sim A_6\eta^8 + B_6\eta^7 + \ldots + I_6 \end{array}\right\} \tag{A-5.6}$$

where the coefficients A_j , B_j , C_j , ... are known.

Substitute (5.6), (5.4a) in (5.4b) to get

$$\begin{aligned} \psi_i \sim\ & \xi^o\left[A_o\left(\tfrac{1}{3}\,Y\right)^2 + A_3\left(\tfrac{1}{3}\,Y\right)^5 + A_6\left(\tfrac{1}{3}\,Y\right)^8 + \ldots\right] \\ & + \xi^1\left[B_o\left(\tfrac{1}{3}\,Y\right) + B_3\left(\tfrac{1}{3}\,Y\right)^4 + B_6\left(\tfrac{1}{3}\,Y\right)^7 + \ldots\right] \\ & + \xi^2\left[C_o + C_3\left(\tfrac{1}{3}\,Y\right)^3 + \ldots\right] + \ldots \end{aligned} \tag{A-5.7}$$

which suggests the large-η expansion

$$\psi_G \sim \psi_e(\xi,Y) \equiv \psi_o(Y) + \psi_1(Y)\xi + \frac{\psi_2(Y)}{2!}\,\xi^2 + \ldots \tag{A-5.8}$$

where the $\psi_j(Y)$ are given by (5.7). Substitute (5.8), (5.3) in (5.2a) to get

$$\left\{\begin{array}{l} \psi_o'\psi_1' - \psi_o''\psi_1 = 0 \\ \psi_o'\psi_2' - \psi_o''\psi_2 = \psi_1\psi_1'' - \psi_1'^2 \\ \vdots \end{array}\right\} \qquad \text{(A-5.9) ode-prob}$$

At the plate trailing-edge $(\xi = 0)$, $\psi_o'(Y) = \frac{1}{2}\zeta'_{B\ell}$ so (5.9) can be integrated sequentially in closed form. Goldstein gives the solution for ψ_o through ψ_8.

Notes:

1. The approximations in the Prandtl boundary-layer eqns. (3.1) apply to the Blasius and Goldstein flows.
2. The dimensional ψ_1 is nondimensionalized by various factors (see eqns. (1.1), (3.2), (4.1), (5.1)) in the different flows.
3. The dimensional coordinate x_1 is nondimensionalized by 4L in the Goldstein flow and by L in the other flows. For simplicity the same symbol x is used for the nondimensional coordinate but the context (eqns. (1.1), (5.1)) clarifies which factor is used.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720