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Entanglement in Finitely Correlated Spin States

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We consider entanglement properties of pure finitely correlated states (FCS). We derive bounds for the entanglement of a spin with an interval of spins in an arbitrary pure FCS. Finitely correlated states are also known as matrix product states or generalized valence-bond states. The bounds become exact in the case where one considers the entanglement of a single spin with a half-infinite chain to the right of it. Our bounds provide a proof of the recent conjecture by Benatti, Hiesmayr, and Narnhofer that their necessary condition for nonvanishing entanglement in terms of a single spin and the memory of the FCS is also sufficient. We also generalize the study of entanglement in the Affleck-Kennedy-Lieb-Tasaki model by Fan, Korepin, and Roychowdhury. Our result permits a more efficient calculation, numerically and in some cases analytically, of the entanglement of arbitrary finitely correlated quantum spin chains.

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Entanglement properties of quantum spin chains have recently attracted attention from researchers in quantum information theory and condensed matter physics. From the perspective of quantum information theory, the distribution of entanglement over long ranges via local operations on a spin chain [1–5] has obvious applications to teleportation-based models of quantum computation [6–8]. Moreover, it has recently been shown that entanglement in finitely correlated chains [9] can be used to achieve universal quantum computation [10] and provide a computational tool for adiabatic quantum computation [11]. On the other hand, the scaling behavior of quantum correlations in infinite spin chains is intimately related to their critical behavior (recent work has established a general mathematical framework for studying entanglement in infinite quantum spin chains [12]).

Finitely correlated states (FCS's) are a generalization of the so-called valence-bond solid states, which arise as the exact ground states of a considerable variety of quantum spin Hamiltonians [13,14]. Interestingly, FCS's also provide approximate ground states of *any* quasi-one-dimensional spin system with finite-range interactions [15]. In particular, density matrix renormalization group calculations produce numerical approximations of the ground state of spin chains that can be regarded as FCS's [16].

Motivated by the potential applications of distributed entanglement in finitely correlated chains, Benatti *et al.* in Ref. [1] give a *necessary* condition for entanglement between a spin and a subset of other spins, namely, that the entanglement between a spin and the “memory” of the finitely correlated state must be nonzero. They, furthermore, conjecture that the same condition is *sufficient*, in the sense that it implies entanglement between a spin and a subset of other spins. We present here a proof of that conjecture by showing that the entanglement between a spin and its neighbors converges exponentially fast (in the number of neighboring spins) to the entanglement between

a spin and the memory of the finitely correlated state. Moreover, we show that entanglement between distant spins vanishes exponentially fast in the length of their separation.

Since finitely correlated states provide the exact ground states for generalized valence-bond solid models [13], our result generalizes the calculation of entanglement [2] for the Affleck-Kennedy-Lieb-Tasaki model [14].

More importantly, our result implies a simple and computationally efficient way for detecting distributed entanglement in finitely correlated states. Namely, the positive partial transpose (PPT or Peres-Horodecki) criterion [17,18] can be applied to the state describing the interactions of a spin with the memory of the finitely correlated state, to detect entanglement between a spin and a subset of other spins.

The setup and main result.—We will work with translation invariant pure FCS's [9] on the infinite one-dimensional lattice. For each $i \in \mathbb{Z}$, the spin at site i of the chain will be described by the algebra \mathcal{A} of $d \times d$ complex matrices. The observables of the spins in an interval $[m, n]$ are given by the tensor product $\mathcal{A}_{[m,n]} = \otimes_{j=m}^n (\mathcal{A})_j$. The algebra $\mathcal{A}_{\mathbb{Z}}$ describing the infinite chain arises as a suitable limit of the *local* tensor-product algebras $\mathcal{A}_{[-n,n]} := \otimes_{j=-n}^n (\mathcal{A})_j$. Any translation invariant state ω over $\mathcal{A}_{\mathbb{Z}}$ is completely determined by a set of density matrices $\rho_{[1,n]}$, $n \geq 1$, which describe the state of n consecutive spins. In the case of a pure FCS, as was shown in Ref. [19], these density matrices can be constructed as follows:

The *memory*, \mathcal{B} , of a FCS is represented by the algebra of $b \times b$ complex matrices. Let $\mathbb{E}: \mathcal{A} \otimes \mathcal{B} \mapsto \mathcal{B}$ be a completely positive unital map of the form $\mathbb{E}(A \otimes B) = V(A \otimes B)V^\dagger$, where $V: \mathbb{C}^d \otimes \mathbb{C}^b \mapsto \mathbb{C}^b$, is a linear map such that $VV^\dagger = \mathbb{1}_{\mathcal{B}}$. We define the completely positive map $\hat{\mathbb{E}}: \mathcal{B} \mapsto \mathcal{B}$, by $\hat{\mathbb{E}}(B) = \mathbb{E}(\mathbb{1}_{\mathcal{A}} \otimes B)$. The condition on V implies that $\hat{\mathbb{E}}$ is unital: $\hat{\mathbb{E}}(\mathbb{1}_{\mathcal{B}}) = \mathbb{1}_{\mathcal{B}}$. In Ref. [19] it is

proved that for any pure translation invariant FCS, it is always possible to choose \mathcal{B} and V such that there is a unique, nonsingular $b \times b$ density matrix, ρ , with the property $\text{Tr} \rho \hat{\mathbb{E}}(B) = \text{Tr} \rho B$, for all $B \in \mathcal{B}$.

We introduce the density matrix $\rho_{\mathcal{A} \otimes \mathcal{B}}$ associated with the state encoding the interaction between the spin at site 1 and the memory of the FCS:

$$\text{Tr}_{\mathcal{A} \otimes \mathcal{B}}(\rho_{\mathcal{A} \otimes \mathcal{B}} A \otimes B) = \text{Tr}_{\mathcal{B}}(\rho \mathbb{E}(A \otimes B)).$$

Using the cyclicity of trace we also have $\rho_{\mathcal{A} \otimes \mathcal{B}} = V^\dagger \rho V$.

We are now ready to define the density matrices $\rho_{[1,n]}$:

$$\rho_{[1,n]} = \text{Tr}_{\mathcal{B}}(V_n^\dagger \rho_{\mathcal{A} \otimes \mathcal{B}} V_n),$$

where $V_n = (\mathbb{1}_{\mathcal{A}} \otimes V)(\mathbb{1}_{\mathcal{A} \otimes 2} \otimes V) \cdots (\mathbb{1}_{\mathcal{A} \otimes n-1} \otimes V)$.

An important property, intimately related to the exponential decay of correlations in a *pure* FCS is that the peripheral spectrum of $\hat{\mathbb{E}}$ is trivial; that is, $\mathbb{1}_{\mathcal{B}}$ is the only eigenvector of $\hat{\mathbb{E}}$ with eigenvalue of modulus 1 [19]. This implies that the iterates of $\hat{\mathbb{E}}$ converge exponentially fast to $\hat{\mathbb{E}}^\infty$ given by $\hat{\mathbb{E}}^\infty(B) = \lim_{n \rightarrow \infty} \hat{\mathbb{E}}^n(B) = \text{Tr}(\rho B) \mathbb{1}_{\mathcal{B}}$. More precisely, for any λ such that $|\lambda_i| < \lambda < 1$, for all eigenvalues λ_i of $\hat{\mathbb{E}}$ different from 1, there exists a constant c such that for all $n \geq 1$:

$$\|\hat{\mathbb{E}}^n - \hat{\mathbb{E}}^\infty\| \leq c \lambda^n, \quad (1)$$

where the norm is the ∞ -norm on \mathcal{B} considered as a Banach space with the 1-norm.

Our object of study is the entanglement of formation (EoF) [20]. The EoF is defined for states of composite systems with a tensor-product algebra of observables $\mathcal{X}_1 \otimes \mathcal{X}_2$.

Definition 1 (Entanglement of Formation).—The entanglement of formation of a bipartite state over $\mathcal{X}_1 \otimes \mathcal{X}_2$ with associated density matrix σ_{12} is given by:

$$E_{[\mathcal{X}_1, \mathcal{X}_2]}(\sigma_{12}) = \inf_i \sum p_i S(\text{Tr}_{\mathcal{X}_2}(\sigma_{12}^i)),$$

where $S(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy and the infimum of the average entropy is taken over all convex decompositions $\sigma_{12} = \sum_i p_i \sigma_{12}^i$ into pure states.

Whenever \mathcal{X}_1 is finite dimensional, as will be the case for us, the infimum can be replaced by a minimum in the above definition, i.e., there is an optimal decomposition, $\{p_i, \sigma_{12}^i\}$, where the infimum is attained (see Ref. [21] for details). We call $\{|\phi_i\rangle\}$ an *ensemble* for the density matrix σ whenever the latter can be decomposed as $\sigma = \sum_i |\phi_i\rangle\langle\phi_i|$. There are an infinite number of ensembles corresponding to a given density matrix. The following lemma provides us with a complete classification:

Lemma 2 (Isometric Freedom in Ensembles, [22,23]).—Let $\{|e_i\rangle\}_{i=1}^d$ be the ensemble corresponding to the eigen-decomposition of the density matrix σ , where $d = \text{rank}(\sigma)$. Then, $\{|\psi_i\rangle\}_{i=1}^m$ is an ensemble for σ if and only if there exists an isometry $U: \mathbb{C}^d \mapsto \mathbb{C}^m$ such that

$$|\psi_i\rangle = \sum_{j=1}^d U_{i,j} |e_j\rangle, \quad 1 \leq i \leq m.$$

The above lemma implies that any two ensembles for the same density matrix, $\{|\psi_j\rangle\}_{j=1}^{M_1}$, and $\{|\phi_i\rangle\}_{i=1}^{M_2}$, are similarly related via a partial isometry $W: \mathbb{C}^{M_1} \mapsto \mathbb{C}^{M_2}$.

Our main result is the following theorem:

Theorem 3.—For any pure translation invariant FCS we have

$$0 \leq E_{[\mathcal{A}, \mathcal{B}]}(\rho_{\mathcal{A} \otimes \mathcal{B}}) - E_{[\mathcal{A}, \mathcal{A}^{\otimes n-1}]}(\rho_{[1,n]}) \leq \epsilon(n), \quad (2)$$

where $\epsilon(n)$ decays exponentially fast in n .

Proof of the theorem.—The lower bound is proven in Ref. [1]. For the sake of completeness, we include here the following proof.

The definition of $\rho_{[1,n]}$ implies that every decomposition of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ into pure states induces a decomposition of $\rho_{[1,n]}$. Moreover, the restrictions to the spin at site 1 of the i th state in the corresponding decompositions of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ and $\rho_{[1,n]}$ are equal. To see this, note that since the operators V_n leave the first spin invariant, the cyclicity of the trace implies

$$\begin{aligned} \text{Tr}_{\mathcal{A}^{\otimes n-1}}(\rho_{[1,n]}^i) &= \text{Tr}_{\mathcal{A}^{\otimes n-1} \otimes \mathcal{B}}(V_n^\dagger \rho_{\mathcal{A} \otimes \mathcal{B}}^i V_n) \\ &= \text{Tr}_{\mathcal{B}}(\rho_{\mathcal{A} \otimes \mathcal{B}}^i), \end{aligned}$$

where we have used $V_n V_n^\dagger = \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}$. It follows that for each decomposition of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ there is a corresponding decomposition of $\rho_{[1,n]}$ with equal average entropy. Since the average entropy of $\rho_{[1,n]}$ is minimized over a (possibly) larger set of decompositions, the lower bound follows.

We now focus on the upper bound. We start with the following decompositions of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ and $\rho_{[1,n]}$ into (unnormalized) pure states:

$$\rho_{\mathcal{A} \otimes \mathcal{B}} = \sum_{i=1}^b V^\dagger |\chi_i\rangle\langle\chi_i| V \quad (3)$$

$$\rho_{[1,n]} = \sum_{i,j=1}^b G_{n,j}^\dagger V^\dagger |\chi_i\rangle\langle\chi_i| V G_{n,j}, \quad (4)$$

where $\{|\chi_i\rangle\}_{i=1}^b$ is the eigen-ensemble of ρ and $G_{n,j} = V_n(\mathbb{1}_{\mathcal{A}^{\otimes n}} \otimes |\chi_j\rangle\langle\chi_j| / \|\chi_j\|)$. The term in parentheses in the expression for $G_{n,j}$ comes from the Kraus operators in the decomposition of the completely positive map $\text{Tr}_{\mathcal{B}}$.

By the observation following Lemma 2, we have that the (unnormalized) states $|\Phi_l^n\rangle$ in the *optimal decomposition* of $\rho_{[1,n]}$ are given by

$$|\Phi_l^n\rangle = \sum_{i,j=1}^b U_{l,(ij)} G_{n,j}^\dagger V^\dagger |\chi_i\rangle, \quad 1 \leq l \leq L, \quad (5)$$

for some partial isometry $U: \mathbb{C}^{b^2} \mapsto \mathbb{C}^L$, whose dependence on n we suppress. Moreover, it is easy to check that

$$|\Psi_l\rangle = \sum_{i,j=1}^b U_{l,(ij)} V^\dagger |\chi_i\rangle, \quad 1 \leq l \leq L, \quad (6)$$

is an ensemble for $\rho_{\mathcal{A} \otimes \mathcal{B}}$.

To calculate the EoF we need the restrictions of $\{|\Phi_l^n\rangle\langle\Phi_l^n|\}$ and $\{|\Psi_l\rangle\langle\Psi_l|\}$ to \mathcal{A} :

$$\tilde{\phi}_l^n = \text{Tr}_{\mathcal{A}^{\otimes n-1}}(|\Phi_l^n\rangle\langle\Phi_l^n|), \quad \tilde{\psi}_l = \text{Tr}_{\mathcal{B}}(|\Psi_l\rangle\langle\Psi_l|). \quad (7)$$

Define the density matrices $\phi_l^n = \tilde{\phi}_l^n / \alpha_l^n$ and $\psi_l = \tilde{\psi}_l / \beta_l$, where $\alpha_l^n \equiv \|\tilde{\phi}_l^n\|_1 = \text{Tr}(\tilde{\phi}_l^n)$, $\beta_l \equiv \|\tilde{\psi}_l\|_1 = \text{Tr}(\tilde{\psi}_l)$.

From the definition of the EoF and the optimality of $\{\tilde{\phi}_l^n\}_{l=1}^L$ we get

$$E_{[\mathcal{A}, \mathcal{B} \otimes \mathcal{B}]}(\rho_{\mathcal{A} \otimes \mathcal{B}}) - E_{[\mathcal{A}, \mathcal{A}^{\otimes n-1}]}(\rho_{[1,n]}) \leq \sum_{l=1}^L \epsilon_l(n), \quad (8)$$

where $\epsilon_l(n) = \beta_l S(\psi_l) - \alpha_l^n S(\phi_l^n)$.

It remains to show that $\sum_{l=1}^L \epsilon_l(n)$ is exponentially small. We estimate each term in the sum as

$$|\epsilon_l(n)| \leq \beta_l |S(\psi_l) - S(\phi_l^n)| + |\beta_l - \alpha_l^n| \log d, \quad (9)$$

since $\text{rank}(\phi_l^n) \leq d$.

To bound $|S(\psi_l) - S(\phi_l^n)|$ we use Fannes's inequality for the continuity of the von Neumann entropy [24]:

$$|S(\psi_l) - S(\phi_l^n)| \leq (\log d + 2) \|\psi_l - \phi_l^n\|_1 + \eta(\|\psi_l - \phi_l^n\|_1), \quad (10)$$

where $\eta(x) = -x \log x$ and \log is the natural logarithm. By the triangle inequality we have

$$|\beta_l - \alpha_l^n| = \|\tilde{\psi}_l\|_1 - \|\tilde{\phi}_l^n\|_1 \leq \|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1. \quad (11)$$

Another application of the triangle inequality gives

$$\|\psi_l - \phi_l^n\|_1 \leq \frac{\|\beta_l \psi_l - \alpha_l^n \phi_l^n\|_1 + \|(\alpha_l^n - \beta_l) \phi_l^n\|_1}{\beta_l},$$

which simplifies, with the use of (11), to the following inequality:

$$\|\psi_l - \phi_l^n\|_1 \leq 2 \frac{\|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1}{\beta_l}. \quad (12)$$

Combining Eqs. (9)–(12) and setting

$$\tau_l^n \equiv \|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1 / \beta_l, \quad (13)$$

we get the following bound for $\epsilon_l(n)$:

$$|\epsilon_l(n)| \leq \beta_l [(\log d^3 + 4) \tau_l^n + \eta(2\tau_l^n)]. \quad (14)$$

where we have assumed that $2\tau_l^n \leq 1/e$, to assure $\eta(x)$ is increasing.

To complete the proof, we show that τ_l^n is exponentially small for large n . Since each $G_{n,j}$ leaves the spin at site 1 invariant, the cyclicity of the trace yields

$$\tilde{\phi}_l^n = \sum_{i,i',j,j'=1}^b U_{l,(i'j')}^* U_{l,(ij)} \text{Tr}_{\mathcal{B}}(V^\dagger |\chi_i\rangle\langle\chi_{i'}| V G_{n,j'} G_{n,j}^\dagger),$$

but $G_{n,j'} G_{n,j}^\dagger = \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathbb{E}}^{n-1}(|\chi_{j'}\rangle\langle\chi_j|) / (\|\chi_{j'}\| \|\chi_j\|)$. Substituting $\hat{\mathbb{E}}^\infty$ for $\hat{\mathbb{E}}^{n-1}$ we get

$$\tilde{\psi}_l - \tilde{\phi}_l^n = \sum_{i,i',j,j'=1}^b U_{l,(i'j')}^* U_{l,(ij)} \text{Tr}_{\mathcal{B}}(X_{i,i'} Y_{j,j'}),$$

where $X_{i,i'} = V^\dagger |\chi_i\rangle\langle\chi_{i'}| V$ and $Y_{j,j'} = \mathbb{1}_{\mathcal{A}} \otimes [\hat{\mathbb{E}}^{n-1} - \hat{\mathbb{E}}^\infty] (|\chi_{j'}\rangle\langle\chi_j|) / (\|\chi_{j'}\| \|\chi_j\|)$.

Like all trace preserving quantum operations, the partial trace is contractive with respect to the 1-norm. Hence, an application of the triangle inequality for the 1-norm gives

$$\|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1 \leq \sum_{i,i',j,j'}^b |U_{l,(i'j')}^*| |U_{l,(ij)}| \|X_{i,i'}\|_1 \|Y_{j,j'}\|_1$$

It is not hard to see that

$$\|X_{i,i'}\|_1 = \|\chi_i\| \|\chi_{i'}\|, \quad \|Y_{j,j'}\|_1 \leq \|\hat{\mathbb{E}}^{(n-1)} - \hat{\mathbb{E}}^\infty\|,$$

and hence

$$\|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1 \leq \left(\sum_{i=1}^b \sum_{j=1}^b |U_{l,(ij)}| \|\chi_i\| \right)^2 \|\hat{\mathbb{E}}^{(n-1)} - \hat{\mathbb{E}}^\infty\|.$$

Since $\sum_{i,j=1}^b |U_{l,(ij)}|^2 \|\chi_i\|^2 = \beta_l$, two applications of Cauchy-Schwarz give

$$\|\tilde{\psi}_l - \tilde{\phi}_l^n\|_1 \leq b^2 \beta_l \|\hat{\mathbb{E}}^{(n-1)} - \hat{\mathbb{E}}^\infty\|. \quad (15)$$

Finally, combining (1) with (15), Eq. (13) becomes

$$\tau_l^n \leq c_1 \lambda^n, \quad c_1 = cb^2/\lambda. \quad (16)$$

To conclude the proof, we note that since the bound for τ_l^n is independent of l , summing over l in Eq. (14) yields

$$\sum_{l=1}^L |\epsilon_l^n| \leq (\log d^3 + 4) c_1 \lambda^n + \eta(2c_1 \lambda^n).$$

It is clear that for $\lambda' > \lambda$ there exists a constant c_2 such that

$$\eta(2c_1 \lambda^n) \leq c_2 (\lambda')^n.$$

The only condition on n was imposed in Eq. (14) where we assumed that $2\tau_l^n \leq 1/e$. Using Eq. (16) we see that there is an n_0 such that the above condition is satisfied for all $n \geq n_0$. The previous observations imply that for all λ' with $\lambda < \lambda' < 1$, there is a constant c_3 such that

$$\epsilon(n) = c_3 (\lambda')^n \geq \sum_{l=1}^L |\epsilon_l^n|, \quad \text{for all } n.$$

Finally, Eq. (8) implies that

$$E_{[\mathcal{A}, \mathcal{B} \otimes \mathcal{B}]}(\rho_{\mathcal{A} \otimes \mathcal{B}}) - E_{[\mathcal{A}, \mathcal{A}^{\otimes n-1}]}(\rho_{[1, n]}) \leq \epsilon(n),$$

and this completes the proof of the theorem.

A natural question to ask at this point is the following: How does the entanglement between the spin at site 1 and spins at sites $[p, n]$, $p \geq 2$, behave as p becomes large? Since the state $\rho_{1, [p, n]}$ factorizes into $\rho_1 \otimes \rho_{[p, n]}$ as $p \rightarrow \infty$ [25], we expect that the bulk of the entanglement is concentrated near site 1. The following theorem confirms this:

Theorem 4.—For any pure translation invariant FCS and $n \geq p \geq 2$, the following bound holds:

$$E_{[\mathcal{A}, \mathcal{A}^{\otimes n-p+1}]}(\rho_{1, [p, n]}) \leq \epsilon(p), \quad (17)$$

where $\epsilon(p)$ decays exponentially fast in p .

Sketch of the proof.—The main observation is that the trace distance between the states $\rho_{1, [p, n]}$ and $\rho_1 \otimes \rho_{[p, n]}$ vanishes exponentially fast with p . This is a consequence of the exponential rate of convergence described in Eq. (1). Since $E_{[\mathcal{A}, \mathcal{A}^{\otimes n-p+1}]}(\rho_1 \otimes \rho_{[p, n]}) = 0$, a straightforward application of Nielsen's inequality for the continuity of the EoF [26] yields the desired result. \square

Discussion.—Having established such a strong connection between the states $\rho_{[1, n]}$ and $\rho_{\mathcal{A} \otimes \mathcal{B}}$, one can apply various entanglement criteria on $\rho_{\mathcal{A} \otimes \mathcal{B}}$ to deduce entanglement properties of the spin chain. To start with, we note that for qubit chains with 2-dimensional memory algebra \mathcal{B} , the entanglement of $\rho_{[1, n]}$ can be computed analytically (in the limit) by evaluating the concurrence [27] of $\rho_{\mathcal{A} \otimes \mathcal{B}}$. For higher dimensions one can apply the PPT criterion to $\rho_{\mathcal{A} \otimes \mathcal{B}}$ to detect distributed entanglement in the finitely correlated state. Specifically, the main theorem in Ref. [28] implies that there can be no PPT bound entanglement in $\rho_{\mathcal{A} \otimes \mathcal{B}}$ since $\text{rank}(\rho_{\mathcal{A} \otimes \mathcal{B}}) = b \leq \max\{d, b\}$. Hence, if the partial transpose of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ is positive, then $\rho_{\mathcal{A} \otimes \mathcal{B}}$ is separable. On the other hand, if the partial transpose of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ is negative, then for n large enough $\rho_{[1, n]}$ becomes entangled. The amount of maximum entanglement in $\rho_{[1, n]}$ depends on the amount of entanglement found in $\rho_{\mathcal{A} \otimes \mathcal{B}}$. From this point of view, it would be very interesting to look at FCS that maximize entanglement of $\rho_{\mathcal{A} \otimes \mathcal{B}}$. Moreover, understanding how entanglement of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ varies with different CP maps \mathbb{E} could lead to a better understanding of how phase transitions occur when we vary the parameters in the underlying Hamiltonian of the system.

To conclude, we note that the conjecture of Benatti *et al.* [1] follows as a corollary of Theorem 3. In particular, our result implies that a spin at site 1 of the chain is entangled with spins at sites $[2, n]$ (for n large enough) if and only if $\rho_{\mathcal{A} \otimes \mathcal{B}}$ is entangled. Moreover, the entanglement of $\rho_{[1, n]}$ approaches the entanglement of $\rho_{\mathcal{A} \otimes \mathcal{B}}$ exponentially fast.

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