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Los Angeles

Essays on Applied Economic Theory

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Economics

by

Mofei Zhao

2016

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2016

Abstract of the Dissertation

Essays on Applied Economic Theory

by

Mofei Zhao

Doctor of Philosophy in Economics

University of California, Los Angeles, 2016

Professor John G. Riley, Co-chair

Professor Moritz Meyer-ter-Vehn, Co-chair

The dissertation consists of three essays on applied theory with a particular focus on industrial organization applications.

The first chapter develops and analyzes a dynamic stochastic model for a firm with limited managerial attention that is spent either on expansion or on improving quality of product. Under a reputational framework, I characterize how the incentives for expansion and innovation depend on a firm's current reputation, quality, and capacity. Intuitively, from the firm's perspective, quality and capacity are complements; the incentive to improve one increases with the level of the other. Thus, the firm innovates when its quality is low, reputation is low, and capacity is high; the firm expands when its quality is high, reputation is high, and capacity is low.

In the second chapter, I investigate several model variants of the reputational model proposed in Chapter 1. First, I introduce a non-trivial cost of innovation and expansion. I show that in equilibrium, the firm's optimal strategy takes an "innovate-shirk-expand" shape. In the second model variant, I replace the strong linearity assumption with increasing/decreasing returns to scale. Consequently, capacity becomes effective in the information structure. I show that, on each equilibrium path, the firm innovates when its quality is low, reputation is low, and capacity is high; it expands when its quality is high, reputation is high,

and capacity is low. Finally, I generalize this model to a continuum of firms and study the steady-state distribution of reputation, quality, and capacity. Both computational and numerical results show that the bulk of low-reputation, low-quality firms lie at the bottom, while a few pioneers with high quality and high capacity are found at the top.

In the third chapter, I study the competition between two firms for a two-stage research and development project, where the difficulty of the first stage is unknown. I assume that each firm holds a belief concerning the difficulty of stage 1 and updates its belief following Bayes' rule. Firm can choose to report or withhold their intermediate results. I show that, as the exit point approaches, the firm has an incentive to conceal its success in the first stage, in the hope that its opponent raises its estimation of the stage's difficulty and soon exits. I demonstrate the existence of a unique equilibrium, characterize the firm's optimal strategy for report, withhold, and exit decisions, and investigate the resulting firm dynamics.

The dissertation of Mofei Zhao is approved.

Sushil Bikhchandani

Hugo Andres Hopenhayn

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John G. Riley, Committee Co-chair

University of California, Los Angeles

2016

To my parents...

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VITA

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CHAPTER 1

A Reputational Model of Firm Size and Product Quality

1.1 Introduction

In 2005, American Apparel ranked 308th on Inc.'s 500 fastest-growing companies in the United States, with 440% three-year growth and revenues of US$211 million. All its figures seemed promising, and it was difficult to imagine that in 2011, 6 years later, this firm found itself in a serious financial crisis. Its stock price fell from 15 dollars per share in 2007 to 52 cents in 2011 and then to 8 cents in 2015. The company's 273 physical stores, of which it was formerly proud, have become substantial burdens, and many are on the verge of closing down.

A closer examination of the tragedy experienced by American Apparel accentuates the discordance between its impetuous expansion and hesitant innovation. While it began with a groundbreaking marketing strategy, American Apparel seems to have focused on opening more shops and obstinately maintaining its initial style. Consider its successful competitors, say, Uniqlo, as a contrast: Uniqlo spent 45 years before expanding to 100 physical stores; it also maintained robust innovation, both in product and in marketing strategy, through its Uniqlo Design Studio and designer invitations.

Quality and capacity are amongst a firm's most prominent concerns. This paper portrays them as two channels for improvement that compete for managerial attention[1] and focuses on a firm's dilemma between the two channels: innovation, to generate quality improvements, and expansion, to gain capacity.

[1] One can also intepret this as, say, fiscal investment, with minor adjustment to the model.

I characterize how the incentives for innovation and expansion depend on a firm's current reputation, quality, and capacity. Intuitively, I explain why a firm focuses on expansion when it has advanced products and low capacity and why it turns to innovation when its products are outdated and existing capacity is high. From the firm's perspective, quality (and reputation) and capacity are complements, where the first determines the price that the firm can charge, and the second determines the quantity of goods sold. Thus, the incentive to improve one increases with the level of the other.

In our baseline model, a firm's value is linear in capacity and, consequently, the firm's decision function is independent of capacity, the firm innovates when its quality and reputation are low and expands when its quality and reputation are high[2]. Our modeling of firm dynamics is inspired by the approach adopted by Board an Meyer-ter-Vehn (2010), where a firm's product quality is a function of its past investments, rather than its current effort. Quality is persistent, and a firm's investment increases its quality and future revenue independent of market beliefs concerning future investment.

The "quality" of a product line as reflected in the willingness of customers to pay is determined by many different characteristics. While some characteristics are observable, others are much less so. Moreover, for many products the market's value of the different characteristics changes over time. Thus a product line that is correctly perceived to be of high quality at time t may not be of high quality at time $t+s$ unless the firm innovates in the interim. To model this, I assume that quality is not publicly observable. Instead, the market learns a firm's quality from publicly observed breakthroughs and forms a belief (namely, reputation) concerning a firm's quality, which relies on beliefs concerning firm's action rather than real action.

Given such asymmetric information, the actual quality of the product will deviate from the reputation. I combine this information structure with firm expansion. By its nature, firm capacity is persistent, i.e., a firm cannot choose its capacity arbitrarily; instead, it must work

[2]In the second Chapter I investigate a model variant in which capacity matters in the information structure.

continuously to develop capacity, and in most cases, downsizing implies a severe problem. Thus, in our model, both quality and capacity are persistent, an important feature that allows us to introduce juxtaposed channels of improvement from which the firm chooses.

Since the reputation lags behind actual quality, a firm may have an incentive to either raise or lower quality over time. If the former it is innovating in the expectation that the reputation will eventually rise as well and so higher prices can be charged. Alternatively the firm may have an incentive to reduce its attention in quality enhancement and so take advantage of its reputation to increase its capacity.

The paper is organized as follows:

In Section 2, I establish the baseline model.

In the model, one long-lived firm sells a product stream that is of high or low quality to a continuum of identical short-lived consumers. The volume of the stream is termed capacity. The firm chooses the intensity of innovation and expansion, which feed into quality and capacity, respectively. The firm has limited managerial attention to spend on innovation and expansion. Thus, the two options compete with one another.

I assume that the firm knows its own quality and production capacity. Consumers, however, observe neither quality nor firm's action directly and learn about firm quality through Poisson signals termed "breakthroughs" that can be generated only by a high-quality firm. The market's belief that firm's quality is high is called the reputation of the firm, which evolves according to Bayes' rule. Absent a breakthrough, reputation depends on the market's beliefs concerning the firm's innovation intensity, rather than actual innovation. Because the firm cannot control these beliefs, innovation incentives are dampened by moral hazard.

In Section 3, I characterize the Markovian equilibrium and study a firm's moral hazard problem.

I first show that, assuming some linear property of innovation and expansion, a firm has constant returns to scale. Therefore, the firm's capacity does not enter its decision function as a state variable. Thus, consumers need not hold an additional belief regarding the firm's capacity. The information structure is simplified (in Section 5, the strong linearity assumption is removed and the equilibrium is re-investigated).

I compare the incentives of two channels: innovation is incentivized by consumers' learning about product quality, which feeds into the firm's reputation and future revenue, whereas expansion is incentivized by direct revenue, which is proportional to the firm's size.

I show that the incentive to innovate is decreasing in firm reputation and that the incentive to expand is increasing in firm reputation. Thus, in any equilibrium, the firm innovates when its reputation lies below some cutoff x_θ and expands when above this cutoff (Figure 1.1).

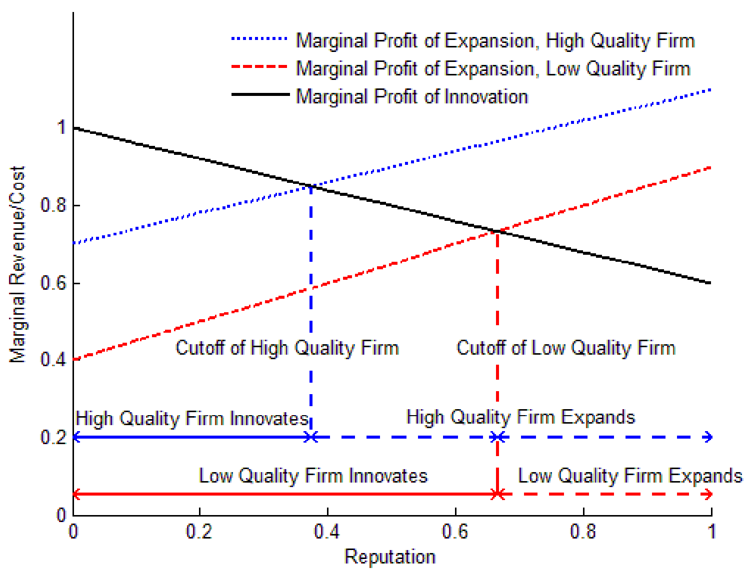


Figure 1.1: Firm's Incentives and Strategy

I call this the "innovate-expand" equilibrium. There are 2 cutoffs, one for each quality type. A high-quality firm has stronger incentives to expand because of its higher value and therefore has a lower cutoff.

In our model, information asymmetry arises both from a firm's product quality and from a firm's decision with respect to innovation/expansion. First, the firm's incentive to innovate is dampened by moral hazard; moreover, due to adverse selection, a high-quality firm's incentive to innovate is reduced to a greater extent than is that of its low-type counterpart. As a firm's quality is undisclosed, consumers' belief concerning the firm's intensity of innovation is an expectation over both types. Thus, a high-quality firm's perceived intensity of innovation is higher than it is in reality, and the opposite is the case for a low-quality firm. Such deviation confirms the adverse selection problem: A high-quality firm expands (partially) at the expense of the low-quality firm, in the form of reduced perceived intensity of innovation and reduced reputation. Unlike the market for lemons, a high-quality firm exploits the pooling effect.

In Section 4, I run a numerical simulation of the model and show the results graphically. Section 5 concludes.

1.1.1 Literature

The paper adopts the reputation framework of Board and Meyer-ter-Vehn (2010, 2013). In contrast to those papers I assume that the firm can expand in capacity, introducing juxtaposed options of improvement competing for managerial attention. This idea of multitask operation draws from Holmstrom and Milgrom (1991), who analyze a principal-agent model that the principal has several different tasks for the agent to perform.

Our model has a close relationship with Mailath and Samuelson (2001) and Holmström (1999), both of which model reputation as the market's belief about an exogenous state variable, while in our model a firm spend managerial attention to improve its endogenous quality type and to gain capacity over time. In Mailath and Samuelson (2001), a competent firm can choose to work so as to distinguish itself from an incompetent firm. In Holmström (1999), the firm works to induce erroneous market beliefs that its exogenous ability type is higher than in reality.

Our analysis depends on the relationship of firm's size and growth, which is the focus of a longtime debate. Pashigian and Hymer (1962) claim no relation between the size of firms and their growth rates. In Lucas and Prescott (1971), adjustment costs with constant return to scale are shown to imply that firms should grow in proportion to their size. On the contrary, Mansfield (1962) finds that smaller firms have higher and more variable growth rates in data. Du Rietz, in a sample of Swedish firms, also finds that smaller firms grow faster.

Finally, Rob and Fishman (2005) used a repeated game with imperfect monitoring to explain the dynamics of firm size. Cabral (2014) and Abito, Besanko, and Diermeier (2012) consider reduced-form models of reputational firm dynamics, whereby reputation is modeled as a state variable akin to capital stock, but is not derived from Bayes' rule. In contrast to these repeated game models, our firm has reputation, quality, capacity, etc. as state variables, enabling us to impose more discipline on equilibria by focusing on Markovian equilbria.

1.2 Model

Firm and Customers: There is one firm and a continuum of identical consumers. Time $t \in [0,+\infty)$ is continuous and infinite. At time t, the firm produces q_t units of a product of quality θ_t, which can be either high or low, i.e., $\theta_t \in \{L=0, H=1\}$. I call q_t the firm's production capacity. The instantaneous value of the firm's production to a consumer is $\theta_t q_t dt$. The firm's value is always non-negative, and the firm shall never exit.

The firm focuses its attention on two channels of improvement: innovation and expansion. They are mutually exclusive. Specifically, I assume that the sum of the two intensities is 1 for $t \in [0,+\infty)$. In this way, the intensity of innovation or expansion represents the percentage of the firm's effort spent on innovation or expansion, respectively, with the sum being equal to 100%. I use η_t to denote the firm's intensity of innovation. Consequently, the intensity of expansion is $1-\eta_t$.

Innovation and expansion feed into the firm's quality and capacity, respectively, in the following way:

At time 0, the firm draws initial quality $\theta_0 \in \{L, H\}$ and initial capacity $q_0 \in (0,+\infty)$.

Product quality θ_t depends on initial quality, past innovation $(\eta_s)_{0\leq s\leq t}$, and technology shocks, which occur according to a Poisson process with arrival rate λ. When a shock occurs, the previous quality becomes obsolete, and the current quality is determined by the current intensity of innovation $\theta_{t+dt} = \eta_t$. Absent a shock, quality is constant, $\theta_{t+dt} = \theta_t$.

Firm capacity q_t is a function of past expansion $1-\eta_s$, $0 \leq s \leq t$, specifically:

$$q_{t+dt} = q_t[1+(1-\eta_t)Mdt] \tag{1.1}$$

where M is a positive parameter measuring the efficiency of managerial attention spent on expansion, or simply the efficiency of expansion. I assume that $M < r$ and, hence, that the value of the firm is always well defined.

The opportunity cost of innovation (or expansion) is the dividends that would otherwise be yielded by the other option.[3]

Information structure: I assume that firm knows its own quality. The firm also knows its capacity from past expansion. However, consumers cannot directly observe innovation η, product quality θ, or firm capacity q.[4]

The consumers learn about quality through signals that arrive to and only to a high-quality firm at Poisson rate μ. I call the signal a product breakthrough, or breakthrough

[3]The above assumption also suffices for the case in which the firm is allowed to shirk: As the firm's value is non-negative, expansion always contributes a non-negative value. Thus, in equilibrium, the firm will always spend its available resources; shirking is never optimal.

[4]Note that we assume that consumers cannot observe the firm's capacity, which can be used to derive the firm's past actions and thus contains additional information on quality. This assumption may initially appear counterintuitive because we know a firm's size from its annual reports. Our considerations are, first, that size only partially reflects capacity; second, similar to the B-M model, the effect of innovation declines over time, and thus, a firm's most informative actions are made during the most recent periods, while there is always a time gap between the latest report and the present.

For example, suppose that Apple Inc. is issuing the latest generation of iPhone. Common consumers have no idea about the number of new iPhones under production. Contrarily, if we know that the amount is, say, less than that of the previous generation, aside from other explanations, an educated guess would be that the new iPhone is not as groundbreaking as its predecessors were. That is, a firm's capacity partially implies the firm's confidence in the new product (which further carries an estimation of its quality) and is kept secret.

for short. The arrival of a breakthrough is statistically independent of the quality-updating process.

The firm conditions its innovation and expansion on its quality history $\{\theta_s\}$, $s \in [0,t)$, capacity history $\{q_s\}$, $s \in [0,t)$, and public information history prior to t, h^{t-}.

I assume that, for every time t, there exists a common belief concerning the firm's past intensity of innovation $\tilde{\eta}_t$. $\tilde{\eta}_t$ is deterministic with respect to public histories. Believed intensity of innovation $\tilde{\eta}_t$ and the exogenous initial belief regarding quality $x_0 \in [0,1]$ control the joint distribution of quality and histories h^t. The market's belief concerning product quality at time t is called the firm's reputation and is denoted by $x_t = E_{\langle\tilde{\eta}\rangle}(\theta_t|h^t)$.Specifically, x_t evolves as follows:

Because breakthroughs only arrive at a high-quality firm, a breakthrough reveals high quality immediately. The firm's reputation jumps to one, $x_{t+dt} = 1$.

Absent a breakthrough, the increase in reputation $dx_t = x_{t+dt} - x_t$ is governed by the market belief concerning the firm's innovation intensity $\tilde{\eta}_t$:

$$x_{t+dt} = x_t + \lambda(\tilde{\eta}_t - x_t)dt - \mu x_t(1-x_t)dt \tag{1.2}$$

The second term on the right-hand side captures the probability that a technology shock arrives within $[t, t+dt]$, and expected quality is updated from x_t to $\tilde{\eta}_t$. The third term is the standard Bayesian increment in the absence of a breakthrough.

Firm's Value: Both the firm and consumers are risk-neutral; the discount rate is $r \in (0,+\infty)$. At time t, the firm sets price equal to the consumers' marginal utility, which is constant, and hence the consumers' expected utility is 0. For simplicity, I assume that consumers' marginal utility is 1, and hence the firm's profit at time t is $x_t q_t dt$. Given the firm's strategy $\langle\eta\rangle$ and market belief $\langle\tilde{\eta}\rangle$, the firm's expected present value is:

$$E_{\langle\theta_t,\eta,\tilde{\eta}\rangle}[\int_t^{+\infty} e^{-r(s-t)} x_s q_s ds] \tag{1.3}$$

(1.3) clearly portrays reputation x_s and capacity q_s as complements. Both contribute positively to the firm's value and the marginal value of one is positively related to the current level of the other. To maximize its value, firm must expend effort on both. For the above reasons, the firm faces the competing goals of investing in reputation, by innovating, and in capacity, by expanding.

1.3 Innovation or Expansion?

I consider a Markovian belief that depends on public history only via the left-sided limit of reputation.

Optimum intensity of innovation: Given such a belief $\langle \tilde{\eta} \rangle$, I can write the firm's continuation value at time t as a function of its current reputation, quality, and capacity:

$$V(x_t, \theta_t, q_t) = \sup_{\{\eta_s\}_{s \geq t}} E_{\langle \theta_t, \eta, \tilde{\eta} \rangle} [\int_t^{+\infty} e^{-r(s-t)} x_s q_s ds] \tag{1.4}$$

I simplify my analysis by focusing on strategies that maximize (1.4) pointwisely.

Lemma 1: A firm's value is homogeneous of degree 1 in the firm's capacity, i.e., $V(x, \theta, q) = qV(x, \theta, 1)$.

Proof: In Appendix A.1.

Lemma 2: A firm's optimum intensity of innovation η is independent of its capacity, i.e., $\forall q_0, q_0' > 0$, $\eta(x, \theta, q_0) = \eta(x, \theta, q_0')$.

Proof: This is a direct application of Lemma 1. Following the last step in Appendix A.1., $\forall q_0, q_0'$, $V(x, \theta, q_0) = \frac{q_0}{q_0'} V(x, \theta, q_0')$ and $\langle \eta \rangle = \langle \eta' \rangle$.

Lemmas 1 and 2 simplify the problem by holding that one state variable, q, is irrelevant to the firm's decision-making function. The firm has constant returns to scale. Most importantly, this means that consumers' belief $\tilde{\eta}$ does not rely on their belief concerning a firm's capacity. For this reason, I do not need to assume the capacity belief held by the consumers because such a belief is payoff irrelevant.

Because capacity does not enter the firm's innovation function, in the following analysis, I drop q from the notation and write intensity of innovation as a function of firm reputation and quality, $\eta(x,\theta)$.

Markov-Perfect-Equilibrium: A Markov-Perfect-Equilibrium $(\eta,\tilde{\eta})$ consists of an decision function $\eta : ([0,1],\{L,H\}) \rightarrow [0,1]$ and market beliefs $\tilde{\eta} : [0,1] \rightarrow [0,1]$, such that: (1) the intensity of innovation maximizes firm value, $V(x,\theta,q)$; (2) market beliefs are correct, $\tilde{\eta}(x) = x\eta(x,H) + (1-x)\eta(x,L)$.

First-best solution: As a benchmark, suppose that product quality θ_t is publicly observed at time t. In equilibrium, price equals quality. The profit yielded by innovation equals the obsolescence rate λ, times the price differential 1, divided by the effective discount rate $r+\lambda$. The profit yielded by expansion is the efficiency of expansion M times the current value of the firm. Thus, first-best strategy is given by:

$$\eta(x,\theta,q) = \begin{cases} 0 \text{ if } \frac{\lambda}{r+\lambda} < MV(x,\theta,1) \\ 1 \text{ if } \frac{\lambda}{r+\lambda} > MV(x,\theta,1) \end{cases} \tag{1.5}$$

In the following section, I characterize a Markovian equilibrium. I pay special attention to the firm's strategy, which is captured in Theorem 1. Theorem 2 calculates a firm's reputational dynamics in closed form with respect to different intensities of innovation.

First, however, I establish some basic properties of the value function that are extensively used in subsequent analysis. First, as profit at time t is bounded by e^{Mt}, the value function is bounded by $[0,\frac{1}{r-M}]$. Second, I demonstrate monotonicity in Lemmas 3 and 4.

Lemma 3: In any equilibrium $\langle \eta, \tilde{\eta} \rangle$, the value function of the firm $V(x,\theta,q)$ is strictly increasing in reputation x.

Proof: Fix the initial reputations $x_0 < x'_0$ of a low-reputation firm ("low firm" for short) and a high-reputation firm ("high firm" for short), respectively. Suppose that both firms have initial quality θ_0 and initial capacity q_0. Suppose that the high firm chooses the non-Markovian strategy $\langle \eta' \rangle$ that mimics the equilibrium intensity of innovation of the low firm, i.e., if at time t after history h^{t-}, the low firm has reputation $x_t = x_t(x_0, h^{t-}, \tilde{\eta})$ then $\eta'_t = \eta(\theta_t, x_t(x_0, h^{t-}, \tilde{\eta}), q_t)$. Adopting this strategy, the high firm's quality θ'_t and capacity q'_t are governed by the same process as the equilibrium quality θ_t and capacity q_t of the low firm. Thus, these firms face the same distribution of public histories, and the reputation of the high firm never falls behind, i.e., $x'_t \geq x_t$ with strict inequality for t close to 0. Then, the profit of the high firm with strategy $\langle \eta' \rangle$ always exceeds the equilibrium profit of the low firm because its revenue is higher by the above argument and the costs are equal by construction. Furthermore, the equilibrium value of the high firm is weakly higher than its value from adopting a mimic strategy; thus $V(x,\theta,q)$ is strictly increasing in reputation x.

Lemma 4: $V(x,H,q) \geq V(x,L,q)$, with strict inequality if $x \neq 1$.

Proof: Suppose that a high-quality firm mimics a low-quality firm's strategy until the first quality shock has an effect, i.e., before the first quality shock $\eta(x,H,q) = \eta(x,L,q)$ [5]. Absent a breakthrough, the two firms always receive the same profit. The difference is that the high-quality firm may receive breakthroughs prior to the first quality shock.

Now, I expand the value difference at time t. I use t_i to denote the time when the ith breakthrough arrives, before the first quality shock. For simplicity, I assume that $t_0 = t$.

[5] This is different from the "high firm" in Lemma 3; in Lemma 3, the high firm mimics the low firm's actions, which is conditional on the history. In Lemma 4, the high-quality firm mimics the low-quality firm's strategy, which is conditional on the states.

$$
\begin{aligned}
\bar{V}(x_t,H,q_t)-V(x_t,L,q_t) &= \int_t^{\infty} e^{-(r+\lambda)(t_1-t)}\mu[\bar{V}(1,H,q_{t_1})-V(x_{t_1},L,q_{t_1})]dt_1 \\
&= \int_t^{\infty} e^{-(r+\lambda)(t_1-t)}\mu[V(1,L,q_{t_1})-V(x_{t_1},L,q_{t_1}) \\
&\quad +\bar{V}(1,H,q_{t_1})-V(1,L,q_{t_1})]ds \\
&= \int_t^{\infty} e^{-(r+\lambda)(t_1-t)}\mu\{V(1,L,q_{t_1})-V(x_{t+s},L,q_{t_1}) \\
&\quad +\int_{t_1}^{\infty} e^{-(r+\lambda)(t_2-t_1)}\mu[\bar{V}(1,H,q_{t_2})-V(x_{t_2},L,q_{t_2})]dw\}ds \\
&= \ldots
\end{aligned}
$$

where x_{t_i} evolves according to $x_{t_{i-1}}=1$, $i=2,3...$

I can infinitely repeat this expansion. By Lemma 3, $V(1,L,q_{t_i})\geq V(x_{t_i},L,q_{t_i})$, with strict inequality if $x_{t_i}\neq 1$. the terms inside the square brackets are the same, while the coefficient is dampened by at least $e^{-(r+\lambda-M)(t_i-t_{i-1})}$. Thus, by mathematical induction, I obtain $\bar{V}(x_t,H,q_t)\geq V(x_t,L,q_t)$, with strict inequality if $x_t\neq 1$.

Moreover, by the definition of equilibrium, $V(x,H,q)\geq\bar{V}(x,H,q)$. In conclusion, $V(x,H,q)\geq V(x,L,q)$, with strict inequality if $x_t\neq 1$.

Intuitively, a firm's problem is to select between two channels of improvement: innovation and expansion. A firm's instantaneous innovation is incentivized by the probability that a technology shock will occur extremely soon in the future, and firm value jumps according to its latest intensity of innovation. Expansion, however, is incentivized by the value gained, which is directly proportional to firm value. Lemma 5 shows the firm's problem, or a comparison of the two channels, in an analytical way.

I define $\Delta(x)$ as the value of quality for a firm with $q=1$, i.e.,

$$
\Delta(x)=V(x,H,1)-V(x,L,1) \tag{1.6}
$$

As $V(x,\theta,q) = qV(x,\theta,1)$, $V(x,H,q) - V(x,L,q) = q\Delta(x)$.

Lemma 5: Equilibrium must satisfy:

$$\eta(x,\theta) = \begin{cases} 0 \text{ if } \lambda\Delta(x) < MV(x,\theta,1) \\ 1 \text{ if } \lambda\Delta(x) > MV(x,\theta,1) \end{cases}$$

Firm strategy and market belief $\langle \eta, \tilde{\eta} \rangle$ form an equilibrium if and only if the above equation holds for all (θ, x).

Proof: Intuitively, this simply indicates that the firm chooses the action that maximizes its value. For a rigorous proof, see Appendix A.2.

When the marginal revenues of the two are equal, i.e., when $\lambda\Delta(x) = MV(x,\theta,1)$, optimal intensity of innovation can take an arbitrary value in the action space.

Lemma 6: In equilibrium, $\eta(x,L) \geq \eta(x,H)$ for $x \in [0,1]$.

Proof: Suppose that $\eta(x,H) = 1$, following Lemma 5, $\lambda\Delta(x) > MV(x,H,1)$; following Lemma 4, $V(x,H,1) > V(x,L,1)$ for $x \in [0,1]$. Thus, $\lambda\Delta(x) > MV(x,H,1) > MV(x,L,1)$, use Lemma 5 again, $\eta(x,L) = 1$. Thus, $\eta(x,H) = 1$ is a sufficient condition for $\eta(x,L) = 1$. As $\eta(x,H)$ only takes value 0 or 1 in equilibrium, I have $\eta(x,L) \geq \eta(x,H)$ for $x \in [0,1]$.

However, firm actions may differ across types. The reputation updating rule x_{t+dt} is the same. x_{t+dt} is deterministically governed by the perceived innovation intensity $\tilde{\eta}_1(x)$, which is the expectation of the innovation of a firm with reputation x and is the same for both types of firms. For instance, consider a candidate equilibrium $\langle \eta, \tilde{\eta} \rangle$; following Lemma 6, suppose that for some value of x, $\eta(x,L) > \eta(x,H)$, by Lemma 5, $\eta(x,L) = 1$, $\eta(x,H) = 0$. In equilibrium, by the definition of perceived innovation intensity, I have:

$$\tilde{\eta}(x) = E[\eta(x,\theta)] = x\eta(x,H) + (1-x)\eta(x,L) = 1 - x \tag{1.7}$$

which is smaller than a low-quality firm's actual intensity of innovation but higher than that of a high-quality firm.

As noted above, the incentive to expand is directly proportional to firm value, which, according to Lemma 3, is increasing in x. Following Lemma 5, the marginal revenue from innovation at time t is $\eta_t q_t \lambda \Delta(x_t) dt$ and identical for high- and low-quality firms. To characterize the incentives, I need to evaluate the value of quality $\Delta(x)$.

Lemma 7: Suppose that $M \leq \lambda$, $\Delta(x)$ is decreasing in x.

Proof: For a rigorous proof, see Appendix A.3. Below, I discuss some intuition and a remark if $M > \lambda$.

From Lemma 4 and Lemma 5, at any reputation level x, the possible firm actions are as follows: (1) both high- and low-quality firms innovate; (2) a high-quality firm expands, and a low-quality firm innovates; and (3) both high- and low-quality firms expand.

In cases 1 and 3, both types take the same action; through $\Delta(x)$, the effect of firm innovation/expansion cancels out. In these cases, intuitively, the value of quality is the discounted integration of a flow of reputational dividends, which is the value of having high quality in the next instant. Specifically, a reputational dividend equals the probability that a breakthrough arrives instantly times the difference between the value of a high-quality firm that receives a breakthrough and its low quality counterpart's continuation value, i.e., $\mu[V(1,H,q) - V(x,L,q)]$. When a firm's reputation is high, its value is already high, and a potential breakthrough thus yields a smaller improvement in value, i.e., $V(1,H,q) - V(x,L,q)$ is smaller. In turn, the value of quality is relatively smaller for high-reputation firms.

Consider case 1 as an example, I expand the firm's current value into its profits over

$[t, t+dt)$ and its expected continuation value:

$$\begin{aligned}
& x_t dt + (1-rdt)[\mu dt V(1,H,1) + (1-\mu dt)V(x_t,H,1)] \\
& -x_t dt - (1-rdt)[\lambda dt V(x_t,H,1) + (1-\lambda dt)V(x_t,L,1)] \\
= \; & \mu dt V(1,H,1) + (1-rdt-\mu dt)V(x_t,H,1) \\
& -\lambda dt V(x_t,H,1) - (1-rdt-\lambda dt)V(x_t,L,1) \\
= \; & \mu dt[V(1,H,1) - V(x_t,L,1)] \\
& +(1-rdt-\lambda dt-\mu dt)[V(x_t,H,1) - V(x_t,L,1)]
\end{aligned}$$

The reputational dividend over $[t, t+dt)$ is $\mu dt[V(1,H,1) - V(x_t,L,1)]$, which is decreasing in x_t.

Case 3 is the same with minor adjustment to the discount rate, which does not affect monotonicity.

In case 2, the same argument holds, except that a high-quality firm is expanding. Intuitively, to maintain monotonicity, I impose some limit on expansion to maintain monotonicity. In this case, I need to more closely examine the coefficient of the continuation terms. A high-quality firm's future dividends yielded by expansion are governed by the efficiency of expansion, in the form of $MV(x,H,q)$, while a low-quality firm's innovation dividends are $\lambda\Delta(x)$. Combine this with case 1, where a firm's actions are cancelled out, the value expansion becomes:

$$\begin{aligned}
& x_t dt + (1-rdt+Mdt)[\mu dt V(1,H,1) + \lambda dt V(x_t,L,1) + (1-\mu dt-\lambda dt)V(x_t,H,1)] \\
& -x_t dt - (1-rdt)[\lambda dt V(x_t,H,1) + (1-\lambda dt)V(x_t,H,1)] \\
= \; & \mu dt V(1,H,1) + \lambda dt V(x_t,L,1) + (1-rdt-\lambda dt-\mu dt+Mdt)V(x_t,H,1) \\
& -\lambda dt V(x_t,H,1) - (1-rdt-\lambda dt)V(x_t,H,1) \\
= \; & \mu dt V(1,H,1) + (\lambda-\mu)dt V(x_t,L,1) + (M-\lambda)dt V(x_t,H,1) \\
& +(1-rdt-\lambda dt-\mu dt)[V(x_t,H,1) - V(x_t,L,1)]
\end{aligned}$$

The reputational dividend over $[t, t+dt)$ is

$$\mu dtV(1,H,1)+(\lambda-\mu)dtV(x_t,L,1)+(M-\lambda)dtV(x_t,H,1)$$

Following the same intuition as in cases 1 and 3, I want reputational dividend decreasing in x_t. A sufficient condition is that all terms are non-increasing in x_t; thus, I require $M \leq \lambda$ (I have $\lambda < \mu$ by definition).

Intuitively, $\Delta(x)$ equals the integration of these reputational dividends over time. An increase in x leads to declines in all of these dividends, and thus, it decreases $\Delta(x)$.

However, if $M > \lambda$, monotonicity may be lost for case 2, where a high-quality firm expands and a low-quality firm does not. Intuitively, a high-quality firm's value may increase too rapidly if it is expanding dramatically. Mathematically, the reputational dividend $\mu V(1,H,q)+(M-\lambda)V(x,H,q)+(\lambda-\mu)V(x,L,q)$ may increase in x if $M > \lambda$.

However, a closer examination reveals that $M > \lambda$ is a rather extreme, if not trivial, case, in which a high-quality firm never innovates in equilibrium, regardless of its reputation. This result is driven by comparing the revenues from investment in capacity and quality: $MV(x,H,1) > M\Delta(x) > \lambda\Delta(x)$. For a high-quality firm, innovation is always inferior to expansion when $M > \lambda$[6].

I assume that a firm's equilibrium strategy $\eta(x,\theta)$ is piecewise continuous for all x and θ while there are a finite number of cutoffs $0 < x_{\theta,1} < x_{\theta,2} < \cdots < x_{\theta,i} < 1$. I call a candidate equilibrium "innovate-expand" if, for each type θ, there exists a single cutoff $x_\theta \in (0,1)$, and hence, $\eta(x,\theta)=1$ (innovate) below x_θ and $\eta(x,\theta)=0$ (expand) above x_θ. Additionally, a candidate equilibrium is full innovate if $\eta(x,\theta)\equiv 1$ and full expand if $\eta(x,\theta)\equiv 0$ for all x and θ.

Lemma 7 sheds light on solving the firm's problem: a firm's incentive to innovate is directly proportional to $\Delta(x)$, which, according to Lemma 7, decreases in x; a firm's incentive

[6]A low-quality firm may still subscribe to another strategy.

to expand is directly proportional to $V(x,\theta)$, which, according to Lemma 3, increases in x. The monotonicity of both incentives supposes either a single or no cross point, at which η and $\tilde{\eta}$ jump according to Lemma 5.

I characterize a candidate equilibrium strategy in Theorem 1 and the perceived intensity of innovation in equilibrium associated with this strategy. The perceived intensity of innovation is a prerequisite for calculating reputational dynamics in closed form.

Theorem 1: Suppose that $M < \lambda$; in equilibrium, (1) the optimal strategy is characterized by cutoffs x_H and x_L, $0 \leq x_H \leq x_L \leq 1$, with the second inequality being strict if $M > 0$ and $\lambda > 0$, such that a firm of quality θ, $\theta \in \{H, L\}$:

$$\begin{aligned} \text{innovate if } x &\in [0, x_\theta] \\ \text{expand if } x &\in [x_\theta, 1] \end{aligned} \tag{1.8}$$

(2) The perceived intensity of innovation is:

$$\tilde{\eta} = \begin{cases} 0 \text{ if } x \in (x_L, 1] \\ 1 - x \text{ if } x \in (x_H, x_L) \\ 1 \text{ if } x \in [0, x_H) \end{cases} \tag{1.9}$$

By the above definition, the firm adopts the "innovate-expand" equilibrium strategy. I call $[0, x_\theta]$ the "innovate region" and $[x_\theta, 1]$ the "expand region". It is possible to have a trivial research region, i.e., $x_\theta = 0$, or a trivial expand region, i.e., $x_\theta = 1$.

Proof: In Appendix A.4.

Intuitively, after a breakthrough success, reputation jumps to 1, both types expand and reputation begins to decline. Because quality/reputation and capacity are complements, as shown in (3), the high-quality firm has greater incentives to expand. Consequently, at intermediate reputation levels, the low-quality firm innovates while the high-quality firm expands. This can lead to quality reversals. Finally, at low reputation levels, both types of firm innovate, in the hope of a new breakthrough.

A high-quality firm would exploit its quality by expanding capacity, rather than waste effort in maintaining quality. This also captures the nature of a real firm: A firm that has just developed a good product (quality jumps to 1) is eager to expand even if the market has yet to fully recognize its advance, which, in our model, means that the breakthrough has yet to arrive and the firm's reputation remains low.

Proposition 1: When M is positive and sufficiently small, any equilibrium value of the firm is less than when $M = 0$.

Proof: See Appendix A.5.

Proposition 1 is a counterintuitive result because M is the efficiency of expansion, and increasing M allows the firm to grow faster. However, at least for a sufficiently small M, improved efficiency damages, rather than improves, a firm's value.

Proposition 1 contrasts the second-best equilibrium with the first best, shown by (1.5). When M is 0, the firm is forced to innovate both in the first best and in equilibrium because there is no second option, although the incentive to do so approaches 0 as x approaches 1, i.e., $\lambda\Delta(x) \to 0$ as $x \to 1$. With the introduction of expansion, resources formerly spent on quality improvement are now reallocated to expansion. By Theorem 1, when M is positive (while sufficiently small), I can rule out the full innovate equilibrium. That is, at least for some firms with specific quality and reputation, the incentive to expand outweighs innovation, thereby drawing the equilibrium strategy away from full innovate. However, as M is small, the actual profit yielded by expansion remains small and, consequently, less than the dividends that would otherwise be yielded by innovation. That is, the first-best equilibrium remains full innovate. The divergence between first-best strategy and equilibrium strategy reflects moral hazard problem, which damages the firm's value in equilibrium.

Now, I analyze firm's reputational dynamics using the perceived intensity of innovation drawn by Theorem 1. Absent a breakthrough, a firm's reputation dynamics are governed by Bayesian updates, as captured by a series of reputational drifts. Following (3.2), reputational drift at x_t is $\lambda(\tilde{\eta}_t - x_t)dt - \mu x_t(1-x_t)dt$, the value of which jumps where perceived innovation is discontinuous in x, i.e., at x_L and x_H.

At a breakthrough, reputation is reset to $x = 1$; no further information arrives until the next breakthrough arrives. For this reason, the firm's reputational dynamics follow a recursive process that depends only on the time elapsed since the last breakthrough. Formally, suppose that I start from $x_0 = 1$; if the last breakthrough before t was at $s < t$, then $x_t = x_{t-s}$.

I focus on the reputational trajectory beginning from $x_0 = 1$, and no breakthrough arrives thereafter. Absent a breakthrough, reputation declines until it reaches x_L, where the reputational drift jumps, resulting in a kink point on the trajectory. Reputation continues to fall until it reaches x_H (or, due to parameters, until the reputation drift approaches 0), where a second kink point occurs, and thereafter, reputation asymptotically arrives $\frac{\lambda}{\mu}$. During this process, every breakthrough returns the equilibrium to the initial state with only one change in the state variable q, which does not enter the firm's decision function.

Finally, beginning from an initial reputation other than 1 is equivalent to starting from another point on this trajectory. For further recursions, suppose that the first breakthrough arrives at s and the last breakthrough before t was at $s' < t$, then $x_t = x_{t-s'+s}$.

The trajectory is given in Theorem 2.

Theorem 2: I can solve for the closed-form trajectory of reputation. Absent a shock, the reputation updating rule follows (3.2), where perceived intensity of innovation is given by Theorem 1.

Case 1: When $x_t \geq x_L$, $\tilde{\eta}_t = 0$; thus, suppose an initial reputation $x_0 \geq x_L$ at $t = 0$, and solve for the differential equation:

$$dx_t = -(\lambda+\mu)x_t dt + \mu x_t^2 dt$$

absent a shock and prior to t_L, where $x_{t_L} = x_L$, I have:

$$x_t = \frac{\lambda+\mu}{(\frac{\lambda+\mu}{x_0} - \mu)e^{(\lambda+\mu)t} + \mu} \tag{1.10}$$

Notice that the inverse of (3.8) is:

$$t = \frac{\ln(\frac{\lambda+\mu}{x} - \mu) - \ln(\frac{\lambda+\mu}{x_0} - \mu)}{\lambda+\mu} \tag{1.11}$$

evaluate (3.8) at $x = x_L$, I have:

$$t_L = \frac{\ln(\frac{\lambda+\mu}{x^*} - \mu) - \ln(\frac{\lambda+\mu}{x_0} - \mu)}{\lambda+\mu} \tag{1.12}$$

Case 2: When $x_H \leq x_t \leq x_L$, $\tilde{\eta}_t = 1 - x$; thus, assume an initial reputation of x_0, $x_0 \geq x_H$, $x_0 \leq x_L$, and solve for the differential equation:

$$dx_t = \lambda dt - (2\lambda+\mu)x_t dt + \mu x_t^2 dt$$

absent a shock, and thus far, as $x_H \leq x_t \leq x_L$, I have:

$$\begin{aligned} x_t &= \frac{1}{\frac{\mu}{\sqrt{4\lambda^2+\mu^2}} - e^{\sqrt{4\lambda^2+\mu^2}t}(\frac{\mu}{\sqrt{4\lambda^2+\mu^2}} - \frac{1}{x_0 - \frac{2\lambda+\mu-\sqrt{4\lambda^2+\mu^2}}{2\mu}})} \\ &\quad + \frac{2\lambda+\mu-\sqrt{4\lambda^2+\mu^2}}{2\mu} \end{aligned} \tag{1.13}$$

Case 3: When $x_t \leq x_H$, $\tilde{\eta}_t = 1$; thus, suppose an initial reputation of $x_0 \leq x_H$ at $t = 0$, and solve for the differential equation:

$$dx_t = \lambda dt - (\lambda + \mu)x_t dt + \mu x_t^2 dt \tag{1.14}$$

absent a shock, and thus far, as $x_t \leq x_H$, I have:

$$x_t = \frac{\frac{\mu}{\lambda-\mu}e^{(\lambda-\mu)t} + \frac{\frac{\mu}{\lambda-\mu}(1-x_0)}{x_0-\frac{\lambda}{\mu}}\frac{\lambda}{\mu}}{\frac{\mu}{\lambda-\mu}e^{(\lambda-\mu)t} + \frac{\frac{\mu}{\lambda-\mu}(1-x_0)}{x_0-\frac{\lambda}{\mu}}} \tag{1.15}$$

Finally, to obtain a clear picture of the reputational dynamics, I provide an example of a simulated trajectory in Figure 1.2.

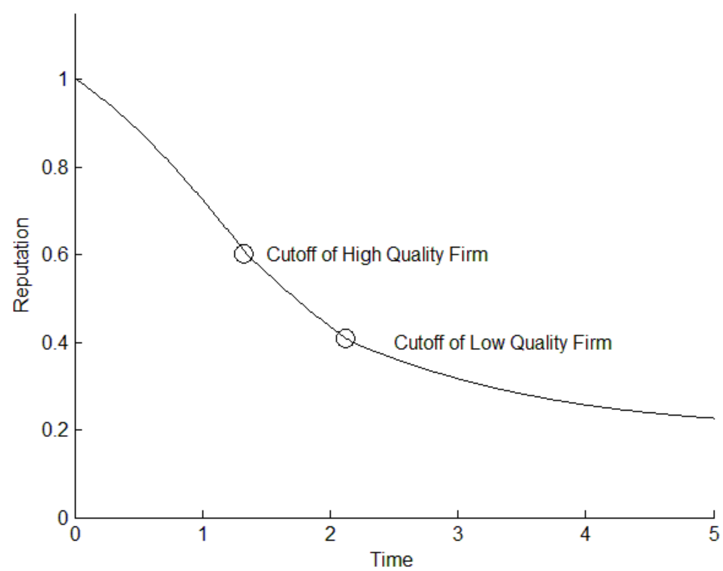


Figure 1.2: An Example of Simulated Trajectory

The closed-form trajectory in Theorem 2 allows us to calculate the value of information regarding capacity. Specifically, I assume that a complete record of firm capacity is collected by a third party and sold to consumers of measure 0 at equilibrium price. Because the dissemination of information is restricted within a small group of consumers, the firm's equilibrium strategy is unchanged. Consumers can use the firm's capacity history to track the firm's innovations, which improves their estimates of the firm's quality. The value, or the equilibrium price, of the record of the firm's capacity history is calculated in Proposition 2.

Proposition 2: In equilibrium, value of the record of the firm's capacity history, denoted by V_c, is:

$$V_c(x) = \begin{cases} 0 \text{ if } x \in (x_L, 1] \\ x(1-x) \text{ if } x \in (x_H, x_L] \\ \frac{x_H(1-e^{-\mu t_H}+e^{-\mu t})}{x_H(1-e^{-\mu t_H}+e^{-\mu t})+(1-P_b)(1-x_H)}(1-x) \text{ if } x \in (\frac{\lambda}{\mu}, x_L] \end{cases}$$

where $P_b = \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t_H} - e^{-\mu t-\lambda t_H} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$, t_L, t_H, and t follows the trajectory given by Theorem 2 such that $x_{t_L} = x_L$, $x_{t_H} = x_H$, and $x_t = x$.

Proof: In Appendix A.6.

1.4 Simulation

I assume that the interest rate $r = 0.2$, technology shock arrival rate $\lambda = 0.2$, breakthrough arrival rate $\mu = 1$, and managerial efficiency $M = 0.1$. To make the figure clear as possible, I assume that $H = 0.8$, $L = -0.2$ (instead of $H = 1$, $L = 0$ as in Section 2 and 3). This assumption does not affect the character of the equilibrium, but it does decrease the value of the firm along the entire spectrum, thus meaning that the high-quality firm does not to shift to expansion too rapidly.

The numerical result is shown in Figure 1.3, which illustrates firm value as a function of firm reputation, for both high- and low-quality firms. The high-quality firm's value is higher than the low-quality firm's value, but its cutoff point is lower than that of the low-quality firm. Both firms exhibit Innovate-expand behavior, $x_H = 0.23$ and $x_L = 0.92$.

I conduct a static analysis with respect to M. Fixing all parameters except M, I examine a firm's value in response to changes in managerial efficiency M (specifically, $M = 0$, 0.02, 0.04, 0.06, 0.07, 0.08, 0.09), the result is as follows:

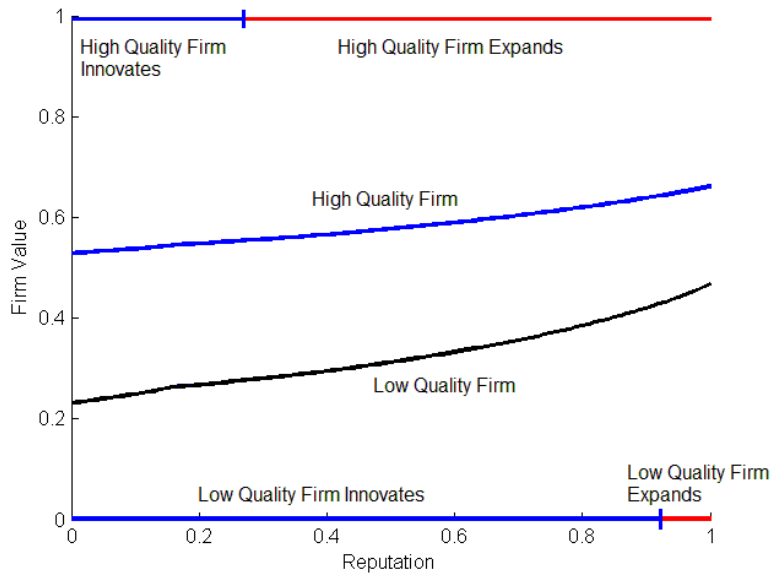


Figure 1.3: Firm's Value and Strategy

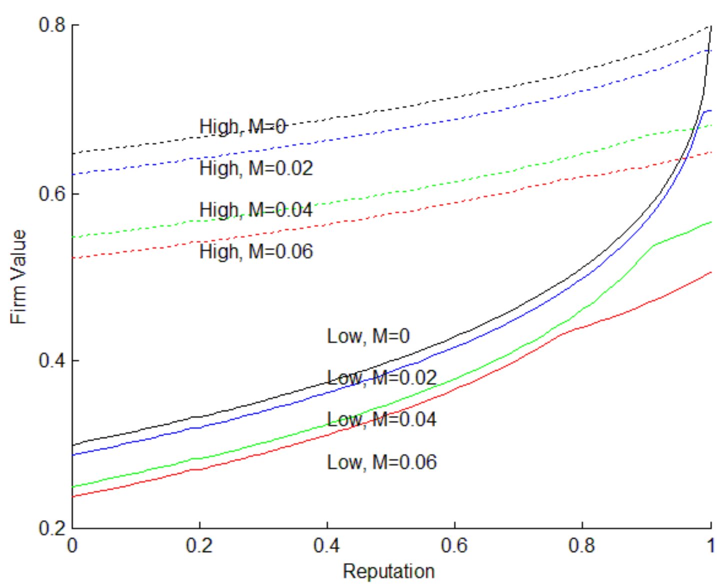


Figure 1.4: Firm Value with Respect to Different M (1)

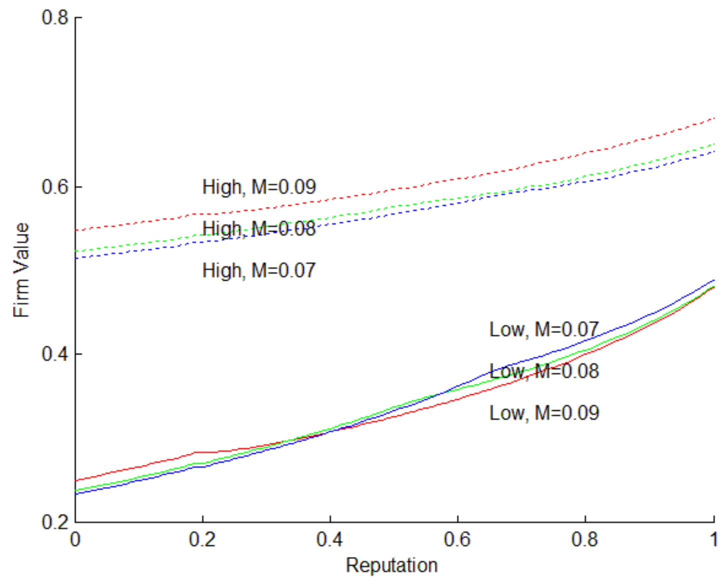


Figure 1.5: Firm Value with Respect to Different M (2)

From Figure 1.4, the value of a firm with perfect reputation is 0.8 when $M = 0$, regardless of firm quality. The firm must invest everything in quality and is certain to receive $x = 1$ thereafter, which yields a dividend of $x - c = 0.8$. As M increases from 0, firm value decreases, which confirms Proposition 1. From Figure 1.5, as M increases further, the value of a high-quality firm and of a low-quality, low-reputation firm increases, while the value of a low-quality, high reputation firm continues to fall.

The intuition for this result can be drawn from Theorem 1.(2): a low-quality firm's expected intensity of innovation is lower than the true value, while for a high-quality firm, the reverse is true. Although a low-quality firm invests while a high-quality firm does not, their reputations decline at the same rate. This reduces the slope of the low-quality firm's value curve (relative to the case in which both types are innovating) and, consequently, reduces the value of the firm.

In our model, information asymmetry exists in both a firm's product quality and in a firm's actions. From a moral hazard perspective, a firm's incentive to innovate is dampened because actions are unobservable. From an adverse selection perspective, a high-quality firm's incentive to innovate is reduced to a greater extent than that of its low-type counterpart, i.e., $x_H < x_L$.

As firms and customers have asymmetric information, high-quality firms will choose the action that benefits them the most, at the expense of low quality firms. Specifically, as a high-quality firm expands, it bears a cost because the perceived intensity of innovation would be reduced. However, the market can't differentiate between high- and low-quality firms. A low-quality firm would share a high-quality firm's cost of expansion because its perceived intensity of innovation is also reduced and its reputation now decreases more rapidly.

Comparative statics for other parameters are incorporated in Appendix A.7.

1.5 Conclusion

I adopt a capital-theoretic model of persistent quality, which allows us to introduce and compare juxtaposed channels of improvement, innovation, which improves quality and reputation, and expansion, which increases capacity. From a firm's perspective, quality (and reputation) and capacity are complements. Thus, the incentive to improve one increases in the level of the other. In our baseline model, in which a firm's value is linear in capacity, a firm's decision function is independent of capacity. I show that any equilibrium is a so-called "Innovate-expand" equilibrium, meaning that the firm innovates when its quality and reputation are low and expands when they are high.

Expansion yields more profit for a high-quality firm, which is therefore more eager to increase capacity. Consequently, at intermediate reputation levels, the low-quality firm innovates while the high-quality firm expands. As quality is unobserved, the market overestimates a high-quality firm's intensity of innovation and underestimates that of a low-quality firm. Thus, the high-quality firm expands at the expense of the low-quality cost and exploits pooling.

The availability of expansion is not always a blessing. Expansion distracts a firm's attention from quality development, hinders innovation and, if the efficiency of expansion is sufficiently low, depreciates the value of reputation throughout the industry and reduces firm value.

Finally, I investigate a model variant, in which I replace the strong linearity assumption with increasing/decreasing returns to scale. Consequently, capacity becomes effective in the information structure. I show that, under a sufficiently low efficiency of expansion, the ”Innovate-expand” feature is maintained. Specifically, on each equilibrium path, the firm innovates when its quality is low, reputation is low, and capacity is high; it expands when its quality is high, reputation is high, and capacity is low.

Referring to the American Apparel case I mentioned at the beginning, after its surge in capacity and size during its first years of operation, the firm's capacity became high, but its reputation and quality declined over time. Thus, it was likely that, from then on, innovation could have yielded higher dividends than continuous expansion, meaning that the firm should have slowed down and shifted more effort into new business and designed new styles. However, American Apparel simply continued to expand. The idea that quality and capacity are complements also implies that size itself will not protect a firm from taking a loss, but a sensibly adjusted flow of effort spent in the two channels will.

Extensions of this model are in progress and capture additional aspects of a firm's decision making. In a following chapter, I generalize this model to a continuum of firms and study the steady-state distribution of reputation, quality, and capacity. Another work in progress also focuses on the problem for a continuum of firms, introducing downward-sloping demand curve, which causes the firms to compete with each other.

This model has various empirical applications. The estimation of model parameters characterizes and records changes in the environment of a certain industry. From the firm's perspective, successful decision-making rules are important in formulating guiding principles for corporate governance.

1.6 Appendix

A.1 Proof of Lemma 1

I consider two independent firms with identical initial quality θ_0 and reputation x_0. Suppose that their initial capacities are q_0 and q_0', respectively.

The proof is in two steps. The first step proves that the value of firm 2, if it mimics the firm 1's strategy, is q_0'/q_0 times the value of its counterpart; the second step proves that if a strategy is the best strategy for firm 1, it is also the best strategy for firm 2.

Step 1: Suppose firm 2 chooses the non-Markovian strategy $\langle\eta\rangle$ that mimics the equilibrium intensity of innovation of firm 1, i.e., for all t and history $h^{t\text{-}}$, $\eta_t'(\theta_t(h^{t\text{-}}), x_t(h^{t\text{-}}), q_t'(h^{t\text{-}})) = \eta_t(\theta_t(h^{t\text{-}}), x_t(h^{t\text{-}}), q_t(h^{t\text{-}}))$. Adopting this strategy, firm 2's quality θ_t' and reputation x_t' are governed by the same process as the equilibrium reputation θ_t and capacity x_t of firm 1; $q_t' = q_t \cdot (q_0'/q_0)$. Then the profit of firm 2 under this strategy is q_0'/q_0 times the profit of firm 1 for all t because its revenue and cost equal q_0'/q_0 times firm 1's revenue and cost, respectively; thus, the value of firm 2 adopting the mimic strategy, denoted by $V\prime$, is q_0'/q_0 times the value of the firm 1, i.e.,

$$\begin{aligned} V_t'(x,\theta,q_t') &= E[\int_0^t e^{-rs} x_s q_s' ds] \\ &= (q_0'/q_0)E[\int_0^t e^{-rs} x_s q_s ds] \\ &= (q_0'/q_0)V_t(x,\theta,q_t). \end{aligned} \tag{1.16}$$

Step 2: Suppose otherwise. Suppose that firm 1's optimum strategy is $\langle\eta\rangle$ at time t; according to step 1, firm 2 can, at least, mimic this strategy to secure a value of $\frac{q_0'V(x,\theta,q_0)}{q_0}$. Thus, $V(x,\theta,q_0') \geq \frac{q_0'V(x,\theta,q_0)}{q_0}$. If $V(x,\theta,q_0') > \frac{q_0'V(x,\theta,q_0)}{q_0}$, I can conclude that firm 2 has a better strategy $\langle\eta'\rangle$, which offers the firm a higher value. However, firm 1 can also mimic firm 2's "better strategy" and secure a value of $V'(x,\theta,q_0) = \frac{q_0V(x,\theta,q_0')}{q_0'} > V(x,\theta,q_0)$, contradicting the statement that $\langle\eta\rangle$ is firm 1's optimum strategy.

In conclusion, both firms adopt the same Markov-Perfect-Equilibrium strategy, and $V(x,\theta,q_0') = \frac{q_0' V(x,\theta,q_0)}{q_0}$; set $q_0' = 1$, I obtain $V(x,\theta,q_0) = q_0 V(x,\theta,1)$.

A.2 Proof of Lemma 5

I expand the firm's current value into its profits over $[t,t+dt)$ and its expected continuation value:

$$\begin{aligned} V(x_t,H,q_t) &= E[\int_t^{+\infty} e^{-rs} x_s q_s ds] \\ &= x_t q_t dt \\ &\quad +(1-rdt)\{\mu dt V(1,H,q_{t+dt}) \\ &\quad +[1-\mu dt-(1-\eta_t)\lambda dt]V(x_{t+dt},H,q_{t+dt}) \\ &\quad +\lambda dt(1-\eta_t)V(x_{t+dt},L,q_{t+dt})\} \end{aligned} \tag{1.17}$$

$$\begin{aligned} V(x_t,L,q_t) &= E[\int_t^{+\infty} e^{-rs} x_s q_s ds] \\ &= x_t q_t dt \\ &\quad +(1-rdt)[(1-\lambda\eta_t dt)V(x_{t+dt},L,q_{t+dt}) \\ &\quad +\lambda\eta_t dt V(x_{t+dt},H,q_{t+dt})\} \end{aligned} \tag{1.18}$$

where $V(x_t,\theta_t,q_t)$ denotes the value of a firm with reputation x_t, quality θ_t, and capacity q_t in equilibrium; x_{t+dt} is the updated reputation without a breakthrough, $x_{t+dt} = x_t + \lambda(\tilde{\eta}_t - x_t)dt - \mu x_t(1-x_t)dt$; $q_{t+dt} = q_t[1+(1-\eta_t)Mdt]$.

The firm's decision to innovate may differ across types. I use η_t^θ, $\theta \in \{L,H\}$ to denote the firm's intensity of innovation at time t, given that the firm's quality at time t is θ, i.e., $\eta_t^\theta = \eta(x_t,\theta)$. Similarly, I use q_{t+dt}^θ, $\theta \in \{L,H\}$, to denote the firm's capacity at time $t+dt$, given that the firm's quality at time t is θ. By definition, $q_{t+dt}^\theta = q_t[1+(1-\eta_t^\theta)Mdt]$. Note that x_{t+dt} is the reputation at $t+dt$ without a breakthrough and thus is deterministic, and does not rely on a firm's quality.

Take the derivative of η. The marginal revenue of innovation over $[t, t+dt)$ is represented by: $(1-rdt)[V(x_{t+dt}, H, q^H_{t+dt}) - V(x_{t+dt}, L, q^L_{t+dt})]\lambda dt$; by taking limits, I obtain:

$$\lambda \Delta(x_t) q_t dt + o(dt) \tag{1.19}$$

which is identical for a high-quality firm and a low-quality firm.

The revenue yielded by expansion over $[t, t+dt)$ is a somewhat more complex to determine; for a high-quality firm, it is $(1-rdt)Mq^H_t dt\{\mu dt V(1, H, q_t) + [1 - \mu dt - (1 - \eta^H_t)\lambda dt]V(x_{t+dt}, H, q_t)] + \lambda dt(1-\eta^H_t)V(x_{t+dt}, L, q_t)\}$; by taking limits, I obtain:

$$V(x_t, H, q_t) M q_t dt + o(dt) \tag{1.20}$$

for a low-quality firm, the revenue is $(1-rdt)Mq^L_t dt[\lambda \eta^L_t dt V(x_{t+dt}, H, q_t) + (1 - \lambda \eta^L_t dt) V(x_{t+dt}, L, q_t)]$; by taking limits I obtain:

$$V(x_t, L, q_t) M q_t dt + o(dt) \tag{1.21}$$

In conclusion, the marginal revenue from expansion for a firm with reputation x, quality θ, and capacity q can be represented by $MV(x, \theta, q)$. Innovation over $[t, t+dt]$ increases the probability of being high quality by λdt and therefore yields the firm a revenue of $\lambda \Delta(x) q dt$. However, expansion over $[t, t+dt]$ yields the firm a revenue of $MV(x, \theta, 1) q dt$. Then, the lemma is reduced to a quick comparison of the marginal profits of the two options.

A.3 Proof of Lemma 7

Due to the complexity of the explicit form of the value function, I do not demonstrate the uniqueness of the equilibrium; instead, I prove that a candidate equilibrium is valid.

To analyze the value of quality $\Delta(x) = V(x, 1, 1) - V(x, 0, 1)$, I expand the value functions into current profits and continuation values (as in the proof of Lemma 5). Current profits cancel out because both current revenue and costs depend on reputation but not on quality. I also retain the terms η_t^θ, $\theta \in \{L,H\}$, and q_{t+dt}^θ, $\theta \in \{L,H\}$ from the proof of Lemma 5.

Namely, the value of quality is:

$$
\begin{aligned}
V(x_t,H,q_t)-V(x_t,L,q_t) &= q_t\Delta(x_t) \\
&= q_t\int_0^{\bar{t}} e^{-[r+\lambda(1-\eta_t^H)+\mu-M(1-\eta_t^H)]s} \\
&\quad [x_{t+s}+\lambda(1-\eta_t^H)V(x_{t+s},L,1)+\mu V(1,H,1)]ds \\
&\quad +e^{-(r+\lambda+\mu)\bar{t}}V(x_{\bar{t}},H,q_{\bar{t}}^H)/q_t \\
&\quad -q_t\int_0^{\bar{t}} e^{-[r+\lambda\eta_t^L-M(1-\eta_t^L)]s} \\
&\quad [x_{t+s}+\lambda\eta_t^L V(x_{t+s},H,1)]ds \\
&\quad -e^{-(r+\lambda+\mu)\bar{t}}V(x_{\bar{t}},L,q_{\bar{t}}^L)/q_t
\end{aligned} \tag{1.22}
$$

where $\bar{t}$ is the time at which reputation, in the absence of a breakthrough, falls below a threshold, and hence that the firm shifts to a different intensity of innovation.

Rearrange (3.9):

$$
\begin{aligned}
V(x_t,H,q_t)-V(x_t,L,q_t) &= q_t\Delta(x_t) \\
&= q_t\int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
&\quad [x_{t+s}+\lambda(1-\eta_t^H)V(x_{t+s},L,1)+\mu V(1,H,1) \\
&\quad +\lambda\eta_t^H V(x_{t+s},H,1)+M(1-\eta_t^H)V(x_{t+s},H,1)]ds \\
&\quad +e^{-(r+\lambda+\mu)\bar{t}}V(x_{\bar{t}},H,q_t) \\
&\quad -q_t\int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
&\quad [x_{t+s}+\lambda\eta_t^L V(x_{t+s},H,1)+\mu V(x_{t+s},L,1) \\
&\quad +\lambda(1-\eta_t^L)V(x_{t+s},L,1)+M(1-\eta_t^L)V(x_{t+s},L,1)]ds \\
&\quad -e^{-(r+\lambda+\mu)\bar{t}}V(x_{\bar{t}},L,q_t)
\end{aligned} \tag{1.23}
$$

By Lemma 6, $\eta_t^H - \eta_t^L \leq 0$, for any reputation x, the firm implements one of the following actions:

$$\begin{aligned}
\eta(x,H) &= 1,\ \eta(x,L) = 1 \\
\eta(x,H) &= 0,\ \eta(x,L) = 1 \\
\eta(x,H) &= 0,\ \eta(x,L) = 0 \qquad (1.24)
\end{aligned}$$

Case 1: $\eta(x,H) = 1$, $\eta(x,L) = 1$.

Use (3.11), plug in the numbers:

$$\begin{aligned}
\Delta(x_t) &= \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
&\quad [x_{t+s} + \mu V(1,H,1) + \lambda V(x_{t+s},H,1)]ds \\
&\quad - \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
&\quad [x_{t+s} + \lambda V(x_{t+s},H,1) + \mu V(x_{t+s},L,1)]ds \\
&\quad + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}},H,1) - V(x_{\bar{t}},L,1)] \\
&= \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \mu[V(1,H,1) - V(x_{t+s},L,1)]ds \\
&\quad + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}},H,1) - V(x_{\bar{t}},L,1)]
\end{aligned}$$

where $\mu[V(1,H,1) - V(x_{t+s},L,1)]$ is decreasing in x_t.

Case 2: $\eta(x,H) = 0$, $\eta(x,L) = 1$.

Use (3.11), plug in the numbers:

$$
\begin{aligned}
\Delta(x_t) &= \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
& [x_{t+s} + \lambda V(x_{t+s}, L, 1) \\
& + \mu V(1, H, 1) + MV(x_{t+s}, H, 1)]ds \\
& - \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s} \\
& [x_{t+s} + \lambda V(x_{t+s}, H, 1) + \mu V(x_{t+s}, L, 1)]ds \\
& + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}}, H, 1) - V(x_{\bar{t}}, L, 1)] \\
&= \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s}[\mu V(1, H, 1) + (\lambda - \mu)V(x_{t+s}, L, 1) \\
& + (M - \lambda)V(x_{t+s}, H, 1)]ds \\
& + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}}, H, 1) - V(x_{\bar{t}}, L, 1)]
\end{aligned}
$$

As $\lambda < \mu$, assuming that $M \leq \lambda$, $\mu V(1,H,1) + (\lambda - \mu)V(x_{t+s}, L, 1) + (M - \lambda)V(x_{t+s}, H, 1)$ is decreasing in x_t.

Case 3: $\eta(x, H) = 0$, $\eta(x, L) = 0$.

Rearrange (3.9) in a slightly different way:

$$
\begin{aligned}
V(x_t, H, q_t) - V(x_t, L, q_t) &= q_t \Delta(x_t) \\
&= q_t \int_0^{\bar{t}} e^{-(r+\lambda+\mu-M(1-\eta_t^H))s} \\
& [x_{t+s} + \lambda(1 - \eta_t^H)V(x_{t+s}, L, 1) \\
& + \mu V(1, H, 1) + \lambda \eta_t^H V(x_{t+s}, H, 1)]ds \\
& + e^{-(r+\lambda+\mu)\bar{t}} V(x_{\bar{t}}, H, q_t) \\
& - q_t \int_0^{\bar{t}} e^{-(r+\lambda+\mu-M(1-\eta_t^L))s} \\
& [x_{t+s} + \lambda \eta_t^L V(x_{t+s}, H, 1) + \mu V(x_{t+s}, L, 1) \\
& + \lambda(1 - \eta_t^L)V(x_{t+s}, L, 1)]ds \\
& - e^{-(r+\lambda+\mu)\bar{t}} V(x_{\bar{t}}, L, q_t) \qquad (1.25)
\end{aligned}
$$

Plug in the numbers:

$$\begin{aligned}
\Delta(x_t) &= \int_0^{\bar{t}} e^{-(r+\lambda+\mu-M)s} \\
&\quad [x_{t+s} + \lambda V(x_{t+s}, L, 1) + \mu V(1, H, 1)]ds \\
&\quad - \int_0^{\bar{t}} e^{-(r+\lambda+\mu-M)s} \\
&\quad [x_{t+s} + \mu V(x_{t+s}, L, 1) + \lambda V(x_{t+s}, L, 1)]ds \\
&\quad + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}}, H, 1) - V(x_{\bar{t}}, L, 1)] \\
&= \int_0^{\bar{t}} e^{-(r+\lambda+\mu)s}(\mu+M)[V(1,H,1) - V(x_{t+s}, L, 1)]ds \\
&\quad + e^{-(r+\lambda+\mu)\bar{t}}[V(x_{\bar{t}}, H, 1) - V(x_{\bar{t}}, L, 1)]
\end{aligned}$$

where $(\mu+M)[V(1,H,1) - V(x_{t+s}, L, 1)]$ is decreasing in x_t.

Notice that $\lambda\Delta(x) = MV(x, H, 1)$ when a firm is in case 1 to one side of the cutoff and in case 2 to the other side; thus, at those cutoffs, I have:

$$\mu V(1,H,1) + (\lambda-\mu)V(x_{t+s}, L, 1) + (M-\lambda)V(x_{t+s}, H, 1) = \mu[V(1,H,1) - V(x_{t+s}, L, 1)]$$

Similarly, $\lambda\Delta(x) = MV(x, L, 1)$ when a firm is in case 2 to one side of the cutoff and in case 3 to the other side; thus, at those cutoffs, I have:

$$\mu V(1,H,1) + (\lambda-\mu)V(x_{t+s}, L, 1) + (M-\lambda)V(x_{t+s}, H, 1) = (\mu+M)[V(1,H,1) - V(x_{t+s}, L, 1)]$$

In conclusion, assuming that $M \leq \lambda$, I can construct a function $f(x_t)$ such that

$$\Delta(x_t) = \int_0^{\infty} e^{-(r+\lambda+\mu)s} f(x_{t+s})ds$$

where $f(x)$ is continuous and decreasing in x.

As x_t increases, $f(x_{t+s})$ decreases for $s \in [0, +\infty)$; thus, $\Delta(x_t)$ is decreasing in x.

A.4 Proof of Theorem 1

Proof of (1): I first prove that the strategy of each type is characterized by one cutoff. By Lemmas 1 and 2, the value function of the firm is homogeneous of degree 1 in quantity, and capacity, as a state variable, does not affect the firm's innovation & expansion decision. By Lemma 7, $\Delta(x)$ is decreasing in x; thus, the marginal profit from innovation $\lambda\Delta(x)$ is decreasing in x. By Lemma 3, $V(x,\theta,q)$ is increasing in x, and thus the marginal profit from capacity expansion is increasing in x.

In summary, $\lambda\Delta'(x) < 0$, $MV'(x,\theta) > 0$ for $\theta \in \{H,L\}$; moreover, when $x = 1$, $\theta = H$, $\lambda\Delta(x) < MV(x,\theta)$, and when $x = 0$, $\theta = L$, $\lambda\Delta(x) > MV(x,\theta)$. As $\Delta(x)$ is decreasing in x, $MV(x,\theta)$ is increasing in x for $\theta = L,H$, $MV(x,H)$, either cross $\lambda\Delta(x)$ once from below ($x \in [0,1]$), or I have $MV(x,H) > \lambda\Delta(x)$ for $x \in [0,1]$; similarly, $MV(x,L)$ either cross $\lambda\Delta(x)$ once from below, or I have $MV(x,L) < \lambda\Delta(x)$ for $x \in [0,1]$.

The corresponding reputations for the two intersecting points are called cutoffs, denoted by x_θ, $\theta = L$, H. The equilibrium can be characterized by $0 \le x_\theta \le 1$ for $\theta = L$, H s.t. $MV(x_\theta,\theta) = \lambda\Delta(x_\theta)$, $\theta = L,H$. As $\Delta(x)$ is decreasing in x and $MV(x,\theta)$ is increasing in x, for $x \in [0,x_\theta]$, $\lambda\Delta(x) \ge MV(x,\theta)$; for $x \in [x_\theta,1]$, $MV(x,\theta) \ge \lambda\Delta(x)$. Following Lemma 5, I have the required result.

Next, I prove $x_H \le x_L$, with strict inequality if $M > 0$, $\lambda > 0$.

When $M = 0$, the firm will always spend everything on innovation because the other option has literally no revenue, and I have a full research equilibrium, $x_H = x_L = 1$.

When $M > 0$, I can rule out the full innovate equilibrium. If $\eta(x,\theta) = 1$ for all x and θ, then $x_t = 1$ implies that $x_s = 1$ hereafter (for all $s \ge t$); thus $\Delta(1) = 0$. However, in this case, $MV(1,H,1) > 0$; following Lemma 5, a firm with perfect reputation prefers expansion. This contradicts the full innovate equilibrium assumption.

When $\lambda > 0$, I can rule out the full expand equilibrium: If $\eta(x,\theta) = 0$ for all x and θ, in the long run, quality becomes 0 almost for certain, and $x_t = 0$ implies $x_s = 0$ hereafter (for all $s \ge t$) because $dx = -\lambda x dt - \mu x(1-x)dt = 0$ for $x = 0$; thus, the revenue earned by a firm with $x = 0$, $\theta = 0$ is 0 for all $s \geq t$, $V(0,L,1) = 0$. However, in this case, $V(0,H,1) > V(0,L,1)$, $\lambda\Delta(x) > 0$ for $x = 0$, following Lemma 5, a low-quality firm with zero reputation would prefer innovation. This contradicts the full expand equilibrium assumption.

For the above reasons, when $M > 0$, $\lambda > 0$, following Lemma 6, $x_H < x_L$.

Proof of (2): For $x \in [0, x_H) \cup (x_L, 1]$, this is straightforward. For $x \in (x_H, x_L)$, high-quality firm expands, while a low-quality firm with identical reputation innovates. The market cannot recognize the firm's type, and the firm's perceived intensity of innovation $\tilde{\eta}(x)$ is the expectation of the intensity of a firm with reputation x; following equation (3.3), the perceived intensity of innovation is as in (1.9).

A.5 Proof of Proposition 1

I refer to the firm's first-best equilibrium as a benchmark. In equilibrium, the first-best strategy is given by (1.5). Suppose that $M = 0$, $MV(x,\theta,1) = 0$, then the firm's first-best strategy is investing in quality regardless of its quality and reputation. Any deviation from that would yield a loss in the firm's expected output, which is value of the firm.

When $M > 0$, following the proof of Theorem 1, I can rule out the full research equilibrium.

Suppose that the firm makes a series of deviations from innovation to expansion. Each deviation, which takes time length dt, yields a loss of:

$$q\lambda dt \int_0^\infty e^{-(r+\lambda)t} dt = \frac{\lambda q}{r+\lambda} dt \tag{1.26}$$

and a revenue of:

$$MdtV(x,\theta,q) \tag{1.27}$$

The instantaneous value of the deviation is $[MV(x,H,q) - \frac{\lambda q}{r+\lambda}]dt$, as $V(x,\theta,q)$ is bounded above by $\frac{q}{r-M}$, for $M < \frac{(r-M)\lambda}{r+\lambda}$, the revenue is negative for any x and θ.

The firm's value can be regarded as the value when $M = 0$ plus the discounted expected value of a series of deviations from innovation, all of which are negative. Thus, for a positive and sufficiently small M, any equilibrium value of the firm is less than when $M = 0$.

A.6 Proof of Proposition 2

Consider $x_0 = 1$; absent a breakthrough, reputation declines to x_L at time t_L, then x_H at time t_H. On the equilibrium path, full information of the history of capacity is not valuable until $x_t \in (x_H, x_L]$ (or, after t_L), when quality is immediately disclosed by firm's action.

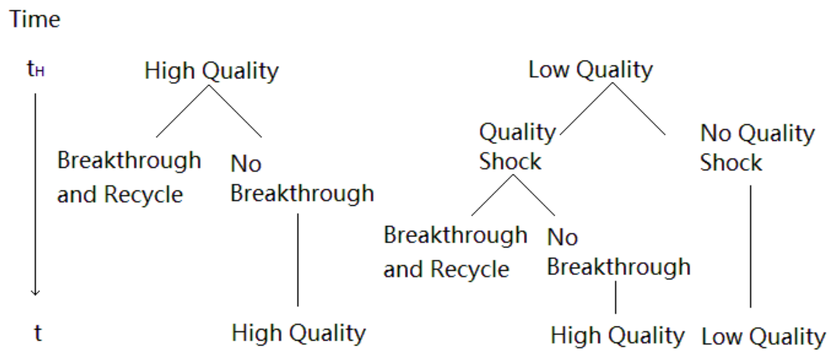
Suppose that the history is sold at price p and that the consumer can choose to buy or not to buy. If the consumer chooses not to buy the history, then he can spend x_t on the product in exchange for 0, with probability $1 - x_t$, or 1, with probability x_t; the consumer's expected payoff is 0. If he buys the history, the firm's quality is revealed. With probability x_t, the firm is good, and the consumer spends x_t on the good to obtain 1; with probability $1 - x_t$, the firm is bad, and the consumer shirks, i.e., spends 0 and obtains 0. The expected payoff is $x_t(1 - x_t) - p$. In equilibrium, $p = x_t(1 - x_t)$.

When firm reputation declines to $x_t \in (\frac{\lambda}{\mu}, x_H]$, the capacity history is valuable in the sense that I can track the accurate quality of the firm up to time t_H. The unobserved history from t_H to t, however, is depicted in the following figure.

The probability that, in $[t_H, t]$, a high-quality firm obtains a breakthrough and returning to the beginning of our recursive equilibrium is $\int_{t_H}^{t} \mu e^{-\mu s} ds$. The probability that a low-quality firm jumps to become a high type and obtains a breakthrough thereafter is:

$$
\begin{aligned}
P_b &= \int_{t_H}^{t} \lambda e^{-\lambda s} \int_{s}^{t} \mu e^{-\mu w} dw ds \\
&= \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t_H} - e^{-\mu t - \lambda t_H} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t}
\end{aligned}
$$

By Bayesian updating, the probability that a firm I investigate at time t (i.e., it has not



received a breakthrough since t_H) was a high-quality one at time t_H is

$$x_{t_H}|h^t = \frac{x_H(1-e^{-\mu t_H}+e^{-\mu t})}{x_H(1-e^{-\mu t_H}+e^{-\mu t})+(1-P_b)(1-x_H)}$$

where $x_{t_H}|h^t$ means updated reputation for the firm at time t_H given the public information available at time t.

Again, suppose that the capacity history record is sold at price p. If a consumer chooses not to buy the history, he can spend x_t on the good in exchange for 0, with probability $1-x_t$, or 1, with probability x_t; the expected payoff is 0. If he buys the history, the firm's reputation is updated as follows:

With probability $x_{t_H}|h^t$, he finds out that the firm was good at time t_H. Because $\eta(x,\theta) = 1$ for $x < x_H$, high quality at time t_H is preserved for certain until the next breakthrough. The consumer then spends x_t on the good to obtain 1.

With probability $1-x_{t_H}|h^t$, he finds out that the firm was bad at t_H, and the updated reputation for the firm at time t is $x_t|(\theta_{t_H}=L) = \frac{e^{-\lambda t_H}-e^{-\lambda t}-P_b}{1-e^{-\lambda t_H}+e^{-\lambda t}}$, where $x_t|(\theta_{t_H}=L)$ denotes updated firm reputation at time t given low quality at time t_H (revealed by the capacity history record). The denominator represents the probability that the firm has jumped to become the high type during $[t_H, t]$ but has not received a breakthrough thereafter; the denominator represents the probability that low quality is preserved for $[t_H, t]$. As $x_t | (\theta_{t_H} = L) < x_t$, the expected payoff of buying the product at time t is negative, and thus the consumer shirks.

The consumer's overall expected payoff from purchasing the capacity history is $(x_{t_H} | h^t)(1 - x_t) - p$. In equilibrium, $p = (x_{t_H} | h^t)(1 - x_t)$.

A.7 Comparative Statics for Parameters other than M

In the following section, I conduct comparative static analysis with respect to different parameters. Due to the lack of uniqueness of the equilibrium, I cannot evaluate analytically the equilibrium strategy or firm value, and all of the following comparative statics are numerical results and thus descriptive rather than analytical. The goal of the following section is to highlights some insights of the model given common parameters, which may be instructive for empirical experiments.

The parameters investigated include r, μ, and λ. The outputs are x_H and x_L.

r : as r increases, incentive to engage in both types of improvement decreases, the value of the firm decreases across the spectrum. Notably, x_H increases. Intuitively, the value of expansion is a series of dividends spread across the entire future, while the dividends yielded by innovation are concentrated in the near future, prior to the next technology shock. Thus, the value of innovation is more prone to interest rate risk.

r	x_H	x_L
0.2	0.19	0.92
0.25	0.37	0.93
0.3	0.44	0.94
0.35	0.48	0.95
0.4	0.50	0.95

μ : as μ increases, x_θ increases. A higher μ makes research more valuable immediately, while changes in the value of expansion result from changes in the value of the firm, which is ambiguous due to complex co-movements among a number of factors.

μ	x_H	x_L
1	0.19	0.92
1.1	0.26	0.94
1.2	0.30	0.96
1.3	0.35	0.97
1.4	0.39	0.99

λ : the result is ambiguous as λ increases. Although λ directly affects the value of research, an increase in λ also narrows the gap between the high type and low type, inducing ambiguity into the direction of changes in incentives.

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CHAPTER 2

Model Variants and Extensions of the Reputational Model of Firm Size and Product Quality

2.1 Introduction

In real businesses, quality and capacity are among a firm's most prominent concerns. To improve quality, a firm conducts research and development programs, develops new marketing strategies, and upgrades its manufacturing line. We term such behaviors innovation. To secure a larger market share, the firm builds more factories and opens more physical stores. We term such behaviors expansion.

Chapter 1 presents a model of firm dynamics based on information asymmetry. Specifically, I study a firm's choice between innovation and expansion. As consumers do not directly observe the firm's quality, the firm has an incentive to cheat. I describe the Markovian equilibrium for the firm's quality/capacity trade-off under information asymmetry in a reputational framework. My main result is that a firm innovates when its quality and reputation are low and when its capacity is high; it expands when its quality and reputation are high and capacity is low.

Following the results in Chapter 1, this second chapter investigates a few model variants and extensions.

In section 2, I investigate a model variant in which firm's actions, i.e., innovation and expansion, involve a non-trivial cost. As taking action is costly, the firm may have an incentive to shirk. I characterize how the incentives for innovation and expansion depend on a firm's current reputation, quality, and capacity. I show that in equilibrium, the firm's optimal strategy takes an "innovate-shirk-expand" shape, in contrast to the "innovate expand" equilibrium observed in the first chapter.

In section 3, I investigate a model variant in which expansion is, in contrast to the assumption in Chapter 1, not linear in capacity. Now, the firm's investment decision relies on both the firm's reputation and capacity. I show that the firm's status can now be characterized by its quality, capacity, its capacity at the last breakthrough, and the time elapsed since the last breakthrough. I show that, on each equilibrium path, the firm innovates when its quality is low, reputation is low, and capacity is high; it expands when its quality is high, reputation is high, and capacity is low.

In section 4, I consider a follow-up question: given the firm's innovation and expansion behavior, what are the implications for the industry-wide equilibrium? In this regard, I study an industry populated by such firms, and I discover that my model can help formulate a more realistic industry equilibrium. Specifically, I explain several stylized facts in the industrial organization literature. First, most research and development efforts are made by large, mature firms rather than by small entrepreneurs. My model shows that higher capacity implies greater incentives to improve quality: a firm leans toward innovation when its capacity is high. Second, within the same generation of firms, the average size of survivors grows over time. In my model, survival implies relatively high quality and reputation through the firm's history; thus, such firms have had more opportunities to grow. The computational results and numerical simulation show that my model reproduces a realistic stationary industry-wide distribution. Finally, both our computational and numerical results show that the distribution of firm size and quality is as follows: a large share of low-reputation, low-quality firms lie at the bottom, while a few pioneers with high quality and high capacity are found at the top.

2.1.1 Literature

The paper adopts the reputation framework of Board and Meyer-ter-Vehn (2010, 2013). In contrast to those papers, I assume that the firm can expand its capacity, introducing juxtaposed options for improvement competing for managerial attention. This idea of multitask operation draws on Holmstrom and Milgrom (1991), who analyze a principal-agent model in which the principal has several different tasks for the agent to perform.

This paper is also related to other studies on firm/industry dynamics. Jovanovic (1982) proposes a selection model for industry evolution. Firms enter without knowing their true type and learn their type as they produce in the industry. Over time, less-efficient types will realize that they are less efficient; they then produce less and, eventually, exit. Ericson and Pakes (1995) analyze a problem in which a countable number of firms with heterogeneous productivity levels serve a single industry. Hopenhayn (1992) assumes that firm capabilities change over time according to a Markov process and examines the resulting entry and exit patterns.

2.2 Model Variant 1: Cost not low

2.2.1 Model

In this section, I closely follow the baseline model from Chapter 1, except for the following changes:

I introduce a positive cost c that is identical for innovation and expansion. As the cost of work (innovation or expansion) is non-trivial, the firm may have an incentive to shirk. To model this, I assume that the firm chooses its intensity of innovation, denoted by $\eta_{1t} \in [0,1]$, and intensity of expansion, denoted by $\eta_{2t} \in [0,1]$, $\eta_{1t} + \eta_{2t} \leq 1$. The total cost of the firm at time t is $q_t c(\eta_{1t} + \eta_{2t})dt$, where q_t is the size of the firm.

Absent a breakthrough, the increase in reputation $dx_t = x_{t+dt} - x_t$ is governed by market beliefs concerning the intensity of innovation $\tilde{\eta}_1$; dx_t is deterministic, and by independence, it can be decomposed additively:

$$x_{t+dt} = x_t + \lambda(\tilde{\eta}_{1t} - x_t)dt - \mu x_t(1 - x_t)dt \tag{2.1}$$

Production capacity at time t is recursively expressed as follows:

$$q_{t+dt} = q_t(1 + \eta_{2t} M dt) \tag{2.2}$$

The firm's profit at time t becomes $[x_t - (\eta_{1t} + \eta_{2t})c]q_t dt$, and its value function is:

$$V(x_t, \theta, q_t) = E[\int_t^{+\infty} e^{-rs}(x_s - c\eta_{1s} - c\eta_{2s})q_s ds] \tag{2.3}$$

2.2.2 Equilibrium

In a Markovian equilibrium, we can write the firm's value as a function of its reputation, quality, and size, $V(x, \theta, q)$. A firm's action in Markov-Perfect Equilibrium $(\eta_1, \tilde{\eta}_1, \eta_2)$ then consists of a function of intensity of innovation and expansion $(\eta_1, \eta_2) : ([0,1], \{0,1\}, [1, +\infty)) \to ([0,1],\ [0,1])$ and market beliefs $(\tilde{\eta}_1) : [0,1] \to [0,1]$, such that: (1) The intensity of innovation and expansion maximizes firm value, $V(x, \theta, q)$; (2) market beliefs are correct, $\tilde{\eta}_1(x,q) = \eta_1(x,H,q) + (1-x)\eta_1(x,L,q)$.

We call x_i^* a cutoff if η_1 or η_2, or both, jumps at x_i^*.

Lemma 1: A firm's value is homogeneous of degree 1 in firm size, i.e., $V(x, \theta, q) = qV(x, \theta, 1)$.

Proof: See Appendix A.1; the proof follows closely the proof of Lemma 1 in the first chapter.

Lemma 2: The Markov-Perfect Equilibrium strategy $(\eta_1, \tilde{\eta}_1, \eta_2)$ is independent of firm size q, i.e., $\forall q \in [1, +\infty)$, $\eta_i(x, \theta, q) = \eta_i(x, \theta, 1)$ for $i = 1, 2$.

Proof: By step 2 of the proof of Lemma 1. As $\forall q_1, q_2 > 0$, $\eta_i(x, \theta, q_1) = \eta_i(x, \theta, q_2)$, we will also use term $\eta_i(x, \theta) = \eta_i(x, \theta, 1)$ to denote the firm's investment decision, $i = 1, 2$.

Lemma 3: In any equilibrium $\langle \eta_1, \eta_2, \tilde{\eta}_1 \rangle$, the value function of the firm $V(x, \theta, q)$ is strictly increasing in reputation x.

Proof: See Appendix A.2

We define $\Delta(x)$ as the value of quality, i.e.,

$$\Delta(x) = V(x, H, 1) - V(x, L, 1) \tag{2.4}$$

Because $V(x, \theta, q) = qV(x, \theta, 1)$, $V(x, H, q) - V(x, L, q) = q\Delta(x)$.

A strategy that maximizes the integrand in (2.3) pointwise must satisfy the following:

$$\begin{aligned} \eta_1(x, \theta) &= \begin{cases} 0 \text{ if } \lambda\Delta(x) < c \text{ or } \lambda\Delta(x) < MV(x, \theta, 1) \\ 1 \text{ if } \lambda\Delta(x) > c \text{ and } \lambda\Delta(x) > MV(x, \theta, 1) \end{cases} \\ \eta_2(x, \theta) &= \begin{cases} 0 \text{ if } MV(x, \theta, 1) < c \text{ or } MV(x, \theta, 1) < \lambda\Delta(x) \\ 1 \text{ if } MV(x, \theta, 1) > c \text{ and } MV(x, \theta, 1) > \lambda\Delta(x) \end{cases} \end{aligned} \tag{2.5}$$

where $\Delta(x) = V(x, H, 1) - V(x, L, 1)$ is the value of quality. A candidate equilibrium $\langle \eta, \tilde{\eta} \rangle$ is an equilibrium if and only if the above equation holds for all (θ, x).

Lemma 4: In equilibrium, $\eta_1(x, 0) \geq \eta_1(x, 1)$, $\eta_2(x, 1) \geq \eta_2(x, 0)$ for $x \in [0, 1]$.

Proof: See Appendix A.3.

Lemma 5: Suppose $M < \mu$; in equilibrium, $\Delta(x)$ is decreasing in x.

Proof: See Appendix A.4.

Theorem 1: Suppose $M < \mu$; in equilibrium, the optimal intensity of innovation/expansion is characterized by cutoffs $0 \leq \underline{x}_H \leq \bar{x}_H \leq 1$ and $0 < \underline{x}_L \leq \bar{x}_L \leq 1$ such that a firm of quality θ

$$
\begin{aligned}
\text{innovates if } x &\in [0, \underline{x}_\theta] \\
\text{shirks if } x &\in [\underline{x}_\theta, \bar{x}_\theta] \\
\text{expands if } x &\in [\bar{x}_\theta, 1]
\end{aligned}
\tag{2.6}
$$

Specifically, $\bar{x}_H \leq \bar{x}_L$, $\underline{x}_H \leq \underline{x}_L$.

We call this equilibrium strategy research-shirk-expand. We call $(0, \underline{x}_\theta)$ the "research region", $(\underline{x}_\theta, \bar{x}_\theta)$ the "shirk region", and $(\bar{x}_\theta, 1)$ the "expand region". It is possible to have a trivial research region, i.e.,. $\underline{x}_\theta = 0$, a trivial shirk region, i.e., $\underline{x}_\theta = \bar{x}_\theta$, or a trivial expansion region, i.e., $\bar{x}_\theta = 1$.

Proof: See Appendix A.5. For a more detailed discussion of the shape of the equilibrium, see Appendix A.6.

2.3 Model Variant 2: Non-Constant Returns to Scale

2.3.1 Model

Our baseline model benefits from assuming the linearity of capacity. The firm has constant returns to scale, and capacity does not enter the firm's decision-making function. Therefore, consumers' belief regarding the firm's capacity is payoff irrelevant and can thus be deleted from the information structure.

In this section, I remove the strong linearity assumption. Specifically, I assume that, given the intensity of expansion $(1-\eta)$, the amount of capacity acquired by the firm is $dq = M(1-\eta)dt$, which is constant, rather than linear in q. The evolution of capacity becomes $q_{t+dt} = q_t + M(1-\eta)dt$.

As expansion is non-linear, Lemma 1 and Lemma 2 are no longer valid, and a firm's action in equilibrium is dependent on its capacity.

To simplify the question, I assume that a breakthrough also reveals a firm's capacity. At a breakthrough, reputation is reset to $x = 1$ and the firm's perceived capacity is reset to its current capacity; no further information arrives between breakthroughs. Further, I maintain the assumption that r measures the aggregate effect of a value discount and a positive death rate, such that a model containing multiple firms converges to a steady state while the individual firm's problem remains unchanged.

Formally, the firm, the consumers, and the action space remain the same as in "Firm and Customers" in Chapter 1, Section 2. The information structure, however, is reorganized as follows:

I use $\langle\hat{\eta}\rangle$ to denote a firm's Markovian strategy. A firm conditions its actions on (1) the public history h^{t-}, (2) θ_t, the firm's current quality, and (3) q_t, the firm's current capacity.

As reputation is reset to 1 and perceived capacity is reset to the actual value at breakthroughs, I consider a recursive strategy that relies only on the public history after the most recent breakthrough and current states, i.e., a firm conditions its actions on (1) t, the time elapsed since the last breakthrough, (2) θ_t, the firm's current quality, (3) q_0, the firm's capacity revealed at the last breakthrough, and (4) q_t, the firm's current capacity. Thus, I focus on a cycle between two breakthroughs: that is, I begin from $t = 0$, $\theta = \theta_0$, $x_0 = 1$, and $q = q_0$, and no breakthrough arrives thereafter. Similar to the case in Theorem 2, suppose that the last breakthrough before t was at $s < t$; I call strategy $\langle\hat{\eta}\rangle$ recursive if there exists strategy $\langle\eta\rangle$ such that $\hat{\eta}(t,\theta,q_0,q) = \eta(t-s,\theta,q_0,q)$ always.

As in Chapter 1, Section 2, I assume that for every public history, there is a market belief concerning the firm's past intensity of innovation. In a Markovian sense, the public history contains (1) t, the time elapsed since the last breakthrough, and (2) q_0, the firm's capacity revealed at the last breakthrough. Thus, I denote the belief regarding the intensity of innovation as $\tilde{\eta}(t,q_0)$, which is deterministic with respect to public histories. $\tilde{\eta}(t,q_0)$ and the exogenous initial belief regarding quality $x_0 = 1$ control the joint distribution of quality and histories h^t.

The firm's reputation is the probability that the firm is of the high type considering the public history and belief regarding the intensity of innovation, i.e., $x_t = E_{\langle\tilde{\eta}\rangle}(\theta_t|h^t)$. Specifically, a firm's reputation depends on the public information described above. Moreover, a firm's reputational dynamics follow a recursive process.

To keep the analysis tractable, I assume that $\lambda < \mu$ and that M is sufficiently small. In this way, I assure that a firm innovates once its reputation falls below some cutoff higher than $\frac{\lambda}{\mu}$, regardless of other state variables (for a rigorous proof see Claim 2 in Appendix A.7). Between breakthroughs, firm reputation is always decreasing in t and finally converges to $\frac{\lambda}{\mu}$.

Before turning to the analysis of the equilibrium, one final remark is that the above assumption of non-linearity would also suffice for other economic contexts with minor adjustments. Most notably, one could extend it to the case in which the cost of innovation does not increase in a firm's capacity, i.e., a large firm enjoys an economy of scale. The firm's problem is to compare the incentives to innovate and expand. Suppose that I rewrite the budget constraint and denote the intensity of innovation as η_1 and the intensity of expansion as η_2. Then, in the baseline model, a firm's budget constraint is $\eta_1 + \eta_2 = 1$. The model in Section 5, however, is the same as if I assume the budget constraint $\eta_1 + q\eta_2 = 1$. Finally, in the case in which the cost of innovation is constant, I have $\frac{1}{q}\eta_1 + \eta_2 = 1$. Note that the equilibrium substitution rate between innovation and expansion should be the same in the later two cases. The only difference is the budget, which affects reputational dynamics.

2.3.2 Equilibrium

In a Markovian equilibrium, a firm's value function is represented by $V(t,\theta,q_0,q)$. A firm's innovation in a Markov-Perfect-Equilibrium $(\eta,\tilde{\eta})$ then consists of an function of intensity of innovation $\eta : ([0,+\infty),\{L,H\},[1,+\infty),[1,+\infty)) \to [0,1]$, and market beliefs $\tilde{\eta} : ([0,+\infty),[1,+\infty)) \to [0,1]$, such that: (1) the intensity of innovation maximizes firm value, $V(t,\theta,q_0,q)$; (2) market beliefs are correct, $\tilde{\eta}(t,q_0) = E_{\theta,q}[\eta(t,\theta,q_0,q)]$.

Define $\Delta(t,q_0,q) = V(t,H,q_0,q) - V(t,L,q_0,q)$, the equilibrium must satisfy:

$$\eta(t,\theta,q_0,q) = \begin{cases} 0 \text{ if } \lambda\Delta(t,q_0,q)dt < MV_q(t,\theta,q_0,q)dt \\ 1 \text{ if } MV_q(t,q_0,q)dt < \lambda\Delta(t,\theta,q_0,q)dt \end{cases}$$

Lemma 6: In equilibrium, $V(t,\theta,q_0,q)$ is continuous in t and q.

Proof:

Continuity in q: I expand the firm's value function with minor adjustments:

$$\begin{aligned} V(t,\theta,q_0,q_t) &\geq x_t q_t dt + (1 - rdt - \lambda dt - \mu\theta dt)V(t,\theta,q_0,q_t + Mdt) \\ &\quad + \lambda V(t+dt,L,q_0,q_t+Mdt)dt + \mu\theta V(0,H,q_t+Mdt,q_t+Mdt)dt \\ &\geq x_t q_t dt + (1 - rdt - \lambda dt - \mu\theta dt)V(t,\theta,q_0,q_t) \end{aligned}$$

The first inequality means that the firm's value conditional on it expanding over $[t,t+dt)$ regardless of its state is not higher than its actual value (adopt optimal strategy). The second inequality means that the firm's value provided that it expands over $[t,t+dt)$ is not lower than it would obtain by shirking. Following an argument similar to the above, I have $V(t,\theta,q_0,q_{t+dt}) \to V(t,\theta,q_0,q_t)$ as $dt \to 0$.

Continuity in t: I expand the firm's current value into its profits over $[t,t+dt)$ and its expected continuation value:

$$\begin{aligned}
V(t,\theta,q_0,q_t) &= x_t q_t dt + (1 - rdt - \lambda dt - \mu\theta dt)V(t+dt,\theta,q_0,q_{t+dt}) \\
&\quad + \lambda\eta_t V(t+dt,H,q_0,q_{t+dt})dt + \lambda(1-\eta_t)V(t+dt,L,q_0,q_{t+dt})dt \\
&\quad + \mu\theta V(0,H,q_{t+dt},q_{t+dt})dt
\end{aligned}$$

where $x_t q_t$, $\lambda\eta_t V(t+dt,H,q_0,q_{t+dt})$, $\lambda(1-\eta_t)V(t+dt,L,q_0,q_{t+dt})$, $\mu(\theta-\alpha)V(0,H,q_{t+dt},q_{t+dt})$ are bounded by q_t, $q_{t+dt}\frac{\lambda}{r-M}$, $q_{t+dt}\frac{\lambda}{r-M}$, and $q_{t+dt}\frac{\mu}{r-M}$, respectively; thus, $V(t+dt,\theta,q_0,q_{t+dt}) \to V(t,\theta,q_0,q_t)$ as $dt \to 0$. Moreover, by continuity in q, $V(t+dt,\theta,q_0,q_{t+dt}) \to V(t+dt,\theta,q_0,q_t) \to V(t,\theta,q_0,q_t)$ as $dt \to 0$, I obtain continuity in t.

Lemma 7: In any equilibrium $\langle\eta,\tilde{\eta}\rangle$, the value function of the firm $V(t,\theta,q_0,q)$ is (a) strictly decreasing in t and (b) strictly increasing in q.

Proof: (a) As reputation decreases in t, I can fix the initial time $t > t'$ of a low-reputation firm (”low firm” for short) and a high-reputation firm (”high firm” for short). Suppose that both firms have the same initial quality $\theta'_{t'} = \theta_t$ and initial capacity $q'_{t'} = q_t$. Suppose that the high firm chooses the non-Markovian strategy $\langle\eta'\rangle$ that mimics the equilibrium innovation of the low firm, i.e., $\eta'_{t'+s} = \eta(t+s,\theta_{t+s},q_0,q_{t+s})$; the high firm's quality $\theta'_{t'+s}$ and capacity $q'_{t'+s}$ are governed by the same process as the equilibrium quality θ_{t+s} and capacity q_{t+s} of the low firm, and the reputation of the high firm never falls behind, i.e., $x'_{t'+s} \geq x_{t+s}$ with strict inequality for s close to 0. Then, the profit of the high firm with strategy $\langle\eta'\rangle$ always exceeds the equilibrium profit of the low firm because its revenue is higher by the above argument, and the costs are equal by construction. Furthermore, the equilibrium value of the high firm is weakly higher than its value from adopting a mimic strategy; thus, $V(t,\theta,q_0,q)$ is strictly decreasing in t.

(b) I expand firm value as follows:

$$\begin{aligned} V(t,\theta,q_0,q) &= E\int_0^{+\infty} e^{-rs} q_{t+s} x_{t+s} ds \\ &= E\int_0^{+\infty} e^{-rs} [q_0 + \int_0^s (1-\eta_{t+w}) M dw] x_{t+s} ds \end{aligned}$$

By the envelope theorem, $V_q(t,\theta,q_0,q_t) = E\int_0^{+\infty} e^{-rs} x^*_{t+s} ds$, where x^*_{t+s} is yielded by the firm's optimal strategy. x^*_{t+s} is always positive, implying that V_q always positive[1].

Theorem 2: For a sufficiently small M, the optimal strategy is characterized by two cutoff functions $t_{H,q_0}(q) \in [0,+\infty)$ and $t_{L,q_0}(q) \in [0,+\infty)$, such that a firm of quality θ, $\theta \in \{H,L\}$, capacity q, $q \in (0,+\infty)$: (1)

$$\begin{aligned} \text{expand if } t &\in [0, t_{\theta,q_0}(q)] \\ \text{innovate if } t &\in [t_{\theta,q_0}(q), 1] \end{aligned}$$

(2) on the equilibrium path, any segment piece where quality θ does not change has at most one cross point with each cutoff function.

Proof: See Appendix A.7.

Note that our result is weaker (assuming that M is sufficiently small) than the strong linear case in Chapter 1 (assuming $M < \lambda$). The reason is that a firm's action is changing in its capacity, as $\partial V/\partial q$ is related with q. Thus, it is difficult to compare the effects of expansion and reputational drift on a firm's value, unless M is sufficiently small.

[1]Here, an implied assumption is that the value function is differentiable. Continuity is given by Lemma 6; suppose that the value function is non-differentiable for a finite number of points. Then, we can "assign" a derivative to this point, the value of which is between the left derivative and the right derivative. The case in which the value function is continuously non-differentiable is only of mathematical significance and not the focus here.

2.4 Extension: Industrial Distribution

2.4.1 Model

I maintain the individual firm-level micro structure in the first two chapters, with the exception that now I assume that there is a continuum of firms and a continuum of identical consumers. Time $t \in [0, +\infty)$ is continuous and infinite. At time t, the firm produces q_t units of a product of quality θ_t, which can be either high or low. The instantaneous value of the firm's production to a consumer is $\theta_t q_t dt$.

The discount rate is $r \in (0, +\infty)$. A firm dies at a Poisson rate ν.

The firm focuses its attention on two channels of improvement: innovation and expansion. They are mutually exclusive. Specifically, I assume that the sum of the two intensities is 1 for $t \in [0, +\infty)$. In this way, the intensity of innovation or expansion represents the percentage of the firm's effort spent on innovation or expansion, respectively, with the sum being equal to 100%. I use η_t to denote the firm's intensity of innovation. Consequently, the intensity of expansion is $1 - \eta_t$.

Innovation and expansion feed into the firm's quality and capacity, respectively, in the following way:

At time 0, the firm draws initial quality $\theta_0 \in \{L, H\}$ and initial capacity $q_0 \in (0, +\infty)$. Product quality θ_t depends on initial quality, past innovation $(\eta_s)_{0 \leq s \leq t}$, and technology shocks, which occur according to a Poisson process with arrival rate λ. When a shock occurs, the previous quality becomes obsolete, and the current quality is determined by the current intensity of innovation $\theta_{t+dt} = \eta_t$. Absent a shock, quality is constant, $\theta_{t+dt} = \theta_t$.

Firm capacity q_t is a function of past expansion $1 - \eta_s$, $0 \leq s \leq t$, specifically:

$$q_{t+dt} = q_t[1 + (1 - \eta_t)Mdt] \tag{2.7}$$

where M is a positive parameter measuring the efficiency of managerial attention spent on expansion, or simply the efficiency of expansion. I assume that $M < r$ and, hence, that the value of the firm is always well defined.

The consumers learn about quality through signals that arrive to and only to a high-quality firm at Poisson rate μ. The reputation-updating process follows that in Chapter 1. Hence, in absence of a breakthrough, the trajectory of reputation follows Theorem 2 in Chapter 1. As established in Chapter 1, the firm's equilibrium strategy follows an innovate-expand style, with cutoffs x_H and x_L for a high-quality firm and a low-quality firm, respectively.

2.4.2 Stationary Distribution

Baseline example:

To preview the stationary equilibrium distribution, we consider a simplified case in which a breakthrough is not possible. Suppose $\mu = 0$; then, any improvement in quality will never be reviewed by the market. It is never the optimum for a firm to innovate, and product quality will gradually decline in expectation. $\forall\, t_1, t_2$, we have

$$\frac{f(t_1)}{f(t_2)} = e^{-\nu(t_1 - t_2)}$$

Thus, we have

$$f(t) = ce^{-\nu t}$$

where c is a constant.

As

$$\int_0^{+\infty} f(t)dt = 1,$$

we have $c = \nu$,

$$f(t) = \nu e^{-\nu t}$$

Moreover,

$$\begin{aligned} f(x) &= f[t(x)]\frac{dx}{dt} \\ f(x,H) &= xf(x) \\ f(x,L) &= (1-x)f(x) \end{aligned} \tag{2.8}$$

For an innovate-expand economy:

$\forall\, t_1, t_2$, we have

$$\frac{f(t_1)}{f(t_2)} = \frac{1-\int_0^{t_1}[x(t)\mu+\nu]e^{-(\mu+\nu)t}dt}{1-\int_0^{t_2}[x(t)\mu+\nu]e^{-(\mu+\nu)t}dt}$$

where $x(t)$ follows Theorem 2 in Chapter 1.

Then, we have

$$f(t) = c\{1-\int_0^t [x(s)\mu+\nu]e^{-(\mu+\nu)s}ds\}$$

where

$$c = \frac{1}{\int_0^\infty\{1-\int_0^t[x(s)\mu+\nu]e^{-(\mu+\nu)s}ds\}dt}$$

and the distribution of $f(x)$, $f(x,H)$, $f(x,L)$ follows (2.8).

Due to the complex form of $x(t)$, it is difficult to analytically solve for $f(t)$ and $f(x)$; instead, I obtain a computational result. Figure 2.1 displays the probability density of a firm's reputation in a stationary distribution. Note that there is a large share of low-reputation firms and a small share of high-reputation firms.

2.4.3 Numerical Simulation

I assume that the interest rate $r = 0.2$, technology shock arrival rate $\lambda = 0.2$, breakthrough arrival rate $\mu = 1$, and managerial efficiency $M = 0.1$. To make the figure clear as possible, I assume that $H = 0.8$, $L = -0.2$. This assumption does not affect the character of the

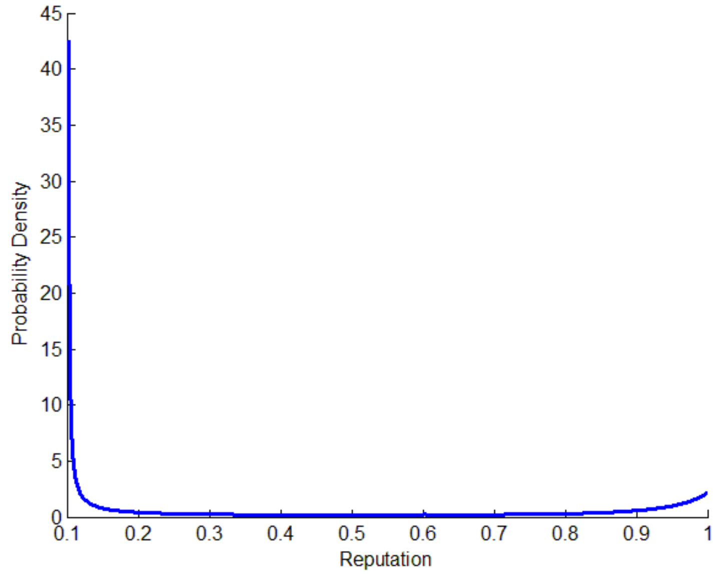

Figure 2.1: Probability Density of Firm's Reputation

equilibrium, but it does decrease the value of the firm along the entire spectrum, thus meaning that the high-quality firm does not to shift to expansion too rapidly. I simulate the reputation and capacity dynamics for 1000 firms. Here, I interpret the discount rate r as the aggregate effect of a value discount and a death rate δ. Specifically, I assume $\delta = 0.1$. If a firm dies, it is replaced with a newborn firm that has high quality, a reputation equal to 1, and a capacity equal to 1. In this way, the market converges to a steady state.

Figures 2.2 and 2.3 show simulated reputation and quality trajectories for two randomly selected firms. Reputation is not penetrable from above at $x = \frac{\lambda}{\mu} = 0.2$ and occasionally jumps up only when quality is high. The first firm survived through $T = 50$; the second firm died at $T = 41.6$.

Figures 2.4, 2.5, and 2.6 show the distribution of the reputation and capacity of 1000 firms at $T = 10$, 50, and 100. At $T = 10$, the distribution is not yet stable, and I observe a group of pioneer firms at $q = 2.5$ (maximum capacity at that time) and backward firms at $q = 1$. As time passes, the distribution becomes smoother, and only a few firms remain on the frontier. At $T = 100$, the distribution is quite smooth, very few firms remain on top, and

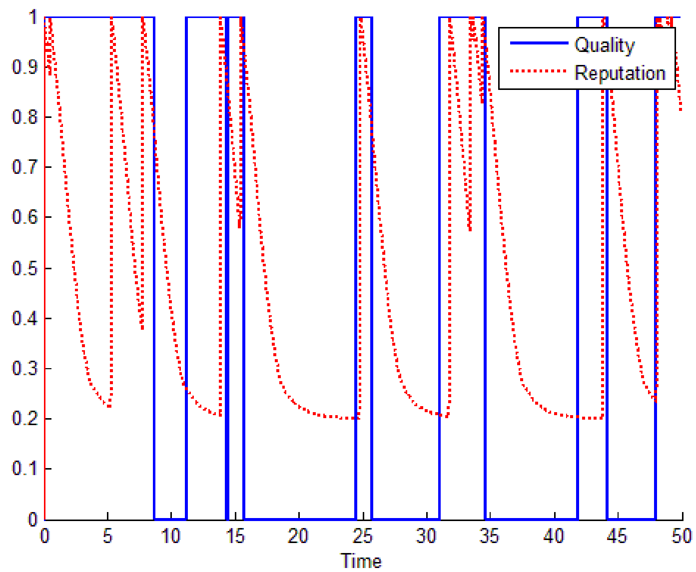

Figure 2.2: Simulated Firm Reputation Trajectory (1)

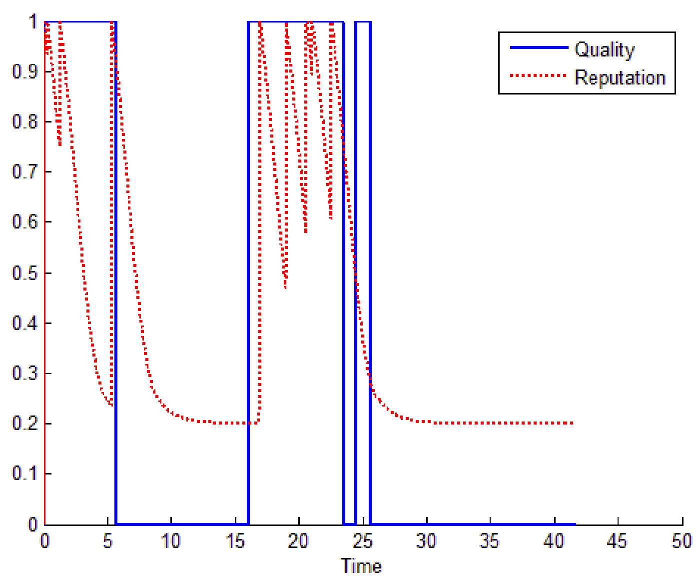

Figure 2.3: Simulated Firm Reputation Trajectory (2)

there is a bulge of firms at $q = 1$ (the maximum is 30), which represents the newborn firms. Comparing $T = 50$ and $T = 100$ reveals that the figures are quite similar, meaning that the market is converging to a steady state.

The simulation result confirms the pyramid structure commonly observed in real industry: a few large firms that have good histories are on top, while a large number of firms never had the opportunity to grow and are ultimately replaced by small, newborn firms.

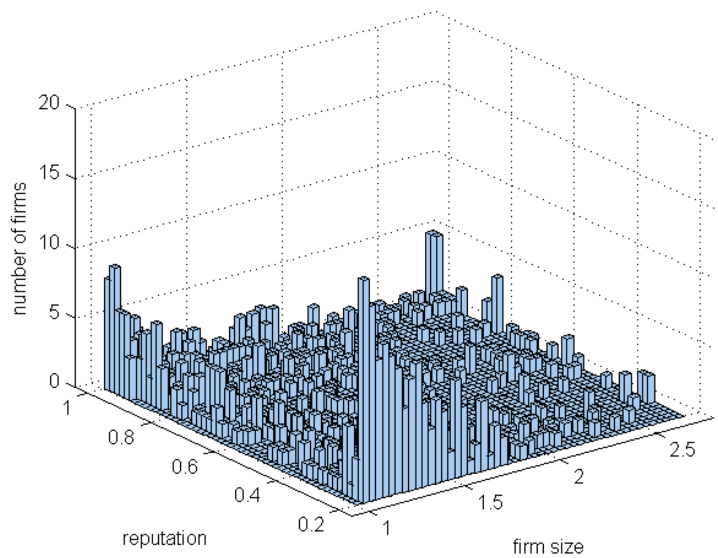

Figure 2.4: Distribution of Firms' Reputation and Capacity, $T = 10$

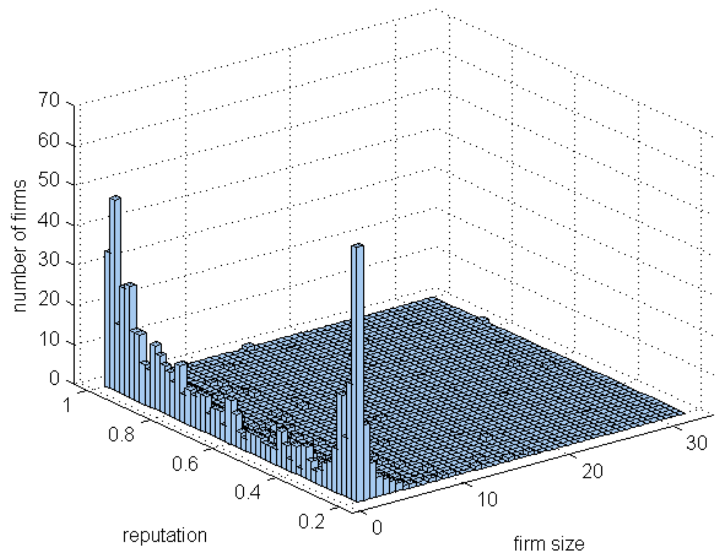

Figure 2.5: Distribution of Firms' Reputation and Capacity, $T = 50$

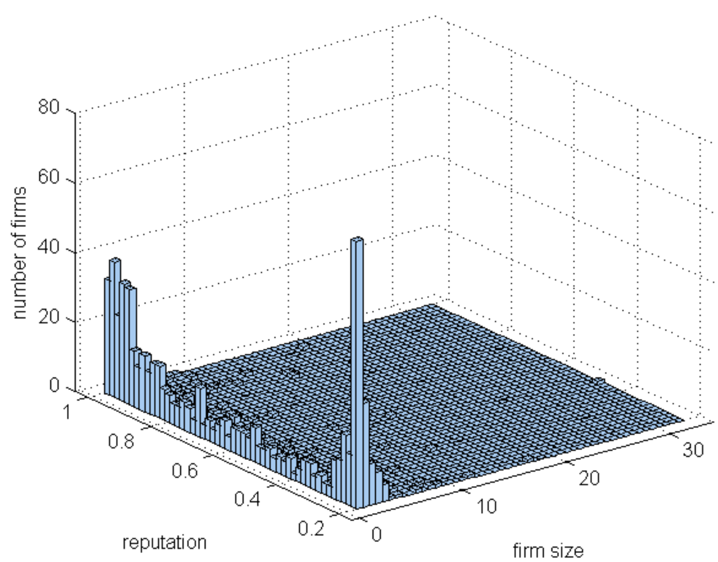

Figure 2.6: Distribution of Firms' Reputation and Capacity, $T = 100$

2.5 Conclusion

In this Chapter, I investigated several model variants of the reputational model proposed in Chapter 1. In the first model variant, I introduced a non-trivial cost of innovation and expansion. As taking action is costly, the firm may have an incentive to shirk. I characterized how the incentives for innovation and expansion depend on a firm's current reputation, quality, and capacity. I showed that in equilibrium, the firm's optimal strategy takes an "innovate-shirk-expand" shape, in contrast to the "innovate expand" equilibrium observed in the first chapter.

In the second model variant, I replaced the strong linearity assumption with increasing/decreasing returns to scale. Consequently, capacity becomes effective in the information structure. I showed that, under a sufficiently low efficiency of expansion, the "innovate-expand" feature is maintained. Specifically, on each equilibrium path, the firm innovates when its quality is low, reputation is low, and capacity is high; it expands when its quality is high, reputation is high, and capacity is low.

Finally, I generalized this model to a continuum of firms and studied the steady-state distribution of reputation, quality, and capacity. Both computational and numerical results reveal that a large share of low-reputation, low-quality firms lie at the bottom, while a few pioneers with high quality and high capacity are found at the top.

2.6 Appendix

A.1 Proof of Lemma 1

The proof is in two steps, the first step proves that: fix initial capacity $q_0 < q_0'$ of a small capacity firm ("small firm" for short) and a large capacity firm ("large firm" for short) with initial quality θ_0 and initial reputation x_0, the value of the larger firm, if it mimics the small firm's investment strategy, is q_0'/q_0 times the value of its small cohort; the second step proves that: if a strategy is the best strategy for the small firm, it is also the best strategy for the large firm.

Step 1: Fix initial capacity $q_0 < q'_0$ of a small capacity firm ("small firm" for short) and a large capacity firm ("large firm" for short) with initial quality θ_0 and initial reputation x_0. Suppose the large firm chooses the non-Markovian strategy $\langle \eta_1, \eta_2, \tilde{\eta}_1(x) \rangle$ that mimics equilibrium investment of the small firm, i.e. at time t after history $h^{t\text{-}}$, $\eta'_{it} = \eta_i(\theta_t, x_t, q_t)$ for $i = 1, 2$. Adopting this strategy, the large firm's quality θ'_t and reputation x'_t is governed by the same process as the equilibrium reputation θ_t and capacity x_t of the small firm; and $q'_t = q_t \cdot (q'_0/q_0)$. Then the profit of the large firm with investment strategy is q'_0/q_0 times the profit of the small firm for all t because its revenue and cost equal to q'_0/q_0 times the small firm's revenue and cost respectively, thus, the value of the large firm is q'_0/q_0 times the value of the small firm, i.e.:

$$
\begin{aligned}
V_t(x,\theta,q'_s) &= E[\int_0^t e^{-rs}(x_s - c\eta_{1s}(x_s,\theta_s,q_s) - c\eta_{2s}(x_s,\theta_s,q_s))q'_s ds] \\
&= (q'_0/q_0)E[\int_0^t e^{-rs}(x_s - c\eta_{1s}(x_s,\theta_s,q_s) - c\eta_{2s}(x_s,\theta_s,q_s))q_s ds] \\
&= (q'_0/q_0)V_t(x,\theta,q_s). \qquad (2.9)
\end{aligned}
$$

Step 2: Suppose otherwise. Suppose that firm 1's optimum strategy is $\langle \eta_1, \eta_2 \rangle$ at time t, according to step 1, firm 2 can, at least, mimic this strategy to secure a value of $qV(x,\theta,1)$. Thus, $V(x,\theta,q) \geq qV(x,\theta,1)$. If $V(x,\theta,q) > qV(x,\theta,1)$, we can conclude that firm 2 has a better strategy $\langle \eta'_1, \eta'_2 \rangle$, which offers the firm a higher value. However, firm 1 can also mimic firm 2's "better strategy" and secure a value of $V(x,\theta,q)/q > V(x,\theta,1)$, contradicting the statement that $\langle \eta_1, \eta_2 \rangle$ is firm 1's optimum strategy.

A.2 Proof of Lemma 3

Fix initial reputations $x_0 < x'_0$ of a low reputation firm ("low firm" for short) and a high reputation firm ("high firm" for short) with initial quality θ_0 and initial capacity q_0. Suppose the high firm chooses the non-Markovian strategy $\langle \eta_1, \eta_2 \rangle$ that mimics equilibrium investment of the low firm, i.e. if at time t after history $h^{t\text{-}}$ the low firm has reputation $x_t = x_t(x_0, h^{t\text{-}}, \tilde{\eta}_1)$ then $\eta'_{it} = \eta_i(\theta_t, x_t(x_0, h^{t\text{-}}, \tilde{\eta}_1), q_t)$ for $i = 1, 2$. Adopting this strategy, the high firm's quality θ'_t and capacity q'_t is governed by the same process as the equilibrium quality θ_t and capacity q_t of the low firm. Thus these firms face the same distribution of public histories and the reputation of the high firm never falls behind, i.e. $x'_t \geq x_t$ with strict inequality for t close to 0. Then the profit of the high firm with investment strategy $\langle \eta'_1, \eta'_2 \rangle$ always exceeds the equilibrium profit of the low firm because the revenue is greater for the high firm by the above argument, and the costs are equal by construction. Furthermore, the equilibrium value of high firm is weakly higher than its value adopt a mimic strategy, thus $V(x, \theta, q)$ is strictly increasing in reputation x.

A.3 Proof of Lemma 4

Suppose $\eta_1(x, 1) = 1$, following Lemma 4, $\lambda\Delta(x) > c$ and $\lambda\Delta(x) > MV(x, 1, 1)$. We also know $V(x, 1, 1) > V(x, 0, 1)$ for $x \in [0, 1]$, thus $\lambda\Delta(x) > MV(x, 1, 1) > V(x, 0, 1)$, use Lemma 4 again, $\eta_1(x, 0) = 1$. $\eta_1(x, 1) = 1$ is a sufficient condition for $\eta_1(x, 0) = 1$. As $\eta_1(x, 1)$ can only take value 0, 1, we have $\eta_1(x, 0) \geq \eta_1(x, 1)$ for $x \in [0, 1]$.

Similarly, suppose $\eta_2(x, 0) = 1$, following Lemma 4, $MV(x, 0, 1) > c$ and $MV(x, 0, 1) > \lambda\Delta(x)$. As $V(x, 1, 1) > V(x, 0, 1)$ for $x \in [0, 1]$, $V(x, 1, 1) > V(x, 0, 1) > \lambda\Delta(x)$; use Lemma 4 again, $\eta_2(x, 1) = 1$. $\eta_2(x, 0) = 1$ is a sufficient condition for $\eta_2(x, 1) = 1$. As $\eta_2(x, 0)$ can only take value 0, 1, we have $\eta_2(x, 1) \geq \eta_2(x, 0)$ for $x \in [0, 1]$.

In order to characterize investment incentives we need to evaluate the value of quality $\Delta(x) = V(x, H, 1) - V(x, L, 1)$

The believed investment $\tilde{\eta}_1(x)$ is the expectation of the investment in quality of a firm whose reputation is x. Consider a candidate equilibrium $\langle \eta, \tilde{\eta} \rangle$, following Lemma 5, suppose that $\eta_1(x, 0) > \eta_1(x, 1)$ for some value of x, then $\eta_1(x, 0) = 1$, $\eta_1(x, 1) = 0$. By definition of believed investment, we have

$$\tilde{\eta}_1(x) = E[\eta_1(x,\theta)] = x\eta_1(x,1) + (1-x)\eta_1(x,0) = 1-x \tag{2.10}$$

which is smaller than a low quality firm's actual investment, but higher than a high quality firm's. Unlike in the benchmark model, the believed investment does not equal to the actual investment of both types in equilibrium. By introducing a second investment option whose benefit depends on the firm's current quality and by limiting the firm's ability in multitask operation, we introduced asymmetric information which arises not only from the unobserved investment into future quality, but also from asymmetric information about product quality.

A.4 Proof of Lemma 5

To analyze the value of quality $\Delta(x) = V(x,1,1) - V(x,0,1)$, we expand the value functions into current profits and continuation values as in the proof of Lemma 4. Current profits cancel because both current revenue and costs depend on reputation but not on quality. We also borrow the terms η_{it}^{θ}, $\theta \in \{L,H\}$, $i = 1,2$ and q_{t+dt}^{θ}, $\theta \in \{L,H\}$ from the proof of Lemma 4.

$$
\begin{aligned}
V(x_t,H,q_t)-V(x_t,L,q_t) &= q_t\Delta(x_t) \\
&= (1-rdt)\{\mu dt V(1,H,q^H_{t+dt}) \\
&\quad +[1-\mu dt-(1-\eta^H_{1t})\lambda dt]V(x_{t+dt},H,q^H_{t+dt}) \\
&\quad +\lambda dt(1-\eta^H_{1t})V(x_{t+dt},L,q^H_{t+dt}) \\
&\quad -(1-\lambda\eta^L_{1t}dt)V(x_{t+dt},L,q^L_{t+dt}) \\
&\quad -\lambda\eta^L_{1t}dtV(x_{t+dt},H,q^L_{t+dt})\} \\
&= (1-rdt)\{(1-\lambda dt+\eta^H_{1t}\lambda dt-\eta^L_{1t}\lambda dt)[V(x_{t+dt},H,q_t)-V(x_{t+dt},L,q_t)] \\
&\quad +\mu dt[V(1,H,q^H_{t+dt})-V(x_{t+dt},H,q^H_{t+dt})] \\
&\quad +Mdt[\eta^H_{2t}V(x_{t+dt},H,q_t)-\eta^L_{2t}V(x_{t+dt},L,q_t)]\}
\end{aligned}
$$

Ignore the discount $1-rdt$. By Lemma 5, $\eta^H_{1t}-\eta^L_{1t}\leq 0$, thus $1-\lambda dt+\eta^H_{1t}\lambda dt-\eta^L_{1t}\lambda dt<1$. We can iterate the above process for the first term of the right hand side (expand $V(x_{t+dt},H,q_t)-V(x_{t+dt},L,q_t)$ on a trajectory of x^{ϕ}_t), and the coefficient of $V(x_{t+s},H,q_t)-V(x_{t+s},L,q_t)$ converges to 0. The next thing is to prove that the sum of the rest two terms is decreasing in x_t.

Rearrange the second and the third term, we get:

$$
\mu dtV(1,H,q_t)+(M\eta^H_{2t}-\mu)dtV(x_{t+dt},H,q_t)-Mdt\eta^L_{2t}V(x_{t+dt},L,q_t)
$$

The first term is constant. Following Lemma 3 and that x_{t+dt} increases in x_t, the third term decreases in x_t. Following the same reason and because $\eta^H_{2t}\leq 1$, if $M<\mu$, the second term is decreasing in x_t.

In conclusion, $q_t\Delta(x_t)$ can be viewed as the sum of a series of functions decreasing (assuming $M<\mu$) in x_t. Since q_t is constant, $\Delta(x_t)$ is decreasing in x_t.

A.5 Proof of Theorem 1

Since the value function of the firm is homogeneous of degree 1 in quantity, the capacity as a state variable doesn't affect the firm's investment decision (Lemma 2).

By Lemma 5, $\Delta(x)$ is decreasing in x, thus the marginal profit of quality investment, $\lambda\Delta(x)$, is decreasing in x. By Lemma 3, $V(x,\theta,q)$ is increasing in x, thus the marginal profit of capacity investment is increasing in x.

We can rule out a full research equilibrium when cost is non trivial: If $\eta_1(x,\theta) = 1$ for all x and θ, then $x_t = 1$ implies $x_s = 1$ hereafter (for all $s \geq t$); thus $\Delta(1) = 0$ and a firm with perfect reputation prefers to shirk (or expand). Following Lemma 4, a high quality firm with perfect reputation $(x = 1)$ either shirks or invests in capacity, i.e. its marginal profit of investing in quality is either smaller than investing in capacity or smaller than cost c.

We can also rule out a full expand equilibrium when cost is non trivial: If $\eta_2(x,\theta) = 1$ for all x and θ, then $\eta_1(x,\theta) = 0$ for all x and θ, in the long run, quality becomes 0 almost for sure, then $x_t = 0$ implies $x_s = 0$ hereafter (for all $s \geq t$) because $dx = -\lambda x dt - \mu x(1-x)dt = 0$ for $x = 0$; thus profit earned by this firm is 0 for all $s \geq t$, $V(0,L,q) \leq 0$ with strict inequality for $\eta_2 > 0$, thus a low quality firm with zero reputation prefers to shirk. Thus, a low quality firm with reputation equal to 0 either shirks or invests in quality. Similarly, a firm with worst reputation $(x = 0)$ has strongest incentive to invest in quality, while it has weakest incentive to invest in capacity.

In summary, $\Delta'(x) < 0$, $MV'(x,\theta) > 0$; besides, when $x = 1$, $\theta = H$, $\Delta(x) < MV(x,\theta)$, and when $x = 0$, $\theta = L$, $\Delta(x) > MV(x,\theta)$. Since $\Delta(x)$ is decreasing in x, $max(\Delta(x),c)$ is weakly decreasing in x. Since $MV(x,\theta)$ is increasing in x for $\theta = L,H$, $max(MV(x,\theta),c)$ is weakly increasing in x for $\theta = L,H$. Thus, by the above argument, $MV(x,H)$ either cross $max(\Delta(x),c)$ once from below ($x \in [0,1]$), or we have $V(x,H) >$ or $< \Delta(x)$ for $x \in [0,1]$; similarly, $MV(x,L)$ either cross $max(\Delta(x),c)$ once from below ($x \in [0,1]$), or we have $MV(x,L) < \Delta(x)$ for $x \in [0,1]$. Moreover, for $x \in [0,1]$, $\Delta(x)$ either cross $max(MV(H,\theta),c)$ once from above, or $\Delta(x) < max(MV(x,H),c)$ for $x \in [0,1]$; similarly, $\Delta(x)$ either cross $max(MV(L,\theta),c)$ once from above, or $\Delta(x) >$ or $< max(MV(x,H),c)$ for $x \in [0,1]$.

Suppose, for a particular type of firm, one (or both) intersecting point(s) doesn't exist, then the equilibrium includes trivial region(s). The corresponding firm reputations for both intersecting points (if exist), along with $x = 0$ and $x = 1$, are candidate cutoffs, namely $\underline{x}_\theta$ and $\bar{x}_\theta$ for $\theta = L$ and H respectively. The equilibrium can be characterized by $0 \leq \underline{x}_\theta \leq \bar{x}_\theta \leq 1$ for $\theta = L,H$ s.t. $\Delta(\underline{x}_\theta) = \max(MV(\underline{x}_\theta, \theta), c)$ and $MV(\bar{x}_\theta, \theta) = \max(\Delta(\bar{x}_\theta), c)$, $\theta = L,H$. Since $\Delta(x)$ is decreasing in x and $MV(x,\theta)$ is increasing in x, for $x \in [0, \underline{x}_\theta]$, $\Delta(x) > \max(MV(x,\theta), c)$; for $x \in [\bar{x}_\theta, 1]$, $MV(x,\theta) > \max(\Delta(x), c)$. Simultaneously, we also have $\underline{x}_\theta \leq \bar{x}_\theta$ for $\theta = L,H$ (otherwise we have contradiction). For $x \in [\underline{x}_\theta, \bar{x}_\theta]$, $MV(x,\theta) < \max(\Delta(x), c)$ and $\Delta(x) < \max(MV(x,\theta), c)$, thus $MV(x,\theta) < c$ and $\Delta(x) < c$.

Firm's incentive of quality investment is, independent of quality, $\lambda\Delta(x)$; however, high quality firm's marginal revenue of investment in capacity is higher than low quality firm's because it has greater firm value, i.e. $MV(x,H,1) > MV(x,L,1)$.

Following Lemma 4, suppose $\bar{x}_H = 1$, $MV(\bar{x}_H, H, 1) \leq \max(\lambda\Delta(\bar{x}_H), c)$, then $MV(\bar{x}_H, L, 1) < \max(\lambda\Delta(\bar{x}_H), c)$, since $MV(x,L,1)$ is increasing in x and $\lambda\Delta(x)$ is decreasing in x, we have $MV(x,L,1) < \max(\lambda\Delta(x), c)$ for $x \in [0,1]$, implying $\bar{x}_L = 1$. In this equilibrium the expand region is trivial.

Suppose $\bar{x}_H < 1$, $MV(\bar{x}_H, H, 1) = \max(\lambda\Delta(\bar{x}_H), c)$, then $MV(\bar{x}_H, L, 1) < \max(\lambda\Delta(\bar{x}_H), c)$, again, since $MV(x,L,1)$ is decreasing in x and $\lambda\Delta(x)$ is increasing in x, we have $MV(x,L,1) < \max(\lambda\Delta(x), c)$ for $x \in [0, \bar{x}_H]$, implying $\bar{x}_H < \bar{x}_L$.

Suppose that M is sufficiently high, a high quality firm whose reputation equals to 1 will always invest in capacity (this can always be achieved by holding a low cost c such that $MV(1,H,1) > c$, as investment in quality is unattractive at $x = 1$, low cost can ensure that high reputation, high quality firm invests in capacity).

Similarly, following Lemma 4, suppose $\underline{x}_H < \bar{x}_H$, a shirk region exists for high quality firm, $\lambda\Delta(\underline{x}_H) = c > MV(\underline{x}_H, H, 1)$, then $\lambda\Delta(\underline{x}_H) = c > MV(\underline{x}_H, L, 1)$, since $MV(x,L,1)$ is increasing in x and $\lambda\Delta(x)$ is decreasing in x, we have $\underline{x}_H = \underline{x}_L$.

Suppose $\underline{x}_H = \bar{x}_H$, then $\lambda\Delta(\underline{x}_H) = MV(\underline{x}_H, H, 1) > c$, we have both $\lambda\Delta(\underline{x}_H) > MV(\underline{x}_H, L, 1)$

and $\lambda\Delta(\underline{x}_H) > c$, again, since $MV(x,L,1)$ is increasing in x and $\lambda\Delta(x)$ is decreasing in x, we have $\underline{x}_H < \underline{x}_L$.

Suppose that M is sufficiently high, a high quality firm whose reputation equals to 1 will always invest in capacity (this can always be achieved by holding a low cost c such that $MV(1,H,1) > c$, as investment in quality is unattractive at $x = 1$, low cost can ensure that high reputation, high quality firm invests in capacity).

A high quality firm would like to take advantage of its quality by enlarging capacity, rather than waste effort maintaining quality. This also captures the nature of a real firm: one who has just came up with a good product (quality jumps to 1) becomes eager to aggrandize even if the market hasn't fully recognized it (breakthrough is yet to arrive and firm's reputation remains unchanged).

A.6 Further Classification of Firm's Investment Strategy:

Following theorem 1, for a particular quality type, firm's behavior falls into one of the following 7 cases: Full research, Full Shirk, Full Expand, Research-shirk, Research-expand, Shirk-Expand, and Research-shirk-expand. We use the initials "R, S, E, RS, RE, SE, RSE" for short. A firm's investment strategy, on the other hand, is a combination of strategy when quality is high and strategy when quality is low, which yields $7 \times 7 = 49$ types of strategy. However, following Lemma 3, 4, 5, and 6, only 16 out of the 49 are possible. To show it more clearly, we construct a matrix in which the rows are high quality firm's possible strategy, and the columns are the low quality firm's. Possible combinations are denoted by "$\surd$" in the corresponding cell while impossible ones are denoted by "$\times$":

H vs L	*R*	*S*	*E*	*RS*	*RE*	*SE*	*RSE*
R	×	×	×	×	×	×	×
S	×	√	×	×	×	×	×
E	√	√	×	√	√	√	√
RS	×	×	×	√	×	×	×
RE	√	×	×	√	√	×	√
SE	×	√	×	×	×	√	×
RSE	×	×	×	√	×	×	√

We first explain why strategies in "×" can't exist in equilibrium.

The first row: Following Theorem 1, a high quality firm can't be Full research, otherwise $\forall x$, $\Delta(x) > \max(MV(x,H,1),c) > \max(MV(x,L,1),c)$, then we have a Full Research equilibrium, which is ruled out by Theorem 1. Thus, strategies in the first row (R vs ?) can't exist in equilibrium.

The second row: Following Proposition 1, as $\underline{x}_H = 0$ and $\bar{x}_H = 1$, we have $\underline{x}_H = \underline{x}_L$ and $\bar{x}_H = \bar{x}_L$. Among strategies in the second row (S vs ?), only "S vs S" suffices the above condition.

The third row: Following Theorem 1, a low quality firm can't be Full expand, otherwise $\forall x$, $MV(x,H,1) > MV(x,L,1) > \max(\Delta(x),c)$, then we have a Full Expand equilibrium, which is ruled out by Theorem 1. Thus, "E vs E" in the third row can't exist in equilibrium.

The fourth row: Following Proposition 1, as $0 < \underline{x}_H < 1$ and $\bar{x}_H = 1$, we have $\underline{x}_L = \underline{x}_H$ and $\bar{x}_L = 1$, meaning that only "RS vs RS" is possible in equilibrium.

The fifth row: Following Proposition 1, as $0 < \underline{x}_H < 1$ and $\underline{x}_H = \bar{x}_H$, we have $\underline{x}_L > \underline{x}_H$, meaning that low quality firm's strategy begins with "R". Low quality firm can still have an expand region, so far as $\bar{x}_L > \bar{x}_H$. We have four possibilities: "RE vs R", "RE vs RS", "RE vs RE", "RE vs RSE".

The sixth row: Following Proposition 1, as $\underline{x}_H = 0$ and $\underline{x}_H < \bar{x}_H < 1$, we have $\underline{x}_L = \underline{x}_H$, meaning that low quality firm's strategy can't begin with "R". We have two possibilities: "SE vs S", "SE vs SE".

The seventh row: Following Proposition 1, as $0 < \underline{x}_H < 1$ and $\underline{x}_H < \bar{x}_H < 1$, we have $\underline{x}_L = \underline{x}_H$ and $\bar{x}_L > \bar{x}_H$, meaning that low quality firm's strategy begins with "R" and a shirk region exists. We have two possibilities: "RSE vs RS", "RSE vs RSE".

The following paragraphs discuss the existence of remaining strategies in equilibrium, among which some can be easily constructed (by manipulating parameters), while a few others heavily rely on parameters. In the following paragraphs, unmentioned parameters are assumed to take value neither close to 0 nor infinite.

Case 1: S vs S

Suppose c is sufficiently high, say, infinite, such that $\forall x \in [0,1]$, $c > \lambda\Delta(x)$ and $c > MV(x,\theta,1)$. In this case, $\underline{x}_\theta = 0$, $\bar{x}_\theta = 1$ for $\theta = L,H$.

Full shirk is the unique equilibrium for both types of firm (high quality and low quality, similarly hereinafter).

Case 2: E vs R

Suppose c is small (close to 0), λ and r are sufficiently low, approaching to 0 from above, and μ is infinite. Low quality firm's value per period (r times firm value) approaches to 0 (regardless of x) while high quality firm's value per period approaches to 1 (regardless of x). Then $\forall x$, $MV(x,H,1)r$ approaches to M, $MV(x,L,1)r$ approaches to 0, $\Delta(x)r$ approaches to 1, c approaches to 0. Suppose $r > M + \varepsilon > \lambda + 2\varepsilon$, high quality firm always invest in capacity. Holding c ultra low (say, $c = o(\lambda)$), the value of low quality firm almost solely comes from the small chance of jumping to high quality status, $V(x,L,1)r$ approaches to λ if low quality firm invests in quality when reputation approaches to 0. As $M < r$ (our initial assumption to prevent firm value from blowing), $MV(x,L,1)r < \lambda\Delta(x)r$, low quality firm always invest in quality.

Case 3: E vs S

Following Case 2, except that now we hold λ, instead of c, ultra low (say, $\lambda = o(c)$). Low quality firm always shirk.

Case 4: E vs RS

Following Case 2, by continuity and the existence of Case 2 and Case 4 we can prove the existence of Case 3, which depends on the relation of λ and c.

Case 5: E vs RE (conjectured)

This strategy heavily depends on parameters thus is conjectured.

Case 6: E vs SE (conjectured)

This strategy heavily depends on parameters thus is conjectured.

Case 7: E vs RSE (conjectured)

This strategy heavily depends on parameters thus is conjectured.

Case 8: RS vs RS

c is sufficiently (but not infinitely) low such that a low reputation firm will prefer investing in quality to shirking. Holding c, M is sufficiently low such that a high quality firm with perfect reputation will not invest in capacity. $\underline{x}_L = \underline{x}_H$; $\bar{x}_\theta = 1$ for $\theta = L, H$. Re-lable $\underline{x}_H = \underline{x}_L = \underline{x}$, the firm:

$$
\begin{aligned}
\text{invests in quality if } x &\in [0, \underline{x}] \\
\text{shirks if } x &\in [\underline{x}, 1]
\end{aligned}
$$

which can be characterized as a ”Research-shirk” equilibrium for both types of firm.

Case 9: RE vs R (conjectured)

This strategy heavily depends on parameters thus is conjectured.

Case 10: RE vs RS

c is sufficiently (but not infinitely) low, M is both sufficiently high such that a high quality firm with perfect reputation will invest in capacity, while still being sufficiently low such that a low quality firm with perfect reputation will not do the same. If M is sufficiently low (within the above range), we have case 15. When M is high (within the above range), it is possible that $\lambda\Delta(\underline{x}_H) \leq MV(\underline{x}_H, H, 1)$ and the shirk region for high quality firm becomes trivial. i.e. $0 < \underline{x}_H = \bar{x}_H < 1$, $0 < \underline{x}_L < \bar{x}_L = 1$. High quality firm adopts a "Research-work" strategy while a low quality firm is doing "Research-shirk". Re-lable $\underline{x}_H = \bar{x}_H = x_H$, the firm:

$$\text{High quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, x_H] \\ \text{invests in capacity if } x \in [x_H, 1] \end{cases}$$

$$\text{Low quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, \underline{x}_L] \\ \text{shirks if } x \in [\underline{x}_L, 1] \end{cases}$$

Numerical examples have proved the existence of such equilibriums, which heavily rely on parameters.

Case 11: RE vs RE

c ultra low. Other parameters are "normal", i.e. neither close to 0 nor infinite. Values of both type of firms and the gap between them are non-trivial, thus a firm either invest in quality or invest in capacity. $0 < \underline{x}_H = \bar{x}_H < \underline{x}_L = \bar{x}_L = 1$. Re-lable $\underline{x}_H = \bar{x}_H = x_H$ and $\underline{x}_L = \bar{x}_L = x_L$, the firm:

$$\text{High quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, x_H] \\ \text{invests in capacity if } x \in [x_H, 1] \end{cases}$$

$$\text{Low quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, x_L] \\ \text{invests in capacity if } x \in [x_L, 1] \end{cases}$$

Case 12: RE vs RSE

Following Case 11, except that now we increase c. As a result value of both type of firms decreases. From Case 1, we know that a shirk zone is possible by changing c solely. By continuity we have Case 12.

Case 13: SE vs S

c is sufficiently high such that $\forall x \in [0,1]$, $c > \lambda\Delta(x)$. Holding c, M is sufficiently high such that a high quality firm with perfect reputation will invest in capacity, while still being sufficiently low such that a low quality firm with perfect reputation will not do the same.

The optimal investment is characterized by cutoff $\underline{x}_H = \underline{x}_L = 0$, $0 < \bar{x}_H < 1$, and $\bar{x}_L = 1$ such that the firm:

$$\begin{aligned} \text{High quality firm} &: \begin{cases} \text{shirks if } x \in [0, \bar{x}_H] \\ \text{invests in capacity if } x \in [\bar{x}_H, 1) \end{cases} \\ \text{Low quality firm} &: \text{always shirks} \end{aligned}$$

High quality firm adopts a "Shirk-expand" strategy; low quality firm adopts full shirk strategy.

Case 14: SE vs SE

c is sufficiently high such that $\forall x \in [0,1]$, $c > \lambda\Delta(x)$. Holding c, M is sufficiently high such that a low quality firm with perfect reputation will invest in capacity. The optimal investment is characterized by cutoffs $\underline{x}_H = \underline{x}_L = 0$; $0 < \bar{x}_H < \bar{x}_L < 1$ such that the firm:

$$\text{High quality firm} \quad : \quad \begin{cases} \text{shirks if } x \in [0, \bar{x}_H] \\ \text{invests in capacity if } x \in [\bar{x}_H, 1) \end{cases}$$

$$\text{Low quality firm} \quad : \quad \begin{cases} \text{shirks if } x \in [0, \bar{x}_L] \\ \text{invests in capacity if } x \in [\bar{x}_L, 1] \end{cases}$$

which is "Shirk-expand" for both types.

Case 15: RSE vs RS

Similar to case 10, c is sufficiently (but not infinitely) low, M is sufficiently high such that a high quality firm with perfect reputation will invest in capacity, while still being sufficiently low such that a low quality firm with perfect reputation will not do the same. Suppose that when $M = \bar{M}$, a high quality firm with perfect reputation is indifferent to invest or not, then we can always find ε sufficiently small such that $M = \bar{M} + \varepsilon$ satisfies the above condition. $0 < \underline{x}_H = \underline{x}_L < \bar{x}_H < 1$, $\bar{x}_L = 1$. High quality firm adopts a "Research-shirk-work" strategy while a low quality firm is doing "Research-shirk". Re-lable $\underline{x}_H = \underline{x}_L = \underline{x}$, the firm:

$$\text{High quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, \underline{x}] \\ \text{shirks if } x \in [\underline{x}, \bar{x}_H] \\ \text{invests in capacity if } x \in [\bar{x}_H, 1] \end{cases}$$

$$\text{Low quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, \underline{x}] \\ \text{shirks if } x \in [\underline{x}, 1] \end{cases}$$

Case 16: RSE vs RSE

Similar to Case 11 and 12, except that now we increase c further than Case 12. By similar argument, the firm:

$$\text{High quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, \underline{x}] \\ \text{shirks if } x \in [\underline{x}, \bar{x}_H] \\ \text{invests in capacity if } x \in [\bar{x}_H, 1] \end{cases}$$

$$\text{Low quality firm} \quad : \quad \begin{cases} \text{invests in quality } x \in [0, \underline{x}] \\ \text{shirks if } x \in [\underline{x}, \bar{x}_L] \\ \text{invests in capacity if } x \in [\bar{x}_L, 1] \end{cases}$$

A.7 Proof of Theorem 2

Suppose that the firm maintains its intensity of innovation over time $[t, \bar{t})$:

$$\begin{aligned} \Delta(t, q_0, q) \quad = \quad & \int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)s} \\ & [q_{t+s}^H x_{t+s} + \lambda(1 - \eta_t^H)V(t+s, L, q_0, q_{t+s}^H) \\ & + \lambda\eta_t^H V(t+s, H, q_0, q_{t+s}^H) + \mu V(0, H, q_{t+s}^H, q_{t+s}^H)]ds \\ & - \int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)s} \\ & [q_{t+s}^L x_{t+s} + \lambda\eta_t^L V(t+s, H, q_0, q_{t+s}^L) \\ & + \lambda(1 - \eta_t^L)V(t+s, L, q_0, q_{t+s}^L) + \mu V(t+s, L, q_0, q_{t+s}^L)]ds \\ & + e^{-(r+\lambda+\mu)\bar{t}}[V(\bar{t}, H, q_0, q_{\bar{t}}^H) - V(\bar{t}, L, q_0, q_{\bar{t}}^L)] \end{aligned}$$

where $\bar{t}$ is the time at which reputation, in the absence of breakthroughs, falls below a breakthrough and hence the firm takes to a different intensity of innovation.

Claim 1: $V_q < \frac{1}{r}$

Proof: $V_q(t, \theta, q_0, q_t) = E\int_0^{+\infty} e^{-rs} x_{t+s}^* ds$ is bounded above by $\frac{1}{r}$.

Claim 2: For an arbitrary constant δ, I can always find an m sufficiently small such that for $M < m$, (1) all cutoffs lie within $(0,\delta)$; (2) the firm innovates on $[\delta,+\infty)$

Proof: To have a cutoff, I require the following:

$$\lambda\Delta(t,q_0,q) = MV_q(t,\theta,q_0,q) \tag{2.11}$$

The left-hand side of (2.11) is bounded below by

$$\int_0^\infty e^{-(r+\lambda)w}\mu\int_0^\delta e^{-(r+\lambda+\mu)s}(x_\delta - x_{2\delta})dsdw$$

which represents forthcoming dividends yielded by a non-trivial probability that a high firm receives a breakthrough. By Claim 1, the right-hand side of (2.11) is bounded above by $\frac{M}{r}$. Suppose that

$$m < r\int_0^\infty e^{-(r+\lambda)w}\mu\int_0^\delta e^{-(r+\lambda+\mu)s}(x_\delta - x_{2\delta})dsdw$$

then for $M < m$, $t \geq \delta$, I have $\lambda\Delta(t,q_0,q) > MV_q(t,\theta,q_0,q)$ always.

For any reputation x, the firm implements one of the following actions:

$$\begin{aligned}
\eta(x,H,q_0,q) &= 1,\ \eta(x,L,q_0,q) = 1 \\
\eta(x,H,q_0,q) &= 0,\ \eta(x,L,q_0,q) = 1 \\
\eta(x,H,q_0,q) &= 0,\ \eta(x,L,q_0,q) = 0
\end{aligned} \tag{2.12}$$

Case 1: $\eta(x,H,q_0,q) = 1$, $\eta(x,L,q_0,q) = 1$.

In this case, $q^H_{t+s} = q^L_{t+s} = q_t$ for s$\in [0,\bar{t}]$. Plug in the numbers:

$$\begin{aligned}\lambda\Delta(t,q_0,q) - MV_q(t,\theta,q_0,q) &= \lambda\int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)s}\mu[V(0,H,q_t,q_t) \\ &\quad -V(t+s,L,q_0,q_t)]ds \\ &\quad +\lambda e^{-(r+\lambda+\mu)\bar{t}}[V(\bar{t},H,q_0,q_t) - V(\bar{t},L,q_0,q_t)] \\ &\quad -MV_q(t,\theta,q_0,q_t)\end{aligned}$$

where $\mu[V(0,H,q_0,q_t) - V(t+s,L,q_0,q_t)]$ is increasing in t. As $\eta^H = \eta^L = 1$, capacity is stable through $[t,\bar{t}]$fix two initial reputations $x_t < x'_t$, for the same history, I have $x^*_{t+s} < x^{*\prime}_{t+s}$ always; thus, $V_q(t,\theta,q_0,q) = \int_0^\infty e^{-rs}x^*_{t+s}ds$ is decreasing in t.

Case 2: $\eta(x,H,q_0,q) = 0$, $\eta(x,L,q_0,q) = 1$.

Plug in the numbers:

$$\begin{aligned}\lambda\Delta(t,q_0,q) - MV_q(t,\theta,q_0,q) &= \lambda\int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)s}[(q^H_{t+s} - q_t)x_{t+s} \\ &\quad +\mu V(0,H,q^H_{t+s},q^H_{t+s}) + \lambda V(t+s,L,q_0,q^H_{t+s}) \\ &\quad -\lambda V(t+s,H,q_0,q_t) - \mu V(t+s,L,q_0,q_t)]ds \\ &\quad +\lambda e^{-(r+\lambda+\mu)\bar{t}}[V(\bar{t},H,q_0,q^H_{\bar{t}}) - V(\bar{t},L,q_0,q_t)] \\ &\quad -MV_q(t,\theta,q_0,q_t) \qquad (2.13)\end{aligned}$$

For a low-quality firm, as t increases, reputation falls and q_t remains unchanged. Specifically, x_{t+s} decreases, $q^H_{t+s} - q_t = Ms$ does not change, $\mu V(0,H,q^H_{t+s},q^H_{t+s})$ does not change, and $-\lambda V(t+s,H,q_0,q_t)$ increases. The only uncertain terms are $\lambda V(t+s,L,q_0,q^H_{t+s}) - \mu V(t+s,L,q_0,q_t)$.

By total differentiation, I have

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}\frac{\partial q}{\partial t}$$

where $V_q(t+s,L,q_0,q^H_{t+s})$ is positive, $\frac{\partial q}{\partial t}$ is M, and $\frac{\partial V}{\partial t}$ is negative. Thus, $(\lambda-\mu)\frac{\partial V}{\partial t}+\lambda\frac{\partial V}{\partial q}\frac{\partial q}{\partial t}$ is positive.

For the above reason, $\lambda V(t+s,L,q_0,q^H_{t+s})-\mu V(t+s,L,q_0,q_t)$ increases in t.

The next thing I need to justify is that the decrease in $(q^H_{t+s}-q_t)x_{t+s}$ is outweighed by increases in other terms, specifically, $-\lambda V(t+s,H,q_0,q_t)$.

Notice that, by Claim 2, all cutoffs lie within $(0,\delta)$ and $\bar{t}$ is finite, i.e., case 2 does not last forever and the firm eventually falls to another decision region. Thus, x will not be infinitely close to $\frac{\mu+2\lambda-\sqrt{\mu^2+4\lambda^2}}{2\mu}$. $d[(q^H_{t+s}-q_t)x_{t+s}]/dt$ is negative and bounded below by

$$Ms\min_{s\in[0,\delta-t]}[\lambda(1-2x_{t+s})-\mu x_{t+s}(1-x_{t+s})]$$

moreover, $d[-\lambda V(t+s,H,q_0,q_t)]/dt$ is positive and bounded below by

$$-\lambda\int_0^T e^{-(r+\lambda+\mu)t}\max_{s\in[0,\delta-t]}[\lambda(1-2x_{t+s})-\mu x_{t+s}(1-x_{t+s})]dt$$

Thus, the ratio between the changes in the values of $(q^H_{t+s}-q_t)x_{t+s}$ and of $-\lambda V(t+s,H,q_0,q_t)$ with respect to changes in t is bounded for finite $\bar{t}$.

In conclusion, I can always find an m smaller than the boundary of this ratio, such that for $M<m$, the sum of the terms in the first square bracket of (2.13) increases in t.

For a high-quality firm, as t increases, reputation decreases and q_t increases. Specifically, x_{t+s} decreases, $q^H_{t+s}-q_t=Ms$ does not change, $\mu V(0,H,q^H_{t+s},q^H_{t+s})$ increases, and $\lambda V(t+s,L,q_0,q^H_{t+s})-\mu V(t+s,L,q_0,q_t)$ increases by the above argument. $\lambda V(t+s,H,q_0,q_t)$ is ambiguous, as the dividends decrease in t while capacity increases in t. However, by Claim 2, all cutoffs lie within $(0,\delta)$; thus by an argument similar to the above, $\frac{\partial V}{\partial t}$ is negative and bounded above by $\int_0^T e^{-(r+\lambda+\mu)t}\max_{s\in[0,\delta-t]}[\lambda(1-2x_{t+s})-\mu x_{t+s}(1-x_{t+s})]dt$ while $V_q(t+$

$s, H, q_0, q_t)$ is bounded above by $\frac{1}{r}$. Thus, for

$$M < m = |r\int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)t} \max_{s\in[0,\delta-t]} [\lambda(1-2x_{t+s}) - \mu x_{t+s}(1-x_{t+s})]dt|$$

$-\lambda V(t+s, H, q_0, q_t)$ increases in t. Moreover, as the sum of two bounded effects is still bounded, I can set m even smaller such that by the same argument as above, a decrease in $(q^H_{t+s} - q_t)x_{t+s}$ is outweighed by an increase in $-\lambda V(t+s, H, q_0, q_t)$. In conclusion, the sum of the terms in the square bracket is increasing in t.

By the same argument as in Case 1, V_q is decreasing in t.

Case 3: $\eta(x, H, q_0, q) = 0$, $\eta(x, L, q_0, q) = 0$.

In this case, $q^H_{t+s} = q^L_{t+s}$ for s$\in [0, \bar{t}]$, and hence I drop those notations and use q_{t+s} instead. Plug in the numbers:

$$\begin{aligned}\lambda\Delta(t, q_0, q) - MV_q(t, \theta, q_0, q) &= \lambda\int_0^{\bar{t}-t} e^{-(r+\lambda+\mu)s}\mu[V(0, H, q_{t+s}, q_{t+s}) \\ &\quad -V(t+s, L, q_0, q_{t+s})]ds \\ &\quad +\lambda e^{-(r+\lambda+\mu)\bar{t}}[V(\bar{t}, H, q_0, q_{\bar{t}}) - V(\bar{t}, L, q_0, q_{\bar{t}})] \\ &\quad -MV_q(t, \theta, q_0, q_t)\end{aligned}$$

where, by the same argument as in Case 2, as M is sufficiently small, $\mu[V(0, H, q_{t+s}, q_{t+s}) - V(t, L, q_0, q_{t+s})]$ is increasing in t. Further, by the same argument as in Case 1, V_q is decreasing in t.

Notice that $\lambda\Delta(t, q_0, q_t) = MV_q(t, H, q_0, q_t)$ when a firm is in case 1 to one side of the cutoff and in case 2 to the other side; thus, at those cutoffs, I have:

$$\begin{aligned}&\mu[V(0, H, q_t, q_t) - V(t, L, q_0, q_t)] \\ =\ &\mu V(0, H, q_t, q_t) - \mu V(t, L, q_0, q_t) - MV_q(t, H, q_0, q_t) + MV_q(t, H, q_0, q_t)\end{aligned}$$

Similarly, $\lambda\Delta(t,q_0,q_t) = MV_q(t,L,q_0,q_t)$ when a firm is in case 2 to one side of the cutoff and in case 3 to the other side; thus, at those cutoffs, I have:

$$\begin{aligned} & \mu V(0,H,q_t,q_t) + MV_q(t,H,q_0,q_t) - \mu V(t+s,L,q_0,q_{t+s}) - MV_q(t,L,q_0,q_t) \\ = \quad & \mu V(0,H,q_t,q_t) - \mu V(t,L,q_0,q_t) - MV_q(t,L,q_0,q_t) + MV_q(t,H,q_0,q_t) \end{aligned}$$

In conclusion, I can construct a function $f(t,q_0,q)$ such that

$$\Delta(x_t) = \int_0^\infty e^{-(r+\lambda+\mu)s} f(t+s,q_0,q_{t+s})ds$$

where $f(t+s,q_0,q_{t+s})$ is continuous and increasing in t for $s \in [0,+\infty)$. Thus, $\lambda\Delta(t,q_0,q_t) - MV_q(t,\theta,q_0,q_t)$ is increasing in t.

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CHAPTER 3

Multistage R&D Competition with Uncertainty

3.1 Introduction

Most important R&D projects are multistage projects. To develop a new model of fighter, a defense company has to invent a new engine for it. A new graphic card requires a new model of chip, or even a new chip architecture. A predominant feature of such projects is that the difficulty of some stages, especially the early ones, is highly uncertain and has to be LEARNED in the actual R&D process. Moreover, the difficulty of early stages determines the expected cost and hence the expected revenue.

The learning dynamics bring about two interesting issues. First, over time, firms may endogenously exit as they learn the difficulty. Second, the announcement of success in early stages creates technology spillovers, with the other firms interpreting the announcement as a signal that those stages are not difficult. In reality, a firm can strategically choose to report or withhold its success in early stages and control the technology spillover. For example, two defense companies are competing for an order of a new model of fighter. To invent the fighter, they must first invent the engine. Suppose that one company has just completed the engine; it can choose to establish its finding and accept the order (of engines) from the Defense Department. The downside is that its competitor can also use this engine to work on the fighter, rather than wait until its own engine is complete. However, if the first company does not disclose its new engine, the other firm may, after some period, invent and establish its own engine and at least obtain the revenue from the engine.

However, these strategic issues are understudied in the existing theoretical literature. For example, in Grossman and Sharprio (1987) and Choi (1991), exits are subject to exogenous shocks in policy or productivity; technology spillover automatically begins whenever an intermediary product is invented. In this paper, we propose the first model to incorporate (1) learning in exit decisions and (2) strategic disclosure of technology in multistage R&D competitions.

We discover new incentives that are governed by the immediate revenue from reporting[1] and the potential loss from technology spillovers. We investigate these incentives under a Poisson learning processes and discover when a firm reports its success in intermediary products, when it withholds this information to control technology spillovers, and when it chooses to exit. Our results suggest that in a symmetric equilibrium, the firm's reporting/withholding behavior is governed by time. We also show that the introduction of learning endogenously determines the time at which the firm abandons the project.

In the model, two suppliers are competing for a two-stage order. A firm must secure access to the intermediary product (namely, stage 1), before working on the final product (namely, stage 2). There are two ways to obtain access to the intermediary product: First, a firm can invent the product itself and earn revenue from it; second, if its opponent has announced a successful intermediary product, then our firm has to use that design without earning any revenue.

The degree of difficulty of stage 1 is unknown. Instead, each firm holds a belief concerning this difficulty, which is updated through the firm's own efforts and information released by its opponent. Absent success, the perceived difficulty increases as time evolves.

When a firm invents the intermediary product, it encounters a tradeoff. On the one hand, the firm may disclose the invention and immediately secure the associated revenue. However, the opponent will be able to borrow the design and save time on stage 1, and then the probability of the first firm losing stage 2 is increased. We call this effect the cost of technology spillover. On the other hand, the firm may conceal its success and research the final

[1] such as patents

product, at the risk that its competitor may soon succeed in stage 1 and share in the revenue from stage 1.

We focus on the above tradeoff and analyze the firm's strategy in equilibrium. First, we demonstrate the existence, in any equilibrium, of an endogenous exit time for the competitor, which gradually becomes increasingly pessimistic about the overall prospects. Thus, if success is achieved at an early time t, the perceived difficulty is low. The firm will disclose its success immediately because the exit point is still far ahead, and the opponent has a good chance to succeed in the first stage before exit.

Second, suppose that failure has has been experienced for a long period; then, the perceived difficulty is high and the other firm is about to abandon the project. A firm that just passed stage 1 will believe that its opponent is unlikely to pass stage 1 before the perceived difficulty becomes too high and triggers an exit. Thus, the firm will withhold its own success and wait until its opponent exits.

Finally, absent success, as time evolves, stage 1 is believed to be considerably difficult, and the firm exits.

In equilibrium, the state of the game is summarized by time, and we prove a report-withhold-exit equilibrium as explained above.

Our results also show that in multistage R&D, competition may not be cost-efficient. An important policy implication is that when the tenderee has considerable bargaining power over the contract terms, it may benefit from assigning the project to one firm instead of inviting multiple firms into competition.

The remainder of the paper is organized as follows. Section 2 establishes the model. Section 3 characterizes the equilibrium. Section 4 presents a numerical example of the model.

3.1.1 Literature

The model most similar to ours is that of Choi (1991), who develops a dynamic model of R&D in which participating firms have imperfect information concerning the true productivity of the R&D process[2]. A rival firm's success has two effects: First, it translates into a larger technological gap, and this effect is always negative; second, the success triggers an optimistic revision of beliefs concerning the difficulty of the task and encourages the firm to stay in the race. This shares similar features with our model. In his model, firms can perfectly observe the R&D activities of their rivals, i.e., technology spillovers occur automatically after any invention. This is in contrast to our model, in which firms cannot observe the status of their opponents, and technology spillovers can be strategically controlled by the inventor.

There are other works investigating dynamic competition, with different emphasis. Harris (1987) considers the technological uncertainty and strategic interaction between competitors as the race unfolds and shows how the efforts of competitors vary with their position in the race. Grossman and Shapiro (1987) introduce an intermediate step in the research program and explore how different institutions alter the dynamics of R&D rivalry. Our model draws merits from both, integrating the uncertainty in the task into a multistage scenario. Compared with this dynamic competition literature, our model is distinguished by imperfect monitoring between competitors. Thus, our focus is on the information disclosure rather than player effort.

Our paper is also related to that of Bloch and Markowitz (1996), who investigate the optimal disclosure delay in a patent race with leaders and followers. Similar to Grossman and Shapiro (1987), there is no uncertainty. In their model, the incentive for disclosure delay is driven by an exogenous protection period following disclosure rather than endogenous information updating as in our model.

[2]This is similar to our idea of "difficulty."

3.2 Model

Players and actions:

In the model, there are two firms, A and B, and one tenderee. The two firms compete for the tender on a continuous timeline. To fulfill the tender, a firm has to invent two types of new product, an intermediate product (stage 1) and a final product (stage 2). The intermediate product is the prerequisite for any research on the final output. At $t = 0$, they begin developing the intermediate product with an i.i.d. Poisson rate of success λ. λ may take one of two values: $H > 0$ or $L = 0$. In other words, a firm can either develop the intermediate product with positive probability, or it can never succeed. The firms cannot observe the value of λ: at $t = 0$, they hold a common prior belief that λ takes either value with equal probability. They update their belief by Bayes' rule over time.

Once a firm succeeds in developing the intermediate product, it can choose whether to report its invention. When it does so, if its competitor has not succeeded, the first firm claims all credit for the invention and receives a reward of $p_1 > 0$. However, if its competitor has also succeeded (but has chosen not to report), the competitor will file a lawsuit for the proprietorship of the invention, and the firms will ultimately each receive $\frac{p_1}{2}$ [3].

Once success on the intermediate product is reported, a technology spillover occurs: the product becomes available to both firms, and they then begin working on the second and final product, with an i.i.d. Poisson rate of success μ. We assume that μ only takes one value and that the value is common knowledge, thereby allowing us to focus on uncertainty over the intermediate product without considering insignificant technical details. However, if a firm chooses not to report after succeeding in developing the intermediate product, it can still work on the final product on its own. Whichever firm develops the final product receives a reward of $p_2 > 0$.

Each firm pays a cost per unit time $c > 0$ during the competition. This cost can be

[3] In principle, whether to file the lawsuit and obtain half of the reward is a choice, but as we will soon see, it must be the optimal action to take. Hence, without loss of generality, we assume that this process is automatic.

interpreted as the opportunity cost of foregoing other possible R&D activities. Once the final product is developed, the game ends and c is no longer incurred. A firm can choose to exit the competition at any time point before it ends. Once a firm exits, it stops paying c but cannot re-enter the competition. Moreover, we assume that exit is observed by the tender but not the opponent firm; this assumption prevents the firm from re-bargaining after the opponent exits.

Finally, we assume that $p_1 H > p_2 \mu$ (otherwise, reporting would never be the optimum action). We also assume that $p_2 \mu > c$ to ensure that a firm will not exit after passing the first stage.

Strategy:

Before finishing the intermediate product, a firm conditions its exit decisions on time elapsed.

After finishing the intermediate product, a firm never exits. It conditions its reporting/withholding decisions on time elapsed. We assume that a firm uses Bayesian updating whenever possible.

We focus solely on symmetric strategies. Specifically, we only refer to strategies that are piecewise continuous. That is, the strategy can be characterized by a number of cutoffs $t_1, t_2 ... t_n$, such that for an arbitrary i, the firm's equilibrium response within time segment (t_i, t_{i+1}) is constant. We call (t_i, t_{i+1}) a "report region" if the firm immediately reports any incoming success in (t_i, t_{i+1}); we call (t_i, t_{i+1}) a "withhold region" if the firm withholds any incoming success in (t_i, t_{i+1}) until t_{i+1}; and we call (t_i, t_{i+1}) an exit region if the firm exits.

Information:

We assume that firms do not directly observe λ; instead, each firm holds a belief $\tilde{\lambda}$ regarding the difficulty of the project $\tilde{\lambda} = \Pr(\lambda = H)$. A firm updates its belief through (1) the result (successful or not) of its own R&D project and (2) its opponent's reported success, silence, or announcement of exit. For simplicity, we assume that at $t = 0$, both firms hold an identical prior $\tilde{\lambda} = \frac{1}{2}$.

The trajectory of $\tilde{\lambda}$ over time is simple for the report region. Note that the game ends when either of the firms reports success. Thus, when reporting is the adopted action, absent a report of success, $\tilde{\lambda}$ evolves as follows:

$$\tilde{\lambda}(t) = \frac{e^{-2Ht}}{e^{-2Ht}+1} \tag{3.1}$$

When success occurs at $\hat{t}$, $\tilde{\lambda}$ jumps to 1 immediately.

For the withhold region, the updating process is somewhat more complex, as silence does not explicitly suggest that the opponent has invented the intermediate product; instead, it implies that the opponent has not succeeded in the second stage, thus undermining the likelihood of its success in the first stage.

Suppose that the last reporting time point is t_1. When withhold is the adopted action, absent a report of successful innovation, $\tilde{\lambda}$ evolves as follows:

$$\tilde{\lambda}(t,t_1) = \frac{e^{-2Ht_1-H(t-t_1)}(e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-sH}e^{-(t-s)\mu}ds)}{e^{-2Ht_1-H(t-t_1)}(e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-sH}e^{-(t-s)\mu}ds)+1} \tag{3.2}$$

Nevertheless, when success occurs at $\hat{t}$, $\tilde{\lambda}$ jumps to 1 immediately.

Lemma 1: $\tilde{\lambda}(t)$ always decreases in t.

Proof: For a detailed proof, see Appendix A.1. The intuition is straightforward but meaningful. In the report region, absent any report, the update of our focal firm is twofold. First, its unsuccessful research decreases its estimation of λ; second, the implied public information that its opponent has not succeeded also decreases its estimation of λ. In withhold region, although the firm no longer observes its opponent's state, it knows that the opponent has not completed stage 2 (otherwise, the opponent will report everything and the game is terminated) and thus is less likely to have completed stage 1. Thus, our focal firm's update draws from: first, its unsuccessful research and, second, the implied public information that the opponent has not completed stage 2. Intuitively, all this information diminishes the likelihood that the task is easy, i.e., $\tilde{\lambda}$ always decreases in t.

3.3 Equilibrium

3.3.1 Baseline models

We first consider a simplified model with perfect information and no exit. Specifically, we assume that λ is publicly known, i.e., $H = L = \lambda$.

Assuming that $\lambda p_1 > c$, $\mu p_2 > c$, firms never exit and the report-withhold decision depends solely on the relative payoffs of the inventions. The payoff for reporting is

$$p_1 + \frac{p_2}{2} \tag{3.3}$$

while the payoff for waiting time t before reporting is

$$\begin{aligned}
& e^{-\lambda t-\mu t}(p_1 + \frac{p_2}{2}) + \int_0^t \mu e^{-\mu s} e^{-\lambda s} ds \cdot (p_1 + p_2) \\
& + \int_0^t \lambda e^{-\lambda s} e^{-\mu s} ds \cdot \frac{(p_1 + p_2)}{2} \\
= \; & e^{-\lambda t-\mu t}(p_1 + \frac{p_2}{2}) + \frac{\mu + \frac{\lambda}{2}}{\mu + \lambda}(1 - e^{-\mu t-\lambda t})(p_1 + p_2) \qquad (3.4)
\end{aligned}$$

If $\frac{\mu+\frac{\lambda}{2}}{\mu+\lambda}(p_1 + p_2) > (p_1 + \frac{p_2}{2})$, then withholding is always the optimal strategy; otherwise, the firm reports.

Next, we maintain perfect information but consider an exogenous deadline for the intermediary good at which point firms that have not succeeded must exit. We assume that the deadline is set by the tender at $\bar{t}$ and is publicly known.

Theorem 1: In equilibrium, the optimal strategy (1) is characterized by a cutoff $\underline{t} = \bar{t}-$ such that a firm

$$\begin{aligned} \text{reports if } t &\in [0, \underline{t}] \\ \text{withholds if } t &\in [\underline{t}, \bar{t}] \end{aligned}$$

which we call the report-withhold strategy.

Proof: See Appendix A.2.

The intuition for Theorem 1 is straightforward. As the deadline approaches, the opponent is unlikely to complete the project in the remaining time and is about to exit. The benefit of reporting vanishes, while the cost does not, and hence one withholds.

This intuition carries over to the general game without an exogenous deadline but with an endogenous exit time for the competitor, which gradually becomes increasingly pessimistic regarding the overall prospects.

3.3.2 Complete model

Lemma 2: In any equilibrium, there exists $\bar{t}$ such that a firm exits when $t \in [\bar{t}, \infty]$.

Proof: As $t \to \infty$, absent success, $\tilde{\lambda}(t) < \frac{e^{-Ht}}{e^{-Ht}+e^{-Lt}} \to 0$, as $\tilde{\lambda}(t) > 0$, we have $\tilde{\lambda}(t) \to 0$. Then, we compare the firm's incentives for staying and exiting. The incentive to stay an additional time increment dt is less than $\tilde{\lambda}(t)H(p_1+p_2)dt$, which approaches $o(dt)$ as $t \to \infty$. By our assumption $c > \varepsilon$, an exit point always exists.

Lemma 2 confirms the existence of an exit region in any equilibrium. As we assumed that a firm can never return after exiting, and supposing that the strategy can be characterized by a number of cutoffs $t_1, t_2 ... t_n$, (t_n, ∞) is the only exit region.

Lemma 3: Suppose that $p_2 > 0$; in any equilibrium, prior to t_n, the firm must be withholding.

Proof: Suppose, alternatively, that the firm is reporting to the left of t_n. At $t_n - dt$, suppose that the probability that the firm's opponent has succeeded in the first phase is p; then, the incentive to report is $p_1(1-p) + \frac{1}{2}p_1 p + \frac{1}{2}p_2$, while the incentive to withhold is $(p_1+p_2)(1-p-o(t)) + \frac{1}{2}(p_1+p_2)(p+o(t)) + o^2(t)$.

However, if $p_2 = 0$, then the second phase provides no value at all, and a firm always reports immediately unless it is exiting.

Lemma 4: In any equilibrium, there exists one and only one withhold region.

Proof: For a detailed proof, see Appendix A.3. The approach here is that we assume a piecewise continuous equilibrium strategy whereby multiple withhold regions exist. We evaluate the incentives for different actions at the time of entering one withhold region and show that it contradicts the condition for, at some time afterward, exiting this withhold region and entering a report region. Thus, exiting a withhold region must be associated with exiting the market and the end of the game.

Lemma 4 suggests that once a firm begins withholding, it never reports.

Theorem 2: In equilibrium, the optimal strategy (1) is characterized by cutoffs t_1 and t_2 such that a firm

$$\begin{aligned} \text{reports if } t &\in [0, t_1] \\ \text{withholds if } t &\in [t_1, t_2] \\ \text{exits if } t &\in [t_2, \infty] \end{aligned} \tag{3.5}$$

which we call the report-withhold-exit strategy.

(2) An equilibrium exists.

(3) The equilibrium is unique.

Proof: For a detailed proof, see Appendix A.4. For (1), a summary of Lemmas 2, 3, and 4 shows that every equilibrium, if any exists, can only yield the report-withhold-exit strategy. For (2), we establish equilibrium existence by finding cutoffs t_1 and t_2 with indifference at the cutoff and proving that the incentives are piecewise monotone to conclude that the firm's action is optimal for all $t \in [0, \infty)$. The proof differs from that of Theorem 1 in that the firm can now choose the time of exit, which adds one dimension to the space of possible deviations. For (3), we prove that for any candidate equilibrium, the length of the withhold region is deterministic, specifically,

$$t_2 - t_1 = \frac{\ln \frac{H(p_1+p_2)}{p_1 H - p_2 \mu}}{H + \mu} \tag{3.6}$$

Then, we prove that the revenue from research is monotone in t_2 (maintaining (3.6)), which is the symmetric cutoff for both firms; given a constant research cost c, we can locate a single crosspoint of the revenue and cost of research, which in turn defines a unique equilibrium.

3.4 Numerical Example

We assume that the arrival rate of innovation is $H = 1$, $L = 0$ for the first stage and $\mu = 0.5$ for the second stage, with prior of difficulty $\tilde{\lambda}(0) = 0.5$, cost of research $c = 0.1$, revenue from the intermediate good $p_1 = 2$, and revenue from the final good $p_2 = 1$.

The numerical result is shown in Figure 3.1 and Figure 3.2. Figure 3.1 shows the trajectory of the firm's belief regarding the difficulty of the first stage; Figure 3.2 depicts the comparison of firm's incentives to "report now" and "withhold until the end", provided that the firm has invented the intermediate product. The firm exhibits report-withhold-exit behavior with cutoffs $t_1 = 1.21$ and $t_2 = 1.67$, i.e., the report region is $[0, 1.21]$; the withhold region is $[1.21, 1.67]$; and the exit point is 1.67.

Intuition can be drawn from Figure 3.2 in support of Theorem 1. Within the report region, the incentive to report remains constant while the incentive to withhold increases due to the upcoming exit point. At cutoff t_1, the incentives are the same and the firm shifts to withholding, in the hope that the opponent fails and exits soon. Then, in the withhold region, both incentives decrease over time, and the incentive to report decreases faster than the incentive to withhold. Here, the intuition is as follows: The incentive to report decreases due to the increasing probability that the opponent has completed stage 1, while the incentive to withhold decreases because it is increasingly unlikely for the firm to finish the final product before its opponent completes the first stage, which is obviously a secondary effect (considering the second stage and is conditional on the opponent's status).

Figure 3.3 compares the immediate revenue from and cost of conducting research on stage 1. The expected revenue decreases over time as $\tilde{\lambda}$ increases through Bayesian updates. The revenue exceeds the cost before t_2 but is lower than the cost after t_2, at which point the firm exits. From Figure 3.1, we know that $\tilde{\lambda}(t_2) = 0.0408$, i.e., the firm is nearly certain that the task is impossible.

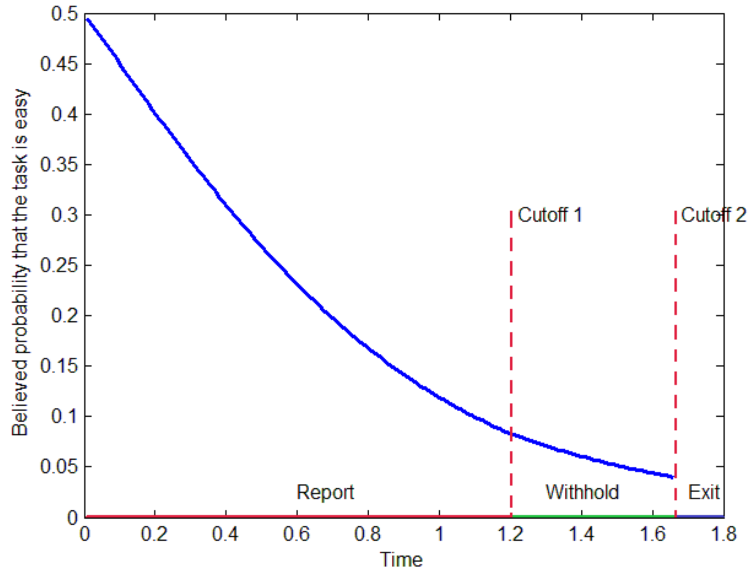

Figure 3.1: Trajectory of $\tilde{\lambda}$

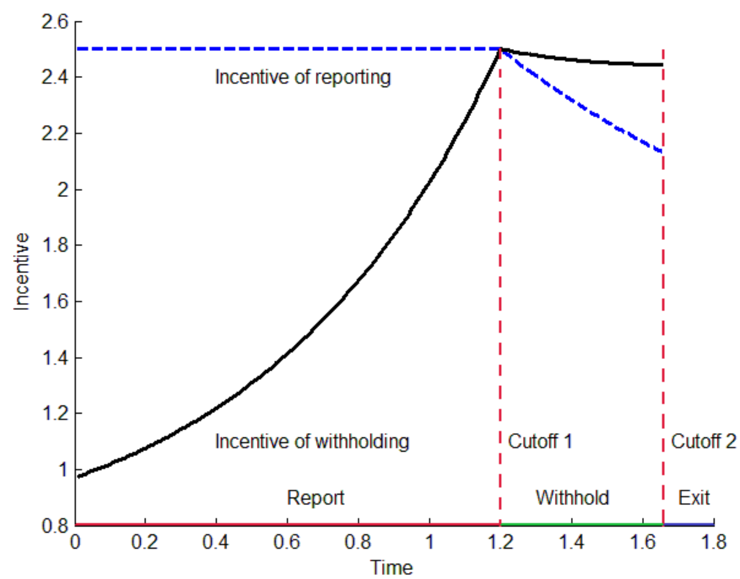

Figure 3.2: Comparison of Incentives

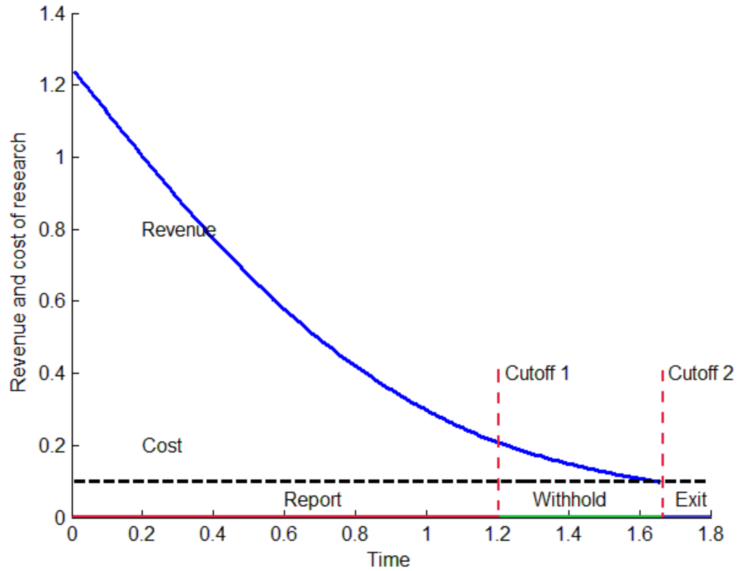

Figure 3.3: Revenue and Cost of Research

3.5 Conclusion

This paper has studied the dynamic competition between two firms for a two-stage R&D project. The difficulty of the first stage is unknown. We assumed that each firm holds a belief regarding the difficulty of stage 1 and updates the belief through its own R&D results and reports from its opponent. We showed that a firm maximizes its value by choosing an exit point and a report-withhold decision as a function of time.

Our result reveals that in multistage R&D, competition may not be cost-efficient. Absent collaboration, the arrival rate of R&D success for a single firm is constant, while competition induces an incentive to conceal the intermediate results, prohibits technology diffusion, and from a social welfare perspective, wastes other firms' resources on a solved problem.

We expect this model to serve as the foundation for several future research topics. For instance, when firms can invest to acquire higher research capability, as reflected by a higher exponential rate, our framework can be used to study possible under- or over-investment in different stages and compare the level of distortion between cases with and without uncertainty. Moreover, it would be worthwhile to investigate equilibrium behavior when there are multiple paths leading to the final solution.

3.6 Appendix

A.1 Proof of Lemma 1

For (3.1), rearrange:

$$1/\tilde{\lambda}(t) = 1 + e^{2Ht}$$

As t increases, $1/\tilde{\lambda}$ increases; thus, $\tilde{\lambda}$ decreases.

For (3.2), rearrange:

$$\begin{aligned}
1/\tilde{\lambda}(t,t_1) &= 1 + \frac{1}{e^{-2Ht_1-H(t-t_1)}\left(e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-sH}e^{-(t-s)\mu}ds\right)} \\
&= 1 + \frac{1}{e^{-2Ht} + e^{-2Ht_1-H(t-t_1)}\int_{t_1}^{t} He^{-sH-(t-s)\mu}ds} \\
&= 1 + \frac{1}{e^{-2Ht} + e^{-2Ht_1-H(t-t_1)}\frac{H}{\mu-H}e^{-sH-(t-s)\mu}|_{t_1}^{t}} \\
&= 1 + \frac{1}{e^{-2Ht} + e^{-2Ht_1-H(t-t_1)}\frac{H}{\mu-H}[e^{-tH} - e^{-t_1H-(t-t_1)\mu}]} \\
&= 1 + \frac{1}{e^{-2Ht} + e^{-2Ht_1-Ht-(t-t_1)\mu}\frac{H}{\mu-H}\left(e^{(t-t_1)(\mu-H)} - 1\right)} \qquad (3.7)
\end{aligned}$$

As t increases, e^{-2Ht} decreases, $e^{-2Ht_1-Ht-(t-t_1)\mu}\frac{H}{\mu-H}\left(e^{(t-t_1)(\mu-H)} - 1\right)$ decreases (for $\mu > H$ and for $\mu < H$; for $\mu = H$, the denominator becomes $e^{-2Ht} + e^{-2Ht_1-Ht-(t-t_1)\mu}H(t-t_1)$, and the same reasoning applies); therefore, as $1/\tilde{\lambda}$ increases, $\tilde{\lambda}$ decreases.

A.2 Proof of Theorem 1

The payoff from reporting is the same as (3.3). For any withholding region $[t_i, t_{i+1}]$ such that $t_{i+1} < \bar{t}$, evaluate the payoff at t_i:

$$\begin{aligned} & e^{-\lambda(t_{i+1}-t_i)-\mu(t_{i+1}-t_i)}(p_1+\frac{p_2}{2})+\int_{t_i}^{t_{i+1}}\mu e^{-\mu(s-t_i)}e^{-\lambda(s-t_i)}ds\cdot(p_1+p_2) \\ & +\int_{t_i}^{t_{i+1}}\lambda e^{-\lambda(s-t_i)}e^{-\mu(s-t_i)}ds\cdot\frac{(p_1+p_2)}{2} \\ = \ & e^{-\lambda(t_{i+1}-t_i)-\mu(t_{i+1}-t_i)}(p_1+\frac{p_2}{2})+\frac{\mu+\frac{\lambda}{2}}{\mu+\lambda}[1-e^{-\lambda(t_{i+1}-t_i)-\mu(t_{i+1}-t_i)}](p_1+p_2) \end{aligned} \tag{3.8}$$

which is essentially the same as (3.4).

For any withholding region $[t_i, t_{i+1}]$ such that $t_{i+1} = \bar{t}$, evaluate the payoff at t_i:

$$\begin{aligned} & e^{-\lambda(\bar{t}-t_i)-\mu(\bar{t}-t_i)}(p_1+p_2)+\int_{t_i}^{\bar{t}}\mu e^{-\mu(s-t_i)}e^{-\lambda(s-t_i)}ds\cdot(p_1+p_2) \\ & +\int_{t_i}^{\bar{t}}\lambda e^{-\lambda(s-t_i)}e^{-\mu(s-t_i)}ds\cdot\frac{(p_1+p_2)}{2} \\ = \ & [\frac{2\mu+\lambda}{\mu+\lambda}+\frac{\lambda}{\mu+\lambda}e^{-\lambda(\bar{t}-t_i)-\mu(\bar{t}-t_i)}]\frac{(p_1+p_2)}{2} \end{aligned} \tag{3.9}$$

Note that (3.9) is always larger than (3.8). Thus, we can exclude any withholding region $[t_i, t_{i+1}]$ with $t_{i+1} < \bar{t}$ from the equilibrium. That is, we can focus on the "report-withhold strategy." Moreover,(3.9) is increasing in t_i, and thus (3.9) and (3.3) has a single crossing point. This ensures the uniqueness of the equilibrium; then, we verify its existence.

Define $\underline{t}$=t_i such that (3.3)=(3.9). For the report region, the payoff from reporting (3.3) is constant, and the payoff from withholding from time t is

$$\begin{aligned} & e^{-\lambda(\underline{t}-t)}e^{-\mu(\underline{t}-t)}(p_1+\frac{p_2}{2})+\int_{t}^{\underline{t}}\mu e^{-\mu(s-t)}e^{-\lambda(s-t)}ds\cdot(p_1+p_2) \\ & +\int_{t}^{\underline{t}}\lambda e^{-\lambda(s-t)}e^{-\mu(s-t)}ds\cdot\frac{(p_1+p_2)}{2} \end{aligned} \tag{3.10}$$

which increases in t. The monotonicity of the payoff ensures that reporting is the best response for the assumed report region.

For the withhold region, at time t, the incentive to report is

$$\begin{aligned} & p_1 e^{-\lambda(t-\underline{t})} + \frac{p_1}{2}[1 - e^{-\lambda(t-\underline{t})}] + \frac{p_2}{2} \\ = \quad & \frac{p_1 + p_2 + p_1 e^{-\lambda(t-\underline{t})}}{2} \end{aligned} \tag{3.11}$$

Similar to (3.9), the incentive to withhold is

$$[\frac{\lambda}{\lambda+\mu} e^{-\lambda(\bar{t}-\underline{t})-\mu(\bar{t}-t)} + \frac{\mu}{\lambda+\mu} e^{-\lambda(t-\underline{t})} + 1]\frac{(p_1+p_2)}{2} \tag{3.12}$$

(3.12)-(3.11) is

$$\begin{aligned} & [\frac{\lambda}{\lambda+\mu} e^{-\lambda(\bar{t}-\underline{t})-\mu(\bar{t}-t)} + \frac{\mu}{\lambda+\mu} e^{-\lambda(t-\underline{t})}]\frac{(p_1+p_2)}{2} - \frac{p_1 e^{-\lambda(t-\underline{t})}}{2} \\ = \quad & \frac{\lambda}{\lambda+\mu} e^{-\lambda(\bar{t}-\underline{t})-\mu(\bar{t}-t)} \frac{(p_1+p_2)}{2} - \frac{1}{2}\frac{1}{\lambda+\mu} e^{-\lambda(t-\underline{t})}(\lambda p_1 - \mu p_2) \end{aligned}$$

which increases in t. Thus, witholding is confirmed to be the best response for the assumed withhold region.

A.3 Proof of Lemma 4

By Lemma 3, there must be at least one withhold region. Specifically, suppose that there is a series of withhold regions, $[t_1, t_2]$, $[t_3, t_4]$, ...,$[t_{2n-1}, t_{2n}]$; we analyze the firm's incentives at t_1.

For the first report region $[0, t_1]$, the payoff from reporting is $p_1 + \frac{p_2}{2}$, which is a constant.

For the withhold region, the expected payoff from withholding from t_1 to t_2 is

$$
\begin{aligned}
& e^{-H(t_2-t_1)}e^{-\mu(t_2-t_1)}(p_1+\frac{p_2}{2})+\int_{t_1}^{t_2}\mu e^{-\mu(s-t_1)}e^{-H(s-t_1)}ds\cdot(p_1+p_2) \\
& +\int_{t_1}^{t_2}He^{-H(s-t_1)}e^{-\mu(s-t_1)}ds\cdot\frac{(p_1+p_2)}{2} \\
= & [-\frac{\mu}{H+\mu}e^{-H(s-t_1)-\mu(s-t_1)}|_{t_1}^{t_2}](p_1+p_2)+e^{-H(t_2-t_1)-\mu(t_2-t_1)}(p_1+\frac{p_2}{2}) \\
& +[-\frac{H}{H+\mu}e^{-H(s-t_1)-\mu(s-t_1)}|_{t_1}^{t_2}]\frac{(p_1+p_2)}{2} \\
= & p_1[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t_1)}+\frac{\frac{1}{2}H+\mu}{H+\mu}]+p_2[\frac{-\frac{1}{2}\mu}{H+\mu}e^{-(H+\mu)(t_2-t_1)}+\frac{\frac{1}{2}H+\mu}{H+\mu}] \quad (3.13)
\end{aligned}
$$

At cutoff t_1, the firm should be indifferent between reporting and withholding:

$$
\begin{aligned}
p_1\frac{\frac{1}{2}H}{H+\mu}[e^{-(H+\mu)(t_2-t_1)}-1]+p_2\frac{\frac{1}{2}\mu}{H+\mu}[1-e^{-(H+\mu)(t_2-t_1)}] &= 0 \\
p_1\frac{\frac{1}{2}H}{H+\mu} &= p_2\frac{\frac{1}{2}\mu}{H+\mu}
\end{aligned}
$$

which contradicts our assumption $p_1H > p_2\mu$.

A.4 Proof of Theorem 2

(1) Combining Lemmas 2, 3, and 4, we prove that any equilibrium is a report-withhold-exit equilibrium. Now, we prove that (3.5) is an equilibrium.

Suppose the assumed equilibrium action is not the firm's best response and the firm wants to deviate. We define two types of deviation. Type-1 deviation means that the firm will withhold for some period and then report; type-2 deviation means that the firm's new strategy is still report-withhold-exit.

We verify that both types of deviation are inferior to the assumed equilibrium action. Then, reporting is confirmed to be the optimal action.

Similar to A.3, in the report region, i.e., $[0,t_1]$, the expected payoff from reporting is

$p_1 + \frac{1}{2}p_2$.

Suppose that the firm withholds for $[t, t']$ and then reports; by the proof of Lemma 4, the payoff is always less than $p_1 + \frac{1}{2}p_2$, and thus any type-1 deviation is not profitable.

Suppose that the firm withholds from t through the end, and suppose that the expected payoff at time t_1 is U; then, the expected payoff at time t is

$$\begin{aligned} & e^{-H(t_1-t)}e^{-\mu(t_1-t)}U + \int_t^{t_1} \mu e^{-\mu(s-t)}e^{-H(s-t)}ds \cdot (p_1+p_2) \\ & + \int_t^{t_1} He^{-H(s-t)}e^{-\mu(s-t)}ds \cdot \frac{(p_1+p_2)}{2} \end{aligned} \tag{3.14}$$

The profit generated by the deviation is (3.14) minus the original payoff $p_1 + \frac{1}{2}p_2$. However, the firm should be indifferent between reporting and withholding at t_1; thus, $U = p_1 + \frac{1}{2}p_2$. Then, following Lemma 4, any type-2 deviation is not profitable.

Hence, reporting is confirmed to be superior to withholding everywhere in the assumed report region.

In the withhold region, at time t, the incentive to report is

$$\begin{aligned} & p_1 e^{-H(t-t_1)} + \frac{p_1}{2}[1 - e^{-H(t-t_1)}] + \frac{p_2}{2} \\ = \quad & \frac{p_1 + p_2 + p_1 e^{-H(t-t_1)}}{2} \end{aligned} \tag{3.15}$$

The incentive to withhold is:

$$
\begin{aligned}
& e^{-H(t_2-t_1)}e^{-\mu(t_2-t)}(p_1+p_2)+\int_t^{t_2}\mu e^{-\mu(s-t)}e^{-H(s-t_1)}ds\cdot(p_1+p_2) \\
& +\int_t^{t_2}He^{-H(s-t_1)}e^{-\mu(s-t)}ds\cdot\frac{(p_1+p_2)}{2}+[1-e^{-H(t-t_1)}]\frac{(p_1+p_2)}{2} \\
= \; & [-\frac{\mu}{H+\mu}e^{-H(s-t_1)-\mu(s-t)}|_t^{t_2}+e^{-H(t_2-t_1)-\mu(t_2-t)}](p_1+p_2) \\
& +[-\frac{H}{H+\mu}e^{-H(s-t_1)-\mu(s-t)}|_t^{t_2}+1-e^{-H(t-t_1)}]\frac{(p_1+p_2)}{2} \\
= \; & [\frac{H}{H+\mu}e^{-H(t_2-t_1)-\mu(t_2-t)}+\frac{\mu}{H+\mu}e^{-H(t-t_1)}+1]\frac{(p_1+p_2)}{2} \qquad (3.16)
\end{aligned}
$$

Then, the difference between withholding and reporting is (3.16)-(3.15)

$$
\begin{aligned}
& [\frac{H}{H+\mu}e^{-H(t_2-t_1)-\mu(t_2-t)}+\frac{\mu}{H+\mu}e^{-H(t-t_1)}]\frac{(p_1+p_2)}{2}-\frac{p_1}{2}e^{-H(t-t_1)} \\
= \; & \frac{H}{H+\mu}e^{-H(t_2-t_1)-\mu(t_2-t)}\frac{(p_1+p_2)}{2}-\frac{1}{2}\frac{1}{H+\mu}e^{-H(t-t_1)}(Hp_1-\mu p_2)
\end{aligned}
$$

and the derivative with respect to t is

$$
\frac{H}{H+\mu}e^{-H(t_2-t_1)-\mu(t_2-t)}\frac{(p_1+p_2)}{2}\mu t+\frac{1}{2}\frac{1}{H+\mu}e^{-H(t-t_1)}(Hp_1-\mu p_2)H
$$

As $Hp_1 > \mu p_2$, the difference is increasing in t through $[t_1, t_2]$. Hence, withholding is confirmed to be superior to reporting everywhere in the assumed withhold region.

The next task is to prove that a unique cutoff t_1 exists. Similar to A.2. for the first report region $[0, t_1]$, the payoff from reporting is $p_1 + \frac{p_2}{2}$; for the withhold region, the expected payoff from withholding from t_1 to t_2 is

$$\begin{aligned}
& e^{-H(t_2-t_1)}e^{-\mu(t_2-t_1)}(p_1+p_2)+\int_{t_1}^{t_2}\mu e^{-\mu(s-t_1)}e^{-H(s-t_1)}ds\cdot(p_1+p_2) \\
& +\int_{t_1}^{t_2}He^{-H(s-t_1)}e^{-\mu(s-t_1)}ds\cdot\frac{(p_1+p_2)}{2} \\
= & [-\frac{\mu}{H+\mu}e^{-H(s-t_1)-\mu(s-t_1)}|_{t_1}^{t_2}+e^{-H(t_2-t_1)-\mu(t_2-t_1)}](p_1+p_2) \\
& +[-\frac{H}{H+\mu}e^{-H(s-t_1)-\mu(s-t_1)}|_{t_1}^{t_2}]\frac{(p_1+p_2)}{2} \\
= & \frac{(p_1+p_2)}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2-t_1)}+\frac{H+2\mu}{H+\mu}]
\end{aligned}$$

At cutoff t_1, the firm should be indifferent between reporting and withholding; otherwise, the firm would deviate:

$$\begin{aligned}
\frac{(p_1+p_2)}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2-t_1)}+\frac{H+2\mu}{H+\mu}] &= p_1+\frac{p_2}{2} \\
t_2-t_1 &= \frac{\ln\frac{\frac{p_1+\frac{p_2}{2}}{\frac{p_1+p_2}{2}}-\frac{H+2\mu}{H+\mu}}{\frac{H}{H+\mu}}}{-(H+\mu)} \\
t_2-t_1 &= \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}
\end{aligned}$$

The gap between t_1 and t_2, or the time length of the withhold region, is constant. Thus, given any t_2, we can always find a unique t_1.

The last message concerns exit. The firm exits when the expected payoff from an incoming success in stage 1 equals the instant cost:

$$\begin{aligned}
& cdt \\
= & \tilde{\lambda}Hdt\{e^{-H(t_2-t_1)}(p_1+p_2)+[1-e^{-H(t_2-t_1)}]\frac{(p_1+p_2)}{2}\} \\
= & \tilde{\lambda}Hdt[e^{-H(t_2-t_1)}+1]\frac{(p_1+p_2)}{2} \qquad (3.17)
\end{aligned}$$

which is decreasing in t_2. Thus, there is a unique solution for t_2.

Now, we need to prove that the firm will not exit before t_2; the proof is trivial for $t \in [0, t_1]$. For $t \in [t_1, t_2)$, the expected payoff from withholding for a short time dt rather than exiting immediately is

$$\begin{aligned} & \tilde{\lambda} H dt \{ e^{-H(t_2-t_1)} e^{-\mu(t_2-t)} (p_1+p_2) + \int_t^{t_2} \mu e^{-\mu(s-t)} e^{-H(s-t_1)} ds \cdot (p_1+p_2) \\ & + \int_t^{t_2} H e^{-H(s-t_1)} e^{-\mu(s-t)} ds \cdot \frac{(p_1+p_2)}{2} + [1 - e^{-H(t-t_1)}] \frac{(p_1+p_2)}{2} \} \\ = & \tilde{\lambda} H dt [\frac{H}{H+\mu} e^{-H(t_2-t_1)-\mu(t_2-t)} + \frac{\mu}{H+\mu} e^{-H(t-t_1)} + 1] \frac{(p_1+p_2)}{2} \end{aligned} \tag{3.18}$$

We prove that this is larger than the cost c. Note that the right-hand side of (3.17) is always larger than cdt for $t \in [t_1, t_2)$, and thus, we only need to prove that (3.18) minus the right-hand side of (3.17) is weakly larger than 0:

$$\begin{aligned} & \tilde{\lambda} H dt [\frac{H}{H+\mu} e^{-H(t_2-t_1)-\mu(t_2-t)} + \frac{\mu}{H+\mu} e^{-H(t-t_1)} + 1] \frac{(p_1+p_2)}{2} \\ & - \tilde{\lambda} H dt [e^{-H(t_2-t_1)} + 1] \frac{(p_1+p_2)}{2} \end{aligned} \tag{3.19}$$

$$= \tilde{\lambda} H dt \frac{(p_1+p_2)}{2} [\frac{H}{H+\mu} e^{-H(t_2-t_1)-\mu(t_2-t)} + \frac{\mu}{H+\mu} e^{-H(t-t_1)} - e^{-H(t_2-t_1)}]$$

$$= \frac{\tilde{\lambda} H dt}{H+\mu} \frac{(p_1+p_2)}{2} [H e^{-H(t_2-t_1)-\mu(t_2-t)} + \mu e^{-H(t-t_1)} - (H+\mu) e^{-H(t_2-t_1)}] \tag{3.20}$$

Taking the derivative with respect to t, we obtain

$$\begin{aligned} & \frac{\tilde{\lambda} H dt}{H+\mu} \frac{(p_1+p_2)}{2} [H e^{-H(t_2-t_1)-\mu(t_2-t)} \mu - \mu e^{-H(t-t_1)} H] \\ = & \frac{\tilde{\lambda} H dt}{H+\mu} \frac{(p_1+p_2)}{2} \mu H e^{-H(t_2-t_1)} [e^{-\mu(t_2-t)} - e^{-H(t-t_2)}] \end{aligned} \tag{3.21}$$

Note that $t < t_2$; thus, $e^{-\mu(t_2-t)} - e^{-H(t-t_2)} < 0$, and hence (3.21)$< 0$. (3.20) equals 0 at $t = t_2$ and decreases in t for $t \in [t_1, t_2)$, and thus (3.20)> 0 for $t \in [t_1, t_2)$.

3.7 References

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