MATHEMATICS *IUXTA COMMUNEM MODUM LOQUENDI*: FORMATION AND USE OF DEFINITIONS IN HEYTESBURY'S *DE MOTU LOCALI*

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One of the most significant aspects of late medieval science was the quantification of physics. The quantification theories of the late thirteenth and fourteenth centuries evolved out of primarily theological discussions such as the weight of sin or how the virtue of grace increased in the soul,[1] as well as statements in Aristotle's *Categories* and the *Commentary on the Categories* of Simplicius. However, the theory of the intension and remission of qualities soon concentrated on matters of natural science such as motion, thermodynamics, light, and medical applications in pharmacology.[2]

By the late thirteenth and early fourteenth centuries, at least four explanations for the differences in quantitative intensities were in circulation. The Thomist position maintained that increases in quality are the result of a subject's varying participation in an unchanged quality. Thus, intension and remission originate not in the quality, but in the subject and its disposition for the quality. Furthermore, increases in intensity of qualities do not occur by addition of one part of the quality to another.[3] By contrast, Henry of Ghent argued that intension of qualities takes place not by participation in the subject, but because the form itself is composed of parts. Intension results when each new part passes from potentiality to act. However, Henry agrees that there can be no part-by-part addition of qualities. The third theory was that of Godefroid de Fontaines and Walter Burley, who argued that intension is accomplished by a succession of forms,

[1] Pierre Duhem, *Études sur Leonard de Vinci* (Paris: De Nobele, 1955), 3:446; A. G. Molland, "The Geometric Background to the Merton School," *British Journal for the History of Science* 4 (1968):113-14. See also Thomas Aquinas, *Summa Theologiae*, II-II, q. 24 a. 4 on the quantification of charity.

[2] Marshall Clagett, "Richard Swineshead and Late Medieval Physics," *Osiris* 9 (1950):131-40; Edith Sylla, "Medieval Quantifications of Qualities: The 'Merton School'," *Archive for the History of Exact Sciences* 8 (1971):9-39.

[3] See Thomas' discussion of the intension of charity cited in note 1. This theory is very close to those of Aristotle and Simplicius.

each more perfect than its predecessor but not containing the predecessor as a numerical part of it. This theory also rejects the part-by-part addition of qualities. This part-by-part addition of qualities—the fourth theory—was supported by Duns Scotus and the Oxford Calculators. According to this theory, a quality is "intended" by the addition of a form while a "remission" takes place when one of the parts is subtracted. The process is analogous to the increase or decrease of mass: just as a quantity of water is increased by the addition of a new part of water, so a quality is made more intense by the addition of a new part of form.[4]

In the fourteenth century, these ontological questions were supplanted by a phenomenological one. Largely under the influence of Ockham, fourteenth-century natural philosophers attempted to describe what happens when a quality becomes more intense; the emphasis shifted from *why* to *how* the phenomenon occurs. The most successful and well-developed subject of this program was the theory of local motion. Beginning with Thomas Bradwardine and continuing with the Oxford School of John Dumbleton, Richard Swineshead, and William Heytesbury, kinematic theories of the early fourteenth century culminated in the Mean Speed Rule of Merton College.[5]

Together with these developments in physics, there was also a development in the theory of logic. In contrast to ancient and modern logic, medieval logic was closely associated with grammar and functioned as a *scientia sermocinalis* for the interpretation of theological texts. The creative period of medieval logic began with Peter Abelard (1079-1142) and reached its completion in the middle of the fourteenth century; it can be divided into two distinct phases. The first extends from Abelard to the middle of the thirteenth century and is characterized by the development of logic along purely formal lines, in isolation from scientific and epistemological problems. Although its development began with the logic of the Ancients (*logica vetus*), it contributed new elements (*logica moderna*) such as theories of properties of terms, *syncategoremata*, and *sophismata*. In the second phase, which extended from about 1250 to 1350, the theories of the earlier period were applied to scientific problems. Within the culture of the universities, the formal linguistic emphasis was replaced by the notion that logic was the means of scientific analysis; in part, this shift was the result of the Western assimilation of the scientific works of Averroes and Avicenna. This development achieved its culmination in the mathematical-logical treatises written in Oxford and Paris in the first half of the fourteenth

[4] Clagett, pp. 131-39. Edith Sylla, "Medieval Concepts of the Latitude of Forms: the Oxford Calculators," *Archives d'histoire doctrinale et littéraire du Moyen Age* 40 (1974):226-33; E. Sylla, "Medieval Quantifications," pp. 9-15, 24-39.

[5] E. J. Dijksterhuis, *The Mechanization of the World Picture* (Oxford: Avendon Press, 1961), pp. 187-93.

century.[6] At Oxford, the foundation of education in the early fourteenth century was the study of the trivium arts represented by the *libri logicales* and the quadrivium studies of physics and mathematics, with the latter functioning as the handmaiden of the *libri naturales*. Although logic and physics were thus formally distinct within the faculty of arts, they became fused in the *sophismata* literature of the period. Conversely, texts such as the *Calculationes de motu* were especially suitable for logical discussions.[7]

The English mathematician-logician William Heytesbury typifies the movement at Merton College in the early fourteenth century. William was probably born before 1313, perhaps in Wiltshire. He first appears in Merton records as a Fellow in 1330, then was named a Fellow of Queen's College in 1340. He became a Master of Theology before 1348 and served as Chancellor of the University on two occasions before he died in December 1372 or January 1373.[8] If Heytesbury composed any theological works, none have survived; his primary work is a logical treatise, the *Regule solvendi sophismata*.[9] Written in 1335, it belongs to the early arts period of Heytesbury's career. According to the *prohemium*, the work was designed as a medium-length *summa* for the *juvenes studio logicalium agentes primum annum*.[10] However, the treatise presupposes considerable knowledge of physics and logic, and therefore probably was designed to aid participants in disputations.

The text is divided into six chapters: (1) *De insolubilibus*, (2) *De scire et dubitare*, (3) *De relativis*, (4) *De incipit et desinit*, (5) *De maximo et minimo*, and (6) *De tribus predicamentis*. Each deals with a particular type of

6 Ernest A. Moody, "The Medieval Contribution to Logic," *Studies in Medieval Philosophy, Science, and Logic* (Berkeley and Los Angeles: University of California Press, 1975), pp. 371-92; Ernest Moody, *Truth and Consequence in Medieval Logic* (Amsterdam: North-Holland Publishing Co., 1953), pp. 1-10; Norman Kretzman, "History of Semantics," *Encyclopedia of Philosophy* (New York: Macmillan, 1967), 7:370; William Kneale, *The Development of Logic* (Oxford: Clarendon Press, 1962), pp. 198-245.

7 James A. Weisheipl, O.P., "Developments in the Arts Curriculum at Oxford in the Early Fourteenth Century," *Medieval Studies* 28 (1966):151-75; A. Maier, *An der Grenze von Skolastik und Naturwissenschaft* (Rome: Edizioni di Storia e letteratura, 1952), p. 264ff.

8 Duhem, *Études*, 3:405-08; James Weisheipl, "Ockham and Some Mertonians," *Medieval Studies* 30 (1968):195-96.

9 For a list of manuscripts of the *Regule* see Curtis Wilson, *William Heytesbury: Medieval Logic and the Rise of Mathematical Physics* (Madison, Wisc.: University of Wisconsin Press, 1960), pp. 206-07 and Marshall Clagett, *The Science of Mechanics in the Middle Ages* (Madison, Wisc.: University of Wisconsin Press, 1959), p. 220 n. 5. See also A. Maier, *An der Grenze*, p. 261, n. 26.

10 *Regule solvendi sophismata* . . . (Venice: Bonetas Locatellus, 1494) fol. 4^{v}. All references to the text refer to this incunable edition; as Maier (loc. cit.) has shown, the text of this edition is practically identical with that of Erfurt, Stadtbibliothek, Amplonian, Fol. MS. 135, ff. 1-17.

sophism: the first three are of a purely logical nature, while the last three discuss sophisms arising out of physical science. The topics discussed in the first five chapters were fairly standard in fourteenth-century Oxford; all arose naturally out of the doctrine of supposition.[11] As will be shown later in this paper, the sixth chapter builds upon the logical demonstrations and concepts of the preceeding chapters.

The sixth chapter is a treatise on kinematics rather than dynamics. It is divided into three sections, (i) *De motu locali*, (ii) *De augmentatione*, and (iii) *De alteratione*, each dealing with the three Aristotelian categories in which motion occurs: place, quantity, and quality. This paper proposes to examine the interrelationship of language analysis and particularly the concept of definition and the theory of motion in Heytesbury's *De motu locali*. Second, I hope to demonstrate some deficiencies in his method and how these deficiencies arose.

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Heytesbury approaches physics problems by examining them through a logical analysis of terms, which in the early fourteenth century included a study of mathematical propositions. According to Duhem, this process was a form of "logical acrobatics": treatises such as the *Regule* were designed to argue sophisms under the guise of physical problems.[12] This seems to misrepresent Heytesbury's program, for the reciprocity of the method prevents such degeneration. On the one hand, physical phenomena are described and analyzed through the logic of terms as prescribed by the *logica moderna*. On the other, mathematical and physical properties are used to analyze logical and semantic problems, particularly the problem of denomination.[13]

Heytesbury establishes two criteria in developing the *Regule*. First, definitions in the text must proceed from the common mode of speech. Thus, although he argues that there is nothing in nature which is an instant *per se*, he concedes that, *iuxta communem modum loquendi*, everything which exists is in an instant, in the sense that everything is measured by an instant.[14] Against the position that the change in augmentation is measured

[11] Weisheipl compares the works of several fourteenth-century logicians at Oxford, including Richard Billingham (fl. 1344-61), William Sutton (fl. 1330-40) and Heytesbury. See "Developments in the Arts Curriculum," pp. 157-61. For a discussion of theories of supposition, see Moody, *Truth and Consequence*, pp. 18-26 and Philotheus Boehner, *Medieval Logic* (Manchester University Press, 1952), pp. 27-51.

[12] Duhem, *Études*, 3:441-48.

[13] Wilson, *Heytesbury*, pp. 21-24.

[14] "Et ideo ad ultimum quod ibidem propositum fuerat, scilicet quod multa incipiunt esse et etiam incipient et desinent esse, quorum nullum erit in instanti, dicitur

only by the absolute change in quantity in a given time, Heytesbury argues that according to this theory a great oak and a small shrub would be augmented equally if they both acquired the same amount of size in a given time, a conclusion which he terms "false and against common speech."[15] Second, the definition should be free from mathematical contradictions.[16] The question of the compatibility of common speech and mathematical rigor is a natural one, one which will concern us as we examine the *De motu locali*.

Heytesbury's *De motu locali* has been previously studied for its statement of the "Merton Mean Speed Rule." In its modern form, the theorem is stated as

$$S = V_o + \left(\frac{V_o \quad + \quad V_f}{2} \right) t$$

where S is the distance traversed in time t, and V_0 and V_f are the initial and final velocities, respectively. Its pre-Mertonian development is uncertain, but the form given in the *De motu locali* (1335) is the earliest statement of the theorem. It is stated without proof, but a contemporary treatise, the *Probationes conclusum*, is generally considered the first proof of the Rule.[17]

Despite the advances made in this text, there are four problems in it. First, the definition of uniform motion is imprecise. Heytesbury defines uniform motion as one in which equal spaces are continually traversed in

negando illam propositionem iuxta communem modum loquendi quia omne quod est, est in instanti, eo quod illud instantanee mensurat instans sive sit tempus vel motus aut etiam instans." *De incipit et desinit*, fol. 26^r.

[15] ". . . ita velociter cresceret una antiqua quercus quasi per totum arida sicut ista herba demonstrata que inequali tempore a minina quantitate quasi insensibili crevit ad bicubitalem seu tricubitalem. Quod est falsum et contra communem modum loquendi." *De tribus predicamentis*, fol. 45^r.

[16] Thus, for example, in *De motu locali*, fol. 38^v, Heytesbury argues that although every magnitude moved by local motion moves as fast as some part of it (in a categorematic sense), it does not follow that it moves as slowly as part of it, for in the case of a rotating cylinder, the center does not move at all.

[17] For detailed studies of the Mean Speed Theorem and its effect on the development of late medieval and early modern kinematics, see Wilson, *Heytesbury*, pp. 122-26; Marshall Clagett, *Giovani Marliani and Late Medieval Physics* (New York: Columbia University Press, 1941), pp. 101-24; William A. Wallace, "Mechanics from Bradwardine to Galileo," *Journal of the History of Ideas* 32 (1971):15-28; and Marshall Clagett, *Science of Mechanics*, pp. 199-329. See also Edith Sylla, "Medieval Quantification," p. 31 for Richard Swineshead's commentary on the application of the theorem of the mean to heat. The authorship of the *Probationes conclusum* (in William Heytesbury, *Hentisberi de sensu composito et diviso* . . . [Venice: Bonetus Locatellus, 1494], fols. 188^v-203^v) has been in dispute since Duhem questioned whether Heytesbury himself was responsible for the text; see *Études*, 3:468-71.

equal parts of time.[18] He fails to say that these distances are traversed in *any* equal parts of time. This deficiency was criticized by contemporaries, including Swineshead, as well as later commentators, notably Galileo, Angelo da Fosambruno, and Gaetano di Thienne. Modern commentators have pointed out that Heytesbury probably realized the need for this condition, because it appears in his definition of uniform acceleration.[19] This seems to avoid the issue; inconsistency in definition is tantamount to *redefining* a concept when the need for greater precision is perceived. Such a redefinition may produce not only ambiguity but also logical or mathematical contradiction.

A portion of the text which has received a great amount of attention from modern scholars is Heytesbury's definition of instantaneous velocity: "In difform motion, the velocity in any instant is measured according to the line which the point moving most rapidly *would* describe, if it were moved uniformly through time at that amount of velocity with which it is moved at that same instant, whatever is given."[20] The modern, mathematically productive definition of instantaneous velocity depends on the derivative dS/dt. Fourteenth-century theorists, constrained by the prescriptions of Euclidean geometry, could not express proportions between dissimilar magnitudes. Because there was no definite idea of velocity as S/t, it is hardly surprising that fourteenth-century authors, including Heytesbury, never defined instantaneous velocity as the limit of such ratios, or in modern terms, dS/dt. This ambiguity in definition, therefore, is fundamentally different from the ambiguity arising from the definition of uniform motion. The former arises from a deficiency in mathematical theory; the latter from a deficiency in the language used to express the theory.[21]

18 "Motuum ergo localium dicitur uniformis quo equali velocitate continue in equali parte temporis spacium pertransitur equale." *De motu locali*, fol. 37^v. The text of the *De motu locali* is taken from the modern critical edition found in Clagett, *Science of Mechanics*, pp. 239-42, 277-83.

19 "Uniformiter enim intenditur motus quicunque, cum in quacunque equali parte temporis, equalem acquirit latitudinem velocitatis." *De motu locali*, fol. 39^v. See Clagett, *Science of Mechanics*, p. 237; Wilson, *Heytesbury*, p. 195, n. 12.

20 My italics. "In motu autem difformi, in quocunque instanti attendetur velocitas penes lineam quam describeret punctus velocissime motus, si per tempus moveretur uniformiter illo gradu velocitatis quo movetur in eodem instanti, quocunque dato." *De motu locali*, fol. 38^v. See Clagett, *Science of Mechanics*, pp. 214-15; Wilson, *Heytesbury*, pp. 120-22.

21 Heytesbury's definition of instantaneous velocity is not altogether sterile, however. It corresponds to the modern heuristic technique used by teachers of the calculus, whereby the derivative of a function is graphically represented as the tangent to the graph of the function at any point. See Clagett, *Science of Mechanics*, pp. 214-15, n. 36 for another modern interpretation. For other examples of this type

The third problem lies within the vocabulary of fourteenth-century kinematics. In the analysis of local motion, fourteenth-century authors developed a distinct vocabulary to express physical concepts; although this terminology had a definite effect on the work of later authors, it was not without defect. The primary difficulty was grounded in the absence of standardization of terms. Thus, Bradwardine uses "*qualitas motus*"[22] and Oresme "*intensio motus*"[23] when referring to the same physical concept. Moreover, even within a single text, vocabulary was not used in a one-to-one correspondence with concepts. On the one hand, the same scholastic term could be used for two different physical concepts: Heytesbury, for example, uses *motus* in two different senses. It is used primarily when speaking of motion in general, as in the opening sentence of the *De motu locali*: "There are three categories or generic ways in which motion, in the strict sense, can occur."[24] On the other hand, *motus* is often used more specifically for speed or velocity: "it is moved more quickly than before and increases its velocity (*motus*)," or "any point of it will continually reduce its velocity (*motus*) throughout the same hour."[25] On occasion, he uses both senses of the word together: "it will be the case that continually throughout the hour, any point *A* of its motion (*motus*) decreases its velocity (*motus*)."[26] On the other hand, Heytesbury often uses two scholastic terms for the same physical concept: *latitudo motus* and *latitudo velocitatis* are both used interchangeably for an increment of velocity.[27]

Finally, Heytesbury's use of the terms *inclusive* and *exclusive* produces problems in the text. He introduces the terms in his treatise *De incipit et desinit*, for they represent in the continuum the same relationship that *incipit* and *desinit* do in time.[28] The definitions closely follow Aristotle's

of intuitive reasoning process, see George Polya, *Mathematics and Plausible Reasoning*, 2 vols. (Princeton, N.J.: Princeton University Press, 1954).

22 *Tractatus de proportionibus*, ed. H. Lamar Crosby, Jr. (Madison, Wisc., University of Wisconsin Press, 1955), p. 118: ". . .sicut non differunt in qualitate resistendi sed in quantitate, sic nec motus per media differunt in qualitate motus (quae est velocitas et tarditas) sed in quantitate motus (quae est longitudo vel brevitas temporis)."

23 "Omnis motus successivus subiecti divisibilis habet partes. . . . tertio modo saltem ymaginative secundum gradum et intensionem velocitatis." *De configuratione qualitatum et motuum*, ed. Marshall Clagett (Madison, Wisc.: University of Wisconsin Press, 1968), 2:i.

24 "Tria sunt predicamenta vel genera in quorom quolibet contingit proprie motum esse," fol: 37^r; the translation is that of Clagett, *Science of Mechanics*, p. 235.

25 ". . . velocius movetur quam prius, et intendit motum suum"; ". . . quilibet punctus illius continue per eandem horam tardabit motum suum," fol. 38^r; my translations.

26 ". . . continue erit ita per eandem horam, quod quilibet punctus motus ipsius *a* remittit motum suum," fol. 38^v; my translation.

27 Fol. 39^v.

28 Fol. 23^v.

theory of time:[29] the instant, as the dividing point of time, can be thought of as both the endpoint of one period of time and as the initial point of another. Yet when one considers the duration of a particular process, the instant must be assigned to one period or the other by a well-defined rule. Similarly, in the continuum, the point ξ determines a Dedikind cut by defining the sets A and B as follows:

$$A = \{x \in R: x \leqslant \xi\} \qquad B = \{x \in R: x > \xi\}$$

or alternatively:

$$A = \{x \in R: x < \xi\} \qquad B = \{x \in R: x \geqslant \xi\}$$

Topologically, this corresponds to half-open intervals $(-\infty, \xi]$, $(\xi, +\infty)$ or $(-\infty, \xi)$, $[\xi, +\infty)$.

Heytesbury also states that in any instant, point *a* will traverse one point of line *b*; therefore, in a finite number of instants, *a* will traverse the same finite number of points of *b*, yet in no finite number of instants will *a* traverse any part of *b*. However, in all the intrinsic instants of the time of its motion, *a* will traverse all of *b*, since there will be an infinite number of instants in this period.[30] Thus, in this system of measurement, uniform or individual degrees are dispensed with, and there is no difference between one line terminated extrinsically at a given point and another terminated intrinsically at the same point.[31]

29 *Physics*, VI:236a7-236b18.

30 ". . . potest probabiliter dici: quod *a* pertransivit *b* in omnibus instantibus intrinsecis istius hore. Et tamen in nullis instantibus finitis pertransivit aliquod illius. Unde dici potest probabiliter quod in quolibet instanti pertransivit *a* unum punctum de *b*. Et ideo libet in nullis instantibus finitis pertransivit *a* aliquod de *b*, tamen in istis instantibus finitis pertransivit *a* tot puncta de *b* quot sunt illa instantia. Et sic per accidens in omnibus instantibus illius hore intrinsecis pertransivit *a* totum *b*. Quod in illis pertransivit *a* omnia puncta illius. Et per consequens pertransivit *a* totam illam lineam." *Sophisma* 29, fol. 152^v-153^r.

31 It is interesting to note that this idea is fundamental to nineteenth-century integration theory. It corresponds to the concept of zero content of a subset. In modern terms, a subset Z of R^p has zero content if for every $\epsilon > 0$, there is a *finite* set $\{J_1, J_2, J_3, \ldots J_n\}$ of closed intervals such that

$$Z \subset \bigcup_{i=1}^{n} J_i$$

and

$$d(J_1) + d(J_2) + d(J_3) + \ldots + d(J_n) < \epsilon$$

where $d(J_i)$ is the content of J_i. According to Riemann integration theory, the

With these definitions in mind, let us examine how Heytesbury uses them in the *De motu locali*. First, Heytesbury states that the velocity of a body is to be measured by the path traversed by the point which is in most rapid motion. If such a point does not exist, as in the case of a spiral which has no ultimate point or in the case of a body whose ultimate point begins to be corrupted continually, the velocity is to be given by the point which would be moved indivisibly more rapidly than any point of the magnitude.[32] This is equivalent to defining the velocity extrinsically by the supremum of the sequence of velocities of the body. Such a statement is consistent with Heytesbury's definitions cited above.

In the development of the Mean Speed Theorem, however, Heytesbury does not fare as well. He begins by stating the general theorem: "Whether it commences from zero degree or some degree, every latitude will correspond to its mean degree, as long as it is terminated by some finite degree acquired or lost uniformly."[33] In the case of a body beginning from rest and terminating at a finite velocity, the mean is one-half the final velocity. With these preliminary remarks let us examine the three passages in which Heytesbury uses the terms *inclusive* and *exclusive*:

> I. From this it follows that the mean degree of any latitude bounded by two degrees (taken either inclusively or exclusively) is more than half the more intense degree bounding that latitude.[34]
>
> II. It also follows in the same way that when any moving body is uniformly accelerated from some degree [of velocity] (taken exclusively) to another degree inclusively or exclusively, it will traverse more than one-half the distance which it would traverse with a uniform motion, in an equal time, at the degree [of velocity] at

integral over a set of zero content is zero. In fact, an integral over a *finite* set would also be zero in the sense of Lebesgue. See Robert G. Bartle, *Elements of Real Analysis* (New York: John Wiley and Sons, 1964), pp. 316-23 and H. L. Royden, *Real Analysis*, 2nd ed. (London: Macmillan, 1968), pp. 52-93.

32 "Posito nempe casu quo mote magnitudinis nullus sit punctus velocissime motus, penes lineam quam describeret punctus quidam qui indivisibiliter velocius moveretur, quam aliquis in magnitudine illa data tota, totius velocitas attendetur: Sicut posito quod continue incipiant corrumpi puncta extrema, aut quod nulla sint ultima puncta illius accidit in linea girativa que ponitur infinita." *De motu locali*, fol. 37ᵛ-38ʳ.

33 "Omnis enim latitudo sive a non gradu incipiat, sive a gradu aliquo, dum tamen ad gradum aliquem terminetur finitum, et uniformiter acquiratur seu deperdatur, correspondebit equaliter gradui medio sui ipsius. . . ." *De motu locali*, fol. 40ʳ.

34 "Ex quo sequitur quod cuiuslibet latitudinis terminate ad duos gradus, inclusive vel exclusive, est gradus medius maior quam subduplus ad gradum intensiorem eandem latitudinem terminantem." *De motu locali*, fol. 40ʳ. The translation is that of Clagett, *Science of Mechanics*, pp. 270-71.

which it arrives in the accelerated motion. For that whole motion will correspond to its mean degree [of velocity], which is greater than one-half of the degree [of velocity] terminating the latitude to be acquired.[35]

III. With respect, however, to the distance traversed in a uniformly accelerated motion commencing from zero degree [of velocity] and terminating at some finite degree [of velocity], it has already been said that the motion as a whole, or its whole acquisition, will correspond to its mean degree [of velocity]. The same thing holds true if the latitude of motion is uniformly acquired from some degree [of velocity] in an exclusive sense, and is terminated at some finite degree [of velocity].[36]

In passage I, Heytesbury is quite certain that he is referring to two non-zero degrees, say a and b. Thus, he feels quite comfortable in stating that for velocities in the sense of closed or open intervals $[a,b]$ or (a,b) or half-open intervals $[a,b)$ and $(a,b]$ (*inclusive vel exclusive*) the mean degree is greater than $1/2b$. Nevertheless, he has not precisely stated that $a \neq 0$; if a should equal zero his statement would be false. In II, he is somewhat less comfortable about this condition and thus removes the statement that the lower bound may be included in the interval; he now requires that the interval be open on the left while either open or closed on the right (i.e., (a,b) or $(a,b]$). Nevertheless, even this condition will not solve his problem, for as we have seen above,[37] if $a = 0$, $m(0,b] = m[0,b]$, where $m(a,b)$ is the measure of (a,b). In III, Heytesbury restates the Mean Speed Theorem, but this time adds the condition that the interval be open to the left for non-zero velocities.

It may be argued that the problem in these passages results from the

[35] "Sequitur etiam consimiliter quod cum aliquod mobile ab aliquo gradu exclusive ad alium gradum inclusive vel exclusive intendet uniformiter motum suum, quod ipsum plus pertransibit quam subduplum ad illud quod ipsum uniformiter pertransiret in equali tempore secundum istum gradum ad quem stabit intensio sui motus; quia totus ille motus correspondebit gradui suo medio, qui maior est quam subduplus ad gradum terminantem illam latitudinem acquirendam." Clagett, *Science of Mechanics*, p. 271.

[36] "Quantum autem ad magnitudinem pertranseundam, uniformiter acquirendo talem latitudinem motus incipientem a non gradu et terminatam in aliquem gradum finitum, dictum est prius quod totus ille motus seu tota illa acquisitio correspondebit gradui suo medio. Et consimiliter etiam, si ab aliquo gradu exclusive uniformiter acquiratur latitudo motus ad aliquem gradum terminata finitum." *De motu locali*, fol. 40^{v}, Clagett, *Science of Mechanics*, p. 272.

[37] See notes 30 and 31 together with the text.

imprecision of the scholastic term for zero. Heytesbury speaks of motions which begin "*a non gradu*" and "*a quiete*"; his use of *non gradu* might give *aliquo gradu* a non-zero definition by contrast. Yet even if this is granted, it does not solve the internal contradiction between passage I and passages II and III. If *aliquo gradu* is a non-zero value in II, Heytesbury has no reason to limit himself to an exclusive (or half-open) interval of values. Similarly, if *aliquo gradu* in III is a non-zero value, Heytesbury has restated the Mean Speed Theorem in a significantly weaker form, with no justifiable reason for doing so. Thus, the terms *inclusive* and *exclusive*, although defined with valuable mathematical insight, have produced serious difficulties in the text when combined with the concept of initial velocity.

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The common element in the four problems discussed in this paper is imprecision in definitions and the use of definitions. We have seen that the imprecision may arise either from a deficiency in mathematical development, as in the case of instantaneous velocity, or from a deficiency in the language used to express the physical and mathematical theory. Both are inextricably bound to the foundations of mathematical theory, and for this reason the problem of ambiguous definition concerned mathematicians of the nineteenth and early twentieth centuries. Gottlob Frege characterized the problem as one of "piecemeal definition":

> . . . the mathematicians' favorite procedure, piecemeal definition, is inadmissible. The procedure is this: First they give the definition for a particular case—e.g., for positive integers—and make use of it; then, many theorems later, there follows a second definition for another case—e.g., for negative integers and zero—; here they often commit the further mistake of making specifications all over again for the case they have already dealt with. Even if in fact they avoid contradictions, in principle their method does not rule them out.[38]

Moreover, in the process of definition, one must avoid statements which are initial approximations of the final definition, even if intended as heuristic devices:

> It is all the more necessary to emphasize that logic cannot recognize as concepts quasi-conceptual constructions that are still fluid and have not yet been given definite and sharp boundaries, and that therefore logic must reject all piecemeal

[38] Gottlob Frege, *Translations from the Writings of Gottlob Frege*, trans. Peter Geach and Max Black (Oxford: Blackwell, 1970), pp. 159-60. The passage cited is from *Grundgesetze der Arithmetik*, 2:57.

> definition. For if the first definition is already complete and has drawn sharp boundaries, then either the second definition draws the same boundaries—and then it must be rejected, because its content ought to be proved as a theorem—or it draws different ones—and then it contradicts the first one.[39]

The problem of redefinition or restatement was particularly apparent in Heytesbury's treatment of uniform motion and the terms *inclusive* and *exclusive*.

Such considerations cast serious doubts upon Wilson's assertion that Heytesbury has succeeded in correcting and elaborating everyday speech through mathematical precision. Wilson states that the problem in the *Regule* is the tension between Heytesbury's emphasis upon mathematical precision in language and his denial of that mathematical "substructure" in the world of physical phenomena which Galileo and his successors later assumed.[40] But there is a more fundamental problem: because he has chosen the common mode of speech as his basis for analysis, Heytesbury has weakened the precision of his descriptions of phenomena. The real problem, then, is not Heytesbury's denial of precise mathematical formulations to the world of sense experience, but rather that the common mode of speech is insufficient for generating mathematical theories.

[39] Ibid., p. 162. The passage is from *Grundgesetze*, 2:58.
[40] Wilson, *Heytesbury*, pp. 148-50.

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