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Li, Li, and Dai's Contribution to the Discussion of "Estimating Means of Bounded Random Variables by Betting" by Waudby-Smith and Aaditya Ramdas

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We congratulate Waudby-Smith and Ramdas for their interesting paper Waudby-Smith and Ramdas (2022) in generating confidence intervals and time-uniform confidence sequences for mean estimation with bounded observations. Their methodology utilizes composite nonnegative martingales and establishes a connection to game-theoretic probability. Our comments will focus on numerical comparisons with alternative methods. The corresponding code is available at `https://github.com/Likelyt/Estimate-mean-with-betting`.

1 Methods

Consider a sequence of random variables $(X_t)_{t=1}^{\infty}$, drawn from a distribution $P \in \mathcal{P}^{\mu}$, where $\mathcal{P}^{\mu}$ represents the set of all distributions on $[0,1]^{\infty}$. We assume $\mathbb{E}_P[X_t|X_1,\ldots,X_{t-1}] = \mu$ for some unknown $\mu \in [0,1]$. The objective is to construct a time-uniform confidence sequence $(C_t)_{t=1}^{\infty}$ that satisfies the condition

$$\sup_{P \in \mathcal{P}^{\mu}} P(\exists t \geq 1 : \mu \notin C_t) \leq \alpha. \tag{1}$$

The confidence sequence guarantees that, for any fixed n, C_n is a $(1-\alpha)$-confidence interval for μ. We review three methods for generating such confidence sequences.

(a) The *predictable plug-in empirical Bernstein* (Pr-EB) method, discussed in Section 3 of Waudby-Smith and Ramdas (2022), combines the Robbins' method of mixture Robbins (1970) with exponential supermartingales.

(b) The *hedged capital process* (Betting) method, introduced in Section 4 of Waudby-Smith and Ramdas (2022), is a *novel* approach that enjoys the interpretation of wealth accumulation in a game and has connections to the game-theoretic probability Shafer and Vovk (2005).

(c) The *Bootstrap* resampling method is implemented using the R package BOOT Canty and Ripley (2022). With $B = 200$ bootstrap replicates and $L = 10$ batches, we calculate a separate confidence interval for data sequence $2^l \leq t < 2^{l+1}$ within each batch $l = 1,\ldots,L$. The confidence intervals are constructed using the $\frac{\alpha}{2L}$- and $(1 - \frac{\alpha}{2L})$-quantiles of the bootstrap means.

2 Numerical Studies

Synthetic Data Example We sequentially generate data from Beta$(10,30)$ distribution for $1 \leq t \leq 10^4$, and construct a 95% confidence sequence C_t in Eq. (1) using methods (a)-(c) in Section 1.

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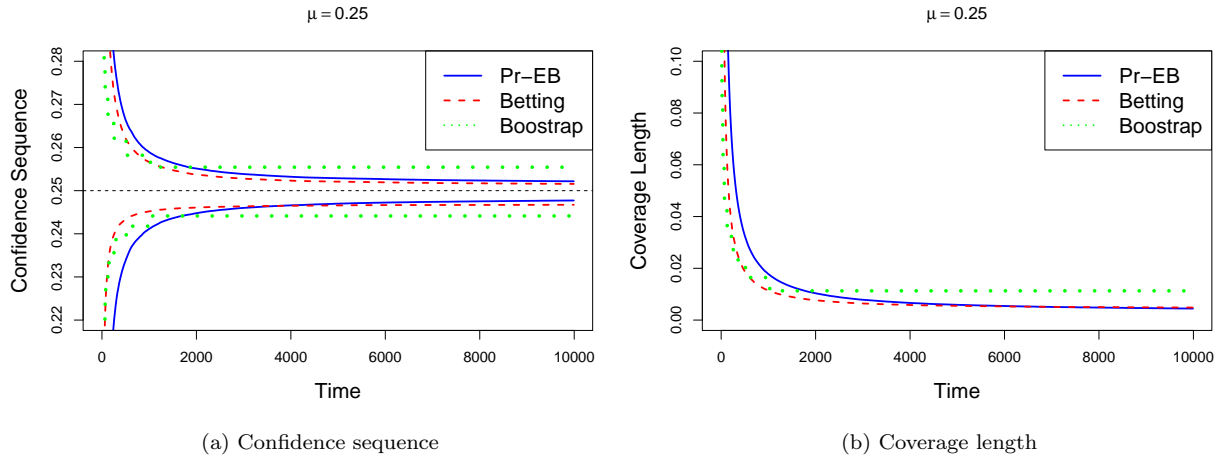| (a) Confidence sequence | (b) Coverage length |

Figure 1: Comparisons based on the synthetic data of Beta distribution.

Figure 1 shows that the Betting method outperforms the Bootstrap method with a higher lower bound in the confidence sequence and consistently tighter intervals for $t \geq 1500$. This aligns with the intuition that the Bootstrap method results in wider intervals due to dividing the confidence budget among data subsets when dealing with a large number of sequential data points. Moreover, the Betting method outperforms the Pr-EB method with consistently tighter confidence sequences for $t \leq 3000$. For $t \geq 3000$, both methods yield comparable coverage lengths. The Betting method's sequence shows a slight shift towards smaller values, reflecting the left-skewed ground truth distribution, while the Pr-EB method produces a symmetric interval around the estimated mean.

Real Data Example We analyze a batting dataset of 18 Major League players from the 1970 season, available in Efron and Hastie (2021) or the R package EFRONMORRIS. The goal is to construct a 95% confidence interval for the true batting level of each player based on their first 45 at-bats. The following results are obtained from 100 data replications.

Figure 2(a) shows the average confidence intervals, where the black circle represents the true batting level of each player. Note that while the Bootstrap method fails to cover the true batting level of player 17, the Betting method successfully includes it. Figure 2(b) presents the coverage probability of both methods. The Betting method achieves higher coverage probability for the true batting levels of players 4 and 17 compared to the Bootstrap method. These results indicate that the Betting method outperforms the Bootstrap method in constructing accurate confidence intervals for this baseball data.

3   Extensions

The work in Waudby-Smith and Ramdas (2022) may inspire several directions for future research. One such direction is the generalization of the Betting method to handle multi-dimensional observations. Currently, the Betting method relies on the assumption that the underlying capital process is a martingale. However, when estimating the mean of multidimensional data, applying Ville's inequality Waudby-Smith and Ramdas (2020) for confident rejection of the null hypothesis becomes more challenging. Defining the hedged capital process in a vector space introduces complexities when simultaneously estimating multiple attributes. Another direction is the extension of the Betting method to online decision-making scenarios, such as dynamic treatment, online recommendation, and dynamic pricing. These tasks frequently involve

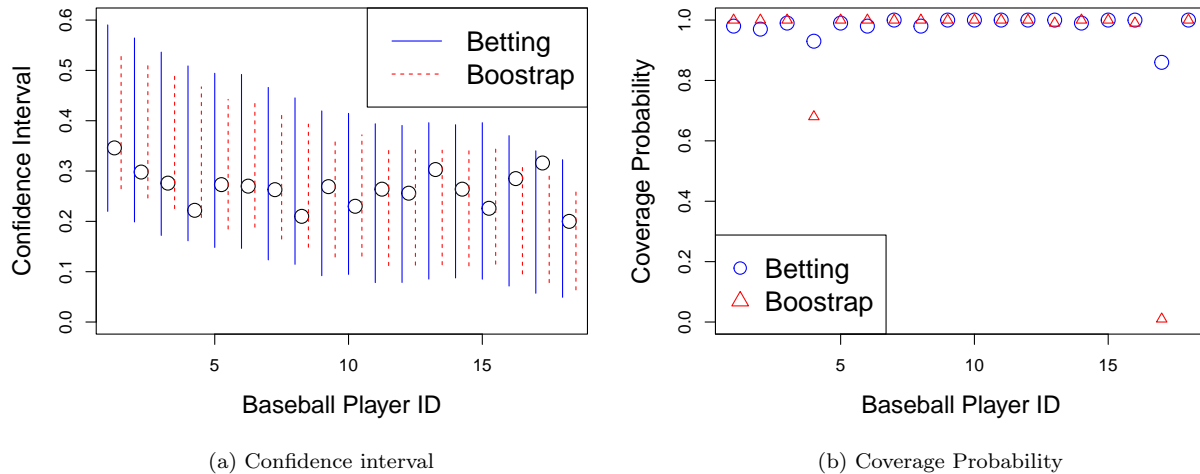(a) Confidence interval   (b) Coverage Probability

Figure 2: Comparisons based on the baseball batting data from Efron and Hastie (2021).

constructing confidence sequences to determine optimal actions. The application of the Betting method in these contexts seems to be interesting.

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