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CONSISTENCY OF SU(N) GAUGE THEORY IN TWO EUCLIDEAN DIMENSIONS

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ABSTRACT

We show that an SU(N) quark-gluon gauge theory is consistent in two Euclidean spacetime dimensions. Using Lorentz-invariant quantization surfaces, an axial gauge, the 1/N expansion and the analog of a principal value infrared cutoff, we solve exactly the Dyson self-dimension equation for a quark with zero bare mass. We thus evade the inconsistency present in the time-like gauge Minkowski-space approach to the theory.

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Introduction. 't Hooft[1] has investigated SU(N) quark-gluon gauge theory in two spacetime dimensions using the 1/N expansion. In the light-like axial gauge $A_a^+(x^+,x^-) = 0$, the Dyson equations for the fermion self-energy are solvable using a principal value cutoff; the Bethe-Salpeter equation for the meson bound-state invariant mass then takes a simple form. Frishman, Sachrajda, Abarbanel and Blankenbecler[2] have pointed out that in the time-like axial gauge, $A_a^1(t,x) = 0$, the Dyson equations are inconsistent for vanishing bare quark mass if the principal value cutoff is employed. Furthermore, they find a noncovariant bound-state equation. Subsequently, Hanson, Peccei and Prasad[3] examined the Dyson equations and the covariance of the bound-state equation for the large bare quark mass (or, equivalently, weak coupling) in the $A_a^1(t,x) = 0$ gauge and found complete consistency with 't Hooft's $A_a^+(x^+,x^-) = 0$ gauge results.

In this letter, we investigate a third approach to SU(N) quark-gluon gauge theory. Using the techniques of Fubini, Hanson and Jackiw,[4] we formulate the theory in two Euclidean spacetime dimensions and carry out equal-radius quantization in the corresponding axial gauge, $A_\theta^a(r,\theta) = 0$. The resulting Dyson equations are quite similar to those occurring in the $A_a^1(t,x) = 0$ approach, except that the continuation from Minkowski to Euclidean space induces an effective sign change in g^2, the coupling constant squared. Because of this sign change, the Dyson equations for the self-dimension of the quark are solvable for zero bare quark mass and the theory is consistent.

The Euclidean Theory. Our continuation from Minkowski to Euclidean space is defined by $t \to -ix_1$, $z \to x_2$. This convention assures us that the Minkowski action functional $i\,S_M$ becomes $-S_E$, where the Euclidean action S_E is positive definite and therefore the Feynman weight $\exp(-S_E)$ is bounded from above. The Lagrangian of our Euclidean SU(N) gauge theory (see Refs. (3) and (4)) is

$$\mathcal{L}_E = \bar\psi\,[\gamma_\mu(\partial_\mu + i\,g\,\tfrac{1}{2}\,\lambda^a A^a_\mu) + m]\psi + \tfrac{1}{2}\,F^a_{12}\,F^a_{12} \tag{1}$$

where $F^a_{12} = \partial_1 A^a_2 - \partial_2 A^a_1 - g\,f^{abc}\,A^b_1 A^c_2$,

and the gamma matrices are related to the Pauli matrices σ_i by $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\gamma_5 = \sigma_3$. For any two-vector V_μ we define the radial component $V_r = \vec{x}\cdot\vec{V}/|\vec{x}|$ together with an angular component V_θ by the transformation

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \begin{pmatrix} V_r \\ V_\theta \end{pmatrix}, \tag{2}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ with $0 \leq \theta \leq 2\pi$ and $0 < r < \infty$.

We now choose to quantize in Euclidean space on the manifestly Lorentz-invariant surfaces $r = |\vec{x}|$ = constant, so that r replaces the time as the dynamical variable and dimension eigenvalues take the place of energy eigenvalues. The natural analog of the axial gauge in this quantization scheme is

$$A^a_\theta(r,\theta) = 0\,. \tag{3}$$

It is now convenient to convert to dimensionless quantities. We define $\tau = \ln r$, replace our Euclidean integration measure by $d\tau\, d\theta$, and redefine $\hat{\mathcal{L}}_E = r^2\,\mathcal{L}_E$, $\hat\psi = r^{\frac{1}{2}}\psi$, giving the effective axial gauge Lagrangian

$$\hat{\mathcal{L}}_E = \hat{\bar\psi}\left[\gamma_r(\partial_\tau - \tfrac{1}{2}) + \gamma_\theta\partial_\theta + mr + igr\,\tfrac{1}{2}\,\lambda^a\,\gamma_r\,A^a_r\right]\hat\psi + \tfrac{1}{2}\,(\partial_\theta A^a_r)^2\,. \tag{4}$$

The dynamical "Hamiltonian" operator generating displacements in the "time" $\tau = \ln r$ is the dilatation operator,[4]

$$\begin{aligned} \Delta(\tau) = i\,D &= -\int_0^{2\pi} d\theta\; x_\mu\, x_\nu\, \Theta_{\mu\nu} \\ &= \int_0^{2\pi} d\theta\; :\hat{\bar\psi}(\tfrac{1}{2}\,\gamma_\theta\,\overrightarrow{\partial}_\theta - \tfrac{1}{2}\,\overleftarrow{\partial}_\theta\,\gamma_\theta + mr)\,\hat\psi: \\ &\quad + \tfrac{1}{2}\,(gr)^2 \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\; :J^a_r(\theta)G(\theta-\theta')J^a_r(\theta'):, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \partial^2_\theta\, A^a_r &= (gr)\,J^a_r \equiv +\,i(gr)\;:\hat{\bar\psi}\;\tfrac{1}{2}\,\lambda^a\,\gamma_r\,\hat\psi: \\ \partial^2_\theta\, G(\theta-\theta') &= \delta(\theta-\theta'). \end{aligned} \tag{6}$$

In deriving Eq. (5), we assumed that $A^a_r(r,\theta)$ and its derivatives were periodic in θ with period 2π. Note that the real eigenvalues of the hermitian operator $\Delta(\tau)$ are the dynamical dimensions of the states examined and that the τ-dependence of $\Delta(\tau)$ reflects the existence of dimensionful parameters in this theory.

The dependent field A^a_r has been eliminated from Eq. (5) using the Green's function $G(\theta-\theta')$. We note, however, that we may replace $G(\theta - \theta')$ by the modified Green's function

$$\tilde{G}(\theta - \theta') = -\frac{1}{2\pi} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{+\infty} \frac{1}{n^2} e^{in(\theta-\theta')}$$

$$= \frac{1}{2} |\theta - \theta'| - \frac{1}{4\pi} (\theta - \theta')^2 - \frac{\pi}{6}, \qquad (7)$$

obeying $\partial_\theta^2 \tilde{G}(\theta - \theta') = \delta(\theta - \theta') - 1/2\pi$, provided we restrict ourselves henceforth to <u>color neutral states</u>:

$$Q^a = \int_0^{2\pi} d\theta \; J_r^a(\theta) = 0 . \qquad (8)$$

We mention the important point that omitting $n = 0$ from the sum in Eq. (7) is our analog of the principal value cutoff procedure used in the Minkowski problem.

The free quark Green's function is found by solving the Dirac equation

$$\left\{ \partial_\tau + \sigma_3 [i\partial_\theta - \tfrac{1}{2} \sigma_3] + mr\gamma_r \right\} \hat{\Psi} = 0 . \qquad (9)$$

The "angular momentum" generator $L = L_{12} = L_{r\theta}$ commutes with the dilatation Δ, so we may expand $\hat{\Psi}$ simulataneously in "angular momentum" and "dilatation" eigenstates:

$$L \hat{\Psi} = [-i\partial_\theta + \tfrac{1}{2} \sigma_3] \hat{\Psi} = (\ell + \tfrac{1}{2}) \hat{\Psi} , \quad \ell = 0, \pm 1, \pm 2, \cdots$$

$$\Delta \hat{\Psi} = \partial_\tau \hat{\Psi} = [+\sigma_3 (\ell + \tfrac{1}{2}) - mr\gamma_r] \hat{\Psi} = \omega \hat{\Psi} , \qquad (10)$$

where $\omega^2 = (\ell + \frac{1}{2})^2 + (mr)^2$ on "dimension-shell." Thus the free spinor propagator in (ω, ℓ) space is

$$S_0 = \left\{ \frac{\omega\gamma_r + i\gamma_\theta(\ell + \frac{1}{2}) - mr}{\omega^2 - (\ell + \frac{1}{2})^2 - (mr)^2 + i\epsilon} \right\} . \qquad (11)$$

Equations (7), (11) and the quark-gluon vertex $(-ig \frac{1}{2} \lambda^a \gamma_r)$, found by expanding $\exp[-S_E]$, define the Feynman rules in the (ω, ℓ) space conjugate to the (r, θ) coordinate space. Explicit functions of r appear in (ω, ℓ) space to provide a length scale in our non-conformally-invariant theory.

<u>Fermion Self-Dimension.</u> The exact fermion propagator is defined as

$$S = \left\{ \frac{\omega\gamma_r + i\gamma_\theta(\ell + \frac{1}{2} + B(\ell,r)) - (A(\ell,r) + m)r}{\omega^2 - (\ell + \frac{1}{2} + B)^2 - (A + m)^2 r^2 + i\epsilon} \right\} . \qquad (12)$$

In the $1/N$ expansion only rainbow diagrams contribute to the Dyson equations for the "self-dimension,"

$$\Sigma(\ell,r) \equiv +A(\ell,r)r + iB(\ell,r)\gamma_\theta = -C \int_{-\infty}^{+\infty} d\omega \sum_{\substack{\ell'=-\infty \\ (\ell'\neq\ell)}}^{+\infty} \frac{\gamma_r S(\ell',r)\gamma_r}{(\ell - \ell')^2} , \qquad (13)$$

where $C = i\, N(gr)^2/8\pi^2$. Thus A and B satisfy

$$A(\ell,r) = -i\pi C \sum_{\ell'\neq\ell} \frac{A(\ell',r) + m}{(\ell - \ell')^2 \omega_0(\ell')}$$

$$B(\ell,r) = -i\pi C \sum_{\ell\neq\ell} \frac{B(\ell',r) + \ell' + \frac{1}{2}}{(\ell - \ell')^2 \omega_0(\ell')} \qquad (14)$$

where $\omega_0(\ell) = + \left\{ [A + m]^2 r^2 + [B + \ell + \frac{1}{2}]^2 \right\}^{\frac{1}{2}}$.

It is instructive to evaluate Eqs. (14) to lowest order in C for $m = 0$. Setting $A = B = 0$ on the right-hand sides, we find that

$$A(\ell,r) = 0, \qquad B(\ell,r) = N(gr)^2\,(4\pi)^{-1}\,f(\ell) \qquad (15)$$

where

$$f(\ell) = \frac{1}{2}\sum_{\substack{n=-\infty\\(n\neq\ell)}}^{\infty} \frac{\epsilon(n+\frac{1}{2})}{(\ell-n)^2} = \begin{cases} +\sum_{1}^{\ell} 1/n^2 & \ell > 0 \\ 0 & \ell = 0,\ -1 \\ -\sum_{1}^{-\ell-1} 1/n^2 & \ell < -1\ . \end{cases}$$

Further iterations produce no change in $A(\ell,r)$ and $B(\ell,r)$. The lowest order solution of the Dyson equations is therefore the exact solution.

The dimension eigenvalues ω_0 appearing in Eqs. (14) thus take the form $\omega_0 = \left\{[B(\ell,r) + \ell + \frac{1}{2}]^2\right\}^{\frac{1}{2}}$ with $B(\ell,r)$ given by Eq. (15). If we expand ω_0 to lowest order in $(gr)^2$, we find

$$\omega_0 = |\ell + \tfrac{1}{2}| + \epsilon(\ell + \tfrac{1}{2})\,B(\ell,r)\ , \qquad (16)$$

which agrees with the lowest order fermion dimension eigenvalue found by using the free field expansion for $\hat{\psi}$ in Eq. (5). This calculation double-checks the crucial signs in Eqs. (13) and (14).

Conclusion. In the time-like gauge treatment, the analogs of Eqs. (14) are inconsistent for vanishing bare fermion mass m. However, as noted by Hanson, Peccei and Prasad,[3] a unique exact solution can be found if one replaces g by ig in the time-like system. Going to Euclidean space effectively accomplishes this replacement. For $m = 0$, we have shown that the Euclidean Dyson equations (14) are consistent and possess the (apparently unique) solution (15). The transition from Minkowski to Euclidean space has essentially Wick-rotated the

coupling constant to avoid the singularities giving the inconsistency discovered by Frishman et al.[2] Our observation supplements the mounting evidence in favor of formulating field theories in Euclidean space. It appears that the Euclidean continuation of a Minkowski space-field theory provides important information not otherwise available.[5]

The Euclidean Bethe-Salpeter equation for the dynamical dimensions of the bound states in the $1/N$ approximation can now be formulated for $m = 0$ without difficulty using our solution for $\Sigma(\ell,r)$. However, it is not especially elegant because the theory is not scale-invariant and factors of r appear even in (ω,ℓ) space. Perhaps conformally invariant theories with essential Euclidean space properties, such as 4-dimensional gauge theories, would be better suited to the quantization scheme examined here.

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