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A refined Gross-Prasad conjecture for unitary groups

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A Refined Gross-Prasad Conjecture for Unitary Groups

A dissertation submitted in partial satisfaction of the
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by

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2011

The dissertation of Richard Neal Harris is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2011

DEDICATION

For Michelle.

EPIGRAPH

Groups, as men, will be known by their actions.

—Guillermo Moreno

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ABSTRACT OF THE DISSERTATION

A Refined Gross-Prasad Conjecture for Unitary Groups

by

Richard Neal Harris
Doctor of Philosophy in Mathematics

University of California San Diego, 2011

Professor Wee Teck Gan, Chair

Let F be a number field, $\mathbb{A}_F$ its ring of adèles, and let π_n and π_{n+1} be irreducible, cuspidal, automorphic representations of $SO_n(\mathbb{A}_F)$ and $SO_{n+1}(\mathbb{A}_F)$, respectively. In 1991, Benedict Gross and Dipendra Prasad conjectured the non-vanishing of a certain period integral attached to π_n and π_{n+1} is equivalent to the non-vanishing of $L(1/2, \pi_n \boxtimes \pi_{n+1})$ [10]. More recently, Atsushi Ichino and Tamotsu Ikeda gave a refinement of this conjecture as well as a proof of the first few cases ($n = 2, 3$) [21]. Their conjecture gives an explicit relationship between the aforementioned L-value and period integral. We make a similar conjecture for unitary groups, and prove the first few cases. The first case of the conjecture will be proved using a theorem of Waldspurger [35], while the second case will use the machinery of the Θ-correspondence.

1 Introduction

1.1 The Gross-Prasad Conjecture

In [10], Benedict Gross and Dipendra Prasad give a conjecture that relates the non-vanishing of a period integral to the non-vanishing of a certain L-value. In this section, we'll discuss that conjecture, as well as a recent refinement due to Atsushi Ichino and Tamotsu Ikeda.

Let F be a number field with adèle ring $\mathbb{A}_F$, and let $V_n \subset V_{n+1}$ be quadratic spaces of dimensions n and $n+1$ over F, respectively. Assume that $n \geq 2$ and that V_n is not a hyperbolic plane. We consider the algebraic groups $\mathrm{SO}(V_n) \subset \mathrm{SO}(V_{n+1})$ defined over F. We denote $G_i := \mathrm{SO}(V_i)$. Let π_n and π_{n+1} be irreducible tempered cuspidal automorphic representations of $G_n(\mathbb{A}_F)$ and $G_{n+1}(\mathbb{A}_F)$ respectively. We fix ismorphisms $\pi_n \cong \otimes_v \pi_{n,v}$ and $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$. Suppose that $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$ for every place v of F. Then the Gross-Prasad Conjecture is the following:

Conjecture 1.1 (Original Gross-Prasad Conjecture)**.** *There exist vectors $\varphi_i \in \pi_i$ such that*

$$\int_{G_n(F)\backslash G_n(\mathbb{A}_F)} \varphi_{n+1}(g_n)\varphi_n(g_n)\ dg_n \neq 0$$

if and only if

$$L(1/2, \pi_{n+1} \boxtimes \pi_n) \neq 0.$$

The L-function here is the product L-function.

Recently in [21], a refinement to this conjecture was proposed which gives a precise relationship between the period integral above and the L-value.

Consider the following $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional

$$\mathcal{P} : (V_{\pi_{n+1}} \boxtimes \bar{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \bar{V}_{\pi_n}) \to \mathbb{C}$$

defined by

$$\mathcal{P}(\phi_1, \phi_2; f_1, f_2) := \left(\int_{[G_n]} \phi_1(g) f_1(g) dg\right) \cdot \left(\int_{[G_n]} \overline{\phi_2(g) f_2(g)} dg\right) \tag{1.1}$$

for $\phi_i \in V_{\pi_{n+1}}$ and $f_i \in V_{\pi_n}$. If $\phi_1 = \phi_2 = \phi$ and $f_1 = f_2 = f$, we simply write $\mathcal{P}(\phi, f) := \mathcal{P}(\phi_1, \phi_2; f_1, f_2)$. We call $\mathcal{P}$ the global period.

We also construct another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional, this time constructed from local integrals. For each place v of F, denote $G_{i,v} := G_i(F_v)$. We fix local pairings

$$\mathcal{B}_{\pi_{i,v}} : \pi_{i,v} \otimes \bar{\pi}_{i,v} \to \mathbb{C}$$

such that

$$\mathcal{B}_{\pi_i} = \prod_v \mathcal{B}_{\pi_{i,v}}$$

where the $\mathcal{B}_{\pi_i}$ are the Petersson pairings

$$\begin{aligned} \mathcal{B}_{\pi_n}(f_1, f_2) &:= \int_{[G_n]} f_1(g_n) \overline{f_2(g_n)} dg_n \\ \mathcal{B}_{\pi_{n+1}}(\phi_1, \phi_2) &:= \int_{[G_{n+1}]} \phi_1(g_{n+1}) \overline{\phi_2(g_{n+1})} dg_{n+1} \end{aligned}$$

and the dg_i are Tamagawa measures on $G_i(\mathbb{A}_F)$. For each v, we define a $G_{n,v} \times G_{n,v}$ invariant functional

$$\mathcal{P}_v^\natural : (\pi_{n+1,v} \boxtimes \bar{\pi}_{n+1,v}) \otimes (\pi_{n,v} \boxtimes \bar{\pi}_{n,v})$$

by

$$\mathcal{P}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) := \int_{G_{n,v}} \mathcal{B}_{\pi_{n+1,v}}(\pi_{n+1,v}(g_{n,v})\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\pi_{n,v}}(\pi_{n,v}(g_{n,v}) f_1, f_2) dg_{n,v}.$$

Here, the $dg_{n,v}$ are local Haar measures, chosen so that $\prod_v dg_{n,v} = dg_n$.

Again, we denote $\mathcal{P}_v^\natural(\phi_v, \phi_v; f_v, f_v) =: \mathcal{P}_v^\natural(\phi_v, f_v)$. We set

$$\begin{aligned} \Delta_{G_i} &:= L(M_i^\vee(1), 0) \\ \Delta_{G_{i,v}} &:= L_v(M_i^\vee(1), 0) \end{aligned}$$

where $M_{n+1}^\vee(1)$ is the twisted dual of the motive M_i associated to G_i by Gross in [9]. It is a result of Ichino and Ikeda (Theorem 1.2 in [21]) that the $\mathcal{P}_v^\natural$ converge absolutely if the $\pi_{i,v}$ are tempered. Furthermore, when the $\mathcal{P}_v^\natural$ converge, we have

$$\mathcal{P}_v^\natural(\phi_v, f_v) = \Delta_{G_{n+1,v}} \frac{L_v(1/2, \pi_{n,v} \boxtimes \pi_{n+1,v})}{L_v(1, \pi_{n,v}, \mathrm{Ad}) L_v(1, \pi_{n+1,v}, \mathrm{Ad})}$$

for unramified data ϕ_v, f_v satisfying conditions $(U1)-(U6)$ in [21]. So we normalize as follows:

$$\mathcal{P}_v := \Delta_{G_{n+1,v}}^{-1} \frac{L_v(1, \pi_{n,v}, \mathrm{Ad}) L_v(1, \pi_{n+1,v}, \mathrm{Ad})}{L_v(1/2, \pi_{n,v} \boxtimes \pi_{n+1,v})} \mathcal{P}_v^\natural.$$

Now we have another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional:

$$\prod_v \mathcal{P}_v : (V_{\pi_{n+1}} \boxtimes \bar{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \bar{V}_{\pi_n}) \to \mathbb{C}.$$

The Refined Gross-Prasad Conjecture gives the explicit constant of proportionality between $\mathcal{P}$ and $\prod_v \mathcal{P}_v$:

Conjecture 1.2 (Refined Gross-Prasad Conjecture)**.**

$$\mathcal{P}(\phi, f) = \frac{\Delta_{G_{n+1}}}{2^\beta} \frac{L(1/2, \pi_n \boxtimes \pi_{n+1})}{L(1, \pi_n, \mathrm{Ad})(1, \pi_{n+1}, \mathrm{Ad})} \prod_v \mathcal{P}_v(\phi_v, f_v).$$

Here, β is an integer such that $2^\beta = |S_{\psi_{n+1}}| \cdot |S_{\psi_n}|$, where ψ_i is the conjectural L-parameter for π_i, and $S_{\psi_i} := \mathrm{Cent}_{\widehat{G_i}}(\mathrm{Im}(\psi_i))$ is the associated component group.

This conjecture is known unconditionally for $n = 2$ [35], for $n = 3$ [20], and for $n = 4$, assuming that π_5 is a Θ-lift of a representation on $SO(4)$ [6].

Our goal is to provide evidence for an analogous conjecture for unitary groups. The conjecture for unitary groups is:

Conjecture 1.3 (Refined Gross-Prasad Conjecture for Unitary Groups)**.** *Let π_n and π_{n+1} be irreducible, cuspidal, tempered, automorphic representations of $G_n(\mathbb{A}_F)$ and $G_{n+1}(\mathbb{A}_F)$, respectively. Let $\mathcal{P}$ be as in line 1.1. Then*

$$\mathcal{P} = \frac{\Delta_{G_{n+1}}}{2^\beta} \frac{L_E(1/2, BC(\pi_{n+1}) \boxtimes BC(\pi_n))}{L_F(1, \pi_{n+1}, \mathrm{Ad}) L_F(1, \pi_n, \mathrm{Ad})} \prod_v \mathcal{P}_v.$$

Here, $BC(\pi_i)$ denotes the quadratic base-change of π_i to a representation of $GL_i(\mathbb{A}_E)$. Also, β is an integer such that $2^\beta = |S_{\psi_{n+1}}| \cdot |S_{\psi_n}|$, where ψ_i is the conjectural L-parameter of π_i, and S_{ψ_i} is the associated component group.

Remark 1.4. *One might ask why the L-values on the RHS above are non-zero. Indeed, in [21] the authors comment that for orthogonal groups it is believed that $L(1, \pi_i, \mathrm{Ad}) \neq 0$ for π_i tempered. However, we can say something stronger for unitary groups. Namely, by results in [5], we see that*

$$L_F(s, \pi_i, \mathrm{Ad}) = L_F(s, BC(\pi_i), \mathrm{As}^{(-1)^i}).$$

Here, we are viewing $BC(\pi_i)$ as a representation of $GL_i(\mathbb{A}_F)$ via $\mathrm{Res}_{E/F}$. The L-function on the RHS above is the 'Asai' (if i is even) or 'twisted Asai' (if i is odd) L-function. Now, since the π_i are assumed to be tempered, by Theorem 5.1 in [30] we have that $L_F(s, BC(\pi_i), \mathrm{As}^{(-1)^i})$ is holomorphic and nonzero at $s = 1$.

1.2 Hermitian Spaces and Unitary Groups

We give a brief review of the unitary groups relevant to this project.

Let F be a number field, and let E be a quadratic extension of F. Let τ be the generator of $\mathrm{Gal}(E/F)$. We will occasionally use the shorthand $\overline{x} := \tau(x)$ for $x \in E$. A hermitian space V over E is an n-dimensional E-vector space equipped with a hermitian form h. The hermitian form is a non-degenerate pairing $h : V \times V \to E$ satisfying

$$\begin{aligned} h(v_1 + v_2, w) &= h(v_1, w) + h(v_2, w) \\ h(ev, w) &= eh(v, w) \\ h(v, w) &= \tau(h(w, v)) \end{aligned}$$

for all $v_1, v_2, v, w \in V$ and $e \in E$. Then we define the unitary group attached to the pair (V, h) as

$$U(V,h) = \{g \in \mathrm{GL}(V) : h(gv, gw) = h(v,w) \text{ for all } v, w \in V\}.$$

We remark that with a choice of basis, h can be represented by an $n \times n$ matrix $J \in GL_n(E)$ ($n = \dim_E(V)$), such that $J^* = J$, (where * indicates conjugate-transpose) such that

$$h(v,w) = w^* Jv.$$

The condition that h be non-degenerate is equivalent to the statement that $\det J \neq 0$. Then note that

$$U(V,h) = \{g \in GL(V) : g^* Jg = J\}.$$

If the reader is unfamiliar with this general notion of unitary groups, they will be comforted by the realization that with $F = \mathbb{R}$ and $E = \mathbb{C}$, and h the usual hermitian inner product, this definition gives the classical unitary groups.

We say that $v \in V$ is an isotropic vector if $h(v,v) = 0$. We say that a subspace $W \subset V$ is an isotropic subspace if $h(w', w_2) = 0$ for all $w', w_2 \in W$. Note that the requirement that h be non-degenerate forces W to be proper. We say that a subspace is anisotropic if it contains no isotropic vectors. When h is understood, we shall omit it from the notation, and shall just refer to the unitary group $U(V,h)$ as $U(V)$.

Note that $U(V)$ is an algebraic group defined over F. The R points of $U(V)$ are given by

$$U(V)(R) = \{g \in GL(V \otimes_F R) : h(gv, gw) = g(v,w) \text{ for all } v, w \in V \otimes_F R\}.$$

For our purposes, we will primarily consider cases where $R = \mathbb{A}_F$, the adèle ring of F, or where $R = F_v$, the completion of F at a prime v. If $G = U(V)$, then we shall denote $G_v := G(F_v)$.

We consider $V_v := V \otimes_F F_v$ as a hermitian space over E_v. If the prime v splits in E/F (say as the product $w_1 w_2$, where the w_i are places of E), then we have

$$E_v \cong E_{w_1} \times E_{w_2} \cong F_v \times F_v.$$

If v is an archimedean place, then the hermitian spaces over F_v are classified by dimension and signature. We omit further discussion of this case, as it has little impact on the sequel.

If v is a non-archimedean place, it is a theorem of Landherr [28] that for each n there are exactly two isomorphism classes of hermitian spaces of dimension n over E_w. (By isomorphism, we mean an isomorphism of vector spaces that preserves the hermitian forms.) If $n = 1$, then V_v can be identified with E_w, and

$$h(e_1, e_2) = ae_1\overline{e_2}$$

for some $a \in E_w^\times$. We denote this space by $E_w(a)$. Then it turns out that $E_w(a) \cong E_w(b)$ if and only if $a/b \in N_{E_w/F_v}(E_w^\times)$. Here, $\mathrm{N_{E_w/F_v}} : E_w \to F_v$ is the relative norm map defined by

$$\mathrm{N_{E_w/F_v}}(e) := e\bar{e}.$$

For $n = 2$, after fixing a basis $\{e_1, e_2\}$ for E_w^2, the hyperbolic plane H over E_w has the hermitian form:

$$h(ae_1 + be_2, ce_1 + de_2) = \bar{a}d + \bar{b}c.$$

Also, for $n = 2$, an anisotropic hermitian space $W_2(a, b)$ is $E_w(a) \oplus E_w(b)$ with $a, b \neq 0$ and $-a/b \notin \mathrm{N_{E_w/F_v}}(E_w^\times)$. All anisotropic planes are isomorphic.

For $n = 2m + 1$, up to isomorphism the two hermitian spaces of dimension n over E_w are $V^\pm \cong mH \oplus W^\pm$ with $W^\pm \cong E_w(a)$ according to whether $a \in \mathrm{N_{E_w/F_v}}(E_w^\times)$ or not.

For $n = 2m$, we have $V^+ \cong mH$ and $V^- \cong (m-1)H \oplus W_2$ where W_2 is an anisotropic plane (known as the anisotropic kernel).

If n is odd and v is non-archimedean, then $U(V^+)$ is isomorphic to $U(V^-)$ and is quasi-split over F_v. This means that it has a Borel-subgroup defined over F_v.[1] If n is even, then $U(V^+)$ is not isomorphic to $U(V^-)$. Furthermore, $U(V^+)$ is

[1]In the context of algebraic groups, a *Borel subgroup* is a maximal connected, closed, solvable subgroup.

quasi-split in this case, while $U(V^-)$ is not. Also, the number of hyperbolic planes appearing in the decompositions above ($= m$ for V^+ and $m-1$ for V^-) is known as the Witt index.

For almost all places v, G_v is *unramified*. We define this term presently:

Definition 1.5. *We call G_v* unramified *if it is quasi-split and split over an unramified extension of F_v.*

If G_v is unramified, then it contains a hyperspecial maximal compact subgroup $K_v \subset G_v$. We have a Borel subgroup $B_v \subset G_v$, and the Iwasawa decomposition gives us

$$G_v = B_v K_v.$$

We can describe the Borel subgroup further; it decomposes as $B_v = T_v N_n$, where T_v is a torus, and N_v is the unipotent radical of B_v. We denote by $A_v \subset T_v$ the maximal split torus of B_v.

Let n be the dimension of V_v over E_w. Then we can describe T_v and A_v abstractly as follows. If $n = 2m$, then we have that $T_v \cong (E_w^\times)^m$ and $A_v \cong (F_v^\times)^m$. If $n = 2m+1$, then we have that $A_v \cong (F_v^\times)^m$ and $T_v \cong (E_w^\times)^m \times E_{w,1}$, where $E_{w,1} \subset E_w^\times$ denotes the elements of relative norm 1.

We make some comments on the case where v splits in E/F, i.e. when $E_v = F_v \oplus F_v$. In this case, it turns out that we have $V_v = V_{v,1} \oplus V_{v,2}$, where each $V_{v,i}$ is an F_v-vector space of dimension n. Furthermore, if h is a hermitian inner product associated to V_v, we have that

$$h(v,w) = (h_1(v_1, w_2), h_2(w_1, v_2))$$

where $h_i : V_{v,1} \times V_{v,2} \to F_v$ is a non-degenerate bilinear pairing. Then we have that

$$G_v = U(V_v) \cong GL(V_{v,1}) \tag{1.2}$$

under the map

$$(g_1, g_2) \mapsto g_1.$$

To see this, one simply notes that the requirement $h(gv, gw) = h(v, w)$ translates to

$$(h_1(g_1v_1, g_2w_2), h_2(g_1w_1, g_2v_2)) = (h_1(v_1, w_2), h_2(w_1, v_2)),$$

and so we see that once g_1 is chosen, g_2 is determined.

1.3 Automorphic L-Functions

We give a brief discussion of automorphic L-functions, with a focus on those attached to unitary groups and general linear groups. The reader should consult [1] for a more complete treatment of the subject.

Let G be a reductive algebraic group defined over a number field F, and let π be an irreducible admissible automorphic representation of $G(\mathbb{A}_F)$. We denote by G_v the F_v points of G, where v is a place of F. We have an isomorphism

$$\pi \cong \otimes_v \pi_v$$

where each π_v is an irreducible admissible representation of G_v.

The goal of this section is to define $L(s, \pi, r)$, where r is a smooth homomorphism $r : {}^LG \to GL_m(\mathbb{C})$, where s is a complex variable, and ${}^LG$ is the so-called L-group of G. We give ad hoc definitions of ${}^LG$ when G is a general linear group or unitary group. If $G = GL(n)$, then ${}^LG = GL_n(\mathbb{C})$. If $G = U(n)$, then ${}^LG = GL_n(\mathbb{C}) \rtimes W_F$, where W_F acts on $GL_n(\mathbb{C})$ through the projection $W_F \to \mathrm{Gal}(E/F)$, and the non-trivial $c \in \mathrm{Gal}(E/F)$ acts on $GL_n(\mathbb{C})$ as follows:

$$c \cdot g := J^t g^{-1} J^{-1}$$

where

$$J := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n+1} & & & \end{pmatrix}.$$

We also have local L-groups ${}^LG_v$. If v is such that E_v is a field, then ${}^LG_v$ is defined analogously with ${}^LG$, in other words ${}^LG_v = GL_n(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F_v}/F_v)$, where the action of the Galois group is analogous to the global case. If v splits in E, then then ${}^LG_v = GL_n(\mathbb{C})$.

As one might expect, the definition of an automorphic L-function is given by an Euler product:

$$L(s,\pi,r) = \prod_v L_v(s,\pi_v,r).$$

So to give a definition of $L(s,\pi,r)$, we must give a definition of $L_v(s,\pi_v,r)$ for each place v of F. This problem is the central issue of the Local Langlands Conjecture, which we discuss now. This conjecture relates equivalence classes of irreducible admissible representations of G_v to equivalence classes of 'admissible' homomorphisms

$$\varphi_v : WD_v \to {}^LG_v.$$

Here, WD_v is the Weil-Deligne group (respectively Weil group) if v is non-archimedean (respectively archimedean), a group closely related to the absolute Galois group $\mathrm{Gal}(\overline{F_v}/F_v)$. For precise definitions of these groups (as well as the definition of the term admissible in the context of these homomorphisms), we refer the reader to [1]. Two such homomorphisms are considered equivalent if they are conjugate via an element of ${}^LG_v^o$, the identity component of the local L-group. We denote the set of equivalence classes of irreducible admissible representations of G_v by $\Pi(G_v)$, and the set of equivalence classes of admissible homomorphisms of WD_v to ${}^LG_v$ by $\Phi(G_v)$. With this, we can state a rough version of the Local Langlands Conjecture:

Conjecture 1.6. *There is a 'natural' finite-to-one map*

$$\mathrm{Rec} : \Pi(G_v) \to \Phi(G_v).$$

Of course, this statement is meaningless without a definition of the word 'natural.' Without giving a complete and precise definition here, we will discuss the case of GL_n. From this, hopefully the reader can at least appreciate the flavor and spirit of the Local Langlands Conjecture. In this case, the conjecture is known, and is a theorem of Harris-Taylor [16] and Henniart [17]:

Theorem 1.7. *There is a unique system of bijections*

$$\mathrm{Rec}_n : \Pi(G_v) \to \Phi(G_v)$$

satisfying

- Rec_1 *is the bijection induced by local class field theory.*
- Rec_n *is compatible with twisting by characters of* F_v *in the following sense:*

$$\mathrm{Rec}_n(\pi \otimes \chi \circ \det) = \mathrm{Rec}_n(\pi) \otimes \mathrm{Rec}_1(\chi)$$

- *Denote by* ω_π *the central character of* π*. Then*

$$\mathrm{Rec}_1(\omega_\pi) = \det \mathrm{Rec}_n(\pi).$$

- $\mathrm{Rec}_n(\pi^\vee) = \mathrm{Rec}_n(\pi)^\vee$.
- Rec_n *preserves* L *and* ε *factors of pairs of representations. More precisely, after fixing an additive character* ψ *of* F_v*, and representations* π_n *and* π_m *of* $GL_n(F_v)$ *and* $GL_m(F_v)$ *respectively:*

$$\begin{aligned} L(s, \pi_n \times \pi_m) &= L(s, \mathrm{Rec}_n(\pi_n) \otimes \mathrm{Rec}_m(\pi_m)) \\ \varepsilon(s, \pi_n \times \pi_m, \psi) &= \varepsilon(s, \mathrm{Rec}_n(\pi_n) \otimes \mathrm{Rec}_m(\pi_m), \psi) \end{aligned}$$

  where the factors on the left were defined by Jacquet, Piatetski-Shapiro and Shalika (see [22]), and those on the right were defined by Artin, Deligne, and Langlands (e.g. as in [2]).

The last item mentioned in the theorem is perhaps the most important: the Local Langlands Correspondence must preserve L- and ε-factors. Let us now return to the setting of general G_v. Notice that we still have not defined the local factors of an *automorphic* L-function (which do not appear in the theorem above). Suppose that $\text{Rec}(\pi_v) = \varphi_v$. Then the requirement that Rec should preserve L-factors leads us to define

$$L_v(s, \pi_v, r) := L_v(s, r \circ \varphi_v)$$

where the L-factor on the right is the local factor of an Artin L-function.

For the time being, this leaves us in a somewhat unsatisfying position. It would seem that we have to rely on a conjecture in order to define the local factors of an automorphic L-function. It turns out that we are in slightly better shape than this; we know how to define the local L-factors for all $v \notin S$, where S is a sufficiently large set of places of F. The main tool that allows us to do this is the Satake Isomorphism. Before discussing this, we describe S.

Definition 1.8. *Let $K_v \subset G_v$ be a hyperspecial maximal compact subgroup. We call π_v K_v-*unramified *if it contains a K_v-fixed vector.*

We know that G_v is unramified for almost all v. Furthermore, we have the following theorem of Flath [3]:

Theorem 1.9. *Let π be an irreducible admissible representation of $G(\mathbb{A}_F)$, then it is uniquely isomorphic to a restricted tensor product $\otimes'_v \pi_v$, where each π_v is an irreducible admissible representation of G_v. For almost all v, π_v is unramified with a distinguished spherical vector f_v^o, and the restricted tensor product is taken with respect to the f_v^o.*

We have already used the first part of this result simply by decomposing π as a tensor product of local representations. The key here is that almost all of the π_v are unramified.

Let S be a finite set of places of F such that both G_v and π_v are unramified for $v \notin S$. We also require that S contain all archimedean places of F.

Without giving a statement of the Satake Isomorphism, we describe how it helps us define the local L-factors for $v \notin S$. A consequence of the Satake

Isomorphism is that to each π_v, we can associate a conjugacy class $[t_v]$ (where $t_v \in {}^LG_v$) and define

$$L_v(s, \pi_v, r) := \frac{1}{\det(I - q_v^{-s} r(t_v))}$$

where q_v is the cardinality of the residue field of F_v.

Furthermore, the ${}^LG_v^o$ component of t_v is semi-simple. We refer to $[t_v]$ as the *Satake parameter* of π_v. We will now compute $L_v(s, \pi_v, r)$ for some unramified representations π_v and for r either the standard or adjoint representation.

For the examples that follow, as well as the rest of the paper, we will find it useful to denote

$$L_v(s, \chi) := \frac{1}{(1 - q_v^{-s}\chi(\varpi))}$$

where χ is some unramified character of $F_v^\times$. When discussing unitary groups in the examples below, we assume that v is inert in E/F. (If v is not inert and $v \notin S$, then we have that v splits in E/F, and we recall that the unitary group is isomorphic to GL_n at this place.) Also, we will denote by χ_{E_v/F_v} the quadratic character attached to the extension E_v/F_v by class field theory.

Example 1.10 (standard L-function for GL_n)**.** *Let $G := GL_n(F_v)$. Let $B \subset G$ be a Borel subgroup, and let K be a hyperspecial maximal compact subgroup. Then we have*

$$G = BK$$

by the Iwasawa decomposition. Furthermore, we have

$$B = TN$$

where $T \cong (F_v^\times)^n$ and N is the associated unipotent radical. Let $\chi = (\chi_1, \ldots, \chi_n)$ be an unramified character of T. (Here, unramified *means 'trivial on K'. We also say that a character of $F_v^\times$ is unramified if it is trivial on $\mathcal{O}_v^\times \subset F_v^\times$.) We extend χ to B by 1 along N. Essentially all unramified irreducible admissible representations of G take the form*

$$\pi \cong \mathrm{Ind}_B^G \chi = \{\textit{smooth } f : G \to \mathbb{C} : f(bg) = \delta_B^{1/2}(b)\chi(b)f(g)\}$$

where δ_B *is the* modular character *of* B*, and* G *acts by the right-translation on* π*. Now, if we let* $\varpi \in F_v^\times$ *be a uniformizer, then a Satake parameter for* π *is*

$$t_v = \mathrm{diag}(\chi_1(\varpi), \chi_2(\varpi), \ldots, \chi_n(\varpi)).$$

Then if we take r *to be the standard representation of* ${}^LG = GL_n(\mathbb{C})$*, then we have*

$$L_v(s, \pi_v, r) = \frac{1}{\det(I_n - q_v^{-s} r(t_v))} = \prod_{i=1}^{n} \frac{1}{1 - q_v^{-s}\chi_i(\varpi)}.$$

When r *is the standard representation of* ${}^LG$*, we will denote* $L_v(s, \pi_v, r)$ *by* $L_v(s, \pi_v)$*.*

Example 1.11 (π_v an unramified principal series for GL_n, r adjoint)**.** *Let* G *and* π_v *be as above. This means that we can take* t_v *to be as above as well. This time,* r *is the adjoint representation of* ${}^LG = GL_n(\mathbb{C})$ *on* $\mathrm{Lie}({}^LG) = M_n(\mathbb{C})$*. Then we have that*

$$\begin{aligned} L_v(s, \pi_v, r) &= \frac{1}{\det(I_n - q_v^{-s} r(t_v))} = \prod_{1 \le i,j \le n} \frac{1}{1 - q_v^{-s}\chi_i(\varpi)\chi_j^{-1}(\varpi)} \\ &= \zeta_{F_v}(s)^n \prod_{i \neq j} \frac{1}{1 - q_v^{-s}\chi_i(\varpi)\chi_j^{-1}(\varpi)} \\ &= \zeta_{F_v}(s)^n \prod_{i \neq j} L_v(s, \chi_i \chi_j^{-1}). \end{aligned}$$

Before we discuss any L-functions attached to unitary groups, we will discuss quadratic base-change. If G_v is a quasi-split unitary group, and π_v is an irreducible admissible unramified representation of G_v, then the Satake isomorphism gives us $\varphi_v \in \Phi(G_v)$ such that $\varphi_v(\mathrm{Frob}) = t_v$, where $\mathrm{Frob} \in WD_v$ is the so-called 'Frobenius' element. Now, by restricting φ_v to WD_v', the Weil-Deligne group of E_v, we obtain $\varphi_v' \in \Phi(G_v(E_v))$. The representation of $G_v(E_v)$ attached to this L-parameter is called the 'base-change' representation of π_v, which we denote by $BC(\pi_v)$.

We remark that if $G_v = U_n(F_v)$, then $G_v(E_v) \cong GL_n(E_v)$.

Example 1.12 (π_v an unramified principal series for U_n, r standard)**.** *Suppose* G *is the quasi-split unitary group attached to a hermitian space* V *of dimension* n

over E_v, and that v does not ramify in E/F. Let $B \subset G$ be a Borel subgroup, and let $K \subset G$ be a hyperspecial maximal compact subgroup. As before, we have the Iwasawa decomposition

$$G = BK.$$

Also, we have $B = TN$, where T and N are described in Section 1.2. As before, we let χ be an unramified character of T, which extend to B by 1 on N. We consider the representation

$$\pi_v := \mathrm{Ind}_B^G \chi$$

where again, the induction is normalized by $\delta_B^{1/2}$. Note that

$$\chi = (\chi_1, \chi_2, \ldots, \chi_{\lfloor n/2 \rfloor}).$$

This is because $T \cong (E^\times)^{\lfloor n/2 \rfloor} \times (E_1)$, where E_1 is as in Section 1.2. A Satake parameter for π_v is given by

$$t_v = \begin{cases} \mathrm{diag}(\chi_1(\varpi)^{\frac{1}{2}}, \ldots, \chi_m(\varpi)^{\frac{1}{2}}, \chi_m(\varpi)^{-\frac{1}{2}}, \ldots, \chi_1(\varpi)^{-\frac{1}{2}}) \rtimes \sigma & \text{if } n = 2m \\ \mathrm{diag}(\chi_1(\varpi)^{\frac{1}{2}}, \ldots, \chi_m(\varpi)^{\frac{1}{2}}, 1, \chi_m(\varpi)^{-\frac{1}{2}}, \ldots, \chi_1(\varpi)^{-\frac{1}{2}}) \rtimes \sigma & \text{if } n = 2m+1 \end{cases}$$

where $\sigma \mapsto c \in \mathrm{Gal}(E_v/F_v)$. But what we really want is a Satake parameter for $BC(\pi_v)$. This turns out to be

$$t'_v = \begin{cases} \mathrm{diag}(\chi_1(\varpi), \ldots, \chi_m(\varpi), \chi_m(\varpi)^{-1}, \ldots, \chi_1(\varpi)^{-1}) & \text{if } n = 2m \\ \mathrm{diag}(\chi_1(\varpi), \ldots, \chi_m(\varpi), 1, \chi_m(\varpi)^{-1}, \ldots, \chi_1(\varpi)^{-1}) & \text{if } n = 2m. \end{cases}$$

(See [28]. Note that the conjugacy class $[t_v]$ is independent of the choice of square-root in the definition of t_v.)

So, if $n = 2m$ we have

$$\begin{aligned} L_v(s, BC(\pi_v)) &= \prod_{i=1}^m \frac{1}{(1 - q_{E_v}^{-s}\chi_i(\varpi))(1 - q_{E_v}^{-s}\chi_i(\varpi)^{-1})} \\ &= \prod_{i=1}^m L_v(s, \chi_i) L_v(s, \chi_i^{-1}) \end{aligned}$$

and if $n = 2m+1$ *we have*

$$L_v(s, BC(\pi_v)) = \zeta_{E_v}(s) \prod_{i=1}^{m} L_v(s, \chi_i) L_v(s, \chi_i^{-1}).$$

Example 1.13 (π_v an unramified principal series for U_n, r adjoint)**.** *Here,* π_v *is as in the previous example, while* r *is the adjoint representation of* ${}^LG_v$ *on* $\mathrm{Lie}({}^LG_v^o) = M_n(\mathbb{C})$*. A Satake parameter for* π_v *is as in the previous example. To compute the local* L*-factor, we must compute*

$$t_v X t_v^{-1}$$

where $X \in M_n(\mathbb{C})$*. If we have* $t_v = d \rtimes \sigma$*, then we have that*

$$t_v X t_v^{-1} = d\sigma(X)d^{-1}$$

where $\sigma(X) = -J^t X J^{-1}$*. From this, we see that if* $n = 2m$ *we have*

$$\begin{aligned} L_v(s, \pi_v, \mathrm{Ad}) &= \zeta_v(s)^m L_v(s, \chi_{E_v/F_v})^m \\ &\times \prod_{1 \le i < j \le m} L_v(2s, \chi_i\chi_j) L_v(2s, \chi_i^{-1}\xi_j) L_v(2s, \chi_i^{-1}\chi_j^{-1}) L_v(2s, \chi_i\chi_j^{-1}) \\ &\times \prod_{i=1}^{m} L_v(s, \chi_i) L_v(s, \chi_i^{-1}) \end{aligned}$$

and if $n = 2m+1$ *we have*

$$\begin{aligned} L_v(s, \pi_v, \mathrm{Ad}) &= \zeta_v(s)^m L_v(s, \chi_{E_v/F_v})^{m+1} \\ &\times \prod_{1 \le i < j \le m} L_v(2s, \chi_i\chi_j) L_v(2s, \chi_i^{-1}\xi_j) L_v(2s, \chi_i^{-1}\chi_j^{-1}) L_v(2s, \chi_i\chi_j^{-1}) \\ &\times \prod_{i=1}^{m} L_v(s, \chi_i\chi_{E_v/F_v}) L_v(s, \chi_i^{-1}\chi_{E_v/F_v}) L_v(2s, \chi_i) L_v(2s, \chi_i^{-1}). \end{aligned}$$

1.4 Outline of Dissertation

In the next chapter, we develop some local theory. We define the unitary group analogue of the $\mathcal{P}_v$ described above. Assuming the local representations are tempered, we use some well-known bounds of matrix coefficients to show that these integrals converge absolutely at every place. Then, we use a result from Michael Khoury's Ph.D. dissertation [24], along with an unpublished result of Kato, Murase, and Sugano [23] to compute the $\mathcal{P}_v$ for $v \notin S$, where S is a finite set of 'bad' places (including all archimedean places), outside of which all relevant data is unramified.

Then, in Chapter 3, we give a proof of Conjecture 1.3 for $U(1) \times U(2)$ (assuming a mild extra hypodissertation). In this case, the conjecture follows easily from a theorem of Waldspurger in [35].

The remainder of the dissertation develops the tools necessary for a proof of Conjecture 1.3 for $U(2) \times U(3)$, assuming the representation on $U(3)$ is a theta-lift from $U(2)$. In Chapter 4 , we review a result of Ichino: the so-called triple product formula. This formula gives an explicit relationship between a period integral attached to three representations of $GL_2(\mathbb{A}_F)$ and an L-value. Through the theory of theta correspondence, we can relate the global period $\mathcal{P}$ in the Refined Gross-Prasad Conjecture to the period considered by Ichino. We describe the theta correspondence for unitary groups in Chapter 5; we also give three versions of a Rallis Inner Product Formula. In Chapter 6, we give a local seesaw identity. This identity relates the local integrals $\mathcal{P}_v$ to the local integrals $\mathcal{I}_v$ in Ichino's formula.

In Chapter 7, we use the results from Chapters 4, 5, and 6 to prove the conjecture for $U(2) \times U(3)$. Using a global seesaw identity, we relate the global period $\mathcal{P}$ to the period $\mathcal{I}$ considered by Ichino. Then, using Ichino's triple product formula, we relate $\mathcal{I}$ to the $\mathcal{I}_v$. The local seesaw identity then relates the $\mathcal{I}_v$ to the $\mathcal{P}_v$, which completes the proof.

We also include an appendix. It contains some identities that relate L-functions of representations of $U(1), U(2)$ to their theta-lifts (to either $U(2)$ or $U(3)$), as well as one that relates L-functions in Ichino's work to those relevant to the Refined Gross-Prasad Conjecture.

2 Local Integrals of Matrix Coefficients

In this section we'll prove the convergence of the $\mathcal{P}_v^\natural$, assuming $\pi_{n,v}$ and $\pi_{n+1,v}$ are tempered. This comes down to some standard arguments based on well-known bounds of matrix coefficients. We'll also compute them for $v \notin S$, where S is a sufficiently large finite set of 'bad' places of F, including all even places and archimedean ones. We also make the following assumptions for $v \notin S$ (cf. $(U1)-(U6)$ on page 5 of [21]):

1. The extension E/F is unramified at v.
2. $G_{i,v}$ is unramified over F_v.
3. $K_{i,v} \subset G_{i,v}$ is a hyperspecial maximal compact subgroup.
4. $K_{i,v} \subset K_{i+1,v}$.
5. $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
6. The local Haar measures $dg_{i,v}$ are chosen so that the $K_{i,v}$ have volume 1.
7. The vectors $f_{i,v} \in \pi_{i,v}$ are $K_{i,v}$-fixed and $||f_{i,v}|| = 1$.

Note that even for $v \in S$, we still fix a maximal compact subgroup $K_i \subset G_i$. For the remainder of this section, we will omit v from the notation, though everything is local.

Put

$$L_{\pi_{i+1},\pi_i}(s) := \frac{L_E(s, BC(\pi_{i+1}) \boxtimes BC(\pi_i))}{L_F(s+1/2, \pi_{i+1}, \mathrm{Ad}) L_F(s+1/2, \pi_i, \mathrm{Ad})}.$$

We also consider the matrix coefficient

$$\Phi_{\varphi_i,\varphi_i'}(g_i) := \mathcal{B}_{\pi_i}(\pi_i(g)\varphi_i, \varphi_i')$$

where the $\mathcal{B}_{\pi_i}$ are pairings with respect to which the π_i are unitary, $g_i \in G_i$, and $K_{i,v}$-finite vectors $\varphi_i, \varphi_i' \in \pi_i$. When $\varphi_i = \varphi_i'$, we simply refer to Φ_{φ_i}. We consider the following integral:

$$\mathcal{P}(\varphi_{n+2}, \varphi_{n+1}) := \int_{G_{n+1}} \Phi_{\varphi_{n+2}}(g) \Phi_{\varphi_{n+1}}(g) dg.$$

We will establish convergence of the integral assuming temperedness of the π_i, and compute the integrals away from S.

2.1 Convergence of Integral

Note that we make no assumption that $v \notin S$, but for now, we assume that v does not split in E/F.

Let V be a hermitian space over E of dimension n, and set $G := U(V)$. Let V_{an} the the anisotropic kernel of V. Let d denote the dimension of V_{an}. We have

$$V = X \oplus V_{\mathrm{an}} \oplus Y$$

where X and Y are totally isotropic subspaces. We set $r = \dim_E X = \dim_E Y$. By fixing a basis for X, we obtain a minimal parabolic subgroup $P \subset G$. The Levi factor $M \subset P$ is isomorphic to $(E^\times)^r \times U(V_{\mathrm{an}})$ with maximal torus T such that $M \supset T \cong (E^\times)^r$. The split component $A \subset T \subset M$ is isomorphic to $(F^\times)^r$. We denote an element $x \in A$ as $x = (x_1, x_2, \ldots, x_r)$. The simple roots of (P, A) are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \ldots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1}$$

and

$$\alpha_r(x) = \begin{cases} x_r & \text{if } n \text{ is odd or } U(V) \text{ is not quasi-split} \\ x_r^2 & \text{otherwise.} \end{cases}$$

We view the α_i as elements of $\mathrm{Hom}(T, \mathbb{G}_m)$. Via the natural projection $M \to T$, we may also view the α_i as characters of M.

If we denote by δ the modulus character of P, then we have

$$\delta(x) = \prod_{i=1}^{r} |x_i|_E^{n+1-2i}.$$

(See Proposition 1.2 in [24].) Fix a special maximal compact subgroup $K \subset G$. Then we have a Cartan decomposition

$$G = KM^+K$$

where

$$M^+ := \{m \in M : |\alpha_i(m)| \leq 1 \text{ for } 1 \leq i \leq r\}$$

and K is in good position relative to M.[1] We also define $T^+ := T \cap M^+$.

We fix an embedding $\eta : G \hookrightarrow GL_m$ for some m. We define a height function

$$\sigma(g) := \max_{1 \leq i,j \leq m} \{\log |\eta(g)_{ij}|, \log |\eta(g^{-1})_{ij}|\}.$$

Let $\Xi(g)$ be Harish-Chandra's spherical function given by

$$\Xi(g) := \int_K h(kg)dk$$

where $h \in \mathrm{ind}_P^G 1$ is the function that is identically 1 on K. It is known that there exist positive constants A, B such that

$$A^{-1}\delta^{1/2}(m) \leq \Xi(m) \leq A\delta^{1/2}(m)(1 + \sigma(m))^B$$

for all $m \in M^+$. Also, recall that a function $f(g)$ on G is said to satisfy the weak inequality if

$$|f(g)| \leq A \cdot \Xi(g)(1 + \sigma(g))^B$$

[1] K is such that the unique point in the building of G fixed by K lies on the apartment of A.

for some positive constants A, B, and all $g \in G$. It is known that a matrix coefficient of a tempered representation satisfies the weak inequality. (See [34], for example.)

Let V_{n+2} and V_{n+1} be hermitian spaces of dimension $n+2$ and $n+1$, respectively. Assume further that we have an embedding $\iota : V_{n+1} \hookrightarrow V_{n+2}$ of hermitian spaces. Let $G_i = U(V_i)$ be their associated unitary groups, and let P_i, M_i, A_i, K_i denote the respective minimal parabolic subgroups, Levi component, maximal split tori, and special maximal compact subgroups of G_i. We view G_{n+1} as a subgroup of G_{n+2} via the embedding ι. Note that we may assume that $T_{n+1} \subset T_{n+2}, T^+_{n+1} \subset T^+_{n+2}, M_{n+1} \subset M_{n+2}, M^+_{n+1} \subset M^+_{n+2}$ in this case.

The main goal of this section is to prove the following:

Proposition 2.1. *The integral $\mathcal{P}(\varphi_{i+1}, \varphi_i)$ converges absolutely.*

Proof. This proof is just an adaptation of the analogous proposition in [21]. First, we note the following result from calculus:

Lemma 2.2. *Let $D, r_1, \ldots, r_i > 0$ and $r_{i+1}, \ldots, r_n < 0$. Then the integral*

$$\int_{|x_1|\le|x_2|\le\cdots\le|x_i|\le 1\le|x_{i+1}|\le\cdots\le|x_n|} |x_1|^{r_1}\ldots|x_n|^{r_n} \times\Big(1-\sum_{j=1}^{i}\log|x_j|+\sum_{k=i+1}^{n}\log|x_k|\Big)^D d^\times x_1\ldots d^\times x_n$$

converges absolutely.

We note that by a theorem of Silberger ([31], page 149), the convergence of the integral above is reduced to the convergence of

$$\int_{M^+_{n+1}} \mu(m) \int_{K_{n+1}\times K_{n+1}} \Phi_{\varphi_{n+2}}(k_1 m k_2)\Phi_{\varphi_{n+1}}(k_1 m k_2)dk_1 dk_2 dm \tag{2.1}$$

where

$$\mu(m) := \mathrm{Vol}(K_{n+1} m K_{n+1}) / \mathrm{Vol}(K_{n+1}).$$

In fact, if either of these two integrals converge, they are equal. Furthermore, we have a positive constant A such that

$$|\mu(m)| \leq A \cdot \delta_{n+1}^{-1}(m) \tag{2.2}$$

for all $m \in M_{n+1}^{+}$ (see [31]).

Since $\Phi_{\varphi_{n+1}}$ and $\Phi_{\varphi_{n+2}}$ are matrix coefficients for tempered representations, they satisfy the so-called weak inequality, which means that there are positive constants B, C such that for all $g_i \in G_i$,

$$|\Phi_{\varphi_i}(g_i)| \leq B \cdot |\Xi_i(g_i)|(1+\sigma(g_i))^C \tag{2.3}$$

It is known that that there are positive constants B', C'

$$|\Xi_i(m)| \leq B' \delta_i^{1/2}(m)(1+\sigma(m))^{C'} \tag{2.4}$$

for all $m \in M_i^{+}$ (see [31]).

Combining lines 2.2, 2.3, and 2.4, we see that the convergence of the integral in 2.1 is reduced to the convergence of

$$\int_{M_{n+1}^{+}} \delta_{n+1}^{-1/2}(m)\delta_{n+2}^{1/2}(m)(1+\sigma(m))^D dm$$

for some positive constant D.

When E is a field, we have $M_{n+1}^{+} = T_{n+1}^{+} \times U(V_{n+1,\mathrm{an}})$ and $U(V_{n+1,\mathrm{an}})$ is compact, so the convergence of the integral above is reduced to the convergence of

$$\int_{T_{n+1}^{+}} \delta_{n+1}^{-1/2}(t)\delta_{n+2}^{1/2}(t)(1+\sigma(t))^D dt.$$

Finally, this is reduced to the convergence of

$$\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{r_{n+1}}|\leq 1} |x_1 x_2 \cdot \cdots \cdot x_{r_{n+1}}|^{1/2} \left(1 - \sum_{i=1}^{r_{n+1}} \log|x_i|\right)^D d^\times x_1 d^\times x_2 \ldots d^\times x_{r_{n+1}}$$

which follows from Lemma 2.2.

Now we suppose that $E = F \times F$. In this case, we have $G_i \cong GL_i(F)$. Note that in this case, we have $T_i^+ = M_i^+$; however, we no longer have $T_{n+1}^+ \subset T_{n+2}^+$. With the right choice of bases, we can view

$$T_{n+1}^+ = \{\mathrm{diag}(x_1, x_2, \ldots, x_{n+1}, 1) : x_i \in F^\times, |x_i| \leq |x_{i+1}|\}$$

and

$$T_{n+2}^+ = \{\mathrm{diag}(x_1, x_2, \ldots, x_{n+2}) : x_i \in F^\times, |x_i| \leq |x_{i+1}|\}$$

both as subgroups of $GL_{n+2}(F)$.

For the moment, set $m := \mathrm{diag}(x_1, x_2, \ldots, x_{n+1}, 1)$. We see that $m \in T_{n+2}^+$ if and only if $|x_{n+1}| \leq 1$. If $m \notin T_{n+2}^+$, then we cannot directly apply the bound on Ξ_{n+2} that we used previously. To remedy this, let i be such that $|x_i| \leq 1 \leq |x_{i+1}|$, and set $m' := \mathrm{diag}(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{n+1})$. Then we see that there are $k_1, k_2 \in K_{n+2}$ such that $m = k_1 m' k_2$, and therefore $\Xi_{n+2}(m) = \Xi_{n+2}(m')$. Furthermore, we see that $m' \in T_{n+2}^+$, and therefore the bound we used previously applies.

So, we see that in this case we are reduced to checking the convergence of

$$\int_{|x_1| \leq \cdots \leq |x_{n+1}| \leq 1} |x_1 \ldots x_{n+1}|^{1/2} \left(1 - \sum_{j=1}^{n+1} \log|x_j|\right)^D d^\times x_1 \ldots d^\times x_{n+1}$$
$$+ \int_{|x_1| \leq \cdots \leq |x_n| \leq 1 \leq |x_{n+1}|} |x_1 \ldots x_n x_{n+1}^{-1}|^{1/2}$$
$$\times \left(1 - \sum_{j=1}^{n} \log|x_j| + \log|x_{n+1}|\right)^D d^\times x_1 \ldots d^\times x_{n+1}$$
$$\vdots$$
$$+ \int_{1 \leq |x_1| \leq \cdots \leq |x_{n+1}|} |x_1^{-1} \ldots x_{n+1}^{-1}|^{1/2}$$
$$\left(1 + \sum_{j=1}^{n+1} \log|x_j|\right)^D d^\times x_1 \ldots d^\times x_{n+1}.$$

The convergence of each of these integrals follows from Lemma 2.2. $\square$

2.2 Calculation of integrals in the unramified case

In what follows, we assume $v \notin S$. The purpose of the remainder of this chapter is to explicitly compute $\zeta(\Xi, \xi)$ and $S_{\Xi^{-1},\xi^{-1}}(1)$ for such v.

Away from places in S, we are either in the non-split, but quasi-split case, or the split case. In the non-split case, E/F is an unramified quadratic extension of p-adic fields. In the split case, we have $E = F \oplus F$. Let $V_{n+1} \subset V_{n+2}$ be (quasi-split) hermitian spaces over E, and let $G_{n+1} \subset G_{n+2}$ be the associated unitary groups.[2]

The calculation of the integrals $\mathcal{P}'(f_{\pi_{n+1}}, f_{\pi_{n+2}})$ will involve splitting them into the product of two pieces, each of which will be computed independently. But first, we give a description of the representations π_i; away from S, these have a concrete description.

We set $l_i := \lfloor i/2 \rfloor$. Let $\xi_1, \ldots, \xi_{l_{n+1}}$ and $\Xi_1, \ldots, \Xi_{l_{n+2}}$ be unramified characters of $E^\times$, and let Ξ_0 and ξ_0 be unramified characters of E_1, where

$$E_1 := \{x \in E^\times : N_{E/F}(x) = 1\}$$

and $N_{E/F}$ is the relative norm map. Note that if E is a field, then E_1 is compact, and ξ_0 and Ξ_0 are trivial (since they're unramified). However, if E is not a field, then $E_1 \cong F^\times$, and Ξ_0, ξ_0 need not be trivial. In the split case, for $i \geq 1$, we have $\Xi_i = (\mu_i, \nu_i)$ and $\xi_i = (\theta_i, \phi_i)$, where each $\mu_i, \nu_i, \theta_i, \phi_i$ are all unramified characters of $F^\times$.

If $n+1$ is odd, then we have $T_{n+1} \cong E_1 \times (E^\times)^{l_{n+1}}$, and $T_{n+2} \cong (E^\times)^{l_{n+2}}$. Then we define characters $\xi := (\xi_0, \xi_1, \ldots, \xi_{l_{n+1}})$ and $\Xi := (\Xi_1, \ldots, \Xi_{l_{n+2}})$ of T_{n+1} and T_{n+2}, respectively. If $n+1$ is even, then $T_{n+1} \cong (E^\times)^{l_{n+1}}$ and $T_{n+2} \cong E_1 \times (E^\times)^{l_{n+2}}$ and we have $\xi = (\xi_1, \ldots, \xi_{l_{n+1}})$ and $\Xi = (\Xi_0, \Xi_1, \ldots, \Xi_{l_{n+2}})$.

Let $B_i = T_i N_i$ be Borel subgroups, where the N_i are unipotent radicals. Then we can view ξ and Ξ as characters of B_{n+1} and B_{n+2} by extending them by 1 to N_i.

[2]The mildly strange choice of notation $n+1$ and $n+2$ will be explained later.

Away from S, we may assume that

$$\begin{aligned}\pi_{n+1} &\cong \operatorname{Ind}_{B_{n+1}}^{G_{n+1}} \xi \\ \pi_{n+2} &\cong \operatorname{Ind}_{B_{n+2}}^{G_{n+2}} \Xi\end{aligned}$$

Note that in the split case, we have $G_i \cong GL_i(F)$, and we have

$$\xi = (\theta_1, \ldots, \theta_{l_{n+1}}, \mu_0, \phi_{l_{n+1}}^{-1}, \ldots, \phi_1^{-1}) \text{ or } (\theta_1, \ldots, \theta_{l_{n+1}}, \phi_{l_{n+1}}^{-1}, \ldots, \phi_1^{-1})$$

and

$$\Xi = (\mu_1, \ldots, \mu_{l_{n+2}}, \nu_{l_{n+2}}^{-1}, \ldots, \nu_1^{-1}) \text{ or } (\mu_1, \ldots, \mu_{l_{n+2}}, \Xi_0, \nu_{l_{n+2}}^{-1}, \ldots, \nu_1^{-1}),$$

according to whether n is odd or even, respectively.

We choose the $\mathcal{B}_{\pi_i}$ so that the $\Phi_{f_{\pi_i}}$ are the spherical matrix coefficients normalized such that $\Phi_{f_{\pi_i}}(k_i) = 1$ for all $k_i \in K_i$. Then we have the following explicit formulae:

$$\Phi_{f_{\pi_i}}(g_i) = \int_{K_i} f_{\pi_i}(k_i g_i)\ dk_i$$

for all $g_i \in G_i$.

We consider the function

$$F(g_{n+2}) := \int_{G_{n+1}} \Phi_{f_{\pi_{n+2}}}(g_{n+2}^{-1} g_{n+1}) \Phi_{f_{\pi_{n+1}}}(g_{n+1})\ dg_{n+1}$$

for $g_{n+2} \in G_{n+2}$. We're interested in computing $F(1)$. While it may seem a bit silly to invent this function if we're only interested in its value at the identity, the reason for this definition will become clear soon.

Let G_{n+1}^{Δ} denote the diagonal copy of G_{n+1} in $G_{n+2} \times G_{n+1}$. Then we note that $G_{n+2} \times G_{n+1}/G_{n+1}^{\Delta}$ is a spherical variety. This means that it has a unique open orbit under the action of $B_{n+2} \times B_{n+1}$ on the left. Also, we note that

$$B_{n+2} \times B_{n+1} \backslash G_{n+2} \times G_{n+1}/G_{n+1}^{\Delta} \cong B_{n+2} \backslash G_{n+2}/B_{n+1}.$$

Recalling that we have Iwasawa decompositions

$$G_i = B_i K_i,$$

we let $\eta_{n+2} \in K_{n+2}$ be a representative for the open $B_{n+2} \times B_{n+1}$ orbit on G_{n+2}. Let $Y_{\Xi,\xi}$ be the function on G_{n+2} determined by the following conditions:

1. $Y_{\Xi,\xi}(b_{n+2}g_{n+2}b_{n+1}) = (\Xi^{-1}\delta_{n+2}^{1/2})(b_{n+2})(\xi\delta_{n+1}^{-1/2})(b_{n+1})Y_{\Xi,\xi}(g_{n+2})$ for all $b_i \in B_i$.

2. $Y_{\Xi,\xi}(\eta_{n+2}) = 1$

3. $Y_{\Xi,\xi}(g_{n+2}) = 0$ if $g_{n+2} \notin B_{n+2}\eta_{n+2}B_{n+1}$.

Here, δ_i denotes the modulus character for B_i. We define the following two functions on G_{n+2}:

$$\begin{aligned}
T_{\Xi,\xi}(g_{n+2}) &:= \begin{cases} \int_{G_{n+1}} f_{\pi_{n+2}}(g_{n+2}g_{n+1})f_{\pi_{n+1}}(g_{n+1})\ d_{g_{n+1}} & \text{if } g_{n+2} \in B_{n+2}\eta_{n+2}B_{n+1} \\ 0 & \text{otherwise} \end{cases} \\
S_{\Xi,\xi}(g_{n+2}) &:= \int_{K_{n+2}}\int_{K_{n+1}} Y_{\Xi,\xi}(k_{n+2}g_{n+2}^{-1}k_{n+1})\ dk_{n+1}d_{k_{n+2}}.
\end{aligned}$$

We have $T_{\Xi,\xi}(g_{n+2}) = T_{\Xi,\xi}(\eta_{n+2})Y_{\Xi^{-1},\xi^{-1}}(g_{n+2})$ since $T_{\Xi,\xi}$ satisfies conditions (1) and (3) for $Y_{\Xi^{-1},\xi^{-1}}$. Also, we note that $T_{\Xi,\xi}(\eta_{n+2})$ does not depend on the choice of representative η_{n+2}. So we denote $\zeta(\Xi,\xi) := T_{\Xi,\xi}(\eta_{n+2})$.

Relating these to the integral that is the subject of this chapter, we have

$$\begin{aligned}
&F(g_{n+2}) \\
&:= \int_{G_{n+1}} \Phi_{f_{\pi_{n+2}}}(g_{n+2}^{-1}g_{n+1})\Phi_{f_{\pi_{n+1}}}(g_{n+1})\ dg_{n+1} \\
&= \int_{G_{n+1}}\int_{K_{n+2}}\int_{K_{n+1}} f_{\pi_{n+2}}(k_{n+2}g_{n+2}^{-1}g_{n+1})f_{\pi_{n+1}}(k_{n+1}g_{n+1})\ dk_{n+1}dk_{n+2}dg_{n+1} \\
&= \int_{G_{n+1}}\int_{K_{n+2}}\int_{K_{n+1}} f_{\pi_{n+2}}(k_{n+2}g_{n+2}^{-1}k_{n+1}g_{n+1})f_{\pi_{n+1}}(g_{n+1})\ dk_{n+1}dk_{n+2}dg_{n+1} \\
&= \int_{K_{n+2}}\int_{K_{n+1}} T_{\Xi,\xi}(k_{n+2}g_{n+2}^{-1}k_{n+1})\ dk_{n+1}dk_{n+2} \\
&= T_{\Xi,\xi}(\eta_{n+2})\int_{K_{n+2}}\int_{K_{n+1}} Y_{\Xi^{-1},\xi^{-1}}(k_{n+2}g_{n+2}^{-1}k_{n+1})\ dk_{n+1}dk_{n+2} \\
&= \zeta(\Xi,\xi)S_{\Xi^{-1},\xi^{-1}}(g_{n+2}).
\end{aligned}$$

Regarding convergence, we note (as in [21]), that $F(g_{n+2})$ is convergent for Ξ and ξ sufficiently close to the unitary axis. (Indeed, Proposition 2.1 holds for such

Ξ, ξ.) So, we see that $T_{\Xi,\xi}(k_{n+2}g_{n+2}^{-1}k_{n+1})$ is convergent for almost all k_{n+2}, k_{n+1} such that $k_{n+2}g_{n+2}^{-1}k_{n+1} \in B_{n+2}\eta_{n+2}B_{n+1}$. But since $T_{\Xi,\xi}(g)$ is convergent for some $g \in B_{n+2}\eta_{n+2}B_{n+1}$ if and only if it is convergent for all $g \in B_{n+2}\eta_{n+2}B_{n+1}$, we see that $T_{\Xi,\xi}(\eta_{n+2})$ is convergent.

2.2.1 Calculation of $\zeta(\Xi, \xi)$

In this section, we will actually have occasion to consider three different hermitian spaces simultaneously; let $V_n \subset V_{n+1} \subset V_{n+2}$ be hermitian spaces of dimensions $n, n+1$ and $n+2$ over E, and let G_i be their respective unitary groups. Until specified otherwise, we do not distinguish between the non-split and split cases. That is, we do not view the split unitary groups as general linear groups over F, but still as isometry groups of hermitian spaces over E.

If $n = 2m$, then we write

$$\begin{aligned} V_n &= \langle e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m \rangle \\ V_{n+1} &= V_n \oplus \langle e_{m+1} + f_{m+1} \rangle \\ V_{n+2} &= V_{n+1} \oplus \langle e_{m+1} - f_{m+1} \rangle \end{aligned}$$

where all of the e_i, f_i are isotropic vectors. Furthermore, if we let h denote the hermitian form on V_{n+2}, we have $h(e_i, f_j) = \delta_{ij}$. Then we have the Borel subgroups $B_i \subset G_i$ where $B_i := \operatorname{Stab}_{G_i} \mathcal{F}_i$ where the $\mathcal{F}_i$ are the following flags of isotropic spaces:

$$\begin{aligned} \mathcal{F}_n &:= \{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \cdots \subsetneq \langle e_1, \ldots e_m \rangle \subsetneq V_n \\ \mathcal{F}_{n+1} &:= \{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \cdots \subsetneq \langle e_1, \ldots e_m \rangle \subsetneq V_{n+1} \\ \mathcal{F}_{n+2} &:= \{0\} \subsetneq \langle e_{m+1} \rangle \subsetneq \langle e_{m+1}, e_1 \rangle \cdots \subsetneq \langle e_{m+1}, e_1, \ldots e_m \rangle \subsetneq V_{n+2}. \end{aligned}$$

If $n = 2m+1$, then we have

$$\begin{aligned} V_n &= \langle e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m, e_{m+1} + f_{m+1} \rangle \\ V_{n+1} &= V_n \oplus \langle e_{m+1} - f_{m+1} \rangle \\ V_{n+2} &= V_{n+1} \oplus \langle e_{m+2} + f_{m+2} \rangle. \end{aligned}$$

Again, all of the e_i, f_i are isotropic, and we have $h(e_i, f_j) = \delta_{ij}$. In this case, the flags are:

$$\begin{aligned}\mathcal{F}_n &:= \{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \cdots \subsetneq \langle e_1, \dots e_m \rangle \subsetneq V_n \\ \mathcal{F}_{n+1} &:= \{0\} \subsetneq \langle e_{m+1} \rangle \subsetneq \langle e_{m+1}, e_1 \rangle \subsetneq \cdots \subsetneq \langle e_{m+1}, e_1 \dots e_m \rangle \subsetneq V_{n+1} \\ \mathcal{F}_{n+2} &:= \{0\} \subsetneq \langle v \rangle \subsetneq \langle v, e_1 \rangle \subsetneq \cdots \subsetneq \langle v, e_1, \dots, e_m \rangle \subsetneq V_{n+2}\end{aligned}$$

where $v = e_{m+2} + f_{m+2} + e_{n+1} - f_{n+1}$.

Let $K_n \subset K_{n+1} \subset K_{n+2}$ be hyperspecial maximal compact subgroups of the G_i, such that $G_i = B_i K_i$. For $i = n, n+1$, we consider the action of $B_{i+1} \times B_i$ on G_{i+1} by $(b_{i+1}, b_i) \cdot g_{i+1} := b_{i+1} g_{i+1} b_i^{-1}$. We let $\eta_{i+1} \in K_{i+1}$ be a representative for the unique open dense orbit. We now prove a proposition that relates the representatives of the open orbits of $B_{n+2} \times B_{n+1}$ on G_{n+2} and $B_{n+1} \times B_n$ on G_{n+1}.

Proposition 2.3. *$\eta_{n+1}^{-1} \in K_{n+1} \subset K_{n+2}$ is a representative for the open orbit of $B_{n+2} \times B_{n+1}$ acting on G_{n+2}.*

Proof. To show that η_{n+1}^{-1} is a representative for the open $B_{n+2} \times B_{n+1}$ orbit in G_{n+2}, it suffices to check that $B_{n+2} \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1}$ is trivial.

Suppose that $n = 2m$. First, we show that $B_{n+2} \cap G_{n+1} \subset B_n$. To see this, take any $b \in B_{n+2} \cap G_{n+1}$. Since $b \in G_{n+1}$, we know that b fixes $e_{m+1} - f_{m+1}$. But since $b \in B_{n+2}$, we know that

$$b \cdot e_{m+1} = a e_{m+1}$$

for some $a \in E^\times$ and

$$b \cdot f_{m+1} = \overline{a}^{-1} f_{m+1} + \sum_{i=1}^{m+1} a_i e_i + \sum_{i=1}^{m} c_i f_i$$

for $a_i, c_i \in E$. This means that we have

$$b \cdot (e_{m+1} - f_{m+1}) = a e_{m+1} - \overline{a}^{-1} f_{m+1} + \sum_{i=1}^{m+1} a_i e_i + \sum_{i=1}^{m} c_i f_i.$$

But since b fixes $e_{m+1} - f_{m+1}$, this means that $a = 1$ and $a_i = c_i = 0$ for all i. So we see that b fixes both e_{m+1}, f_{m+1} and is therefore in G_n. But since it preserves $\mathcal{F}_{n+2}$ and e_{m+1}, we see that it preserves $\mathcal{F}_n$, and is therefore in B_n.

Now we have that

$$B_{n+2} \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} = B_{n+2} \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} \cap G_{n+1} = B_n \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} = \{1\}.$$

Now suppose that $n = 2m+1$. Again, we will show that $B_{n+2} \cap G_{n+1} \subset B_n$. First, we show that $B_{n+2} \cap G_{n+1} \subset G_n$. To see this, we need only show that $b \in B_{n+2} \cap G_{n+1}$ fixes $e_{m+1} - f_{m+1}$. Well, since $b \in G_{n+1}$, we know that it fixes $e_{m+2} + f_{m+2}$. Also, since $b \in B_{n+2}$, we know that it scales v. So we have:

$$\begin{aligned}
b \cdot v &= b(e_{m+2} + f_{m+2} + e_{m+1} - f_{m+1}) \\
&= b(e_{m+2} + f_{m+2}) + b(e_{m+1} - f_{m+1}) \\
&= e_{m+2} + f_{m+2} + b(e_{m+1} - f_{m+1}) \\
&= av \quad \text{for some } a \in E^\times \\
&= ae_{m+2} + af_{m+2} + ae_{m+1} - af_{m+1}.
\end{aligned}$$

But since $b \in G_{n+1}$, we know that $b \cdot (e_{m+1} - f_{m+1}) \in V_{n+1}$, and so $a = 1$. This gives that b fixes $e_{m+1} - f_{m+1}$ and is therefore in G_{n+1}. We also see that it fixes v. This, together with the fact that it preserves $\mathcal{F}_{n+2}$ immediately gives that it preserves $\mathcal{F}_n$, and is therefore in B_n. Then just as in the case where $n = 2m$, we see that

$$B_{n+2} \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} = B_{n+2} \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} \cap G_{n+1} = B_n \cap \eta_{n+1}^{-1} B_{n+1} \eta_{n+1} = \{1\}.$$

□

Now, we take V to be an n-dimensional hermitian space with associated hermitian form $\langle \cdot, \cdot \rangle_V$. Let V^- be the same space as V, but with the hermitian form $-\langle \cdot, \cdot \rangle_V$. Also, let $Q = \operatorname{Span}\{e, f\}$ be a hyperbolic plane, where e and f are isotropic vectors and $\langle e, f \rangle_Q = 1$. We also consider the hermitian space $W := V \oplus V^- \oplus Q$.

We will be considering the following groups defined over F:

$$\begin{aligned} G_n &:= U(V) \cong U(V^-) \\ G_{n+1} &:= U(V^- \oplus \langle e+f \rangle) \\ G_{n+2} &:= U(V^- \oplus Q) \\ G &:= U(W) \end{aligned}$$

where in the first line, we identity V and V^- as vector spaces (but not as hermitian spaces) via the identity map.

Note that we have inclusions $G_n \subset G_{n+1} \subset G_{n+2}$. Let B_n, B_{n+1}, B_{n+2} be Borel subgroups as in the beginning of this section. Let T_i, N_i be the corresponding tori and unipotent radicals, and let K_i be hyperspecial maximal compact subgroups such that $G_i = B_i K_i$. Let $\tilde{\Xi}, \xi, \Xi$ be characters of T_n, T_{n+1}, T_{n+2} respectively, where

$$\begin{aligned} \tilde{\Xi} &:= (\Xi_1, \Xi_2, \ldots, \Xi_{\lfloor n/2 \rfloor}) \\ \xi &:= (\xi_1, \xi_2, \ldots, \xi_{\lfloor (n+1)/2 \rfloor}) \\ \Xi &:= (\Xi_1, \ldots, \Xi_{\lfloor (n+2)/2 \rfloor}) \end{aligned}$$

and each Ξ_i, ξ_i is an unramified character of $E^\times$. Recall that $l := \lfloor (n+2)/2 \rfloor$.

As before, we extend these characters to B_i by 1 along N_i. Denote by π_i the corresponding unramified principal series representation, and let $f_{\pi_i} \in \pi_i$ be the corresponding spherical vector, normalized so that $f_{\pi_i}(k_i) = 1$ for all $k_i \in K_i$.

Let $\iota : V \to V^-$ be the identity map. Then we define the following subspace of W:

$$V^\dagger := \operatorname{Span}\{v_i - \iota(v_i), e\}.$$

Note that $V^\dagger$ is a maximal isotropic subspace of W. Denote by $G \supset P := \operatorname{Stab}_G V^\dagger$ the corresponding maximal parabolic subgroup. The Levi subgroup $M \subset P$ is isomorphic to $GL(V^\dagger)$. Denote by $N \subset P$ the unipotent radical. We consider the following induced representation of G:

$$I(\Xi_l) := \operatorname{Ind}_P^G(\Xi_l \circ \det_{V^\dagger})$$

where the induction is normalized. Let $f_0 \in I(\Xi_l)$ be the spherical vector, normalized so that $f_0(1) = 1$. We consider $f_0|_{G_n \times G_{n+2}}$, and denote this by $\tilde{f}_0$. We define the following integral:

$$\Lambda(f_0, f_{\pi_n})(g_{n+2}) := \int_{G_n} \tilde{f}_0(g_n, g_{n+2})\pi_n(g_n)f_{\pi_n}\ dg_n.$$

We remark that since Ξ_l is unramified, we have $\Xi_l = |\cdot|_E^s$ for some $s \in \mathbb{C}$. The integral Λ converges for $\mathrm{Re}(s) >> 0$, and we use analytic continuation to define Λ elsewhere.

Note that

$$\Lambda(f_0, f_{\pi_n}) \in \mathrm{Ind}_{GL(V^\dagger \cap Q)\times G_n}^{G_{n+2}} \Xi_l \otimes \pi_n,$$

and so by transitivity of induction, we may view $\Lambda(f_0, f_{\pi_n})$ as an element of π_{n+2}.

Proposition 2.4. *For any $k \in K_{n+2}$ and $n \geq 1$ we have*

$$\Lambda(f_0, f_{\pi_n})(k) = \frac{L_E(1, BC(\pi_n)\otimes \Xi_l)}{\prod_{j=1}^n L_F(1+j, \Xi_l \chi_{E/F}^{n-j})} f_{\pi_n},$$

where $\chi_{E/F}$ is the quadratic character attached to the extension E/F by class field theory. Recall that in the case where E is not a field, this is the trivial character.

Proof. First, we note that Λ is constant as a function on K_{n+2}. To see this, we simply note that f_0 is a spherical vector.

Now, we consider $\widehat{f}_0 := f_0|_{G_n \times G_n}$. Then we see that

$$\widehat{f}_0 \in \mathrm{Ind}_{P\cap(G_n\times G_n)}^{G_n\times G_n} \left(\frac{\delta_P}{\delta_{P\cap(G_n\times G_n)}}\right)^{1/2} (\Xi_l \circ \det_{V^\dagger})$$

Now, we know that

$$\delta_P(p) = |\det_{V^\dagger}(p)|^{n+1}$$

and

$$\delta_{P\cap(G_n\times G_n)}(p) = |\det_{V^\dagger}(p)|^{n}$$

for $p \in P\cap(G_n \times G_n)$. So, by taking $G_n = G$ in Proposition 3 in [26], we see that

$$\Lambda(f_0, f_{\pi_n})(1) = \Lambda(\widehat{f}_0, f_{\pi_n})(1) = \frac{L_E(1, BC(\pi_n)\otimes \Xi_l)}{\prod_{j=1}^n L_F(1+j, \Xi_l \chi_{E/F}^{n-j})} f_{\pi_n}.$$

(This is analogous to Theorem 1.1 on page 16 of [8].) □

We recall the action of $B_{i+1} \times B_i$ on G_{i+1} by $(b_{i+1}, b_i)g_{i+1} := b_{i+1}g_{i+1}b_i^{-1}$. As we mentioned before, there is a unique open and dense orbit under this action. We let $\eta_{i+1} \in K_{i+1}$ be a representative of this orbit.

Using the previous proposition, we have the following inductive relationship between $\zeta(\Xi, \xi)$ and $\zeta(\xi, \tilde{\Xi})$:

Proposition 2.5. *For $n \geq 1$ we have*

$$\zeta(\Xi, \xi) = \frac{L_E(1/2, BC(\pi_{n+1}) \otimes \Xi_l)}{L_E(1, BC(\pi_n) \otimes \Xi_l) L_F(1, \chi_{E/F}^n \otimes \Xi_l)} \zeta(\xi, \tilde{\Xi}).$$

Proof. First, we note the following:

$$\begin{aligned} &\int_{G_{n+1}} \tilde{f}_0(1, g_{n+1}) \int_{G_n} f_{\pi_{n+1}}(\eta_{n+1} g_n g_{n+1}) f_{\pi_n}(g_n) dg_n dg_{n+1} \\ &= \frac{L_E(1/2, BC(\pi_{n+1}) \otimes \Xi_l)}{\prod_{j=1}^{n+1} L_F(j, \Xi_l \chi_{E/F}^{n+1-j})} \zeta(\xi, \tilde{\Xi}) \end{aligned}$$

which follows from Proposition 3 in [26].[3] To see this, we consider the pairing $\mathcal{T} : \pi_{n+1} \otimes \pi_n \to \mathbb{C}$ given by

$$\mathcal{T}(f_1, f_2) := \int_{G_n} f_1(\eta_{n+1} g_n) f_2(g_n) dg_n.$$

Clearly, we have $\mathcal{T}(f_{\pi_{n+1}}, f_{\pi_n}) = \zeta(\xi, \tilde{\Xi})$. Now, note that $g_{n+1} \mapsto \tilde{f}_0(1, g_{n+1})$ is bi-invariant under K_{n+1}. Choosing dk so that $\int_{K_{n+1}} dk = 1$, we have

$$\begin{aligned} &\int_{G_{n+1}} \tilde{f}_0(1, g_{n+1}) \mathcal{T}(g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dg_{n+1} \\ &= \int_{G_{n+1}} \int_{K_{n+1}} \tilde{f}_0(1, k g_{n+1}) \mathcal{T}(g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dk dg_{n+1} \\ &= \int_{G_{n+1}} \tilde{f}_0(1, g_{n+1}) \int_{K_{n+1}} \mathcal{T}(k^{-1} g_{n+1} f_{\pi_{n+1}}, f_{\pi_n}) dk dg_{n+1}. \end{aligned}$$

The inner integral gives a smooth linear form on π_{n+1}; more specifically,

$$f \mapsto \int_{K_{n+1}} \mathcal{T}(k^{-1} f, f_{\pi_n}) dk$$

[3] Note that in [26] the authors actually consider an integral over $G_{n+1}^{\Delta} \backslash (G_{n+1} \times G_{n+1})$, where G_{n+1}^{Δ} is the diagonally embedded copy of G_{n+1}. By identifying this with G_{n+1}, we are led to consider the integral above instead.

gives a map $L \in \pi_{n+1}^{\vee}$. In fact, $L(g_{n+1}f_{\pi_{n+1}})$ gives a matrix coefficient on π_{n+1} which is bi-invariant under K_{n+1}. So, we see that there is a constant A such that

$$L(g_{n+1}f_{\pi_{n+1}}) = A \cdot \mathcal{B}_{\pi_{n+1}}(g_{n+1}f_{\pi_{n+1}}, f_{\pi_{n+1}}).$$

To compute A, we simply take $g_{n+1} = 1$, and we obtain $\zeta(\xi, \tilde{\Xi}) = A$. Now we use the result in [26] to obtain the claim at the beginning of the proof.

Now, using Proposition 2.4, we observe the following:

$$\begin{aligned}
&\int_{G_{n+1}} \tilde{f}_0(1, g_{n+1}) \int_{G_n} f_{\pi_{n+1}}(\eta_{n+1}g_n g_{n+1}) f_{\pi_n}(g_n) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} \int_{G_n} f_{\pi_{n+1}}(\eta_{n+1}g_n g_{n+1}) \tilde{f}_0(g_n, g_n g_{n+1}) f_{\pi_n}(g_n) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} \int_{G_n} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \tilde{f}_0(g_n, g_{n+1}) f_{\pi_n}(g_n) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \int_{G_n} \tilde{f}_0(g_n, g_{n+1}) f_{\pi_n}(g_n) dg_n dg_{n+1} \\
&= \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) \Lambda(f_0, f_{\pi_n})(g_{n+1})(1) dg_{n+1} \\
&= \frac{L_E(1, BC(\pi_n) \otimes \Xi_l)}{\prod_{j=1}^n L_F(1+j, \Xi_l \chi_{E/F}^{n-j})} \int_{G_{n+1}} f_{\pi_{n+1}}(\eta_{n+1}g_{n+1}) f_{\pi_{n+2}}(g_{n+1}) dg_{n+1} \\
&= \frac{L_E(1, BC(\pi_n) \otimes \Xi_l)}{\prod_{j=1}^n L_F(1+j, \Xi_l \chi_{E/F}^{n-j})} \int_{G_{n+1}} f_{\pi_{n+2}}(\eta_{n+1}^{-1}g_{n+1}) f_{\pi_{n+1}}(g_{n+1}) dg_{n+1} \\
&= \frac{L_E(1, BC(\pi_n) \otimes \Xi_l)}{\prod_{j=1}^n L_F(1+j, \Xi_l \chi_{E/F}^{n-j})} \zeta(\Xi, \xi).
\end{aligned}$$

Note that we've used the fact that η_{n+1}^{-1} is a representative for the open $B_{n+2} \times B_{n+1}$-orbit in G_{n+2}. Also, we remark that the calculation above is similar to that carried out for orthogonal groups in [8].

Combining this with the first identity mentioned completes the proof. □

By induction on n, this gives us the following in the non-split (but quasi-split) case:

Corollary 2.6. *Suppose E is a field. If n is even, then*

$$\begin{aligned}\zeta(\Xi,\xi) &= \prod_{1\le i<j\le n/2+1} L_E(1/2,\xi_i\Xi_j)L_E(1/2,\xi_i^{-1}\Xi_j)L_E(1,\Xi_i\Xi_j)^{-1}L_E(1,\Xi_i^{-1}\Xi_j)^{-1}\\ &\quad\times\prod_{1\le i\le j\le n/2} L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i^{-1}\xi_j)\\ &\quad\times\prod_{1\le i<j\le n/2} L_E(1,\xi_i\xi_j)^{-1}L_E(1,\xi_i^{-1}\xi_j)^{-1}\\ &\quad\times\prod_{i=1}^{n/2} L_E(1/2,\chi_{E/F}\xi_i)^{-1}L_E(1,\xi_i)^{-1}.\end{aligned}$$

If n is odd, then

$$\begin{aligned}\zeta(\Xi,\xi) &= \prod_{1\le i\le j\le (n+1)/2} L_E(1/2,\xi_i\Xi_j)L_E(1/2,\xi_i^{-1}\Xi_j)\\ &\quad\times\prod_{1\le i<j\le (n+1)/2} L_E(1,\Xi_i\Xi_j)^{-1}L_E(1,\Xi_i^{-1}\Xi_j)^{-1}L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i^{-1}\xi_j)\\ &\quad\times\prod_{1\le i<j\le (n+1)/2} L_E(1,\xi_i\xi_j)^{-1}L_E(1,\xi_i^{-1}\xi_j)^{-1}\\ &\quad\times\prod_{i=1}^{(n+1)/2} L_E(1/2,\chi_{E/F}\Xi_i)^{-1}L_E(1,\Xi_i)^{-1}.\end{aligned}$$

Proof. We check the base cases. The inductive steps follow from the previous proposition.

The base case is computing $\zeta((\Xi_1),(\xi_0))$, where ξ_0 is the trivial character. The two groups involved in this calculation are G_2 and G_1 with principal series representations π_2 and π_1 respectively. Let $f_{(\Xi_1)}$ be the normalized spherical vector in π_2, and let $f_{(\xi_0)}$ be the normalized spherical vector in π_1. Now, since $G_1 = K_1$ is compact, we see that $f_{(\xi_0)}$ is the constant function equal to 1. Now, let $\eta_2 \in K_2$

be a representative for the open $B_2 \times B_1$ orbit in G_2. Then we have that

$$\begin{aligned}
\zeta((\Xi_1),(\xi_0)) &= \int_{G_1} f_{(\Xi_1)}(\eta_2 g_1) f_{(\xi_0)}(g_1)\ dg_1 \\
&= \int_{K_1} f_{(\Xi_1)}(\eta_2 g_1)\ dg_1 \\
&= \int_{K_1} dg_1 \text{ (since } \eta_2 g_1 \in K_2) \\
&= 1.
\end{aligned}$$

□

We state the result in the split case as a separate corollary.

Corollary 2.7. *Suppose that $E = F \times F$. If n is even, then*

$$\begin{aligned}
\zeta(\Xi,\xi) &= \prod_{1 \le i < j \le l_{n+2}} L_F(1/2, \theta_i \mu_j)_F(1/2, \phi_i^{-1}\mu_j) L_F(1/2, \theta_i^{-1}\nu_j) L_F(1/2, \phi_i \nu_j) \\
&\times \prod_{1 \le i \le j \le l_{n+1}} L_F(1/2, \mu_i \theta_j) L_F(1/2, \nu_i^{-1}\theta_j) L_F(1/2, \mu_i^{-1}\phi_j) L_F(1/2, \nu_i \phi_j) \\
&\times \prod_{i=1}^{l_{n+2}} L_F(1/2, \xi_0 \mu_i) L_F(1/2, \xi_0^{-1}\nu_i) \\
&\times \prod_{1 \le i < j \le l_{n+2}} L_F(1, \mu_i^{-1}\mu_j)^{-1} L_F(1, \nu_i \mu_j)^{-1} L_F(1, \mu_i \nu_j)^{-1} L_F(1, \nu_i^{-1}\nu_j)^{-1} \\
&\times \prod_{1 \le i < j \le l_{n+1}} L_F(1, \theta_i^{-1}\theta_j)^{-1} L_F(1, \phi_i \theta_j)^{-1} L_F(1, \theta_i \phi_j)^{-1} L_F(1, \phi_i^{-1}\phi_j)^{-1} \\
&\times \prod_{i=1}^{l_{n+2}} L_F(1, \mu_i \nu_i)^{-1} \prod_{i=1}^{l_{n+1}} L_F(1, \xi_0^{-1}\theta_i)^{-1} L_F(1, \xi_0 \phi_i)^{-1} L_F(1, \theta_i \phi_i)^{-1}.
\end{aligned}$$

If n is odd, then

$$\begin{aligned}
\zeta(\Xi,\xi) &= \prod_{1\le i\le j\le l_{n+2}} L_F(1/2,\theta_i\mu_j)L_F(1/2,\phi_i^{-1}\mu_j)L_F(1/2,\theta_i^{-1}\nu_j)L_F(1/2,\phi_i\nu_j)\\
&\quad\times \prod_{1\le i<j\le l_{n+1}} L_F(1/2,\mu_i\theta_j)L_F(1/2,\nu_i^{-1}\theta_j)L_F(1/2,\mu_i^{-1}\phi_j)L_F(1/2,\nu_i\theta_j)\\
&\quad\times \prod_{i=1}^{l_{n+1}} L_F(1/2,\Xi_0\theta_i)L_F(1/2,\Xi_0^{-1}\phi_i)\\
&\quad\times \prod_{1\le i<j\le l_{n+2}} L_F(1,\mu_i^{-1}\mu_j)^{-1}L_F(1,\nu_i\mu_j)^{-1}L_F(1,\mu_i\nu_j)^{-1}L_F(1,\nu_i^{-1}\nu_j)^{-1}\\
&\quad\times \prod_{1\le i<j\le l_{n+1}} L_F(1,\theta_i^{-1}\theta_j)^{-1}L_F(1,\phi_i\theta_j)^{-1}L_F(1,\theta_i\phi_j)^{-1}L_F(1,\phi_i^{-1}\phi_j)^{-1}\\
&\quad\times \prod_{i=1}^{l_{n+2}} L_F(1,\mu_i\nu_i)^{-1}L_F(1,\Xi_0^{-1}\mu_i)^{-1}L_F(1,\Xi_0\nu_i)^{-1}\prod_{i=1}^{l_{n+1}} L_F(1,\theta_i\phi_i)^{-1}.
\end{aligned}$$

Proof. Again, we need only check the base case. This time, we're computing $\zeta((\mu_1,\nu_1^{-1}),(\xi_0))$ or $\zeta((\theta_1,\phi_1^{-1}),(\Xi_0))$, depending on whether n is even or odd, respectively.

We compute $\zeta((\theta_1,\phi_1^{-1}),(\Xi_0))$. We have

$$\zeta((\theta_1,\phi_1^{-1}),(\Xi_0)) = \int_{G_1} f_{(\theta_1,\phi_1^{-1})}(a)f_{(\Xi_0)}(a)\ da.$$

Now, we now that $G_1\cong F^\times$. Also, G_1 embeds in G_2 in the following way:

$$F^\times \ni a \mapsto \begin{pmatrix} a+1/2 & a-1/2\\ a-1/2 & a+1/2\end{pmatrix}\in G_2.$$

(Note that since $B_2\cap B_1=\{1\}$, we take η_2 to be trivial.)

Now, if $a\in\mathcal{O}_F^\times$, then we have that

$$\begin{pmatrix} a+1/2 & a-1/2\\ a-1/2 & a+1/2\end{pmatrix}\in K_2\supset K_1$$

and therefore $f_{(\theta_1,\phi_1^{-1})}(a)=f_{(\Xi_0)}(a)=1$ in this case.

If $a \in \mathcal{O}_F - \mathcal{O}_F^\times$, then we have the following Iwasawa decomposition:

$$\begin{pmatrix} a+1/2 & a-1/2 \\ a-1/2 & a+1/2 \end{pmatrix} = \begin{pmatrix} -a & -1 \\ & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ a-1/2 & a+1/2 \end{pmatrix}.$$

If $a \notin \mathcal{O}_F$, then we have the following Iwasawa decomposition:

$$\begin{pmatrix} a+1/2 & a-1/2 \\ a-1/2 & a+1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & a \\ & a \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1-\frac{1}{2a} & 1+\frac{1}{2a} \end{pmatrix}.$$

Using this, we can realize the integral as a geometric series:

$$\begin{aligned} \int_{G_1} f_{(\theta_1,\phi_1^{-1})}(a) f_{(\Xi_0)}(a)\, da &= 1 + \sum_{n=1}^{\infty} \left(\frac{\theta_1 \Xi_0}{q_F^{1/2}}\right)^n + \left(\frac{\phi_1}{\Xi_0 q_F^{1/2}}\right)^n \\ &= \frac{1 - q_F^{-1}\theta_1\phi_1}{(1-q_F^{-1/2}\phi_1\Xi_0^{-1})(1-q_F^{-1/2}\theta_1\Xi_0)} \\ &= L_F(1/2, \phi_1\Xi_0^{-1}) L_F(1/2, \theta_1\Xi_0) L_F(1, \theta_1\phi_1)^{-1}. \end{aligned}$$

□

2.2.2 Calculation of $S_{\Xi^{-1},\xi^{-1}}(1)$

The calculation of $S_{\Xi^{-1},\xi^{-1}}(1)$ follows from Michael Khoury's Ph.D. dissertation if the unitary groups are non-split (but still quasi-split). If the groups are split, then the calculation follows from unpublished work of Kato, Murase, and Sugano.

Quasi-Split Case

Recall that in this case, E is the quadratic unramified extension of F. Let ϖ be a uniformizer for F, which – since E is unramified over F – is also a uniformizer for E. Also, we remind the reader that $l_n := \lfloor n/2 \rfloor$. Let ξ and Ξ be as before. Denote by $A_i \subset T_i$ the maximal split tori.

We begin the calculation by defining some convenient members of $\mathbb{Z}[q_E^{\pm 1/2}, \Xi_1(\varpi)^{\pm 1}, \Xi_2(\varpi)^{\pm 1}, \ldots, \xi_1(\varpi)^{\pm 1}, \xi_2(\varpi)^{\pm 1}, \ldots]$. (Recall that q_E is the cardinality of the residue field of E.) If $n+1$ is even (hereafter known as Case A), then

$A_{n+1} \cong A_{n+2} \cong (F^\times)^{l_{n+1}}$, and we set:

$$\begin{aligned}
b(\Xi,\xi)^{-1} &:= \prod_{i=1}^{l_{n+1}} L_E(1/2,\xi_i) \prod_{1\le i\le j\le l_{n+1}} L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i\xi_j^{-1}) \\
&\quad \times \prod_{1\le j<i\le l_{n+1}} L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i^{-1}\xi_j) \\
d_1(\Xi)^{-1} &:= \prod_{i=1}^{l_{n+1}} L_E(0,\Xi_i^2) \prod_{1\le i<j\le l_{n+1}} L_E(0,\Xi_i\Xi_j)L_E(0,\Xi_i\Xi_j^{-1}) \\
d_0(\xi)^{-1} &:= \prod_{i=1}^{l_{n+1}} L_E(0,\xi_i) \prod_{1\le i<j\le l_{n+1}} L_E(0,\xi_i\xi_j)L_E(0,\xi_i\xi_j^{-1}).
\end{aligned}$$

We remark that while we've given formulae for $b^{-1}, d_1^{-1}, d_0^{-1}$ above, it's actually b, d_1, d_0 that are in the ring $\mathbb{Z}[q_E^{\pm 1/2}, \Xi_1(\varpi)^{\pm1}, \Xi_2(\varpi)^{\pm1}, \ldots, \xi_1(\varpi)^{\pm1}, \xi_2(\varpi)^{\pm1}, \ldots]$.

If $n+1$ is odd (hereafter known as Case B), then we have $A_{n+1} \cong (F^\times)^{l_{n+1}}$ and $A_{n+2} \cong (F^\times)^{l_{n+2}}$. We set:

$$\begin{aligned}
b(\Xi,\xi)^{-1} &:= \prod_{i=1}^{l_{n+2}} L_E(1/2,\Xi_i) \prod_{1\le i\le j\le l_{n+1}} L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i\xi_j^{-1}) \\
&\quad \times \prod_{1\le j<i\le l_{n+2}} L_E(1/2,\Xi_i\xi_j)L_E(1/2,\Xi_i^{-1}\xi_j) \\
d_1(\Xi)^{-1} &:= \prod_{i=1}^{l_{n+2}} L_E(0,\Xi_i) \prod_{1\le i<j\le l_{n+2}} L_E(0,\Xi_i\Xi_j)L_E(0,\Xi_i\Xi_j^{-1}) \\
d_0(\xi)^{-1} &:= \prod_{i=1}^{l_{n+1}} L_E(0,\xi_i^2) \prod_{1\le i<j\le l_{n+1}} L_E(0,\xi_i\xi_j)L_E(0,\xi_i\xi_j^{-1}).
\end{aligned}$$

Recall that the local L_E factors are defined as $L_E(s,\chi) := \left(1 - q_E^{-s}\chi(\varpi)\right)^{-1}$.

We also define

$$c(\Xi,\xi) := \frac{b(\Xi,\xi)}{d_1(\Xi)d_0(\xi)}$$

as an element of $\mathbb{Q}(q_E^{1/2}, \Xi_1(\varpi), \ldots, \xi_1(\varpi), \ldots)$. Let W_{n+2} and W_{n+1} be the Weyl groups $W(G_{n+2}, A_{n+2})$ and $W(G_{n+1}, A_{n+1})$. If $n+1$ is even, both of these groups are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{l_{n+1}} \rtimes S_{l_{n+1}}$. If $n+1$ is odd, then $W_{n+2} \cong (\mathbb{Z}/2\mathbb{Z})^{l_{n+2}} \rtimes$

$S_{l_{n+2}} \not\cong W_{n+1} \cong (\mathbb{Z}/2\mathbb{Z})^{l_{n+1}} \rtimes S_{l_{n+1}}$. These groups act on elements of A_i (and therefore unramified characters of A_i) by permutation (the S_l factor) and inversion (the $(\mathbb{Z}/2\mathbb{Z})^l$) factor.

Finally, define

$$A_{\Xi,\xi} := \sum_{w' \in W_{n+2}, w \in W_{n+1}} c(w'\Xi, w\xi)$$

as an element of $\mathbb{Q}(q_E^{1/2}, \Xi_1(\varpi), \ldots, \xi_1(\varpi), \ldots)$.

Theorem 11.4 in [24] says the following:

Theorem 2.8. *Let $w_\ell \in W_{n+1}$ and $w'_\ell \in W_{n+2}$ be the long elements, and let $\mathcal{B}_i \subset K_i$ be Iwahori subgroups. (Recall that the K_i were fixed in defining $S_{\Xi^{-1},\xi^{-1}}$.) Then*

$$S_{\Xi^{-1},\xi^{-1}}(1) = \zeta(\Xi^{-1}, \xi^{-1}) q_E^{l(w_\ell)+l(w'_\ell)} \operatorname{Vol}(\mathcal{B}_{n+1}) \operatorname{Vol}(\mathcal{B}_{n+2}) A_{\Xi^{-1},\xi^{-1}}.$$

Here, the volumes are computed with respect to the Haar measures which give the K_i volume 1.

Having introduced them already, we remind the reader that $\mathcal{B}_i = N^-_{i,(1)} T_{i,(0)} N_{i,(0)}$, where $N_{i,(0)} = N_i \cap K_i$, $T_{i,(0)} = T_i \cap K_i$, N_i^- is the unipotent radical of the parabolic opposite that of N_i, and $N^-_{i,(1)}$ is the subgroup of $N_i^- \cap K_i$ whose elements' off-diagonal entries lie in the ideal generated by ϖ. The volumes are

$$\operatorname{Vol}(\mathcal{B}_i) = \frac{\prod_{j=1}^{i}(q_F - (-1)^j)}{\prod_{j=1}^{i}(q_F^j - (-1)^j)}.$$

Lemma 2.9. *$A_{\Xi,\xi}$ is independent of Ξ and ξ.*

Proof. Adapting the proof from [21], we first consider Case A. We define the following Weyl vectors:

$$\begin{aligned} \rho_{n+2} &:= (l_{n+1}, l_{n+1} - 1, \ldots, 1) \\ \rho_{n+1} &:= (l_{n+1} - 1/2, l_{n+1} - 3/2, \ldots, 1/2). \end{aligned}$$

Note that ρ_{n+1} is half the sum of the positive roots of type $B_{l_{n+1}}$, while ρ_{n+2} is half the sum of the positive roots of type $C_{l_{n+1}}$. In what follows, we use the notation $\Xi^\rho = \prod \Xi_i^{\rho_i}$.

We define more members of $\mathbb{Z}[q_E^{\pm 1/2}, \Xi_1(\varpi), \ldots, \xi_1(\varpi), \ldots]$. We set:

$$\begin{aligned} \mathcal{D}_\Xi &:= \Xi^{-\rho_{n+2}} d_1(\Xi) = \sum_{w' \in W_{n+2}} \operatorname{sgn}(w') \cdot (w'\Xi)^{-\rho_{n+2}} \\ \mathcal{D}_\xi &:= \xi^{-\rho_{n+1}} d_0(\xi) = \sum_{w \in W_{n+1}} \operatorname{sgn}(w) \cdot (w\xi)^{-\rho_{n+1}}. \end{aligned}$$

Then we see that $\mathcal{D}_{w'\Xi} = \operatorname{sgn}(w')\mathcal{D}_\Xi$ and $\mathcal{D}_{w\xi} = \operatorname{sgn}(w)\mathcal{D}_\xi$. We introduce one more member of $\mathbb{Z}[q_E^{\pm 1/2}, \Xi_1(\varpi), \ldots, \xi_1(\varpi), \ldots]$. Setting:

$$B_{\Xi,\xi} := \Xi^{-\rho_{n+2}} \xi^{-\rho_{n+1}} b(\Xi, \xi)$$

we see that

$$A_{\Xi,\xi} = (\mathcal{D}_\Xi \mathcal{D}_\xi)^{-1} \sum_{w' \in W_{n+2}, w \in W_{n+1}} \operatorname{sgn}(w) \operatorname{sgn}(w') B(w'\Xi, w\xi). \tag{2.5}$$

Now write

$$B_{\Xi,\xi} = \sum_{\lambda \in \mathbb{Z}^{l_{n+1}}, \mu \in (\frac{1}{2}\mathbb{Z})^{l_{n+1}}} c_{\lambda,\mu} \Xi^\lambda \xi^\mu$$

for some coefficients $c_{\lambda,\mu} \in \mathbb{Z}[q_E^{\pm 1/2}]$ (almost all of which are 0, of course). We say a monomial is *regular* if its stabilizer under the action of the Weyl group is trivial; otherwise we call a monomial *singular*. We show that all regular monomials in $B_{\Xi,\xi}$ are in the orbit of $\Xi^{\rho_{n+2}}\xi^{\rho_{n+1}}$.

Note that it is sufficient to show that that we have $|\lambda_i| \leq l_{n+1}$ and $|\mu_i| \leq l_{n+1} - 1/2$ and that none of the μ_i are integral. (All such monomials are either singular or in the orbit of $\Xi^{\rho_{n+2}}\xi^{\rho_{n+1}}$.)

Note that

$$\begin{aligned} B_{\Xi,\xi} &= \prod_{1 \leq j \leq l_{n+1}} (\xi_j^{-1/2} - q_E^{-1/2}\xi_j^{1/2}) \prod_{1 \leq i,j \leq l_{n+1}} (1 - q_E^{-1/2}\Xi_i\xi_j) \\ &\quad \times \prod_{1 \leq i \leq j \leq l_{n+1}} (\Xi_i^{-1} - q_E^{-1/2}\xi_j^{-1}) \prod_{1 \leq j < i \leq l_{n+1}} (\xi_j^{-1} - q_E^{-1/2}\Xi_i^{-1}). \end{aligned}$$

It is clear from this that all μ_i are half-integral but not integral.

We check that $|\lambda_i| \leq l_{n+1}$. Choose $i_0 \in \{1, 2, \ldots, l_{n+1}\}$. The positive contribution of Ξ_{i_0} comes from

$$\prod_{1 \leq j \leq l_{n+1}} (1 - q_E^{-1/2} \Xi_{i_0} \xi_j)$$

and the negative contribution comes from

$$\prod_{i_0 \leq j \leq l_{n+1}} (\Xi_{i_0}^{-1} - q_E^{-1/2} \xi_j^{-1}) \prod_{1 \leq j < i_0} (\xi_j^{-1} - q_E^{-1/2} \Xi_{i_0}^{-1}).$$

From this we see that $|\lambda_{i_0}| \leq l_{n+1}$.

Now we check that $|\mu_j| \leq l_{n+1} - 1/2$. Pick some $j_0 \in \{1, 2, \ldots, l_{n+1}\}$. The positive contribution of μ_{j_0} comes from

$$(\xi_{j_0}^{-1/2} - q_E^{-1/2} \xi_{j_0}^{1/2}) \prod_{1 \leq i \leq l_{n+1}} (1 - q_E^{-1/2} \Xi_i \xi_{j_0}).$$

From this we can see that $|\mu_{j_0}| \leq l_{n+1} + 1/2$. Since we know that μ_{j_0} is not integral, we now must show that $|\mu_{j_0}| \neq l_{n+1} + 1/2$.

Suppose there is some regular monomial $c_{\lambda,\mu} \Xi^\lambda \xi^\mu$ such that $|\mu_{j_0}| = l_{n+1} + 1/2$. We will show that $|\lambda_i| < l_{n+1}$ for all i. This will contradict the fact that the monomial is regular.

A monomial with $\mu_{j_0} = l_{n+1} + 1/2$ appears in the the following product

$$\begin{aligned} &c_{j_0} \xi_{j_0}^{l_{n+1}+1/2} \prod_{j \neq j_0} (\xi_i^{-1/2} - q_E^{-1/2} \xi_i^{1/2}) \prod_{\substack{j \neq j_0 \\ 1 \leq i \leq l_{n+1}}} (1 - q_E^{-1/2} \Xi_i \xi_j) \\ &\times \prod_{\substack{j \neq j_0 \\ 1 \leq i \leq j \leq l_{n+1}}} (\Xi_i^{-1} - q_E^{-1/2} \xi_j^{-1}) \prod_{\substack{j \neq j_0 \\ 1 \leq j < i \leq l_{n+1}}} (\xi_j^{-1} - q_E^{-1/2} \Xi_i^{-1}) \end{aligned}$$

where $c_{j_0} \in \mathbb{Z}[q_E^{\pm 1/2}]$.

We claim that for any i_0, we cannot have $|\lambda_{i_0}| = l_{n+1}$. If $i_0 \leq j_0$, we get $l_{n+1} - i_0$ copies of $\Xi_{i_0}^{-1}$ from the third product, and $i_0 - 1$ copies of $\Xi_{i_0}^{-1}$ from the fourth product, and no more such factors anywhere else. Furthermore, we only get at most $l_{n+1}-1$ copies of Ξ_{i_0} from the second product, and no more such factors anywhere else.

A similar argument works to show that we cannot have $\mu_{j_0} = -l_{n+1} - 1/2$. So, all regular monomials in $B_{\Xi,\xi}$ are in the orbit of $\Xi^{\rho_{n+2}}\xi^{\rho_{n+1}}$.

Now, all singular monomials are stabilized by a collection of pairs of Weyl-group elements of opposite sign, and will therefore vanish from (2.5). Furthermore, since $A_{\Xi,\xi}$ is clearly Weyl-invariant, all regular monomials will appear with the same constant coefficient c, which is independent of both Ξ and ξ. So we have

$$A_{\Xi,\xi} = c \cdot (\mathcal{D}_\Xi \mathcal{D}_\xi)^{-1} \sum_{w' \in W_{n+2}, w \in W_{n+1}} \operatorname{sgn}(w) \operatorname{sgn}(w')(w'\Xi)^{-\rho_{n+1}}(w\xi)^{-\rho_n} = c.$$

The proof in Case B proceeds the same as in Case A. One needs to take

$$\rho_{n+2} := (l_{n+2} - 1/2, l_{n+2} - 3/2, \ldots, 1/2)$$

and

$$\rho_{n+1} := (l_{n+2} - 1, l_{l+2} - 2, \ldots, 1)$$

in this case. □

Now we compute $A_{\Xi,\xi}$.

Proposition 2.10. *We have*

$$A_{\Xi,\xi} = (L(1,\chi)\zeta(2)L(3,\chi)\ldots L(n,\chi)\zeta(n+1))^{-1}$$

if $n+1$ is even and

$$A_{\Xi,\xi} = (L(1,\chi)\zeta(2)L(3,\chi)\ldots L(n+1,\chi))^{-1}$$

if $n+1$ is odd. Here, χ is the quadratic character associated to the extension E/F and all L and ζ factors are with respect to q_F.

Note that this is simply saying that $A_{\Xi,\xi}^{-1} = L\big(0, M_{n+1}^\vee(1)\big)$, where $M_{n+1}^\vee(1)$ is the twisted dual of the motive M_{n+1} associated to G_{n+1} by Gross [9].

Proof. We prove this only in Case A. The proof proceeds similarly in Case B.

We set

$$\begin{aligned}\widehat{\Xi} &= (q_E^{-l_{n+2}}, q_E^{-(l_{n+2}-1)}, \ldots, q_E^{-1}) \\ \widehat{\xi} &= (q_E^{-l_{n+2}+1/2}, q_E^{-l_{n+2}+3/2}, \ldots, q_E^{-1/2})\end{aligned}$$

and compute $A_{\widehat{\Xi},\widehat{\xi}}$.

Note that $b(w'\widehat{\Xi}, w\widehat{\xi}) = 0$ if and only if at least one of the following is true:

$$\begin{aligned}(w'\widehat{\Xi})_i(w\widehat{\xi})_j &= q_E^{1/2} \text{ for some } i, j \\ (w'\widehat{\Xi})_i(w\widehat{\xi})_j^{-1} &= q_E^{1/2} \text{ for some } i \leq j \\ (w'\widehat{\Xi})_i^{-1}(w\widehat{\xi})_j &= q_E^{1/2} \text{ for some } i > j \\ (w\widehat{\xi})_j &= q_E^{1/2} \text{ for some } j.\end{aligned}$$

We show that $b(w'\widehat{\Xi}, w\widehat{\xi}) \neq 0 \implies w', w = 1$. To see this, note that there are elements σ, τ of the symmetric group $S_{l_{n+2}}$ and $\varepsilon_i, \varepsilon'_i \in \{\pm 1\}$ such that

$$\begin{aligned}w'\widehat{\Xi} &= \left(\widehat{\Xi}_{\sigma(1)}^{\varepsilon'_1}, \widehat{\Xi}_{\sigma(2)}^{\varepsilon'_2}, \ldots, \widehat{\Xi}_{\sigma(l_{n+2})}^{\varepsilon'_{l_{n+2}}}\right) \\ w\widehat{\xi} &= \left(\widehat{\xi}_{\tau(1)}^{\varepsilon_1}, \widehat{\xi}_{\tau(2)}^{\varepsilon_2}, \ldots, \widehat{\xi}_{\tau(l_{n+2})}^{\varepsilon_{l_{n+2}}}\right).\end{aligned}$$

(Note that $l_{n+2} = l_{n+1} = \frac{n+1}{2}$ in this case.) Set $r_a := \sigma^{-1}(l_{n+2} + 1 - a)$ and $s_b := \tau^{-1}(l_{n+2} + 1 - b)$. To avoid the fourth condition above, we see that we must have $\varepsilon_{s_1} = 1$. But then, to avoid the first condition, we must have $\varepsilon'_{r_1} = 1$ as well. Continuing to avoid the first condition, we see that $\varepsilon_i = \varepsilon'_j = 1$ for all $1 \leq i, j, \leq l_{n+2}$. We are left to show that both σ and τ are trivial. This is equivalent to showing that $r_i \leq s_i$ for all i, and that $s_{i+1} < r_i$. Suppose that $s_i < r_i$ for some i. Then we'd have $(w'\widehat{\Xi})_{r_i}^{-1}(w\widehat{\xi})_{s_i} = q_E^{1/2}$, which is the third condition above. Similarly, if $s_{i+1} \geq r_i$, then we'd have $(w'\widehat{\Xi})_{r_i}(w\widehat{\xi})_{s_{i+1}}^{-1} = q_E^{1/2}$, which is the second condition above. So we have

$$s_1 \geq r_1 > s_2 \geq r_2 > \cdots > s_{l_{n+2}} \geq r_{l_{n+2}}$$

which means that both w and w' are trivial. This means that

$$A_{\widehat{\Xi},\widehat{\xi}} = \frac{b(\widehat{\Xi},\widehat{\xi})}{d_1(\widehat{\Xi})d_0(\widehat{\xi})}. \tag{2.6}$$

We denote $A_l := A_{\widehat{\Xi^{(l)}},\widehat{\xi^{(l)}}}$ where $\widehat{\Xi}^{(l)} := (q_E^{-l}, q_E^{-(l-1)}, \ldots, q_E^{-1})$ and $\widehat{\xi}^{(l)} := (q_E^{-l+1/2}, \ldots, q_E^{-1/2})$. By (2.6), and a straightforward calculation, we see that

$$A_{l+1} = A_l \left(L(2l+1,\chi)\zeta(2l+2)\right)^{-1}.$$

By induction on l, the proof is complete.

The proof in Case B proceeds in the same way by setting

$$\widehat{\Xi} := (q_E^{-(l_{n+2}-1/2)}, q_E^{-(l_{n+2}-3/2)}, \ldots, q_E^{-1/2})$$

and

$$\widehat{\xi} := (q_E^{-(l_{n+2}-1)}, q_E^{-(l_{n+2}-2)}, \ldots, q_E^{-1}).$$

□

The Split Case

Recall that in the split case, the unitary groups are just general linear groups. So, we consider the groups G_{n+2} and G_{n+1} where $G_i := GL_i(F)$.[4] Let B_i, T_i and N_i denote the standard Borel subgroups of upper triangular matrices, tori of diagonal matrices, and subgroups of upper triangular unipotent matrices (unipotent radicals). Now, let

$$\xi = (\xi_1, \ldots, \xi_{n+1})$$

and

$$\Xi = (\Xi_1, \ldots, \Xi_{n+2})$$

[4]At this point, the reason for sticking with the choice of $n+1$ and $n+2$ is only for consistency with the non-split case.

where each ξ_i, Ξ_i are unramified characters of $F^\times$. We see that ξ and Ξ can be viewed as characters of T_{n-1} and T_n in the obvious manner; we extend them to characters of B_{n+1} and B_{n+2} by triviality on the N_i. Then we define

$$I(\xi) := \mathrm{Ind}_{B_{n+1}}^{G_{n+1}} \xi$$

and

$$I(\Xi) := \mathrm{Ind}_{B_{n+2}}^{G_{n+2}} \Xi$$

where the induction is normalized. Let η be a representative for the unique open dense orbit of $B_{n+2} \times B_{n+1}$ on G_{n+2}. Now, we recall the function $S_{\Xi,\xi}$ on G_{n+2}. We have

$$S_{\Xi,\xi}(g) := \int_{K_{n+1}\times K_{n+2}} Y_{\Xi,\xi}(k_{n+2}g^{-1}k_{n+1})\ dk_{n+1}dk_{n+2}$$

where $Y_{\Xi,\xi}$ is the function defined on G_{n+2} by

1. $Y_{\Xi,\xi}(b_{n+2}gb_{n+1}) = (\Xi^{-1}\delta_{n+2}^{1/2})(b_{n+2})(\xi\delta_{n+1}^{-1/2})(b_{n+1})Y_{\Xi,\xi}(g)$ for all $b_{n+2} \in B_{n+2}$ and $b_{n+1} \in B_{n+1}$.

2. $Y_{\Xi,\xi}(\eta) = 1$

3. $Y_{\Xi,\xi}(g) = 0$ for $g \notin B_{n+2}\eta B_{n+1}$.

As mentioned earlier, we're interested in computing $S_{\Xi,\xi}$ at the identity. We have the following result of [23]:

Theorem 2.11.

$$S_{\Xi,\xi}(1) = q_F^{l(w_\ell)+l(w'_\ell)} \frac{\prod_{1\le i<j\le n+2} L_F(1/2,\xi_i\Xi_{n-j+3}) \prod_{1\le j\le i<n+2} L_F(1/2,\xi_i^{-1}\Xi_{n-j+3}^{-1})}{\prod_{i=1}^{n+1}\zeta_F(i)\prod_{1\le i<j\le n+1} L_F(1,\xi_i\xi_j^{-1})\prod_{1\le i<j\le n+2} L_F(1,\Xi_i\Xi_j^{-1})}.$$

2.2.3 Concluding the unramified calculations

Now that we've computed both $S_{\Xi,\xi}(1)$ and $\zeta(\Xi,\xi)$, we have actually computed the local integrals in the unramified case. In this section, we show that they are essentially a product of local L-factors.

If we let π_n and π_{n+1} be unramified principal series representations of G_n and G_{n+1}, respectively. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_{\lfloor n/2 \rfloor})$ and $\Xi = (\Xi_1, \ldots, \Xi_{\lfloor (n+1)/2 \rfloor})$ be the relevant characters of B_n and B_{n+1}. We first consider the standard local L-factor. In the quasi-split case, for $n = 2l$, we have

$$\begin{aligned} & L_E(s, BC(\pi_n) \otimes BC(\pi_{n+1}), \mathrm{st}) \\ & = \prod_{1 \le i < j \le l} L_E(s, \xi_i \Xi_j) L_E(s, \xi_i^{-1} \Xi_j) L_E(s, \xi_i \Xi_j^{-1}) L_E(s, \xi_i^{-1} \Xi_j^{-1}) \\ & \quad \times \prod_{1 \le j \le i \le l} L_E(s, \xi_i \Xi_j) L_E(s, \xi_i^{-1} \Xi_j) L_E(s, \xi_i \Xi_j^{-1}) L_E(s, \xi_i^{-1} \Xi_j^{-1}) \\ & \quad \times \prod_{i=1}^{l} L_E(s, \xi_i) L_E(s, \xi_i^{-1}). \end{aligned}$$

If $n = 2l - 1$, we have

$$\begin{aligned} & L_E(s, BC(\pi_n) \otimes BC(\pi_{n+1}), \mathrm{st}) \\ & = \prod_{1 \le i < j \le l} L_E(s, \xi_i \Xi_j) L_E(s, \xi_i^{-1} \Xi_j) L_E(s, \xi_i \Xi_j^{-1}) L_E(s, \xi_i^{-1} \Xi_j^{-1}) \\ & \quad \times \prod_{1 \le j \le i \le l-1} L_E(s, \xi_i \Xi_j) L_E(s, \xi_i^{-1} \Xi_j) L_E(s, \xi_i \Xi_j^{-1}) L_E(s, \xi_i^{-1} \Xi_j^{-1}) \\ & \quad \times \prod_{i=1}^{l} L_E(s, \Xi_i) L_E(s, \Xi_i^{-1}). \end{aligned}$$

Now we consider the adjoint local L-factors. In the quasi-split case, for $n = 2l$, we have

$$\begin{aligned} L_F(s, \pi_n, \mathrm{Ad}) &= \zeta_F(s)^l L_F(s, \chi_{E/F})^l \\ & \quad \times \prod_{1 \le i < j \le l} L_F(2s, \xi_i \xi_j) L_F(2s, \xi_i^{-1} \xi_j) L_F(2s, \xi_i^{-1} \xi_j^{-1}) L_F(2s, \xi_i \xi_j^{-1}) \\ & \quad \times \prod_{i=1}^{l} L_F(s, \xi_i) L_F(s, \xi_i^{-1}) \end{aligned}$$

and

$$\begin{aligned}
&L_F(s,\pi_{n+1},\mathrm{Ad})\\
&\quad= \zeta_F(s)^l L_F(s,\chi_{E/F})^{l+1}\\
&\qquad\times\prod_{1\le i<j\le l} L_F(2s,\Xi_i\Xi_j)L_F(2s,\Xi_i^{-1}\Xi_j)L_F(2s,\Xi_i^{-1}\Xi_j^{-1})L_F(2s,\Xi_i\Xi_j^{-1})\\
&\qquad\times\prod_{i=1}^{l} L_F(s,\chi_{E/F}\Xi_i)L_F(s,\chi_{E/F}\Xi_i^{-1})L_F(2s,\Xi_i)L_F(2s,\Xi_i^{-1}).
\end{aligned}$$

In the quasi-split case, for $n=2l-1$, then we have

$$\begin{aligned}
&L_F(s,\pi_n,\mathrm{Ad})\\
&\quad= \zeta_F(s)^{l-1} L_F(s,\chi_{E/F})^{l}\\
&\qquad\times\prod_{1\le i<j\le l-1} L_F(2s,\xi_i\xi_j)L_F(2s,\xi_i^{-1}\xi_j)L_F(2s,\xi_i^{-1}\xi_j^{-1})L_F(2s,\xi_i\xi_j^{-1})\\
&\qquad\times\prod_{i=1}^{l-1} L_F(s,\chi_{E/F}\xi_i)L_F(s,\chi_{E/F}\xi_i^{-1})L_F(2s,\xi_i)L_F(2s,\xi_i^{-1})
\end{aligned}$$

and

$$\begin{aligned}
&L_F(s,\pi_{n+1},\mathrm{Ad})\\
&\quad= \zeta_F(s)^l L_F(s,\chi_{E/F})^{l}\\
&\qquad\times\prod_{1\le i<j\le l} L_F(2s,\Xi_i\Xi_j)L_F(2s,\Xi_i^{-1}\Xi_j)L_F(2s,\Xi_i^{-1}\Xi_j^{-1})L_F(2s,\Xi_i\Xi_j^{-1})\\
&\qquad\times\prod_{i=1}^{l} L_F(s,\Xi_i)L_F(s,\Xi_i^{-1}).
\end{aligned}$$

Now we discuss the split case. Recall that at a split place, we have $E=F\oplus F$, and $V=V_1\oplus V_2$, where each V_i is an F-vector space. Also, we have $U(V)\cong GL(V_1)$ via the map $(g_1,g_2)\mapsto g_1$. So, the representations π_n and π_{n+1} are unramified spherical series representations of $GL_n(F)$ and $GL_{n+1}(F)$, respectively. If $\Xi=(\Xi_1,\ldots,\Xi_{n+1})$ and $\xi=(\xi_1,\ldots,\xi_n)$, where each of the Ξ_i and ξ_i are unramified characters of $F^\times$, then we have

$$\begin{aligned} L_E(s, BC(\pi_n) \otimes BC(\pi_{n+1}), \mathrm{st}) &= L_F(s, \pi_n \otimes \pi_{n+1}, \mathrm{st}) L_F(s, \pi_n^\vee \otimes \pi_{n+1}^\vee, \mathrm{st}) \\ &= \prod_{\substack{1 \le i \le n \\ 1 \le j \le n+1}} L_F(s, \xi_i \Xi_j) L_F(s, \xi_i^{-1} \Xi_j^{-1}). \end{aligned}$$

The adjoint L-factors are as follows:

$$L_F(s, \pi_n, \mathrm{Ad}) = \zeta_F(s)^n \prod_{1 \le i \ne j \le n} L_F(s, \xi_i \xi_j^{-1})$$

and

$$L_F(s, \pi_{n+1}, \mathrm{Ad}) = \zeta_F(s)^{n+1} \prod_{1 \le i \ne j \le n+1} L_F(s, \Xi_i \Xi_j^{-1}).$$

So using the results from the previous sections, we have the following:

Theorem 2.12. *For $v \notin S$*

$$\begin{aligned} \mathcal{P}'(f_{\pi_{n+2}}, f_{\pi_{n+1}}) &= \zeta(\Xi, \xi) S_{\Xi^{-1}, \xi^{-1}}(1) \\ &= L\big(M_{n+2}^\vee(1), 0\big) \frac{L_E(1/2, BC(\pi_{n+2}) \boxtimes BC(\pi_{n+1}))}{L_F(1, \pi_{n+2}, \mathrm{Ad}) L_F(1, \pi_{n+1}, \mathrm{Ad})} \\ &= \Delta_{G_{n+2}} L_{\pi_{n+2}, \pi_{n+1}}(1/2). \end{aligned}$$

3 The Refined Gross-Prasad Conjecture for $U(1) \times U(2)$

In this chapter, we give a proof of Conjecture 1.3 for $U(1) \times U(2)$. As mentioned before, in this case the conjecture follows almost immediately from a theorem of Waldspurger. However, this theorem does not deal directly with representations of unitary groups, but instead of a quaternion algebra.

To bridge this gap, we use a result that says that any irreducible, cuspidal, automorphic representation π of $U(2)$ can be lifted (in the appropriate sense of 'lift') to an irreducible, cuspidal, automorphic representation $\tilde{\pi}$ of $GU(2)$. Then, by using a an isomorphism of algebraic groups that relates $GU(2)$ to a quaternion algebra B, we have that $\tilde{\pi} \cong \Sigma \boxtimes \eta$, where Σ is a representation of $B^\times$, and η is a Hecke character of $\mathbb{A}_E^\times$. After getting our hands on a representation of a quaternion algebra, we let Waldspurger's theorem finish the job.

3.1 The groups $U(1) \subset U(2) \subset GU(2)$

Let F be a number field, and let E be a quadratic extension of F. Let B be a quaternion algebra defined over F, with a fixed embedding $E \hookrightarrow B$ of F-algebras. We view B as a 2-dimensional vector space over E via left multiplication. Let $^- : B \to B$ be the standard involution. Let $\langle \tau \rangle = \mathrm{Gal}(E/F)$. We note that $\overline{e} = \tau(e)$ for all $e \in E$. With this involution, we define trace and norm maps in the usual way:

$$\begin{aligned} N_B(x) &:= x\overline{x} \in F \\ Tr_B(x) &:= x + \overline{x} \in F \end{aligned}$$

for all $x \in B$. We have $b \in B$ of trace 0 which normalizes E, and whose conjugation action on E is τ. Any other member of B with these properties is of the form λb, with $\lambda \in E$. This gives us

$$B = E \oplus E \cdot b.$$

Now, we can define a non-degenerate hermitian form on B as follows:

$$\langle x, y\rangle_B := \text{projection of } x\overline{y} \text{ onto the } E \text{ factor via the decomposition above.}$$

Since we have a hermitian form on B, we can consider the unitary group $U(B)$ and the similitude group $GU(B)$. Furthermore, we have

$$GU(B) \cong (B^\times \times E^\times)/(\Delta F^\times)$$

where $\Delta F^\times$ denotes the diagonally embedded copy of $F^\times$ in $B^\times \times E^\times$. The action of $B^\times \times E^\times$ on B is given by

$$(b, e)(x) := exb^{-1}$$

The similitude character is given by

$$(b, e) \mapsto N_{E/F}(e)N_B(b)^{-1}.$$

So, we see that

$$U(B) = \{\widehat{(b, e)} : N_B(b) = N_{E/F}(e)\}.$$

Here, $\widehat{(b, e)}$ denotes the equivalence class of (b, e) modulo the diagonally embedded $F^\times$. We also see the center is given by

$$Z_{U(B)} = \{\widehat{(f, e)} : N_B(f) = N_{E/F}(e), f \in F^\times\} = \{\widehat{(1, e)} : N_{E/F}(e) = 1\}. \tag{3.1}$$

We also consider the line $L_B := E \cdot b \subset B$, and we view the associated unitary group $U(L_B)$ as E_1 in $GU(B)$.

Now, for any pair of unitary groups $U(1) \subset U(2)$ defined over F, there is a quaternion algebra B over F and embedding $E \hookrightarrow B$ such that $U(1) \cong U(L_B)$ and $U(2) \cong U(B)$. For ease of notation, we will refer to the unitary groups as G_1 and G_2, and the unitary similitude group as $\widetilde{G_2}$.

We view G_1, G_2 and $\widetilde{G_2}$ as algebraic groups over F. Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be irreducible, cuspidal, tempered automorphic representations of $G_1(\mathbb{A}_F)$ and $G_2(\mathbb{A}_F)$, respectively.

3.2 Extending Cusp Forms

By Theorem 4.13 in [18], we have the following result about extending cusp forms from $G_2(\mathbb{A}_F)$ to $\widetilde{G_2}(\mathbb{A}_F)$.

Theorem 3.1. *There is an irreducible, cuspidal, automorphic representation $(\widetilde{\pi_2}, V_{\widetilde{\pi_2}})$ of $\widetilde{G_2}(\mathbb{A}_F)$ such that $V_{\widetilde{\pi_2}}|_{G_2(\mathbb{A}_F)} \supset V_{\pi_2}$.*

Caution 3.2. *Note that the restriction above is that of functions, and not restriction of the representation.*

Let $(\widetilde{\pi_2}, V_{\widetilde{\pi_2}})$ be a representation of $\widetilde{G_2}(\mathbb{A}_F)$ given by the theorem above. Then we have

$$\widetilde{\pi_2} \cong \Sigma \boxtimes \eta$$

where Σ is a cuspidal irreducible automorphic representation of $B^\times(\mathbb{A}_F)$, and η is a Hecke character of $\mathbb{A}_E^\times$, and $\omega_\Sigma \eta|_{\mathbb{A}_F^\times} \equiv 1$, where ω_Σ is the central character of Σ. As usual, fix isomorphisms $\Sigma \cong \otimes_v \Sigma_v$ and $\eta \cong \otimes_v \eta_v$.

We observe that we have

$$\omega_{\pi_2} = \eta|_{\mathbb{A}_{E,1}^\times}.$$

We denote by Σ' the representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to Σ by the Jacquet-Langlands correspondence.

Now, for $f \in V_{\pi_2}$, we denote by $\widetilde{f} \in V_{\widetilde{\pi_2}}$ a cusp form such that $\widetilde{f}|_{G_2(\mathbb{A}_F)} = f$. We write

$$\tilde{f} = \tilde{f}_\Sigma \otimes \eta.$$

As a consequence of this decomposition, we note that for any $f_1, f_2 \in V_{\pi_2}$, the corresponding $\tilde{f}_1, \tilde{f}_2 \in V_{\widetilde{\pi_2}}$ satisfy $\tilde{f}_1(z)\overline{\tilde{f}_2(z)} = 1$ for all $z \in Z_{GU(B)}$, so that $\tilde{f}_1\overline{\tilde{f}_2}$ is a function on $Z_{GU(B)}\backslash GU(B)$. Furthermore, since $\tilde{f}_{1,\Sigma}(z) = \tilde{f}_{2,\Sigma}(z)$ for all z in the center $Z_{B^\times}(\mathbb{A}_F)$, we see that $\tilde{f}_{1,\Sigma}\overline{\tilde{f}_{2,\Sigma}}$ is a function on $\mathbb{P}B^\times(\mathbb{A}_F) := (Z_{B^\times}\backslash B^\times)(\mathbb{A}_F)$.

3.3 Waldspurger's Theorem

We give a brief discussion of Waldspurger's theorem to be used in the proof of Conjecture 1.3 for $n = 1$. We take B, Σ, and Σ' as defined above. Let T be the torus given by the embedding $E \hookrightarrow B$, so that $T(F)$ is identified with $E^\times \subset B^\times(F)$. Let χ be a Hecke character of $T(\mathbb{A}_F)$, and $f \in \Sigma$. Let Z_B denote the center of $B^\times$. The period that Waldspurger considers is

$$\widetilde{\mathcal{P}}(f,\chi) := \left| \int_{Z_B(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} f(t)\chi^{-1}(t)dt \right|^2$$

where dt is the Tamagawa measure, which gives $\mathrm{Vol}(Z_B(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)) = 2$. Let $\mathcal{B}_{\Sigma_v}$ and $\mathcal{B}_{\chi_v}$ be local pairings. We also choose local measures dt_v so that $dt = \prod_v dt_v$, as usual. Then set

$$\alpha_v(f_v, \chi_v) := \left(\frac{\zeta_{F_v}(2) L_{F_v}(1/2, \Sigma'_v \otimes \chi_v^{-1})}{L_{F_v}(1, \Sigma'_v, \mathrm{Ad}) L_{F_v}(1, \chi_{E_v/F_v})} \right)^{-1} \int_{T_v} \mathcal{B}_{\Sigma_v}(\Sigma_v(t_v)f_v, f_v)\mathcal{B}_{\chi_v}(\chi_v^{-1}(t_v)\chi_v, \chi_v)dt_v.$$

We let $\mathcal{B}_\Sigma$ be the Petersson inner product on Σ, where the integral is taken over $[\mathbb{P}B^\times]$. That is

$$\mathcal{B}_\Sigma(f_1, f_2) := \int_{[\mathbb{P}B^\times]} f_1(b)\overline{f_2(b)}db$$

where db is the Tamagawa measure.

Then Waldspurger's theorem (see [35], page 222) is the following:

Theorem 3.3. *Suppose that Σ has trivial central character. Then*

$$\frac{\widetilde{\mathcal{P}}(f,\chi)}{\mathcal{B}_\Sigma(f,f)} = \frac{\zeta_F(2)L_E(1/2, BC(\Sigma')\otimes\chi^{-1})}{2L_F(1,\Sigma',\mathrm{Ad})L_F(1,\chi_{E/F})}\prod_v \frac{\alpha_v(f_v,\chi_v)}{\mathcal{B}_{\Sigma_v}(f_v,f_v)\mathcal{B}_{\chi_v}(\chi_v,\chi_v)}.$$

Remark 3.4. *The reader will notice that the formulation of Waldspurger's theorem we give looks slightly different than that given by Waldspurger himself. In [35], he chooses the global Haar measure*[1] *such that* $\mathrm{Vol}(Z_B(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)) = 2L_F(1,\chi_{E/F})$, *and he chooses local measures compatibly with respect to this. With our choice of measures, the formulation above is equivalent. He also does not include the local pairing $\mathcal{B}_{\chi_v}$ anywhere in the result. Our inclusion of $\mathcal{B}_{\chi_v}$ – both in the definition of the α_v and in the denominator of the product on the RHS of the theorem – does not change anything.*

3.4 Proof of Conjecture 1.3 for $U(1)\times U(2)$

Waldspurger's theorem does the bulk of the work in proving Conjecture 1.3 for $n = 1$. As usual, we let π_i denote an irreducible, tempered, cuspidal, automorphic representation of $G_i(\mathbb{A}_F)$.

Let $\mathcal{B}_\Sigma$ be as above, and $\mathcal{B}_{\pi_i}, \mathcal{B}_{\widetilde{\pi_2}}$ are defined as follows, where all global measures are the appropriate Tamagawa measure:

$$\begin{aligned}
\mathcal{B}_{\pi_i}(f_1,f_2) &:= \int_{[G_i]} f_1(g)\overline{f_2(g)}dg \\
\mathcal{B}_{\widetilde{\pi_2}}(\tilde{f}_1,\tilde{f}_2) &:= \int_{Z_{\widetilde{G_2}}(\mathbb{A}_F)\widetilde{G_2}(F)\backslash\widetilde{G_2}(\mathbb{A}_F)} \tilde{f}_1(g)\overline{\tilde{f}_2(g)}dg.
\end{aligned}$$

Note that for $\tilde{f}_1, \tilde{f}_2 \in \widetilde{\pi_2}$, we have $\mathcal{B}_{\widetilde{\pi_2}}(\tilde{f}_1,\tilde{f}_2) = \mathcal{B}_\Sigma(\tilde{f}_{1,\Sigma},\tilde{f}_{2,\Sigma})$.

We choose local pairings $\mathcal{B}_{\pi_{i,v}}$ compatibly with the associated global pairings, so that $\prod_v \mathcal{B}_{\pi_{i,v}} = \mathcal{B}_{\pi_i}$. However, we choose the local pairings $\mathcal{B}_{\Sigma_v}$ so that for $f_i \in \pi_2$ – which we extend to $\tilde{f}_i \in \widetilde{\pi_2}$ by Theorem 3.1 – we have $\mathcal{B}_{\Sigma_v}(\tilde{f}_{1,\Sigma_v},\tilde{f}_{2,\Sigma_v}) = \mathcal{B}_{\pi_{2,v}}(f_{1,v},f_{2,v})$, where $\tilde{f}_{i,v} = \tilde{f}_{i,\Sigma_v}\otimes\eta$.

[1] Waldspurger also refers to this as the Tamagawa measure. This collision of terminology is unfortunate, and we hope the reader suffers minimal confusion as a result.

We now have enough in place to prove Conjecture 1.3 for $n = 1$. Recall that Waldspurger's theorem assumes that the central character of Σ is trivial. However, in [37], the authors remove this assumption.

Theorem 3.5. *Let $f \in \pi_2$ and $\tilde{f} = \tilde{f}_\Sigma \otimes \eta \in \widetilde{\pi}_2$ such that $\tilde{f}|_{G_2} = f$. Let $\theta \in \pi_1$ be a unitary character of $G_1(\mathbb{A}_F)$, which we are viewing as the norm-one elements of $\mathbb{A}_E^\times$. Then*

$$\mathcal{P}(f|_{G_1}, \theta) = \frac{\Delta_{G_2} L_E(1/2, BC(\pi_2) \boxtimes BC(\pi_1))}{4|X(\pi_2)| L_F(1, \pi_2, \mathrm{Ad}) L_F(1, \theta, \mathrm{Ad})} \prod_v \mathcal{P}_v(f_v, \theta_v)$$

where $X(\pi_2)$ is the set of automorphic characters ω of $GU(2)(\mathbb{A}_F)/U(2)(\mathbb{A}_F)$ such that $\widetilde{\pi}_2 \otimes \omega \cong \widetilde{\pi}_2$. We remind the reader that $\Delta_{G_2} := L_F(1, \chi_{E/F})\zeta_F(2)$.

Proof. We have the following:

$$\begin{aligned}
\int_{[G_1]} f(g)\overline{\theta(g)}\, dg &= \int_{[G_1]} \tilde{f}(g)\overline{\theta(g)}\, dg \\
&= \int_{Z_B(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \tilde{f}_\Sigma(g)\eta(g)\overline{\theta(g)}\, dg \\
&= \int_{Z_B(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \tilde{f}_\Sigma(g)\overline{\eta^{-1}(g)\theta(g)}\, dg.
\end{aligned}$$

So, Waldspurger's theorem gives

$$\frac{\mathcal{P}(f|_{G_1}, \overline{\theta})}{\mathcal{B}_\Sigma(\tilde{f}_\Sigma, \tilde{f}_\Sigma)} = \frac{\zeta_F(2) L_E(1/2, BC(\Sigma') \otimes \eta BC(\theta^{-1}))}{2L_F(1, \Sigma', \mathrm{Ad}) L_F(1, \chi_{E/F})} \prod_v \frac{\alpha_v(f_{\Sigma,v}, \overline{\eta_v^{-1}\theta_v})}{\mathcal{B}_{\Sigma_v}(\tilde{f}_{\Sigma_v}, \tilde{f}_{\Sigma_v}) \mathcal{B}_{\eta_v}(\eta_v^{-1}\theta_v, \eta_v^{-1}\theta_v)}.$$

Noting that

$$\alpha_v(\tilde{f}_{\Sigma_v}, \overline{\eta_v^{-1}\theta_v}) = \mathcal{P}_v(f_v, \overline{\theta_v})$$

and

$$\prod_v \mathcal{B}_{\eta_v}(\eta_v^{-1}\theta_v, \eta_v^{-1}\theta_v) = 2$$

we have

$$\frac{\mathcal{P}(f|_{G_1}, \overline{\theta})}{\mathcal{B}_\Sigma(\tilde{f}_\Sigma, \tilde{f}_\Sigma)} = \frac{\zeta_F(2) L_E(1/2, BC(\Sigma') \boxtimes \eta BC(\theta^{-1}))}{4L_F(1, \Sigma', \mathrm{Ad}) L_F(1, \chi_{E/F})} \prod_v \frac{\mathcal{P}_v(f_v, \overline{\theta_v})}{\mathcal{B}_{\Sigma_v}(\tilde{f}_{\Sigma_v}, \tilde{f}_{\Sigma_v})}.$$

We remark that $BC(\theta^{-1})$ is the character of $\mathbb{A}_E^\times/\mathbb{A}_F^\times$ given by

$$BC(\theta^{-1})(x) = \frac{\theta^{-1}(x)}{\theta^{-1}(\tau(x))}$$

where $\mathrm{Gal}(E/F)$ is generated by τ.

Recall that $\mathcal{B}_\Sigma(\tilde{f}_\Sigma, \tilde{f}_\Sigma) = \mathcal{B}_{\widetilde{\pi_2}}(\tilde{f}, \tilde{f})$ and $\mathcal{B}_{\Sigma_v}(\tilde{f}_{\Sigma_v}, \tilde{f}_{\Sigma_v}) = \mathcal{B}_{\pi_{2,v}}(f_v, f_v)$. As for the L-values, we have

$$L_F(s, \Sigma', \mathrm{Ad}) = L_F(s, \pi_2, \mathrm{Ad}) L_F(s, \chi_{E/F})^{-1}$$

and

$$L_E(1/2, BC(\Sigma') \otimes \eta BC(\theta^{-1})) = L_E(s, BC(\pi_2) \boxtimes BC(\pi_1^\vee)).$$

This gives:

$$\frac{\mathcal{P}(f|_{G_1}, \overline{\theta})}{\mathcal{B}_{\widetilde{\pi_2}}(\tilde{f}, \tilde{f})} = \frac{\Delta_{G_2} L_E(s, BC(\pi_2) \boxtimes BC(\pi_1^\vee))}{4 L_F(1, \pi_2, \mathrm{Ad}) L_F(1, \chi_{E/F})} \prod_v \frac{\mathcal{P}_v(f_v, \overline{\theta_v})}{\mathcal{B}_{\pi_{2,v}}(f_v, f_v)}.$$

By Remark 4.20 of [18], we have:

$$\frac{\mathcal{B}_{\pi_2}(f, f)}{\mathrm{Vol}(G_2(F)\backslash G_2(\mathbb{A}_F))} = |X(\pi_2)| \cdot \frac{\mathcal{B}_{\widetilde{\pi_2}}(\tilde{f}, \tilde{f})}{\mathrm{Vol}(Z_{\widetilde{G_2}}(\mathbb{A}_F)\widetilde{G_2}(F)\backslash \widetilde{G_2}(\mathbb{A}_F))}.$$

The volumes are:

$$\mathrm{Vol}(G_2(F)\backslash G_2(\mathbb{A}_F)) = \mathrm{Vol}(Z_{\widetilde{G_2}}(\mathbb{A}_F)\widetilde{G_2}(F)\backslash \widetilde{G_2}(\mathbb{A}_F)) = 2.$$

Finally, we note that $|S_{\psi_1}| = 2$ and $|S_{\psi_2}| = 2 \cdot |X(\pi_2)|$, so that

$$|S_{\psi_1}| \cdot |S_{\psi_2}| = 4 \cdot |X(\pi_2)|$$

as stated in Conjecture 1.3. This completes the proof. □

4 Ichino's Triple Product Formula

We now begin introducing the machinery necessary to prove Conjecture 1.3 for $n = 2$. The first tool that we introduce is a result due to Ichino: the so-called triple product formula. Like Waldspurger's theorem and the Refined Gross-Prasad Conjectures already mentioned, Ichino's formula gives an explicit relationship between a period integral and a particular L-value.

Let τ_1, τ_2 and τ_3 be irreducible, cuspidal representations of G_2. Denote by ω_i the central character of τ_i. We require that $\omega_1\omega_2\omega_3 \equiv 1$. Recall that from Theorem 3.1 we have corresponding representations $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3$ of $\widetilde{G_2}$. Also recall that we have

$$\tilde{\tau}_i \cong \Sigma_i \boxtimes \eta_i.$$

In order to make use of Ichino's formula, we must ensure that the central characters of the Σ_i multiply to give the trivial character. The following lemma ensures that we can choose the $\tilde{\tau}_i$ extending the τ_i such that this holds.

Lemma 4.1. *There exist $\tilde{\tau}_i$ extending the τ_i such that the corresponding η_i satisfy $\eta_1\eta_2\eta_3 \equiv 1$.*

Proof. We note that since $\omega_1\omega_2\omega_3 \equiv 1$, we already have that $\eta_1\eta_2\eta_3|_{\mathbb{A}_{E,1}^\times} = 1$. But this means that $\eta_1\eta_2\eta_3 = \chi \circ N_{E/F}$ for some Hecke character χ of $\mathbb{A}_F^\times$. So, by twisting one of the η_i – say η_1 – by $\chi^{-1} \circ N_{E/F}$, we can achieve the desired result. Note that in order to ensure that $\tilde{\tau}_1$ still extends τ_1, we must also twist Σ_1 by

$\chi \circ N_B$. □

An immediate consequence of the previous result is that we can choose the $\tilde{\tau}_i$ such that the central characters ω_{Σ_i} of the Σ_i satisfy $\omega_{\Sigma_1}\omega_{\Sigma_2}\omega_{\Sigma_3} \equiv 1$. To see this, we simply note that

$$\omega_{\Sigma_1}\omega_{\Sigma_2}\omega_{\Sigma_3} = (\eta_1\eta_2\eta_3)^{-1}|_{\mathbb{A}_F^\times} \equiv 1.$$

So, having chosen the $\tilde{\tau}_i$ in this way, we are entitled to make use of Ichino's formula.

Let $\tilde{f}_i \in \tilde{\tau}_i$. In [20], Ichino proves:

$$\left|\int_{\mathbb{P}B^\times(F)\backslash\mathbb{P}B^\times(\mathbb{A}_F)} (\tilde{f}_{1,\Sigma_1}\tilde{f}_{2,\Sigma_2}\tilde{f}_{3,\Sigma_3})(b)\ db\right|^2 = \frac{\zeta_F(2)^2 L_F(1/2,\Sigma')}{8\left(\prod_{i=1}^3 |X(\tau_i)|\right) L_F(1,\Sigma',\mathrm{Ad})} \prod_v \mathcal{J}_v \tag{4.1}$$

where

$$L_F(s,\Sigma') := L_F(s,\Sigma'_1 \boxtimes \Sigma'_2 \boxtimes \Sigma'_3)$$

is the triple-product L-function, and

$$L_F(s,\Sigma',\mathrm{Ad}) := L_F(s,\Sigma'_1,\mathrm{Ad})L_F(s,\Sigma'_2,\mathrm{Ad})L_F(s,\Sigma'_3,\mathrm{Ad}).$$

and the $\mathcal{J}_v$ are defined as follows:

$$\begin{aligned}\mathcal{J}_v \quad := \quad & \frac{L_{F_v}(1,\Sigma'_v,\mathrm{Ad})}{\zeta_{F_v}(2)^2 L_{F_v}(1/2,\Sigma'_v)} \times \\ & \int_{\mathbb{P}B_v^\times} \mathcal{B}_{\Sigma_{1,v}}(\Sigma_{1,v}(b_v)\tilde{f}_{\Sigma_1,v},\tilde{f}_{\Sigma_1,v})\mathcal{B}_{\Sigma_{2,v}}(\Sigma_{2,v}(b_v)\tilde{f}_{\Sigma_2,v},\tilde{f}_{\Sigma_2,v}) \\ & \mathcal{B}_{\Sigma_{3,v}}(\Sigma_{3,v}(b_v)\tilde{f}_{\Sigma_3,v},\tilde{f}_{\Sigma_3,v})\ db_v\end{aligned}$$

where we make what will seem – for the moment – to be a strange normalization; we require that $|X(\tau_i)| \cdot \prod_v \mathcal{B}_{\Sigma_i,v}$ is the Petersson inner product on $\mathbb{P}B^\times$.

We will spend the remainder of this chapter using this formula to derive one more suited for our purposes. For the remainder of the chapter, we assume that at least one of the $\tilde{\tau}_i$ – say $\tilde{\tau}_3$ – is dihedral with respect to E/F; in other words, we assume that $\Sigma_3 \cong \Sigma_3 \otimes \chi_{E/F}$.

At every place v of F, we consider the following subgroup of $B_v^\times$:

$$(B_v^\times)^+ := \{b_v \in B_v^\times : N_{B_v}(b_v) \in N_{E_v/F_v}(E_v^\times)\}.$$

Note that $(B_v^\times)^+$ is not the F_v-points of an algebraic group over F. We also write

$$B^\times(\mathbb{A}_F)^+ := \{(b_v)_v \in B^\times(\mathbb{A}_F) : b_v \in (B_v^\times)^+ \text{ for every } v\}.$$

We will denote

$$(\mathbb{P}B_v^\times)^+ := \{\hat{b}_v \in \mathbb{P}B_v^\times : b_v \in (B_v^\times)^+\}.$$

It is easy to see that this is well-defined. Note that $(B_v^\times)^+$ contains the center Z_{B_v} of $B_v^\times$, so that

$$(\mathbb{P}B_v^\times)^+ \cong (B_v^\times)^+/Z_{B_v}.$$

We also set

$$\mathbb{P}B^\times(\mathbb{A}_F)^+ := \{\widehat{(b_v)}_v \in \mathbb{P}B^\times(\mathbb{A}_F) : b_v \in (B_v^\times)^+ \text{ for all } v\}.$$

We now give two lemmas toward converting Ichino's triple product formula to a formula involving only data for $U(2)$. The first of these is a local result which will allow us to relate integrals of matrix coefficients of the $\tilde{\tau}_i$ over $GU(2)_v$ to integrals of matrix coefficients of the τ_i over $U(2)_v$.

Lemma 4.2. *Let $\tilde{\tau}_v = \Sigma_v \boxtimes \eta_v$ be an irreducible representation of $GU(2)_v$. Suppose that Σ_v is dihedral with respect to E_v/F_v. Viewing $\tilde{\tau}_v$ as a representation of $U(2)_v \subset GU(2)_v$ by restriction, we have*

$$\tilde{\tau}_v = \tau_v^+ \oplus \tau_v^-$$

where τ_v^+ and τ_v^- are inequivalent and both irreducible. Furthermore, for a vector $x^+ \in \tau^+, \langle\cdot,\cdot\rangle$ any $GU(2)_v$-invariant inner product on $\tilde{\tau}_v$, and f a function on $GU(2)_v$ such that $\langle g\cdot x^+, x^+\rangle \cdot f(g)$ is a function on $Z_{GU(2)_V}\backslash GU(2)_v$, we have

$$\int_{Z_{GU(2)_v}\backslash GU(2)_v} \langle g\cdot x^+, x^+\rangle f(g)dg = \int_{Z_{GU(2)_v}\backslash GU(2)_v^+} \langle g\cdot x^+, x^+\rangle f(g)dg$$

where $GU(2)_v^+ := Z_{GU(2)_v} U(2)_v$.

Proof. The fact that the restriction of $\tilde{\tau}_v$ to $U(2)_v$ decomposes as described follows from the fact that Σ_v is dihedral. Now, setting $GU(2)_v^- := GU(2)_v - GU(2)_v^+$ we have

$$GU(2)_v = GU(2)_v^+ \cup GU(2)_v^+ c$$

for some $c \in GU(2)_v^-$ that interchanges τ_v^+ and τ_v^-. (The fact that c interchanges τ_v^+ and τ_v^- follows from the fact that neither is invariant under $GU(2)_v$.) So, we have the following identity:

$$\int_{Z_{GU(2)_v} \backslash GU(2)_v} \langle g \cdot x^+, x^+ \rangle f(g) dg$$
$$= \int_{Z_{GU(2)_v} \backslash GU(2)_v^+} \langle g \cdot x^+, x^+ \rangle f(g) dg + \int_{Z_{GU(2)_v} \backslash GU(2)_v^+} \langle gc \cdot x^+, x^+ \rangle f(g) dg.$$

We claim the second integral on the RHS vanishes. Indeed, $c \cdot x^+ \in \tau_v^-$, and $g \cdot x^- \in \tau_v^-$ for any $x^- \in \tau_v^-$ and $g \in GU(2)_v^+$. Since τ_v^- and τ_v^+ are orthogonal, this completes the proof. □

The next result is a global analogue of the previous one. While we already know that for a cusp form f on $U(2)(\mathbb{A}_F)$, there is a cusp form $\tilde{f}$ on $GU(2)(\mathbb{A}_F)$ which restricts to f, this is not quite good enough to compare Ichino's triple product integral to one over $[U(2)]$. We need a $\tilde{f}$ which vanishes away from 'the complement of $U(2)$ in $GU(2)$'. The following lemma says that such a $\tilde{f}$ exists.

Lemma 4.3. *Let τ be an irreducible, cuspidal, automorphic representation of $U(2)(\mathbb{A}_F)$, and let $\tilde{\tau} = \Sigma \boxtimes \eta$ be an irreducibile, cuspidal, automorphic representation of $GU(2)(\mathbb{A}_F)$ extending τ. Suppose that Σ is dihedral with respect to E/F. Then for $f \in \tau$, there is a $\tilde{f} = \tilde{f}_\Sigma \otimes \eta \in \tilde{\tau}$ such that $\tilde{f}_\Sigma$ vanishes away from $B^\times(F)B^\times(\mathbb{A}_F)^+$.*

Proof. First, we fix a decomposition $\tilde{\tau} \cong \otimes_v \tilde{\tau}_v$. Now, as in the previous lemma, we have $\tilde{\tau}_v = \tau_v^+ \oplus \tau_v^-$ as representations of $U(2)_v$. We adjust the labelings so that for almost all v, τ_v^+ contains the spherical vector of $\tilde{\tau}_v$. We note that as a $U(2)$-module we have

$$\tilde{\tau} \cong \bigoplus_S \tau_S$$

where S runs over all finite sets of places of F, and

$$\tau_S := (\otimes_{v\in S}\ \tau_v^-) \otimes (\otimes_{v\notin S}\ \tau_v^+).$$

We note that $S \neq S' \implies \tau_S \ncong \tau_{S'}$. So, there is a unique S_0 such that $\tau \cong \tau_{S_0}$.

We denote by $\mathcal{A}_0(GU(2))$ and $\mathcal{A}_0(U(2))$ the spaces of cusp forms on $GU(2)(\mathbb{A}_F)$ and $U(2)(\mathbb{A}_F)$, respectively. Under the natural restriction map $\mathrm{Res} : \mathcal{A}_0(GU(2)) \to \mathcal{A}_0(U(2))$, we see that τ_{S_0} is sent isomorphically to τ. So, there is a unique $\tilde{f} = \tilde{f}_\Sigma \otimes \eta \in \tau_{S_0} \subset \tilde{\tau}$ such that $\mathrm{Res}(\tilde{f}) = f$. We now claim that $\tilde{f}_\Sigma$ vanishes away from $B^\times(F)B^\times(\mathbb{A}_F)^+$.

Toward proving this claim, we consider the map

$$\begin{aligned} \phi : \mathcal{A}_0(GU(2)) &\to \mathcal{A}_0(GU(2)) \\ f_\Sigma \otimes \eta &\mapsto f_\Sigma \chi_{E/F} \otimes \eta. \end{aligned}$$

If we let R be the action given by right translation of $GU(2)(\mathbb{A}_F)$ on $\mathcal{A}_0(GU(2))$, then we see that ϕ intertwines the action of R and $R \otimes \chi_{E/F}$. In particular, since $\chi_{E/F}$ is trivial on $U(2)(\mathbb{A}_F) \subset GU(2)(\mathbb{A}_F)$, we see that ϕ commutes with the action of $U(2)(\mathbb{A}_F)$.

Now, let $\tilde{V} \subset \mathcal{A}_0(GU(2))$ denote the underlying space of functions for $\tilde{\tau}$. We claim that

$$\phi(\tilde{V}) = \tilde{V}.$$

This follows from the fact that $\tilde{\tau}$ is dihedral with respect to E/F, and that $GU(2)$ enjoys the multiplicity-one property.

So, we see that we have a $GU(2)(\mathbb{A}_F)$-equivariant isomorphism

$$\phi : (\tilde{V}, R) \to (\tilde{V}, R \otimes \chi_{E/F}).$$

Noting that $\phi^2 = 1$, and that ϕ is clearly not a scalar, we have a decomposition

$$\tilde{V} = \tilde{V}^+ \oplus \tilde{V}^-$$

where $\tilde{V}^+$ is the $+1$ eigenspace for ϕ, and $\tilde{V}^-$ is the -1 eigenspace.

Now, note that if $\tilde{f} \in \tilde{V}^+$, $\tilde{f}_\Sigma$ vanishes away from $B^\times(F)B^\times(\mathbb{A}_F)^+$. Similarly, if $\tilde{f} \in \tilde{V}^-$, $\tilde{f}_\Sigma$ vanishes on $B^\times(F)B^\times(\mathbb{A}_F)^+$. Note that this is simply saying that the kernel of $\mathrm{Res}\,|_{\tilde{V}}$ is precisely $\tilde{V}^-$.

Let $\tilde{V}_{S_0} \subset \tilde{V}$ be the space of functions affording the representation τ_{S_0} described above. To finish the proof, we must show that $\tilde{V}_{S_0} \subset \tilde{V}^+$. Since ϕ commutes with the action of $U(2)(\mathbb{A}_F)$ on $\tilde{V}_{S_0}$, we see that either $\tilde{V}_{S_0} \subset \tilde{V}^+$ or $\tilde{V}_{S_0} \subset \tilde{V}^-$. But the latter is impossible, since $\mathrm{Res}(\tilde{V}^-) = 0$. □

Remark 4.4. *Note that the map ϕ in the preceding proof has a local analogue. Indeed: if τ is dihedral, then for each v, we have $GU(2)_v$-equivariant isomorphisms*

$$\phi_v : \tilde{\tau}_v \longrightarrow \tilde{\tau}_v \otimes \chi_{E_v/F_v}.$$

Note that we may normalize these ϕ_v so that they are well-determined up to a sign. By Schur's Lemma, we see that $\phi = \pm \otimes_v \phi_v$. By adjusting the sign of one of the ϕ_v, we may assume that $\phi = \otimes_v \phi_v$. Let τ_v^+ and τ_v^- denote the $+1$ and -1 eigenspaces for ϕ_v, respectively. Recall that there is a unique set S_0 of places of F such that $(V_{S_0}, \tau_{S_0}) \subset \tilde{V}^+$ affords the representation τ. Note that by the above, $|S_0|$ is even. By adjusting the ϕ_v for $v \in S_0$, we can assume $S_0 = \emptyset$, and we can still have $\phi = \otimes_v \phi_v$. The fact that $S_0 = \emptyset$ means that we have $\tau_v = \tau_v^+$ for all v.

We observe that $\mathbb{P}B^\times(F)\mathbb{P}B^\times(\mathbb{A}_F)^+$ is a subgroup of index 2 in $\mathbb{P}B^\times(\mathbb{A}_F)$. Similarly, the space $\mathbb{P}B^\times(F)\backslash\mathbb{P}B^\times(F)\mathbb{P}B^\times(\mathbb{A}_F)^+$ has 'index 2' in $\mathbb{P}B^\times(F)\backslash\mathbb{P}B^\times(\mathbb{A}_F)$. Denoting $\mathbb{P}B^\times(F)^+ := \mathbb{P}B^\times(F) \cap \mathbb{P}B^\times(\mathbb{A}_F)^+$, we see that we may identify the spaces $\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)^+$ and $\mathbb{P}B^\times(F)\backslash\mathbb{P}B^\times(F)\mathbb{P}B^\times(\mathbb{A}_F)^+$. With this in mind, we may view $\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)^+$ as a space of 'index 2' in $\mathbb{P}B^\times(F)\backslash\mathbb{P}B^\times(\mathbb{A}_F)$.

With this is mind, we have the following consequence of Ichino's formula and the preceding lemma:

Proposition 4.5. *If $\tilde{f}_3$ is chosen such that $\tilde{f}_{3,\Sigma_3}$ vanishes off $B^\times(F)B^\times(\mathbb{A}_F)^+$, then*

$$\left|\int_{\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)^+} (\tilde{f}_{1,\Sigma_1}\tilde{f}_{2,\Sigma_2}\tilde{f}_{3,\Sigma_3})(b)\ db\right|^2 = \frac{\zeta_F(2)^2 L_F(1/2,\Sigma')}{8\left(\prod_{i=1}^3 |X(\tau_i)|\right) L_F(1,\Sigma',\mathrm{Ad})} \prod_v \mathcal{J}_v.$$

Our final task for this chapter is to reinterpret the result above completely in terms of data on unitary groups. Having chosen extensions $\tilde{\tau}_i$ of the τ_i, we have the decompositions

$$\tilde{\tau}_i \cong \otimes_v \tilde{\tau}_{i,v}$$

and

$$\tau_i \cong \otimes_v \tau_{i,v}.$$

We see that as representations of $U(2)$, we have $\tau_{i,v} \subset \tilde{\tau}_{i,v}$. Furthermore, by Remark 4.4 we have that $\tau_{3,v} = \tau_{3,v}^{+}$.

We have already fixed the pairings $\mathcal{B}_{\Sigma_{i,v}}$; now we also fix pairings $\mathcal{B}_{\eta_{i,v}}$ on the $\eta_{i,v}$ such that $\prod_v \mathcal{B}_{\eta_{i,v}}$ gives the Petersson pairing on η_i. We see that the tensor product of $\mathcal{B}_{\Sigma_{i,v}}$ and $\mathcal{B}_{\eta_{i,v}}$ yields a pairing on $\tilde{\tau}_{i,v}$, which we denote by $\mathcal{B}_{\tilde{\tau}_{i,v}}$. We also consider its restriction to the subspace $\tau_{i,v}$, which we denote by $\mathcal{B}_{\tau_{i,v}}$.

Remark 4.6. *By Remark* 4.20 *of [18] and our choice of $\mathcal{B}_{\Sigma_{i,v}}$ earlier, we have that the products $\prod_v \mathcal{B}_{\tau_{i,v}}$ give the respective Petersson inner products on τ_i.*

Let $\tilde{f}_i \in \tilde{\tau}_i$ be cusp forms extending $f_i \in \tau_i$. Suppose that $f_i = \otimes_v f_{i,v}$. Without loss of generality, we may assume that the $\tilde{f}_i$ are also pure tensors (because Res restricts to an isomorphism on the subspaces of $\mathcal{A}_0(GU(2))$ that afford the τ_i). Suppose that $\tilde{f}_3$ is chosen such that $\tilde{f}_{3,\Sigma_3}$ vanishes away from $B^\times(F)B^\times(\mathbb{A}_F)^+$.

We can now give a reinterpretation of the previous proposition, using data for G_2 instead of $\widetilde{G_2}$. We remark that since τ_3 is assumed to be dihedral with respect to E/F, we have that $BC(\tau_3)$ is isomorphic to the principal series representation $\pi(\chi_1, \chi_2)$ of $GL_2(\mathbb{A}_E)$ for some Hecke characters χ_i of $\mathbb{A}_E^\times$.

Note that with the choices we have made, we can rewrite the $\mathcal{J}_v$ as integrals of matrix coefficients of the τ_i. Indeed, we define

$$\begin{aligned}\mathcal{I}_v \;:=\; & \frac{L_{F_v}(1,\tau_{1,v},\mathrm{Ad})L_{F_v}(1,\tau_{2,v},\mathrm{Ad})L_{F_v}(1,\tau_{3,v},\mathrm{Ad})}{\zeta_{F_v}(2)^2 L_{E_v}(1/2, BC(\tau_{1,v})\boxtimes BC(\tau_{2,v})\boxtimes \chi_{1,v})L_{F_v}(1,\chi_{E_v/F_v})^3} \times \\ & \int_{Z_{U(2)_v}\backslash U(2)_v} \mathcal{B}_{\tau_{1,v}}(\tau_{1,v}(g_v)f_{1,v}, f_{1,v})\mathcal{B}_{\tau_{2,v}}(\tau_{2,v}(g_v)f_{2,v}, f_{2,v}) \\ & \mathcal{B}_{\tau_{3,v}}(\tau_{3,v}(g_v)f_{3,v}, f_{3,v})dg_v.\end{aligned}$$

We remark that this is well-defined since $\omega_1\omega_2\omega_3 \equiv 1$. We also note that the constants in front of the respective integrals in the definitions of $\mathcal{J}_v$ and $\mathcal{I}_v$ agree by Propositions A.1, A.2, A.3, and A.4 in the appendix. Recall that we have chosen $\tilde{f}_3$ so that $\tilde{f}_{3,v} \in \tau_v^+$ for all v. Then by Lemma 4.2 we see that

$$\begin{aligned}
&\int_{\mathbb{P}B_v^\times} \mathcal{B}_{\Sigma_{1,v}}(\Sigma_{1,v}(b_v)\tilde{f}_{\Sigma_{1,v}}, \tilde{f}_{\Sigma_{1,v}})\mathcal{B}_{\Sigma_{2,v}}(\Sigma_{2,v}(b_v)\tilde{f}_{\Sigma_{2,v}}, \tilde{f}_{\Sigma_{2,v}}) \\
&\mathcal{B}_{\Sigma_{3,v}}(\Sigma_{3,v}(b_v)\tilde{f}_{\Sigma_{3,v}}, \tilde{f}_{\Sigma_{3,v}})\, db_v \\
= &\int_{(\mathbb{P}B_v^\times)^+} \mathcal{B}_{\Sigma_{1,v}}(\Sigma_{1,v}(b_v)\tilde{f}_{\Sigma_{1,v}}, \tilde{f}_{\Sigma_{1,v}})\mathcal{B}_{\Sigma_{2,v}}(\Sigma_{2,v}(b_v)\tilde{f}_{\Sigma_{2,v}}, \tilde{f}_{\Sigma_{2,v}}) \\
&\mathcal{B}_{\Sigma_{3,v}}(\Sigma_{3,v}(b_v)\tilde{f}_{\Sigma_{3,v}}, \tilde{f}_{\Sigma_{3,v}})\, db_v \\
= &\int_{Z_{U(2)_v}\backslash U(2)_v} \mathcal{B}_{\tau_{1,v}}(\tau_{1,v}(g_v)f_{1,v}, f_{1,v})\mathcal{B}_{\tau_{2,v}}(\tau_{2,v}(g_v)f_{2,v}, f_{2,v}) \\
&\mathcal{B}_{\tau_{3,v}}(\tau_{3,v}(g_v)f_{3,v}, f_{3,v})dg_v.
\end{aligned}$$

Here, dg_v is the Haar measure derived from db_v under the isomorphism $(\mathbb{P}B^\times)^+ \cong Z_{U(2)_v}\backslash U(2)_v$. So, we see that $\mathcal{J}_v = \mathcal{I}_v$, and the $\mathcal{I}_v$ are defined completely in terms of data for $U(2)_v$.

Finally, we give the following restatement of Ichino's triple product formula. Note that since τ_3 is assumed to be dihedral with respect to E/F, we have $BC(\tau_3) \cong \pi(\chi_1, \chi_2)$, the principal series representation on $GL_2(\mathbb{A}_E)$.

Corollary 4.7. *Let $f_i \in \tau_i$ and $\tilde{f}_i \in \tilde{\tau}_i$ be as above. Then*

$$\left|\int_{[G_2]}(f_1f_2f_3)(g)\right|^2 = \frac{\zeta_F(2)^2 L_E(1/2, BC(\tau_1)\boxtimes BC(\tau_2)\boxtimes\chi_1)L_F(1,\chi_{E/F})^3}{2\prod_{i=1}^3 |X(\tau_i)|L_F(1,\tau_1,\mathrm{Ad})L_F(1,\tau_2,\mathrm{Ad})L_F(1,\tau_3,\mathrm{Ad})}\prod_v \mathcal{I}_v.$$

Proof. First we show that

$$\int_{[G_2]}(f_1f_2f_3)(g)dg = 2\int_{\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)^+}(\tilde{f}_{1,\Sigma_1}\tilde{f}_{2,\Sigma_2}\tilde{f}_{3,\Sigma_3})(b)db.$$

Since the product of the central characters of the τ_i is trivial, we see that $f_1f_2f_3$ is really a function on $G_2(F)Z_{G_2}(\mathbb{A}_F)\backslash G_2(\mathbb{A}_F)$. So we have

$$\int_{[G_2]}(f_1f_2f_3)(g)dg = \mathrm{Vol}([Z_{G_2}])\int_{G_2(F)Z_{G_2}(\mathbb{A}_F)\backslash G_2(\mathbb{A}_F)}(f_1f_2f_3)(g)dg.$$

Here, $\mathrm{Vol}(Z_{G_2})$ is computed with respect to the Tamagawa measure on Z_{G_2}. Note there is a natural identification of the spaces $G_2(F)Z_{G_2}(\mathbb{A}_F)\backslash G_2(\mathbb{A}_F)$ and $\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)^+$. This gives

$$\int_{G_2(F)Z_{G_2}(\mathbb{A}_F)\backslash G_2(\mathbb{A}_F)}(f_1f_2f_3)(g)dg = \int_{\mathbb{P}B^\times(F)^+\backslash\mathbb{P}B^\times(\mathbb{A}_F)}(\tilde{f}_{1,\Sigma_1}\tilde{f}_{2,\Sigma_2}\tilde{f}_{3,\Sigma_3})(b)db.$$

Since $\mathrm{Vol}([Z_{G_2}]) = 2$, this proves the claim.

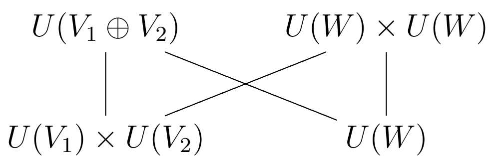
Finally, by invoking Propositions A.1, A.2, A.3, and Corollary A.5 in the appendix, the proof is complete. $\square$

5 The Θ-Correspondence for Unitary Groups

While the first case ($n = 1$) of Conjecture 1.3 follows rather easily from Waldspurger's theorem, a proof of the conjecture for $n = 2$ will require significantly more work. In fact, we will only be able to prove the conjecture for a restricted class of representations.

In this section, we will first introduce the Weil Representation and Θ-correspondence. Given two unitary groups $U(V)$ and $U(W)$ and a cuspidal automorphic representation π of $U(V)(\mathbb{A}_F)$, the Θ-correspondence constructs for us a cuspidal automorphic representation of $U(W)(\mathbb{A}_F)$.

The Θ-correspondence is useful for us because it relates the period integral we wish to compute to a more familiar integral: Ichino's triple product integral. The relevant cartoon is the following seesaw diagram:



Here, the vertical bars denote containment, and the oblique bars denote members of a so-called *dual reductive pair* in $Sp((V_1 \oplus V_2) \otimes W)$, which we define presently:

Definition 5.1. *Let $G, H \subset Sp(W)$ be subgroups. Then (G, H) is called a* dual reductive pair *if*

- *G is the centralizer of H in $Sp(W)$, and vice versa.*

- *The actions of G and H on W are completely reducible.*

Let $\pi_W, \pi_{V_1}, \pi_{V_2}$ be cuspidal automorphic representations of $U(W), U(V_1)$ and $U(V_2)$. Then the Θ-correspondence gives us representations $\Theta(\pi_W)$ of $U(V_1 \oplus V_2)$ and $\Theta(\pi_{V_1}) \otimes \Theta(\pi_{V_2})$ of $U(W) \times U(W)$.

Seesaw duality tells us that integrating a cusp form $\theta(f_W) \in \Theta(\pi_W)$ against a pair of cusp forms f_{V_1} and f_{V_2} is the same as integrating $\theta(f_{V_1})$ and $\theta(f_{V_2})$ against f_W. The first integral described is essentially our period integral, while the second is the triple-product integral. This can be succinctly described by the following commutative diagram:

$$\begin{array}{ccc} \omega_{\psi,\gamma} \otimes \bar{\pi}_{V_1} \otimes \bar{\pi}_{V_2} \otimes \bar{\pi}_W & \xrightarrow{\mathcal{T}} & \Theta(\pi_{V_1}) \otimes \Theta(\pi_{V_2}) \otimes \bar{\pi}_W \\ \downarrow \mathcal{T}' & & \downarrow \mathcal{I} \\ \bar{\pi}_{V_1} \otimes \bar{\pi}_{V_2} \otimes \Theta(\pi_W) & \xrightarrow{\mathcal{P}} & \mathbb{C} \end{array} \quad (5.1)$$

Here, $\mathcal{T}$ and $\mathcal{T}'$ denote Θ lifts, $\mathcal{P}$ is our global period integral, and $\mathcal{I}$ is the triple product integral.

Recall that the Refined Gross-Prasad Conjecture relates $\mathcal{P}$ to $\prod_v \mathcal{P}_v$, the product of integrals of local matrix coefficients. We know from various multiplicity-one results that:

$$\begin{aligned} \mathcal{T} &\approx \otimes_v \mathcal{T}_v \\ \mathcal{T}' &\approx \otimes_v \mathcal{T}'_v \\ \mathcal{I} &\approx \prod_v \mathcal{I}_v \end{aligned}$$

where $\approx$ denotes equality up to a constant of proportionality. If we can compute the exact constants of proportionality, then this and diagram 5.1 will provide us with the constant of proportionality between $\mathcal{P}$ and $\prod_v \mathcal{P}_v$.

So how do we find these constants of proportionality? For $\mathcal{T}$ and $\mathcal{T}'$, we need several incarnations of the Rallis inner-product formula. For two of these Rallis inner-product formulae, we simply invoke results of Michael Harris. For the other, we will give a proof, which follows from Victor Tan's regularized Siegel-Weil Formula. For $\mathcal{I}$, we use Ichino's work on the triple product integral.

The first part of this chapter will be spent introducing both the local and global theta correspondences. For the local correspondence, we refer the reader to [12] and [14]. However, before discussing the Θ-correspondence, we'll give a brief overview of the Weil Representation, both globally and locally.

5.1 The Weil Representation for Unitary Groups

As before, E/F is a quadratic extension of number fields. Let V be a hermitian space over E of dimension m, and W a skew-hermitian space over E of dimension n.

For brevity, we will treat the global and local Weil representation simultaneously. For an algebraic group G, where the same statements can be made globally and locally, and if there is no risk of confusion, we will not make reference to both $G(F_v)$ and $G(\mathbb{A}_F)$, but only to G.

We denote

$$\begin{aligned} G &:= U(V) \\ H &:= U(W) \end{aligned}$$

viewed as algebraic groups over F. We also consider the space

$$\mathbb{W} := \mathrm{Res}_{E/F}\, V \otimes_E W$$

along with a complete polarization

$$\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}.$$

Denote by $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ the hermitian and skew-hermitian forms for V and W respectively. We equip $\mathbb{W}$ with the following symplectic form:

$$\langle \cdot, \cdot \rangle_{\mathbb{W}} := \mathrm{tr}_{E/F}\left(\langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_W \right).$$

So, we consider the associated isometry group $Sp(\mathbb{W})$, along with the metaplectic cover $\widetilde{Sp}(\mathbb{W})$. We have the following short exact sequence:

$$1 \to \mathbb{C}^\times \to \widetilde{Sp}(\mathbb{W}) \to Sp(\mathbb{W}) \to 1.$$

Now, after fixing a an additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$ (globally) or $\psi : F_v \to \mathbb{C}^\times$ (locally), we have a Schrödinger model of the Weil Representation of $\widetilde{Sp}(\mathbb{W})$ on $\mathcal{S}(\mathbb{X})$, where $\mathcal{S}$ denotes the space of Schwartz-Bruhat functions.

A priori, we have embeddings

$$\iota_W : G \hookrightarrow Sp(\mathbb{W}) \tag{5.2}$$

$$\iota_V : H \hookrightarrow Sp(\mathbb{W}) \tag{5.3}$$

which induces a map

$$\iota_{W,V} : G \times H \to Sp(\mathbb{W}).$$

Remark 5.2. *While the maps ι_W and ι_V are embeddings, the induced map $\iota_{W,V}$ is never an embedding.*

In order to obtain a splitting homomorphism $G \times H \to \widetilde{Sp}(\mathbb{W})(\mathbb{A}_F)$, we need a pair of characters (γ_V, γ_W) of $\mathbb{A}_E^\times/E^\times$ (or E_v in the local case) such that

$$\gamma_V|_{\mathbb{A}_F^\times \text{ or } F_v^\times} = \chi_{E/F}^m$$

$$\gamma_W|_{\mathbb{A}_F^\times \text{ or } F_v^\times} = \chi_{E/F}^n$$

where, as usual, the character $\chi_{E/F}$ is the quadratic character of $\mathbb{A}_F^\times/F^\times$ or F_v associated to E/F by class field theory. These characters give us splitting homormorphisms

$$\iota_{W,\gamma_W} : G \to \widetilde{Sp}(\mathbb{W})$$

$$\iota_{V,\gamma_V} : H \to \widetilde{Sp}(\mathbb{W})$$

which then induces

$$\iota_{W,\gamma_W,V,\gamma_V} : G \times H \to \widetilde{Sp}(\mathbb{W}).$$

With this splitting map, we can compose with ω_ψ to get a Weil representation of $G \times H$ realized on $\mathcal{S}(\mathbb{X})$:

$$\omega_{W,\gamma_W,V,\gamma_V,\psi} := \omega_\psi \circ \iota_{W,\gamma_W,V,\gamma_V}.$$

For simplicity, we shall abide by a certain convention when choosing splitting characters. Globally, let γ be a character of $\mathbb{A}_E^\times/E^\times$ such that $\gamma|_{\mathbb{A}_F^\times} = \chi_{E/F}$. When choosing γ_W and γ_V as above, we simply take $\gamma_W := \gamma^{\dim W}$ and $\gamma_V = \gamma^{\dim V}$. We also follow the analogous local convention.

Remark 5.3. *The convention above does not lead to any real loss of generality, if we also consider twists of the theta-lifts by characters.*

5.2 The Local Θ-Correspondence

For this section, we fix a place v of F and omit it from the notation, so that $F = F_v$. As usual, E is a quadratic extension of F; in the case that v splits, we have $E = F \oplus F$. Let $\chi_{E/F}$ be the character associated to E/F by class field theory. In the split case, $\chi_{E/F}$ is trivial.

5.2.1 Howe Duality

Suppose that (G, G') is a dual reductive pair of unitary groups in some symplectic group $Sp(\mathbb{W})$. After fixing the characters ψ and γ as described above, we obtain a Weil representation $(\omega_{\psi,\gamma}, \mathcal{S})$ of $G \times G'$. Let π be an irreducible admissible representation of G. We let $\mathcal{S}(\pi)$ be the maximal quotient of $\mathcal{S}$ on which G acts as a multiple of π. Then we have:

$$\mathcal{S}(\pi) \cong \pi \otimes \Theta(\pi)$$

where $\Theta(\pi)$ is a representation of G'. We simply set $\Theta(\pi) = 0$ if π does not occur as a quotient of $\mathcal{S}$. The *Howe Duality Principle* states:

1. $\Theta(\pi)$ is a finitely generated admissible representation of G'.

2. $\Theta(\pi)$ has a unique proper maximal G'-invariant subspace and a unique irreducible quotient $\theta(\pi)$.

3. The correspondence $\pi \mapsto \theta(\pi)$ gives a bijection between the irreducible admissible representations of G and G' that occur as quotients of $\mathcal{S}$.

The first assertion is known, due to [25] and [29]. The last two assertions are known for $v \neq 2$, due to [36]. Furthermore, for $v = 2$, the second two assertions are easily checked in the low rank examples considered here.

5.3 The Global Θ-Correspondence

Globally, the Θ-correspondence is realized using Θ-series. For any $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_F))$, we define the theta kernel as

$$\theta(g, h, \varphi) := \sum_{\lambda \in \mathbb{X}(F)} \omega_{W,\gamma_W,V,\gamma_V,\psi}(g,h)(\varphi)(\lambda).$$

If f is some cusp form on $G(\mathbb{A}_F)$, we define:

$$\theta(f, \varphi)(h) := \int_{[G]} \theta(g, h, \varphi)\overline{f(g)}\, dg \tag{5.4}$$

where dg is the Tamagawa measure.

With all of this, we can define the Θ-lift of a cuspidal representation of G.

Definition 5.4. *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$, then*

$$\Theta_{V,W,\gamma_W,\gamma_V,\psi}(\pi) = \{\theta(f, \varphi) : f \in \pi, \varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_F))\}$$

is the Θ-lift of π with data $(\gamma_W, \gamma_V, \psi)$.

Remark 5.5. *The reader may balk at the definition given in line 5.4. By integrating $\overline{f}$ (instead of merely f) against the theta series, we ensure that π and $\Theta(\pi)$ have the same central characters.*

5.4 The Rallis Inner Product Formula

The goal of this section is to compute the constant of proportionality between $\mathcal{T}$ and $\prod_v \mathcal{T}_v$. This involves several versions of the Rallis Inner Product

Formula, which relates the Petersson inner product of two vectors to that of their Θ-lifts. We will need to use three different versions of the Rallis Inner Product formula, one for lifts from $U(1)$ to $U(2)$, one for lifts from $U(2)$ to $U(2)$, and one for lifts from $U(2)$ to $U(3)$.

Before giving the Rallis Inner Product Formulae we shall need, we give a brief discussion of the doubling method.

5.4.1 The Doubling Method

We remind the reader that V is a hermitian space over E of dimension m, and W is a skew-hermitian space of dimension n. We will also consider the space V^-, which is the same space as V, but with hermitian form $-\langle \cdot, \cdot \rangle_V$. We note that $U(V) = U(V^-)$.

The seesaw diagram relevant to this discussion is the following:

$$\begin{array}{ccc} U(V \oplus V^-) & & U(W) \times U(W) \\ | & \times & | \\ U(V) \times U(V^-) & & U(W)^{\Delta} \end{array} \tag{5.5}$$

which we shall call the *doubling seesaw*. $U(W)^{\Delta}$ simply denotes the diagonally embedded copy of $U(W)$ in $U(W) \times U(W)$. We shall also occasionally use the shorthand $G := U(V) = U(V^-), H := U(W)$ and $G^{\diamond} := U(V \oplus V^-)$.

The dual reductive pairs above live inside $\widetilde{Sp}(2\mathbb{W})$, where

$$2\mathbb{W} := \mathbb{W} \oplus \mathbb{W}^-$$

and $\mathbb{W}^- := V^- \otimes_E W$.

In order to have a Weil representation, we will have to fix splitting characters for each of the dual reductive pairs in the doubling seesaw. After choosing splitting characters for one of the diagonal segments above, the splitting characters will be fixed for the other diagonal segment. We shall choose splitting characters for the dual reductive pair $(U(V) \times U(V^-), U(W) \times U(W))$.

Note that this pair can be viewed as two dual reductive pairs: $(U(V), U(W))$ in $\widetilde{Sp}(\mathbb{W})$ and $(U(V^-), U(W))$ in $\widetilde{Sp}(\mathbb{W}^-)$. Suppose that we choose splitting characters (γ_W, γ_V) for $(U(V), U(W))$ and (γ_W, γ'_V) for $(U(V^-), U(W))$. With these choices made, we are forced to take $(\gamma_W, \gamma_V\gamma'_V)$ as our splitting characters for the dual reductive pair $(U(V \oplus V^-), U(W)^\Delta)$.[1]

Suppose that π and π' are cuspidal, irreducible, automorphic representations of $U(V)$ and $U(V^-)$, respectively. Let $f \in \pi$ and $f' \in \pi'$. With the splitting data (γ_W, γ_V), (γ_W, γ'_V) and the additive character ψ, and Schwartz functions ϕ and ϕ', we can consider the Θ-lifts $\theta(f, \varphi)$ and $\theta(f', \varphi')$, which are both cusp forms on $U(W)$.

The Rallis Inner Product Formula computes the following ratio:

$$\frac{\langle \theta(f, \varphi), \theta(f', \varphi') \rangle}{\langle f, f' \rangle}$$

where the pairings are the respective Petersson inner products, which are defined using the respective Tamagawa measures. However, the RHS of the Rallis Inner Product Formula will contain division by $\prod_v \langle f_v, f'_v \rangle_v$. Since we make the assumption that the global and local inner products are chosen compatibly, for our purposes the Rallis Inner Product Formula simply computes $\langle \theta(f, \varphi), \theta(f', \varphi) \rangle$ in terms of an L-value.

Note that unless $\theta(f, \varphi)$ is in the contragredient of $\Theta(\pi)$, the Petersson pairing $\langle \theta(f, \varphi), \theta(f', \varphi) \rangle$ will vanish. So, we see that π and π' cannot be chosen independently.

What then, is the required relationship between π and π' to ensure the non-triviality of the inner product? We note that

$$\begin{aligned} \theta(f, \varphi) &\in \Theta_{V,W,\gamma_W,\gamma_V,\psi}(\pi) \\ \theta(f', \varphi') &\in \Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi'). \end{aligned}$$

We have the following technical result:

[1] This choice is 'forced' in the sense that seesaw duality does not hold otherwise.

Lemma 5.6.

$$\Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi') = (\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V,W,\gamma_W,\gamma_V,\psi}((\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)}) \cdot \overline{\pi'})}$$

where $i_{U(1)}$ *is the inverse of the isomorphism* $i_{U(1)}^{-1} : E^\times/F^\times \xrightarrow{\sim} U(1) \subset E^\times$ *given by* $e \mapsto \frac{e}{e^\tau}$*, where* τ *is the generator of* $\mathrm{Gal}(E/F)$*.*

Proof. We begin by considering the Weil representation $\omega_{V^-,W,\gamma_W,\gamma'_V,\psi}$ as a representation of $U(V) \times U(W)$. (A priori, it is a representation of $U(V^-) \times U(W)$. We are identifying $U(V)$ with $U(V^-)$ via the identity map on $GL(V)$.) We note that

$$\begin{aligned}
\omega_{V^-,W,\gamma_W,\gamma'_V,\psi} &\cong \omega_{V^-,W,\gamma_W,\gamma_V,\psi} \otimes \left(1_{U(V)} \boxtimes \frac{\gamma'_V}{\gamma_V} \circ i_{U(1)} \circ \det_{U(W)}\right) \\
&\cong \omega_{V,W,\gamma_W,\gamma_V,\psi^{-1}} \otimes \left(1_{U(V)} \boxtimes \frac{\gamma'_V}{\gamma_V} \circ i_{U(1)} \circ \det_{U(W)}\right) \\
&\cong \omega_{V,W,\gamma_W^{-1},\gamma_V^{-1},\psi^{-1}} \otimes (\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)} \boxtimes \gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \\
&\cong \overline{\omega_{V,W,\gamma_W,\gamma_V,\psi}} \otimes (\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)} \boxtimes \gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}).
\end{aligned}$$

There is a surjection

$$\omega_{V^-,W,\gamma_W,\gamma'_V,\psi} \twoheadrightarrow \pi' \boxtimes \Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi')$$

and therefore, from the series of isomorphisms above, we have

$$\overline{\omega_{V,W,\gamma_W,\gamma_V,\psi}} \twoheadrightarrow (\gamma_W^{-2} \circ i_{U(1)} \circ \det_{U(V^-)}) \cdot \pi' \boxtimes ((\gamma_V\gamma'_V)^{-1} \circ i_{U(1)} \circ \det_{U(W)}) \cdot \Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi')$$

which immediately gives

$$\omega_{V,W,\gamma_W,\gamma_V,\psi} \twoheadrightarrow (\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)}) \cdot \overline{\pi'} \boxtimes (\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi')}.$$

The surjection above tells us that

$$(\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi')} \cong \Theta_{V,W,\gamma_W,\gamma_V,\psi}(\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)} \cdot \overline{\pi'})$$

and therefore

$$\Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi') = (\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V,W,\gamma_W,\gamma_V,\psi}((\gamma_W^2 \circ i_{U(1)} \circ \det_{U(V^-)}) \cdot \overline{\pi'})}.$$

□

Now, if $e \in U(1) \subset E^\times$, we observe that $i_{U(1)}^{-1}(e) = e^2$, so that $\gamma_W(e) = \gamma_W(i_{U(1)}(e^2)) = \gamma_W^2(i_{U(1)}(e))$. This means that

$$\Theta_{V^-,W,\gamma_W,\gamma'_V,\psi}(\pi') = (\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V,W,\gamma_W,\gamma_V,\psi}((\gamma_W|_{U(1)} \circ \det_{U(V^-)}) \cdot \overline{\pi'})}.$$

So if we take

$$\pi' = (\gamma_W \circ \det_{U(V^-)}) \cdot \overline{\pi}$$

then

$$\theta(f', \varphi') \in (\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}) \cdot \overline{\Theta_{V,W,\gamma_W,\gamma_V,\psi}(\pi)}.$$

Therefore, the integral

$$\int_{[H^\Delta]} \theta(f,\varphi)(h)\theta(f',\varphi')(h) \cdot ((\gamma_V\gamma'_V)^{-1} \circ i_{U(1)} \circ \det_{U(W)})(h)dh$$

is the Petersson inner product of two vectors in $\Theta_{V,W,\gamma_W,\gamma_V,\psi}(\pi)$. Seesaw duality is the statement that this integral is equal to

$$\int_{[G\times G]} \overline{f(g_1)f'(g_2)}\theta_{\varphi\otimes\varphi'}((\gamma_V\gamma'_V) \circ i_{U(1)} \circ \det_{U(W)})(g_1, g_2)dg_1dg_2 \tag{5.6}$$

where

$$\theta_{\varphi\otimes\varphi'}((\gamma_V\gamma'_V) \circ i_{U(1)} \circ \det_{U(W)}) \in \Theta_{V\oplus V^-,W,\gamma_W,\gamma_V\gamma'_V,\psi}(\gamma_V\gamma'_V \circ i_{U(1)} \circ \det_{U(W)}).$$

Since $f' \in \pi' = (\gamma_W|_{U(1)} \cdot \det_{U(V^-)}) \cdot \pi^\vee$, we can take $f' = (\gamma_W|_{U(1)} \cdot \det_{U(V^-)}) \cdot \overline{f_2}$ for some $f_2 \in \pi$. (For consistency of notation, $f_1 := f$.)

Now, the Θ-lift in the integral in line 5.6 can actually be identified with a certain Eisenstein series $E(\Phi_s, g)$. More specifically, the integral above can be identified with

$$\int_{[G\times G]} \overline{f_1(g_1)}f_2(g_2)E(\Phi_s,(g_1,g_2))\gamma_W^{-1}(\det_{U(V^-)} g_2)dg_1dg_2 \tag{5.7}$$

where Φ_s is a member of the degenerate principal series $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G^\diamond(\mathbb{A}_F)}(\gamma_W \circ \det) \cdot |\det|^s$, with P the Siegel parabolic that preserves the diagonal $V^\Delta \subset V \oplus V^-$, the determinants are taken with respect to $GL(V^{\Delta})$ (which is isomorphic to the Levi of P), and

$$E(\Phi_s, g) := \sum_{x \in P(F) \backslash G^{\diamond}(F)} \Phi_s(xg)$$

for $g \in G^{\diamond}$.

Definition 5.7. *The Piatetski-Shapiro-Rallis zeta integral is defined as*

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) := \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}E(\Phi_s, \iota(g_1, g_2)), \gamma_W^{-1}(\det_{U(V^-)} g_2) dg_1 dg_2.$$

We remark that the integral in line 5.7 is equal to $Z(s, \bar{f}_1, \bar{f}_2, \Phi_s, \gamma_W)$.

It is important to note that invoking seesaw duality requires one to deal with convergence of the relevant integrals. Unfortunately, these integrals do not always converge. Certain regularization methods are sometimes required to relate the Petersson inner product of the Θ-lifts to the zeta integral. However, once this relationship between the Petersson inner product and zeta integral are established, there is a very nice result which relates Z to an L-function. Before giving this result, we need a bit more discussion.

Recall that $m := \dim_E V$ and $n := \dim_E W$. Set

$$d_m(s, \gamma_W) := \prod_{r=0}^{m-1} L(2s + m - r, \chi_{E/F}^{n+r}).$$

We assume that $\Phi_s = \otimes_v \Phi_{s,v}$ and $f_i = \otimes_v f_{i,v}$. We take S to be a sufficiently large finite set of places of F such that for all $v \notin S$, all relevant data is unramified, and the local vectors $f_{i,v}$ are normalized spherical vectors, with the additional property that $\langle f_{1,v}, f_{2,v} \rangle_{\pi_v} = 1$.[2] Let the $d_{m,v}$ be the local factors of d_m. We have the following factorization of the zeta-integral (see [12] as well as [13]):

Theorem 5.8. *For* $\mathrm{Re}(s) >> 0$*,*

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) = \prod_v Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}, \gamma_{W,v})$$

[2]We remind the reader that the local pairings $\langle \cdot, \cdot \rangle_{\pi_v}$ are chosen so that the product over all places v gives the Petersson inner product $\langle \cdot, \cdot \rangle_{\pi}$.

where

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}, \gamma_{W,v}) := \int_{U(V)_v} \Phi_{s,v}((g_v, 1))\langle \pi_v(g_v) f_{1,v}, f_{2,v}\rangle_{\pi_v}(\gamma_{W,v}^{-1} \det g_v) dg_v.$$

We note that the integral defining the Z_v in the theorem above only converges for $\mathrm{Re}(s)$ sufficiently large. The definition of the Z_v is extended to all of $\mathbb{C}$ by meromorphic continuation.

Now, if we set $Z_S := \prod_{v \in S} Z_v$, then we have the following so-called 'basic identity' (again, see [12] and [13]):

Theorem 5.9 (Basic Identity)**.** *For $f_1, f_2 \in \pi$, we have:*

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) = [d_m(s, \gamma_W)]^{-1} Z_S(s, f_1, f_2, \Phi_s, \gamma_W) \cdot L^S(s + 1/2, \pi \otimes \gamma_W).$$

5.4.2 Lifting from $U(1)$ to $U(2)$

Here, $\dim V = 1$ and $\dim W = 2$. In this case, π is just a character, which we will denote by μ. In this section, we will not only consider $\Theta_{V,W,\gamma_W,\gamma_V,\psi}(\mu)$, but also its transfer to $GU(2)$, à la Theorem 3.1. Denote this by $\widetilde{\Theta(\mu)}$. We remind the reader that $\gamma_W = \gamma^2$ and $\gamma_V = \gamma$ for some Hecke character γ of $\mathbb{A}_E^\times$ extending $\chi_{E/F}$, per our convention.

The first incarnation of the Rallis Inner Product Formula that we will need is as follows:

Theorem 5.10 (RIPF for Θ-lifts from $U(1)$ to $U(2)$)**.** *Suppose that $f_i = \otimes_v f_{i,v}, \varphi_i = \otimes_v \varphi_{i,v}, \Phi_s = \otimes_v \Phi_{s,v}$, and that Φ is a holomorphic section given by $[\delta(\varphi_1, \varphi_2)]$ in the notation of [27], page 182. Then:*

$$\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2)\rangle_{\Theta(\bar{\mu})} = |X(\Theta(\bar{\mu}))| \cdot \frac{L_E(1, BC(\mu) \otimes \gamma^2)}{\zeta_F(2)} \prod_v Z_v^\sharp(1/2, f_{1,v}, f_{2,v}, \Phi_{1/2,v}, \gamma_v^2)$$

where

$$Z_v^\sharp := \frac{\zeta_{F_v}(2)}{L_{E_v}(1, BC(\mu_v) \otimes \gamma_v^2)} \cdot Z_v$$

and $X(\Theta(\bar{\mu}))$ is the set of automorphic characters ω of $GU(2)(\mathbb{A}_F)/U(2)(\mathbb{A}_F)$ such that $\widetilde{\Theta(\bar{\mu})} \otimes \omega \cong \widetilde{\Theta(\bar{\mu})}$.

Proof. First, we remark that the presence of complex conjugation bars over the f_i is due to our normalization of the theta-correspondence. In line 6.4.8 on page 710 of [11], Harris has the following:

$$\langle \widetilde{\theta(\bar{f}_1, \varphi_1)}, \widetilde{\theta(\bar{f}_2, \varphi_2)} \rangle_{\widetilde{\Theta(\bar{\mu})}} = \frac{L_E^S(1, BC(\mu))}{\zeta_F^S(2)} \prod_{v \in S} Z_v(1/2, f_{1,v}, f_{2,v}, \Phi_{1/2,v}) \tag{5.8}$$

Using Remark 4.20 of [18], we have

$$\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2) \rangle_{\Theta(\bar{\mu})} = |X(\Theta(\bar{\mu}))| \cdot \frac{L_E(1, BC(\mu))}{\zeta_F(2)} \prod_v Z_v^\sharp(1/2, f_{1,v}, f_{2,v}, \Phi_{1/2,v}).$$

Now, this appears different from the claimed result for a simple (albeit subtle) reason: Harris has used a particular normalization of the theta correspondence that is slightly different from ours. When choosing a pair of splitting characters (γ_V, γ_W), he chooses γ_W to be the trivial character. However, there is no real loss of generality here, since we can merely replace μ with $\mu \otimes \gamma|_{U(1)}$ and then follow Harris' conventions.

□

5.4.3 Lifting from $U(2)$ to $U(2)$

Here, $\dim W = \dim V = 2$. With a sufficiently large set of places S, and the same assumptions made for $v \notin S$ as in the previous section, the Rallis Inner Product Formula is stated in line 1.3.5 of [15]. Once again, the presence of complex conjugation bars in the result below is due to our normalization of the theta-correspondence.

Theorem 5.11 (RIPF for lifts from $U(2)$ to $U(2)$)**.** *Suppose that $f_i = \otimes_v f_{i,v}, \varphi_i = \otimes_v \varphi_{i,v}, \Phi_s = \otimes_v \Phi_{s,v}$, and that Φ is a holomorphic section given by $[\delta(\varphi_1, \varphi_2)]$ in the notation of [27], page 182. Then we have*

$$\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2) \rangle_{\Theta(\bar{\pi})} = \frac{L_E(1/2, BC(\pi) \otimes \gamma^2)}{L_F(1, \chi_{E/F}) \zeta_F(2)} \prod_v Z_v^\sharp(0, f_{1,v}, f_{2,v}, \Phi_{0,v}, \gamma_v^2)$$

where

$$Z_v^\sharp := \frac{L_{F_v}(1, \chi_{E_v/F_v}) \zeta_{F_v}(2)}{L_{E_v}(1/2, \pi_v \otimes \gamma_v^2)} Z_v.$$

5.4.4 Lifting from $U(2)$ to $U(3)$

We have $\dim_E V = 2$ and $\dim_E W = 3$. If W is anisotropic, then $[H]$ is compact, and the Rallis Inner Product Formula we need follows from a Siegel-Weil Formula (Theorem 1.1 in [19]). However, if W is not anisotropic (so that $H(\mathbb{A}_F)$ is quasi-split), the theta integral in the Siegel-Weil formula does not converge. In this case, the Rallis Inner Product Formula follows from Tan's Regularized Siegel-Weil formula. What follows is a brief summary of [32]. We encourage the interested reader to consult Tan's paper for further details.

We again have occasion to consider the following degenerate principal series representation of $G^\diamond(\mathbb{A}_F)$:

$$I(s,\gamma) := \mathrm{Ind}_{P(\mathbb{A}_F)}^{G^\diamond(\mathbb{A}_F)} \gamma^3 || \cdot ||_{\mathbb{A}_E}^s \circ \det$$

where we remind that reader that P is the parabolic preserving the diagonal

$$V^\Delta := \{(v,v) : v \in V\} \subset V \oplus V^-.$$

Given $\Phi_s \in I(s,\gamma)$, we define the Siegel-Eisenstein series

$$\mathcal{E}(g,\Phi_s) := \sum_{\varepsilon \in P(F)\backslash G^\diamond(F)} \Phi_s(\varepsilon g).$$

There is a maximal compact subgroup $K \subset G^\diamond(\mathbb{A}_F)$ such that we have the decomposition:

$$G^\diamond(\mathbb{A}_F) = P(\mathbb{A}_F)K.$$

We call Φ_s a *standard section* if its restriction to K is independent of s. For a standard section Φ_s, the Siegel-Eisenstein series $\mathcal{E}(g,\Phi_s)$ converges for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation to $\mathbb{C}$. Furthermore, for each $s \in \mathbb{C}$ where it is holomorphic, $\mathcal{E}(g,\Phi_s)$ is an automorphic form on $G^\diamond(\mathbb{A}_F)$. We take Φ_s to be a standard section in the sequel.

The Eisenstein series $\mathcal{E}(g,\Phi_s)$ has at worst a simple pole at $s = 1/2$. So we write its Laurent expansion as:

$$\mathcal{E}(g,\Phi_s) = \frac{A_{-1}(g,\Phi)}{s-1/2} + A_0(g,\Phi) + \dots$$

Before defining the theta integral, we remind the reader of the setup for the Weil representation. We have

$$\mathbb{W} := \operatorname{Res}_{E/F} 2V \otimes_E W$$

where we remind the reader that $2V := V \oplus V^-$. We set

$$V^\nabla := \{(v, -v) : v \in V\} \subset V \oplus V^-.$$

Having fixed the characters ψ and γ, we have a Schrödinger model of the Weil representation of $G^\diamond(\mathbb{A}_F) \times H(\mathbb{A}_F)$ realized on $\mathcal{S}((V^\nabla \otimes W)(\mathbb{A}_F))$.

Now, if we fix polarizations

$$\begin{aligned} V &= X^+ \oplus Y^+ \\ V^- &= X^- \oplus Y^- \end{aligned}$$

and denote

$$\begin{aligned} X &:= X^+ \oplus X^- \\ Y &:= Y^+ \oplus Y^- \end{aligned}$$

then we obtain another polarization of $\mathbb{W}$. We have:

$$2V = X \oplus Y$$

and therefore

$$\mathbb{W} = (X \otimes W) \oplus (Y \otimes W).$$

We denote

$$\begin{aligned} \mathbb{X} &:= X \otimes W \\ \mathbb{X}^+ &:= X^+ \otimes W \\ \mathbb{X}^- &:= X^- \otimes W. \end{aligned}$$

There is a $U(V)(\mathbb{A}_F) \times U(V^-)(\mathbb{A}_F)$-intertwining map

$$\sigma : \mathcal{S}(\mathbb{X}^+(\mathbb{A}_F)) \otimes \mathcal{S}(\mathbb{X}^-(\mathbb{A}_F)) \to \mathcal{S}(\mathbb{X}(\mathbb{A}_F)) \to \mathcal{S}((V^\nabla \otimes W)(\mathbb{A}_F))$$

where the first map is the obvious one, and the second map is given by a Fourier transform.

We now define the *theta integral* as follows:

$$I(g,\varphi) := \int_{[H]} \theta(g,h,\varphi)\gamma^{-2}(\det h)\ dh$$

where $\varphi \in \mathcal{S}(V^{\nabla} \otimes W)(\mathbb{A}_F)), g \in G^{\diamond}(\mathbb{A}_F)$, and dh is the Tamagawa measure on $H(\mathbb{A}_F)$.

For $g \in G^{\diamond}(\mathbb{A}_F)$, we have $g = pk$, with $p \in P(\mathbb{A}_F)$, and $k \in K$. With the right choice of basis, we have $p = \begin{pmatrix} a & * \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix}$ for some $a \in GL_2(\mathbb{A}_E)$. Write $|a(g)| := |\det a|_{\mathbb{A}_E}$. If W is anisotropic, and Φ_s is chosen such that $\Phi_s(g) = |a(g)|^{s-1/2}\omega_{\psi,\gamma}(g)\varphi(0)$, then $\mathcal{E}(g,\Phi_s)$ is holomorphic at $s = 1/2$, and the theta integral defined above converges. Indeed, Theorem 1.1 of [19] says that

$$\mathcal{E}(g,\Phi_{1/2}) = I(g,\varphi).$$

However, as mentioned above, if $H(\mathbb{A}_F)$ is quasi-split (i.e. W is not anisotropic), the theta integral does not necessarily converge. We'll have to 'regularize' it so that we can think of it as a meromorphic function of a complex variable. The Regularized Siegel-Weil Formula relates the Laurent expansions of this yet-to-be-defined regularized theta integral and the Siegel Eisenstein series.

Let v be an odd place of F such that all relevant data is unramified. Tan finds a Hecke operator z in the Hecke algebra of $G_v^{\diamond}$ that is used in the definition of the regularized theta integral.

We also need an auxiliary Eisenstein series. Let B_H be a Borel subgroup of H. Then we have

$$B_H = M_H N_H$$

where M_H is the Levi component of B_H, and N_H is the unipotent radical. We know that $M_H(\mathbb{A}_F) \cong \mathbb{A}_E^{\times} \times \mathbb{A}_E^{\times,1}$. For $s \in \mathbb{C}$, let μ_s be the character of $M_H(\mathbb{A}_F)$ defined by $\mu_s(x,u) := ||x||_{\mathbb{A}_E}^s$. We extend μ_s to all of B_H by triviality on N_H. We consider the induced representation

$$I^{Aux}(s) := \mathrm{Ind}_{B_H}^{H} \mu_s.$$

Let $K_H \subset H$ by a maximal compact subgroup such that $H = B_H K_H$. Let $\Phi_s^{Aux} \in I^{Aux}(s)$ be the normalized K_H-fixed vector (i.e. $\Phi_s^{Aux}(k) = 1$ for all $k \in K_H$). Then the auxiliary Eisenstein series we need is defined by

$$E(h, \Phi_s^{Aux}) := \sum_{\varepsilon \in B_H(F)\backslash H(F)} \Phi_s^{Aux}(\varepsilon h).$$

It is known that $E(h, \Phi_s^{Aux})$ converges for $\mathrm{Re}(s)$ sufficiently large, and has meromorphic continuation to all of $\mathbb{C}$. Furthermore, $E(h, \Phi_s^{Aux})$ has a simple pole at $s = 1$ which is independent of h; we denote this residue by

$$\kappa := \operatorname*{Res}_{s=1} E(h, \Phi_s^{Aux}).$$

We now define a new theta integral which incorporates both the auxiliary Eisenstein series and Hecke operator:

$$I(g, s, \omega_{\psi,\gamma}(z)\varphi) := \int_{[H]} \theta(g, h, \omega_{\psi,\gamma}(z)\varphi) E(h, \Phi_s^{Aux}) \gamma^{-2}(\det h)\, dh.$$

With all of this in place, we can define the *regularized theta integral*. The only modification from the theta integral above is that we multiply by an appropriate factor to cancel the effect of the Hecke operator.

Definition 5.12. *For $g \in G^{\diamond}(\mathbb{A}_F), s \in \mathbb{C}$, and $\varphi \in \mathcal{S}(V^{\nabla} \otimes W)(\mathbb{A}_F))$, the* regularized theta integral *is given by*

$$\mathcal{I}(g, s, \varphi) := \frac{1}{\kappa} \cdot \frac{I(g, s, \omega_{\psi,\gamma}(z)\varphi)}{P_z(s)}$$

where

$$P_z(s) := q_F^{s} + q_F^{-s} - q_F - q_F^{-1}.$$

The observant reader will notice that this definition differs by a constant from the definition given by Tan in [32].

The regularized integral $\mathcal{I}(g,s,\varphi)$ has a double pole at $s=1$; so we write the Laurent expansion as

$$\mathcal{I}(g,s,\varphi)=\frac{B_{-2}(g,\varphi)}{(s-1)^2}+\frac{B_{-1}(g,\varphi)}{s-1}+B_0(g,\varphi)+\ldots$$

where the $B_i(g,\varphi)$ are automorphic forms on $G^\diamond(\mathbb{A}_F)$.

In order to prove the version of the Rallis Inner Product Formula that we'll use later, we need a result of Tan which relates the second terms in the Laurent expansions of $\mathcal{I}(g,s,\varphi)$ and $\mathcal{E}(g,\Phi_s)$.

Theorem 5.13 (Second term identity)**.** *Suppose that* $\Phi_s(k)=(\omega(k)\varphi)(0)$ *for all* $k\in K$ *Then*

$$A_0(g,\Phi)=B_{-1}(g,\varphi)+\Psi(g)$$

where Ψ *is an automorphic form on* $G^\diamond(\mathbb{A}_F)$ *which satisfies*

$$\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\Psi(\iota(g_1,g_2))dg_1dg_2=0$$

for cusp forms f_i *on* $G(\mathbb{A}_F)$*.*

Proof. This follows from Theorem 1.2 and Proposition 5.1.1 in [32], as well as Lemma 4.9 in [33]. Note that our renormalization in the definition of the regularized theta integral eliminates the need for the constant c found in [32]. □

Remark 5.14. *Instead of using the Tamagawa measure on* H *to define the theta integral (and regularized theta integral), Tan uses the measure which gives* $[H]$ *volume* 1*. However, he also takes* $c=\frac{2}{\kappa}$*. Since the Tamagawa measure gives* $[H]$ *volume* 2*, and we have renormalized the regularized theta integral by* $\frac{1}{\kappa}$*, the theorem above is correct.*

We are now equipped to state and prove the Rallis Inner Product Formula.

Theorem 5.15 (Rallis Inner Product Formula)**.** *Let* π *be an irreducible, cuspidal, automorphic representation of* $G(\mathbb{A}_F)$*. Let* $f_1,f_2\in\pi$*. Let* $\varphi_1\in\mathcal{S}(\mathbb{X}^+(\mathbb{A}_F))$ *and* $\varphi_2\in\mathcal{S}(\mathbb{X}^-(\mathbb{A}_F))$*. Let* $\theta(\bar{f}_1,\varphi_1)$ *and* $\theta(\bar{f}_2,\varphi_2)$ *be the theta-lifts of* $\bar{f}_1$ *and* $\bar{f}_2$

to $H(\mathbb{A}_F)$, and suppose these lifts are cuspidal. (Once again, we have included complex conjugation here to compensate for our different normalization of the theta correspondence.) Set $\varphi := \sigma(\varphi_1 \otimes \overline{\varphi_2})$, and let $\Phi_s \in I(s, \gamma)$ be such that $\Phi_s(k) = (\omega(k)\varphi)(0)$ for all $k \in K$. Then the following equality holds:

$$\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2) \rangle_{\Theta(\bar{\pi})} = \frac{L_E(1, BC(\pi) \otimes \gamma^3)}{\zeta_F(2) L_F(3, \chi_{E/F})} \prod_v Z_v^\sharp(1/2, f_{1,v}, f_{2,v}, \Phi_{1/2,v}, \gamma_v^3)$$

where

$$Z_v^\sharp := \frac{\zeta_{F_v}(2) L_{F_v}(3, \chi_{E_v/F_v})}{L_{E_v}(1, BC(\pi_v) \otimes \gamma_v^3)} Z_v.$$

Proof. We begin with the argument in the case that W is not anisotropic. In this case we require the regularized Siegel-Weil formula.

We have the following equalities from the definitions of I and $\mathcal{I}$ given above:

$$\begin{aligned} (*) := \operatorname*{Res}_{s=1} & \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)} \mathcal{I}(\iota(g_1, g_2), s, \varphi) dg_1 dg_2 \\ = & \frac{1}{\kappa P_z(s)} \operatorname*{Res}_{s=1} \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)} I(g, s, \omega_{\psi,\gamma}(z)\varphi) dg_1 dg_2 \\ = & \frac{1}{\kappa P_z(s)} \operatorname*{Res}_{s=1} \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)} \int_{[H]} \theta(\iota(g_1, g_2), h, \omega_{\psi,\gamma}(z)\varphi) \\ & \times E(h, \Phi_s^{Aux}) \gamma^{-2}(\det h) dh dg_1 dg_2. \end{aligned}$$

By Corollary 2.3.2 in [32], $\theta(\iota(g_1, g_2), h, \omega_{\psi,\gamma}(z)\varphi)$ is rapidly decreasing on $[H]$, and we may change the order of integration, so that we have

$$\begin{aligned} (*) \quad = \quad & \frac{1}{\kappa P_z(s)} \operatorname*{Res}_{s=1} \int_{[H]} \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)} \theta(\iota(g_1, g_2), h, \omega_{\psi,\gamma}(z)\varphi) \\ & \times E(h, \Phi_s^{Aux}) \gamma^{-2}(\det h) \ dg_1 dg_2 dh. \end{aligned}$$

Now, there is a Hecke operator z' in H_v (for some place v of F) corresponding to z such that $\omega_{\psi,\gamma}(z)$ and $\omega_{\psi,\gamma}(z')$ have the same action on φ. (See section 2.2 of [32] for details.) So we have

$$\begin{aligned} (*) \quad = \quad & \frac{1}{\kappa P_z(s)} \operatorname*{Res}_{s=1} \int_{[H]} \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)} \theta(\iota(g_1, g_2), h, \omega_{\psi,\gamma}(z')\varphi) \\ & \times E(h, \Phi_s^{Aux}) \gamma^{-2}(\det h) \ dg_1 dg_2 dh. \end{aligned}$$

We have

$$\theta(\iota(g_1,g_2),h,\omega_{\psi,\gamma}(z')\varphi)=\int_{H_v} z'(h_v)\theta(\iota(g_1,g_2),hh_v,\varphi)dh_v.$$

By plugging this in to the previous equation for $(*)$ and making a change of variables, we have

$$\begin{aligned}(*) \quad = \quad & \frac{1}{\kappa P_z(s)}\operatorname*{Res}_{s=1}\int_{[H]}\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\theta(\iota(g_1,g_2),h,\varphi)\gamma^{-2}(\det h)\\ & \times\int_{H_v} z'(h_v)E(hh_v^{-1},\Phi_s^{Aux})\gamma^2(\det h_v)dh_v\ dg_1dg_2dh.\end{aligned}$$

But since

$$\int_{H_v} z'(h_v)E(hh_v^{-1},\Phi_s^{Aux})\gamma^2(\det h_v)dh_v = P_z(s)E(h,\Phi_s^{Aux})$$

(see the top of page 351 of [32]) we have

$$(*)=\frac{1}{\kappa}\operatorname*{Res}_{s=1}\int_{[H]}\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\theta(\iota(g_1,g_2),h,\varphi)E(h,\Phi_s^{Aux})\gamma^{-2}(\det h)dg_1dg_2dh.$$

Now we use a Poisson summation formula

$$\theta(\iota(g_1,g_2),h,\varphi)=\theta(g_1,h,\varphi_1)\theta(g_2,h,\overline{\varphi_2})\gamma^{-2}(\det h)$$

to obtain

$$(*)=\frac{1}{\kappa}\operatorname*{Res}_{s=1}\int_{[H]}\theta(\bar{f}_1,\varphi_1)(h)\overline{\theta(\bar{f}_2,\varphi_2)(h)}E(h,\Phi_s^{Aux})dh.$$

Then, since $\kappa:=\operatorname*{Res}_{s=1} E(h,\Phi_s^{Aux})$, we see that

$$(*)=\langle\theta(\bar{f}_1,\varphi_1),\theta(\bar{f}_2,\varphi_2)\rangle_{\Theta(\pi)}.$$

Now, returning to the definition of $(*)$, we see that

$$(*)=\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}B_{-1}(\iota(g_1,g_2),\varphi)dg_1dg_2$$

and by Theorem 5.13 we have

$$\begin{aligned}(*) \quad = \quad & \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}A_0(\iota(g_1,g_2),\Phi)dg_1dg_2\\ & -\int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}\Psi(\iota(g_1,g_2))dg_1dg_2\\ = \quad & \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}A_0(\iota(g_1,g_2),\Phi)dg_1dg_2.\end{aligned}$$

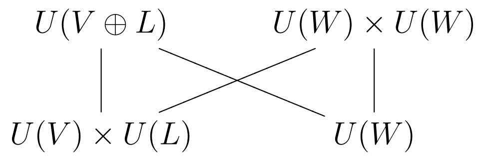
Finally, Lemma 4.7 and line 4.8 of [33] gives that the integral above is just $Z(1/2, f_1, f_2, \Phi_{1/2}, \gamma^3)$. Therefore, after applying the basic identity, we are done.

If W is anisotropic (so that $[H]$ is compact), then the proof is much simpler; there is no need to regularize the Siegel-Weil formula in this case. The result follows from Theorem 1.1 of [19]. □

6 A local seesaw identity

Everything in this chapter is local in nature, though we omit v from the notation. Whenever we refer to a group in this chapter, we really mean the F_v points of the underlying algebraic group.

The purpose of this chapter is to provide an identity between the local integrals $\mathcal{P}_v$ considered in Conjecture 1.3 and the $\mathcal{J}_v$ from Ichino's triple product formula. The seesaw diagram which motivates the identity is:



Here, V is a two-dimensional hermitian space, W is two-dimensional skew-hermitian spaces, and L is a one-dimensional hermitian space.

We fix representations for the groups on the 'bottom row' of the seesaw. That is, let π, μ, and σ be irreducible, cuspidal representations of $U(V), U(L)$, and $U(W)$, respectively. After fixing the appropriate splitting characters (which we suppress, for now), we also consider the Weil representation ω of $U(V \oplus L) \times U(W)$.

Let $\mathcal{B}_\pi, \mathcal{B}_\mu, \mathcal{B}_\sigma$ and $\mathcal{B}_\omega$ be pairings for the relevant representation. Inspired by the analogous global seesaw duality property, one hopes to consider matrix coefficients for σ, π, μ, and ω. Then, by showing that the integral

$$\int_{U(V)\times U(L)\times U(W)} \mathcal{B}_\pi(\pi(g)f_\pi, f_\pi)\mathcal{B}_\mu(\mu(\ell)f_\mu, f_\mu)\mathcal{B}_\sigma(\sigma(h)f_\sigma, f_\sigma) \\ \mathcal{B}_\omega(\omega((g,\ell)h)f_\omega, f_\omega)\ dg d\ell dh$$

converges absolutely, once can use Fubini's theorem to arrive at a local seesaw identity. On one side of this hypothetical local seesaw identity would be an integral of matrix coefficients for $\Theta(\sigma), \pi$, and μ, and on the other would be an integral of matrix coefficients for $\sigma, \Theta(\pi)$, and $\Theta(\mu)$. Alas, the convergence of the integral above does not hold. However, all is not lost. By ignoring μ and $U(L)$, and integrating only over $U(V) \times U(W)$, we obtain a convergent integral.

Proposition 6.1. *Suppose that π and σ are tempered. Then the integral*

$$\int_{U(V)\times U(W)} \mathcal{B}_\pi(\pi(g)f_\pi, f_\pi)\mathcal{B}_\sigma(\sigma(h)f_\sigma, f_\sigma)\mathcal{B}_\omega(\omega(g,h)f_\omega, f_\omega)dgdh$$

converges absolutely.

Proof. First, we suppose that E is a quadratic field extension of F. We also suppose that neither V nor W is anisotropic. We recall that we have Cartan decompositions

$$\begin{aligned} U(V) &= K_V M_V^+ K_V \\ U(W) &= K_W M_W^+ K_W \end{aligned}$$

where in this case we have

$$M_V^+ \cong M_W^+ \cong \{x \in E^\times : |x| \leq 1\}.$$

Following the proof of Proposition 2.1, we see that the integral above is reduced to the convergence of

$$\int_{M_V^+\times M_W^+} \mu_1(m_1)\mu_2(m_2)\mathcal{B}_\pi(\pi(m_1)f_\pi, f_\pi)\mathcal{B}_\sigma(\sigma(m_2)f_\sigma, f_\sigma)\mathcal{B}_\omega(\omega(m_1,m_2)f_\omega, f_\omega)dm_1dm_2$$

where

$$\mu_1(m) := \mathrm{Vol}(K_V m K_V)/\,\mathrm{Vol}(K_V)$$

for $m \in M_V^+$, and

$$\mu_2(m) := \mathrm{Vol}(K_W m K_W)/\,\mathrm{Vol}(K_W)$$

for $m \in M_W^+$. We know that $|\mu_1(m_1)| \leq A_1|m_1|^{-1}$ and $|\mu_2(m_2)| \leq A_2|m_2|^{-1}$, where A_1, A_2 are positive constants. Furthermore, since π and σ are tempered, we know that

$$|\mathcal{B}_\pi(\pi(m_1)f_\pi, f_\pi)| \leq C_1|m_1|^{1/2}(1 - \log|m_1|)^{r_1}$$

and

$$|\mathcal{B}_\sigma(\pi(m_2)f_\sigma, f_\sigma)| \leq C_2|m_2|^{1/2}(1 - \log|m_2|)^{r_2}$$

where the C_i and r_i are positive constants.

For any $x \in E^\times$, we set

$$\Upsilon(x) := \min(1, |x|^{-1}).$$

We recall that for $\phi, \phi' \in \mathcal{S}(E)$, there is some $C > 0$ such that for any $a \in E^\times$ we have

$$\left| \int_E \phi(ax)\overline{\phi'(x)}dx \right| \leq C \cdot \Upsilon(a).$$

Realizing ω on $\mathcal{S}(V \oplus L)$, we write $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in V \oplus L$ with $x_2 \in L, (x_1, x_3) \in V$, and note that $\{(x_1, 0)\}, \{(0, x_3)\} \subset V$ are isotropic lines. Then we have

$$\omega(m_1, m_2)\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \gamma(m_1)|m_1|^{3/2}\Phi\left(\begin{bmatrix} m_1m_2^{-1}x_1 \\ m_1x_2 \\ m_1\overline{m_2}x_3 \end{bmatrix}\right).$$

So, we see that there is a positive constant C such that for all $m_1 \in M_V^+$ and $m_2 \in M_W^+$ we have

$$|B_\omega(\omega(m_1, m_2))f_\omega, f_\omega| \leq C|m_1|^{3/2}\Upsilon(m_1m_2^{-1})\Upsilon(m_1)\Upsilon(m_1\overline{m_2}).$$

Note that for $|m_1|, |m_2| \leq 1$, we have $\Upsilon(m_1) = \Upsilon(m_1\overline{m_2}) = 1$.

So, we are reduced to checking the convergence of

$$\int_{|m_1|,|m_2|\leq 1} |m_1| \cdot |m_2|^{-1/2} \cdot \Upsilon(m_1m_2^{-1})(1 - \log|m_1|)^{r_1}(1 - \log|m_2|)^{r_2}d^\times m_1d^\times m_2.$$

When $|m_1| \geq |m_2|$, the integrand is

$$|m_2|^{1/2}(1-\log|m_1|)^{r_1}(1-\log|m_2|)^{r_2}$$

which is bounded above by, say

$$|m_1|^{1/4}|m_2|^{1/4}(1-\log|m_1|)^{r_1}(1-\log|m_2|)^{r_2}.$$

When $|m_1| \leq |m_2|$, the integrand is

$$|m_1|\cdot|m_2|^{-1/2}(1-\log|m_1|)^{r_1}(1-\log|m_2|)^{r_2}$$

which is also bounded above by

$$|m_1|^{1/4}|m_2|^{1/4}(1-\log|m_1|)^{r_1}(1-\log|m_2|)^{r_2}.$$

The convergence of

$$\int_{|m_1|,|m_2|\leq 1}|m_1|^{1/4}|m_2|^{1/4}(1-\log|m_1|)^{r_1}(1-\log|m_2|)^{r_2}d^\times m_1 d^\times m_2.$$

follows from Lemma 2.2.

Now we suppose that W is anisotropic, but V is not. In that case, we need only check the convergence of

$$\int_{|m_1|\leq 1}|m_1|(1-\log|m_1|)^{r_1}d^\times m_1,$$

which follows from Lemma 2.2.

If V is anisotropic, but W is not, then we are reduced to checking the convergence of

$$\int_{|m_2|\leq 1}|m_2|^{1/2}(1-\log|m_2|)^{r_2}d^\times m_2,$$

which also follows from Lemma 2.2.

If both V and W are anisotropic, then there is nothing to check.

We now assume that $E = F \times F$. In this case, we recall that both $U(V)$ and $U(W)$ are isomorphic to $GL_2(F)$. With the right choice of bases, we have

$$M_V^+ \cong M_W^+ \cong \{\mathrm{diag}(x,y) : |x| \leq |y|\}.$$

The proof in this case requires us to check many cases. Before we can make use of Lemma 2.2, we note that because both π and σ are tempered, we have constants $A, A' > 0$ such that

$$\begin{aligned} |\mathcal{B}_\pi(\pi(g)f_\pi, f_\pi)| &\leq A \cdot |g_1|^{1/2}|g_2|^{-1/2} \\ |\mathcal{B}_\sigma(\sigma(h)f_\sigma, f_\sigma)| &\leq A' \cdot |h_1|^{1/2}|h_2|^{-1/2} \end{aligned}$$

where $g = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \in M_V^+$ and $h = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \in M_W^+$. Recall that

$$\begin{aligned} \mu_1(g) &= \mathrm{Vol}(K_V g K_V)/\,\mathrm{Vol}(K_V) \\ \mu_2(2) &= \mathrm{Vol}(K_W h K_W)\,\mathrm{Vol}(K_W) \end{aligned}$$

and that we have constants $B, B' > 0$ such that

$$\begin{aligned} |\mu_1(g)| &\leq B \cdot |g_1|^{-1}|g_2| \\ |\mu_2(h)| &\leq B' \cdot |h_1|^{-1}|h_2|. \end{aligned}$$

Realizing ω on $\mathcal{S}(M_{2,3}(F))$, we have

$$\omega(g,h)\phi\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}\right) = \det(g)^{-3/2}\det(h)\phi\left(\begin{bmatrix} g_1^{-1}h_1x_1 & g_1^{-1}x_2 & g_1^{-1}h_2x_3 \\ g_2^{-1}h_1x_4 & g_2^{-1}x_5 & g_2^{-1}h_2x_3 \end{bmatrix}\right).$$

So, combining the various bounds mentioned above, we see that the integral whose convergence we must check is

$$\int_{|g_1|\leq|g_2|,|h_1|\leq|h_2|} \begin{aligned} &|g_1|^{-2}|g_2|^{-1}|h_1|^{1/2}|h_2|^{3/2}(1-\sum\log|g_i|)^r(1-\sum\log|h_i|)^s \\ &\Upsilon(h_1g_1^{-1})\Upsilon(h_1g_2^{-1})\Upsilon(h_2g_1^{-1})\Upsilon(h_2g_2^{-1})\Upsilon(g_1^{-1})\Upsilon(g_2^{-1}) \\ &d^\times g_1 d^\times g_2 d^\times h_1 d^\times h_2 \end{aligned}$$

where $r, s > 0$. We will cut the region of integration into thirty regions, and verify the convergence of the integral in each of these regions.

1. $1 \leq |g_1| \leq |g_2| \leq |h_1| \leq |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1-\sum\log|g_i|)^r(1-\sum\log|h_i|)^s.$$

2. $|g_1| \le 1 \le |g_2| \le |h_1| \le |h_2|$. In this region, the integrand is bounded by

$$|g_1||g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/2}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

3. $|g_1| \le |g_2| \le 1 \le |h_1| \le |h_2|$. In this region, the integrand is bounded by

$$|g_1||g_2|^2|h_1|^{-3/2}|h_2|^{-1/2}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

4. $|g_1| \le |g_2| \le |h_1| \le 1 \le |h_2|$. In this region, the integrand is bounded by

$$|g_1||g_2|^{1/4}|h_1|^{1/4}|h_2|^{-1/2}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

5. $|g_1| \le |g_2| \le |h_1| \le |h_2| \le 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

6. $1 \le |g_1| \le |h_1| \le |g_2| \le |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

7. $|g_1| \le 1 \le |h_1| \le |g_2| \le |h_2|$. In this region, the integrand is bounded by

$$|g_1||g_2|^{-1/4}|h_1|^{-1/2}|h_2|^{-1/4}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

8. $|g_1| \le |h_1| \le 1 \le |g_2| \le |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1/4}|h_1|^{1/4}|h_2|^{-1/4}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

9. $|g_1| \le |h_1| \le |g_2| \le 1 \le |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2||h_1|^{1/4}|h_2|^{-1/2}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

10. $|g_1| \le |h_1| \le |g_2| \le |h_2| \le 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1 - \sum \log|g_i|)^r(1 - \sum \log|h_i|)^s.$$

11. $1 \le |g_1| \le |h_1| \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

12. $|g_1| \le 1 \le |h_1| \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1||g_2|^{-1/4}|h_1|^{-1/2}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

13. $|g_1| \le |h_1| \le 1 \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1/4}|h_1|^{1/4}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

14. $|g_1| \le |h_1| \le |h_2| \le 1 \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1}|h_1|^{1/4}|h_2|^{1/2}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

15. $|g_1| \le |h_1| \le |h_2| \le |g_2| \le 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

16. $1 \le |h_1| \le |g_1| \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

17. $|h_1| \le 1 \le |g_1| \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1}|g_2|^{-1/4}|h_1|^{1/2}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

18. $|h_1| \le |g_1| \le 1 \le |h_2| \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1/4}|h_1|^{1/4}|h_2|^{-1/4}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

19. $|h_1| \le |g_1| \le |h_2| \le 1 \le |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1}|h_1|^{1/4}|h_2|^{1/2}(1-\sum \log|g_i|)^r(1-\sum \log|h_i|)^s.$$

20. $|h_1| \leq |g_1| \leq |h_2| \leq |g_2| \leq 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

21. $1 \leq |h_1| \leq |g_1| \leq |g_2| \leq |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

22. $|h_1| \leq 1 \leq |g_1| \leq |g_2| \leq |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1}|g_2|^{-1/4}|h_1|^{1/2}|h_2|^{-1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

23. $|h_1| \leq |g_1| \leq 1 \leq |g_2| \leq |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1/4}|h_1|^{1/4}|h_2|^{-1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

24. $|h_1| \leq |g_1| \leq |g_2| \leq 1 \leq |h_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2||h_1|^{1/4}|h_2|^{-1/2}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

25. $|h_1| \leq |g_1| \leq |g_2| \leq |h_2| \leq 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

26. $1 \leq |h_1| \leq |h_2| \leq |g_1| \leq |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1/4}|h_1|^{-1/4}|h_2|^{-1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

27. $|h_1| \leq 1 \leq |h_2| \leq |g_1| \leq |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-1/4}|g_2|^{-1}|h_1|^{1/2}|h_2|^{-1/4}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

28. $|h_1| \leq |h_2| \leq 1 \leq |g_1| \leq |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{-2}|g_2|^{-1}|h_1|^{1/2}|h_2|^{3/2}(1 - \sum \log |g_i|)^r(1 - \sum \log |h_i|)^s.$$

29. $|h_1| \leq |h_2| \leq |g_1| \leq 1 \leq |g_2|$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{-1}|h_1|^{1/4}|h_2|^{1/2}(1-\sum \log |g_i|)^r(1-\sum \log |h_i|)^s.$$

30. $|h_1| \leq |h_2| \leq |g_2| \leq |g_2| \leq 1$. In this region, the integrand is bounded by

$$|g_1|^{1/4}|g_2|^{1/4}|h_1|^{1/4}|h_2|^{1/4}(1-\sum \log |g_i|)^r(1-\sum \log |h_i|)^s.$$

By applying Lemma 2.2 in each of these cases, we see that the integral converges.

□

Having settled the issue of convergence, our local seesaw identity follows from Fubini's theorem. In addition to the pairings already discussed, we consider pairings $\mathcal{B}_{\Theta(\sigma)}, \mathcal{B}_{\Theta(\pi)}$, and $\mathcal{B}_{\Theta(\mu)}$ which are 'inherited' from the pairings $\mathcal{B}_\sigma, \mathcal{B}_\pi, \mathcal{B}_\mu, \mathcal{B}_\omega$ and the local theta-correspondence. For $\tau = \mu, \pi, \sigma$, we have surjective maps

$$\omega \to \tau \boxtimes \Theta(\tau)$$

(defined up to scaling) where $\Theta(\tau)$ is the 'big' theta-lift from the previous chapter. This induces a map

$$\Theta : \tau^\vee \otimes \omega \to \Theta(\tau).$$

Note that for the τ considered above, we have $\tau^\vee = \bar{\tau}$ since τ is unitary. Now, we set

$$\begin{aligned}
\mathcal{B}_{\Theta(\mu)}(\Theta(f_1,\varphi_1),\Theta(f_2,\varphi_2)) &:= \int_{U(L)} \overline{\mathcal{B}_\mu(\mu(z)f_1,f_2)}\mathcal{B}_\omega(\omega(z)\varphi_1,\varphi_2)dz \\
\mathcal{B}_{\Theta(\pi)}(\Theta(f_1,\varphi_1),\Theta(f_2,\varphi_2)) &:= \int_{U(V)} \overline{\mathcal{B}_\pi(\pi(g)f_1,f_2)}\mathcal{B}_\omega(\omega(g)\varphi_1,\varphi_2)dg \\
\mathcal{B}_{\Theta(\sigma)}(\Theta(f_1,\varphi_1),\Theta(f_2,\varphi_2)) &:= \int_{U(W)} \overline{\mathcal{B}_\sigma(\sigma(h)f_1,f_2)}\mathcal{B}_\omega(\omega(h)\varphi_1,\varphi_2)dh
\end{aligned}$$

where $f_i \in \mu, \pi$ or σ, respectively, and $\varphi_i \in \omega$.

We emphasize that the pairings above are defined on the 'big' theta-lifts. In order to make use of the following theorem, some work is required to show that these pairings descend to pairings on the 'small theta-lifts. This will be addressed in the following chapter.

Theorem 6.2. *Let $Z_W \subset U(W)$ denote the center. Then with the pairings as above, we have*

$$\int_{U(V)} \mathcal{B}_{\Theta(\sigma)}(\Theta(\sigma)(g)\Theta(f_\sigma,\varphi),\Theta(f_\sigma,\varphi))\overline{\mathcal{B}_\pi(\pi(g)f_\pi,f_\pi)}dg$$
$$= \int_{Z_W\backslash U(W)} \mathcal{B}_{\Theta(\pi)}(\Theta(\pi)(h)\Theta(f_\pi,\varphi),\Theta(f_\pi,\varphi))\mathcal{B}_{\Theta(\mu)}(\Theta(\mu)(h)\Theta(f_\mu,\varphi),\Theta(f_\mu,\varphi))$$
$$\overline{\mathcal{B}_\sigma(\sigma(h)f_\sigma,f_\sigma)}dh.$$

Proof. We start with

$$\int_{U(V)\times U(W)} \overline{\mathcal{B}_\pi(\pi(g)f_\pi,f_\pi)\mathcal{B}_\sigma(\sigma(h)f_\sigma,f_\sigma)}\mathcal{B}_\omega(\omega(g,h)f_\omega,f_\omega)dgdh.$$

By Proposition 6.1, we may view the integral above as an iterated integral. By first integrating out the $U(W)$ variable, we have that the above is equal to

$$\int_{U(V)} \mathcal{B}_{\Theta(\sigma)}(\Theta(\sigma)(g)\Theta(f_\sigma,\varphi),\Theta(f_\sigma,\varphi))\overline{\mathcal{B}_\pi(\pi(g)f_\pi,f_\pi)}dg.$$

Before we proceed to integrate the $U(V)$ variable, we remind the reader that ω is a representation of $U(V\oplus L)\times U(W)$ realized on the space of Schwartz functions $\mathcal{S}(V\oplus L)\cong\mathcal{S}(V)\otimes\mathcal{S}(L)$. Noting this decomposition, we see that the pairing $\mathcal{B}_\omega$ can be written as the product $\mathcal{B}_{\omega,V}\cdot\mathcal{B}_{\omega,L}$.

If we instead integrate out the $U(V)$ variable, we obtain

$$\int_{U(W)} \mathcal{B}_{\Theta(\pi)}(\Theta(\pi)(h)\Theta(f_\pi,\varphi),\Theta(f_\pi,\varphi))\overline{\mathcal{B}_\sigma(\sigma(h)f_\sigma,f_\sigma)}\mathcal{B}_{\omega,L}(\omega(h)f_{\omega,L},f_{\omega,L})dh,$$

which we rewrite as

$$\int_{Z_W\backslash U(W)} \mathcal{B}_{\Theta(\pi)}(\Theta(\pi)(h)\Theta(f_\pi,\varphi),\Theta(f_\pi,\varphi))\overline{\mathcal{B}_\sigma(\sigma(h)f_\sigma,f_\sigma)}$$
$$\int_{Z_W} \omega_\pi(z)\omega_\sigma^{-1}(z)\mathcal{B}_{\omega,L}(\omega(zh)f_{\omega,L},f_{\omega,L})dz\ dh.$$

Noting that $\omega_\pi\omega_\sigma^{-1}=\mu^{-1}$, we see that this is equal to

$$\int_{Z_W\backslash U(W)} \mathcal{B}_{\Theta(\pi)}(\Theta(\pi)(h)\Theta(f_\pi,\varphi),\Theta(f_\pi,\varphi))\overline{\mathcal{B}_\sigma(\sigma(h)f_\sigma,f_\sigma)}$$
$$\int_{Z_W} \overline{\mu(z)}\mathcal{B}_{\omega,L}(\omega(zh)f_{\omega,L},f_{\omega,L})dz\ dh.$$

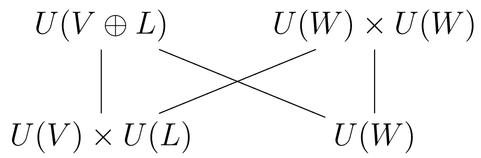
Finally, we see that the inner integral gives the pairing $\mathcal{B}_{\Theta(\mu)}$ on $\Theta(\mu)$. This completes the proof. $\square$

7 The Refined Gross-Prasad Conjecture for $U(2) \times U(3)$

With everything that we've developed so far, we can now prove Conjecture 1.3 for $n = 2$, provided that $\pi_{n+1} = \Theta(\bar{\sigma})$, where σ is a cuspidal, irreducible automorphic representation of $U(2)$. (The theta-lift is $\Theta(\bar{\sigma})$ because of our normalization of the theta correspondence, and the fact that the seesaw identities we use do not involve complex conjugation.) We will employ the various Rallis inner product formulae developed in Chapter 5, as well as Ichino's triple product formula from Chapter 4.

7.1 The Setup

We remind the reader of the following seesaw diagram:



(7.1)

Here, V is a 2-dimensional hermitian space over E/F, W is a 2 dimensional skew-hermitian space over E/F, and L is a hermitian line over E/F. Using the theory of Θ-correspondence and seesaw duality, we will relate the period integral in Conjecture 1.3 (with $n = 2$) to the so-called triple product integral considered by Ichino in [20].

We fix the following:

- π is an irreducible, cuspidal, tempered, automorphic representation of $U(V)(\mathbb{A}_F)$.
- σ is an irreducible, cuspidal, tempered, automorphic representation of $U(W)(\mathbb{A}_F)$.
- $\mu := \omega_\sigma \omega_\pi^{-1}$ is an automorphic character of $U(L)(\mathbb{A}_F)$, where ω_σ and ω_π are the central characters of σ and π, respectively.
- $(\omega_\psi, \mathcal{S})$ is a Weil representation of $\widetilde{Sp}(\mathbb{W})(\mathbb{A}_F)$. (See Chapter 5 for notation.)

We also fix local pairings $\mathcal{B}_{\pi_v}, \mathcal{B}_{\sigma_v}, \mathcal{B}_{\mu_v}$ such that $\prod_v \mathcal{B}_{\pi_v}, \prod_v \mathcal{B}_{\sigma_v}$ and $\prod_v \mathcal{B}_{\mu_v}$ give the respective Petersson inner products on the global representation.

After fixing splitting data $(\gamma_V, \gamma_L, \gamma_W)$ as in Chapter 5, we consider $\Theta(\bar{\pi}) := \Theta_{V,W\gamma_W,\gamma_V,\psi}(\bar{\pi})$ on $U(W)(\mathbb{A}_F)$, $\Theta(\bar{\sigma}) := \Theta_{W,V\oplus L,\gamma_V\gamma_L,\gamma_W,\psi}(\bar{\sigma})$ on $U(V\oplus L)(\mathbb{A}_F)$, and $\Theta(\bar{\mu}) := \Theta_{L,W\gamma_W,\gamma_L,\psi}(\bar{\mu})$ on $U(W)(\mathbb{A}_F)$. We take $\gamma_W, \gamma_V = \gamma^2$ and $\gamma_L = \gamma$, where γ is a character of $\mathbb{A}_E^\times/E^\times$ such that $\gamma|_{\mathbb{A}_F^\times} = \chi_{E/F}$. We assume that these Θ-lifts are cuspidal.

By using Ichino's triple product formula from Chapter 4, the various Rallis Inner Product formulae from Chapter 5, the explicit local seesaw identity from Chapter 6, and some L-function identities in the appendix, we can establish the Refined Gross-Prasad Conjecture for $n = 2$ with $\pi_2 = \pi$ and $\pi_3 = \Theta(\bar{\sigma})$:

Theorem 7.1. *Let μ, π, σ and $\Theta(\bar{\sigma})$ be as above. Let $f_{\Theta(\bar{\sigma})} \in \Theta(\bar{\sigma})$ and $f_\pi \in \pi$ be cusp forms such that $f_{\Theta(\bar{\sigma})} = \otimes_v f_{\Theta(\bar{\sigma})_v}$ and $f_\pi = \otimes_v f_{\pi_v}$. Then*

$$\mathcal{P}(f_{\Theta(\bar{\sigma})}, f_\pi) = \frac{\Delta_{G_3} L_E(1/2, BC(\Theta(\bar{\sigma})) \boxtimes BC(\pi))}{|S_{\psi_{\Theta(\bar{\sigma})}}| \cdot |S_{\psi_\pi}| L_F(1, \Theta(\bar{\sigma}), \mathrm{Ad}) L_F(1, \pi, \mathrm{Ad})} \prod_v \mathcal{P}_v(f_{\Theta(\bar{\sigma})_v}, f_{\pi_v}).$$

Here, $\psi_{\Theta(\bar{\sigma})}$ and ψ_π are the L-parameters of $\Theta(\bar{\sigma})$ and π, and $S_{\psi_{\Theta(\bar{\sigma})}}$ and S_{ψ_π} are the associated component groups.

The proof of the above will occupy the rest of this chapter.

7.2 Proof of Theorem 7.1

The proof of Theorem 7.1 involves using both a global and local seesaw identity, as well as all of the various Rallis Inner Product Formulae.

The first global seesaw identity that we need is

$$\mathcal{P}' \circ \mathcal{T}_1' = \mathcal{I} \circ \mathcal{T}_2' \tag{7.2}$$

where the maps are defined as follows:

$$\mathcal{T}_1' : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\mu \boxtimes \bar{V}_\mu) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\mathcal{S} \boxtimes \bar{\mathcal{S}}) \to (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\mu \boxtimes \bar{V}_\mu) \otimes (V_{\Theta(\bar{\sigma})} \boxtimes \bar{V}_{\Theta(\bar{\sigma})})$$

is the map induced by the global theta integral for σ. Similarly,

$$\mathcal{T}_2' : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\mu \boxtimes \bar{V}_\mu) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\mathcal{S} \boxtimes \bar{\mathcal{S}}) \to (V_{\Theta(\bar{\pi})} \boxtimes \bar{V}_{\Theta(\bar{\pi})}) \otimes (V_{\Theta(\bar{\mu})} \boxtimes \bar{V}_{\Theta(\bar{\mu})}) \otimes (V_\sigma \boxtimes \bar{V}_\sigma)$$

is the map induced by the global theta integrals for π and μ. Also, the global period

$$\mathcal{P}' : (V_{\Theta(\bar{\sigma})} \boxtimes \bar{V}_{\Theta(\bar{\sigma})}) \otimes (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\mu \boxtimes \bar{V}_\mu) \to \mathbb{C}$$

is defined by

$$\begin{aligned} \mathcal{P}'(f_{\Theta(\bar{\sigma})}, \bar{f}'_{\Theta(\bar{\sigma})}, f_\pi, \bar{f}'_\pi, f_\mu, \bar{f}'_\mu) &:= \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}((g_1, g_2)) f_\pi(g_1) f_\mu(g_2) dg_1 dg_2 \\ &\quad \times \int_{[U(V)\times U(L)]} \overline{f'_{\Theta(\bar{\sigma})}((g_1, g_2)) f'_\pi(g_1) f'_\mu(g_2)} dg_1 dg_2. \end{aligned}$$

Here, we view $(g_1, g_2) \in U(V \oplus L)$ via the natural embedding $U(V) \times U(L) \hookrightarrow U(V \oplus L)$. Finally, we consider the map

$$\mathcal{I} : (V_{\Theta(\bar{\pi})} \boxtimes \bar{V}_{\Theta(\bar{\pi})}) \otimes (V_{\Theta(\bar{\mu})} \boxtimes \bar{V}_{\Theta(\bar{\mu})}) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \to \mathbb{C}$$

which is given by

$$\begin{aligned} \mathcal{I}(f_{\Theta(\bar{\pi})}, \bar{f}'_{\Theta(\bar{\pi})}, f_{\Theta(\bar{\mu})}, \bar{f}'_{\Theta(\bar{\mu})}, f_\sigma, \bar{f}'_\sigma) &:= \int_{[U(W)]} f_{\Theta(\bar{\pi})}(g) f_{\Theta(\bar{\mu})}(g) f_\sigma(g) dg \\ &\quad \times \int_{[U(W)]} \overline{f'_{\Theta(\bar{\pi})}(g) f'_{\Theta(\bar{\mu})}(g) f'_\sigma(g)} dg \end{aligned}$$

and is closely related to Ichino's triple product integral.

We follow the convention of Chapter 1 and set

$$\mathcal{P}'(f_{\Theta(\bar{\sigma})}, f_\pi, f_\mu) := \mathcal{P}'(f_{\Theta(\bar{\sigma})}, \bar{f}_{\Theta(\bar{\sigma})}, f_\pi, \bar{f}_\pi, f_\mu, \bar{f}_\mu).$$

We follow a similar convention for $\mathcal{I}$ and set

$$\mathcal{I}(f_{\Theta(\bar{\pi})}, f_{\Theta(\bar{\mu})}, f_\sigma) := \mathcal{I}(f_{\Theta(\bar{\pi})}, \bar{f}_{\Theta(\bar{\pi})}, f_{\Theta(\bar{\mu})}, \bar{f}_{\Theta(\bar{\mu})}, f_\sigma, \bar{f}_\sigma).$$

We note that $\mathcal{P}'$ is not quite the period integral in Theorem 7.1; however, the two are related by a constant. Indeed, if we denote by $\mathcal{P}$ the LHS of Theorem 7.1, then we have the following:

Lemma 7.2.

$$\mathcal{P}'(f_{\Theta(\bar{\sigma})}, f_\pi, \mu) = 4 \cdot \mathcal{P}(f_{\Theta(\bar{\sigma})}, f_\pi).$$

Proof. We see that by the change of variables $g_1 \mapsto g_1 g_2$ we have

$$\begin{aligned}&\int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}((g_1, g_2)) f_\pi(g_1)\mu(g_2) dg_1 dg_2 \\ &= \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}((g_1 g_2, g_2)) f_\pi(g_1 g_2)\mu(g_2) dg_1 dg_2.\end{aligned}$$

Note that $\Theta(\bar{\sigma})$ has central character $\omega_{\Theta(\bar{\sigma})} = \omega_{\Theta(\sigma)}^{-1} = \omega_\sigma^{-1}$. So, after observing that (g_2, g_2) is in the center of $U(V \oplus L)$ and g_2 is in the center of $U(V)$, we have

$$\begin{aligned}&\int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}((g_1 g_2, g_2)) f_\pi(g_1 g_2)\mu(g_2) dg_1 dg_2 \\ &= \int_{[U(V)\times U(L)]} \omega_{\Theta(\bar{\sigma})}(g_2)\omega_\pi(g_2)\mu(g_2) f_{\Theta(\bar{\sigma})}|_{U(V)}(g_1) f_\pi(g_1) dg_1 dg_2 \\ &= \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g_1) f_\pi(g_1) dg_1 dg_2 \\ &= \mathrm{Vol}([U(L)]) \int_{[U(V)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_\pi(g) dg \\ &= 2\int_{[U(V)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_\pi(g) dg.\end{aligned}$$

□

With this, we can restate our global seesaw identity as

$$4 \cdot \mathcal{P} \circ \mathcal{T}_1 = \mathcal{I} \circ \mathcal{T}_2 \tag{7.3}$$

where

$$\mathcal{T}_1 : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\mathcal{S} \boxtimes \bar{\mathcal{S}}) \to (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\Theta(\bar{\sigma})} \boxtimes \bar{V}_{\Theta(\bar{\sigma})})$$

is the map induced by the global theta-lift of σ, and

$$\mathcal{T}_2 : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\mathcal{S} \boxtimes \bar{\mathcal{S}}) \to (V_{\Theta(\bar{\pi})} \boxtimes \bar{V}_{\Theta(\bar{\pi})}) \otimes (V_{\Theta(\bar{\mu})} \boxtimes \bar{V}_{\Theta(\bar{\mu})}) \otimes (V_\sigma \boxtimes \bar{V}_{\bar{\sigma}})$$

is the map induced from $\mathcal{T}_2'$ by fixing the argument $\mu \otimes \bar{\mu} \in V_\mu \boxtimes \bar{V}_\mu$. Let $\mathcal{T}_{i,v}$ be maps such that $\mathcal{T}_i = \otimes_v \mathcal{T}_{i,v}$.

Before proceeding, we note that $\Theta(\bar{\mu})$ is dihedral with respect to E/F. Indeed, we have $BC(\Theta(\bar{\mu})) \cong \pi(\gamma^{-1}BC(\bar{\mu}), \gamma)$, the principal series representation of $GL_2(\mathbb{A}_E)$. By Corollary 4.7 and line 7.3 we have

$$\begin{aligned} \mathcal{P} \circ \mathcal{T}_1 \quad = \quad & \frac{\zeta_F(2)^2 L_F(1, \chi_{E/F})^3}{8 \cdot |X(\Theta(\bar{\pi})| \cdot |X(\sigma)| \cdot |X(\Theta(\bar{\mu}))|} \times \\ & \frac{L_E(1/2, BC(\Theta(\bar{\pi})) \boxtimes BC(\sigma) \boxtimes \gamma^{-1})}{L_F(1, \Theta(\bar{\pi}), \mathrm{Ad}) L_F(1, \sigma, \mathrm{Ad}) L_F(1, \Theta(\bar{\mu}), \mathrm{Ad})} \prod_v \mathcal{I}_v \circ \mathcal{T}_{2,v} \end{aligned} \tag{7.4}$$

where the $\mathcal{I}_v$ are defined with suitably chosen local pairings as in Chapter 4.

The next step in the argument is to use the Rallis inner product formulae from Chapter 5 as well as the local seesaw identity from Chapter 6. However, note that the matrix coefficients in Chapter 6 are attached to the 'big' theta-lifts. Before we can make use of our local seesaw identity, we must prove a lemma which allows us to relate local integrals of matrix coefficients of 'big' theta-lifts to those for the corresponding 'small' theta-lifts.

Recall that in Chapter 6 we considered the following pairings $\mathcal{B}_{\Theta(\bar{\tau}_v)}$ on the big local theta-lifts $\Theta(\bar{\tau}_v)$ for $\tau = \mu, \pi, \sigma$:

$$\mathcal{B}_{\Theta(\bar{\tau}_v)}(\Theta(\bar{f}_{1,v}, \varphi_{1,v}), \Theta(\bar{f}_{2,v}, \varphi_{2,v})) := \int_{G_v} \mathcal{B}_{\omega_v}(\omega_v(g_v)\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\tau_v}(\tau_v(g_v) f_{1,v}, f_{2,v}) dg_v.$$

Here, G_v is $U(1)_v$ if $\tau = \mu$, and $G_v = U(2)_v$ if $\tau = \pi$ or $\tau = \sigma$. Now, we observe that

$$\mathcal{B}_{\Theta(\bar{\tau}_v)}(\Theta(\bar{f}_{1,v}, \varphi_{1,v}), \Theta(\bar{f}_{2,v}, \varphi_{2,v})) = Z_v(s_0, f_{1,v}, f_{2,v}, \Phi_{s_0,v}, \chi_v)$$

where $s_0 = 0$ if $\tau = \pi$ and $s_0 = 1/2$ if $\tau = \mu$ or $\tau = \sigma$, $\Phi_{s,v} = \delta(\varphi_{1,v}, \varphi_{2,v})$, and χ_v is the appropriate power of γ_v. Fix an isomorphism $\Theta(\bar{\tau}) \cong \otimes_v \theta(\bar{\tau}_v)$. The observation above and the Rallis inner product formulae from Chapter 5 give

$$\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2) \rangle_{\Theta(\bar{\tau})} = \prod_v \mathcal{B}_{\Theta(\bar{\tau}_v)}(\Theta(\bar{f}_{1,v}, \varphi_{1,v}), \Theta(\bar{f}_{2,v}, \varphi_{2,v})). \tag{7.5}$$

Since we are assuming that the $\Theta(\bar{\tau})$ are cuspidal, and therefore semisimple, we know that $\langle \theta(\bar{f}_1, \varphi_1), \theta(\bar{f}_2, \varphi_2) \rangle_{\Theta(\bar{\tau})}$ factors as a map

$$\bar{\tau} \times \omega \times \tau \times \bar{\omega} \to \Theta(\bar{\tau}) \times \overline{\Theta(\bar{\tau})}$$

composed with the Petersson inner product on $\Theta(\bar{\tau})$. This, along with 7.5 above implies that $\mathcal{B}_{\Theta(\bar{\tau}_v)}$ descends to a pairing on the small theta-lift.

So, at each place v, we are entitled to define the following pairings on the small theta-lifts:

$$\mathcal{B}^\sharp_{\theta(\bar{\sigma}_v)}(\theta(\bar{f}_{1,v}, \varphi_{1,v}), \theta(\bar{f}_{2,v}, \varphi_{2,v})) := \\ \left(\frac{L_{E_v}(1, BC(\sigma_v) \otimes \gamma_v^3)}{\zeta_{F_v}(2) L_{F_v}(3, \chi_{E_v/F_v})} \right)^{-1} \int_{U(W)_v} \mathcal{B}_{\omega_v}(\omega_v(h)\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\sigma_v}(\sigma_v(h) f_{1,v}, f_{2,v}) dh,$$

for $\varphi_{i,v} \in \omega_v$ and $f_{i,v} \in \sigma_v$,

$$\mathcal{B}^\sharp_{\theta(\bar{\pi}_v)}(\theta(\bar{f}_{1,v}, \varphi_{1,v}), \theta(\bar{f}_{2,v}, \varphi_{2,v})) := \\ \left(\frac{L_{E_v}(1/2, BC(\pi_v) \otimes \gamma_v^2)}{L_{F_v}(1, \chi_{E_v/F_v}) \zeta_{F_v}(2)} \right)^{-1} \int_{U(V)_v} \mathcal{B}_{\omega_v}(\omega_v(g)\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\pi_v}(\pi_v(g) f_{1,v}, f_{2,v}) dg,$$

for $\varphi_{i,v} \in \omega_v$ and $f_{i,v} \in \pi_v$,

$$\mathcal{B}^\sharp_{\theta(\bar{\mu}_v)}(\theta(\bar{f}_{1,v}, \varphi_{1,v}), \theta(\bar{f}_{2,v}, \varphi_{2,v})) := \\ \left(\frac{L_{E_v}(1, BC(\mu_v) \otimes \gamma_v^2)}{\zeta_{F_v}(2)} \right)^{-1} \int_{U(L)} \mathcal{B}_{\omega_v}(\omega_v(g), \varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\mu_v}(\mu_v(g) f_{1,v}, f_{2,v}) dg,$$

for $\varphi_{i,v} \in \omega_v$ and $f_{i,v} \in \mu_v$. We remark that we have made normalizations so that the parings take value 1 for unramified data.

With these in place, we set $\mathcal{I}_v^\sharp(f_{\sigma_v}, f_{\theta(\bar{\pi}_v)}, f_{\theta(\bar{\mu}_v)}) := \mathcal{I}_v^\sharp$ where

$$\begin{aligned} \mathcal{I}_v^\sharp \quad := \quad & \left(\frac{\zeta_{F_v}(2)^2 L_{E_v}(1/2, BC(\bar{\sigma}_v) \boxtimes BC(\pi_v) \boxtimes \gamma_v^{-1}) L_{F_v}(1, \chi_{E_v/F_v})^3}{L_{F_v}(1, \sigma_v, \mathrm{Ad}) L_{F_v}(1, \theta(\bar{\pi}_v), \mathrm{Ad}) L_{F_v}(1, \theta(\bar{\mu}_v), \mathrm{Ad})} \right)^{-1} \\ & \int_{Z_W \backslash U(W)_v} \mathcal{B}_{\sigma_v}(\sigma(g_v) f_{\sigma_v}, f_{\sigma_v}) \\ & \mathcal{B}^\sharp_{\theta(\bar{\pi}_v)}(\theta(\bar{\pi}_v)(g) f_{\theta(\bar{\pi}_v)}, f_{\theta(\bar{\pi}_v)}) \mathcal{B}^\sharp_{\theta(\bar{\mu}_v)}(\theta(\bar{\mu}_v)(g_v) f_{\theta(\bar{\mu}_v)}, f_{\theta(\bar{\mu}_v)}) dg_v. \end{aligned}$$

We also set $\mathcal{P}_v^\sharp(f_{\theta(\bar{\sigma}_v)}, f_{\pi_v}) := \mathcal{P}_v^\sharp$ where

$$\begin{aligned} \mathcal{P}_v^\sharp \quad := \quad & \left(\Delta_{G_{3,v}} \frac{L_{E_v}(1/2, BC(\theta(\bar{\sigma}_v)) \boxtimes BC(\pi_v))}{L_{F_v}(1, \theta(\bar{\sigma}_v), \mathrm{Ad}) L_{F_v}(1, \pi_v, \mathrm{Ad})} \right)^{-1} \\ & \int_{U(V)_v} \mathcal{B}^\sharp_{\theta(\bar{\sigma}_v)}(\theta(\bar{\sigma}_v)(g_v) f_{\theta(\bar{\sigma}_v)}, f_{\theta(\bar{\sigma}_v)}) \mathcal{B}_{\pi_v}(\pi_v(g_v) f_{\pi_v}, f_{\pi_v}) dg_v. \end{aligned}$$

Once again, the normalizations made in the definitions above ensure that the functionals $\mathcal{P}_v^\sharp$ and $\mathcal{I}_v^\sharp$ take value 1 for unramified data.

The local seesaw identity of the previous chapter, along with the L-function identities in the appendix give:

$$\mathcal{P}_v^\sharp \circ \mathcal{T}_{1,v} = \mathcal{I}_v^\sharp \circ \mathcal{T}_{2,v}. \tag{7.6}$$

However, neither side of line 7.6 appears in line 7.4. This is where we use two of the three Rallis inner product formulae from Chapter 5. Indeed, by Theorems 5.10 and 5.11, we have:

$$\prod_v \mathcal{I}_v \circ \mathcal{T}_{2,v} = |X(\Theta(\bar{\mu}))| \frac{L_E(1, BC(\mu) \otimes \gamma^2) L_E(1/2, BC(\pi) \otimes \gamma^2)}{\zeta_F(2)^2 L_F(1, \chi_{E/F})} \prod_v \mathcal{I}_v^\sharp \circ \mathcal{T}_{2,v}.$$

So, using this, along with lines 7.4 and 7.6, we have

$$\begin{aligned} \mathcal{P} \circ \mathcal{T}_1 \quad = \quad & \frac{L_F(1, \chi_{E/F})^2 L_E(1/2, BC(\Theta(\pi)) \boxtimes BC(\sigma) \boxtimes \gamma^{-1})}{8 \cdot |X(\Theta(\bar{\pi}))| \cdot |X(\sigma)|} \\ & \times \frac{L_E(1, BC(\mu) \otimes \gamma^2) L_E(1/2, BC(\pi) \otimes \gamma^2)}{L_F(1, \sigma, \mathrm{Ad}) L_F(1, \Theta(\bar{\pi}), \mathrm{Ad}) L_F(1, \Theta(\bar{\mu}), \mathrm{Ad})} \\ & \times \prod_v \mathcal{P}_v^\sharp \circ \mathcal{T}_{1,v}. \end{aligned}$$

To finish the proof, we need to use the third Rallis inner product formula to replace the equation above with one involving $\prod_v \mathcal{P}_v \circ \mathcal{T}_{1,v}$ instead of $\prod_v \mathcal{P}_v^\sharp \circ \mathcal{T}_{1,v}$. From Theorem 5.15, we have

$$\prod_v \mathcal{P}_v \circ \mathcal{T}_{1,v} = \frac{L_E(1, BC(\sigma) \otimes \gamma^3)}{\zeta_F(2) L_F(3, \chi_{E/F})} \prod_v \mathcal{P}_v^\sharp \circ \mathcal{T}_{1,v}.$$

This, along with the L-function identities in the appendix gives

$$\mathcal{P} \circ \mathcal{T}_1 = \frac{\Delta_{G_3}}{8 \cdot |X(\Theta(\bar{\pi}))| \cdot |X(\sigma)|} \cdot \frac{L_E(1/2, BC(\Theta(\bar{\sigma})) \boxtimes BC(\pi))}{L_F(1, \Theta(\bar{\sigma}), \mathrm{Ad}) L_F(1, \pi, \mathrm{Ad})} \prod_v \mathcal{P}_v \circ \mathcal{T}_{1,v}.$$

Finally, we note that

$$8 \cdot |X(\Theta(\bar{\pi}))| \cdot |X(\sigma)| = |S_{\psi_{\Theta(\bar{\sigma})}}| \cdot |S_{\psi_\pi}|,$$

which completes the proof.

A L-function identities

We collect several L-function identities used above. In particular, by using known relationships between L-parameters of representations and their Θ-lifts, we compare some associated L-functions.

As in Chapter 7, we take σ and π to be irreducible, tempered, cuspidal automorphic representations of $U(2)$, and $\mu = \omega_\sigma \omega_\pi^{-1}$. As in Chapter 7, we fix splitting characters which are powers of a fixed character γ, and consider the Θ-lifts $\Theta(\bar{\pi})$ and $\Theta(\bar{\mu})$ on $U(2)$, and $\Theta(\bar{\sigma})$ on $U(3)$. Once again, we remind the reader that the need to consider $\Theta(\bar{\pi}), \Theta(\bar{\sigma})$, and $\Theta(\bar{\mu})$ comes from our normalization of the theta correspondence, and the fact that in the seesaw identities we used, there was no complex conjugation.

The first identity we record is the following:

Proposition A.1.

$$L_F(s, \Theta(\bar{\pi}), \mathrm{Ad}) = L_F(s, \pi, \mathrm{Ad}).$$

Proof. By Theorem 11.2 in [4], we see that $\bar{\pi}$ and $\Theta(\bar{\pi})$ have the same L-parameters. The result above then follows from the fact that $L_F(s, \bar{\pi}, \mathrm{Ad}) = L_F(s, \pi, \mathrm{Ad})$. $\square$

Now we compute the adjoint L-function for $\Theta(\sigma)$:

Proposition A.2.

$$L_F(s, \Theta(\bar{\sigma}), \mathrm{Ad}) = L_F(s, \chi_{E/F}) L_F(s, \sigma, \mathrm{Ad}) L_E(s, BC(\sigma) \otimes \gamma^3).$$

Proof. Let M and N denote the restrictions of the L-parameters of $\bar{\sigma}$ and $\Theta(\bar{\sigma})$ to $WD(E)$. By Theorem 8.1 in [5], this restriction inflicts no loss of information.

From [7], we have that M and N are related as follows:

$$N = \gamma^{-1} M \oplus \gamma^2.$$

The result follows from this and the fact that $L_F(s, \bar{\sigma}, \mathrm{Ad}) = L_F(s, \sigma, \mathrm{Ad})$. $\square$

Finally, we compute the adjoint L-function for $\Theta(\mu)$:

Proposition A.3.

$$L_F(s, \Theta(\bar{\mu}), \mathrm{Ad}) = L_F(s, \chi_{E/F})^2 L_E(s, BC(\mu) \otimes \gamma^2).$$

Proof. Let M and N be as in the previous proposition. Then it is easy to see that once again we have

$$N = \gamma^{-1} M \oplus \gamma.$$

As before, the result follows. $\square$

The last identity we record is one that relates the L-function in Ichino's triple product formula to L-functions attached to representations of $U(2)$. In the notation of Chapter 5, we have $\tau_1 = BC(\Theta(\bar{\pi})), \tau_2 = BC(\sigma)$, and $\tau_3 = BC(\Theta(\bar{\mu}))$.

Proposition A.4.

$$L_F(s, \Sigma') = L_E(s, BC(\pi) \boxtimes BC(\bar{\sigma}) \boxtimes \gamma^{-1}).$$

Proof. We compare L-parameters. Let $N_{\Theta(\bar{\pi})}, N_\sigma$, and $N_{\Theta(\bar{\mu})}$ denote the L-parameters for $\Sigma'_{\Theta(\bar{\pi})}, \Sigma'_\sigma$, and $\Sigma'_{\Theta(\bar{\mu})}$, respectively. Now, since $\Sigma_{\Theta(\bar{\mu})}$ is dihedral with respect to E/F, we know that

$$N_{\Theta(\bar{\mu})} = \mathrm{Ind}_{WD(E)}^{WD(F)} M$$

for some one-dimensional representation M of $WD(E)$. Now observe that we have the following:

$$N_{\Theta(\bar{\pi})} \otimes N_\sigma \otimes \mathrm{Ind}_{WD(E)}^{WD(F)} M = \mathrm{Ind}_{WD(E)}^{WD(F)} \left(N_{\Theta(\bar{\pi})}|_{WD(E)} \otimes N_\sigma|_{WD(E)} \otimes M \right).$$

Now, identifying automorphic representations with their L-parameters, we have

$$\begin{aligned} L_F(s,\Sigma') &= L_F(s, N_{\Theta(\bar{\pi})} \otimes N_\sigma \otimes \mathrm{Ind}_{WD(E)}^{WD(F)} M) \\ &= L_F\left(s, \mathrm{Ind}_{WD(E)}^{WD(F)}\left(N_{\Theta(\bar{\pi})}|_{WD(E)} \otimes N_\sigma|_{WD(E)} \otimes M\right)\right) \\ &= L_F\left(s, \mathrm{Ind}_{WD(E)}^{WD(F)}\left(N_{\Theta(\bar{\pi})}|_{WD(E)}\eta_{\Theta(\bar{\pi})} \otimes N_\sigma|_{WD(E)}\eta_\sigma \otimes M\eta_{\Theta(\bar{\mu})}\right)\right) \\ &= L_E(s, BC(\Theta(\bar{\pi})) \boxtimes BC(\sigma) \boxtimes \gamma) \\ &= L_E(s, BC(\bar{\pi}) \boxtimes BC(\sigma) \boxtimes \gamma) \\ &= L_E(s, BC(\pi) \boxtimes BC(\bar{\sigma}) \boxtimes \gamma^{-1}). \end{aligned}$$

Note that we have used several facts above. We have used the fact that $\Theta(\bar{\pi})$ and $\bar{\pi}$ have the same L-parameters. We have also used the fact that $M\eta_{\Theta(\bar{\mu})}$ is one of the summands in the L-parameter for $BC(\Theta(\bar{\mu}))$, which we know to be $\gamma^{-1}M_{\bar{\mu}} \oplus \gamma$, where $M_{\bar{\mu}}$ is the L-parameter for $\bar{\mu}$. We note that we can replace $M\eta_{\Theta(\bar{\mu})}$ with either of these two summands without changing the L-function. Finally, for the last equality above we have used Lemma 3.5 in [5], which simply says that if M is conjugate-self dual, then $\mathrm{Ind}_{WD(E)}^{WD(F)} M$ is self-dual. □

Finally, we have a corollary of the result above.

Corollary A.5.

$$L_F(s,\Sigma')L_E(s, BC(\pi) \otimes \gamma^2) = L_E(s, BC(\Theta(\bar{\sigma})) \boxtimes BC(\pi)).$$

Proof. This follows from the previous result, along with the fact that the L-parameters N of $\Theta(\bar{\sigma})$ and M of $\bar{\sigma}$ are related by:

$$N = \gamma^{-1}M \oplus \gamma^2.$$

□

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