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# The Bochner Technique and Weighted Curvatures

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**Abstract.** In this note we study the Bochner formula on smooth metric measure spaces. We introduce weighted curvature conditions that imply vanishing of all Betti numbers.

*Key words:* Bochner technique; smooth metric measure spaces; Hodge theory

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## 1 Introduction

Let  $(M, g)$  be an oriented Riemannian manifold, let  $\text{vol}_g$  denote its volume form and let  $f$  be a smooth function on  $M$ . The triple  $(M, g, e^{-f} \text{vol}_g)$  is called a smooth metric measure space. Based on considerations from diffusion processes, Bakry–Émery [1] introduced the tensor

$$\text{Ric}_f = \text{Ric} + \text{Hess } f$$

as a weighted Ricci curvature for a geometric measure space. In fact, this tensor appeared earlier in work of Lichnerowicz [3]. Volume comparison theorems for smooth metric measure spaces with  $\text{Ric}_f$  bounded from below have been established by Qian [7], Lott [4], Bakry–Qian [2] and Wei–Wylie [8].

In this note we study the Bochner technique on smooth metric measure spaces. The distortion of the volume element introduces a diffusion term to the Bochner formula

$$\Delta_f \omega = (dd_f^* + d_f^* d)\omega = \nabla_f^* \nabla \omega + \text{Ric}(\omega) - (\text{Hess } f)\omega,$$

where  $\text{Ric}$  is the Bochner operator on  $p$ -forms. Lott [4] proved that if  $\text{Ric}_f \geq 0$ , then all  $\Delta_f$ -harmonic 1-forms are parallel and, for compact manifolds,  $H^1(M; \mathbb{R})$  is isomorphic to the space of all parallel 1-forms  $\omega$  which satisfy  $\langle \nabla e^{-f}, \omega \rangle = 0$ . Moreover, if  $\text{Ric}_f > 0$ , then all  $\Delta_f$ -harmonic 1-forms vanish.

We introduce new weighted curvature conditions that imply rigidity and vanishing results for  $\Delta_f$ -harmonic  $p$ -forms for  $p \geq 1$ . We can restrict to  $p$ -forms  $\omega$  for  $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$  since  $\omega$  is parallel if and only if  $*\omega$  is parallel, where  $*$  denotes the Hodge star.

By convention, we will refer to the eigenvalues of the curvature operator simply as the eigenvalues of the associated curvature tensor.

**Theorem.** *Let  $(M^n, g, e^{-f} \text{vol}_g)$  be a smooth metric measure space. For  $1 \leq p < \frac{n}{2}$  set*

$$h = \frac{1}{n-2p} \text{Hess } f - \frac{\Delta f}{2(n-p)(n-2p)} g.$$

*Let  $\omega$  be a  $\Delta_f$ -harmonic  $p$ -form with  $|\omega| \in L^2(M, e^{-f} \text{vol}_g)$  for  $1 \leq p < \frac{n}{2}$ . Let  $\lambda_1 \leq \dots \leq \lambda_{\binom{n}{2}}$  denote the eigenvalues of the weighted curvature tensor  $\text{Rm} + h \otimes g$ .*

If  $\lambda_1 + \cdots + \lambda_{n-p} \geq 0$ , then  $\omega$  is parallel. If in addition  $M$  is compact, then  $H^p(M) = \{\omega \in \Omega^p(M) \mid \nabla\omega = 0 \text{ and } i_{\nabla f}\omega = 0\}$ .

If  $\lambda_1 + \cdots + \lambda_{n-p} > 0$ , then  $\omega$  vanishes. If in addition  $M$  is compact, then the Betti numbers  $b_p(M)$  and  $b_{n-p}(M)$  vanish for  $1 \leq p < \frac{n}{2}$ .

For  $p = 1$  the Ricci curvature of the modified curvature tensor is the Bakry–Émery Ricci tensor, and the assumption in the Theorem implies that it is nonnegative. In this sense the Theorem is a generalization of Lott’s [4] results for 1-forms.

A stronger curvature assumption also allows control in the middle dimension  $p = \frac{n}{2}$ . Recall that a curvature tensor is  $l$ -nonnegative (positive) if the sum of its lowest  $l$  eigenvalues is nonnegative (positive).

**Proposition.** *Let  $(M^n, g, e^{-f} \text{vol}_g)$  be a smooth metric measure space. Let  $\mu_1 \leq \cdots \leq \mu_n$  denote the eigenvalues of Hess  $f$  and let  $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ .*

*Let  $\omega$  be a  $\Delta_f$ -harmonic  $p$ -form with  $|\omega| \in L^2(M, e^{-f} \text{vol}_g)$ . If the weighted curvature tensor*

$$\text{Rm} + \frac{\sum_{i=1}^p \mu_i}{2p(n-p)} g \otimes g$$

*is  $(n-p)$ -nonnegative, then  $\omega$  is parallel. If it is  $(n-p)$ -positive, then  $\omega$  vanishes.*

*In particular, if  $M$  is compact, then  $H^p(M) = \{\omega \in \Omega^p(M) \mid \nabla\omega = 0 \text{ and } i_{\nabla f}\omega = 0\}$  and in case the weighted curvature tensor is  $(n-p)$ -positive, the Betti numbers  $b_p(M)$  and  $b_{n-p}(M)$  vanish.*

The notation in this paper builds up on the presentation in [5, Chapter 9] and [6].

## 2 Preliminaries

### 2.1 Algebraic curvature tensors

For an  $n$ -dimensional Euclidean vector space  $(V, g)$  let  $\mathcal{T}^{(0,k)}(V)$  denote the vector space of  $(0, k)$ -tensors and  $\text{Sym}^2(V)$  the vector space of symmetric  $(0, 2)$ -tensors on  $V$ .

Let  $\mathcal{C}(V)$  denote the vector space of  $(0, 4)$ -tensors with  $T(X, Y, Z, W) = -T(Y, X, Z, W) = T(Z, W, X, Y)$ . If  $T$  also satisfies the algebraic Bianchi identity, then  $T$  is called algebraic curvature tensor,  $T \in \mathcal{C}_B(V)$ .

The Kulkarni–Nomizu product of  $S_1, S_2 \in \text{Sym}^2(V)$  is given by

$$\begin{aligned} (S_1 \otimes S_2)(X, Y, Z, W) &= S_1(X, Z)S_2(Y, W) - S_1(X, W)S_2(Y, Z) \\ &\quad + S_1(Y, W)S_2(X, Z) - S_1(Y, Z)S_2(X, W). \end{aligned}$$

With this convention the algebraic curvature tensor  $I = \frac{1}{2}g \otimes g$  corresponds to the curvature tensor of the unit sphere.

Recall that the decomposition of  $\mathcal{C}(V)$  into  $O(n)$ -irreducible components is given by

$$\mathcal{C}(V) = \langle I \rangle \oplus \langle \mathring{\text{Ric}} \rangle \oplus \langle W \rangle \oplus \Lambda^4 V,$$

where  $\langle \mathring{\text{Ric}} \rangle = S_0^2(V) \otimes g$  is the subspace of algebraic curvature tensors of trace-free Ricci type,  $S_0^2(V) = \{h \in \text{Sym}^2(V) \mid \text{tr}(h) = 0\}$ , and  $\langle W \rangle$  denotes the subspace of Weyl tensors.

Explicitly, every algebraic curvature tensor decomposes as

$$\text{Rm} = \frac{\text{scal}}{2(n-1)n} g \otimes g + \frac{1}{n-2} \mathring{\text{Ric}} \otimes g + W.$$

## 2.2 Lichnerowicz Laplacians on smooth metric measure spaces

Let  $(M, g, f)$  be a smooth metric measure space. The formal adjoints of the exterior and covariant derivative with respect to the measure  $e^{-f} \text{vol}_g$  are given by

$$d_f^* = d^* + i_{\nabla f} \quad \text{and} \quad \nabla_f^* = \nabla^* + i_{\nabla f}.$$

More generally, for a vector field  $U$  on  $M$ , we will consider

$$d_U^* = d^* + i_U \quad \text{and} \quad \nabla_U^* = \nabla^* + i_U.$$

The associated generalized Lichnerowicz Laplacian on  $(0, k)$ -tensors is given by

$$\Delta_U T = \nabla_U^* \nabla T + \text{Ric}(T) - (\nabla U)T,$$

where the curvature term is given by

$$\text{Ric}(T)(X_1, \dots, X_k) = \sum_{i=1}^k \sum_{j=1}^n (R(X_i, e_j)T)(X_1, \dots, e_j, \dots, X_k).$$

A tensor  $T$  is called  $U$ -harmonic if  $\Delta_U T = 0$ .

To emphasize that the curvature term is calculated with respect to the curvature tensor  $\text{Rm}$ , we will also write  $\text{Ric}_{\text{Rm}}(T)$  for  $\text{Ric}(T)$ .

Recall that for an endomorphism  $L$  of  $V$  and a  $(0, k)$ -tensor  $T$  we have

$$(LT)(X_1, \dots, X_k) = - \sum_{i=1}^k T(X_1, \dots, L(X_i), \dots, X_k).$$

In particular, the Ricci identity implies that the definition of the curvature term in the Lichnerowicz Laplacian naturally carries over to algebraic curvature tensors.

**Proposition 2.1.** *Let  $(M, g)$  be a Riemannian manifold and  $U$  a vector field on  $M$ . For a  $(0, k)$ -tensor  $T$  on  $M$  set  $\text{Ric}_U(T) = \text{Ric}(T) - (\nabla U)T$ .*

(a) *Every  $p$ -form satisfies*

$$(dd_U^* + d_U^* d)\omega = \nabla_U^* \nabla \omega + \text{Ric}_U(\omega).$$

(b) *Every symmetric  $(0, 2)$ -tensor satisfies*

$$(\nabla_X \nabla_U^* T)(X) + (\nabla_U^* d^\nabla T)(X, X) = (\nabla_U^* \nabla T)(X, X) + \frac{1}{2}(\text{Ric}_U T)(X, X),$$

$$\text{where } d^\nabla T(Z, X, Y) = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z).$$

**Proof.** (a) The case  $U = 0$  recovers the well-known Bochner formula. The generalized Hodge Laplacian satisfies

$$dd_U^* + d_U^* d = dd^* + d^* d + di_U + i_U d = \Delta + L_U.$$

In addition to the classical Lichnerowicz Laplacian we have on the right hand side

$$\nabla_U - (\nabla U) = L_U$$

and thus all diffusion terms balance out.

(b) As in (a), it suffices to consider all terms that depend on  $U$  and show that

$$(\nabla_X i_U h)(X) + (i_U d^\nabla h)(X, X) = (\nabla_U h)(X, X) - \frac{1}{2}((\nabla U)h)(X, X).$$

This is a straightforward calculation

$$\begin{aligned} & (\nabla_X i_U h)(X) + (i_U d^\nabla h)(X, X) \\ &= (\nabla_X h)(U, X) + h(\nabla_X U, X) + (\nabla_U h)(X, X) - (\nabla_X h)(U, X) \\ &= (\nabla_U h)(X, X) + h(\nabla_X U, X) \\ &= (\nabla_U h)(X, X) - \frac{1}{2}((\nabla U)h)(X, X). \quad \blacksquare \end{aligned}$$

**Remark 2.2.** The curvature tensor  $\text{Rm}$  of a Riemannian manifold satisfies

$$\begin{aligned} \nabla_U^* \nabla \text{Rm} + \frac{1}{2} \text{Ric}_U(\text{Rm}) &= \frac{1}{2}(\nabla_X \nabla_U^* \text{Rm})(Y, Z, W) - \frac{1}{2}(\nabla_Y \nabla_U^* \text{Rm})(X, Z, W) \\ &\quad + \frac{1}{2}(\nabla_Z \nabla_U^* \text{Rm})(W, X, Y) - \frac{1}{2}(\nabla_W \nabla_U^* \text{Rm})(Z, X, Y). \end{aligned}$$

A straightforward computation based on the second Bianchi identity shows that all terms that involve  $U$  cancel.

The Bochner technique with diffusion relies on the following basic observations. Firstly, the maximum principle implies:

**Lemma 2.3.** *Let  $(M, g)$  be a Riemannian manifold,  $U$  a vector field on  $M$ . Let  $T$  be a tensor such that*

$$g(\nabla_U^* \nabla T, T) \leq 0.$$

*If  $|T|$  has a maximum, then  $T$  is parallel.*

**Remark 2.4.** Note that a  $p$ -form  $\omega$  satisfies  $(dd_U^* + d_U^* d)\omega = 0$  if and only if  $d\omega = 0$  and  $d_U^* \omega = 0$ .

As in [4], if  $M$  is compact and oriented, standard elliptic theory implies that

$$H^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0 \text{ and } d_U^* \omega = 0\}.$$

Suppose that  $\text{Ric}_U \geq 0$  on  $p$ -forms. It follows that a  $p$ -form  $\omega$  is  $U$ -harmonic if and only if  $\omega$  is parallel and  $i_U \omega = 0$ . Thus,

$$H^p(M) = \{\omega \in \Omega^p(M) \mid \nabla \omega = 0 \text{ and } i_U \omega = 0\}.$$

If  $U = \nabla f$ , then we can use integration to conclude:

**Lemma 2.5.** *Let  $(M, g, f)$  be a smooth metric measure space with  $\int_M e^{-f} \text{vol}_g < \infty$ . If  $T$  is a  $(0, k)$ -tensor with  $|T| \in L^2(M, e^{-f} \text{vol}_g)$  and*

$$g(\nabla_f^* \nabla T, T) \leq 0,$$

*then  $T$  is parallel.*

### 3 Weighted Lichnerowicz Laplacians

The idea of this section is to define a weighted curvature tensor  $\widetilde{\text{Rm}}$  so that for a given symmetric tensor  $S$  the curvature term of the Lichnerowicz Laplacian satisfies

$$g(\text{Ric}_{\text{Rm}}(T) - (S)T, T) = g(\text{Ric}_{\widetilde{\text{Rm}}}(T), T).$$

This will be achieved by adding a weight to the Ricci tensor of  $\text{Rm}$ , leaving the Weyl curvature unchanged. The specific weight will depend on the irreducible components of the tensors of type  $T$ , e.g., it is different for forms and symmetric tensors.

Let  $T$  be a  $(0, k)$ -tensor. For  $\tau_{ij} \in S_k$  let  $T \circ \tau_{ij}$  denote the transposition of the  $i$ -th and  $j$ -th entries of  $T$  and for  $h \in \text{Sym}^2(V)$  let  $c_{ij}(h \otimes T)$  denote the contraction of  $h$  with the  $i$ -th and  $j$ -th entries of  $T$ .

**Proposition 3.1.** *For  $h \in \text{Sym}^2(V)$  let  $H: V \rightarrow V$  denote the associated symmetric operator. If  $T \in \mathcal{T}^{(0,k)}(V)$ , then*

$$\begin{aligned} \text{Ric}_{h \otimes g}(T)(X_1, \dots, X_k) &= 2 \sum_{i \neq j} (T \circ \tau_{ij})(X_1, \dots, H(X_i), \dots, X_k) \\ &\quad - \sum_{i \neq j} g(X_i, X_j) c_{ij}(h \otimes T)(X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \\ &\quad - \sum_{i \neq j} h(X_i, X_j) c_{ij}(g \otimes T)(X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \\ &\quad - (n-2)(HT)(X_1, \dots, X_k) + k \cdot \text{tr}(h)T(X_1, \dots, X_k). \end{aligned}$$

**Proof.** The algebraic curvature tensor  $R = h \otimes g$  satisfies

$$\begin{aligned} R(X, Y, Z, W) &= g(H(X), Z)g(Y, W) - g(Y, Z)g(H(X), W) \\ &\quad + g(X, Z)g(H(Y), W) - g(H(Y), Z)g(X, W) \end{aligned}$$

and hence

$$R(X, Y)Z = (H(X) \wedge Y + X \wedge H(Y))Z$$

is the corresponding  $(1, 3)$ -tensor. It follows that

$$\begin{aligned} \text{Ric}_{h \otimes g}(T)(X_1, \dots, X_k) &= \sum_{i=1}^k \sum_{a=1}^n (R(X_i, e_a)T)(X_1, \dots, e_a, \dots, X_k) \\ &= \sum_{i=1}^k \sum_{a=1}^n ((H(X_i) \wedge e_a)T)(X_1, \dots, e_a, \dots, X_k) \\ &\quad + \sum_{i=1}^k \sum_{a=1}^n ((X_i \wedge H(e_a))T)(X_1, \dots, e_a, \dots, X_k). \end{aligned}$$

It is straightforward to calculate

$$\begin{aligned} &\sum_{i=1}^k \sum_{a=1}^n ((X_i \wedge H(e_a))T)(X_1, \dots, e_a, \dots, X_k) \\ &= \sum_{i \neq j} \sum_{a=1}^n T(X_1, \dots, (H(e_a) \wedge X_i)X_j, \dots, e_a, \dots, X_k) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^k \sum_{a=1}^n T(X_1, \dots, (H(e_a) \wedge X_i)e_a, \dots, X_k) \\
& = \sum_{i \neq j} \sum_{a=1}^n T(X_1, \dots, g(H(e_a), X_j)X_i - g(X_i, X_j)H(e_a), \dots, e_a, \dots, X_k) \\
& \quad + \sum_{i=1}^k \sum_{a=1}^n T(X_1, \dots, g(H(e_a), e_a)X_i - g(e_a, X_i)H(e_a), \dots, X_k) \\
& = \sum_{i \neq j} \sum_{a=1}^n T(X_1, \dots, g(e_a, H(X_j))X_i, \dots, e_a, \dots, X_k) \\
& \quad - \sum_{i \neq j} \sum_{a=1}^n g(X_i, X_j)T(X_1, \dots, H(e_a), \dots, e_a, \dots, X_k) \\
& \quad + \sum_{i=1}^k \sum_{a=1}^n h(e_a, e_a)T(X_1, \dots, X_k) - \sum_{i=1}^k \sum_{a=1}^n T(X_1, \dots, H(g(e_a, X_i)e_a), \dots, X_k) \\
& = \sum_{i \neq j} T(X_1, \dots, X_i, \dots, H(X_j), \dots, X_k) \text{ [here } X_i \text{ is in the } j\text{-th position]} \\
& \quad - \sum_{i \neq j} \sum_{a,b=1}^n g(X_i, X_j)h(e_a, e_b)T(X_1, \dots, e_b, \dots, e_a, \dots, X_k) + k \cdot \text{tr}(h)T(X_1, \dots, X_k) \\
& \quad - \sum_{i=1}^k T(X_1, \dots, H(X_i), \dots, X_k) \\
& = \sum_{i \neq j} (T \circ \tau_{ij})(X_1, \dots, H(X_j), \dots, X_i, \dots, X_k) \text{ [here } H(X_j) \text{ is in the } j\text{-th position]} \\
& \quad - \sum_{i \neq j} g(X_i, X_j)c_{ij}(h \otimes T)(X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \\
& \quad + k \cdot \text{tr}(h)T(X_1, \dots, X_k) + (HT)(X_1, \dots, X_k).
\end{aligned}$$

Similarly one computes

$$\begin{aligned}
& \sum_{i=1}^k \sum_{a=1}^n ((H(X_i) \wedge e_a)T)(X_1, \dots, e_a, \dots, X_k) \\
& = \sum_{i \neq j} (T \circ \tau_{ij})(X_1, \dots, X_j, \dots, H(X_i), \dots, X_k) \text{ [here } X_j \text{ is in the } j\text{-th position]} \\
& \quad - \sum_{i \neq j} h(X_i, X_j)c_{ij}(g \otimes T)(X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) - (n-1)(HT)(X_1, \dots, X_k).
\end{aligned}$$

Adding up both terms yields  $\text{Ric}_{h \otimes g}(T)$  as claimed.  $\blacksquare$

**Proposition 3.2.** *Let  $(V, g)$  be an  $n$ -dimensional Euclidean vector space and  $h \in \text{Sym}^2(V)$ . The following hold:*

1. Every  $T \in \text{Sym}^2(V)$  satisfies

$$\begin{aligned}
\text{Ric}_{h \otimes g}(T) & = -nHT - 2\langle T, h \rangle g - 2\text{tr}(T)h + 2\text{tr}(h)T, \\
g(\text{Ric}_{h \otimes g}(T), T) & = -ng(HT, T) - 4\text{tr}(T)\langle T, h \rangle + 2\text{tr}(h)|T|^2.
\end{aligned}$$

2. Every  $p$ -form  $\omega$  satisfies

$$\begin{aligned}\operatorname{Ric}_{h \otimes g}(\omega) &= -(n-2p)H\omega + p \operatorname{tr}(h)\omega, \\ g(\operatorname{Ric}_{h \otimes g}(\omega), \omega) &= -(n-2p)g(H\omega, \omega) + p \operatorname{tr}(h)|\omega|^2.\end{aligned}$$

3. Every algebraic  $(0, 4)$ -curvature tensor  $\operatorname{Rm}$  satisfies

$$\operatorname{Ric}_{h \otimes g}(\operatorname{Rm}) = -2(h \otimes \operatorname{Ric}) - 2g \otimes (c_{24}(h \otimes \operatorname{Rm})) - (n-2)H \operatorname{Rm} + 4 \operatorname{tr}(h) \operatorname{Rm}.$$

**Proof.** (a) Due to the symmetry of  $T$  it follows that

$$\begin{aligned}\operatorname{Ric}_{h \otimes g}(T)(X_1, X_2) &= 2\{T(H(X_1), X_2) + T(X_1, H(X_2))\} \\ &\quad - 2\{g(X_1, X_2)\langle h, T \rangle + h(X_1, X_2) \operatorname{tr}(T)\} \\ &\quad - (n-2)(HT)(X_1, X_2) + 2 \operatorname{tr}(h)T(X_1, X_2).\end{aligned}$$

(b) Since  $\omega \circ \tau_{ij} = -\omega$  for every transposition  $\tau_{ij}$  it follows that

$$\begin{aligned}\sum_{i \neq j} (\omega \circ \tau_{ij})(X_1, \dots, H(X_i), \dots, X_p) &= -\sum_{i \neq j} \omega(X_1, \dots, H(X_i), \dots, X_p) \\ &= -(p-1) \sum_{i=1}^p \omega(X_1, \dots, H(X_i), \dots, X_p) \\ &= (p-1)(H\omega)(X_1, \dots, X_p)\end{aligned}$$

and furthermore  $c_{ij}(g \otimes \omega) = c_{ij}(h \otimes \omega) = 0$  for all  $i \neq j$ . This implies the claim.

(c) The symmetries of the curvature tensor imply that

$$\begin{aligned}\sum_{i \neq j} (\operatorname{Rm} \circ \tau_{ij})(X_1, \dots, H(X_i), \dots, X_4) \\ = (H \operatorname{Rm})(X_1, X_2, X_3, X_4) + (H \operatorname{Rm})(X_2, X_3, X_1, X_4) + (H \operatorname{Rm})(X_3, X_1, X_2, X_4) = 0\end{aligned}$$

due to the first Bianchi identity.

Computing with respect to an orthonormal eigenbasis of  $H$  it follows that

$$\begin{aligned}(g(\cdot, \cdot)c_{12}(h \otimes \operatorname{Rm}))(X, Y, Z, W) &= 0, \\ (g(\cdot, \cdot)c_{13}(h \otimes \operatorname{Rm}))(X, Y, Z, W) &= \sum_{a,b=1}^n g(X, Z) \operatorname{Rm}(g(H(e_a), e_b)e_b, Y, e_a, W) \\ &= \sum_{a=1}^n g(X, Z) \operatorname{Rm}(H(e_a), Y, e_a, W) \\ &= \sum_{a=1}^n g(Z, X) \operatorname{Rm}(e_a, Y, H(e_a), W) \\ &= (g(\cdot, \cdot)c_{31}(h \otimes \operatorname{Rm}))(X, Y, Z, W).\end{aligned}$$

This implies

$$\begin{aligned}\sum_{i \neq j} (g(\cdot, \cdot)c_{ij}(h \otimes \operatorname{Rm}))(X, Y, Z, W) \\ = 2 \sum_{i=1}^n \{g(X, Z) \operatorname{Rm}(H(e_i), Y, e_i, W) + g(X, W) \operatorname{Rm}(H(e_i), Y, Z, e_i)\}\end{aligned}$$



$$\begin{aligned}
& + g(Y, Z) \operatorname{Rm}(X, H(e_i), e_i, W) + g(Y, W) \operatorname{Rm}(X, H(e_i), Z, e_i)\} \\
& = 2 \sum_{i=1}^n \{g(X, Z) \operatorname{Rm}(Y, H(e_i), W, e_i) - g(X, W) \operatorname{Rm}(Y, H(e_i), Z, e_i) \\
& \quad - g(Y, Z) \operatorname{Rm}(X, H(e_i), W, e_i) + g(Y, W) \operatorname{Rm}(X, H(e_i), Z, e_i)\} \\
& = 2 \left( g \otimes \left[ \sum_{i=1}^n \operatorname{Rm}(\cdot, H(e_i), \cdot, e_i) \right] \right) (X, Y, Z, W) \\
& = 2 (g \otimes c_{24}(h \otimes \operatorname{Rm})) (X, Y, Z, W).
\end{aligned}$$

Similarly it follows that

$$\sum_{i \neq j} (h(\cdot, \cdot) c_{ij}(g \otimes \operatorname{Rm})) = 2 (h \otimes c_{24}(g \otimes \operatorname{Rm})) = 2 (h \otimes \operatorname{Ric}).$$

This completes the proof. ■

**Remark 3.3.** For a Weyl tensor  $W$  and  $h$  a symmetric  $(0, 2)$ -tensor it is not hard to check that  $\operatorname{Ric}_{h \otimes g}(W)$  satisfies

$$\begin{aligned}
g(\operatorname{Ric}_{h \otimes g}(W), W) & = -(n-2)g(HW, W) + 4 \operatorname{tr}(h)|W|^2, \\
g(\operatorname{Ric}_{h \otimes g}(W), g \otimes \overset{\circ}{\operatorname{Ric}}) & = -8(n-2)\langle c_{24}(h \otimes W), \operatorname{Ric} \rangle = -8(n-2)\langle c_{24}(\overset{\circ}{h} \otimes W), \overset{\circ}{\operatorname{Ric}} \rangle, \\
g(\operatorname{Ric}_{h \otimes g}(W), g \otimes g) & = 0.
\end{aligned}$$

It is worth noting that there are trace-free symmetric  $(0, 2)$ -tensors  $h_1, h_2$  such that the curvature tensor  $h_1 \otimes h_2$  is Weyl.

The main Theorem follows as in Proposition 3.4 below by using Lemma 2.5 instead of Lemma 2.3. The description of the de Rham cohomology groups follows from Remark 2.4.

**Proposition 3.4.** *Let  $(M, g)$  be a Riemannian manifold and let  $U$  be a vector field on  $M$ . Set  $S = \nabla U$  and for  $1 \leq p < \frac{n}{2}$  set*

$$H = \frac{1}{n-2p} S - \frac{1}{2(n-p)(n-2p)} \operatorname{tr}(S)I,$$

where  $I: TM \rightarrow TM$  denotes the identity operator.

Suppose that the eigenvalues  $\lambda_1 \leq \dots \leq \lambda_{\binom{n}{2}}$  of the weighted curvature tensor  $\operatorname{Rm} + h \otimes g$  satisfy

$$\lambda_1 + \dots + \lambda_{n-p} \geq 0$$

and let  $\omega$  be a  $U$ -harmonic  $p$ -form for  $1 \leq p < \frac{n}{2}$ .

If  $|\omega|$  achieves a maximum, then  $\omega$  is parallel. If in addition the inequality is strict, then  $\omega$  vanishes.

**Proof.** Proposition 3.2 (b) and  $-I\omega = p\omega$  imply that

$$\begin{aligned}
g(\operatorname{Ric}_{h \otimes g} \omega, \omega) & = -(n-2p)g(H\omega, \omega) + p \operatorname{tr}(h)|\omega|^2 = -g(((n-2p)H + \operatorname{tr}(h)I)\omega, \omega) \\
& = -g\left(\left(S - \frac{\operatorname{tr}(S)}{2(n-p)}I + \frac{\operatorname{tr}(S)}{2(n-p)}I\right)\omega, \omega\right) = -g(S\omega, \omega).
\end{aligned}$$

Thus the Bochner formula takes the form

$$\Delta_U \omega = \nabla_U^* \nabla \omega + \operatorname{Ric}(\omega) - (\nabla U)\omega = \nabla_U^* \nabla \omega + \operatorname{Ric}_{\operatorname{Rm} + h \otimes g}(\omega).$$

The argument in [6, proof of Theorem A] shows that  $\text{Ric}_{\text{Rm}+h\odot g}(\omega) \geq 0$ . Lemma 2.3 implies the claim.

If the inequality is strict, then the same argument shows that  $\text{Ric}_{\text{Rm}+h\odot g}(\omega) > 0$  unless  $\omega = 0$ . ■

The above approach only works for  $p = \frac{n}{2}$  if  $S$  is a multiple of the identity. However, we have

**Proposition 3.5.** *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold and let  $U$  be a vector field on  $M$ . Set  $S = \nabla U$  and fix  $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ . Let  $\mu_1 \leq \dots \leq \mu_n$  denote the eigenvalues of  $S$ . Suppose that the weighted curvature tensor*

$$\text{Rm} + \frac{\sum_{i=1}^p \mu_i}{2p(n-p)} g \odot g$$

*is  $(n-p)$ -nonnegative. If  $\omega$  is a  $U$ -harmonic  $p$ -form  $\omega$  such that  $|\omega|$  has a maximum, then  $\omega$  is parallel. If in addition the weighted curvature tensor is  $(n-p)$ -positive, then  $\omega$  vanishes.*

**Proof.** Calculating with respect to an orthonormal eigenbasis for  $S$  it follows that

$$-g((S\omega), \omega) = - \sum_{i_1 < \dots < i_p} (S\omega)_{i_1 \dots i_p} \omega_{i_1 \dots i_p} = \sum_{i_1 < \dots < i_p} \left( \sum_{j=1}^p \mu_{i_j} \right) (\omega_{i_1 \dots i_p})^2 \geq \left( \sum_{i=1}^p \mu_i \right) |\omega|^2.$$

Let  $\{\lambda_\alpha\}$  denote the eigenvalues of (the curvature operator associated to)  $\text{Rm}$  and let  $\{\Xi_\alpha\}$  be an orthonormal eigenbasis. It follows from [6, Proposition 1.6] that

$$g(\text{Ric}_{\text{Rm}}(\omega), \omega) - g(S\omega, \omega) \geq \sum_{\alpha} \lambda_{\alpha} |\Xi_{\alpha} \omega|^2 + \left( \sum_{i=1}^p \mu_i \right) |\omega|^2 = \sum_{\alpha} \left( \lambda_{\alpha} + \frac{\sum_{i=1}^p \mu_i}{p(n-p)} \right) |\Xi_{\alpha} \omega|^2.$$

The proof can now be completed as in Proposition 3.4. ■

This principle can also be applied to  $(0, 2)$ -tensors.

**Proposition 3.6.** *Let  $T \in \text{Sym}^2(V)$  with  $\text{tr}(T) = 0$ , let  $S = \nabla U$  and set*

$$H = \frac{S}{n} - \frac{\text{tr}(S)}{2n^2} I.$$

*Let  $\lambda_1 \leq \dots \leq \lambda_{\binom{n}{2}}$  denote the eigenvalues of the weighted curvature tensor  $\text{Rm} + h \odot g$  and suppose that*

$$\lambda_1 + \dots + \lambda_{\lfloor \frac{n}{2} \rfloor} \geq 0.$$

*If  $T$  is  $U$ -harmonic and  $|T|$  has a maximum, then  $T$  is parallel. If in addition the inequality is strict, then  $T$  vanishes.*

**Proof.** Proposition 3.2(a) implies that

$$\begin{aligned} g(\text{Ric}_{h\odot g}(T), T) &= -ng \left( \left( H + \frac{\text{tr}(h)}{n} I \right) T, T \right) \\ &= -ng \left( \left( \frac{S}{n} - \frac{\text{tr}(S)}{2n^2} I + \frac{\text{tr}(S)}{2n^2} I \right) T, T \right) = -g(ST, T). \end{aligned}$$

It follows from Proposition 2.1(b) that

$$(\nabla_X \nabla_U^* T)(X) + (\nabla_U^* d^\nabla T)(X, X) = (\nabla_U^* \nabla T)(X, X) + \frac{1}{2} (\text{Ric}_{\text{Rm}+h\otimes g} T)(X, X).$$

As in [6, Lemma 2.1 and Proposition 2.9] we conclude that  $\text{Ric}_{\text{Rm}+h\otimes g}(T) \geq 0$ . When the inequality is strict, the argument shows moreover  $\text{Ric}_{\text{Rm}+h\otimes g}(T) > 0$  unless  $T = 0$ . This uses again that  $T$  is trace-less.

An application of Lemma 2.5 as before implies the claim. ■

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