FINITE-PARAMETERS FEEDBACK CONTROL FOR STABILIZING DAMPED NONLINEAR WAVE EQUATIONS

VARGA K. KALANTAROV AND EDRISS S. TITI

Abstract. In this paper we introduce a finite-parameters feedback control algorithm for stabilizing solutions of various classes of damped nonlinear wave equations. Specifically, stabilization the zero steady state solution of initial boundary value problems for nonlinear weakly and strongly damped wave equations, nonlinear wave equation with nonlinear damping term and some related nonlinear wave equations, introducing a feedback control terms that employ parameters, such as, finitely many Fourier modes, finitely many volume elements and finitely many nodal observables and controllers. In addition, we also establish the stabilization of the zero steady state solution to initial boundary value problem for the damped nonlinear wave equation with a controller acting in a proper subdomain. Notably, the feedback controllers proposed here can be equally applied for stabilizing other solutions of the underlying equations.

1. Introduction

This paper is devoted to the study of finite-parameters feedback control for stabilizing solutions of initial boundary value problems for nonlinear damped wave equations. Feedback control, stabilization and control of wave equations is a well established area of control theory. Many interesting results were obtained, in the last dacades, on stabilization of linear and nonlinear wave equations (see , e.g., [4], [17], [20],[23],[25],[27],[28] and references therein). Most of the problems considered are problems of stabilization by interior and boundary controllers involving linear or nonlinear damping terms. However, only few results are known on feedback stabilization of linear hyperbolic equation by employing finite-dimensional controllers (see, e.g., [3] and references therein).
We study the problem of feedback stabilization of initial boundary value problem for damped nonlinear wave equation

$$\partial_t^2 u - \Delta u + b\partial_t u - au + f(u) = -\mu w,\ x \in \Omega, t > 0, \tag{1.1}$$

nonlinear wave equation with nonlinear damping term

$$\partial_t^2 u - \Delta u + bg(\partial_t u) - au + f(u) = -\mu w,\ x \in \Omega, t > 0, \tag{1.2}$$

and the strongly damped wave equation

$$\partial_t^2 u - \Delta u - b\Delta\partial_t u - \lambda u + f(u) = -\mu w,\ x \in \Omega, t > 0. \tag{1.3}$$

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Here and in what follows μ, a, b, ν are given positive parameters, w is a feedback control input (different for different problems), $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function such that $f(0) = 0$ and

$$f(s)s - \mathcal{F}(s) \geq 0, f'(s) \geq 0, \ \forall s \in \mathbb{R}, \quad \mathcal{F}(s) := \int_0^s f(\tau)d\tau, \quad \forall s \in \mathbb{R}. \tag{1.4}$$

We show that the feedback controller proposed in [1] for nonlinear parabolic equations can be used for stabilization of above mentioned wide class of nonlinear dissipative wave equations. We also show stabilization of the initial boundary value problem for equation (1.1), with the control input acting on some proper subdomain of Ω.
Our study, as well as the results obtained in [1], are inspired by the fact that dissipative dynamical systems generated by initial boundary value problems, such as the 2D Navier - Stokes equations, nonlinear reaction-diffusion equation, Cahn-Hilliard equation, damped nonlinear Schrödinger equation, damped nonlinear Klein - Gordon equation, nonlinear strongly damped wave equation and related equations and systems have a finite dimensional asymptotic (in time) behavior (see, e.g., [2],[6]- [10],[12]-[14],[21],[22],[29] and references therein). This property has also been implicitly used in the feedback control of the Navier-Stokes equations overcoming the spillover phenomenon in [5]
Motivated by this observation, we specifically show

(1) stabilization of $1D$ damped wave equation (1.1), under Neuman boundary conditions, when the feedback control input employs observables based on measurement of *finite volume elements*,

(2) stabilization of equation (1.1), under homogeneous Dirichlet boundary condition, with one feedback controller supported on some proper subdomain of $\Omega \subset \mathbb{R}^n$,

(3) stabilization of equation (1.1) and equation (1.3), under the Dirichlet boundary condition, when the feedback control involves *finitely many Fourier modes of the solution*, based on eigenfunctions of the Laplacian subject to homogeneous Dirichlet boundary condition,
(4) stabilization of the $1D$ equation (1.3), when the feedback control incorporates observables at *finitely many nodal points*.

In the sequel we will use the notations:

- $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and the norm of $L^2(\Omega)$,
 and the following inequalities:

- Young's inequality
$$ab \leq \varepsilon a^2 + \frac{\varepsilon}{4}b^2, \tag{1.5}$$
 that is valid for all positive numbers a, b and ε
- the Poincaré inequality
$$\|u\|^2 \leq \lambda_1^{-1}\|\nabla u\|^2, \tag{1.6}$$

which holds for each $u \in H_0^1(\Omega)$.

2. Feedback control of damped nonlinear wave equations

In this section, we show that the initial boundary value problem for nonlinear damped wave equation can be stabilized by employing finite volume elements feedback controller, feedback controllers acting in a subdomain of Ω, or feedback controllers involving finitely many Fourier modes.

1. Stabilization employing finite volume elements feedback control. We consider the following feedback control problem

$$\begin{cases} \partial_t^2 u - \nu\partial_x^2 u + b\partial_t u - \lambda u + f(u) = -\mu \sum\limits_{k=1}^{N} \overline{u}_k \chi_{J_k}(x), \quad x \in (0,L),\ t > 0, \\ \partial_x u(0,t) = \partial_x u(L,t) = 0,\ t > 0, \\ u(x, 0 = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad x \in (0,L). \end{cases} \tag{2.1}$$

Here $J_k := \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right)$, for $k = 1, 2, \cdots N-1$ and $J_N = [\frac{N-1}{N}L, L]$, $\overline{\phi}_k := \frac{1}{|J_k|}\int\limits_{J_k} \phi(x)dx$, and $\chi_{J_k}(x)$ is the characteristic function of the interval J_k. In what follows we will need the following lemma

Lemma 2.1. *(see* [1]*) Let $\phi \in H^1(0,L)$. Then*

$$\|\phi - \sum_{k=1}^{N} \overline{\phi}_k \chi_{J_k}(\cdot)\| \leq h\|\phi_x\|, \tag{2.2}$$

and

$$\|\phi\|^2 \leq h \sum_{k=1}^{N} \overline{\phi}_k^2 + \left(\frac{h}{2\pi}\right)^2 \|\phi_x\|^2, \tag{2.3}$$

where $h := \frac{L}{N}$.

By employing this Lemma, we proved the following theorem:

Theorem 2.2. *Suppose that the nonlinear term $f(\cdot)$ satisfies the condition* (1.4) *and that μ and N are large enough satisfying*

$$\mu \geq 2\left(\lambda + \frac{\delta_0 b}{2}\right) \ \text{and}\ \ N^2 > \frac{L^2}{2\nu\pi^2}\left(\lambda + \frac{\delta_0 b}{2}\right), \tag{2.4}$$

where

$$\delta_0 = \frac{b}{2}\min\{1, \nu\}. \tag{2.5}$$

Then each solution of the problem (2.1) *satisfies the following decay estimate:*

$$\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 \leq K(\|u_1\|^2 + \|\partial_x u_0\|^2)e^{-\delta_0 t}, \tag{2.6}$$

where K is some positive constant depending on b, λ and L.

Proof. First observe that one can use standard tools of the theory of nonlinear wave equations to show global existence and uniqueness of solution to problem (2.1) (see, e.g., [24]).
Taking the $L^2(0,L)$ inner product of (2.1) with $\partial_t u + \varepsilon u$, where $\varepsilon > 0$ is a parameter, to be determined later, gives us the following relation:

$$\frac{d}{dt}\left[\frac{1}{2}\|\partial_t u\|^2 + \frac{\nu}{2}\|\partial_x u\|^2 - \frac{1}{2}(\varepsilon b - \lambda)\|u\|^2 + \int_0^L \mathcal{F}(u)dx + \frac{1}{2}h\mu\sum_{k=1}^N \overline{u}_k^2 + \varepsilon(u,\partial_t u)\right]$$
$$+ (b-\varepsilon)\|\partial_t u\|^2 + \varepsilon\nu\|\partial_x u\|^2 - \varepsilon\lambda\|u\|^2 + \varepsilon(f(u),u) + \varepsilon\mu h\sum_{k=1}^N \overline{u}_k^2 = 0. \quad (2.7)$$

It follows from (2.7) that

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta\Phi_\varepsilon(t) + (b-\varepsilon)\|\partial_t u\|^2 + \varepsilon\nu\|\partial_x u\|^2 - \varepsilon\lambda\|u\|^2 + \varepsilon(f(u),u)$$
$$+ \varepsilon\mu h\sum_{k=1}^N \overline{u}_k^2 - \frac{\delta}{2}\|\partial_t u\|^2 - \frac{\delta\nu}{2}\|\partial_x u\|^2 - \frac{\delta}{2}(\varepsilon b - \lambda)\|u\|^2 - \delta(\mathcal{F}(u),1)$$
$$- \frac{\delta}{2}h\mu\sum_{k=1}^N \overline{u}_k^2 - \delta\varepsilon(u,\partial_t u) = 0. \quad (2.8)$$

Here

$$\Phi_\varepsilon(t) := \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{\nu}{2}\|\partial_x u(t)\|^2 + \frac{1}{2}(\varepsilon b - \lambda)\|u(t)\|^2 + \int_0^L \mathcal{F}(u(x,t))dx$$
$$+ \frac{1}{2}h\mu\sum_{k=1}^N \overline{u}_k^2(t) + \varepsilon(u(t),\partial_t u(t)).$$

Due to condition (1.4) and the Cauchy-Schwarz inequality we have the following lower estimate for $\Phi_\varepsilon(t)$

$$\Phi_\varepsilon(t) \geq \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{2}\|\partial_x u\|^2 + (\frac{b\varepsilon}{2} - \frac{\lambda}{2} - \varepsilon^2)\|u\|^2 + \frac{1}{2}h\mu\sum_{k=1}^N \overline{u}_k^2.$$

By choosing $\varepsilon \in (0, \frac{b}{2}]$ and by employing inequality (2.3), we get from the above inequality:

$$\Phi_\varepsilon(t) \geq \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{2}\|\partial_x u\|^2 - \lambda\left[h\sum_{k=1}^N \overline{u}_k^2 + \left(\frac{h}{2\pi}\right)^2\|\partial_x u\|^2\right]$$
$$+ \frac{1}{2}\mu h\sum_{k=1}^N \overline{u}_k^2 = \frac{1}{4}\|\partial_t u\|^2 + \left(\frac{\nu}{2} - \lambda\frac{L^2}{4\pi^2 N^2}\right)\|\partial_x u\|^2 + h(\frac{\mu}{2} - \lambda)\sum_{k=1}^N \overline{u}_k^2, \quad (2.9)$$

Thanks to the condition (2.4) we also have

$$\Phi_\varepsilon(t) \geq \frac{1}{4}\|\partial_t u(t)\|^2 + d_0\|\partial_x u(t)\|^2, \quad (2.10)$$

where $d_0 = \frac{\delta_0 b L^2}{4N^2\nu\pi^2}$, and δ_0 is defined in (4.21).
Let $\delta \in (0, \varepsilon)$, to be chosen below. According to condition (1.4) $f(u)u \geq \mathcal{F}(u), \forall u \in \mathbb{R}$. Therefore (2.8) implies

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta\Phi_\varepsilon(t) + \frac{b}{2}\|\partial_t u\|^2 + (\frac{b\nu}{2} - \frac{\delta}{2})\|\partial_x u\|^2 + \left(\frac{\delta\lambda}{2} - \frac{\delta b^2}{4} - \frac{b\lambda}{2}\right)\|u\|^2 + \mu h\left(\frac{b}{2} - \frac{\delta}{2}\right)\sum_{k=1}^N \overline{u}_k^2 \leq 0. \quad (2.11)$$

We choose here $\delta = \delta_0 := \frac{b}{2}\min\{\nu, 1\}$ and employ inequality (2.3) to obtain:

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta_0\Phi_\varepsilon(t) + \frac{b}{2}\|\partial_t u\|^2 + \left[\frac{b\nu}{4} - \frac{b}{2}\left(\lambda + \frac{\delta_0 b}{2}\right)\frac{L^2}{4\pi^2 N^2}\right]\|\partial_x u\|^2 + \frac{bh}{2}\left[\frac{\mu}{2} - \left(\lambda + \frac{\delta_0 b}{2}\right)\right]\sum_{k=1}^N \overline{u}_k^2 \leq 0.$$

Finally by using condition (2.4) we get:

$$\frac{d}{dt}\Phi_\varepsilon(t) + \delta_0\Phi_\varepsilon(t) \leq 0.$$

Thus by Gronwalls inequality and thanks (2.10) we have

$$\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 \leq K(\|u_1\|^2 + \|\partial_x u_0\|^2)e^{-\delta_0 t}, \quad (2.12)$$

where δ_0 is defined in (4.21), and K is a positive constant, depending on b, λ and L. □

2.1. **Stabilization with feedback control on a subdomain.** In this section we study the problem of internal stabilization of initial boundary value problem for nonlinear damped wave equation on a bounded domain. We show that the problem can be exponentially stabilized by a feedback controller acting on a strict subdomain. So, we consider the following feedback control problem:

$$\partial_t^2 u - \Delta u + b\partial_t u - au + |u|^{p-2}u = -\mu\chi_\omega(x)u, \ x \in \Omega, t > 0, \quad (2.13)$$

$$u(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \quad (2.14)$$

$$u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x), \ x \in \Omega. \quad (2.15)$$

Here $a > 0, p \geq 2$ are given numbers, and $\mu > 0$ is a parameter to be determined, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\chi_\omega(x)$ is the characteristic function of the subdomain $\omega \subset \Omega$ with smooth boundary and $\overline{\omega} \subset \Omega$.
Let us denote by $\lambda_1(\mu, \Omega)$ the first eigenvalue of the elliptic problem

$$-\Delta v + \mu\chi_\omega(x)v = \lambda v; \ x \in \Omega, \ v = 0, \ x \in \partial\Omega,$$

and denote by $\lambda_1(\Omega_\omega)$ the first eigenvalue of the problem

$$-\Delta v = \lambda v, \ x \in \Omega_\omega; \ v = 0, \ x \in \partial\Omega_\omega,$$

where $\Omega_\omega := \Omega \setminus \overline{\omega}$. We will need the following Lemma in the proof of the main result of this section:

Lemma 2.3. *(see, e.g.,[30]) For each $d > 0$ there exists a number $\mu_0(d) > 0$ such that the following inequality holds true*

$$\int_\Omega \left(|\nabla v(x)|^2 + \mu\chi_\omega(x)v^2(x)\right) dx \geq (\lambda_1(\Omega_\omega) - d) \int_\Omega v^2(x)dx, \ \forall v \in H_0^1(\Omega), \tag{2.16}$$

whenever $\mu > \mu_0$.

Theorem 2.4. *Suppose that*

$$\lambda_1(\Omega_\omega) \geq 4a + \frac{3b^2}{2}, \quad \text{and} \ \ \mu > \mu_0, \tag{2.17}$$

where μ_0 is the parameter stated in Lemma 2.3 , corresponding to $d = \frac{1}{2}\lambda_1(\Omega_\omega)$. Then the energy norm of each weak solution of the problem (2.13)-(2.15) tends to zero with an exponential rate. More precisely, the following estimate holds true:

$$\|\partial_t u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^p_{L^p(\Omega)} \leq C_0 e^{-\frac{b}{2}t}, \tag{2.18}$$

where C_0 is a positive constant depending on initial data.

Proof. Taking the $L^2(\Omega)$ inner product of (2.13) with $\partial_t u + \frac{b}{2}u$ we get

$$\frac{d}{dt}E_b(t) + \frac{b}{2}\left[\|\partial_t u\|^2 + \|\nabla u\|^2 - a\|u\|^2 + \|u\|^p_{L^p(\Omega)} + \mu\int_\Omega \chi_\omega(x)u^2(x)dx\right] = 0, \tag{2.19}$$

where

$$E_b(t) := \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{a}{2}\|u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} + \frac{\mu}{2}\int_\Omega \chi_\omega(x)u^2(x)dx + \frac{b}{2}(u, \partial_t u) + \frac{b^2}{4}\|u\|^2.$$

By Cauchy-Schwarz and Young inequalities we have

$$\frac{b}{2}|(u, \partial_t u)| \leq \frac{1}{4}\|\partial_t u\|^2 + \frac{b^2}{4}\|u\|^2. \tag{2.20}$$

By employing (2.20), we obtain the lower estimate for $E_b(t)$:

$$E_b(t) \geq \frac{1}{4}\|\partial_t u\|^2 + \frac{1}{4}\|\nabla u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} + \frac{1}{4}\left[\|\nabla u\|^2 + 2\mu\|u\|^2_{L^2(\omega)} - 2a\|u\|^2\right]. \tag{2.21}$$

According to Lemma 2.3

$$\|\nabla u\|^2 + \mu\|u\|^2_{L^2(\omega)} \geq \frac{1}{2}\lambda_1(\Omega_\omega)\|u\|^2, \quad \text{for every} \ \ \mu > \mu_0. \tag{2.22}$$

Thus, thanks to condition (2.17), we have

$$E_b(t) \geq \frac{1}{4}\|\partial_t u\|^2 + \frac{1}{4}\|\nabla u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)}. \tag{2.23}$$

Adding to the left-hand side of (2.19) the expression $E_b(t) - \delta E_b(t)$ with some $\delta > 0$ (to be chosen below), we get

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \frac{1}{2}(b-\delta)\|\partial_t u\|^2 + \frac{1}{2}(b-\delta)\|\nabla u\|^2 + \frac{1}{2}(a\delta - ab - \frac{\delta b^2}{2})\|u\|^2 + \frac{\mu}{2}(b-\delta)\|u\|^2_{L^2(\omega)} - \frac{1}{2}\delta b(u, \partial_t u) = 0$$

We use here the inequality

$$\frac{1}{2}\delta b|(u, \partial_t u)| \le \frac{b}{4}\|\partial_t u\|^2 + \frac{b\delta^2}{4}\|u\|^2,$$

then in the resulting inequality we choose $\delta = \frac{b}{2}$, and get:

$$\frac{d}{dt}E_b(t) + \frac{b}{2}E_b(t) + \frac{b}{4}\left[\|\nabla u\|^2 + \mu\|u\|^2_{L^2(\omega)} - (2a + \frac{3}{4}b^2)\|u\|^2\right].$$

Finally, by using the condition (2.17), thanks to Lemma 2.3 we obtain

$$\frac{d}{dt}E_b(t) + \frac{b}{2}E_b(t) \le 0.$$

Integrating the last inequality and taking into account (2.23), we arrive at the desired estimate (2.18).

□

3. Stabilization employing finitely many Fourier modes feedback controls. In this section we consider the feedback control problem for damped nonlinear wave equation based on finitely many Fourier modes, i.e. we consider the feedback system of the following form:

$$\partial_t^2 u - \nu\Delta u + b\partial_t u - au + |u|^{p-2}u = -\mu\sum_{k=1}^{N}(u, w_k)w_k, \ x \in \Omega, t > 0, \tag{2.24}$$

$$u = 0, \ x \in \partial\Omega, \ t > 0, \tag{2.25}$$

$$u(x, 0 = u_0(x), \ \partial_t u(x, 0) = u_1(x), \quad x \in \Omega, \ t > 0. \tag{2.26}$$

Here $\nu > 0, a > 0, b > 0, \mu > 0, p \ge 2$ are given numbers; $w_1, w_2, ..., w_n, ...$ is the set of orthonormal (in $L^2(\Omega)$) eigenfunctions of the Laplace operator $-\Delta$ under the homogeneous Dirichlet boundary condition, corresponding to eigenvalues $0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_n \le, \cdots$.

Theorem 2.5. *Suppose that μ and N are large enough such that*

$$\nu \ge (2a + 3b^2/4)\lambda_{N+1}^{-1}, \quad \textit{and} \quad \mu \ge a + 3b^2/4. \tag{2.27}$$

Then the following decay estimate holds true

$$\|\partial_t u(t)\|^2 + \|\nabla u(t)\|^2 + \int_\Omega |u(x,t)|^p dx \le E_0 e^{-\frac{b}{2}t}, \tag{2.28}$$

where

$$E_0 := \frac{1}{2}\|u_1\|^2 + \frac{\nu}{2}\|\nabla u_0\|^2 + (\frac{b^2}{4} - \frac{a}{2})\|u_0\|^2 + \frac{1}{p}\int_\Omega |u_0(x)|^p dx + \frac{\mu}{2}\sum_{k=1}^N (u_0, w_k)^2 + \frac{b}{2}(u_0, u_1).$$

Proof. Multiplication of (2.24) by $\partial_t u + \frac{b}{2}u$ and integration over Ω gives

$$\frac{d}{dt}E_b(t) + \frac{b}{2}\left[\|\partial_t u\|^2 + \nu\|\nabla u\|^2 - a\|u(t)\|^2 + \int_\Omega |u|^p dx + \mu\sum_{k=1}^N (u, w_k)^2\right] = 0, \qquad (2.29)$$

where

$$E_b(t) := \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{\nu}{2}\|\nabla u(t)\|^2 + (\frac{b^2}{4} - \frac{a}{2})\|u(t)\|^2 + \frac{1}{p}\int_\Omega |u(x,t)|^p dx + \frac{\mu}{2}\sum_{k=1}^N (u(t), w_k)^2 + \frac{b}{2}(u(t), \partial_t u(t)).$$

Thanks to the Cauchy-Schwarz and Young inequalities we have

$$\frac{b}{2}|(u(t), \partial_t u(t))| \le \frac{1}{4}\|\partial_t u\|^2 + \frac{b^2}{4}\|u\|^2.$$

Consequently,

$$E_b(t) \ge \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 - \frac{a}{2}\|u\|^2 + \frac{1}{p}\int_\Omega |u|^p dx + \frac{\mu}{2}\sum_{k=1}^N (u, w_k)^2.$$

Since

$$-\frac{a}{2}\|u\|^2 + \frac{\mu}{2}\sum_{k=1}^N (u, w_k)^2 = \frac{1}{2}(\mu - a)\sum_{k=1}^N (u, w_k)^2 - \frac{a}{2}\sum_{k=N+1}^\infty (u, w_k)^2,$$

by using the Poincaré -like inequality

$$\|\phi - \sum_{k=1}^N (\phi, w_k)w_k\|^2 \le \lambda_{N+1}^{-1}\|\nabla\phi\|^2, \qquad (2.30)$$

which is valid for each $\phi \in H_0^1(\Omega)$, we get

$$E_b(t) \ge \frac{1}{4}\|\partial_t u\|^2 + \frac{1}{2}\left(\nu - a\lambda_{N+1}^{-1}\right)\|\nabla u\|^2 + \frac{1}{2}(\mu - a)\sum_{k=1}^N (u, w_k)^2 + \frac{1}{p}\int_\Omega |u|^p dx. \qquad (2.31)$$

Thus,

$$E_b(t) \ge \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{4}\|\nabla u\|^2 + \frac{1}{p}\int_\Omega |u|^p dx. \qquad (2.32)$$

Adding to the left-hand side of (2.29) the expression $\delta E_b(t) - \delta E_b(t)$ (with δ to be chosen later), we obtain

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \frac{1}{2}(b-\delta)\|\partial_t u\|^2 + \frac{\nu}{2}(b-\delta)\|\nabla u\|^2 + \left(-\frac{ba}{2} + \frac{\delta a}{2} - \frac{\delta b^2}{4}\right)\|u^2\| + \left(\frac{b}{2} - \frac{\delta}{2}\right)\|u\|^p_{L^p(\Omega)} + \frac{\mu}{2}(b-\delta)\sum_{k=1}^{N}(u,w_k)^2 - \frac{1}{2}b\delta(u,\partial_t u) = 0.$$

We choose here $\delta = \frac{b}{2}$, and obtain

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \frac{b}{4}\|\partial_t u\|^2 + \frac{\nu b}{4}\|\nabla u\|^2 - \left(\frac{ba}{4} + \frac{b^3}{8}\right)\|u\|^2 + \frac{\mu b}{4}\sum_{k=1}^{N}(u,w_k)^2 - \frac{b^2}{4}(u,\partial_t u) = 0. \quad (2.33)$$

Employing the inequality

$$\frac{b^2}{4}|(u,\partial_t u)| \le \frac{b}{4}\|\partial_t u\|^2 + \frac{b^3}{16}\|u\|^2,$$

and inequality (2.30), we obtain from (2.34):

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \frac{b}{4}\left[\nu - (a + 3b^2/4)\lambda_{N+1}^{-1}\right]\|\nabla u\|^2 + \frac{b}{4}\left[\mu - (a + 3b^2/4)\right]\sum_{k=1}^{N}(u,w_k)^2. \quad (2.34)$$

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \left(\frac{b}{2} - \frac{\delta}{2}\right)\|\partial_t u\|^2 + \left(\frac{b}{2}(\nu - a\lambda_{N+1}^{-1}) - \frac{\delta\nu}{2}\right)\|\nabla u\|^2 + \left(\frac{b}{2}(\mu - a) - \frac{\delta\mu}{2}\right)\sum_{k=1}^{N}(u,w_k)^2 + \left(\frac{b}{2} - \frac{\delta}{p}\right)\int_\Omega |u|^p dx - \delta\left(\frac{b^2}{4} - \frac{a}{2}\right)\|u\|^2 + \frac{\delta b}{2}(u,\partial_t u) \le 0. \quad (2.35)$$

By using the inequalities

$$\frac{\delta b}{2}|(u,\partial_t u)| \le \frac{b}{4}\|\partial_t u\|^2 + \frac{\delta^2 b}{4}\|u\|^2,$$

and (2.30) in (2.35) we get

$$\frac{d}{dt}E_b(t) + \delta E_b(t) + \left(\frac{b}{4} - \frac{\delta}{2}\right)\|\partial_t u\|^2 + \left(\frac{b}{2}(\nu - a\lambda_{N+1}^{-1}) - \frac{\delta^2 b}{4}\lambda_{N+1}^{-1} - \frac{\delta\nu}{2}\right)\|\nabla u\|^2 + \left(\frac{b}{2}(\mu - a) - \frac{\delta\mu}{2}\right)\sum_{k=1}^{N}(u,w_k)^2 + \left(\frac{b}{2} - \frac{\delta}{p}\right)\int_\Omega |u|^p dx - \delta\left(\frac{b^2}{4} - \frac{a}{2} - \frac{\delta^2 b}{4}\right)\|u\|^2 \le 0.$$

By choosing $\delta = \frac{b}{2}$ we obtain :

$$\frac{d}{dt}E_b(t) + \frac{b}{2}E_b(t) + \frac{b}{2}\left[\nu - (\frac{b^2}{8} + \frac{3}{2}a)\lambda_{N+1}^{-1}\right]\|\nabla u\|^2 + \frac{b}{2}\left[\frac{\mu}{2} - \frac{3}{2}a - \frac{b^2}{8}\right]\sum_{k=1}^{N}(u, w_k)^2 \leq 0.$$

Taking into account conditions (2.27) we deduce from the last inequality the inequality

$$\frac{d}{dt}E_b(t) + \frac{b}{2}E_b(t) \leq 0.$$

Integrating the last inequality we get the desired estimate (2.28) thanks to (2.32). □

Remark 2.6. We would like to note that estimate (2.28) allows us to prove existence of a weak solution to the problem (2.24), (2.25) such that (see [24])

$$u \in L^\infty\left(\mathbb{R}^+; H_0^1(\Omega) \cap L^p(\Omega)\right),\ u \in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\right)$$

Note that there are no restrictions on the spatial dimension of the domain Ω or the growth of nonlinearity.

3. Nonlinear Wave Equation with Nonlinear Damping term:Stabilization with finitely many Fourier modes

In this section we consider the initial boundary value problem for a nonlinear wave equation with nonlinear damping term with a feedback controller involving finitely many Fourier modes:

$$\partial_t^2 u - \nu\Delta u + b|\partial_t u|^{m-2}\partial_t u - au + |u|^{p-2}u = -\mu\sum_{k=1}^{N}(u, w_k)w_k,\ x \in \Omega, t > 0, \tag{3.1}$$

$$u = 0,\ x \in \partial\Omega,\ t > 0, \tag{3.2}$$

$$u(x, 0) = u_0(x),\quad \partial_t u(x, 0) = u_1(x),\quad x \in \Omega, \tag{3.3}$$

where $\nu > 0, b > 0, a > 0, p \geq m > 2$ are given parameters.
We show stabilization of solutions of this problem to the zero with a polynomial rate. Our main result is the following theorem:

Theorem 3.1. *Suppose that μ ad N are large enough such that*

$$\nu > 2a\lambda_{N+1}^{-1},\quad \text{and}\quad \mu > a. \tag{3.4}$$

Then for each solution of the problem (3.1)-(3.3) *following estimate holds true*

$$\|\partial_t u(t)\|^2 + \|\nabla u(t)\|^2 + \int_\Omega |u(x, t)|^p dx \leq Ct^{-\frac{m-1}{m}}, \tag{3.5}$$

where C is a positive constant depending on initial data.

Proof. Taking the inner product of equation (3.1) in $L^2(\Omega)$ with $\partial_t u$ we get

$$\frac{d}{dt}\mathcal{E}(t) + b\|\partial_t u(t)\|^m = 0, \tag{3.6}$$

where

$$\mathcal{E}(t) := \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 - \frac{a}{2}\|u(t)\|^2 + \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)} + \frac{\mu}{2}\sum_{k=1}^{N}(u(t), w_k)^2. \quad (3.7)$$

Similar to (2.31) we have

$$\mathcal{E}(t) \geq \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}\left(\nu - a\lambda_{N+1}^{-1}\right)\|\nabla u\|^2 + \frac{1}{2}(\mu - a)\sum_{k=1}^{N}(u, w_k)^2 + \frac{1}{p}\int_\Omega |u|^p dx$$
$$\geq \frac{1}{2}\|\partial_t u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 + \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)}. \quad (3.8)$$

This estimate implies that the function $\mathcal{E}(t)$ is non-negative, for $t \geq 0$. Let us integrate (3.6) over the interval $(0, t)$:

$$\mathcal{E}(0) - \mathcal{E}(t) = \int_0^t \|\partial_t u(\tau)\|_m^m d\tau. \quad (3.9)$$

Taking the inner product of (3.1) in $L^2(\Omega)$ with u gives

$$\frac{d}{dt}(u, \partial_t u) = \|\partial_t u\|^2 - \nu\|\nabla u\|^2 - b\int_\Omega u|\partial_t u|^{m-2}\partial_t u dx + a\|u\|^2 - \int_\Omega |u|^p dx - \mu\sum_{k=1}^{N}(u, w_k)^2.$$

By using notation (3.7) we can rewrite the last relation in the following form:

$$\frac{d}{dt}(u, \partial_t u) = \frac{3}{2}\|\partial_t u\|^2 - \mathcal{E}(t) - \frac{\nu}{2}\|\nabla u\|^2$$
$$+ \frac{a}{2}\|u\|^2 - \frac{\mu}{2}\sum_{k=1}^{N}(u, w_k)^2 - \frac{p-1}{p}\int_\Omega |u|^p dx - b\int_\Omega u|\partial_t u|^{m-2}\partial_t u dx. \quad (3.10)$$

By using inequality (2.30) we obtain from (3.10)

$$\frac{d}{dt}(u, \partial_t u) \leq -\mathcal{E}(t) + \frac{3}{2}\|\partial_t u\|^2 - \frac{1}{2}(\mu - a)\sum_{k=1}^{N}(u, w_k)^2$$
$$+ \frac{a}{2}\sum_{k=N+1}^{\infty}(u, w_k)^2 - \frac{\nu}{2}\|\nabla u\|^2 + b\int_\Omega |u||\partial_t u|^{m-1}dx \leq -\mathcal{E}(t) + \frac{3}{2}\|\partial_t u\|^2$$
$$- \frac{1}{2}(\mu - a)\sum_{k=1}^{N}(u, w_k)^2 - \frac{1}{2}(\nu - a\lambda_{N+1}^{-1})\|\nabla u\|^2 + b\int_\Omega |u||\partial_t u|^{m-1}dx.$$

Taking into account condition (3.4) we deduce the inequality

$$\mathcal{E}(t) \leq -\frac{d}{dt}(u(t), \partial_t u(t)) + \frac{3}{2}\|\partial_t u(t)\|^2 + b\int_\Omega |u(x,t)|\partial_t u(x,t)|^{m-1}dx.$$

After integration over the interval $(0,t)$, we obtain

$$\int_0^t \mathcal{E}(\tau)d\tau \leq (u(0),\partial_t u(0)) - (u(t),\partial_t u(t)) + \frac{3}{2}\int_0^t \|\partial_t u(\tau)\|^2 d\tau + b\int_0^t\int_\Omega |u(x,\tau)||\partial_t u(x,\tau)|^{m-1}dxd\tau. \quad (3.11)$$

Since $\mathcal{E}(t)\leq \mathcal{E}(0)$ and $p\geq 2$, then due to (3.8), we have

$$|(u(0),\partial_t u(0)) - (u(t),\partial_t u(t))| \leq C, \quad (3.12)$$

where C depends on the initial data.
By using the Hölder inequality and estimate (3.9), we estimate the second term on the right-hand side of (3.11):

$$\int_0^t \|\partial_t u(\tau)\|^2 d\tau = \int_0^t\int_\Omega |\partial_t u(x,\tau)|^2\cdot 1dxd\tau \leq \left(\int_0^t \|\partial_t u(\tau)\|^m_{L^m(\Omega)}d\tau\right)^{\frac{2}{m}} (|\Omega|t)^{\frac{m-2}{m}} \leq Ct^{\frac{m-2}{m}}. \quad (3.13)$$

The third term on the right-hand side of (3.11) we estimate again by using the estimates (3.9) similarly (recalling that $m\leq p$):

$$\int_0^t\int_\Omega |\partial_t u(x,\tau)|^{m-1}|u(x,\tau)|dxd\tau \leq \left(\int_0^t\int_\Omega |\partial_t u(x,\tau)|^m dxd\tau\right)^{\frac{m-1}{m}}\left(\int_0^t\int_\Omega |u(x,\tau)|^m dxd\tau\right)^{\frac{1}{m}} \leq Ct^{\frac{1}{m}}. \quad (3.14)$$

Since $\mathcal{E}(t)$ is non-increasing, positive function we have:

$$t\mathcal{E}(t)\leq \int_0^t \mathcal{E}(\tau)d\tau. \quad (3.15)$$

Thus employing (3.12)-(3.15) we obtain from (3.11)

$$\mathcal{E}(t)\leq Ct^{-\frac{m-1}{m}}.$$

Hence

$$\|\partial_t u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^p_{L^p(\Omega)} \leq Ct^{-\frac{m-1}{m}}. \quad (3.16)$$

□

Remark 3.2. We would like to note that unlike the result on finite set of functionals determining long time behavior of solutions to equation

$$\partial_t^2 u - \nu\Delta u + b|\partial_t u|^{m-2}\partial_t u - au + |u|^{p-2}u = 0,\ x\in\Omega, t>0,$$

under the homogeneous Dirichlet's boundary condition, established in [7], (as in the case of the equation (2.24)), we do not require restrictions neither on the dimension of the domain Ω nor on the parameters $m>0, p>0$. It suffices to know that problem (2.24)-(2.26) has a global solution such that (see, e.g., [24]):

$$u\in L^\infty\left(\mathbb{R}^+; H_0^1(\Omega)\cap L^p(\Omega)\right)\cap L^{m+2}(\mathbb{R}^+; L^{m+2}(\Omega)),\ u\in L^\infty\left(\mathbb{R}^+; L^2(\Omega)\right).$$

4. Nonlinear Strongly Damped Wave equation

In this section, we study the problem of feedback control of initial boundary value problem for nonlinear strongly damped equation with controllers involving finitely many Fourier modes and by nodal observables.

1. Feedback control employing finitely many nodal valued observables.

First we consider first the following problem

$$\partial_t^2 u - \partial_x^2 u - b\partial_x^2\partial_t u - au + f(u) = -\mu \sum_{k=1}^{N} hu(\bar{x}_k)\delta(x - x_k),\ x \in (0, L), t > 0, \tag{4.1}$$

$$u(0, t) = u(L, t) = 0,\ x \in (0, L), t > 0, \tag{4.2}$$

where $x_k, \bar{x}_k \in J_k = [(k-1)\frac{L}{N}, k\frac{L}{N}], k = 1, ..., N, h = \frac{L}{N}$, $f(\cdot)$ is continuously differentiable function that satisfies the conditions (1.4), $\delta(x - x_k)$ is the Dirac delta function, a, b and μ are given positive parameters.

Our estimates will be based on the following lemma:

Lemma 4.1. *(see,e.g., [1]) Let $x_k, \overline{x}_k \in J_k = [(k-1)\,h, k\,h], k = 1, .., N$, where $h = \frac{L}{N}$, $N \in \mathbb{Z}^+$. Then for every $\varphi \in H^1(0, L)$ the following inequalities hold true*

$$\sum_{k=1}^{N} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 \le\ h\,\|\varphi_x\|_{L^2}^2, \tag{4.3}$$

and

$$\|\varphi\|^2 \le\ 2\left[h\sum_{k=1}^{N}|\varphi(x_k)|^2 + h^2\,\|\varphi_x\|^2\right]. \tag{4.4}$$

Taking the H^{-1} action of (4.1) on $\partial_t u + \varepsilon u \in H^1$, where $\varepsilon > 0$, to be determined, we get

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}(1+\varepsilon b)\|\partial_x u\|^2 - a\|u\|^2 + (F(u), 1) + \varepsilon(u, \partial_t u)\right] + \\ b\|\partial_t\partial_x u\|^2 + \varepsilon\|\partial_x u\| + \varepsilon(f(u), u) - a\varepsilon\|u\|^2 - \varepsilon\|\partial_t u\|^2 = \\ -\mu h\sum_{k=1}^{N} u(\bar{x}_k)\partial_t u(x_k) - \varepsilon\mu h\sum_{k=1}^{N} u(\bar{x}_k)u(x_k).\end{aligned} \tag{4.5}$$

By using the equalities

$$\sum_{k=1}^{N} u(\bar{x}_k)\partial_t u(x_k) = \frac{1}{2}\frac{d}{dt}\sum_{k=1}^{N} u^2(\bar{x}_k) + \sum_{k=1}^{N} u(\bar{x}_k)\left(\partial_t u(x_k) - \partial_t u(\bar{x}_k)\right)$$

and

$$\sum_{k=1}^{N} u(\bar{x}_k)u(x_k) = \sum_{k=1}^{N} u^2(\bar{x}_k) + \sum_{k=1}^{N} u(\bar{x}_k)\left(\partial_t u(x_k) - u(\bar{x}_k)\right),$$

we can rewrite (4.5) in the following form

$$\frac{d}{dt}E_\varepsilon(t) + b\|u_{xt}\|^2 + \varepsilon\|\partial_x u\| + \varepsilon(f(u),u) - a\varepsilon\|u\|^2 - \varepsilon\|\partial_t u\|^2 + \varepsilon\mu h\sum_{k=1}^N u^2(\bar{x}_k) =$$
$$- \mu h\sum_{k=1}^N u(\bar{x}_k)\left(\partial_t u(x_k) - \partial_t u(\bar{x}_k)\right) - \varepsilon\mu h\sum_{k=1}^N u(\bar{x}_k)\left(u(x_k) - u(\bar{x}_k)\right), \quad (4.6)$$

where

$$E_\varepsilon(t) := \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}(1+\varepsilon b)\|\partial_x u\|^2 - a\|u\|^2 + (F(u),1) + \varepsilon(u,\partial_t u) + \frac{\mu h}{2}\sum_{k=1}^N u^2(\bar{x}_k).$$

By using inequality (4.4) we obtain

$$E_\varepsilon(t) \geq \frac{1}{4}\|\partial_t u\|^2 + \left[\frac{1}{2}(1+\varepsilon b) - 2(a+\varepsilon^2)h^2\right]\|\partial_x u\|^2 + \varepsilon(F(u),1) +$$
$$\left[\frac{\mu h}{2} - 2(a+\varepsilon^2)h\right]\sum_{k=1}^N u^2(\bar{x}_k). \quad (4.7)$$

Employing (4.3) we get

$$\mu h\sum_{k=1}^N u(\bar{x}_k)\left(\partial_t u(x_k) - \partial_t u(\bar{x}_k)\right) \leq \frac{\varepsilon\mu h}{4}\sum_{k=1}^N u^2(\bar{x}_k) + \frac{\mu h^2}{\varepsilon}\|\partial_x\partial_t u\|^2, \quad (4.8)$$

$$\mu h\sum_{k=1}^N u(\bar{x}_k)\left(u(x_k) - u(\bar{x}_k)\right) \leq \frac{\varepsilon\mu h}{4}\sum_{k=1}^N u^2(\bar{x}_k) + \frac{\mu h^2}{\varepsilon}\|\partial_x u\|^2. \quad (4.9)$$

Now we choose $\varepsilon = \frac{b\lambda_1}{2}$, use the Poincaré inequality, (1.6), and inequalities (4.8) and (4.9) in (4.6) and obtain:

$$\frac{d}{dt}E_\varepsilon(t) + \frac{b}{2}\|\partial_x\partial_t u\|^2 + \left(\varepsilon - \frac{4}{\varepsilon}\mu h^2\right)\|\partial_x u\|^2 + \varepsilon(F(u),1) - a\varepsilon\|u\|^2 +$$
$$\frac{1}{2}\varepsilon\mu h\sum_{k=1}^N u^2(\bar{x}_k) \leq 0.$$

We employ here inequality (4.4) to obtain

$$\frac{d}{dt}E_\varepsilon(t) + \frac{b}{2}\|u_{xt}\|^2 + \left(\varepsilon - \frac{4}{\varepsilon}\mu h^2 - 4a\varepsilon h^2\right)\|\partial_x u\|^2 + \varepsilon(F(u),1) +$$
$$(\frac{1}{2}\varepsilon\mu h - 2a\varepsilon h)\sum_{k=1}^N u^2(\bar{x}_k) \leq 0. \quad (4.10)$$

It is not difficult to see that if μ is large enough such that

$$\mu > 4\left(a + \frac{\lambda_1^2 b^2}{4}\right), \quad (4.11)$$

and $N = \frac{L}{h}$ is large enough such that

$$\frac{\lambda_1 b}{2} - 2h^2\left(\frac{\mu}{\lambda_1 b} - a\lambda_1 b\right) > 0, \tag{4.12}$$

$$\frac{b^2\lambda_1^2}{4} - a^2\lambda_1^2 b^2 h^2 - \mu h^2 > 0, \tag{4.13}$$

then there exists $d_1 > 0$ such that

$$E_\varepsilon(t) \geq d_1\left(\|\partial_t u\|^2 + \|\partial_x u\|^2\right), \tag{4.14}$$

and that there exists a positive number δ such that

$$\frac{b}{2}\|u_{xt}\|^2 + \left(\varepsilon - \frac{4}{\varepsilon}\mu h^2\right)\|\partial_x u\|^2 + \varepsilon(F(u),1) - a\varepsilon\|u\|^2 + \frac{1}{2}\varepsilon\mu h\sum_{k=1}^N u^2(\bar{x}_k) \geq \delta E_\varepsilon(t). \tag{4.15}$$

By virtue of (4.15) we deduce from (4.10) the inequality

$$\frac{d}{dt}E_\varepsilon(t) + \delta E_\varepsilon(t) \leq 0.$$

This inequality and inequality (4.14) imply the exponential stabilization estimate

$$\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 \leq D_0 e^{-\delta t}. \tag{4.16}$$

Consequently we have proved the following:

Theorem 4.2. *Suppose that conditions* (4.11)-(4.13) *are satisfied. Then all solutions of the problem* (4.1)-(4.2) *tend to zero with an exponential rate, as* $t \to \infty$.

Remark 4.3. By using similar arguments we can prove an analog to Theorem 4.2 for solutions of the semilinear pseudo-hyperbolic equation

$$\partial_t^2 u - u_{xxtt} - \nu\partial_x^2 u - bu_{xxt} - au + f(u) = -\mu\sum_{k=1}^N hu(\bar{x}_k)\delta(x - x_k),\ x \in (0,L), t > 0, \tag{4.17}$$

under the boundary conditions (4.2) or under the periodic boundary conditions. Here a, b, ν are positive parameters, and f satisfies conditions (1.4), μ and N should be chosen appropriately large enough.

4.1. **Feedback control employing finitely many Fourier modes.** The second problem we are going to study in this section is the following feedback problem

$$\begin{cases} \partial_t^2 u - \nu\Delta u - b\Delta\partial_t u - au + |u|^{p-2}u = -\mu\sum\limits_{k=1}^N (u, w_k)w_k,\ x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ x \in \Omega, \\ u = 0,\ x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.18}$$

where $\nu > 0, b > 0, a > 0, p > 2$ are given parameters.
The following theorem guarantees the exponential feedback stabilization of solutions of problem (4.18)

Theorem 4.4. *Suppose that μ is large enough such that*

$$\mu > 2a + \frac{1}{4}\delta_0\lambda_1 b, \tag{4.19}$$

and N is large enough such that

$$\nu \geq \left(2a + \frac{1}{4}b\lambda_1\delta_0\right)\lambda_{N+1}^{-1}, \tag{4.20}$$

where

$$\delta_0 := \frac{b\lambda_1\nu}{2\nu + b^2\lambda_1}. \tag{4.21}$$

Then the solution of (4.18) *satisfies the following exponential decay estimate:*

$$\|\partial_t u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^p_{L^p(\Omega)} \leq E_0 e^{-\delta_0 t}, \text{ for all} t > 0. \tag{4.22}$$

with a constant E_0 depending on initial data.

Proof. The proof of this theorem is similar to the proof of Theorem 2.5.
The energy equality in this case has the form

$$\frac{d}{dt}E_\varepsilon(t) + b\|\nabla\partial_t u(t)\|^2 - \varepsilon\|\partial_t u(t)\|^2 + \varepsilon\nu\|\nabla u(t)\|^2 + \varepsilon\int_\Omega |u(x,t)|^p dx - \varepsilon a\|u\|^2 + \varepsilon\mu\sum_{k=1}^N (u(t), w_k)^2 = 0, \tag{4.23}$$

where $\varepsilon = \frac{b\lambda_1}{2}$, and

$$E_\varepsilon(t) := \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}(\nu + \varepsilon b)\|\nabla u\|^2 - \frac{a}{2}\|u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} + \frac{\mu}{2}\sum_{k=1}^n (u, w_k)^2 + \varepsilon(u, \partial_t u).$$

Employing the Young inequality (1.5), the Poincaré inequality (1.6), the Poincaré-like inequality (1.6), and the conditions (4.19) and (4.20) we get:

$$\begin{aligned}
E_\varepsilon(t) &\geq \frac{1}{4}\|\partial_t u\|^2 + \frac{1}{2}\left(\nu + \frac{b^2\lambda_1}{2}\right)\|\nabla u\|^2 - \left(\frac{a}{2} + \frac{b^2\lambda_1^2}{4}\right)\|u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} \\
&\quad + \frac{\mu}{2}\sum_{k=1}^n (u, w_k)^2 \geq \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 - \frac{a}{2}\|u\|^2 + \frac{\mu}{2}\sum_{k=1}^n (u, w_k)^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} \\
&\geq \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{4}\|\nabla u\|^2 + \left(\frac{\nu}{4} - \frac{a}{2}\lambda_{N+1}^{-1}\right)\|\nabla u\|^2 + \left(\frac{\mu}{2} - \frac{a}{2}\right)\sum_{k=1}^n (u, w_k)^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)} \\
&\geq \frac{1}{4}\|\partial_t u\|^2 + \frac{\nu}{4}\|\nabla u\|^2 + \frac{1}{p}\|u\|^p_{L^p(\Omega)}.
\end{aligned} \tag{4.24}$$

Let $\delta \in (0, \varepsilon)$, be another parameter to be chosen below. Then employing the Poincaré inequality, and the fact that $\delta < \varepsilon = \lambda_1 b/2$, and $p > 2$, we obtain from (4.23)

$$\frac{d}{dt}E_\varepsilon(t) + \delta E_\varepsilon(t) + \left(\frac{b}{2} - \frac{\delta}{2\lambda_1}\right)\|\nabla\partial_t u\|^2 + \left(\frac{b\lambda_1\nu}{2} - \frac{\delta}{2}(\nu + \frac{b^2\lambda_1}{2})\right)\|\nabla u\|^2 - \frac{ab\lambda_1}{4}\|u\|^2 + \left(\frac{b\lambda_1}{2} - \frac{\delta}{p}\right)\|u\|^p_{L^p(\Omega)} + \frac{\mu}{2}(b\lambda_1 - \delta)\sum_{k=1}^n (u, w_k)^2 - \frac{\delta\lambda_1 b}{2}(u, \partial_t u) \leq 0. \quad (4.25)$$

By using the inequality

$$\frac{1}{2}\delta\lambda_1 b|(u, \partial_t u)| \leq \frac{1}{2}\delta\lambda_1^{\frac{1}{2}} b\|\nabla\partial_t u\|\|u\| \leq \frac{\delta}{2\lambda_1}\|\nabla\partial_t u\|^2 + \frac{\delta}{8}\lambda_1^2 b^2\|u\|^2,$$

and assuming here that $\delta \in (0, \varepsilon)$, we get

$$\frac{d}{dt}E_\varepsilon(t) + \delta E_\varepsilon(t) + \left(\frac{b}{2} - \frac{\delta}{\lambda_1}\right)\|\nabla\partial_t u\|^2 + \frac{1}{2}\left[\nu b\lambda_1 - \delta(\nu + \frac{b^2\lambda_1}{2})\right]\|\nabla u\|^2 - \frac{1}{2}\left(ab\lambda_1 + \frac{\delta}{8}\lambda_1^2 b^2\right)\|u\|^2 + \frac{\mu}{2}(b\lambda_1 - \delta)\sum_{k=1}^n (u, w_k)^2. \quad (4.26)$$

We choose in the last inequality $\delta = \delta_0$, defined in (4.21), and obtain the inequality

$$\frac{d}{dt}E_\varepsilon(t) + \delta_0 E_\varepsilon(t) + \frac{1}{4}\nu b\lambda_1\|\nabla u\|^2 - \frac{b\lambda_1}{2}\left(a + \frac{\delta}{8}\lambda_1 b\right)\|u\|^2 + \frac{\mu}{4}b\lambda_1\sum_{k=1}^n (u, w_k)^2. \quad (4.27)$$

Finally, employing in (4.27) inequality (2.30), and the conditions (4.19) and (4.20) we obtain the desired inequality

$$\frac{d}{dt}E_\varepsilon(t) + \delta_0 E_\varepsilon(t) \leq 0.$$

Thanks to (4.24), this inequality implies the desired decay estimate (4.22). □

Remark 4.5. In a similar way we can prove exponential stabilization of solutions to strongly damped Boussinesq equation, with homogeneous Dirichlet boundary conditions

$$\begin{cases} \partial_t^2 u - \nu\partial_x^4 u - b\partial_x^2\partial_t u + \partial_x^2\left(au - |u|^{p-2}u\right) = -\mu w, \ x \in (0, L), t > 0, \\ u(0, t) = u(L, t) = \partial_x^2 u(0, t) = \partial_x^2 u(L, t) = 0, \quad t > 0, \end{cases} \quad (4.28)$$

where a, ν, b are given positive parameters, and w is a controller of the

$$w = \sum_{k=1}^N \lambda_k(u, w_k)w_k, \quad \lambda_k = \frac{k^2\pi^2}{L^2}, \quad w_k(x) = \sin\frac{k\pi}{L}.$$

Here also we can find N and μ large enough such that

$$\|\partial_t u(t)\|^2_{H^{-1}(0,L)} + \|\partial_x^2 u(t)\|^2 \leq E_0 e^{-\delta t}$$

with some $\delta > 0$, depending on parameters of the problem, i.e. ν, b, p and L.

Remark 4.6. It is worth noting that the estimates we obtained in Theorem 2.5 and Theorem 4.4 are valid for weak solutions of the corresponding problems from the class of functions such that

$$\partial_t u \in L^\infty(\mathbb{R}^+; L^2(\Omega)),\ u \in L^\infty(\mathbb{R}^+; H_0^1(\Omega) \cap L^p(\Omega)).$$

Existence of a weak solution for each of these problems, as well as justification of our estimates can be done by using the Galerkin method (see e.g. [24]). We would like also to note that the feedback stabilization estimate for problem (2.24)-(2.25) is established, for arbitrary $p > 2$, without any restrictions on the spatial dimension of the domain Ω. As far as we know, even uniqueness of a weak solution in the case $p > 5, \Omega \subset \mathbb{R}^3$, is an open problem.

Remark 4.7. It is also worth mentioning that the feedback stabilization of nonlinear damped wave equation, nonlinear strongly damped wave equation, nonlinear wave equation with nonlinear damping term with controllers involving finitely many parameters, can be shown by employing the concept of general determining functionals (projections) introduced in [8],[9], then exploited and developed for the study of dissipative wave equations [6].
Recently it was shown in [15], [16] and [18] that the approach of a new feedback controlling of dissipative PDEs using finitely many determining parameters can be used to show that the global (in time) dynamics of the 2D Navier - Stokes equations, and of that of the 1D damped driven nonlinear Schrödinger equation

$$i\partial_t u - \partial_x^2 u + i\gamma u + |u|^2 u = f, \quad b > 0,$$

can be embedded in an infinite-dimensional dynamical system induced by an ordinary differential equation, called *determining form*, governed by a global Lipschitz vector field. The existence of determining form for the long-time dynamics of nonlinear damped wave equation and nonlinear strongly damped wave equation is a subject of a future research.

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(V.K.Kalantarov) Department of mathematics, Koç University,
Rumelifeneri Yolu, Sariyer 34450
Sariyer, Istanbul, Turkey
E-mail address: vkalantarov@ku.edu.tr

(E.S.Titi) Department of Mathematics, Texas AM University, 3368 TAMU,
College Station, TX 77843-3368, USA. **ALSO**,
Department of Computer Science and Applied Mathematics, Weizmann
Institute of Science, Rehovot 76100, Israel.
E-mail address: titi@math.tamu.edu
E-mail address: edriss.titi@weizmann.ac.il