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I. INTRODUCTION

A number of articles have been written on the general theory of accelerators based on the synchrotron principle.[1] The design of the proton synchrotron at Berkeley has raised several specific problems which require a more detailed analysis. In this paper we shall discuss the treatment of some of these problems and the results obtained.

The work can be divided into two main sections. The first, concerning injection, will deal with two schemes for preventing the ions from striking the injector structure and with a system for focusing the incoming beam. The second, concerning the orbits during the acceleration period, will deal with certain effects of the straight sections, stray fields, magnetic inhomogeneities, resonant coupling between the various modes of oscillation, and with the damping effects introduced by the accelerating electrode.

We shall not discuss the important question of how many ions are lost through scattering in the residual gas. Our results on this point are essentially the

1. See, for example, Bohm and Foldy, Phys. Rev. 70, 249 (1946). An extension to the "racetrack" design has been made by Blachmann and Courant, RSI 20, 596 (1949).

same as those obtained at Brookhaven,[2] and agree reasonably well with measure-

2. Blachman and Courant, Phys. Rev. 74, 140 (1948)

ments made on several machines.

The calculations will be carried through explicitly for machines of the type described by W. H. Brobeck.[3] Fig. 1 shows the typical design. There are four mag-

3. W. H. Brobeck, RSI 19, 545 (1948)

netic quadrants connected by field free straight sections in which the accelerating electrode, injector structure, and auxiliary equipment are located. The magnetic field rises more or less linearly during the acceleration period, the frequency of the accelerating voltage varying in such a way that the equilibrium orbit remains at the center of the vacuum chamber. A machine intended to serve as a model is now in operation. A machine four times as large with a maximum proton energy of 3 Bev is under construction; it is planned eventually to raise this maximum to 6 Bev by decreasing the aperture and thereby increasing the maximum magnetic field obtainable. Our numerical results will apply to these three variations, which will be referred to hereafter as machines A, B, and C. Table I is a summary of the design constants.

II. INJECTION PROBLEMS

A. Injection of the ions.

Since the ions available for injection generally will come from a constant current source, it is desirable to make the time during which the machine will accept them as long as possible. Also, one must be sure that a reasonable fraction of the ions clears the injector structure on subsequent turns. The simplest method of injection, which is to introduce the ions at the appropriate time in the magnetic

TABLE I

Summary of Design Constants for Berkeley Machines

(Figures are for protons except where otherwise stated)

General			A	B	C
Radius to center of field		ft.	11.5	48.5	50
Length of straight sections		ft.	5	20	20
Inside height vacuum chamber at center of gap		in.	9.8	24	11.5
Acceleration time		sec.	0.3	1.75	1.85
Repetition rate at full energy		pulses/min.	16	10	10
Fraction remaining after scattering at 10^{-5} mm Hg air pressure (approx.)		%	.16	75	40
Energy gain per turn (equilibrium)	initial	ev	43.5	1142	1,750
	final	ev		684	1,075
Maximum energy	protons	Bev	.0059	3.65	6.44
	deuterons	Bev	.0029	3.0	5.71
	α-particles	Bev	.0059	6.0	11.4
Magnet					
Magnet gap		in.	13.5	24	14
Useful width of field ($1/2 < n < 1$)	at injection	in.	36	68	48
	at final energy	in.	36	60	20
Injection field		gauss	326	303	299
Final field		gauss	1,000	9,800	16,000
Rate of rise of field	initial	gauss/sec.	4,450	6,610	9,600
	final	gauss/sec.		3,950	5,900
Magnetic field exponent "n"			0.60	0.60	0.60
Injection					
Injection energy		Mev	0.625	10	10
Injection time (for beam to reach center of chamber)		ms	3.94	1.090	.498
Turns during injection time			1535	405	181
Radial motion per turn during injection (r.f. off)		in.	0.012	0.084	0.133

cycle, with the accelerating voltage on, has disadvantages on both counts. The acceptance time corresponds to the radial half-width of the phase stable region, which is less than the half-width of the chamber, and the damping of the phase and free oscillations is not sufficient to prevent the ions from eventually striking the injector.

The method which has been adopted here is to allow the ions to spiral in to the center of the chamber under the influence of the increasing magnetic field before the accelerating voltage is turned on. In this way, the acceptance time is made to correspond to the full half-width of the chamber, and at the same time the ions stay well clear of the injector because the phase oscillations during the acceleration period occupy a small central region in the chamber.

The success of this method depends on the ions spiralling in fast enough to clear the injector during the injection period. In the present machines, the instantaneous circle moves in at a rate much less than an inch per turn, so that the ions would surely strike the injector on the first turn were it not for the free radial and vertical oscillations. Since these oscillations may have amplitudes of the order of a few feet, there is a good chance that the ions will pass around and above the injector until the instantaneous circle has moved out of the dangerous region.

We shall proceed to estimate the fraction of ions which clears the injector because of the free oscillations. For the purpose let us consider a circular machine in which the radial and vertical oscillations are of the forms

$$x = a_r \cos \sqrt{1 - n}\ \theta$$

$$z = a_z \sin \sqrt{n}\ \theta$$

respectively, $\theta = 0$ corresponding to the instant of injection. These equations describe the motion of an ion which is injected tangentially, but at an angle to the median plane, the most favorable starting condition. If h_z is the height of the injector and h_r is the radial extension of the injector inward from the point of injection, then the ion will strike the injector on the k^{th} turn if both

$$a_r (1 - \cos 2\pi k \sqrt{1-n}) \leq h_r - k \Delta r$$

$$a_z \left| \sin 2\pi k \sqrt{n} \right| \leq h_z/2$$

where Δr is the change per turn in the radius of the instantaneous circle. Whether or not an ion of given amplitudes and frequencies would strike the injector could be determined by examining these conditions on each turn until the instantaneous circle has moved in a distance equal to h_r. However, the fulfillment of the conditions depends critically upon the frequencies, since the phases of the oscillations on successive turns are involved, and the frequencies will vary considerably with amplitude because of changes in the average value of n and because the oscillations are not strictly linear, even for constant n. We shall attempt to solve the problem by assuming that in a small range of amplitudes a certain fraction of the ions will clear the injector and introduce a probability method.

If the probability that an ion in the amplitude range da_r miss the injector on the k^{th} turn be called p_k, then

$$p_k \, da_r = \left[1 - \frac{1}{\pi^2} \frac{h_z}{a_z} \sqrt{\frac{2(h_r - k\Delta r)}{a_r}} \right] da_r$$

The probability of an ion missing the injector on all turns is then

$$P(a_r)da_r = \prod_{k=1}^{h_r/\Delta r} p_k da_r \cong \exp. \left(- \frac{2\sqrt{2}}{3\pi^2} \frac{h_z}{a_z} \sqrt{\frac{h_r}{\Delta r}, \frac{h_r}{a_r}} \right) da_r$$

Since the radial amplitude is given by the distance from the injector to the instantaneous circle at the time of injection, a_r for successive ions will vary linearly in time from zero to half the width of the chamber, if all the time available for injection is used. The over-all efficiency, η, will be the average of $P(a_r)$ over all amplitudes. For machine A, η = 70 percent, indicating that a sufficient fraction of the ions should be available for acceleration. For the larger machines the efficiency should be greater, since the injector will be relatively smaller.

The influence of the straight sections on the above calculation is small, for although the oscillations are not sinusoidal in the presence of the straight sections, it can be shown that in the neighborhood of the injector the displacement of the ion from its instantaneous cirlce does vary sinusoidally, but with different frequency (see Section IIIA).

If the ion is not injected tangentially, the amplitude is greater than a_r and the ion is in danger for a greater number of turns. An estimate shows that the efficiency drops off sharply with increasing angle of injection. In machine A, for example, the efficiency is down by a factor two for an injection angle of $\sim \pm .3$ degrees.

B. Injection from beyond n = 1.

The possibility of a quite different mechanism for missing the injector is suggested by experience with the betatron. It was found that in certain machines the best yields were obtained with the injector not in the "good" region of the field ($0 < n < 1$), but outside the radius corresponding to $n = 1$, where the orbits are not stable according to the theory of the machine. That the yield should be poor when the electrons are injected in the good region would follow from the argument of the preceding section, for in these betatrons, the injector structure almost fills the vertical aperture. That some yield is obtained when the electrons (or in our case, ions) are injected beyond the $n = 1$ point can be understood qualitatively from a consideration of the forces involved. Fig. 2 shows, as a function of radius, the magnetic force at three successive times during the injection period, together with the centrifugal force, which is independent of time for constant injection energy. The net radial force gives rise to the three "potential" curves shown in Fig. 2b. At time t_1 the magnetic force is equal to the centrifugal force at the n = 1 point; the corresponding potential curve has a horizontal point of inflection there. At the later time t_2 the forces are equal at two points and the potential curve exhibits a well in which stable oscillations can occur. In the original theory, only oscillations near the bottom of this well were considered. At the time t_3 the rim of the well reaches the radius of the injector; at subsequent times injection takes place inside the well, the case discussed in the preceding section.

The ions which are caught in the time interval t_3-t_1 are those which leave the injector with just sufficient radial momentum to clear the rim of the well, cross the well, and return to find that the barrier has risen to prevent escape. They will then stay in the well, oscillating with a moderate amplitude, but never again coming close to the injector.

The efficiency of this mechanism may be estimated in the following way. Let us consider a machine of the bevatron type, in which the energy of the particles does not vary during the injection period. The radial equation of motion is, to lowest order,

$$\frac{d^2x}{\omega^2 dt^2} + \frac{ax^2}{2} = -\frac{\dot{H}_z}{H_z}(t-t_1)$$

where
$$x = \frac{r - r_1}{r_1} \qquad a = -\frac{1}{H_z}\frac{\partial^2}{\partial x^2}\left[(1 + x)H_z\right]$$

$$\omega = \frac{v}{r_1} \qquad r_1 = \text{radius at which } n = 1$$

neglecting the effect of the straight sections.

Since the change in the magnetic field is small during a few revolutions of the ion, an adequate solution may be obtained by calculating first the time required for an ion to travel from the injector across the well and back to the rim, neglecting the variation in the shape of the well, and then calculate the amount by which the barrier has risen during this time. In this way is established a range of initial radial momenta in which ions will be caught. The efficiency is then expressible in terms of the acceptable interval in angle of injection (corresponding to the calculated radial momentum range) and the time interval, $t_3 - t_1$. These two quantities are given by

$$\Delta a = \frac{\overline{\Delta \dot{r}}}{v} = \frac{b}{a}\,\frac{1}{\omega}\cdot\frac{\dot{H}_z}{H_z}\cdot\frac{1}{x_i}$$

$$t_3 - t_1 = \frac{a}{2}\left(\frac{H_z}{\dot{H}_z}\right) x_i^{\,2}$$

where b is a dimensionless constant independent of the various parameters. Its value, obtained by numerical integration, is of the order of 20. x_i is the value of x corresponding to the position of the injector. These expressions are valid for any machine in which the increase in magnetic field per revolution is small and the acceleration due to the changing flux is negligible.

For machine A,

$$\Delta a \cong 0.02 \text{ degrees}$$

$$t_3 - t_1 = 0.31 \text{ millisec.}$$

These numbers may be compared with the acceptable angular range, $\pm 0.3^\circ$ (Section IIA) and the time available, 4 ms., (Table I) under normal injection conditions, remembering that in the normal case only the fraction of ions given in Section IIA succeeds in clearing the injector. We may conclude that as long as the injector structure is small, it should be better to inject inside the n = 1 point. Some rough measurements with machine A confirm this conclusion.

C. Focusing of the incoming beam.

As remarked in Section A, the incoming beam must be well collimated. From this point of view the most promising device for preacceleration is the linear accelerator, which is capable of supplying a beam of high energy and small divergence. The possibility of using a cyclotron has been seriously considered, however, because of its relative simplicity of operation--in fact machine A is now being fed by a .6 Mev cyclotron. Some means of focusing the widely divergent beam must be provided before bringing the ions into the machine, in order to make efficient use of the available current.

To determine whether satisfactory collimation can be obtained by a reasonably simple arrangement, we have investigated the focusing properties of a system consisting of a uniform-field magnetic wedge and the 90° electrostatic deflector used to bring the beam into the bevatron tank (see Fig. 3). Focusing action in the horizontal plane occurs in the wedge because ions entering at different points and angles have different path lengths in the magnetic field. The electrostatic deflector also contributes appreciably to the horizontal focusing because of the non-uniformity of the electric field and because of the variation in kinetic energy as the ions cross the equipotential lines. Vertical focusing takes place only in the fringing field of the magnet. Since the actions are different in the two directions, it is possible to adjust the focal properties more or less independently.

We shall assume that the ions approach the wedge as though emanating from a point source although the distances from source to wedge used in calculating vertical and horizontal effects need not be equal. Such a representation of the source is consistent with tests of beam structure made with the present cyclotron injector. The analysis of orbits in the system is straightforward but messy and we shall only quote the pertinent results. The method is identical to that used by Herzog[4] in a paper concerning orbits in mass spectrographs.

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4. Herzog, Zeits. f. Phys. 89, 447 (1934)

It is possible to adjust the parameters for first order focusing in a way which will satisfy the following three conditions simulataneously:

1) Ions of a given energy shall leave the deflector with no vertical angular divergence.

2) Ions of a given energy shall leave the deflector with no horizontal angular divergence.

3) Ions of different energies entering along the central ray shall leave the deflector parallel to the central ray.

In the notation of Fig. 4, these conditions are represented by the equations:

$$\ell_H'' = L - \frac{R_E}{\sqrt{2}} \cot \sqrt{2}\ \phi_E$$

$$\ell_V' = R_M(1 - \phi_M \tan \varepsilon'')/\left[\tan \varepsilon' + \tan \varepsilon'' - \phi_M \tan \varepsilon' \tan \varepsilon''\right]$$

$$\ell_H'' = \left[R_E - R_M(1 - \cos \phi_M)\right]/\left[\tan \varepsilon''(1 - \cos \phi_M) + \sin \phi_M\right]$$

where

$$\ell_H'' = \frac{R_M\left\{\ell_H'(1 + \tan \varepsilon' \tan \phi_M) + R_M \tan \phi_M\right\}}{\ell_H'\left[\tan \phi_M - \tan\varepsilon' - \tan \varepsilon'' - \tan\phi_M \tan\varepsilon' \tan\varepsilon''\right] - R_M\left[1 + \tan \phi_M \tan\varepsilon''\right]}$$

is the image distance of the wedge alone.

If these conditions are satisfied, the emerging beam will have the following characteristics:

1) horizontal width $\Delta_H = \alpha_H' R_E \cos \phi_M \left[R_M \tan \phi_M + \ell_H'(1 + \tan \phi_M \tan \varepsilon')\right]/\sqrt{2}\,\ell_H'' \sin\sqrt{2}\ \phi_E$

2) vertical width $\Delta_V = \alpha_V'\left[\ell_V'(1 - \phi_M \tan \varepsilon') + R_M \phi_M\right]$

3) displacement due to energy spread $\Delta_E = \Delta R_M \frac{R_E}{R_M}\left[1 + \frac{\sin \phi_M + (1 - \cos \phi_M) \tan \varepsilon''}{\sqrt{2} \sin \sqrt{2}\ \phi_E}\right]$

where α_H' and α_V' are the angular divergences at the effective source positions and ΔR_M is the deviation in radius of curvature in the magnetic field due to an energy spread.

In Table II are listed three sets of values of the parameters which satisfy the three conditions for the cyclotron used with machine A. Since no attempt has been made as yet to focus the beam properly, the usefulness of these figures is not established. A glance at Table II, however, does indicate that the narrowness of the beam is not as good as one might wish, especially since the numerous aberrations will certainly make the collimation much poorer than the table indicates.

Plans are in progress for a 10 Mev linear accelerator which should give at least 20 ma of protons which would enter machine B or C in an area about 1/4 in. in diameter with an angular divergence of about 1 part in 500. Such an injector should be adequate without the aid of additional focusing devices, and so cyclotron injection has been tentatively abandoned.

III. ION ORBITS DURING ACCELERATION

A. Effects of stray fields in the straight sections.

It has been pointed out[5] that the introduction of straight sections influences

5. R. Serber, Phys. Rev. 70, 434 (1946), and D. M. Dennison and T. Berlin, Phys. Rev. 70, 654 (1946)

the frequencies and amplitudes of the free oscillations. In these discussions the magnetic field was assumed to end abruptly at the ends of the circular sections. We have found it necessary to include the effect of stray fields in the straight sections, particularly because of machine A, which was designed conservatively with a large relative aperture. It will, in fact, turn out that the correction in frequency in the case of machine A is of the same order of magnitude as the change due to the straight sections themselves.

There are several ways of attacking the problem. We shall present what seems to us the most direct method, which consists of setting up and solving recursion relations between the amplitudes and phases of the oscillations in

TABLE II

Solutions of the focusing equations for three sets of parameters. The values

$$R_M = 10'' \quad R_E = 25'' \quad \ell_H' = 35'' \quad \ell_V' = 30'' \quad \phi_E = 90^\circ$$

are appropriate to the cyclotron injector used with machine A.

ϕ_M	L	ε'	ε''	ℓ_H''	$\Delta_H(\alpha_H' = 1^\circ)$	$\Delta_V(\alpha_V' = 1^\circ)$	$\Delta_E(\frac{\Delta R_M}{R_M} = .01)$
45°	20"	24.8°	-8.5°	33"	.52"	.47"	.40"
30°	39"	37.2°	-26.6°	52"	.32"	.40"	.35"
20°	68"	42.0°	-35.0°	81"	.2"	.42"	.32"

successive quadrants.

In treating the free radial oscillations, the phase oscillations may be neglected because of their long period, and the variation in magnetic field with time is negligible during several periods of free oscillation. Under these conditions, time may be eliminated from the equations of motion by making use of the energy, which is a constant of the motion. Then

$$(1)\quad \frac{d^2x}{d\Theta^2} = \frac{2x'^2}{1+x} + 1 + x - (1+x)^{2-n}\left[1 + \left(\frac{x'}{1+x}\right)^2\right]^{3/2}$$

in the region of each quadrant where the stray field is negligible, where

$$x = \frac{r-r_o}{r_o}, \quad x' = \frac{dx}{d\Theta}, \quad H = H_o\left(\frac{r_o}{r}\right)^n$$

and r_o is the radius of the instantaneous circle. We cannot neglect quadratic terms in x, because these terms give an effect comparable to the effect of the stray fields. To this approximation (1) becomes

$$(2)\quad x'' + (1-n)\,x = \frac{x'^2}{2} - \frac{(2-n)(1-n)}{2}\,x^2$$

of which the solution for the k^{th} quadrant, to the same approximation, is

$$(3)\quad x_k(\Theta_k) = A_k e^{i\sqrt{1-n}\,\Theta_k} + B_k e^{-i\sqrt{1-n}\,\Theta_k} + \frac{3-n}{6}A_k^2 e^{2i\sqrt{1-n}\,\Theta_k} + \frac{3-n}{6}B_k^2 e^{-2i\sqrt{1-n}\,\Theta_k} - (1-n)A_kB_k$$

where the angle Θ_k is measured from the center of the k^{th} quadrant.

In order to set up recursion relations between A_k, B_k and A_{k+1}, B_{k+1}, we must evaluate the change in position and slope in traversing the fringing fields and the straight sections. In the notation of Fig. 4

$$(4)\quad x_{k+1}(-\pi/4) - x_k(\pi/4) = \frac{x_k'(\pi/4) + x_{k+1}'(-\pi/4)}{2}\,L/r_o$$

$$(5)\quad x'_{k+1}(-\pi/4) - x'_k(\pi/4) = -2\delta - \delta_r\left(x_{k+1}(-\pi/4) + x_k(\pi/4)\right)$$

where
$$\delta = \int_0^{L/2} \frac{H(s)}{H_o}\frac{ds}{r_o} - \int_0^{\pi/4} \frac{H_o - H(\theta)}{H_o}\, d\theta$$

and
$$\delta_r = (1 - \bar{n}_s)\int_0^{L/2} \frac{H(s)}{H_o}\frac{ds}{r_o} + \bar{n}_o \int_0^{\pi/4} \frac{H_o - H(\theta)}{H_o}\, d\theta$$

$\bar{n}_o$ and $\bar{n}_s$ are the average values of n in the respective regions. δ and δ_r, which measure the stray field, are both small compared to unity. (4) and (5) are correct up to terms of order δ^2 and $\frac{\delta L}{r_o}$.

Since the equilibrium orbit will not be exactly at the center of the chamber because of the stray field, we shall assume a solution of the form

$$A_k = a + Ae^{ik\frac{\pi}{2}\mu_r}$$
$$B_k = b + Be^{ik\frac{\pi}{2}\mu_r}$$

a and b represent the displaced equilibrium orbit, and A and B represent oscillations about that orbit of frequency μ_r. a, b, and μ_r are determined by the recursion relations (4) and (5), while A and B remain arbitrary until the initial conditions are specified.

Since A = B = 0 must be a solution of (4) and (5), a and b are determined by requiring that the terms in those equations independent of A and B be equal to zero; that is,

$$(4a)\quad (a-b)\left[\sin\Delta + \frac{\sqrt{1-n}L}{2r_o}\cos\Delta\right] + \frac{3-n}{6}(a^2-b^2)\left[\sin 2\Delta + \frac{\sqrt{1-n}L}{r_o}\cos 2\Delta\right] = 0$$

$$(5a)\quad (a+b)\left[\sqrt{1-n}\sin\Delta + \delta_r\cos\Delta\right] + \frac{3-n}{6}(a^2+b^2)\left[2\sqrt{1-n}\sin 2\Delta + \delta_r\cos 2\Delta\right] = -\delta$$

where $\Delta = \frac{\pi}{4}\sqrt{1-n}$.

Therefore
$$a = b = -\frac{\delta}{2\sqrt{1-n}\sin\Delta}$$

to the correct order of approximation.

The linear terms in A and B yield the equations

$$(4b)\quad Ae^{i\Delta}\left[\alpha-\alpha^{*}e^{i(\frac{\pi}{2}\mu_r-2\Delta)}\right]+Be^{-i\Delta}\left[\alpha^{*}-\alpha e^{i(\frac{\pi}{2}\mu_r+2\Delta)}\right]=0$$

$$(5b)\quad Ae^{i\Delta}\left[\beta-\beta^{*}e^{i(\frac{\pi}{2}\mu_r-2\Delta)}\right]-Be^{-i\Delta}\left[\beta^{*}-\beta e^{i(\frac{\pi}{2}\mu_r+2\Delta)}\right]=0$$

where $\alpha = 1 + \frac{i\sqrt{1-n}L}{2r_o} - \frac{\delta}{2\sqrt{1-n}\,\sin\Delta}\left[\frac{3-n}{3}e^{i\Delta} - (1-n)e^{-i\Delta}\right]$,

$$\beta = 1 + \frac{i\delta_r}{\sqrt{1-n}} - \frac{3-n}{3}\,\frac{\delta}{\sqrt{1-n}\,\sin\Delta}$$

Terms quadratic in A and B may be neglected because we are now interested only in small oscillations about the displaced orbit.

Now μ_r must have a value such that the determinant of the coefficients of A and B is zero. μ_r is then determined by the equation

$$(6)\qquad \cos\frac{\pi}{2}\mu_r = \cos\frac{\pi}{2}\sqrt{1-n} - \left[\frac{L\sqrt{1-n}}{2r_o}+\frac{\delta_r}{\sqrt{1-n}}-\frac{(2-n)\delta}{\sqrt{1-n}}\right]\sin\frac{\pi}{2}\sqrt{1-n}$$

If there is no stray field $\delta = \delta_r = 0$, and

$$(7)\qquad \cos\frac{\pi}{2}\mu_r = \cos\frac{\pi}{2}\sqrt{1-n} - \frac{L\sqrt{1-n}}{2r_o}\sin\frac{\pi}{2}\sqrt{1-n},$$

which is the result obtained by Serber, and by Dennison and Berlin. Its validity is not limited to small values of L/r_o, for equations (4) and (5) are exact in the absence of stray field, though, of course, (6) and (7) are both limited to small amplitudes of oscillation.

It is of interest to compute the average displacement of the orbits from r_o.

$$\overline{\Delta r_o} = \frac{2r_o}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(ae^{i\sqrt{1-n}\theta}+be^{-i\sqrt{1-n}\theta}\right)d\theta = -\frac{4}{\pi}\,\frac{\delta r_o}{(1-n)}$$

Because of the fringing field, the ions will run on the average at a radius smaller than that corresponding to their velocity. Since the true orbit should be at the center of the chamber, the magnetic field for a given velocity must be reduced by an amount

$$\frac{\Delta H}{H_o} = (1 - n)\frac{\Delta r_o}{r_o} = -\frac{4\delta}{\pi}$$

This is just the reduction needed to make $\oint$Hds along the center line of the chamber correspond to an angle of 2π.

The vertical oscillations can be treated more simply, because there is no vertical displacement of the orbit due to the fringing field. The results for the frequency of oscillation is

$$\cos\frac{\pi}{2}\mu_v = \cos\frac{\pi}{2}\sqrt{n} - \left[\frac{L\sqrt{n}}{2r_o} - \frac{2\sqrt{n}}{\pi}\delta_z\right]\sin\frac{\pi}{2}\sqrt{n}$$

where

$$\delta_z = \int_0^{\frac{\pi}{4}}\left(1 - \frac{n(\Theta)}{n}\right)\frac{H(\Theta)}{H_o}\,d\Theta + \int_0^{\frac{L}{2}}\left(1 - \frac{n(s)}{n}\right)\frac{H(s)}{H_o}\,\frac{ds}{r_o}$$

in which n again describes the field shape at the center of a quadrant, and $n(\Theta)$ or $n(s)$ describes the field shape in the fringing region. δ_z is different from zero only through the variation in the shape of the field at the ends of the quadrants.

In Table III are listed the values of the frequencies for the three machines, together with the values obtained by neglecting the fringing field, for comparison.

In calculating the influence of the straight sections on the amplitudes of oscillation, we shall simplify the discussion by omitting stray field effects, since the corrections to the amplitudes are predominantly due to the straight sections alone. The oscillations will be approximately sinusoidal of frequency μ_r or μ_z, with an admixture of a fourth harmonic component. The interesting question is whether the mixture will result in displacements large enough to cause the ions to strike the walls or the injector structure.

Let us consider an ion which leaves the injector tangentially with an initial displacement, x_o, from its instantaneous circle. Its maximum displacement will be

TABLE III

Frequencies of Radial and Vertical Oscillations

Machine	n	$\sqrt{1-n}$	μ_r	μ_r if $\delta = \delta_r = 0$	$\sqrt{n}$	μ_z	μ_z if $\delta_z = 0$
A	.6	.63	.68	.71	.78	.86	.88
B	.6	.63	.69	.71	.78	.87	.88
C	.6	.63	.70	.71	.78	.88	.88

$$x_{max} = x_o \frac{\cos \frac{\pi}{4} \frac{\sqrt{1-n}}{\mu r}}{\cos \frac{\pi}{4}} \sim x_o \left(1 + \frac{L\sqrt{1-n}}{4r_o} \tan \frac{\pi}{4} \quad \sqrt{1-n}\right)$$

For the Berkeley machines, the increase is about 4 percent. Although the maximum displacement is greater than x_o, the displacement at the injector is given by

$$x_I = x_o \cos 2\pi \ell \mu_r$$

on the ℓ^{th} turn. This is the basis for the assumption of a sinusoidal variation used in Section IIB in calculating the number of ions missing the injector.

For the vertical motion, consider an ion which leaves the injector at an angle α from the median plane. Its maximum displacement will be

$$z_{max} = \frac{\alpha r_o}{\sqrt{n}} \frac{\sin \frac{\pi}{4} \frac{\mu_v}{\sqrt{n}}}{\sin \frac{\pi}{4}} \sim \frac{\alpha r_o}{\sqrt{n}} \left(1 + \frac{L\sqrt{n}}{4r_o} \cot \frac{\pi}{4} \quad \sqrt{n}\right)$$

an increase due to the straight sections of 12 percent. Both corrections are sufficiently small so that it should be safe to increase the length of the straight sections if necessary.

B. Effects of magnetic inhomogeneities.

If we again neglect phase oscillations and the variation of the magnetic field with time, the complete equations of motion are

$$\text{(1a)} \quad x'' = (1+2L/\pi r_o)^2(1+x) + \frac{2x'^2}{1+x} + \frac{er_o}{mcv}\left[1 + \frac{x'^2+z'^2}{(1+x)^2(1+2L/\pi r_o)^2}\right]^{1/2}\Big[-(1+x)^2(1+2L/\pi r_o)^2 H_z - x'^2 H_z + x'z' H_x + (1+x)(1+2L/\pi r_o) z' H_\phi\Big]$$

$$\text{(1b)} \quad z'' = \frac{2x'z'}{1+x} + \frac{er_o}{mcv}\left[1 + \frac{x'^2+z'^2}{(1+x)^2(1+2L/\pi r_o)^2}\right]^{1/2}\Big[(1+x)^2(1+2L/\pi r_o)^2 H_x - (1+x)(1+2L/\pi r_o) x' H_\phi - x'z' H_z + z'^2 H_x\Big]$$

in the quadrants, and

$$(2a)\quad x'' = \frac{er_o}{mcv}\left[1+\frac{x'^2+z'^2}{(1+2L/\pi r_o)^2}\right]^{1/2}\left[x'z'H_x-\left(x'^2+(1+2L/\pi r_o)^2\right)H_z+z'(1+2L/\pi r_o)H_\phi\right]$$

$$(2b)\quad z'' = \frac{er_o}{mcv}\left[1+\frac{x'^2+z'^2}{(1+2L/\pi r_o)^2}\right]^{1/2}\left[-x'z'H_z+\left(z'^2+(1+2L/\pi r_o)^2\right)H_x-x'(1+2L/\pi r_o)H_\phi\right]$$

in the straight sections. x and z are the lateral and vertical displacements of the ion from the center line of the chamber in units of the radius at the center of the chamber. Primes denote differentiation with respect to the independent variable, ϕ, defined by

$$\phi = \frac{s}{r_o(1+2L/\pi r_o)}$$

where s is distance measured along the center line of the chamber.

To use these equations, the field components are expanded in power series in x and z and in Fourier series in ϕ. (1) and (2) are then written in ascending powers of x, x', z, z' and combined into one pair of equations by Fourier analyzing the coefficients of the various terms. As in the preceding section, the motion of the ions will in general consist of oscillations about a distorted equilibrium orbit, so that it is convenient to make the change of variables.

$$x = \xi + \sum a_\ell \cos(\ell\phi + \alpha_\ell)$$

$$z = \zeta + \sum b_\ell \cos(\ell\phi + \beta_\ell)$$

and choose a_ℓ, α_ℓ and b_ℓ, β_ℓ in such a way that the equations of motion for ξ and ζ contain no constant terms. a_ℓ, α_ℓ, b_ℓ, and β_ℓ then represent the Fourier analysis of the distorted orbit. The resulting equations for ξ and ζ are of the form:

(3a) $$\xi'' + \mu_r^2 \xi = \sum A^r_{ghjk\ell} \xi^g \xi'^h \zeta^j \zeta'^k \cos(\ell\phi + \gamma^r_{ghjk\ell})$$

(3b) $$\zeta'' + \mu_v^2 \zeta = \sum A^z_{ghjk\ell} \xi^g \xi'^h \zeta^j \zeta'^k \cos(\ell\phi + \gamma^v_{ghjk\ell})$$

where

$$A^r_{oooo\ell} = A^z_{oooo\ell} = A^r_{1oooo} = A^z_{oo1oo} = 0$$

and where μ_r and μ_v correspond to the frequencies found in the special case of Section IIIA. We have here, in fact, an alternate way of determining μ_r and μ_v, but the method of Section IIIA is preferable since it avoids the complicated Fourier analyses.

Actually, the distortion of the equilibrium orbit is of primary importance, for unless this orbit lies entirely within the chamber, no ions will survive for many turns. It is possible to set a tolerance on a given type of inhomogeneity by considering only the distorted orbit, because the accelerated beam will be more or less proportional to the smallest separation between orbit and chamber wall.

As an example of the above procedure, let us determine the amount by which the magnetic field in one quadrant may differ from the others without seriously decreasing the beam. The straight sections shall be neglected for simplicity. For the vertical motion, the net restoring force will be changed somewhat, but the equilibrium orbit remains in the central plane. However, (1a) becomes

$$x'' + (1-n)\, x = -\frac{3}{4}\varepsilon \qquad \text{in the bad quadrant}$$

$$x'' + (1-n)\, x = \frac{1}{4}\varepsilon \qquad \text{in the other quadrants}$$

where ε is the fractional increase in magnetic field in the bad quadrant.

By Fourier analyzing the right hand side of these equations, a single equation for x is obtained:

$$x'' + (1-n)x = -\frac{2\varepsilon}{\pi}\sum_{1}^{\infty}\frac{1}{\ell}\sin\frac{\ell\pi}{4}\cos\ell\phi$$

where ϕ is measured from the center of the bad quadrant. With the substitution

$$(4) \qquad x = \xi + \frac{2\varepsilon}{\pi}\sum_{1}^{\infty}\frac{\sin \ell\pi/4}{\ell\left[\ell^2 - (1-n)\right]}\cos\ell\phi$$

(3a) becomes

$$\xi'' + (1-n)\,\xi = 0$$

so that the distorted orbit is described by the series in (4). Keeping only the first term in the series, we get the condition on ε:

$$\varepsilon < \frac{\pi}{2}\,\frac{n}{\sin \pi/4}\,\frac{d}{r_o}$$

where d is the half-width of the chamber. If ε is held to, say, 10 percent of this upper limit, the loss of beam should be of the order of 10 percent.

C. Resonance effects.

The equations for ξ and ζ do become important in the event that there exists a commensurability relation between radial and vertical oscillation frequencies and the rotation frequency, for then one or more of the small "driving" terms on the right hand sides of (3) will be in resonance with the linear left hand sides. A solution by successive approximations would indicate an unbounded increase in amplitude. This apparent catastrophe enters because the variation of frequency with amplitude has so far been ignored. These cases are of considerable importance in this type of machine, since the ions must travel a long distance in a field whose shape is virtually constant, so that even a very slow increase of amplitude may be fatal. It is, of course, not possible to shape the magnetic field in such a way that all commensurability relations are avoided, and the question arises of the seriousness of such effects.

We shall use the method of variation of parameters to investigate the case of resonance coupling; i.e., when

(5) $\ell \pm m\mu_r \pm n\mu_v = \delta$, where ℓ, m, and n are integers and $\delta << 1$

Solutions of (3) can be written in the form

$$\xi = A_r(\phi) \sin\left[\mu_r \phi + \Psi_r(\phi)\right]$$
$$\zeta = A_v(\phi) \sin\left[\mu_z \phi + \Psi_v(\phi)\right] .$$

With the supplementary conditions

$$A_r \Psi_r' \cos(\mu_r\phi + \Psi_r) + A_r' \sin(\mu_r\phi + \Psi_r) = 0$$
$$A_v \Psi_v' \cos(\mu_v\phi + \Psi_v) + A_v' \sin(\mu_v\phi + \Psi_v) = 0$$

(3) become four first order equations

$$A_r' = \frac{1}{\mu_r} \sum P_{ghjk} A^r_{ghjk\ell} \cos(\ell\phi + \gamma^r_{ghjk\ell}) \cos(\mu_r\phi + \Psi_r)$$

(6)
$$A_r\Psi_r' = \frac{-1}{\mu_r} \sum PA^r \cos(\ell\phi + \gamma^r) \sin(\mu_r\phi + \Psi_r)$$
$$A_v' = \frac{1}{\mu_v} \sum PA^v \cos(\ell\phi + \gamma^v) \cos(\mu_v\phi + \Psi_v)$$
$$A_v\Psi_v' = \frac{-1}{\mu_v} \sum PA^v \cos(\ell\phi + \gamma^v) \sin(\mu_v\phi + \Psi_v)$$

where

$$P_{ghjk} = \mu_r^h \mu_v^k A_r^{g+h} A_v^{j+k} \sin^g(\mu_r\phi + \Psi_r) \cos^h(\mu_r\phi + \Psi_r) \sin^j(\mu_v\phi + \Psi_v) \cos^k(\mu_v\phi + \Psi_v)$$

and where the subscripts have been omitted in the last three equations for clarity.

Most of the terms will induce rapid fluctuation in the amplitudes and phases which are not of interest. Before attempting to solve the equations, therefore, the right hand side should be averaged over a period in order to retain only those terms which give rise to net variations in the amplitudes and phases. These terms will be just the ones which would have given an infinite amplitude by the method of successive approximations, plus terms which will represent the non-linear dependence of frequency on amplitude.

The solutions differ depending on whether the frequency shift terms or the resonant terms predominate. We shall try to demonstrate the two cases by examples.

Suppose first that

$$2\mu_r - \mu_v = \delta \qquad \delta << 1$$

and that equations (3) have the form

$$\xi'' + \mu_r^2 \xi = a\xi^2 + b\xi\zeta$$

$$\zeta'' + \mu_v^2 \zeta = \alpha\zeta^2 + \beta\xi^2$$

Then, after averaging over a period, (6) becomes

(7a) $$A_r' = c_1 A_v A_r \cos u$$

(7b) $$\Psi_r' = c_2 A_r^2 - c_1 A_v \sin u$$

(7c) $$A_v' = -c_3 A_r^2 \cos u$$

(7d) $$\Psi_v' = c_4 A_v^2 - c_3 \frac{A_r^2}{A_v} \sin u$$

where the c_n's are of the order of unity and depend on the field shape and where $u = 2\Psi_r - \Psi_v + \delta\phi$. The c_2 and c_4 terms represent the dependence of frequency on amplitude, and the c_1 and c_3 terms are the resonance terms. Since the amplitudes are in units of the equilibrium radius, the quadratic frequency shift is negligible compared to the resonance terms.

From (7a) and (7c)

$$A_v^2 + \frac{c_3}{c_1} A_r^2 = A^2$$

where A is an arbitrary constant; the total "energy" of the system is conserved. The amplitudes are therefore bounded, but this is not sufficient to prevent loss of ions, since the amplitudes may vary by an order of magnitude.

Ignoring the frequency shift terms in (7b) and (7d), we obtain a second integral

$$\sin u = \frac{A_{v_o}(A^2 - A_{v_o}^2)}{A_v(A^2 - A_v^2)} \sin u_o - \frac{\delta}{A_v} \frac{(A_v^2 - A_{v_o}^2)}{(A^2 - A_v^2)} \tag{8}$$

Equation (8) determines the limits of variation of A_v, since sin u cannot exceed one. If δ is large (off resonance), then $\Delta A_v \sim \frac{A_r^2}{\delta}$ in agreement with the method of successive approximations. On the other hand, if $\delta = 0$, the energy may pass completely from one mode to the other. Since the vertical aperture is much smaller than the radial, this could mean a loss of most of the ions.

The rate at which the exchange takes place is indicated by a special solution of (7). If $\delta = u_o = A_{v_o} = 0$, then sin u = 0 at all times, and by (7c)

$$A_v \sim \sqrt{\frac{c_3}{c_1}}\, A_{r_o} (1 - e^{-A_{r_o}\phi})$$

For $A_{r_o} \sim .02$, the vertical amplitude would build up in a few dozen turns.

The situation is different in the case

$$2\mu_v - \mu_r - 1 = \delta \qquad\qquad \delta << 1$$

Here equations (6) after averaging have the form

$$\begin{aligned} A_r' &= d_1 \varepsilon_1 A_v^2 \cos u \\ A_r \Psi_r' &= d_2 A_r^3 + d_1 \varepsilon_1 A_v^2 \sin u \\ A_v' &= -d_3 \varepsilon_1 A_v A_r \cos u \\ A_v \Psi_v' &= d_4 A_v^3 + d_3 \varepsilon_1 A_v A_r \sin u, \quad u = 2\Psi_v - \Psi_r - \delta\phi \end{aligned} \tag{9}$$

where the d_n's are again of order unity and ε_1 is the amplitude of the first harmonic component of the magnetic field. Since we are interested in a case in which the distortion of the equilibrium orbit is not important, ε_1 can be assumed to be smaller than A_r and A_z.

The first integral is $A_r^2 + \frac{d_1}{d_3} A_v^2 = A^2$ where A is again an arbitrary constant.

The second integral is

$$(10)\quad \sin u = \frac{A_{r_o}}{A_r}\left(\frac{A^2-A_r^2}{A^2-A_{r_o}^2}\right)\sin u_o + \frac{(d_1d_2-2d_3d_4)}{2d_1{}^2d_3\mathcal{E}_1}(A^2-A_r{}^2)\ln\frac{A^2-A_r^2}{A^2-A_{r_o}^2} + \frac{d_2A^2+\delta}{2d_1d_3\mathcal{E}_1A_r}\,\frac{A_r{}^2-A_{r_o}^2}{A^2-A_{r_o}^2}$$

In Fig. 5 sin u is plotted as a function of A_r/A for various initial conditions. The variation in the radial amplitude for a given set of initial conditions determined by the differences of the abscissae of the corresponding curve at $\sin u = \pm 1$. The fractional variation is of the order of $\frac{\mathcal{E}_1}{A}$, except inside the region bounded by the heavy line. This region represents a "locking in" of the frequencies, giving rise to a variation of the order of $\left(\frac{\mathcal{E}_1}{A}\right)^{1/2}$. The locking in region is therefore the more dangerous one so that we must require that $\mathcal{E}_1$ be small enough so that the change in amplitude in this region is not serious.

This second example is typical of those cases in which the frequency shift terms play a significant role. The solutions indicate two regions, one in which the variation in the amplitudes is of the order of $\mathcal{E}_\ell A^{m+n-4}$ (l, m, and n are defined in (5), and $\mathcal{E}_\ell$ is the amplitude of the ℓ^{th} harmonic in the field), and a more dangerous region in which the amplitude varies by an amount $(\mathcal{E}_\ell A^{m+n-4})^{1/2}$. Each situation, of course, contains special cases which must be examined individually, but these general results are correct whenever the initial amplitudes are comparable with each other and the dimensions of the chamber. For large m and n the variations in amplitude are negligible; for instance, even though the straight sections introduce a strong fourth harmonic in the field, m+n would have to be at least five to satisfy the relation (5), so that any resonance effects involving the fourth harmonic should be of no importance.

On the other hand, resonances in which the frequency shift terms are negligible, such as (0,2,1)(the first example), (0,1,1), and (1,0,1) can easily be avoided, by a proper choice of field shape and certainly should be avoided, although in a given case a catastrophe may be averted because of the natural damping of the free oscillations or because the oscillation frequencies vary somewhat under the influence of the phase oscillations.

It is of interest to know by what amount δ must differ from zero to avoid a resonance; i.e., to prevent the occurrence of a locking in region. The answer is contained in equation (10) of the second example, for the last term on the right controls the position of the region. Since δ in this term combines additively with A^2, the region can be moved away from the range of initial amplitudes under consideration by a change in δ of the order of

$$\delta \sim A^2 \sim 10^{-3}$$

so that a very small change in the shape of the field should eliminate the resonance effect. For resonances of higher order, the required change is smaller yet; if one is faced with low order resonances, the required change in field shape is in general too great to be obtained by such devices as correcting coils.

D. Damping effects.

The damping effect of an increasing magnetic field on both the free and the phase oscillations in a circular machine is well established.[1,6] It has recently

6. Kerst and Serber, Phys. Rev. 60, 53 (1941)

been suggested that a radial variation in the accelerating voltage could be used to introduce an additional damping of the phase oscillations. We reinvestigated the problem of damping for the machine with straight sections, including the effect of such a variation. It will appear that the straight sections have no influence on the damping, but that radial variation of the accelerating field can indeed increase the damping of the phase oscillations, though only at the expense of decreasing the damping of the free radial oscillations.

We consider first the phase equations in the non-relativistic case. The phase velocity is

$$(11) \qquad \mathring{\phi} = \omega - \omega_s = \frac{2\pi v}{4L + 2\pi r} - \frac{2\pi v_s}{4L + 2\pi r_s}$$

where ω_s, v_s, r_s refer to the synchronous orbit. The velocity is related to the synchronous velocity by

$$v = v_s\,(1 + (1-n)\,x)$$

where $x = \dfrac{r - r_s}{r_s}$

and r_s is the radius of the instantaneous circle.

We then have for (11)

$$(12) \qquad \mathring{\phi} = -\omega_s \left(n - \frac{2L/\pi r_s}{1 + 2L/\pi r_s} \right) x$$

The rate of change of velocity is

$$(13) \qquad \mathring{v} = \frac{eV \sin \phi}{2\pi r_s m (1 + 2L/\pi r_s)} + \frac{e(\mathring{\phi}_s + 2\pi r_s^{\,2} \mathring{H}_s x)}{mc(2\pi r_s + 4L)}$$

where V is the accelerating voltage, ϕ_s the flux enclosed by the synchronous orbit, and H_s is the magnetic field at the synchronous orbit. The second term on the right represents the betatron effect.

Before going further we must discuss the form of V. To date two different types of accelerating electrodes have been considered practicable: the resonant cavity, and the dee type electrode, in which the voltage gain depends on the time of flight. In the first case the voltage gain will necessarily be a function of radius because of leakage effects; a radial variation may be introduced in the latter case by shaping the dee faces. Because of the dependence on time of flight, the voltage gain will also be a function of velocity in the dee type accelerator. We shall accordingly write V as

(14) $$V \approx V_s\left(1 + \mathcal{E} x - \beta \frac{(v - v_s)}{v_s}\right)$$

where V_s is the maximum voltage gain at the synchronous radius, $\mathcal{E}$ is a measure of the variation of voltage with radius, and β is zero for the cavity type and 1 for the dee type accelerator. If the magnetic field varies linearly (12), (13) and (14) combine to give, for small amplitudes of oscillation,

(15) $$\ddot{\phi} + \left(\beta - \frac{\mathcal{E}}{1-n}\right)\dot{\phi}/t + \frac{eV_s}{2\pi m r_s^2}\,\frac{\left(n - \frac{2L/\pi r_s}{1+2L/\pi r_s}\right)\cos\phi_s}{(1+2L/\pi r_s)^2(1-n)}(\phi - \phi_s) = 0$$

where

$$\sin\phi_s = \frac{\Delta E - e/c\,\dot{\Phi}_s}{eV_s}$$

and

$$\Delta E_s = e2\pi r_s^2\,\dot{H}_s(1 + 2L/\pi r_s),$$

the synchronous voltage gain per turn. This corresponds to the phase equation of Bohm and Foldy[1] in the non-relativistic range if we set $L/r_s = \mathcal{E} = \beta = 0$. The asymptotic solution of (15), valid since injection occurs many periods of phase oscillation after the magnetic field is turned on, represents a motion whose phase and radial amplitudes vary in time according to

(16a) $$\phi_m \sim t^{-\beta/2 + \mathcal{E}/2(1-n)}$$

(16b) $$x_m \sim t^{-1-\beta/2 + \mathcal{E}/2(1-n)}$$

We see that the damping is increased if $\mathcal{E}$ is negative; that is, if the accelerating voltage decreases with increasing radius. On the other hand there is a definite upper limit permissible for $\mathcal{E}$. If the oscillations are to be damped at all, it is necessary that

(17) $$\frac{\mathcal{E}}{1-n} < \beta$$

The calculation for the extreme relativistic case is similar to the above, except that the mass varies instead of the velocity. The solutions will be the same for both types of electrodes because of the almost constant velocity. The phase and radial amplitudes vary as

$$\phi_m \sim t^{-1/4 + \mathcal{E}/2(1-n)}$$
$$x_m \sim t^{-3/4 + \mathcal{E}/2(1-n)}.$$

Therefore condition (17) must be supplemented by

$$(18) \qquad \frac{\mathcal{E}}{1-n} < 1/2$$

In order to demonstrate the effect of a radial variation of the accelerating voltage on the free radial oscillations, we shall consider an ion which is being accelerated at synchronous phase. If the ion is performing free radial oscillations it will traverse the accelerating chamber inside the synchronous orbit as often as outside, so that on the average it receives the proper energy increase per turn. The phase oscillations induced in this way will be small since the variations in velocity occur in a time short compared to the period of a phase oscillation.

The free radial oscillations, however, are seriously influenced. If the ion traverses the accelerating electrode with a relative displacement x from the synchronous orbit, its instantaneous circle will be moved suddenly by an amount

$$(19) \qquad \frac{\Delta r}{r_s} = \frac{-\Delta p}{(1-n)p_s} = -\frac{(1+\epsilon x)\,\Delta E_s}{(1-n)\,p_s\,v_s}$$

where p_s is the synchronous momentum. The resulting increase in the amplitude of oscillation depends on the phase of the oscillation at the electrode and is given by

$$(20) \qquad \Delta x_m = x/x_m \,\frac{\Delta r}{r_s} = -\frac{\cos\psi\,(1+\epsilon x_m \cos\psi)}{1-n}\,\frac{\Delta E_s}{p_s\,v_s}$$

where x_m is the relative amplitude and Ψ the phase of the oscillation. Since the frequency of oscillation is not commensurable with the rotation frequency, (20) may be averaged with respect to Ψ :

$$(21) \qquad \frac{\overline{\Delta x_m}}{x_m} = -\frac{\mathcal{E}}{2(1-n)} \frac{\Delta E_s}{p_s\, v_s} = -\frac{\mathcal{E}}{2(1-n)} \frac{\Delta H_s}{H_s}$$

If the damping due to the changing magnetic field is added to (21), the total variation of amplitude with time will be given by

$$(22) \qquad x_m \sim t^{-1/2-\mathcal{E}/2(1-n)}$$

for all energies.

Since $\mathcal{E}$ appears in the damping exponent with opposite signs in (16) and (22), either type of oscillation can be damped only at the expense of the other. Therefore we must extend (17) and (18) to include a lower limit on $\mathcal{E}$:

$$(23) \qquad -1 < \frac{\mathcal{E}}{1-n} < \begin{cases} 0 & \text{cavity type} \\ 1/2 & \text{dee type} \end{cases}$$

Condition (23) is actually quite restrictive. For machine C (using a dee type electrode), (23) requires that the voltage variation between the center and outer edge of the accelerating electrode be not less than -1.6 percent and not greater than 0.8 percent.

If the dependence of voltage on radius is not linear, the average of $\epsilon x_m \cos^2 \Psi$ in (20) must be replaced by

$$\int \cos \Psi \, f(x_m \cos \Psi) \, d\Psi$$

where

$$V = V_s(1 + f(x)) \quad .$$

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