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MEMORIAL VOLUME FOR SASHA ANAN'IN

A note on properly discontinuous actions

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Abstract
We compare various notions of proper discontinuity for group actions. We also discuss fundamental domains and criteria for cocompactness.

1 Introduction

This note is meant to clarify the relation between different commonly used definitions of proper discontinuity without the local compactness assumption for the underlying topological space. Much of the discussion applies to actions of nondiscrete locally compact Hausdorff topological groups, but, since my primary interest is geometric group theory, I will mostly work with discrete groups. All group actions are assumed to be continuous, in other words, for discrete groups, these are homomorphisms from abstract groups to groups of homeomorphisms of topological spaces. This combination of *continuous* and *properly discontinuous*, sadly, leads to the ugly terminology "a continuous properly discontinuous action." A better terminology might be that of a *properly discrete* action, since it refers to proper actions of discrete groups.

Throughout this note, I will be working only with topological spaces which are 1st countable, since spaces most common in metric geometry, geometric topology, algebraic topology and geometric group theory satisfy this property. One advantage of this assumption is that if (x_n) is a sequence converging to a point $x \in X$, then the subset $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, which is not true if we work with nets instead of sequences. However, I will try to avoid the local compactness assumption whenever possible, since many spaces appearing in metric geometry and geometric group theory (e.g. asymptotic cones) and algebraic topology (e.g. CW complexes) are not locally compact. (Recall that topological space X is *locally compact* if every point

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has a basis of topology consisting of relatively compact subsets.) In the last two sections of the note I also discuss cocompact group actions and fundamental sets/domains of properly discontinuous group actions.

2 Group actions

A *topological group* is a group G equipped with a topology such that the multiplication and inversion maps

$$G \times G \to G, (g, h) \mapsto gh, G \to G, g \mapsto g^{-1}$$

are both continuous. A *discrete group* is a group with discrete topology. Every discrete group is clearly a topological group.

A *left continuous action* of a topological group G on a topological space X is a continuous map

$$\lambda : G \times X \to X$$

satisfying

1. $\lambda(1_G, x) = x$ for all $x \in X$.
2. $\lambda(gh, x) = \lambda(g, \lambda(h, x))$, for all $x \in X$, $g, h \in G$.

From this, it follows that the map $\rho : G \to Homeo(X)$

$$\rho(g)(x) = \lambda(g, x),$$

is a group homomorphism, where the group operation $\phi\psi$ on $Homeo(X)$ is the composition $\phi \circ \psi$.

If G is discrete, then every homomorphism $G \to Homeo(X)$ defines a left continuous action of G on X.

The shorthand for $\rho(g)(x)$ is gx or $g \cdot x$. Similarly, for a subset $A \subset X$, GA or $G \cdot A$, denotes the *orbit of* A under the G-action:

$$GA = \bigcup_{g \in G} gA.$$

The *quotient space* X/G (also frequently denoted $G\backslash X$), of X by the G-action, is the set of G-orbits of points in X, equipped with the quotient topology: The elements of X/G are equivalence classes in X, where $x \sim y$ when $Gx = Gy$ (equivalently, $y \in Gx$).

The *stabilizer* of a point $x \in X$ under the G-action is the subgroup $G_x < G$ given by

$$\{g \in G : gx = x\}.$$

An action of G on X is called *free* if $G_x = \{1\}$ for all $x \in X$. Assuming that X is Hausdorff, G_x is closed in G for every $x \in X$.

Example 1 An example of a left action of G is the action of G on itself via left multiplication:

$$\lambda(g, h) = gh.$$

In this case, the common notation for $\rho(g)$ is L_g. This action is free.

3 Proper maps

Properness of certain maps is the most common form of defining proper discontinuity; sadly, there are two competing notions of properness in the literature.

A continuous map $f : X \to Y$ of topological spaces is *proper in the sense of Bourbaki*, or simply *Bourbaki–proper* (cf. [3, Ch. I, §10, Theorem 1]) if f is a closed map (images of closed subsets are closed) and point–preimages $f^{-1}(y), y \in Y$, are compact. A continuous map $f : X \to Y$ is *proper* (and this is the most common definition) if for every compact subset $K \subset X$, $f^{-1}(K)$ is compact. It is noted in [3, Ch. I, §10; Prop. 7] that if X is Hausdorff and Y is locally compact, then f is Bourbaki–proper if and only if f is proper.

The advantage of the notion of Bourbaki-properness is that it applies in the case of Zariski topology, where spaces tend to be compact[1] (every subset of a finite-dimensional affine space is Zariski-compact) and, hence, the standard notion of properness is useless.

Since our goal is to trade local compactness for 1st countability, I will prove a lemma which appears as a Corollary in [12]:

Lemma 2 *If $f : X \to Y$ is proper, and X, Y are Hausdorff and 1st countable, then f is Bourbaki-proper.*

Proof We only have to verify that f is closed. Suppose that $A \subset X$ is a closed subset. Since Y is 1st countable, it suffices to show that for each sequence (x_n) in A such that $(f(x_n))$ converges to $y \in Y$, there is a subsequence (x_{n_k}) which converges to some $x \in A$ such that $f(x) = y$. The subset $C = \{y\} \cup \{f(x_n) : n \in \mathbb{N}\} \subset Y$ is compact. Hence, by properness of f, $K = f^{-1}(C)$ is also compact. Since X is Hausdorff, and K is compact, follows that (x_n) subconverges to a point $x \in K$. By continuity of f, $f(x) = y$. Since A is closed, $x \in A$. □

Remark 3 This lemma still holds if one were to replace the assumption that X is 1st countable by surjectivity of f, see [12].

The converse (each Bourbaki–proper map is proper) is proven in [3, Ch. I, §10; Prop. 6] without any restrictions on X, Y. Hence:

[1] quasicompact in the Bourbaki terminology.

Corollary 4 *For maps between 1st countable Hausdorff spaces, Bourbaki-properness is equivalent to properness.*

4 Proper discontinuity

Suppose that X is a 1st countable Hausdorff topological space, G a discrete group and $G \times X \to X$ a (continuous) action. I use the notation $g_n \to \infty$ in G to indicate that g_n converges to ∞ in the 1-point compactification $G \cup \{\infty\}$ of G, i.e. for every finite subset $F \subset G$,

$$\mathrm{card}(\{n : g_n \in F\}) < \infty.$$

Given a group action $G \times X \to X$ and two subsets $A, B \subset X$, the *transporter subset* $(A|B)_G$ is defined as

$$(A|B)_G := \{g \in G : gA \cap B \neq \emptyset\}.$$

Properness of group actions is (typically) stated using certain transporter sets.

Definition 5 Two points $x, y \in X$ are said to be *G-dynamically related* if there is a sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ in X such that $g_n x_n \to y$.

A point $x \in X$ is said to be a *wandering point* of the G-action if there is a neighborhood U of x such that $(U|U)_G$ is finite.

Lemma 6 *Suppose that the action $G \times X \to X$ is wandering at a point $x \in X$. Then the G-action has a G-slice at x, i.e. a neighborhood $W_x \subset U$ which is G_x-stable and for all $g \notin G_x$, $gW_x \cap W_x = \emptyset$.*

Proof For each $g \in (U|U)_G - G_x$ we pick a neighborhood $V_g \subset U$ of x such that

$$gV_g \cap V_g = \emptyset.$$

Then the intersection

$$V := \bigcap_{g \in (U|U)_G - G_x} V_g$$

satisfies the property that $(V|V)_G = G_x$. Lastly, take

$$W_x := \bigcap_{g \in G_x} V.$$

□

The next lemma is clear:

Lemma 7 *Assuming that X is Hausdorff and* 1*st countable, the action* $G \times X \to X$ *is wandering at x if and only if x is not dynamically related to itself.*

Given a group action $\alpha : G \times X \to X$, we have the natural map

$$\hat{\alpha} := \alpha \times \mathrm{id}_X : G \times X \to X \times X$$

where $\mathrm{id}_X : (g, x) \mapsto x$.

Definition 8 An action α of a discrete group G on a topological space X is *Bourbaki–proper* if the map $\hat{\alpha}$ is Bourbaki-proper.

Lemma 9 *If the action* $\alpha : G \times X \to X$ *of a discrete group G on a Hausdorff topological space X is Bourbaki-proper, then the quotient space X/G is Hausdorff.*

Proof The quotient map $X \to X/G$ is an open map by the definition of the quotient topology on X/G. Since α is Bourbaki-proper, the image of the map $\hat{\alpha}$ is closed in $X \times X$. This image is the equivalence relation on $X \times X$ which use used to form the quotient X/G. Now, Hausdorffness of X/G follows from [3, Proposition 8 in I.8.3]. □

Definition 10 An action α of a discrete group G on a topological space X is *proper* if the map $\hat{\alpha}$ is proper.

Note that the equivalence of (1) and (5) in the following theorem is proven in [3, Ch. III, §4.4, Proposition 7] without any assumptions on X.

Theorem 11 *Assuming that X is Hausdorff and* 1*st countable, the following are equivalent*:

(1) *The action* $\alpha : G \times X \to X$ *is Bourbaki-proper*.
(2) For every compact subset $K \subset X$,

$$\mathrm{card}((K|K)_G) < \infty.$$

(3) The action $\alpha : G \times X \to X$ is proper, i.e. the map $\hat{\alpha}$ is proper.
(4) For every compact subset $K \subset X$, there exists an open neighborhood U of K such that $\mathrm{card}((U|U)_G) < \infty$.
(5) For any pair of points $x, y \in X$ there is a pair of neighborhoods U_x, V_x (of x, y respectively) such that $\mathrm{card}((U_x|V_y)_G)) < \infty$.
(6) There are no G-dynamically related points in X.
(7) Assuming, that G is countable and X is completely metrizable:[2] The G-stabilizer of every $x \in X$ is finite and for any two points $x \in X, y \in X - Gx$, there exists a pair of neighborhoods U_x, V_y (of x, resp. y) such that $\forall g \in G, gU_x \cap V_y = \emptyset$.

[2] It suffices to assume that X is *hereditarily Baire*: Every closed subset of X is Baire.

(8) Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous:[3] There is no $x \in X$ and a sequence $h_n \to \infty$ in G such that $h_n x \to x$.
(9) Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous: Every $x \in X$ is a wandering point of the G-action.
(10) Assuming that X is a CW complex and the action $G \times X \to X$ is cellular: Every point of X is wandering.
(11) *Assuming that* X *is a CW complex the action* $G \times X \to X$ *is cellular*: *Every cell in* X *has finite* G*-stabilizer*.

Proof The action α is Bourbaki-proper if and only if the map $\hat{\alpha}$ is proper (see Corollary 4) which is equivalent to the statement that for each compact $K \subset X$, the subset $(K|K)_G \times K$ is compact. Hence, (1) $\iff$ (2).

Assume that (3) holds, i.e. α is proper, equivalently, the map $\hat{\alpha}$ is proper. This means that for each compact $K \subset X$, $\hat{\alpha}^{-1}(K \times K) = \{(g, x) \in G \times K : x \in K, gx \in K\}$ is compact. This subset is closed in $G \times X$ and projects onto $(K|K)_G$ in the first factor and to the subset

$$\bigcup_{g \in (K|K)_G} g^{-1}(K) \tag{$\star$}$$

in the second factor. Hence, properness of the action α implies finiteness of $(K|K)_G$, i.e. (2). Conversely, if $(K|K)_G$ is finite, compactness of $g^{-1}(K)$ for every $g \in G$ implies compactness of the union ($\star$). Thus, (2) $\iff$ (3).

In order to show that (2)⇒(6), suppose that x, y are G-dynamically related points: There exists a sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ such that $g_n(x_n) \to y$. The subset

$$K = \{x, y\} \cup \{x_n, g_n(x_n) : n \in \mathbb{N}\}$$

is compact. However, $y_n \in g_n(K) \cap K$ for every n. A contradiction.

(6)⇒(5): Suppose that the neighborhoods U_x, V_y do not exist. Let $\{U_n\}_{n \in \mathbb{N}}$, $\{V_n\}_{n \in \mathbb{N}}$ be countable bases at x, y respectively. Then for every n there exists $g_n \in G$, such that $g_n(U_n) \cap V_n \neq \emptyset$ for infinitely many g_n's in G. After extraction, $g_n \to \infty$ in G. This yields points $x_n \in U_n, y_n = g_n(x_n) \in V_n$. Hence, $x_n \to x, y_n \to y$. Thus, x is G-dynamically related to y. A contradiction.

(5) ⇒ (4). Consider a compact $K \subset X$. Then for each $x \in K, y \in K$ there exist neighborhoods U_x, V_y such that $(U_x|V_y)_G$ is finite. The product sets $U_x \times V_y, x, y \in K$ constitute an open cover of K^2. By compactness of K^2, there exist $x_1, ..., x_n, y_1, ..., y_m \in K$ such that

$$K \subset U_{x_1} \cup ... \cup U_{x_n}$$
$$K \subset V_{y_1} \cup ... \cup V_{y_m}$$

and for each pair (x_i, y_j),

[3] E.g. an isometric action.

$$\mathrm{card}(\{g \in G : gU_{x_i} \cap V_{y_j} \neq \emptyset\}) < \infty.$$

Setting

$$W := \bigcup_{i=1}^{n} U_{x_i}, V := \bigcup_{j=1}^{m} V_{y_j},$$

we see that

$$\mathrm{card}((W|V)_G) < \infty.$$

Taking $U := V \cap W$ yields the required subset U.

The implication (4) ⇒ (2) is immediate.

This concludes the proof of equivalence of the properties (1)—(6).

(5) ⇒ (7): Finiteness of G-stabilizers of points in X is clear. Let x, y be points in distinct G-orbits. Let U'_x, V'_y be neighborhoods of x, y such that $(U'_x|V'_y)_G = \{g_1, ..., g_n\}$. For each i, since X is Hausdorff, there are disjoint neighborhoods V_i of y and W_i of $g_i(x_i)$. Now set

$$V_y := \bigcap_{i=1}^{n} V_i, \quad U_x := \bigcap_{i=1}^{n} g_i^{-1}(W_i).$$

Then $gU_x \cap V_y = \emptyset$ for every $g \in G$.

(7) ⇒ (6): It is clear that (7) implies that there are no dynamically related points with distinct G-orbits. In particular, every G-orbit in X is closed.

Assume now that X is completely metrizable and G is countable. Suppose that a point $x \in X$ is G-dynamically related to itself. Since the stabilizer G_x is finite, the point x is an accumulation point of Gx; moreover, Gx is closed in X. Hence, Gx is a closed perfect subset of X. Since X admits a complete metric, so does its closed subset Gx. Thus, for each $g \in G$, the complement $U_g := Gx - \{gx\}$ is open and dense in Gx. By the Baire Category Theorem, the countable intersection

$$\bigcap_{g \in G} U_g$$

is dense in Gx. However, this intersection is empty. A contradiction.

It is clear that (6) ⇒ (8) (without any extra assumptions).

(8) ⇒ (6). Suppose that X is a metric space and the G-action is equicontinuous. Equicontinuity implies that for each $z \in X$, a sequence $z_n \to z$ and $g_n \in G$,

$$g_n z_n \to gz.$$

Suppose that there exist a pair of G-dynamically related points $x, y \in X$: $\exists x_n \to x, g_n \in G,\ g_n x_n \to y$. By the equicontinuity of the action, $g_n x \to y$. Since $g_n \to \infty$, there exist subsequences $g_{n_i} \to \infty$ and $g_{m_i} \to \infty$ such that the products $h_i := g_{n_i}^{-1} g_{m_i}$ are all distinct. Then, by the equicontinuity,

$$h_i x \to x.$$

A contradiction.

The implications (5) $\Rightarrow$ (9)$\Rightarrow$ (8) and (5) $\Rightarrow$ (10)$\Rightarrow$ (11) are clear.

Lastly, let us prove the implication (11) $\Rightarrow$ (2). We first observe that every CW complex is Hausdorff and 1st countable. Furthermore, every compact $K \subset X$ intersects only finitely many open cells e_λ in X. (Otherwise, picking one point from each nonempty intersection $K \cap e_\lambda$ we obtain an infinite closed discrete subset of K.) Thus, there exists a finite subset $E := \{e_\lambda : \lambda \in \Lambda\}$ of open cells in X such that for every $g \in (K|K)_G$, $gE \cap E \neq \emptyset$. Now, finiteness of $(K|K)_G$ follows from finiteness of cell-stabilizers in G. □

Unfortunately, the property that every point of X is a wandering point is frequently taken as the definition of proper discontinuity for G-actions, see e.g. [9, 11]. Items (8) and (10) in the above theorem provide a (weak) justification for this abuse of terminology. I feel that the better name for such actions is *wandering actions*.

Example 12 Consider the action of $G = \mathbb{Z}$ on the punctured affine plane $X = \mathbb{R}^2 - \{(0,0)\}$, where the generator of $\mathbb{Z}$ acts via $(x, y) \mapsto (2x, \frac{1}{2}y)$. Then for any $p \in X$, the G-orbit Gp has no accumulation points in X. However, any two points $p = (x, 0), q = (0, y) \in X$ are dynamically related. Thus, the action of G is not proper.

This example shows that the quotient space of a wandering action need not be Hausdorff.

Lemma 13 *Suppose that $G \times X \to X$ is a wandering action. Then each G-orbit is closed and discrete in X. In particular, the quotient space X/G is $T1$.*

Proof Suppose that Gx accumulates at a point y. Then $Gx \cap W_y$ is nonempty, where W_y is a G-slice at y. It follows that all points of $Gx \cap W_y$ lie in the same W_y-orbit, which implies that $Gx \cap W_y = \{y\}$. □

There are several reasons to consider proper actions of discrete (and, more generally, locally compact) groups; one reason is that such each proper action of a discrete group yields an *orbi-covering map* in the case of smooth group actions on manifolds: $M \to M/G$ is an orbi-covering provided that the action of G on M is smooth (or, at least, locally smoothable). Another reason is that for a proper action on a Hausdorff space, $G \times X \to X$, the quotient X/G is again Hausdorff, see Lemma 9.

Question 14 Suppose that G is a discrete group, $G \times X \to X$ is a free continuous action on an n-dimensional topological manifold X such that the quotient space X/G is a (Hausdorff) n-dimensional topological manifold. Does it follow that the G-action on X is proper?

The answer to this question is negative if one merely assumes that X is a locally compact Hausdorff topological space and X/G is Hausdorff, see [7] (the action given there was even cocompact). Below is a different example. We begin by constructing a non-proper free continuous $\mathbb{R}$-action on a manifold, such that the quotient space is not just Hausdorff but is a manifold with boundary.

Example 15 This is a variation on Example 12. We start with the space

$$Z = \{(x, y) : x, y \in [0, \infty), (x, y) \neq (0, 0)\}.$$

Take the quotient space X of Z by the equivalence relation $(x, 0) \sim (0, \frac{1}{x})$. The space X is homeomorphic to the open Moebius band. The group $G = \mathbb{R}$ acts on Z continuously by

$$(t, (x, y)) \mapsto (2^t x, 2^{-t} y).$$

The above equivalence relation on X is preserved by the G-action and, hence, the G-action descends to a continuous G-action on X. It is easy to see that this action is free but not proper: The equivalence class of (1, 0) is dynamically related to itself. Lastly, the quotient X/G is Hausdorff, homeomorphic to [0, 1) (the equivalence class of (1, 0) maps to $0 \in [0, 1)$).

Lastly, we use Example 15 to construct a non-proper free $\mathbb{Z}$-action with Hausdorff quotient. We continue with the notation of the previous example.

Example 16 Let $Y \subset Z$ denote the following subset of Z (with the subspace topology):

$$Y = \{(2^m, 0) : m \in \mathbb{Z}\} \cup \{(0, 2^n) : n \in \mathbb{Z}\} \cup \{(2^m, 2^n) : (m, n) \in \mathbb{Z}^2\}.$$

Let W denote the projection of Y to X. We take $\Gamma = \mathbb{Z} < G = \mathbb{R}$. This subgroup preserves Y and, hence, W. The quotient W/Γ is homeomorphic to $Y \cap \{(0, y) : y \in \mathbb{R}\}$, hence, is Hausdorff. At the same time, the Γ-action on W is non-proper.

5 Cocompactness

There are two common notions of cocompactness for group actions:

(1) $G \times X \to X$ is cocompact if there exists a compact $K \subset X$ such that $G \cdot K = X$.
(2) $G \times X \to X$ is cocompact if X/G is compact.

It is clear that (1)⇒ (2), as the image of a compact under the continuous (quotient) map $p : X \to X/G$ is compact.

Lemma 17 *If X is locally compact then* (2)⇒ (1).

Proof For each $x \in X$ let U_x denote a relatively compact neighborhood of x in X. Then

$$V_x := p(U_x) = p(G \cdot U_x),$$

is compact since $G \cdot U_x$ is open in X. Thus, we obtain an open cover $\{V_x : x \in X\}$ of X/G. Since X/G is compact, this open cover contains a finite subcover

$$V_{x_1}, \dots, V_{x_n}.$$

It follows that

$$p(\bigcup_{i=1}^{n} U_{x_i}) = X/G.$$

The set

$$K = \bigcup_{i=1}^{n} \overline{U_{x_i}}$$

is compact and $p(K) = X/G$. Hence, $G \cdot K = X$. □

Lemma 18 *Suppose that X is normal and Hausdorff, $G \times X \to X$ is a proper action of a discrete group, such that X/G is locally compact. Then X is locally compact.*

Proof Pick $x \in X$. Let W_x be a slice for the G-action at x; then $W_x/G_x \to X/G$ is a topological embedding. Thus, our assumptions imply that W_x/G_x is compact for every $x \in X$. Let (x_α) be a net in W_x. Since W_x/G_x is compact, the net $(x_\alpha)/G$ contains a convergent subnet. Thus, after passing to a subnet, there exists $g \in G_x$ such that (gx_α) converges to some $x \in \overline{W_x}$. Hence, (x_α) subconverges to $g^{-1}(x)$. Thus, W_x is relatively compact. Since X is assumed to be normal, x admits a basis of relatively compact neighborhoods. □

Corollary 19 *For normal Hausdorff spaces X the two notions of cocompactness agree for proper discrete group actions on X.*

On the other hand, if the drop the properness condition, the two notions are not equivalent even for $\mathbb{Z}$-actions with Hausdorff quotients, see the example by R. de la Vega in [16].

6 Fundamental sets

Definition 20 A closed subset $F \subset X$ is a *fundamental set* for the action of G on X if $G \cdot F = X$ and there exists an open neighborhood $U = U_F$ of F such that for every compact $K \subset X$, the transporter set $(U|K)_G$ is finite (the *local finiteness condition*).

Fundamental sets appear naturally in the reduction theory of arithmetic groups (Siegel sets), see [13] and [2].

There are several existence theorems for fundamental sets. The next proposition, proven in [10, Lemma 2], guarantees the existence of fundamental sets under the *paracompactness assumption* on X/G.

Proposition 21 *Each proper action $G \times X \to X$ of a discrete group G on a locally compact Hausdorff space X with paracompact quotient X/G admits a fundamental set.*

One frequently encounters a sharper version of fundamental sets, called *fundamental domains*. A *domain* in a topological space X is an open connected subset $U \subset X$ which equals the interior of its closure.

Definition 22 Suppose that $G \times X \to X$ is a proper action of a discrete group. A subset F in X is called a *fundamental domain* for an action $G \times X \to X$ if the following hold:

(1) F is a domain in X.
(2) $G \cdot \overline{F} = X$.
(3) $gF \cap F \neq \emptyset$ if and only if $g = 1$.
(4) For every compact subset $K \subset X$, the transporter set $(\overline{F}|K)_G$ is finite, i.e. the family $\{g\overline{F}\}_{g\in G}$ of subsets in X is *locally finite*.

Suppose that (X, d) is a proper geodesic metric space, i.e. a space where every closed metric ball is compact and every two points are connected by a geodesic segment. Suppose, furthermore, that $G \times X \to X$ is a proper isometric action of a discrete group, $x \in X$ is a point which is fixed only by the identity element.

Remark 23 If G is countable and fixed point sets in X of nontrivial elements of G are nowhere dense, then Baire's Theorem implies existence of such x.

One defines the *Dirichlet domain* of the action as

$$D = D_x = \{y \in X : d(y, x) < d(y, gx) \quad \forall g \in G \setminus G_x\}.$$

Note that $gD_x = D_{gx}$.

Proposition 24 *Each Dirichlet domain D is a fundamental domain for the G-action.*

Proof 1. The closure $\overline{D}$ is contained in

$$\hat{D} = \hat{D}_x = \{y \in X : d(y, x) \leq d(y, gx) \quad \forall g \in G \setminus G_x\}.$$

As before, $g\hat{D}_x = \hat{D}_{gx}$. I claim that $\hat{D}$ is the closure of D and D is the interior of $\hat{D}$; this will prove that D is a domain. Clearly, D is contained in the interior of $\hat{D}$ and $\hat{D}$

is closed. Hence, it suffices to prove that each point of $\hat{D}$ is the limit of a sequence in D. Consider a point $z \in \hat{D} \setminus D$ and let $c : [0, T] \to X$ be a geodesic connecting x to z. Then for each $t \in [0, T)$ and $g \in G \setminus \{1\}$,

$$d(x, c(t)) < d(x, c(t)) + d(c(t), z) = d(x, z) \leq d(z, gx),$$

i.e. $c(t) \in D$. Thus, indeed, z lies in the closure of D, as claimed. This argument also proves that D is connected.

2. Let us prove that $g\hat{D} = X$. For each $y \in X$ the function $g \mapsto d(z, gx)$ is a proper function on G, hence, it attains its minimum at some $g \in G$. Then, clearly, $y \in \hat{D}_{gx}$, hence, $y \in g\hat{D}_x$. Thus, $g\overline{D} = X$.

3. Suppose that $g \in G \setminus \{1\}$ is such that $gD = D_{gx} \cap D \neq \emptyset$. Then each point y of intersection is closer to x than to gx (since $y \in D_x$) and also y is closer to gx than to $g^{-1}gx = x$ (since $y \in D_{gx}$). This is clearly impossible.

4. Lastly, we verify local finiteness. Consider a compact $K \subset X$. Then $K \subset B = B(x, R)$ for some R. For every $g \in G$ such that $gB \cap B \neq \emptyset$, $d(x, gx) \leq 2R$. Since (X, d) is a proper metric space and the action of G on X is proper, the set of such elements of G is finite. ◻

We will now prove existence of fundamental domains for proper discrete group actions on a certain class of topological spaces, cf. [14].

Theorem 25 *Suppose that X is a 2nd countable, connected and locally connected locally compact Hausdorff topological space. Suppose that $G \times X \to X$ is a proper action of a discrete countable group such that the fixed-point set of each nontrivial element of G is nowhere dense in X. Then this action admits a fundamental domain.*

Proof Our goal is to construct a G-invariant geodesic metric metrizing X. Then the result will follow from the proposition.

Lemma 26 *The quotient space $Y = X/G$ is locally compact, connected, locally connected and metrizable.*

Proof Local compactness and connectedness of Y follows from that of X. The 2nd countability of X implies the 2nd countability of Y. By Lemma 9, Y is Hausdorff. Since Y is locally compact and Hausdorff, its one-point compactification is compact and Hausdorff, hence, regular. It follows that Y itself is regular. In view of the 2nd countability of Y, Urysohn's metrization theorem implies that Y is metrizable. ◻

Remark 27 Note that each locally compact metrizable space is also locally path-connected.

It is proven in [15] that each locally compact, connected, locally connected metrizable space, such as Y, admits a complete geodesic metric d_Y which we fix from now on. Consider the projection $p : X \to Y$. According to [4, Theorem 6.2] (see also [1, Lemma 2]), the map p satisfies the path-lifting property: Given any path $c : [0, 1] \to Y$, a point $x \in X$ satisfying $p(x) = c(0)$, there exists a path $\tilde{c} : [0,1] \to X$ such that $p \circ \tilde{c} = c$. (This result is, of course, much easier if the G-action is free, i.e. $p : X \to Y$ is a covering map.) We let $\mathcal{L}_X$ denote the set of paths in X which are lifts of rectifiable paths $c : [0,1] \to Y$. Clearly, the postcomposition of $\tilde{c} \in \mathcal{L}_X$ with an element of G is again in $\mathcal{L}_X$. Our next goal is to equip X with a G-invariant *length structure* using the family of paths $\mathcal{L}_X$. Such a structure is a function on $\mathcal{L}_X$ with values in $[0, \infty)$, satisfying certain axioms that can be found in [5, Section 2.1]. Verification of most of these axioms is straightforward, I will check only some (items 1, 2, 3 and 4 below).

1. If $\tilde{c} \in \mathcal{L}_X$ is a lift of a path c in Y, then we declare $\ell(\tilde{c})$ to be equal to the length of c.

2. If $\tilde{c}_i, i = 1, 2$, are paths in $\mathcal{L}_X$ (which are lifts of the paths c_1, c_2 respectively) whose concatenation $b = \tilde{c}_1 \star \tilde{c}_2$ is defined, then b is a lift of the concatenation $c_1 \star c_2$. Clearly, $\ell(b) = \ell(\tilde{c}_1) + \ell(\tilde{c}_2)$.

3. Let U be a neighborhood of some $x \in X$. We need to prove that

$$\inf_{\gamma}\{\ell(\gamma)\} > 0, \tag{1}$$

where the infimum is taken over all $\gamma = \tilde{c} \in \mathcal{L}_X$ connecting x to points of $X \setminus U$. It suffices to prove this claim in the case when U is G_x-invariant, satisfies

$$\overline{U} \cap g\overline{U} \neq \emptyset \iff g \in G_x, \tag{2}$$

and γ connects x to points of ∂U. Then $V = p(U)$ is a neighborhood of $y = p(x)$ in Y and the paths $c = p \circ \gamma$ connect y to points in ∂V. But the lengths of the paths c are clearly bounded away from zero and are equal to the lengths of their lifts $\tilde{c}$. Thus, we obtain the required bound (1).

4. Let us verify that any two points in X are connected by a path in $\mathcal{L}_X$. Since X is connected, it suffices to verify the claim locally. Let U is G_x-invariant neighborhood of x satisfying (2), such that $V = p(U)$ is an open metric ball in Y centered at $y = p(x)$. Take $u \in U$, $v := p(u) \in V$. Let $c : [0, T] \to V$ be a geodesic connecting v to y. Then there exists a lift $\tilde{c} : [0, T] \to U$ of c with $\tilde{c}(0) = u$. Since $x \in U$ is the only point projecting to y, we get $\tilde{c}(T) = x$. By taking concatenations of pairs of such radial paths in U, we conclude that any two points in U are connected by a path $\tilde{c} \in \mathcal{L}_X$.

Given a length structure on X, one defines a path-metric (metrizing the topology of X) by

$$d_X(x_1, x_2) = \inf_{\gamma}\{\ell(\gamma)\}$$

where the infimum is taken over all $\gamma \in \mathcal{L}_X$ connecting x_1 to x_2. By the construction, the projection $p : (X, d_X) \to (Y, d_Y)$ is 1-Lipschitz.

Lemma 28 *The metric d_X is complete.*

Proof Let (x_n) be a Cauchy sequence in (X, d_X). By the construction of the metric d_X, there exists a finite length path $\tilde{c} : [0, 1) \to (X, d_X)$ and a sequence $t_n \in [0, 1)$ such that $\tilde{c}(t_n) = x_n, \tilde{c}(0) = x = x_1$. Since the map p is 1-Lipschitz, the path

$c = p \circ \tilde{c} : [0, 1) \to (Y, d_Y)$ also has finite length. Since the metric d_Y was complete to begin with, the path c extends to a path $\bar{c} : [0, 1] \to Y$; set $y' := \bar{c}(1)$.

Assume for a moment that G acts freely on X. Then we have the *uniqueness* of lifts of paths from Y to X. Thus, the unique lift $\tilde{\bar{c}}$ of $\bar{c}$ starting at the point x satisfies the property that its restriction to [0, 1) equals $\tilde{c}$. It follows that the sequence (x_n) converges to $\tilde{\bar{c}}(1)$. Below we generalize this argument to the case of non-free actions.

Let U be a neighborhood of $y' = \bar{c}(1)$ which is the projection to Y of a relatively compact slice neighborhood $\tilde{U}$ of some $x' \in p^{-1}(y')$. Without loss of generality (by removing finitely many initial terms of the sequence (x_n)) we can assume that the image of the path c lies entirely in U. Applying the path-lifting property to the path c with the prescribed *terminal* point x', we obtain a lift of the path $\bar{c}$ that terminates at x'. This lift has to be entirely contained in $\tilde{U}$ and its initial point has to be of the form $g(x)$ for some $g \in G$. Applying g^{-1} to this lift, we obtain another lift of $\bar{c}$, denoted $\tilde{\bar{c}}$, which starts at x and terminates at $g^{-1}(x')$.

Consider the restriction of $\tilde{\bar{c}}$ to [0, 1). This restriction is also a lift to the path $c|_{[0,1)}$ and the image of the latter lies entirely in U. Hence, the image of $\tilde{\bar{c}}|_{[0,1)}$ lies entirely in the relatively compact subset $g^{-1}(\tilde{U}) \subset X$. Thus, the Cauchy sequence (x_n) lies in a relatively compact subset of X, and it follows that this sequence converges in X.

□

Since (X, d_X) is locally compact and complete, by Theorem 2.5.28 (and Remark 2.5.29) in [5], (X, d_X) is a geodesic metric space. Lastly, we note that, by the construction, the length structure on X and, hence, the metric d_X, is G-invariant. This concludes the proof of the theorem. □

Question 29 Local compactness and local connectivity were critical for the proof of the theorem. Does the theorem hold without these assumptions?

For each fundamental set F of a G-action on a topological space X we define its quotient space F/G as the quotient space of the equivalence relation $x \sim y \iff (\{x\}|\{y\})_G \neq \emptyset$. The following proposition explains why fundamental sets are useful: They allow one to describe quotient spaces of proper actions by discrete groups using less information than is contained in the description of that action.

Proposition 30 *Suppose that F is a fundamental set for proper action by discrete group G on a 1st countable and Hausdorff space X. Then the natural projection map $p : F/G \to X/G$ is a homeomorphism.*

Proof The map p is continuous by the definition of the quotient topology. It is also obviously a bijection. It remains to show that p is a closed map. Since F is closed, it suffices to show that the projection $q : F \to X/G$ is a closed map. Suppose that (x_n) is a sequence in F such that $q(x_n)$ converges to some $y \in X/G$, y is represented by a point $x \in F$. Then there is a sequence $g_n \in G$ such that $g_n(x_n)$ converges to x. Since $\{g_n(x_n) : n \in \mathbb{N}\} \cup \{x\}$ is compact which, without loss of generality is contained in U_F, the local finiteness assumption implies that the sequence (g_n) is finite. Hence, after extraction, $g_n = g$ for all n. The fact that F is closed then implies that $x \in F$. It follows that x is an accumulation point of (x_n). Thus, $q : F \to F/G$ is a closed map. ☐

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Declarations

Conflicts of interest The author has no conflicts of interest to disclose.

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