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Publication Date

1985-12-01



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Submitted to Physics of Fluids

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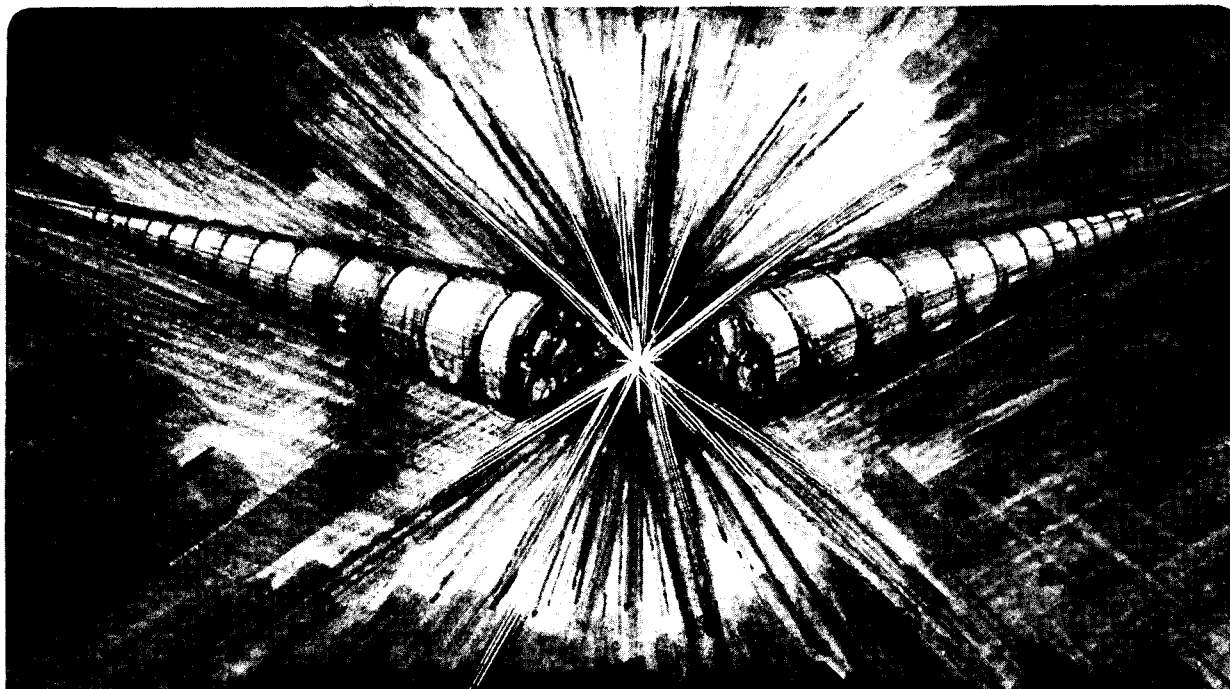
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LBL-19664

WAVE ENTROPY: A DERIVATION BY JAYNES' PRINCIPLE*

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Abstract

The Jaynes maximum-entropy principle is used to derive the standard expression for the entropy of a set of weakly interacting waves.

* This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

In discussions of classical wave kinetic equations, use is made of an expression for the wave entropy, as a functional of the mean density of wave action in ray phase space, $\bar{J}(\underline{k}, \underline{x})$, and having the property of increasing monotonically in time as \bar{J} evolves. The standard choice¹ is

$$S(\bar{J}) = \int \ln \bar{J}(\underline{k}, \underline{x}), \quad (1)$$

with the integration element $d^3x d^3k / (2\pi)^3$.

It would clearly be desirable to have a classical derivation of this expression. Up to now, one has appealed to the quantum expression,² and taken the classical limit of large occupation number. A classical derivation can be based on Jaynes' maximum entropy principle,³ and provides a striking illustration of its utility. In this note, we review the Jaynes algorithm in performing this derivation.

We begin by recognizing that $(\underline{k}, \underline{x})$ space is not a continuum, but rather a set of cells, whose size is determined by Fourier's uncertainty principle. Hence we rewrite Eq. (1) as

$$S(\bar{J}) = \sum \ln \bar{J}_i, \quad (2)$$

where \bar{J}_i is the mean action in the i th cell.

Following Planck, we characterize each cell as an oscillator, possibly nonlinear. With (J_i, Θ_i) as the action-angle variables for the i th oscillator, we introduce the corresponding probability density $\rho_i(J_i, \Theta_i)$,

and have

$$\begin{aligned}\bar{J}_i &= \int dJ_i \int d\theta_i \rho_i(J_i, \theta_i) J_i \\ &= \int dJ_i \rho_i(J_i) J_i.\end{aligned}\tag{3}$$

The Jaynes prescription³ is to introduce the information-theoretic (Gibbs-Shannon) entropy $S(\rho)$, as a functional of the system distribution function ρ :

$$S(\rho) = - \int \rho \ln \rho,\tag{4}$$

and maximize it with respect to ρ , subject to the constraint (3). This determines the "best" ρ , as a parametric function of the given data $\{\bar{J}_i\}$. When we carry out this procedure, we obtain, not surprisingly, the Gibbs distribution:

$$\rho(J; \bar{J}) = \pi_i(\bar{J}_i)^{-1} \exp(-J_i/\bar{J}_i).\tag{5}$$

Since Eq. (4) can be considered as $S = - \langle \ln \rho \rangle$, we form $\langle \ln \rho \rangle$ from (5):

$$\langle \ln \rho \rangle (\bar{J}) = \sum_i (- \ln \bar{J}_i - 1),\tag{6}$$

and finally obtain the desired expression (2), after discarding the constant term in (6). Thus the entropy associated with the data $\{\bar{J}_i\}$ is the information-theoretic entropy of the best distribution consistent with those data.

I am indebted to Steven M. Omohundro, not only for stressing the importance of Jaynes' principle, but also for providing, in his Ph.D. thesis,⁴ the basic mathematical underpinning for that principle.

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4. S.M. Omohundro, "Geometric Perturbation Theory and Plasma Physics," Ph.D. thesis, University of California, Berkeley, 1985 (LBL-20147), Chapter 16.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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