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November 1998

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Abstract

Although the Lévy (stable-Paretian) distribution of stock returns was first observed by Mandelbrot 35 years ago, an explanation for this phenomenon has not yet been found. Several extensive studies have recently shown that short-term rates of return on stock indices and on single stocks are distributed according to the <u>truncated</u> Lévy distribution. An apparently unrelated but well-documented fact is that wealth is distributed according to the Pareto-law distribution at high wealth levels. In this paper we suggest that the Lévy distribution of rates of return originates from the Pareto-law wealth distribution among investors. We present a model which assumes i)a Pareto distribution of wealth and ii)that the effect an investor has on the price of a stock is proportional (in a stochastic sense) to the investor's wealth. This model generates a truncated Lévy distribution of stock returns. This result is robust to many variations of the basic model. The model leads to the prediction that the parameter α_L of the Lévy rate of return distribution should be equal to the Pareto constant α_w of the Pareto wealth distribution. Empirical evidence from the U.S., the U.K. and France reveals a striking agreement between these a-priori unrelated parameters (U.S.: $\alpha_L = 1.39$, $\alpha_w = 1.35$; U.K.: $\alpha_L = 1.12$, $\alpha_w = 1.06$; France: $\alpha_L = 1.82$, $\alpha_w = 1.83$).

Keywords: Distribution of returns, Lévy, Stable-Paretian, Wealth Distribution, Pareto.

Introduction

Review

It has been long known that the distributions of returns on stocks, especially when calculated for short time intervals, are not fitted well by the normal distribution. Rather, the distributions of stock returns are leptokurtic, or "fat tailed". Thirty five years ago Benoit Mandelbrot (1963a, 1963b) proposed an exact functional form for return distributions. Namely, he has suggested that log-returns are distributed according to the symmetrical Lévy probability distribution defined by:

$$L_{\alpha_L}^{\gamma}(x) \equiv \frac{1}{\pi} \int_0^{\infty} \exp(-\gamma \Delta t q^{\alpha_L}) \cos(qx) dq \qquad (1)$$

where $L_{\alpha_L}^{\gamma}(x)$ is the Lévy probability density function at x, α_L is the characteristic exponent of the distribution, $\gamma \Delta t$ is a general scale factor, γ is the scale factor for $\Delta t = 1$, and q is an integration variable. This distribution is also known as the "stable-Paretian" distribution. In order to avoid confusion, we will use only the term "Lévy distribution" throughout this paper.

Mandelbrot's pioneering work gained support in the first few years after it's publication (Fama 1963a, Fama 1963b, Fama 1965a, Roll 1968, Teichmoeller 1971, and Officer 1972). Fama and Roll (1968, 1971) developed methodologies in order to estimate the parameters of the Lévy distribution. Efficient portfolio selection in a market with Lévy distributions was analyzed by Fama (1965b) and Samuelson (1967). Subsequent works, however, have questioned the Lévy distribution hypothesis (Hsu, Miller and Wichern, 1974, Joyce, Brorsen and Irwin, 1989). By the definition of the stability property of the Lévy distribution, the sum of independent Lévy random variables is also distributed according to Lévy distribution (Lévy, 1925). Thus, if short-term log-returns

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are distributed according to a Lévy distribution, longer term log-returns (which are the sums of the short-term log-returns), should retain the Lévy distribution functional form. However, from the empirical data it is evident that as the time interval between price observations grows longer, the distribution of log-returns deviates from the short-term Lévy distribution, and converges to the normal distribution. This means that the log-return distribution is not truly stable. Over the next few years, the Lévy distribution temporarily lost favor, and alternative processes where suggested as mechanisms generating stock returns. The two main alternatives which where proposed are mixtures of distributions (Clark 1973, Hsu, Miller and Wichern 1974, Pedrosa and Roll 1998), and ARCH/GARCH processes (Engle 1982, Bollerslev, Chou and Kroner, 1990).

In the 90's the Lévy distribution hypothesis has made a dramatic comeback, and with a twist. Recent extensive analysis of short-term returns has shown that price differences, log-returns, and rates of return [1] are described extremely well by a <u>truncated</u> Lévy distribution. This is not a sharp truncation in the usual mathematical sense, but rather, it describes a smooth fall-off of the empirical distribution from the Lévy distribution at some value, (for a general picture of the empirical short-term rate of return distribution, see figure 2). Mantegna and Stanley (1995) analyze tick-by-tick data on the S&P 500 index and find excellent agreement with a Lévy distribution up to six standard deviations away from the mean (in both directions). For more extreme observations, the distribution they find falls off approximately exponentially. Similar results have been found in the examination of the Milan stock exchange (Mantegna 1991), the CAC40 index (Zajdenweber 1994), individual French stocks (Belkacem 1996),

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[1]Some studies examine the distribution of price differences, some of log-returns, some of rates of return. As the focus is on a very sort term (a few seconds to a few days) all of the above are very closely related. See also the discussion in section I.B.4.

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and foreign exchange markets (Pictet and Muller 1995, Guillaume et al. 1997, and Cont, Potters and Bouchaud 1997).

Because the empirical log-return distribution is truncated, it is not precisely a stable distribution. Rather, it is a "pseudo-stable" distribution - a distribution which is fitted well by the Lévy distribution in the central part of the distribution, but falls off, or is truncated, at some value

Such a distribution exhibits the following behavior: for short time scales it behaves like a stable distribution, but for longer time scales the truncation causes the distribution to converge to a normal distribution (Mantegna and Stanley 1994, Cont, Potters and Bouchaud 1997). Thus, the observation that the empirical log-return distribution is a <u>truncated</u> Lévy distribution explains both the short-term stable properties of the distribution, and the long-run convergence to the normal distribution (Akgiray and Booth 1988, Mantegna and Stanley 1995). For an extensive review of evidence and applications of the Lévy distribution in finance, see Mandelbrot (1997).

The Problem

The revival of the (truncated) Lévy distribution awakens an old question: **Why are returns distributed in this very specific way?** The first, and to the best of our knowledge, the only explanation was suggested by Fama (1963a) and Mandelbrot (1963). Their explanation is based the following mathematical result, proved independently by Doblin and Gnedenko (1954):

<u>Definition</u>: A random variable $\tilde{x}$ is said to be <u>asymptotically Paretian</u> if:

$$1 - F(x) \longrightarrow D_1 x^{-\alpha} \quad as \ x \longrightarrow \infty$$
$$F(-x) \longrightarrow D_2 x^{-\alpha} \quad as \ x \longrightarrow \infty$$

(2)

where F is the cumulative distribution of x, and D_1 and D_2 are constants (Fama, 1963, pg. 423).

<u>Doblin-Gnedenko Result:</u>

The sum of n i.i.d. random variables which are asymptotically Paretian with $0 < \alpha \leq 2$ tends to the Lévy distribution with characteristic exponent α as n tends to infinity (Gnedenko and Kolmogorov 1954, Feller 1971).

<u>Remarks:</u>

1. Equations (2) is equivalent to stating that the tails of the density function $f(x)$ decrease asymptotically as a power-law: $f(x) \longrightarrow D_1 \alpha x^{-(1+\alpha)}$, and $f(-x) \longrightarrow D_2 \alpha x^{-(1+\alpha)}$, as $x \longrightarrow \infty$.

2. The Doblin-Gnedenko result depends only on the properties of the "tails" of $f(x)$. The central part of the distribution does not matter.

3. For $\alpha = 2$ the variance of x is finite and the Doblin-Gnedenko result reduces to the familiar central limit theorem (the Lévy distribution with a characteristic exponent $\alpha_L = 2$ is the normal distribution). For $0 < \alpha < 2$ the variance of x is infinite. Thus, one can view the Doblin-Gnedenko result as an extension of the central limit theorem to cases where the variance of the random variable is infinite.

Fama and Mandelbrot suggest that in order to explain the Lévy distribution of returns one should model changes in prices as the result of many i.i.d effects that are asymptotically Paretian distributed. In the spirit of the efficient market hypothesis, Fama and Mandelbrot suggest that the underlying forces which drive price changes are bits of new information. In the words of Fama:

"As long as the effects of individual bits of information are asymptotically Paretian, various types of complicated combinations of these effects will also be asymptotically Paretian... As long as the effects of individual bits of information combine in a way which makes the price changes from transaction to transaction asymptotically Paretian with exponent α, then according to the conditions of Doblin and Gnedenko, the price changes for longer differencing intervals will be stable Paretian with the same value of α." (Fama 1963a, pg. 426).

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Thus, if the effects of bits of information are distributed according to some asymptotically Paretian distribution with exponent α, and if price changes are due to the aggregate effect of many independent bits of information, then it follows from the Doblin-Gnedenko result that price changes should be distributed according to the Lévy distribution with a characteristic exponent α. This is a big step towards understanding the distribution of price changes. However, following this track, one is left with the problem of explaining why the effects of bits of new information should be asymptotically Paretian.

The Contribution of this Paper

In this paper we suggest an explanation for the exact functional form of the distribution of stock returns. We propose that the observed truncated Lévy distribution of stock returns originates from the well-documented Pareto-law distribution of wealth. Our explanation produces a surprising theoretical prediction which is empirically testable. Namely, we argue that the characteristic exponent of the Lévy return distribution, α_L, and the Pareto constant of the wealth distribution, α_w, should be equal. Note that a-priori these two parameters do not seem to be related at all: one has to do with a description of stock returns, while the other is typically used as a measure of wealth inequality in a society. An empirical comparison of these parameters reveals a striking agreement between α_L and α_w, for each of the three countries investigated.

The explanation that we suggest for the truncated Lévy distribution of stock returns extends and complements the Fama-Mandelbrot approach. We model price changes as resulting from the aggregate effect of the actions of many investors (which can be driven either by new information or by other reasons). We assume that the wealth of investors is distributed according to the Pareto-law, and that the effect the action of an

investor has on the stock price is proportional (in a stochastic sense) to that investor's wealth. Both of these assumptions conform with evidence which has been previously empirically documented (Steindl 1965, Atkinson and Harrison 1978, Persky 1992, Hausman, Lo and MacKinlay, 1992). These assumptions lead to an asymptotically Paretian distribution of single-trade price changes. The price change measured over a certain time interval is the aggregate of many of these single-trade changes, and according to the Doblin-Gnedenko result it will be distributed according to the Lévy distribution. Thus, in our model the Lévy distribution of stock price changes is driven by the Pareto wealth distribution. As a result, the characteristic exponent of the Lévy price change distribution α_L, should be equal to the Pareto constant α_w of the Pareto-law wealth distribution. The model we suggest is very generic and robust, and it explains not only the distribution of price changes, but also the distribution of rates of return, and log-returns.

Our model also provides an explanation for the truncation of the return distribution. The truncation of the return distribution in our model is a consequence of the truncation of the wealth distribution - in any market with a finite number of individuals, the wealth distribution is truncated at the wealthiest individual.

The paper is organized as follows. In section I we present the basic model, and several generalizations. In section II we present empirical estimations of the characteristic exponent of the Lévy rate of return distribution, and of the wealth distribution Pareto constant, for three different countries. Section III concludes the paper with a discussion of our main results.

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I. The Model

The two basic assumptions that we make are:

i) Pareto-law wealth distribution at high wealth ranges:

$$for \ W > W_0 \quad f(W) = CW^{-(1+\alpha_w)}$$

where W is wealth, and $f(W)$ is the probability density function. The exponent α_w is positive, and it is known as the Pareto constant. C is a normalization constant. Notice that the Pareto-law distribution describes the distribution of wealth only at high wealth ranges, $W > W_0$. It does not deal with the form of the wealth distribution for $W < W_0$.

ii) The effect that the action of an investor has on the stock price is proportional (in a stochastic sense) to the investor's wealth.

Assumption i), the Pareto-law wealth distribution, has been first empirically observed by Vilfredo Pareto over one hundred years ago (Pareto 1897). It has been since verified in numerous empirical studies, conducted in various countries at different times (see, for example, Steindl 1965, Atkinson and Harrison 1978, Persky 1992, Levy and Solomon 1997). For a review of the properties of the Pareto-law distribution, see Johnson and Kotz (1970). The Pareto-law provides a very good description of the empirical wealth distribution in the range of the top 5-10% of the population (for lower wealth levels, the empirical density function is typically described by the log-normal distribution). In this paper we are interested only in individuals who are wealthy enough to actively invest in the stock market. The group of investors typically corresponds to individuals in the top-wealth range, which is described by the Pareto-law [2]. Thus, we assume a

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[2]Mankiw and Zeldes (1991) report that only 11.9% of U.S. households hold more than $10,000 in stocks. Only 1.8% of households hold more than $100,000 in stocks. Haliassos and Bertaut (1995) find that the percentage of households owning stock increases dramatically with household income. Their data implies that the top 1% of the population (by income) holds more than 57% of the stock market. The top 5% of the population holds about 80% of the market.

Pareto-law distribution of wealth among investors [3]. We do not make any assumption regarding the form of the wealth distribution at the lower wealth range ($W < W_0$).

The second assumption seems natural: it is intuitive that on average the actions of an investor with \$100 million will effect prices roughly 10 times as much as the actions of a similar investor with only \$10 million. This is also consistent with models of constant relative risk aversion, which imply that investors make decisions regarding proportions of their wealth (see, for example, Levy and Markowitz 1979, Samuelson 1989), and with the finding that the price impact of a trade is roughly proportional to the volume of the trade, especially for high volume trades (see figure 4 in Hausman, Lo and MacKinlay, 1992). To be specific, if the volume traded by an investor is (stochastically) proportional to the investor's wealth, and if the effect of a trade on the price is proportional to the volume of the trade, then the effect that an investor's trade has on the price is (stochastically) proportional to the investor's wealth.

The emergence of a Lévy distribution of returns is not sensitive to the specifics of the model. In order to present the main idea of the proposed mechanism we would like to start by considering an over-simplified model. We then move on to discuss more realistic versions.

I.A The Over-Simplified Model

Assume:

i) The wealth of investors is distributed according to the Pareto-law, with a Pareto constant α_w:

$$for \ W > W_0, \quad f(W) = CW^{-(1+\alpha_w)} \tag{3}$$

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[3]The effects of institutional investors are discussed in section III.

Individuals with $W < W_0$ do not trade in the stock market.

ii) There is one traded stock in the market (which serves as a proxy for the index). At every time period t, an investor is chosen at random, and she trades with the specialist. With probability 1/2 the investor buys, and with probability 1/2 she sells. If the investor buys the stock price goes up by $0.1 * W_t$, where W_t is the wealth of the investor chosen to trade at time t. If the investor sells the stock price goes down by $0.1 * W_t$.

Let us denote the single-trade price change at period t as z_t. In this model we have:

$$\underline{\text{probability}}$$

$$z_t = \begin{cases} +0.1W_t, & 1/2 \\ -0.1W_t, & 1/2 \end{cases} \tag{4}$$

Notice that W_t is a random variable with a continuous distribution, and therefore so is z_t. The price change z_t will be in the range $(z, z + dz)$, only if the wealth of the investor chosen to trade at time t is in the range $(10z, 10(z+dz))$. An additional condition is that the trade is in the appropriate direction (i.e., if z is positive it is not enough that the wealth of the investor is in the range $(10z, 10(z + dz))$, we also require that the investor buys, rather than sells.) Thus, we have:

$$for \ z \leq 0 \quad Prob[z < z_t < z + dz] = 1/2 Prob[10z < W_t < 10(z + dz)] \tag{5}$$

$$for \ z > 0 \quad Prob[z < z_t < z + dz] = 1/2 Prob[10z < W_t < 10(z + dz)],$$

where the 1/2 comes from the equal probabilities of buying or selling. If we denote the probability density function of a single-trade price change by $g(z)$, equation (5) can be written as:

$$for \ z \leq 0 \quad g(z)dz = 1/2f(10z)10dz \tag{6}$$

$$for \ z > 0 \quad g(z)dz = 1/2f(10z)10dz$$

where f is the wealth probability density function. Writing f explicitly (using eq.(3)), we obtain:

$$for \ z \leq 0 \quad g(z) = 1/2f(10z)10 = 1/2C(10z)^{-(1+\alpha_w)}10 = C'z^{-(1+\alpha_w)}, \tag{7}$$

$$for \ z > 0 \quad g(z) = 1/2f(10z)10 = 1/2C(10z)^{-(1+\alpha_w)}10 = C'z^{-(1+\alpha_w)},$$

where C' is a new constant ($C' = 1/2C10^{-\alpha_w}$). Notice that the distribution of $g(z)$ is a special case of an asymptotically Paretian distribution with $D_1 = D_2 = C'$, and exponent α_w (see the definition in eq (2), and remark 1). Let us denote the price change after n independent trades by ΔP. This price change is simply:

$$\Delta P = \sum_{t=1}^{n} z_t. \tag{8}$$

As the z_t's are asymptotically Paretian with exponent α_w, according to the Doblin-Gnedenko result, for a large number of trades, n, the distribution of ΔP converges to the Lévy distribution with the same exponent α_w, which is the Pareto constant of the wealth distribution. Thus, the price change distribution is a Lévy distribution with characteristic exponent $\alpha_L = \alpha_w$.

I.B Generalizations

I.B.1 Truncated Wealth Distribution $\longrightarrow$ Truncated Lévy Distribution

In the preceding analysis we have assumed a Pareto-law wealth distribution (eq.3). This distribution goes all the way up to infinite levels of wealth. In any real market,

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however, the wealth distribution is truncated at some value - the wealth of the richest individual. In this section we show that this leads to a corresponding truncation of the Lévy price change distribution.

Assume that the wealth distribution is truncated at some maximal wealth W_{max}:

$$f(W) = \begin{cases} C''W^{-(1+\alpha_w)}, & \text{for} \quad W_0 < W < W_{max}; \\ 0, & \text{for} \quad W_{max} < W \end{cases} \tag{9}$$

As result of this, the single-trade price change z_t is bounded:

$$max|z_t| = 0.1W_{max}. \tag{10}$$

(We do not change our assumption regarding the effect of investors on the price, equation (4)). The truncation in the distribution of z_t, in turn, leads to a truncation of the Lévy distribution of ΔP, the price change after n trades:

$$max|\Delta P| = n0.1W_{max}. \tag{11}$$

While the truncation of the Lévy price change distribution is a general result, the sharp nature of the truncation described in equation (11) is a consequence of our simplified assumption regarding the effect of investors on price (equation (4)). A general stochastic effect of investors on price, as discussed in the next section, tends to "smooth out" this truncation.

I.B.2 Stochastic Effect on Price

In the over-simplified model above we assume that the investor's effect on the price is a deterministic proportion of the investor's wealth. This, of course, is unrealistic. There are many reasons why the effect an investor has on the price is stochastic: investors may act upon new information which is of variable relevance and of variable precision.

Liquidity needs may also be stochastic. In this subsection we show that our result is robust to investors' actions having a stochastic effect on price.

Before deriving the general result consider a simple example: assume that investors act on the arrival of new information, and suppose that there are two kinds of signals: strong signals (which can be positive or negative), and weak signals (which again can be positive or negative). Assume that all four possible signals are equally likely. If an investor with wealth W_t receives a strong signal she effects the price by $0.1W_t$. If she receives a weak signal she effects the price by $0.05W_t$. Thus, we have:

$$\underline{\text{probability}}$$

$$z_t = \begin{cases} +0.05W_t, & 1/4 \\ +0.10W_t, & 1/4 \\ -0.10W_t, & 1/4 \\ -0.05W_t, & 1/4 \end{cases} \tag{12}$$

The price change z_t will be in the range $(z, z + dz)$ in the following two circumstances:

a) the signal is strong, <u>and</u> the wealth of the investor chosen to trade is in the range $(10z, 10(z + dz))$.

or:

b) the signal is weak, <u>and</u> the wealth of the investor chosen to trade is in the range $(20z, 20(z + dz))$.

For a positive price change z, we must add the requirement that the investor buys, and vice versa for negative z. Since these two cases are symmetric (as in the previous example), let us develop the price change probability distribution only for $z > 0$. We have:

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$$for \ z > 0 \quad Prob[z < z_t < z+dz] = 1/2 Prob[strong signal] Prob[10z < W_t < 10(z+dz)]$$
$$+1/2 Prob[weak \ signal] Prob[20z < W_t < 20(z+dz)] \tag{13}$$

Which can be rewritten as:

$$for \ z > 0 \quad g(z)dz = 1/4 f(10z)10dz + 1/4 f(20z)20dz. \tag{14}$$

Writing f explicitly we get:

$$for \ z > 0 \quad g(z) = 1/4 C(10z)^{-(1+\alpha_w)}10 + 1/4 C(20z)^{-(1+\alpha_w)}20 = C'z^{-(1+\alpha_w)}, \tag{15}$$

where $C' = 1/4 C(10^{-\alpha_w} + 20^{-\alpha_w})$. Again, the single trade price change distribution $g(z)$ is asymptotically Paretian with the exponent α_w, and therefore the price change after a certain period, which is the sum of many independent z_t's, will be distributed according to the Lévy distribution with the same exponent α_w.

The same result holds for the general case of a stochastic effect on prices. In the general case we have:

$$z_t = \tilde{q}_t W_t, \qquad \qquad \cdot \tag{16}$$

where $\tilde{q}_t$ is a stochastic proportionality factor, which is distributed according to some distribution $h(\tilde{q})$, and is uncorrelated with W_t. In this case there is a continuum of combinations that can lead to a price change z. A price change z will occur whenever $\tilde{q}_t W_t = z$. Generalizing equation(12) we get:

$$for \ z > 0 \quad g(z) = 1/2 \int_0^\infty h(q)f(\frac{z}{q})\frac{1}{q}dq = 1/2 \int_0^\infty h(q)C(\frac{z}{q})^{-(1+\alpha_w)}\frac{1}{q}dq = C'z^{-(1+\alpha_w)} \tag{17}$$

where

$$C' = 1/2C \int_0^\infty h(q)(\frac{1}{q})^{-\alpha_w} dq.$$

Once more, the single trade price change distribution is asymptotically Paretian with exponent α_w, and the Doblin-Gnedenko result tells us that the sum of many such independent price changes will be distributed according to the Lévy distribution with $\alpha_L = \alpha_w$.

I.B.3 Variable Trading Frequency

In the preceding sections we have assumed that at time period t only one investor is chosen to trade with the specialist. This assumption can be relaxed to allow for a stochastic number of investors to trade at each time period.

The stochastic price change due to the effect of a single trade is given by:

$$z = \tilde{q}W. \tag{18}$$

In section I.B.2 it was shown that $g(z)$, the probability density function of z, is asymptotically Paretian with exponent α_w. Let us denote the sum of m i.i.d. random variables z by S_m: $S_m \equiv z^1 + z^2 + ... + z^m$. Notice that because the z's are asymptotically Paretian with exponent α_w, so is S_m (Gnedenko and Kolmogorov, 1954). Let us denote the density function of S_m by k_m. The number of trades taking place at a given time period is a discrete random variable which we denote by $\tilde{m}$:

<div align="center">probability</div>

$$\tilde{m} = \begin{cases} 1, & \pi_1 \\ 2, & \pi_2 \\ \vdots & \vdots \\ m, & \pi_m \\ \vdots & \vdots \end{cases} \tag{19}$$

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The probability density function of an aggregate price change z_t at time period t is given by:

$$g(z_t) = \sum_{m=1}^{\infty} \pi_m k_m(z_t) \tag{20}$$

where m is the number of trades, $k_m(z_t)$ is the density function of S_m at z_t, and the summation is over all possible numbers of trades in a single time period. As the k_m's are asymptotically Paretian distributions with an exponent α_w, so is z_t, their weighted average (if $k_i(z_t) \longrightarrow d_i z_t^{-(1+\alpha_w)}$ as $z_t \longrightarrow \infty$, and $k_j(z_t) \longrightarrow d_j z_t^{-(1+\alpha_w)}$ as $z_t \longrightarrow \infty$, then $\pi_i k_i(z_t) + \pi_j k_j(z_t) \longrightarrow (\pi_i d_i + \pi_j d_j) z_t^{-(1+\alpha_w)}$ as $z_t \longrightarrow \infty$, which implies that the weighted average is also asymptotically Paretian).

As before, the price change after n time periods is:

$$\Delta P = \sum_{t=1}^{n} z_t. \tag{21}$$

As the z_t's have been shown to be asymptotically Paretian with exponent α_w, according to the Doblin-Gnedenko result, for a large number of trades, n, the distribution of ΔP converges to the Lévy distribution with the same exponent α_w, which is the Pareto constant of the wealth distribution. Thus, the result of the model is robust to a stochastic number of trades at each time period.

I.B.4 Rates of Return Versus Price Differentials

Up to this point we have discussed only the distribution of price differences. What about the distribution of rates of return, or log-returns? We would like to argue that considerations similar to those discussed in the previous sections lead to a Lévy distribution of rates of return and of log-returns as well.

This is pretty obvious if the price remains at a fairly constant level during the

The price change due to a trade is in nominal terms, and it is therefore proportional to nominal wealth:

$$z_t \equiv P_{t+1} - P_t = \tilde{q}_t W_t^n. \tag{24}$$

Substituting W_t^n from equation(23) we obtain:

$$P_{t+1} - P_t = \tilde{q}_t W_t^r \left(\frac{P_t}{P_{base}} \right), \tag{25}$$

or:

$$r_t \equiv \left(\frac{P_{t+1} - P_t}{P_t} \right) = \left(\frac{\tilde{q}_t}{P_{base}} \right) W_t^r. \tag{26}$$

Following the derivation in section I.B.2 we conclude that the rate of return r_t is an asymptotically Paretian variable with exponent α_w. From the Doblin-Gnedenko result it follows that the sum of many i.i.d rates of return, $\sum_{t=1} r_t$, is distributed according to the Lévy distribution with characteristic exponent α_w. The rate of return over an interval of T time periods, R_T, is given by:

$$(1 + R_T) = \Pi_{t=1}^{T}(1 + r_t). \tag{27}$$

As we are considering short time periods (up to about a month), the one-period rates of return r_t are small, and R_T can be approximated by:

$$R_T \approx \sum_{t=1}^{T} r_t. \tag{28}$$

The distribution of rates of return R_t is be closely approximated by the distribution of $\sum_{t=1}^{T} r_t$, which is the Lévy distribution with characteristic exponent α_w. Thus, the

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Lévy distribution of rates of return, and the equality $\alpha_L = \alpha_w$ for this distribution are natural extensions of the same result which was derived for price changes.

As for log-returns, since we are considering short time periods, rates of return are small, and log-returns are very closely approximated by:

$$\log(\frac{P_{t+1}}{P_t}) = \log(1 + \frac{P_{t+1} - P_t}{P_t}) \approx \frac{P_{t+1} - P_t}{P_t}. \tag{29}$$

If rates of return are distributed according to the Lévy distribution with $\alpha_L = \alpha_w$ as argued above, the same is true for log-returns.

I.B.5 Asymmetric Price Processes

Up to this point we have assumed that the price process is symmetric. Namely, we have assumed equal probabilities for buying and selling the stock, and as a consequence the singe-trade price change distribution, $g(z)$, was not only asymptotically Paretian, it was also symmetric. This lead to a longer-term symmetric Lévy distribution of price changes. Our result is robust to the relaxation of the symmetry assumption. If, for example, the probability for buying is greater than the probability of selling, $g(z)$ would no longer be symmetric. However, it would still be asymptotically Paretian - it would satisfy the conditions of equation (2), (with $D_1 > D_2$). As a result, the price change distribution after many trades will tend to the a-symmetric Lévy distribution (see Mandelbrot 1963).

We are focused on the distribution of short-term price changes (or rates of return). In the short term, we do not expect a strong drift in prices, and as a consequence, the short-term price change distribution is not expected to be very a-symmetric. Indeed, when examining the short-term empirical data, one finds that the a-symmetry negligible, and that the symmetric Lévy distribution is a very good approximation for the empirical distribution (see figure 2).

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II. Empirical Evidence

The model presented in this paper predicts that the characteristic exponent of the truncated Lévy return distribution α_L should be equal to the wealth distribution Pareto constant α_w. In this section we estimate these parameters empirically for three countries: the U.S. the U.K. and France.

II.A Estimation of α_L

For the estimation of α_L we follow the methodology used by Mantegna and Stanley (1995). They note that the density at 0 of the symmetric Lévy distribution given in equation (1) is:

$$p_0 \equiv L_{\alpha_L}^{\gamma}(0) = \frac{\Gamma(1/\alpha_L)}{\pi \alpha_L (\gamma \Delta t)^{1/\alpha_L}} \tag{30}$$

where Γ is the Gamma function, Δt is the horizon for which rates of return are being calculated, and γ is the scale factor for $\Delta t = 1$ (for proof, see appendix A). Thus, the probability density at 0 decays as $\Delta t^{-1/\alpha_L}$. Mantegna and Stanley estimate the density at 0 for various time intervals Δt in order to estimate α_L. They run the regression:

$$\log[p_0(\Delta t)]_i = A + B \log[\Delta t]_i + \epsilon_i, \tag{31}$$

and estimate α_L as $-\frac{1}{B}$. We employ the same technique.

Our data sets consists of:

S&P 500: all 1,780,752 records of the cash index between 1990-1995, obtained from the Chicago Mercantile Exchange.

FTSE 100: all 75,606 records of the cash index between January 1997 and August 1997, obtained from the Futures Industry Institute.

CAC 40: all 234,501 records of the cash index in 1996, obtained from the Bourse de Paris.

For each of these series we estimate the density $p_0(\Delta t)$ by going over the entire data set and calculating the rates of return for intervals of Δt. We count the number of rates of return within the range $-0.0001 - +0.0001$ and divide this number by the total number of observations, in order to get the probability of rate of return in this range. Then we divide the result by the size of the range, 0.0002, in order to get the probability density. We repeat this procedure for different sampling time intervals Δt. Our results are reported in Figures 1-4. For the S&P 500 we obtain $\alpha_L = 1.39$ (Figure 1). The standard error of this estimation is 0.04, the correlation coefficient is -0.987, and the t-value is -33.8. This estimated value that we find for α_L is close to the value of 1.40 reported by Mantegna and Stanley (1995) for the S&P 500, during the sample period 1984-89.

In order to verify that the empirical rate of return distribution is indeed fitted well by the symmetric Lévy distribution with the estimated α_L of 1.39, we compare these two distributions in figure 2. The empirical distribution is calculated for 1-minute rates of return, and the plot is semi-logarithmic [4]. Figure 2 shows an excellent agreement between the empirical and theoretical distributions up to rates of return in the order of 0.001, which are about six standard deviations away from the mean. For more extreme returns, the empirical distribution falls off from the Lévy distribution. This is the so-called truncation, which will be discussed below. It is interesting to note the secondary peak of the empirical distribution at a rate of return of about -0.001. A similar bimodel distribution was observed by Jackwerth and Rubinstein (1996). We do not have an explanation for this phenomenon in the framework of the present model.

For the FTSE 100 we find $\alpha_L = 1.12$ (Figure 3). The standard error of this

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[4]The empirical distribution is estimated by a non-parametric density estimate with a Gaussian kernel and the "normal reference rule", Scott (1992, pg. 131).

estimation is 0.03, the correlation coefficient is -0.996, and the t-value is -59.7. This number is close to the α_L value of 1.10 which is calculated from the 1993 FTSE 100 data reported by Abhyankar, Copeland, and Wong (1995).

For the CAC 40 we find $\alpha_L = 1.82$ (Figure 4). The standard error of this estimation is 0.05, the correlation coefficient is -0.978, and the t-value is -35.9.

II.B Estimation of α_w

Estimating the Pareto wealth distribution exponent α_w requires data regarding the "right-tail" of the distribution, i.e. data about the wealth of the wealthiest individuals.

The French almanac Quid provides wealth data on the top 162,370 individuals in France. This data is in aggregate form, i.e. the numbers of individuals with wealth exceeding certain wealth levels are reported. According to the Pareto-law (eq. (3)), the number of individuals with wealth exceeding a certain level W_x should be proportional to $W_x^{-\alpha_w}$:

$$N(W > W_x) = N \int_{W_x}^{\infty} f(W)dW = NC \int_{W_x}^{\infty} W^{-(1+\alpha_w)}dW = \frac{NC}{\alpha_w}W_x^{-\alpha_w} \qquad (32)$$

where $N(W > W_x)$ is the number of individuals with wealth exceeding W_x, and N is the total number of individuals. When plotting $N(W > W_x)$ as a function of W_x on a double-logarithmic graph, one expects the data points to fall on a straight line with slope $-\alpha_w$. Figure 5 is a double logarithmic plot of $N(W > W_x)$ as a function of W_x for the French data provided by Quid. This figure shows an excellent agreement between the empirical wealth distribution and the Pareto-law. In order to estimate α_w for France we run the regression:

$$\log[N(W > W_x)]_i = A + B\log[W_x]_i + \epsilon_i \qquad (33)$$

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The absolute value of the slope of the regression line, which is the estimate for α_w is 1.82. (Standard error =0.03, correlation coefficient =-0.999 t-value =-59.5).

For the U.S and the U.K. the available data regarding the wealthiest people is more detailed but also more limited in scope. For these countries, lists with the ranking and wealth of the several hundreds of wealthiest individuals are published. We use the methods suggested by Levy and Solomon (1997) in order to estimate α_w from these data. A Pareto law distribution of wealth with exponent α_w implies the following relation between the rank of an individual in the wealth hierarchy and her wealth:

$$W(n) = An^{-1/\alpha_w} \tag{34}$$

where n is the rank (by wealth), and W is wealth. The constant A is given by $A = \left(\frac{\alpha_w}{NC}\right)^{-1/\alpha_w}$, where N is the total number of individuals in the population, and C is the normalization constant from equation (3); (for a mathematical derivation of this relation see appendix B).

For the U.S. we obtain data from the 1997 Forbes 400 list. Wealth as a function of rank is plotted in double logarithmic form in figure 6. Running the regression:

$$\log[W(n)]_i = A + B \log[n]_i + \epsilon_i \tag{35}$$

we estimate the slope of the regression as -0.74. This is the estimation of $-1/\alpha_w$, and it corresponds to an estimation of $\alpha_w = 1.35$ for the U.S. The standard error of this estimation is 0.005. We would like to clarify that we do not assume that only the wealthiest 400 individuals determine the S&P rate of return distribution. Rather, we use the data that we are able to obtain in order to estimate the Pareto constant for the entire upper wealth range. Our estimation of $\alpha_w = 1.35$ for the U.S., is close to the estimate

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of $\alpha_w = 1.35 - 1.42$ which is obtained by the data provided by (1996) regarding the percentage of wealth held by the top 1%, 5%, and 10% of the population (see appendix C).

For the U.K. we obtain data from the Sunday Times Rich List 1997. The data are plotted in Figure 7. We obtain a slope of -0.94 which corresponds to a value of $1/0.94 = 1.06$ for α_w (standard error 0.004).

The summary of our empirical results appears in Table I. This evidence shows a striking agreement between the values of α_w and α_L for the three countries investigated.

Table I

	α_L	α_w
U.S.	1.39 ± 0.04	1.35 ± 0.005
U.K.	1.12 ± 0.03	1.06 ± 0.004
France	1.82 ± 0.05	1.83 ± 0.030

III. Discussion

In this paper we show that a (truncated) Pareto-law wealth distribution and a proportionality between the wealth of an investor and the effect the investor has on stock price, lead to a (truncated) Lévy distribution of stock returns. This result is based on a mathematical theorem by Doblin and Gnedenko. Our result is robust to different variations of the model, such as a stochastic effect of a trade on price, variable trading frequency, and asymmetric price processes. The model suggested leads to a surprising prediction of an equality between the a-priori unrelated wealth distribution Pareto constant α_w, and the Lévy return distribution exponent α_L. Empirical evidence from the U.S., the U.K., and France, provides striking confirmation of this prediction.

Our model suggests that the cut-off of the Lévy return distribution is due to the cut-off in the wealth distribution. This is hard to confirm empirically, since we do not

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know the distribution of the stochastic proportionality factor, $\tilde{q}$ (see equation (16)). However, a rough back-of-the-envelope calculation shows that the orders of magnitude are right. Suppose that the wealthiest person in the economy has a wealth W_{max} and is invested 100% in stocks. If this investor would take all of her money out of the market in a "single trade", the market capitalization would change from M to $M - W_{max}$, and the rate of return generated would be $-\frac{W_{max}}{M}$. Thus, the maximal single trade rate of return should be in the order of magnitude of $\frac{W_{max}}{M}$. The wealth cut-off in the U.S. is in the order of magnitude of about \$10 billion, and the U.S. stock market capitalization is about \$10 trillion. Thus, we would expect the fall-off from the Lévy return distribution to occur at rates of return in the order of 0.001. Figure 2, shows that the fall-offs do occur roughly in this range.

This paper suggests an explanation for the Lévy rate of return distribution in terms of the distribution of wealth among individual investors. How do institutional investors fit into this picture? Institutional investors usually have freedom to choose the stocks which go into their portfolio, but typically they do not have a great deal of discretion about the stock-bond mix of their portfolio. As a consequence, the main effect institutional investors have on the level of the index, is when they increase or decrease the overall <u>size</u> of their stock portfolio. This happens when individual investors deposit or withdraw money to/from their accounts. Thus, in the context our model, the institution serves as a middle-man: instead of trading directly with the specialist, individuals may trade through an institution. The return generating mechanism is not altered: changes in the index are still due to the actions of the <u>individual</u> investors. Hence, the results derived in this paper are robust to the introduction of institutional investors.

The Doblin-Gnedenko result holds for random variables which are independent.

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Therefore, in order to employ this powerful result, in this paper we have assumed that the single-trade price changes are independent. However, the Doblin-Gnedenko result can be generalized in some circumstances to random variables which are dependent (Davis 1983, Avram and Taqqu 1986, Jakubowski 1993, Samorodnitsky and Taqqu 1994). These generalizations may allow the extension of the results presented in this paper to cases in which the stock's volatility is auto-correlated. It would be very interesting to explore the conditions under which these results can be generalized to include ARCH/GARCH processes. Extensions of the model in this direction is left for future work.

Appendix A

<u>Lemma:</u>

$$p_0 \equiv (0) = \frac{\Gamma(1/\alpha_L)}{\pi \alpha_L (\gamma \Delta t)^{1/\alpha_L}} \tag{36}$$

<u>Proof:</u>

From the definition of $L_{\alpha_L}^\gamma(x)$ in equation (1):

$$L_{\alpha_L}^\gamma(0) = \frac{1}{\pi} \int_0^\infty \exp(-\gamma \Delta t q^{\alpha_L}) \cos(q0) dq = \frac{1}{\pi} \int_0^\infty \exp(-\gamma \Delta t q^{\alpha_L}) dq. \tag{37}$$

Define a new variable $u \equiv \gamma \Delta t q^{\alpha_L}$, and note that $dq = \alpha_L^{-1}(\gamma \Delta t)^{-1/\alpha_L} u^{(\frac{1}{\alpha_L}-1)} du$. Substituting in equation (37) we obtain:

$$L_{\alpha_L}^\gamma(0) = \frac{1}{\pi \alpha_L (\gamma \Delta t)^{1/\alpha_L}} \int_0^\infty \exp(-u) u^{(\frac{1}{\alpha_L}-1)} du. \tag{38}$$

Recalling that the integral is the definition of $\Gamma(\frac{1}{\alpha_L})$, we have:

$$L_{\alpha_L}^\gamma(0) = \frac{\Gamma(1/\alpha_L)}{\pi \alpha_L (\gamma \Delta t)^{1/\alpha_L}} \tag{39}$$

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Appendix B

The relationship

$$W(n) = An^{-1/\alpha_w} \tag{40}$$

holds, if and only if the wealth probability density function is given by the Pareto-law:

$$for \ W > W_0 \quad f(W) = CW^{-(1+\alpha_w)} \tag{41}$$

where n is the rank, W is wealth, the constant A is given by $A = \left(\frac{\alpha_w}{NC}\right)^{-1/\alpha_w}$, and N is the total number of individuals in the population.

$\underline{(41) \longrightarrow (40)}$

Assuming the Pareto-law (eq. 41), the number of individuals with wealth exceeding W is given by:

$$n(W) = N \int_W^\infty f(x)dx = NC \int_W^\infty x^{-(1+\alpha_w)} = \frac{NC}{\alpha_w}W^{-\alpha_w}. \tag{42}$$

The wealth of the n'th wealthiest individual is therefore given by:

$$W(n) = \left(\frac{\alpha_w}{NC}\right)^{-1/\alpha_w} n^{-1/\alpha_w} \tag{43}$$

$\underline{(40) \longrightarrow (41)}$

The relation (40) can be written as:

$$n = \left(\frac{W}{A}\right)^{-\alpha_w}. \tag{44}$$

The number of individuals with wealth in the range $(W, W + dW)$ is $-\frac{dn}{dW}$ (as the wealth goes up, the rank goes down). Thus, the probability density function $f(W)$ is given by:

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$$f(W) = -\frac{1}{N}\frac{dn}{dW}. \tag{45}$$

Writing the derivative $\frac{dn}{dW}$ explicitly (from equation (40)), and using the definition of A, $\left(A = \left(\frac{\alpha_w}{NC}\right)^{-1/\alpha_w}\right)$, we obtain:

$$f(W) = \left(\frac{\alpha_w}{N}A^{\alpha_w}\right)W^{-(1+\alpha_w)} = CW^{-(1+\alpha_w)}. \tag{46}$$

For other derivations of this relationship, see, for example Takayasu (1990).

Appendix C

Wolff's (1996) findings regarding the holdings of the top 1%, top 5%, and top 10% of the U.S. population in 1992 are reported in table II below:

Table II

k	P_k
1%	37.2%
5%	60.0%
10%	71.8%

In this table P_k denotes the percentage of total wealth held by the top k percent of the population. A Praeto-law wealth distribution with exponent α_w implies the following relationship for any two k's:

$$\frac{P_{k_1}}{P_{k_2}} = \left(\frac{k_1}{k_2}\right)^{1-\frac{1}{\alpha_w}} \tag{47}$$

(See proof below). Employing this relationship we can estimate α_w for the U.S. by using the data in table II. By comparing the holdings of the top 1% with the holdings of the top 5% we obtain:

$$\frac{.372}{.600} = \left(\frac{0.01}{0.05}\right)^{1-\frac{1}{\alpha_w}},$$

which yields $\alpha_w = 1.42$. A similar comparison of the holdings of the top 1% with the holdings of the top 10% yields $\alpha_w = 1.40$. Comparing the holdings of the top 5% with the holdings of the top 10% yields $\alpha_w = 1.35$.

<u>Lemma:</u>

A Praeto-law wealth distribution with exponent α_w implies the following relationship for any two k's:

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$$\frac{P_{k_1}}{P_{k_2}} = \left(\frac{k_1}{k_2}\right)^{1-\frac{1}{\alpha_w}}.$$

Proof:

The number of individuals with wealth exceeding a certain value W_k is given by $\frac{NC}{\alpha_w}W_k^{-\alpha_w}$, where N is the total number of individuals (see equation (42) in appendix B). This number of individuals corresponds to a percentage $k = \frac{C}{\alpha_w}W_k^{-\alpha_w}$ of the population. The above result can be restated in the following way: the wealth of the poorest individual in the top $k\%$ of the population is given by:

$$W_k = \left(\frac{k\alpha_w}{C}\right)^{-\frac{1}{\alpha_w}}. \tag{48}$$

The aggregate wealth held by the top $k\%$ of the population is given by:

$$N\int_{W_k}^{\infty} f(W)WdW = NC\int_{W_k}^{\infty} W^{-(1+\alpha_w)}WdW = = \frac{NC}{\alpha_w - 1}W_k^{1-\alpha_w} = \left(\frac{NC^{1/\alpha_w}}{\alpha_w - 1}\right)\alpha_w^{(1-1/\alpha_w)}k^{(1-1/\alpha_w)}, \tag{49}$$

where the last equality is obtained by substituting W_k from equation (48). The percentage of wealth held by the top $k\%$ of the population is:

$$P_k = \frac{1}{W_{total}}\left(\frac{NC^{1/\alpha_w}}{\alpha_w - 1}\right)\alpha_w^{(1-1/\alpha_w)}k^{(1-1/\alpha_w)}, \tag{50}$$

where W_{total} is the total wealth of all the population. Comparing the percentage of wealth held by the top $k_1\%$ of the population, with the percentage of wealth held by the top $k_2\%$ of the population, we obtain:

$$\frac{P_{k_1}}{P_{k_2}} = \left(\frac{k_1}{k_2}\right)^{1-\frac{1}{\alpha_w}}. \tag{51}$$

Note: In the above proof we have assumed that the Pareto-law wealth distribution holds for all wealth levels. However, it is straightforward to show that the above result is also valid if one assumes a Pareto-law distribution in the high wealth range but a different wealth distribution in the low wealth range (as long as the k's are in the high wealth range in which the Pareto-law wealth distribution holds).

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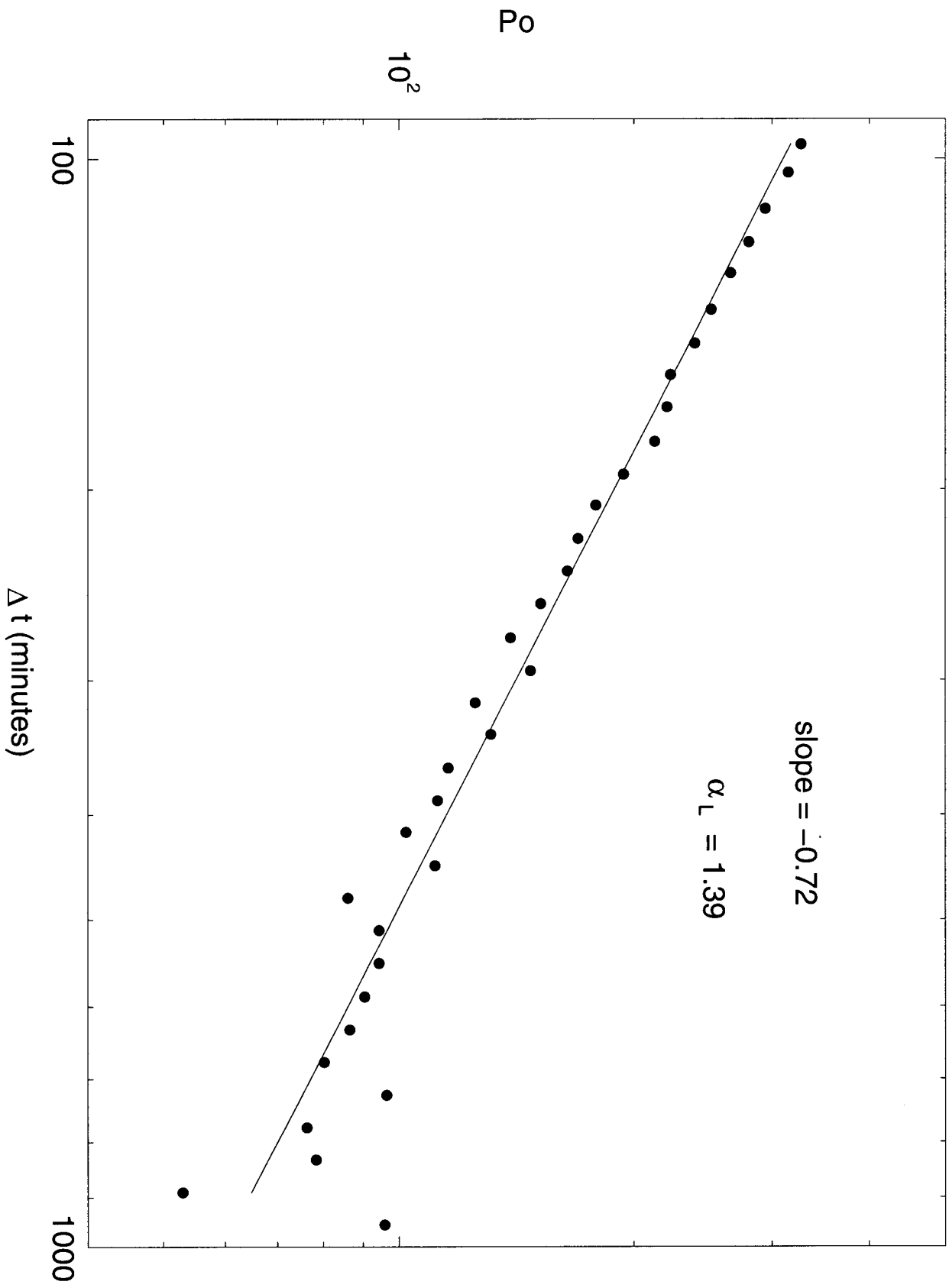Figure 1: S&P 500 Return Density at 0

Po

10^2

100

1000

Δ t (minutes)

slope = −0.72

α_L = 1.39

Figure 2: S&P 500 1 min rate of return distribution
1990–1995

Legend:
——— Levy dist. $\alpha_L = 1.39$
•——• S&P data
— – — best normal fit

Axis labels:
f(rate of return)
rate of return
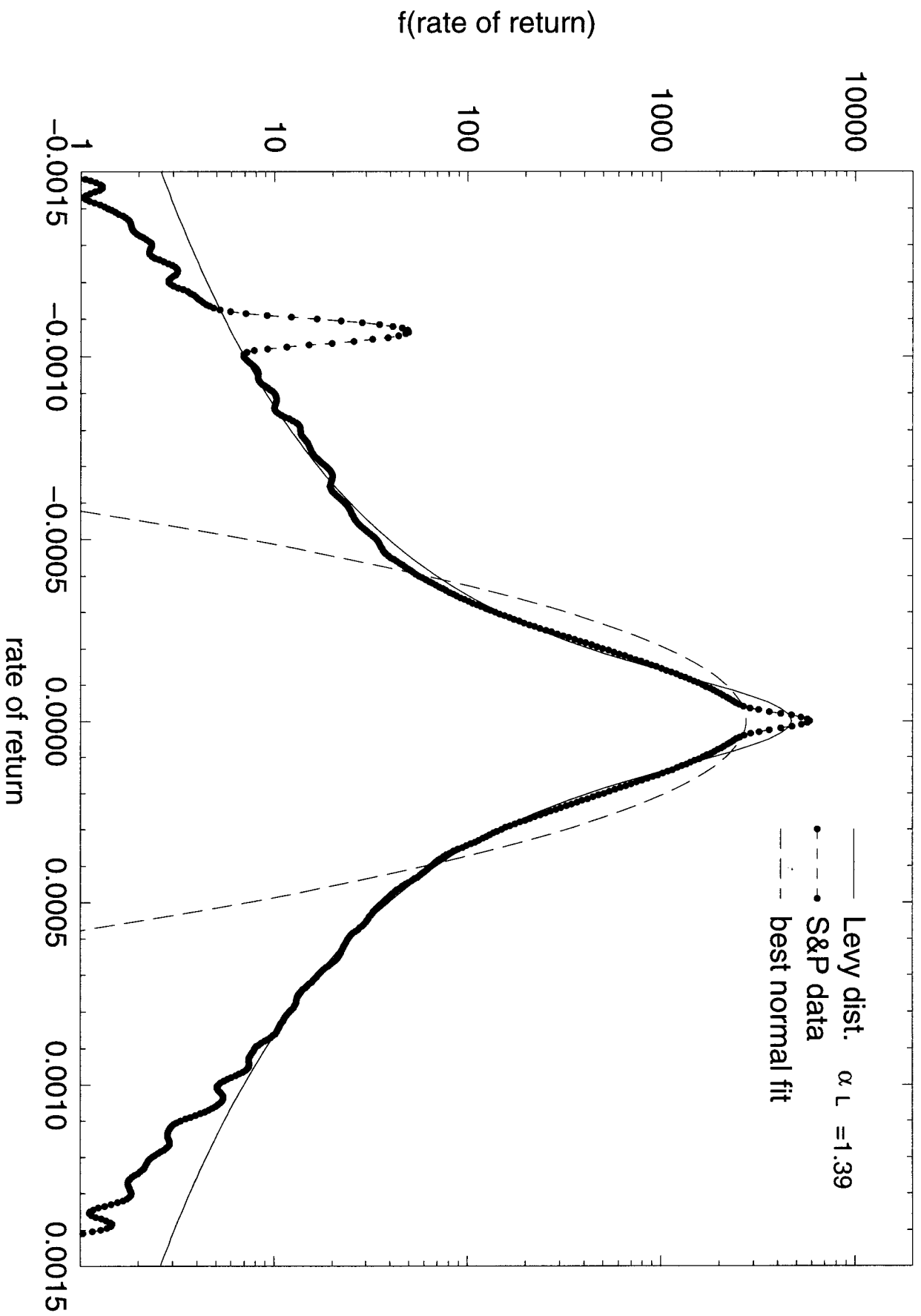
Figure 3: FTSE 100 Return Density at 0

slope = −0.89

$\alpha_L = 1.12$

Po

Δt (minutes)
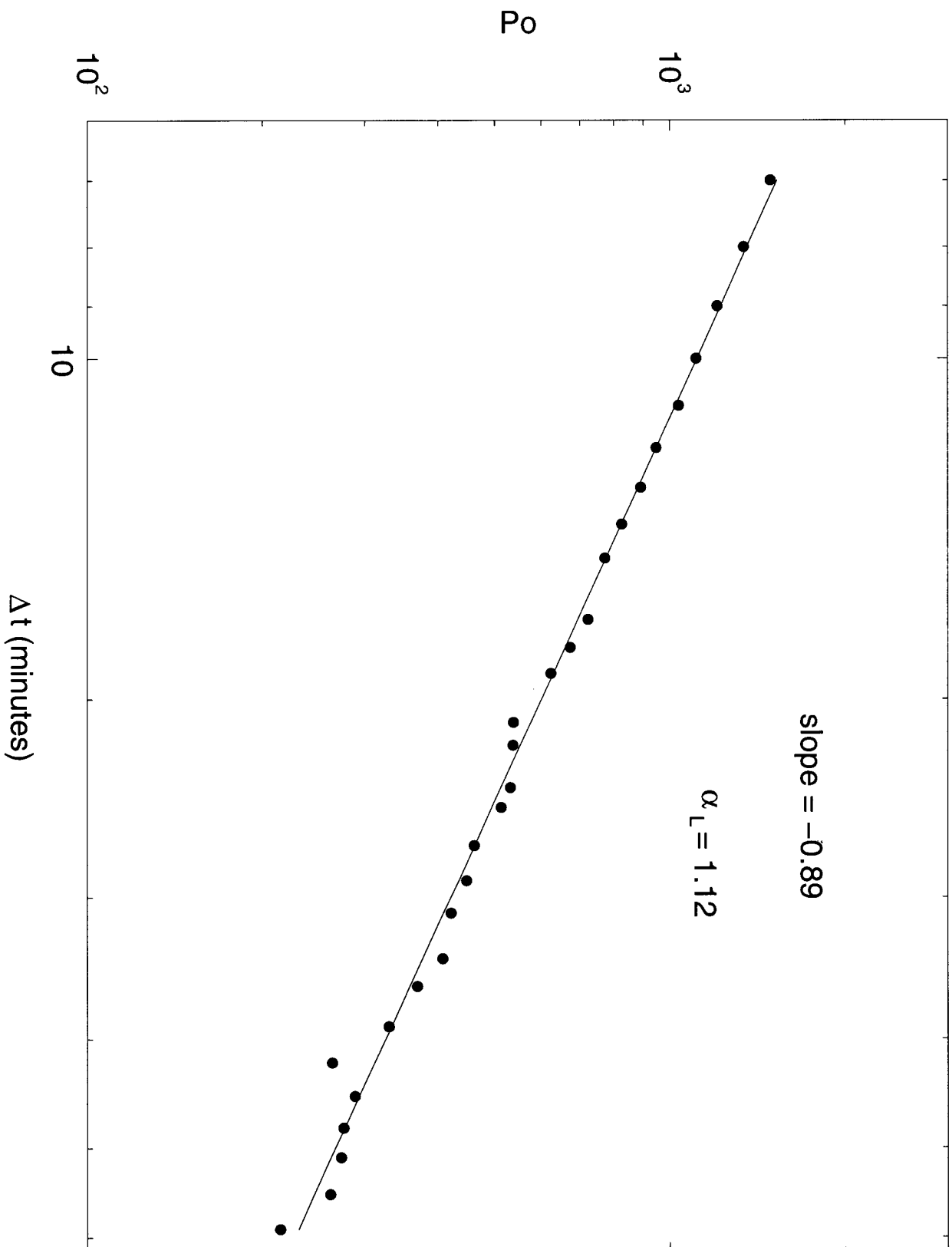
Figure 4: CAC40 Return Density at 0

slope = −0.55

$\alpha_L = 1.82$

number of people with wealth exceeding w x
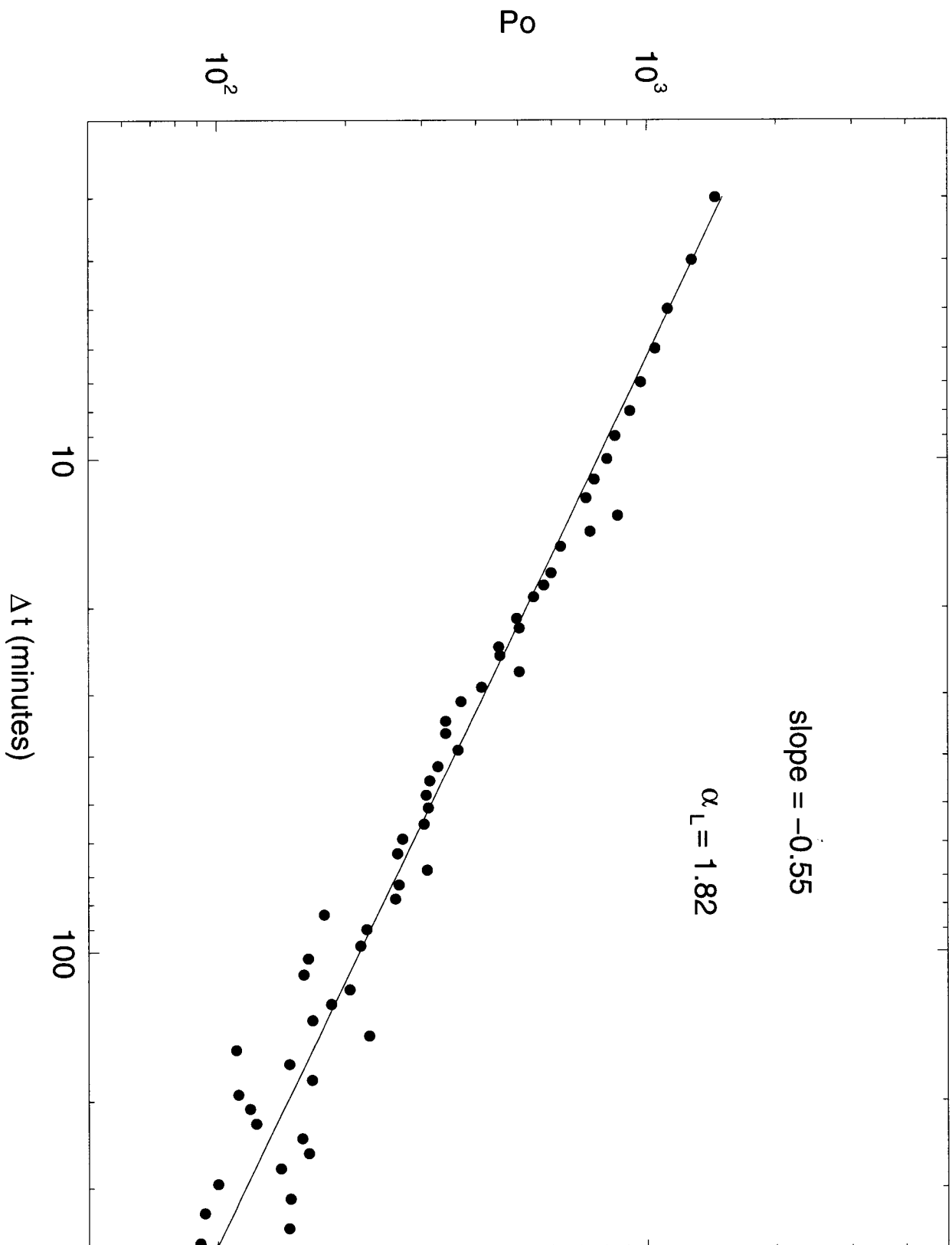
Figure 5: Wealth Distribution in France
(Quid 1994)

slope = −1.83

$\alpha_w = 1.83$

100000

10000

1000

100

10

100

w_x(millions of Francs)
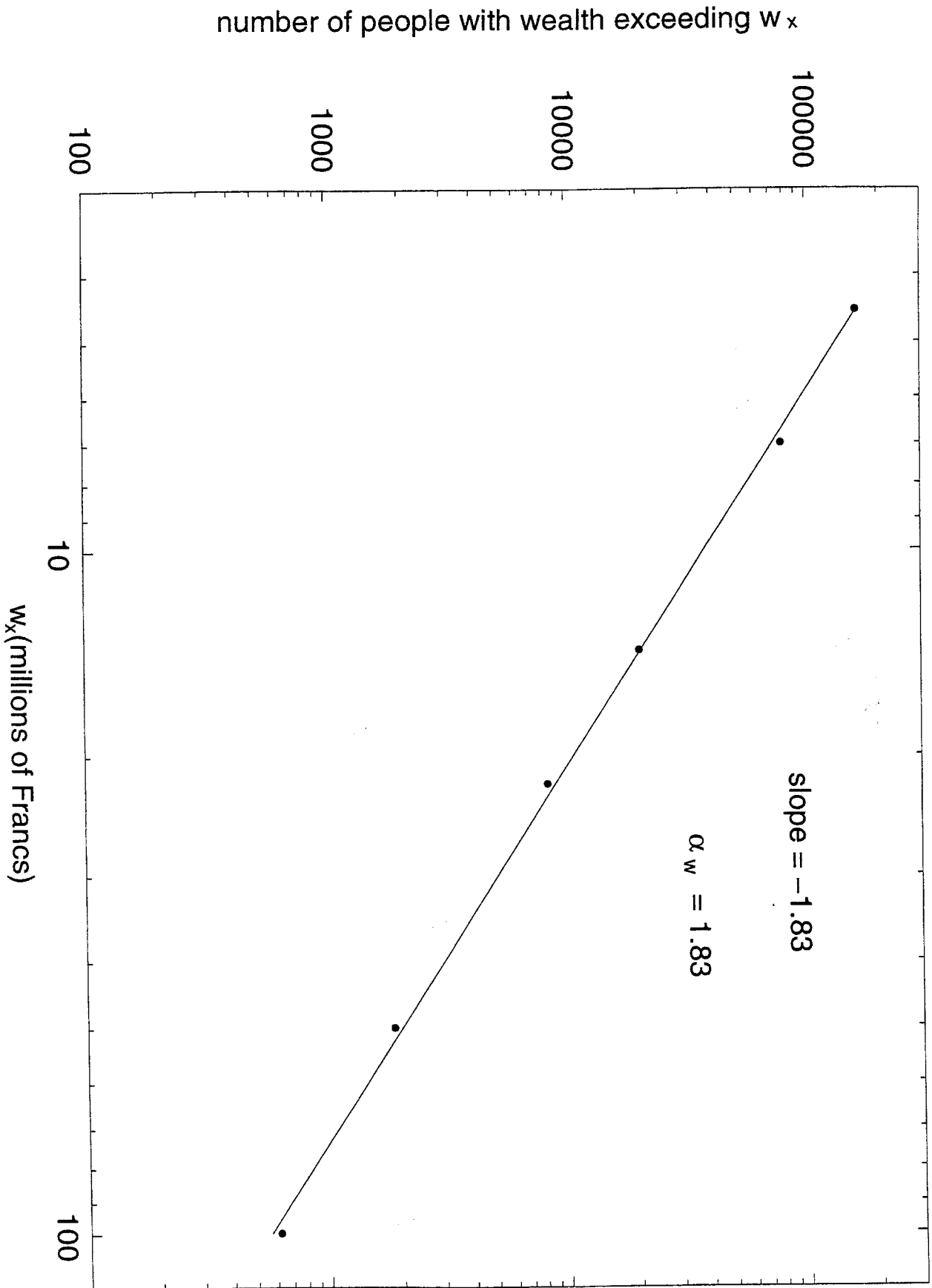
Figure 6: Wealth Distribution in US (Forbes 1997)

slope = −0.74

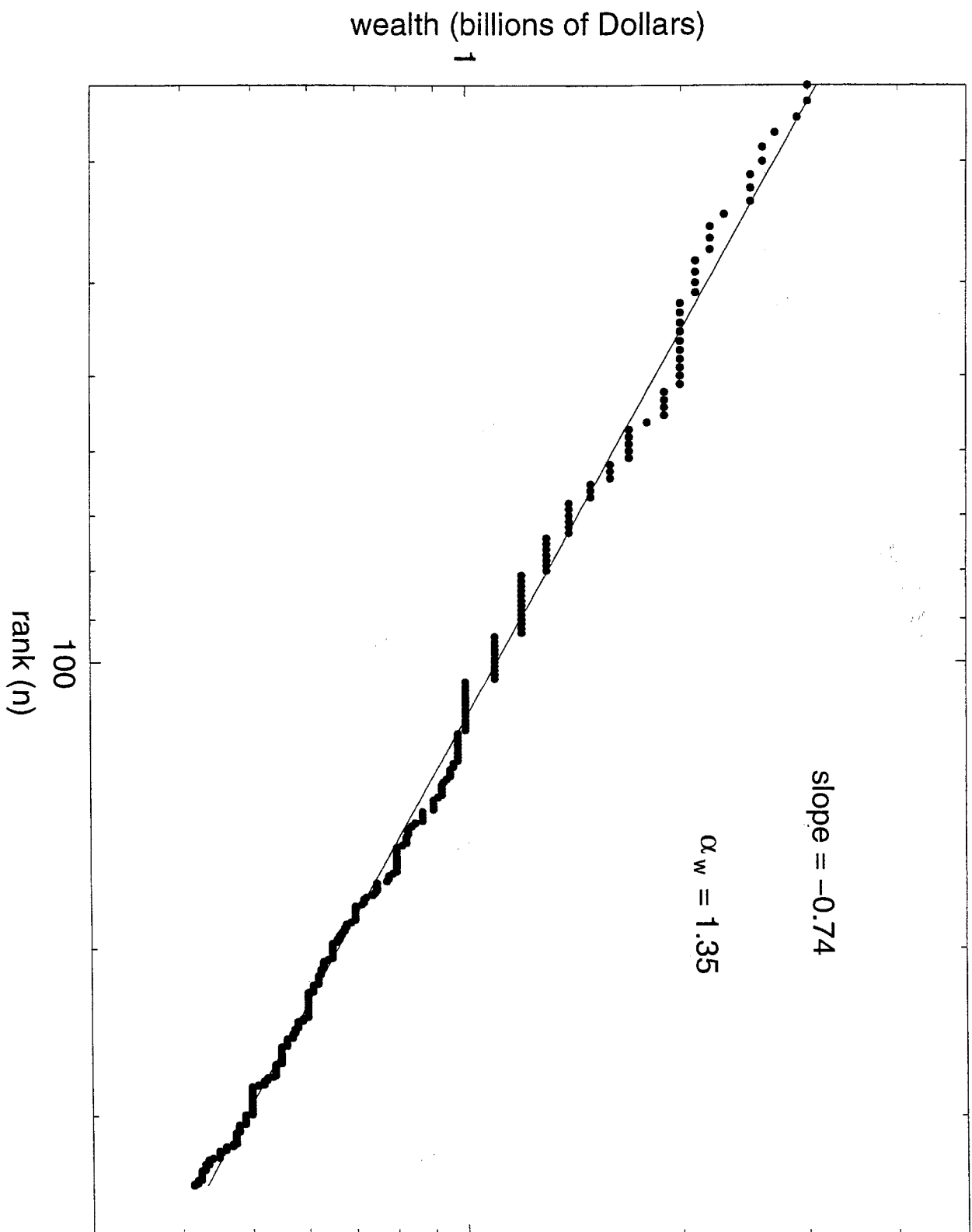$\alpha_w = 1.35$

wealth (billions of Dollars)

rank (n)

100

1

Figure 7: Wealth Distribution in UK
(Sunday Times 1997)

slope = –0.94

$\alpha_w = 1.06$