Title
Combinatorics of Macdonald polynomials and extensions

Author
Bandlow, Jason

Publication Date
2007

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Combinatorics Of Macdonald Polynomials And Extensions

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Jason Bandlow

Committee in charge:

Professor Adriano Garsia, Chair
Professor Walt Burkhard
Professor Ron Graham
Professor Jeffrey Remmel
Professor Hans Wenzl

2007

The dissertation of Jason Bandlow is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2007

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

This work would not have been possible without the encouragement, guidance, and boundless enthusiasm of my academic advisor, Adriano Garsia. The hours of discussion in offices, coffee shops, and, of course, on the beach, allowed me to develop the intellectual framework upon which this dissertation is based. My gratitude is unending.

I would also like to thank the following people, conversations (or entire courses) with whom have been particularly illuminating while learning this subject: Jim Haglund, Mark Haiman, Allen Knutson, Nick Loehr, Gregg Musiker, and Jeffrey Remmel. Thanks also to Walt Burkhard, Ron Graham, and Hans Wenzl for graciously agreeing to serve on my committee.

I would like the National Science Foundation, from which I received support every summer since 2003. Much of this work was done in that time, in particular with grant DMS-0500557 awarded to Adriano Garsia.

Thanks to Dave, Tom, Kristin, Larissa, Stephen, Henning, Maia, Jonathon, Urška, and all others with whom I've shared a trail. Thanks to Justin, Ryan, Andre, Kyle, and all others with whom I've shared a tune. Thanks to Mark, Jaime, Oded, Kelly, Andy, Nate, Ross, Meghan, Eric, Laura, and all others with whom I've shared a beer.

Finally, I would like to thank my family, Rich, Diane, Jennifer, David, and Laurel Bandlow, for their constant support and faith in me. I am the person I am today because of them.

VITA

1997	B.S. in Mathematics, Computer Science *summa cum laude*, Alma College, Alma, MI
1997-2000	Systems Analyst for Ford Motor Company, Dearborn, MI
1999	M.S. in Computer and Information Science, University of Michigan, Dearborn
2000-2002	Graduate Teaching Assistant, Colorado State University, Fort Collins
2002	M.S. in Mathematics, Colorado State University, Fort Collins
2002-2007	Graduate Teaching Assistant, University of California, San Diego
2005	Adjunct Instructor, City College, San Diego, CA
2007	Ph.D. in Mathematics, University of California, San Diego

PUBLICATIONS

J. Bandlow and G. Musiker, "Quasiinvariants of S_3", *Journal of Combinatorial Theory, Series A*, 109, 2005.

"A Variety of Combinatorial Interpretations of the q-Catalan Polynomial", Master's thesis, Colorado State University, 2002.

J. Bandlow, E. Egge, and K. Killpatrick, "A Weight-Preserving Bijection Between Schröder Paths and Schröder Permutations", *Annals of Combinatorics*, 6, 2002.

J. Bandlow and K. Killpatrick, "An area-to-inv bijection between Dyck paths and 312-avoiding permutations", *Electronic Journal of Combinatorics*, 8, 2001.

J. Bandlow and R. Molina, "Reconstruction of bit strings from their substrings", *Proceedings of the Twenty-eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, Congressus Numerantium, 127, 1997.

ABSTRACT OF THE DISSERTATION

Combinatorics Of Macdonald Polynomials And Extensions

by

Jason Bandlow
Doctor of Philosophy in Mathematics

University of California San Diego, 2007

Professor Adriano Garsia, Chair

The theory of symmetric functions is ubiquitous throughout mathematics. They arise naturally in combinatorics, algebra, and geometry, and as a result have been studied intensively for many years. This classical area was revitalized in 1988, with Ian Macdonald's description of what are now known as the Macdonald polynomials. These are a two parameter basis for the space of symmetric functions, which specialize to many of the well-known one parameter and classical bases.

Macdonald conjectured that when a certain normalization of these polynomials were expanded in terms of the classical Schur functions, the coefficients would always be polynomials in $\mathbb{N}[q,t]$. He called these coefficients q,t-*Kostka functions*, and the conjecture became known as the Macdonald positivity conjecture. It was proved in 2001 by Mark Haiman.

While attempting to prove the positivity conjecture, Garsia and Haiman conjectured the existence of a larger class of symmetric functions, satisfying certain properties and indexed by finite subsets of $\mathbb{N} \times \mathbb{N}$ (usually thought of as a collection of 1×1 squares in the first quadrant of the plane). Computer generated data strongly suggested the existence of these polynomials in general.

In 2003, Jim Haglund proposed a purely combinatorial description of the Macdonald polynomials. This description, the generating function for a particular pair of statistics, was soon proved correct by Haiman, Haglund and Nick Loehr.

In this dissertation, we show that the statistics of Haglund allow us to construct

the polynomials of Garsia and Haiman for a particular class of diagrams; namely, skew shapes with no column of height greater than two. The proof of this fact involves a new and careful analysis of these statistics.

1 Introduction and Basic Definitions

The theory of symmetric functions is ubiquitous throughout mathematics. They arise naturally in combinatorics, algebra, and geometry, and as a result have been studied intensively for many years. This classical area was revitalized in 1988, with Ian Macdonald's description of what are now known as the Macdonald polynomials. These are a two parameter basis for the space of symmetric functions, which specialize to many of the well-known one parameter and classical bases.

These polynomials, indexed by partitions, were first described as the unique symmetric functions satisfying certain triangularity and orthogonality conditions. With this definition, a fairly lengthy proof was required just to show their existence. Unfortunately, the proof gave very little insight into an explicit form for the polynomials. Nevertheless, Macdonald was able to compute many examples. In particular, after applying a simple transformation he obtained what he referred to as the "integral form" of these polynomials, denoted $J_\mu[X;q,t]$. He conjectured that when the $J_\mu[X;q,t]$ were expanded in terms of certain functions $s_\lambda[X(1-t)]$, the coefficients would always be polynomials in $\mathbb{N}[q,t]$, despite the fact that, *a priori*, these coefficients could only be said to be rational functions in q and t. These coefficients were called the *q,t-Kostka functions*, and the statement became known as the Macdonald positivity conjecture. (The $s_\lambda[X(1-t)]$ are a "plethystic transformation" of the classical Schur functions $s_\lambda(x_1, x_2, \cdots)$. A precise description will be given in section 1.2.6.)

Throughout the 1990s, Adriano Garsia and Mark Haiman approached the positivity conjecture by relating the q,t-Kostka functions to certain representations of the symmetric group. In particular, they described certain subspaces of the ring of polynomials in the two sets of variables, $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$. These subspaces are indexed by partitions, and called M_μ. The symmetric group acts on M_μ, and Garsia and Haiman conjectured that the bigraded Frobenius character of this representation was related to $J_\mu[X; q, t]$ in a precise way, which would immediately imply the positivity conjecture. They were later able to show that this would follow if it could only be shown that $\dim_{\mathbb{Q}}(M_\mu) = n!$ (where μ is a partition of n). This became known as the $n!$ conjecture.

While attempting to prove the $n!$ conjecture, Garsia and Haiman generated an entire framework of conjectures (implying, but not all implied by, the $n!$ conjecture) which became known as "science fiction". Considerations of the representation theory suggested these conjectures, all of which were supported by an enormous amount of computer generated data.

In 2001, following a geometric approach suggested by Claudio Procesi, Haiman proved what is now the $n!$ theorem. This finally established Macdonald's positivity conjecture, but many questions remained open. For example, the positivity result implies that there should be a purely combinatorial description of the q, t-Kostka polynomials; however, such a description remains unknown. Many of the "science fiction" conjectures also remain open.

An advance in the combinatorial direction was made in 2003 which Jim Haglund proposed a purely combinatorial description of the modified Macdonald polynomials. This was soon proved correct by Haiman, Haglund and Nick Loehr. This description does not give a combinatorial description of the q, t-Kostka polynomials, but it has provided an approach for studying this and other open combinatorial questions concerning Macdonald polynomials.

In particular, as part of the "science fiction" conjectures, Garsia and Haiman suggested that there should exist an extension of the modified Macdonald polynomials to a family of polynomials indexed not only by partitions, but by any finite collection of "cells" corresponding to points in the lattice of ordered pairs of natural numbers. They proposed a collection of axioms that such polynomials should satisfy, and were able to use these axioms to compute several examples. However their existence in general, remained a conjecture.

In this dissertation, we will see that the statistics of Haglund allow us to construct these polynomials for a particular class of diagrams; namely, skew shapes with no column of height greater than two. The proof of this fact involves a careful analysis of these statistics particularly in the case of rectangular or almost rectangular shapes.

We begin by defining many of the basic objects of study, along with giving some of their fundamental properties. For more information, see one of the standard texts on the subject, including [Ful91] [Mac], [Sta97], and [Sta99].

1.1 Basic Combinatorial Objects

1.1.1 Partitions

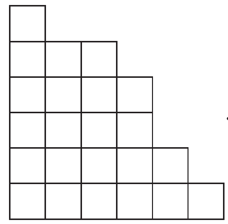
Definition 1. A *partition* λ of a positive integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers satisfying

1. $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i < k$ and
2. $\sum_{i=1}^{k} \lambda_i = n$.

We write $\lambda \vdash n$ to say λ is a partition of n. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of n, we say the *length* of λ is k (written $l(\lambda) = k$) and the *size* of λ is n (written $|\lambda| = n$). The numbers λ_i are referred to as the *parts* of λ. We write $Par(n)$ for the set of all partitions of n.

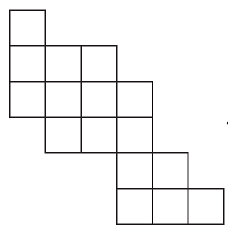
Definition 2. The *Young diagram* (also called a *Ferrers diagram*) of a partition λ is a collection of boxes (or *cells*), left justified and with λ_i cells in the i^{th} row from the bottom. The cells are indexed by pairs (i, j), with i being the row index (the bottom row is row 0), and j being the column index (the leftmost column is column 0).

Example 1. The Young diagram of the partition $\lambda = (6, 5, 4, 4, 3, 1)$ is



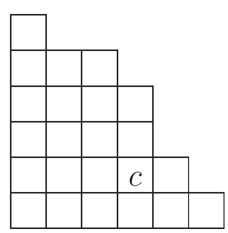
We say the partition μ is contained in λ, and write $\mu \subseteq \lambda$, when the Young diagram of λ contains the Young diagram of μ, *i.e.*, $\mu_i \leq \lambda_i$ for all i. When $\mu \subseteq \lambda$, we also have a Young diagram for the shape $\lambda \setminus \mu$, given by removing the cells in the diagram of μ from the diagram of λ. Such a diagram is called *skew*.

Example 2. The Young diagram for the skew shape $(6, 5, 4, 4, 3, 1) \setminus (3, 3, 1)$ is



We now define some statistics on the cells of a Young diagram. For a given cell $c \in \lambda$, the *arm* (respectively *leg*, *coarm*, *coleg*) of c is the number of cells in the diagram and strictly to the right (resp. above, to the left, below) c. It is denoted by $a(c)$ (resp. $l(c)$, $a'(c)$, $l'(c)$).

Example 3. In the following diagram, we have $a(c) = 1$, $l(c) = 2$, $a'(c) = 3$, $l'(c) = 1$.

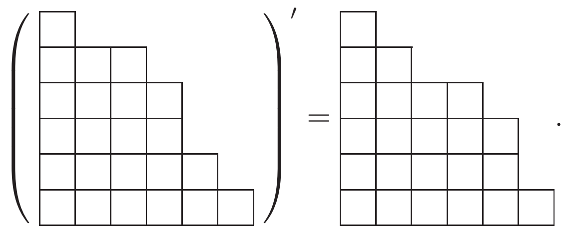


Another important statistic is defined here:

$$n(\lambda) = \sum_{c \in \lambda} l(c) = \sum_i (i-1)\lambda_i.$$

Definition 3. The *transpose* of a partition λ, indicated by λ', is given by reflecting the Young diagram of λ about the main diagonal, $i = j$.

Example 4. The transpose of the partition $(6, 5, 4, 4, 3, 1)$ is $(6, 5, 5, 4, 2, 1)$:



For fixed n, there is a total ordering on $Par(n)$, called *lexicographic ordering*. It is denoted by $<_L$, and defined by setting $\lambda <_L \mu$ if there exists a k such that

1. $\lambda_i = \mu_i$ for $i < k$ and

2. $\lambda_k < \mu_k$.

There is also a commonly used partial ordering on the set of partitions, called *dominance order*, and denoted by $<_D$, or just $<$ This is given by defining $\lambda <_L \mu$ if for all positive integers k:

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i.$$

It is easy to see that lexicographic order refines dominance order. We also have the following standard fact, the proof of which can be found in [Sta99].

Proposition 1. *Transposition reverses dominance order. In symbols,* $\lambda \leq_D \mu$ *if and only if* $\mu' \leq_D \lambda'$.

1.1.2 Tableaux

Definition 4. A *tableau* of shape $\lambda \vdash n$ is a function from the cells of the Young diagram of λ to the positive integers. The shape of T is denoted by $sh(T)$. The size of a tableau T, $|T|$, is the size of $sh(T)$. A tableau which is weakly increasing across rows and strictly increasing up columns is called *semi-standard*. A tableau which is strictly increasing across rows and up columns is called *standard*. The *weight* of a tableau is the vector

$$wt(T) = (|T^{-1}(1)|, |T^{-1}(2)|, \dots).$$

Analogous definitions apply for *skew* tableaux of shape $\lambda \setminus \mu$.

Throughout this document, for any vector of integers α, we use the notation

$$x^\alpha = \prod_{i \geq 1} x_i^{\alpha_i}$$

For T a tableau, we use the shorthand notation x^T for $x^{wt(T)}$.

We often depict a tableau T by a diagram in which the (i, j) cell contains the number $T(i, j)$.

Example 5. Let

$$T = \begin{array}{|c|c|c}\hline 3 & 5 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}.$$

This T is a semi-standard tableau with

1. $|T| = 5$,
2. $sh(T) = (3, 2)$,
3. $wt(T) = (2, 0, 1, 1, 1, 0, 0, \dots)$ and
4. $x^T = x_1^2 x_3 x_4 x_5$.

1.1.3 Compositions

Definition 5. A *composition* of a number n is a vector of positive integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $\sum_{i=1}^k \alpha_i = n$. We write $\alpha \models n$ to say α is a composition of n. We say the length of α is k (written $l(\alpha) = k$) and the size of α is n (written $|\alpha| = n$). The numbers α_i are called the *parts* of α. We write $Comp(n)$ for the set of compositions of n.

There is a well-known bijection between compositions of n and subsets of the set $\{1, \dots, n-1\}$. This bijection is given by the following two functions:

- For $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, the subset associated to α is given by

$$\mathbb{S}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}$$

- For $S = \{s_1, \dots, s_k\} \subseteq \{1, \dots, n-1\}$, with $s_1 < s_2 < \cdots < s_k$, the composition of n associated to S is

$$\mathrm{co}_n(S) = \{s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_k\}$$

It is not hard to see that these functions are indeed inverses. This shows that $|Comp(n)| = 2^{n-1}$.

1.1.4 Words

Definition 6. A *word* of length n is a function from $\{1, \dots, n\}$ to the positive integers. The *weight* or *content* of a word w is the vector

$$wt(w) = ct(w) = \left\{|w^{-1}(1)|, |w^{-1}(2)|, \dots, \right\}.$$

We will think of words as vectors

$$w = (w(1), w(2), \dots) = (w_1, w_2, \dots)$$

and when we can do so without ambiguity, we write the word $w = (w_1, w_2, \dots, w_n)$ as simply $w_1 w_2 \dots w_n$. A word with weight $(1, 1, \dots, 1)$ is called a *permutation*.

Definition 7. The set of *rearrangements* of a word w is denoted $\mathcal{R}(w)$ and is defined by

$$\mathcal{R}(w) = \{v : ct(v) = ct(w)\}$$

Example 6. The rearrangements of the word $(1, 1, 3, 3)$ are given by

$$\mathcal{R}(1,1,3,3) = \{(1,1,3,3), (1,3,1,3), (1,3,3,1), (3,1,1,3), (3,1,3,1), (3,3,1,1)\}$$

We associate a permutation to every word through a map called standardization.

Definition 8. A word w with content $\{m_1, \ldots, m_k\}$ has *standardization* ($std(\sigma)$) given by replacing the 1s from left to right with $1, \ldots, m_1$, replacing the 2s with $m_1 + 1, \ldots, m_1 + m_2$, etc.

Example 7. The standardization of $(1, 5, 3, 3, 2, 4, 1, 2, 1)$ is given by

$$std(153324121) = 196748253.$$

Definition 9. Given a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, the *descent set* $d(\sigma)$ is given by

$$d(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$$

For example, we have $d(53412) = \{1, 3\}$. Note that in general we have $d(\sigma) \subseteq \{1, \ldots, n-1\}$. For σ a permutation of length n, we will write $\mathrm{co}(\sigma)$ for $\mathrm{co}_n(d(\sigma))$.

The following proposition characterizes the set of permutations which are the image under standardization of the set of words with a given content.

Proposition 2. *For a fixed composition $\alpha \models n-1$ we have*

$$\{std(w) : ct(w) = \alpha\} = \{\sigma \in S_n : d(\sigma^{-1}) \subseteq \mathbb{S}(\alpha)\}$$

Proof. Given a word w of content α, the standardization $std(w)$ will have $i+1$ to the left of i only if i and $i+1$ came from different letters in w. This can only happen if $i \in \mathbb{S}(\alpha)$. The following example may make this more clear:

$$\begin{aligned} \alpha &= (3, 1, 2) \\ \mathbb{S}(\alpha) &= \{3, 4\} \\ w &= 313121 \\ std(w) &= 516243 \end{aligned}$$

So we have

$$\{std(w) : ct(w) = \alpha\} = \{\sigma \in S_n : \ i \text{ is left of } i+1 \text{ only if } i \in \mathbb{S}(\alpha)\}$$

Also, a descent occurs at position i in σ^{-1} if and only if $i+1$ is to the left of i in σ. Again, we consider our simple example:

$$\sigma = 516243$$
$$\sigma^{-1} = 246513$$
$$d(\sigma^{-1}) = \{3, 4\}$$

Thus

$$\{\sigma \in S_n : d(\sigma^{-1}) \subseteq \mathbb{S}(\alpha)\} = \{\sigma \in S_n : \ i \text{ is left of } i+1 \text{ only if } i \in \mathbb{S}(\alpha)\}$$

which completes the proof. □

1.2 Symmetric Functions

1.2.1 Formal Power Series

We begin this section by saying a bit about the rings in which we work. We let X be the infinite set of variables $\{x_1, x_2, \cdots\}$ The "base" ring for most calculations will be the ring $\mathbb{Q}[\![X]\!]$ of formal power series in infinitely many variables. We use common shorthand to refer to elements of this ring; for example

$$\frac{1}{1-x_i} = \sum_{j\geq 0} x_i^j = (1-x_i)^{-1}$$
$$exp(x_i) = \sum_{j\geq 0} \frac{x_i^j}{j!}$$

Many of the following definitions use the ring $\mathbb{Q}[\![X]\!][\![t]\!]$. In this ring, we denote by $f\Big|_t$ the coefficient of t in f, which in general is an element of $\mathbb{Q}[\![X]\!]$.

A *symmetric function* in the variables X is an element of $\mathbb{Q}[\![X]\!]$ which is invariant under permutation of subscripts. We $\mathbb{Q}$ as our coefficient field; however, any field of characteristic 0 would do just as well. The symmetric functions form a subring $\Lambda \subset \mathbb{Q}[\![X]\!]$. We now define several important elements of Λ.

1. For $i > 0$ we define

$$e_i = \prod_{j>0}(1+tx_j)\Big|_{t^i}$$

For λ a partition of length k, we set

$$e_\lambda := \prod_{i=1}^{k} e_{\lambda_i}.$$

These functions are known as the *elementary* symmetric functions.

2. For $i > 0$ we define

$$h_i = \prod_{j>0} \frac{1}{1 - tx_j}\Big|_{t^i}$$

For λ a partition of length k, we set

$$h_\lambda := \prod_{i=1}^{k} h_{\lambda_i}.$$

These functions are known as the *complete homogeneous* symmetric functions or, less precisely but more conventially, as the *homogeneous* symmetric functions.

3. For $i > 0$ we define

$$p_i = \sum_{j>0} {x_j}^i$$

For λ a partition of length k, we set

$$p_\lambda := \prod_{i=1}^{k} p_{\lambda_i}.$$

These functions are known as the *power* symmetric functions.

4. For λ any partition, we think of λ as an infinite length vector by appending 0s, so $(\lambda_1, \cdots, \lambda_k)$ becomes $(\lambda_1, \cdots, \lambda_k, 0, 0, \cdots)$. We then define

$$m_\lambda = \sum_{\alpha \in \mathcal{R}(\lambda)} x^\alpha$$

These functions are known as the *monomial* symmetric functions.

5. For any partition λ we define

$$s_\lambda = \sum_T x^T$$

where the sum is over all semi-standard tableaux of shape λ. We extend this definition to all skew shapes $\lambda \setminus \mu$. These are known as the *Schur* symmetric functions.

The importance of these examples can be seen in the following proposition.

Proposition 3. *The collections* $\{e_\lambda\}_{\lambda \vdash n}$, $\{h_\lambda\}_{\lambda \vdash n}$, $\{p_\lambda\}_{\lambda \vdash n}$, $\{m_\lambda\}_{\lambda \vdash n}$, $\{s_\lambda\}_{\lambda \vdash n}$ *are all bases over* $\mathbb{Q}$ *for the vector space of homogeneous symmetric functions of degree* n.

Proof. See [Mac] or [Sta99]. □

Note that this proposition gives that the collection $\{e_i\}_{i \geq 0}$ is algebraically independent (and similarly for the h_i and p_i).

1.2.2 The Hall Inner Product

The algebra Λ has a useful inner product, known as the *Hall* inner product, defined by making the Schur functions orthonormal. Precisely, we set

$$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

and extend linearly.

We now state some general facts about inner product spaces of formal power series. Two bases $\{f_\lambda\}_\lambda$ and $\{g_\lambda\}_\lambda$ of such a space are called *dual* if they have the property that for all power series $P(x)$

$$P(x) = \sum_\lambda \langle P(y), f_\lambda(y) \rangle . g_\lambda(x)$$

Of course, by linearity we then have

$$P(x) = \langle P(y), \sum_\lambda f_\lambda(y) g_\lambda(x) \rangle .$$

The quantity $\sum_\lambda f_\lambda(y)g_\lambda(x)$ is called the *reproducing kernel* of the inner product space.

Proposition 4. *Two bases $\{f_\lambda\}_\lambda$ and $\{g_\lambda\}_\lambda$ are dual if and only $\sum_\lambda f_\lambda(y)g_\lambda(x)$ is the reproducing kernel.*

Proof. From the comments above, we must only verify that the reproducing kernel is independent of the choice of dual bases. This calculation is straightforward. □

Remark 1. Of course, we are only using polynomiality here to apply the labels x and y, and we are only using multiplication in a purely formal sense. For an arbitrary inner product space V, we can think of the reproducing kernel as an element of $V \otimes V$, but the above notation will be more convenient for our purposes.

Using the Hall inner product, we have the following relations among the bases of Λ.

Proposition 5. *1. The basis $\{s_\lambda\}$ is self-dual.*

2. The reproducing kernel is

$$K(x,y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

This is known as the Cauchy *kernel.*

3. The bases $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual.

4. The basis $\left\{\frac{p_\lambda}{\sqrt{z_\lambda}}\right\}$ is self-dual. We define z_λ by setting m_i to be the number of times i occurs as a part of λ, and

$$z_\lambda = \prod_{i=1}^{|\lambda|} i^{m_i}(m_i)!$$

5. The adjoint operation to multiplication by s_λ is skewing by s_λ. That is

$$\langle s_\lambda s_\mu, s_\nu \rangle = \langle s_\mu, s_{\nu \backslash \lambda} \rangle$$

for all λ, μ, ν with $|\lambda| + |\mu| = |\nu|$.

6. The adjoint of multiplication by p_1 is differentiation by p_1.

Proof. See [Mac]. □

1.2.3 The Involution ω

We define the function ω on Λ by setting

$$\omega(e_i) = h_i$$

and extending algebraically. This function has the following properties.

Proposition 6. *1. $\omega(h_i) = e_i$ (so ω is an involution).*

2. $\omega(s_\lambda) = s_{\lambda'}$

3. $\omega(p_k) = (-1)^{k-1} p_k$.

Proof. See [Mac]. □

1.2.4 Symmetric Function Relations

Some details about transitions between the various bases are given here. For a more complete account, see, *e.g.*, [Mac], or [Sta99].

We define the following generating functions for the elementary, homogeneous and power bases:

$$H(t) = \sum_{j \geq 0} h_j t^j$$

$$E(t) = \sum_{j \geq 0} e_j t^j$$

$$P(t) = \sum_{j \geq 1} \frac{p_j}{j}$$

Proposition 7. *We have the following relationships among these generating functions:*

$$H(t) = \prod_i \frac{1}{1 - tx_i}$$

$$E(t) = \prod_i (1 + tx_i)$$

$$H(t)E(-t) = 1$$

$$H(t) = exp\left(P(t)\right)$$

Proof. These follow immediately from the definitions and the identity

$$\frac{1}{1-x} = exp\left(\sum_{i\geq 1} \frac{x^i}{i}\right) \qquad \square$$

Definition 10. The number of semi-standard tableaux of shape λ and weight μ is called the *Kostka number* indexed by λ and μ. It is written $K_{\lambda,\mu}$. By definition, this is the coefficient of m_μ in s_λ.

We sometimes write relationships between bases in matrix notation. For example, we order the partitions lexicographically, and let $\mathbf{K}$ be the $|Par(n)| \times |Par(n)|$ matrix with $\mathbf{K}(\lambda, \mu) = K_{\lambda,\mu}$. We set, for b a basis of the symmetric functions, $\langle \mathbf{b} \rangle$ to be the row vector with entries b_λ in lexicographic order. This allows us to write

$$\langle \mathbf{s} \rangle^{\mathbf{t}} = \mathbf{K} \langle \mathbf{m} \rangle^{\mathbf{t}}$$

We also have

$$\langle \mathbf{h} \rangle^{\mathbf{t}} = \mathbf{K}^{\mathbf{t}} \langle \mathbf{s} \rangle^{\mathbf{t}} \text{ or, more simply,}$$
$$\langle \mathbf{h} \rangle = \langle \mathbf{s} \rangle \mathbf{K}$$

The following rule, for multiplying Schur functions by elementary functions is known as the *Pieri* rule.

Proposition 8. *We have*

$$e_i s_\lambda = \sum_\mu s_\mu$$

where the sum is over all partitions μ *of size* $|\lambda| + i$ *which contain* λ *and with* $\mu \setminus \lambda$ *having no two cells in the same row. Similarly,*

$$h_i s_\lambda = \sum_\mu s_\mu$$

where here the sum is over all partitions μ *of size* $|\lambda| + i$ *which contain* λ *and with* $\mu \setminus \lambda$ *having no two cells in the same column.*

Proof. See [Mac]. $\square$

1.2.5 Quasisymmetric functions

There is an important algebra which contains the symmetric functions and is contained in the polynomial ring. This is the algebra of *quasisymmetric* functions, first introduced by Gessel.

Definition 11. Let $\alpha = (\alpha_1, \alpha_2, \cdots)$ be a vector of nonnegative integers, with only a finite number of nonzero entries. We say another such vector α' is a *weak rearrangement* of α if the non-zero entries are identical to those of α and in the same relative order. We denote by $W(\alpha)$ the set of *weak rearrangements* of α.

Example 8.

$$W(2,3,0,0,\cdots) = \{(2,0,3,0,\cdots),(0,2,3,0,\cdots),(2,0,0,3,0,\cdots),\\ (0,2,0,3,0,\cdots),(0,0,2,3,0,\cdots),\cdots\}$$

Definition 12. A function f is *quasisymmetric* if for any composition α,

$$f|_{x^\alpha} = f_{x^{\alpha'}} \quad \text{for all} \quad \alpha' \in W(\alpha).$$

It is not hard to verify that sums and products of quasisymmetric functions are quasisymmetric and to see that all symmetric functions are also quasisymmetric. The quasisymmetric functions have two important bases.

Definition 13. Given a composition α of length n, the *monomial quasisymmetric function* indexed by α is given by thinking of α as an infinite length vector by appending 0s, and then setting

$$M_\alpha = \sum_{\beta \in W(\alpha)} x^\beta$$

It is clear that these form a basis for the quasisymmetric functions.

Definition 14. The *fundamental quasisymmetric function* indexed by α, a composition of n, is given by

$$Q_\alpha = \sum_S M_{\mathrm{co}_n(S)}$$

where the sum is over all $S \subseteq \{1, \ldots, n-1\}$ which contain $\mathbb{S}(\alpha)$.

These are triangular with respect to the monomial basis, and thus also form basis.

1.2.6 Plethysm

The operation of *plethysm* is one which generalizes symmetric functions, so that we may apply them to any formal power series, not just a set of variables. The precise definition follows.

Definition 15. For $E = E(t_1, t_2, \dots)$ a formal power series, and $f = Q(p_1, p_2, \dots)$ a polynomial in the power symmetric functions, we set

$$f[E] = Q(p_1, p_2, \dots)\Big|_{p_k \to E(t_1^k, t_2^k, \dots)}$$

Example 9. For $X = x_1 + x_2 + x_3 + \dots$ and $f \in \Lambda$, we have

$$f[X] = f(x_1, x_2, x_3, \dots)$$

For this reason we abuse notation and use the same symbol (*e.g.,* X) for an alphabet ($\{x_1, x_2, \dots\}$) and for the formal sum of all variables in that alphabet ($x_1 + x_2 + \dots$).

We now establish the most important properties of plethysm.

Proposition 9. *For any two formal power series* A, B *and any partition* λ*, we have*

$$p_\lambda[AB] = p_\lambda[A]p_\lambda[B]$$

Proof. If A and B are power series in different sets of variables, we can consider them to be formal power series in the combined set of variables $\{t_1, t_2, t_3, \dots\}$. Thus we can write $A = \sum_\alpha a_\alpha t^\alpha$ and $B = \sum_\beta b_\beta t^\beta$. The proof is now a straightforward calculation:

$$\begin{aligned}
p_\lambda[AB] &= \prod_i p_{\lambda_i}[AB] \\
&= \prod_i p_{\lambda_i}\left[\sum_\gamma \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta t^\gamma\right] \\
&= \prod_i \left(\sum_\gamma \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta t^{\lambda_i \gamma}\right) \\
&= \prod_i \left(\sum_\alpha a_\alpha t^{\lambda_i \alpha}\right)\left(\sum_\beta b_\beta t^{\lambda_i \beta}\right) \\
&= \prod_i p_{\lambda_i}[A] p_{\lambda_i}[B] \\
&= p_\lambda[A] p_\lambda[B]
\end{aligned}$$

□

It is worth noting that this property does not extend to symmetric functions in general. It is easy to verify, for example, that $h_2[XY] \neq h_2[X]h_2[Y]$.

A very common use of plethystic substitution is to place inside the brackets an alphabet X multiplied by some invertible formal power series in the variables q and t. The following proposition shows that this operation is invertible.

Proposition 10. *Let $X = x_1 + x_2 + \dots$ as usual, and let E be an invertible formal power series in some set of variables $\{t_1, t_2, \dots\}$ disjoint from X. Let $f \in \Lambda$, and set $g = f[XE]$, considered as a symmetric function in the variables X with coefficients in the ring $\mathbb{Q}[\![t_1, t_2, \dots]\!]$. Then $g[XE^{-1}] = f[X]$.*

Proof. We write $f = \sum_\lambda f_\lambda p_\lambda$. We have

$$\begin{aligned}
f[XE] &= \sum_\lambda f_\lambda p_\lambda[XE] \\
&= \sum_\lambda f_\lambda p_\lambda[E] p_\lambda[X]
\end{aligned}$$

so we can write $g = \sum_\lambda g_\lambda p_\lambda$ where

$$g_\lambda = f_\lambda p_\lambda[E]$$

Thus we have

$$g[XE^{-1}] = \sum_\lambda f_\lambda p_\lambda[E] p_\lambda[XE^{-1}]$$
$$= \sum_\lambda f_\lambda p_\lambda[1] p_\lambda[X]$$
$$= f[X]$$

since $p_\lambda[1] = 1$ for any λ. □

It is important to note that if we specialize the variables involved in a plethystic expression, we must generally specify whether this specialization is to be done before or after the plethystic operation. For example,

$$p_2 \left[t \Big|_{t \to -x} \right] = p_2[(-1)x] = (-1)x^2$$

while

$$p_2[t]\Big|_{t \to -x} = t^2\Big|_{t \to -x} = x^2$$

There are important exceptions to this: if the specialization we are considering (say to the variable x) commutes with the homomorphism $x \mapsto x^k$ for all k, then it does not matter whether we do so before or after evaluating the plethystic bracket. In particular, the specializations $x \mapsto 0$, $x \mapsto 1$, and $x \mapsto x^{-1}$ can be done inside or outside the plethystic brackets with equivalent results.

However, the operation of specializing a variable to its negative, as we saw above, does not commute with taking powers. Performing this specialization *after* a plethystic evaluation is common enough that we have a special notation for it. We write $f[^-x]$ for $f[x]\Big|_{x \to -x}$.

Proposition 11. *We have* $f[-^-X] = (\omega f)[X]$.

Proof. Since ω is a algebra homomorphism, it is enough to check this for the p_k.

$$p_k[-^-X] = p_k[-X]\Big|_{X \to -X}$$
$$= -p_k(-x_1, -x_2, \dots)$$
$$= (-1)^{k-1} p_k[X]$$
$$= (\omega p_k)[X]$$

□

Example 10. For any formal power series E, any variable u, and any partition λ, we have

$$\begin{aligned} s_\lambda[-tX] &= \omega s_\lambda[{}^{-}tX] \\ &= (-t)^{|\lambda|}\omega s_\lambda[X] \\ &= (-t)^{|\lambda|} s_{\lambda'}[X] \end{aligned}$$

We give one more formula which will be useful for us.

Proposition 12. *For any partition λ and the infinite alphabets $X = x_1 + x_2 + \ldots$ and $Y = y_1 + y_2 + \ldots$ we have*

$$s_\lambda[X+Y] = \sum_{\mu \subseteq \lambda} s_\mu[X] s_{\lambda\backslash\mu}[Y]$$

Proof. We think of the alphabet Y as being strictly greater than the alphabet X, and the result follows from the combinatorial definition of the Schur functions. □

1.3 Basic S_n Representation Theory

A *representation* of a group G is a homomorphism $G \to GL(V)$ for some finite dimensional vector space V. To avoid explicitly naming this homomorphism, we will abuse notation and refer to V as the representation and to the image of an element $g \in G$ as $g\Big|_V$.

A *subrepresentation* of a representation V is a subspace $V \subseteq W$ which is invariant under the action of G. A representation V is called *irreducible* if the only subrepresentations of V are $\{0\}$ and V.

It is well known that every finite-dimensional representation of a finite group is isomorphic to the direct sum of a finite number of irreducible representations. Furthermore, the number of these irreducible representations is the number of conjugacy classes of the group.

We will be concerned here with representations of the symmetric group, S_n. The irreducible representations of S_n are indexed by partitions; there is a well-known, conventional construction of the irreducible representation V^λ for any $\lambda \vdash n$.

The *character* of a representation of a finite group G is a map $G \to \mathbb{C}$ defined by $g \mapsto tr(g\Big|_V)$. The fact which forms the basis for the study of representation theory is the following.

Proposition 13. *Every representation of a finite group is determined (up to isomorphism) by its character.*

The characters of S_n form a basis for the center of the group algebra of S_n. There is an isomorphism between the center of the group algebra of S_n, and the degree n symmetric functions. This map, $\mathcal{F}$ sends $\chi(V^\lambda) \mapsto s_\lambda$. The map $\mathcal{F}$ is called the *Frobenius map* and the image of a character of some representation V under the Frobenius map is known as the *Frobenius character* of V.

The representations we will be concerned with will be contained in the ring $R_n = \mathbb{Q}[x_1, x_2, \cdots, x_n]$. They will not, in general, be finite dimensional, but they will decompose as the direct sum of finite dimensional subrepresentations. These subrepresentations will consist of the homogeneous polynomials of fixed degree.

Precisely, we write $\mathcal{H}_k[R_n]$ for the set of homogeneous polynomials of degree k in R_n. We call a representation $V \subseteq R_n$ of S_n *homogeneous* if for each k, we have $\mathcal{H}_k[V]$ (defined by $V \cap \mathcal{H}_k[R_n]$) a subrepresentation of V. In this case, we define the *Hilbert series* of V by

$$Hilb(V) = \sum_{k \geq 0} (\dim_{\mathbb{Q}} \mathcal{H}_k[V]) q^k.$$

Similarly, we define the *graded Frobenius character* of V by

$$\mathcal{F}_g(V) = \sum_{k \geq 0} \mathcal{F}(\chi(\mathcal{H}_k[V])).$$

2 Macdonald Polynomials

The Macdonald polynomials are a family of symmetric polynomials with coefficients which are, *a priori*, in the field $\mathbb{Q}(q,t)$. The goals of this chapter are to define these polynomials and some well-known modifications, and discuss some of their properties and implications.

Theorem 1 (Macdonald)**.** *There exists a unique family of symmetric polynomials indexed by partitions,* $\{P_\lambda[X;q,t]\}$ *such that*

1. $P_\lambda = s_\lambda + \sum_{\mu<\lambda} \xi_{\mu,\lambda}(q,t) s_\mu$
2. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ *if* $\lambda \neq \mu$.

where

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \begin{cases} z_\lambda p_\lambda\left[\frac{1-t}{1-q}\right] & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

and $\xi_{\mu,\lambda}(q,t) \in \mathbb{Q}(q,t)$.

Proof. See [Mac]. □

Note that the Macdonald polynomials are triangularly related to the Schur basis, and thus also form a basis for the symmetric functions. We begin our discussion of this basis by giving some comparisons with the classical theory of symmetric functions. As the inner product is central to this definition, we begin by establishing some some results concerning it. We first notice that if we set

$$z_\lambda(q,t) = z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

we can write the inner product as

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \begin{cases} z_\lambda(q,t) & \text{if } \lambda = \mu \\ 0 & \text{ortherwise} \end{cases}$$

This is compared with the Hall inner product which can be given by

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} z_\lambda & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

We now explore the reproducing kernel with respect to the inner product $\langle \cdot, \cdot \rangle_{q,t}$. Plethystic notation provides a convenient way to compare this with the Cauchy kernel. We begin by defining the function

$$\begin{aligned} \Omega[X] &:= exp\left(\sum_{k\geq 1} \frac{p_k[X]}{k}\right) \\ &= \prod_i exp\left(\sum_{k\geq 1} \frac{x_i^k}{k}\right) \\ &= \prod_i \frac{1}{1-x_i} \\ &= \sum_\lambda \frac{p_\lambda[X]}{z_\lambda} \end{aligned}$$

A plethystic substitution gives

$$\begin{aligned} \Omega[XY] &= exp\left(\sum_{k\geq 1} \frac{p_k[X]p_k[Y]}{k}\right) \\ &= \prod_{i,j} \frac{1}{1-x_iy_j} \end{aligned}$$

which, recall, is the Cauchy kernel; the reproducing kernel for the Hall inner product. We also have

$$\begin{aligned} \Omega[X(1-t)] &= exp\left(\sum_{k\geq 1} \frac{1}{k}\left(\sum_i x_i^k - \sum_i (tx_i)^k\right)\right) \\ &= \prod_i \frac{1-tx_i}{1-x_i} \end{aligned}$$

and similarly,

$$\Omega\left[XY\frac{1-t}{1-q}\right] = exp\left(\sum_{k\geq 1}\frac{p_k[X]p_k[Y]}{k}\frac{1-t^k}{1-q^k}\right)$$
$$= \prod_{s\geq 0}\prod_i\prod_j\frac{1-tx_iy_jq^s}{1-x_iy_jq^s}$$

We call this last quantity *Macdonald's kernel* and denote it by $\Omega_{q,t}[XY]$.

It is now easy to see that the reproducing kernel for the inner product $\langle\cdot,\cdot\rangle_{q,t}$ is given by

$$\sum_\lambda \frac{p_\lambda[X]}{z_\lambda p_\lambda[\frac{1-q}{1-t}]}p_\lambda[Y] = \sum_\lambda \frac{p_\lambda\left[XY\frac{1-t}{1-q}\right]}{z_\lambda}$$
$$= \Omega_{q,t}\left[XY\right]$$

Macdonald noticed that the Macdonald polynomials are invariant under inverting q and t, as made precise in the following proposition.

Proposition 14.

$$P_\lambda[X;q,t] = P_\lambda\left[X;q^{-1},t^{-1}\right]$$

Proof. We first consider the scalar product we get by inverting q and t.

$$\langle p_\lambda,p_\mu\rangle_{q^{-1},t^{-1}} = \chi(\lambda=\mu)z_\lambda\prod_i\frac{1-q^{-\lambda_i}}{1-t^{-\lambda_i}}$$
$$= \chi(\lambda=\mu)z_\lambda\prod_i\frac{t^{\lambda_i}}{q^{\lambda_i}}\prod_i\frac{q^{\lambda_i}-1}{t^{\lambda_i}-1}$$
$$= \chi(\lambda=\mu)z_\lambda\left(\frac{t}{q}\right)^n p_\lambda\left[\frac{1-q}{1-t}\right]$$
$$= \left(\frac{t}{q}\right)^n\langle p_\lambda,p_\mu\rangle_{q,t}$$

Setting $t\to t^{-1}$ and $q\to q^{-1}$ in property (1) of Macdonald's polynomials, gives

$$P_\lambda\left[X;q^{-1},t^{-1}\right] = s_\lambda + \sum_{\mu<\lambda}s_\mu\xi_{\mu,\lambda}(q^{-1},t^{-1})$$

so the $P_\lambda\left[X;q^{-1},t^{-1}\right]$ also satisfy (1).

Similarly, setting $t \to t^{-1}$ and $q \to q^{-1}$ in property (2) gives, for $\lambda \neq \mu$,

$$\begin{aligned} 0 &= \langle P_\lambda \left[X; q^{-1}, t^{-1}\right], P_\mu \left[X; q^{-1}, t^{-1}\right] \rangle_{q^{-1}, t^{-1}} \\ &= \left(\frac{t}{q}\right)^n \langle P_\lambda \left[X; q^{-1}, t^{-1}\right], P_\mu \left[X; q^{-1}, t^{-1}\right] \rangle_{q,t} \end{aligned}$$

so the $P_\lambda \left[X; q^{-1}, t^{-1}\right]$ satisfy property (2) as well. The uniqueness of the P_λ gives the desired result. □

Macdonald also computes the following specializations of the $P_\lambda[X; q, t]$, which we do not prove here.

Proposition 15. *We have the following specializations:*

$$\begin{aligned} P_\lambda[X; t, t] &= s_\lambda[X] \\ P_\lambda[X; q, 1] &= m_\lambda[X] \\ P_\lambda[X; 1, t] &= e_{\lambda'}[X] \\ P_{(1^n)}[X; q, t] &= e_n[X] \end{aligned}$$

Proof. See [Mac]. □

Much of the interest in Macdonald polynomials has not been in the functions $P_\lambda[X; q, t]$ themselves, but rather in certain modifications of them. In this section, we give these modifications, and describe how some of the properties of the P_λ apply to them.

In order to simplify the notation for these modifications, we use the following common abbreviations.

$$\begin{aligned} h_\lambda(q, t) &= \prod_{c \in \lambda} (1 - q^{a(c)} t^{l(c)+1}) \\ h'_\lambda(q, t) &= \prod_{c \in \lambda} (1 - t^{l(c)} q^{a(c)+1}) \\ d_\lambda(q, t) &= \frac{h_\lambda(q, t)}{h'_\lambda(q, t)} \end{aligned}$$

(Recall the definitions of *arm* and *leg* given earlier.)

By definition, the $P_\lambda[X; q, t]$ are orthogonal; however, they are not orthonormal. Macdonald computed the inner product of P_λ with itself.

Proposition 16 (Macdonald)**.** *With respect to the Macdonald inner product, we have*

$$\langle P_\lambda[X;q,t], P_\lambda[X;q,t]\rangle_{q,t} = \frac{1}{d_\lambda(q,t)}$$

Proof. See [Mac]. □

This fact allows us to define the dual basis to the $P_\lambda[X;q,t]$:

$$Q_\lambda[X;q,t] = \frac{P_\lambda[X;q,t]}{d_\lambda(q,t)}$$

We now introduce an important q,t analog of the ω involution.

Definition 16. We define the homomorphism $\omega_{q,t}$ on symmetric functions by

$$\omega_{q,t} f[X] = (\omega f)\left[X\frac{1-q}{1-t}\right]$$

Note that $\omega_{q,t}$ is not an involution, but that $(\omega_{q,t})^{-1} = \omega_{t,q}$. The action of $\omega_{q,t}$ on the $P_\lambda[X;q,t]$ is very nice.

Proposition 17.

$$\begin{aligned}\omega_{q,t} P_\lambda[X;q,t] &= Q_{\lambda'}[X;t,q] \\ &= \frac{1}{d_{\lambda'}(q,t)} P_{\lambda'}[X;t,q]\end{aligned}$$

Proof. See [Mac]. □

We are now in a position to define what Macdonald refers to as the "integral form" of these polynomials. These are modifications of the $P_\lambda[X;q,t]$ which have "nice" expansions in terms of modified Schur functions. Precisely, we begin with the following definition:

$$\begin{aligned}J_\mu[X;q,t] &= h_\mu P_\mu[X;q,t] \\ &= h'_\mu Q_\mu[X;q.t]\end{aligned}$$

We then define the q,t-*Kostka* functions by means of the following expansion:

$$J_\mu[X;q,t] := \sum_{\lambda \vdash |\mu|} K_{\lambda,\mu}(q,t) s_\lambda[X(1-t)]$$

In fact, we will find it more convenient to work with the functions whose ordinary Schur function coefficients are given by the $K_{\lambda,\mu}(q,t)$. To that end we define

$$\begin{aligned} H_\mu[X;q,t] &= J_\mu\left[\frac{X}{1-t};q,t\right] \\ &= \sum_{\lambda \vdash |\mu|} K_{\lambda,\mu}(q,t) s_\lambda[X] \end{aligned}$$

For representation theoretical reasons to be discussed later, we define one final modification:

$$\begin{aligned} \widetilde{H}_\mu[X;q,t] &= t^{n(\mu)} H_\mu[X;q,t^{-1}] \\ &= \sum_{\lambda \vdash |\mu|} \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda[X] \end{aligned}$$

where the functions

$$\widetilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,t^{-1})$$

are the called the *modified* q,t*-Kostka functions.*

Macdonald introduced the $J_\lambda[X;q,t]$ and the q,t-Kostka functions in 1988; he conjectured at that time that the q,t-Kostka functions were polynomials in $\mathbb{N}[q,t]$. After much work, this was finally proved in 2001 by Mark Haiman.

Theorem 2 (Haiman)**.** *The* $K_{\lambda,\mu}(q,t)$ *are polynomials in* q *and* t *with non-negative integer coefficients.*

Proof. See [Hai01]. □

In fact, Haiman proved a much stronger result, about which more will be said in a later section.

2.1 Modified Macdonald Polynomials

From a representation theoretical standpoint, the most important "flavor" of the Macdonald polynomials is the $\widetilde{H}_\mu$. Correspondingly, this section is devoted to establishing the significant properties of this modification. The ultimate goal is to establish conditions describing the $\widetilde{H}_\mu$ without reference to the P_μ.

Before proceeding with this, we first establish that the $\widetilde{H}[X;q,t]$ satisfy a kind of "transpose symmetry". Precisely, we have $\widetilde{H}_\mu[X;q,t] = \widetilde{H}_{\mu'}[X;t,q]$. We show this by finding two separate relations among the q,t-Kostka polynomials and combining them.

Proposition 18. *The q,t-Kostka polynomials satisfy*

$$K_{\lambda,\mu}(q,t) = q^{n(\mu'}t^{n(\mu)}K_{\lambda',\mu}\left(q^{-1},t^{-1}\right).$$

Proof. We start with the definition of the $J_\mu[X;q,t]$ and use Proposition 14 to obtain

$$\begin{aligned} J_\mu\left[X;q^{-1},t^{-1}\right] &= h_\mu\left(q^{-1},t^{-1}\right)P_\mu\left[X;q^{-1},t^{-1}\right] \\ &= \prod_{c\in\mu}\left(1-q^{-a(c)}t^{-l(c)-1}\right)P_\mu[X;q,t] \end{aligned}$$

Since the sum of the $a(c)$ is $n(\mu')$ and the sum of the $l(c)$ is $n(\mu)$, we can make the exponents positive by pulling out a factor:

$$\begin{aligned} J_\mu\left[X;q^{-1},t^{-1}\right] &= (-1)^{|\mu|}q^{-n(\mu')}t^{-n(\mu)-|\mu|}h_\mu(q,t)P_\mu[X;q,t] \\ &= (-1)^{|\mu|}q^{-n(\mu')}t^{-n(\mu)-|\mu|}J_\mu[X;q,t] \\ &= \sum_{\lambda\vdash|\mu|}(-1)^{|\mu|}q^{-n(\mu')}t^{-n(\mu)-|\mu|}K_{\lambda,\mu}(q,t)s_\lambda[X(1-t)] \qquad (2.1) \end{aligned}$$

Alternatively, the definition of the q,t-Kostka polynomials gives

$$J_\mu\left[X;q^{-1},t^{-1}\right] = \sum_{\lambda\vdash|\mu|}K_{\lambda,\mu}\left(q^{-1},t^{-1}\right)s_\lambda\left[X\left(1-\frac{1}{t}\right)\right]$$

Now it follows from Example 10 that

$$s_\lambda\left[X\left(1-\frac{1}{t}\right)\right] = (-t)^{-|\lambda|}s_{\lambda'}\left[X\left(1-t\right)\right]$$

which gives

$$J_\mu\left[X;q^{-1},t^{-1}\right]=\sum_{\lambda\vdash|\mu|}(-t)^{-|\mu|}K_{\lambda',\mu}\left(q^{-1},t^{-1}\right)s_\lambda[X(1-t)].$$

Equating coefficients with 2.1 and simplifying gives the desired result:

$$K_{\lambda,\mu}(q,t)=q^{n(\mu')}t^{n(\mu)}K_{\lambda',\mu}\left(q^{-1},t^{-1}\right). \qquad \square$$

A similar result is obtained by applying $\omega_{q,t}$ to the J_μ.

Proposition 19. *The q,t-Kostka polynomials satisfy*

$$K_{\lambda,\mu}(q,t)=K_{\lambda',\mu'}(t,q).$$

Proof. The q,t version of ω applied to $J_\mu[X;q,t]$ gives

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t]&=h_\mu(q,t)\omega_{q,t}P_\mu[X;q,t]\\&=h_\mu(q,t)Q_{\mu'}[X;t,q]\end{aligned}$$

Using the fact that $h'_{\mu'}(t,q)=h_\mu(q,t)$, we get

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t]&=h_{\mu'}(t,q)P_{\mu'}[X;t,q]\\&=J_{\mu'}[X;t,q]\\&=\sum_{\lambda\vdash|\mu|}K_{\lambda,\mu'}(t,q)s_\lambda[X(1-q)]\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\omega_{q,t}J_\mu[X;q,t]&=\sum_{\lambda\vdash|\mu|}K_{\lambda,\mu}(q,t)\omega_{q,t}s_\lambda\left[X(1-t)\right]\\&=\sum_{\lambda\vdash|\mu|}K_{\lambda,\mu}(q,t)s_{\lambda'}[X(1-q)].\end{aligned}$$

Once again, we equate the coefficients in the two different expansions to obtain the result. $\square$

We can put these two results together to obtain the important "transpose symmetry" property of the $\widetilde{H}_\mu$.

Proposition 20. *The modified Macdonald polynomials* $\widetilde{H}_\mu[X;q,t]$ *satisfy*

$$\widetilde{H}_\mu[X;q,t] = \widetilde{H}_{\mu'}[X;t,q].$$

Proof. We begin by expanding both sides. The left hand side becomes

$$\begin{aligned}\widetilde{H}_\mu[X;q,t] &= \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t)s_\lambda[X] \\ &= \sum_\lambda t^{n(\mu)}K_{\lambda,\mu}(q,t^{-1})s_\lambda[X].\end{aligned}$$

Similarly, the right hand side is

$$\begin{aligned}\widetilde{H}_{\mu'}[X;t,q] &= \sum_\lambda \widetilde{K}_{\lambda,\mu'}(t,q)s_\lambda[X] \\ &= \sum_\lambda q^{n(\mu')}K_{\lambda,\mu'}(t,q^{-1})s_\lambda[X].\end{aligned}$$

Equating coefficients of s_λ, we see that it is enough to show

$$t^{n(\mu)}K_{\lambda,\mu}(q,t^{-1}) = q^{n(\mu')}K_{\lambda,\mu'}(t,q^{-1}). \tag{2.2}$$

By Proposition 19 we have

$$t^{n(\mu)}K_{\lambda,\mu}(q,t^{-1}) = t^{n(\mu)}K_{\lambda',\mu'}(t^{-1},q)$$

and by Proposition 18 this is equal to

$$t^{n(\mu)}K_{\lambda,\mu'}(t,q^{-1})(t^{-1})^{n(\mu)}q^{n(\mu')} = q^{n(\mu')}K_{\lambda,\mu'}(t,q^{-1})$$

which establishes 2.2 and completes the proof. □

This symmetry provides the means to characterize the $\widetilde{H}_\mu[X;q,t]$ independently of the $P_\lambda[X;q,t]$.

Proposition 21. *The functions* $\widetilde{H}_\mu[X;q,t]$ *are the unique functions in* $\mathbb{Q}(q,t)\Lambda$ *satisfying the following conditions:*

1. $\left\langle \widetilde{H}_\mu[X;q,t], s_\lambda\left[\frac{X}{t-1}\right]\right\rangle = 0$ *if* $\lambda > \mu$,

2. $\left\langle \widetilde{H}_\mu[X;q,t], s_\lambda\left[\frac{X}{1-q}\right]\right\rangle = 0$ *if* $\lambda < \mu$, *and*

3. $\left\langle \widetilde{H}_\mu[X;q,t], s_{(n)}\right\rangle = 1$ *(where* $n = |\mu|$*).*

Proof. We first show the $\widetilde{H}_\mu[X;q,t]$ do, in fact, satisfy these conditions. The relationship between the $\widetilde{H}_\mu[X;q,t]$ and $P_\mu[X;q,t]$ can be succinctly stated as

$$\widetilde{H}_\mu[X;q,t] = t^{n(\mu)} h_\mu\left(q,t^{-1}\right) P_\mu\left[\frac{X}{1-t^{-1}};q,t^{-1}\right].$$

Some plethystic manipulation simplifies the right hand side.

$$\begin{aligned}\widetilde{H}_\mu[X;q,t] &= t^{n(\mu)} h_\mu\left(q,t^{-1}\right) P_\mu\left[t\frac{X}{t-1};q,t^{-1}\right]\\ &= t^{n(\mu)} h_\mu\left(q,t^{-1}\right) P_\mu\left[t;q,t^{-1}\right] P_\mu\left[\frac{X}{t-1};q,t^{-1}\right].\end{aligned}$$

In matrix form, we have

$$\langle P[X;q,t]\rangle = \langle s\rangle \xi$$

where ξ is uppper triangular, and

$$\langle \widetilde{H}[X;q,t]\rangle = Diag\left(t^{n(\mu)} h_\mu\left(q,\frac{1}{t}\right) P_\mu[t;q,t^{-1}]\right)\left\langle P\left[\frac{X}{t-1};q,t^{-1}\right]\right\rangle$$

Thus we can write

$$\langle \widetilde{H}[X;q,t]\rangle = \left\langle s\left[\frac{X}{t-1}\right]\right\rangle \mathbf{U}$$

for some upper triangular matrix $\mathbf{U}$. This is condition (1). It is transpose symmetry which gives (2). From (1) we have

$$\widetilde{H}_\mu[X;q,t] = \sum_{\lambda\le\mu} u_{\lambda,\mu}(q,t) s_\lambda\left[\frac{X}{t-1}\right]$$

and thus also

$$\widetilde{H}_{\mu'}[X;t,q] = \sum_{\lambda\le\mu} u_{\lambda,\mu}(q,t) s_\lambda\left[\frac{X}{t-1}\right].$$

Swapping q and t gives

$$\widetilde{H}_{\mu'}[X;q,t] = \sum_{\lambda \leq \mu} u_{\lambda,\mu}(t,q) s_\lambda \left[\frac{X}{q-1}\right]$$

which can be rewritten using Proposition 11 as

$$\widetilde{H}_{\mu'}[X;q,t] = \sum_{\lambda \leq \mu} \tilde{u}_{\lambda,\mu}(t,q) s_{\lambda'} \left[\frac{X}{1-q}\right].$$

Replacing μ' with μ gives

$$\widetilde{H}_{\mu}[X;q,t] = \sum_{\lambda \leq \mu'} \tilde{u}_{\lambda,\mu'}(t,q) s_{\lambda'} \left[\frac{X}{1-q}\right].$$

Now, the set of partitions $\{\lambda' \mid \lambda \leq \mu'\}$ is exactly the set of partitions $\{\rho \mid \rho \geq \mu\}$. This establishes (2).

We do not prove (3) here; see [Hai99] for a proof.

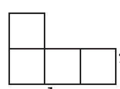
For uniqueness, let $G_\mu[X;q,t]$ be a set of polynomials satisfying the given conditions. Condition (1) implies, for each μ, $G_\mu[X;q,t]$ is a linear combination of $s_\lambda[X/(t-1)]$ where $\lambda \leq \mu$. Thus G_μ is a linear combination of $\widetilde{H}_\mu[X;q,t]$ where $\lambda \leq \mu$. Similarly, condition (2) implies that $G_\mu[X;q,t]$ is a linear combination of $\widetilde{H}_\mu[X;q,t]$ where $\lambda \geq \mu$. The only possibility, therefore, is that $G_\mu[X;q,t]$ is a constant times $\widetilde{H}_\mu[X;q,t]$ and by condition (3) that constant is 1. $\square$

3 The modules of Garsia-Haiman

Throughout the 1990s, Garsia and Haiman carried out a program to prove the Macdonald positivity conjecture by realizing the polynomials $\widetilde{H}[X; q.t]$ as the bigraded character of an S_n module. While a full survey of their results with this approach would be impossible in this space, we give the important definitions and survey some of the results here.

Definition 17. For $\mu \vdash n$, we label the cells (i, j) in the diagram of μ by $\{(p_1, q_1), \cdots, (p_n, q_n)\}$ by starting in the cell $(0, 0)$ and working left to right, bottom to top. We then set

$$\Delta_\mu = \det ||x_i^{p_j} y_i^{q_j}||_{i,j=1}^n$$

Example 11. For $\mu =$ , we have $(p_1, q_1) = (0, 0), (p_2, q_2) = (0, 1), (p_3, q_3) = (0, 2), (p_4, q_4) = (1, 0)$ and

$$\Delta_\mu = \det \begin{pmatrix} x_1^0 y_1^0 & x_1^0 y_1^1 & x_1^0 y_1^2 & x_1^1 y_1^0 \\ x_2^0 y_2^0 & x_2^0 y_2^1 & x_2^0 y_2^2 & x_2^1 y_2^0 \\ x_3^0 y_3^0 & x_3^0 y_3^1 & x_3^0 y_3^2 & x_3^1 y_3^0 \\ x_4^0 y_4^0 & x_4^0 y_4^1 & x_4^0 y_4^2 & x_4^1 y_4^0 \end{pmatrix}$$

We define the "diagonal action" of S_n on $\mathbb{Q}[X, Y]$ by setting $\sigma(x_i) = x_{\sigma(i)}, \sigma(y_i) = y_{\sigma(i)}$ and extending algebraically. With this action, it is clear that Δ_μ is an alternant; that is, that $\sigma\Delta_\mu = sgn(\sigma)\Delta_\mu$ for all $\sigma \in S_n$. This allows us to define the following S_n module.

Definition 18. Let M_μ be the linear span of all partial derivatives of Δ_μ. In symbols,

$$M_\mu = \mathbb{Q}\left\{\partial_x^p \partial_y^q \Delta_\mu\right\}_{|p|,|q|=0}^n$$

In [GH93] Garsia and Haiman describe this module and conjectured that it had bigraded character given by $\widetilde{H}_\mu[X; q, t]$. The validity of this would immediately imply the positivity conjecture of Macdonald. In particular, they conjectured that the dimension of M_μ was $n!$; this became known as the "$n!$ conjecture". In [Hai99], Haiman proved the surprising result that these conjectures are equivalent. In [Hai01], Haiman resolved both of these conjectures positively.

Haiman's proof of what is now the $n!$ theorem came about by relating the modules M_μ to the geometry of the Hilbert scheme H_n of points in the complex plane $\mathbb{C}^2$. A purely algebraic proof of the fact is still not known.

While studying the modules M_μ, Garsia and Haiman gave a large collection of conjectures, suggested by certain representation theoretical heuristics, and supported by overwhelming experimental evidence. These heuristics became known as "science fiction", and the associated conjectures are, for the most part, still open.

In particular, in [GH95] they conjectured the existence of a family of polynomials, indexed by certain lattice diagrams L in the plane, satisfying certain conditions. To state these conditions we give some further definitions. We say that two lattice square diagrams L_1, L_2 are equivalent if they differ by a series of row and column swaps, and we write $L_1 \equiv L_2$ in this case. As with Young diagrams, we will define the conjugate of a lattice diagram by reflecting the diagram about the main diagonal $i = j$. We write L' for the conjugate of L. Finally, if L can be decomposed into two diagrams L_1, L_2 so that no rook placed on a cell in L_1 attacks any cell in L_2, we say that L is *decomposable* and write $L = L_1 \times L_2$.

Given a cell in a diagram, the *arm* (respectively *leg*, *coarm*, *coleg*) are defined by the number of cells in the diagram and strictly to the East (resp. North, West, South) of the given cell. This agrees with the previously established notation for Young diagrams.

These definitions given, they conjectured the existence of polynomials $G_L[X; q, t]$ indexed by diagrams L equivalent to skew Young diagrams, and satisfying the following conditions:

1. For μ a Young diagram, the polynomial $G_\mu[X; q, t]$ is the modified Macdonald polynomial $\widetilde{H}_\mu[X; q, t]$.

2. Polynomials indexed by equivalent diagrams are equal.

3. If a diagram $L = L_1 \times L_2$ then $G_L[X; q, t] = G_{L_1}[X; q, t] G_{L_2}[X; q, t]$.

4. $G_L[X; q, t] = G_{L'}[X; t, q]$.

5. The polynomials G_L satisfy the following equation:

$$\partial_{p_1} G_L[X; q, t] = \sum_{c \in L} q^{a'(c)} t^{l(c)} G_{L \backslash c}[X; q, t] \tag{3.1}$$

The existence of polynomials satisfying 1 through 4 is clear. Including condition 5 allows us to derive another condition which the polynomials must satisfy.

Proposition 22. *Suppose the polynomials $G_L[X; q, t]$ described above exist. Then they must also satisfy*

$$\partial_{p_1} G_L[X; q, t] = \sum_{c \in L} q^{a(c)} t^{l'(c)} G_{L \backslash c}[X; q, t] \tag{3.2}$$

Proof. By condition 5 we have

$$\partial_{p_1} G_{L'}[X; t, q] = \sum_{c \in L'} t^{a'(c)} q^{l(c)} G_{L' \backslash c}[X; t, q]$$

Applying condition 4 to the left hand side gives

$$\partial_{p_1} G_L[X; q, t] = \sum_{c \in L'} t^{a'(c)} q^{l(c)} G_{L' \backslash c}[X; t, q]$$

We note that transposition sends coleg to arm and leg to coarm. Thus we can apply condition 4 to the right hand side to get

$$\partial_{p_1} G_L[X; q, t] = \sum_{c \in L} q^{a(c)} t^{l'(c)} G_{L \backslash c}[X; q, t]$$

as desired. □

The inclusion of conditions 3.1 and 3.2 raise the question of whether such polynomials exist. Nevertheless, it is possible to use these conditions to explicitly determine the polynomials $G_L[X; q, t]$ in many special cases. However, it has never been proven that this can be done in general. In the following sections, we show how a combinatorial description of the Macdonald polynomials give an explicit determination of the $G_L[X; q, t]$ where L is equivalent to a skew diagram with no column of height greater than or equal to 2.

4 The combinatorial description of Haiman, Haglund and Loehr

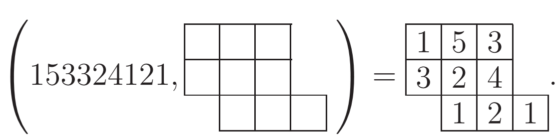
A major advance was made in the theory of Macdonald polynomials in [Hag04] when Jim Haglund proposed a purely combinatorial description of the polynomials $\widetilde{H}[X;q,t]$. (This was quickly proved correct, in [HHL05].) It is this combinatorial approach that is at the heart of the proof of the existence of the $G_L[X;q,t]$ for the diagrams we consider. To give this description, we need to introduce some further notation.

4.1 Diagrams and Statistics

Throughout this section, the diagrams under consideration will always be skew Young diagrams.

Definition 19. A *filling* of a diagram L of size n with a word σ of length n (written (σ, L)) is a function from the cells of the diagram to $\mathbb{Z}_+$ given by labeling the cells from top to bottom and left to right within rows by 1 to n in order, then applying σ.

Example 12. Let L be the shape $(4,3,3)\setminus(1)$. We have the following filling, where $(1,5,3,3,2,4,1,2,1)$ is written 153324121 for brevity:



The value of a filling of a particular cell is denoted by $\sigma(c)$, *e.g.,* $\sigma(1,0) = 3$ in the filled diagram above.

Definition 20. This ordering (top to bottom and left to right within rows) is called the *reading order*, and we write $c <_R c'$ if c precedes c' in this order.

Definition 21. The *descent set* of a filled diagram is defined as follows:

$$Des(\sigma, L) = \{(i, j) \in L : (i-1, j) \in L \text{ and } \sigma(i, j) > \sigma(i-1, j)\}$$

Example 13. In our running example, we have

$$Des\left(\begin{array}{|c|c|c|c}\cline{1-3} 1&5&3\\ \cline{1-3} 3&2&4\\ \cline{1-4} \multicolumn{1}{c|}{}&1&2&\multicolumn{1}{c|}{1}\\ \cline{2-4}\end{array}\right) = \left\{\begin{array}{|c|c|c|c}\cline{1-3} &5&\\ \cline{1-3} &&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&\\ \cline{1-3} &2&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&\\ \cline{1-3} &&4\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}\right\}$$

Definition 22. Two distinct cells $(i, j) <_R (i', j')$ in a filled diagram form an *attacking pair* if

- $i = i'$ OR
- $i = i' + 1$ and $j' \leq j$

Definition 23. The *Inversion set* of a filled diagram is given by

$$Inv(\sigma, L) = \{(c, c') : (c, c') \in \text{an attacking pair in } L, c <_R c', \sigma(c) > \sigma(c')\}.$$

Example 14. In the running example, we have

$$Inv\left(\begin{array}{|c|c|c|c}\cline{1-3} 1&5&3\\ \cline{1-3} 3&2&4\\ \cline{1-4} \multicolumn{1}{c|}{}&1&2&\multicolumn{1}{c|}{1}\\ \cline{2-4}\end{array}\right) = \left\{\begin{array}{|c|c|c|c}\cline{1-3} &5&3\\ \cline{1-3} &&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &5&\\ \cline{1-3} 3&&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&3\\ \cline{1-3} &2&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&\\ \cline{1-3} 3&2&\\ \cline{1-4} \multicolumn{1}{c|}{}&&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&\\ \cline{1-3} &&4\\ \cline{1-4} \multicolumn{1}{c|}{}&1&&\multicolumn{1}{c|}{}\\ \cline{2-4}\end{array}, \begin{array}{|c|c|c|c}\cline{1-3} &&\\ \cline{1-3} &&\\ \cline{1-4} \multicolumn{1}{c|}{}&&2&\multicolumn{1}{c|}{1}\\ \cline{2-4}\end{array}\right\}.$$

Definition 24. The *maj* statistic of a filled diagram is given by

$$\text{maj}(\sigma, L) = \sum_{c \in Des(\sigma, L)} (\text{leg}(c) + 1)$$

Definition 25. The *inv* statistic of a filled diagram is given by

$$\text{inv}(\sigma, L) = |Inv(\sigma, L)| - \sum_{c \in Des(\sigma, L)} \text{arm}(c)$$

Example 15. In our ongoing example, we have

$$\text{maj}\left(\begin{array}{|c|c|c|c}\cline{1-3} 1&5&3\\ \cline{1-3} 3&2&4\\ \cline{1-4} \multicolumn{1}{c|}{}&1&2&\multicolumn{1}{c|}{1}\\ \cline{2-4}\end{array}\right) = (0+1) + (1+1) + (1+1) = 5$$

$$\text{inv}\left(\begin{array}{|c|c|c|c}\cline{1-3} 1&5&3\\ \cline{1-3} 3&2&4\\ \cline{1-4} \multicolumn{1}{c|}{}&1&2&\multicolumn{1}{c|}{1}\\ \cline{2-4}\end{array}\right) = 6 - (1+1+0) = 4$$

Proposition 23. *The statistic* $\operatorname{inv}(\sigma, L) \geq 0$ *for all* (σ, L).

Proof. Every cell in the arm of a descent attacks both of the cells involved in the descent, and must form an inversion with at least one. □

We have a different characterization of inv, in the case when the filling σ of L is a permutation.

Proposition 24. *The statistic* $\operatorname{inv}(\sigma, L)$ *is given by the number of triples* $\begin{smallmatrix} x & \quad & y \\ z & & \end{smallmatrix}$ *which have a counter-clockwise orientation when orderd from smallest to largest, plus the number of pairs* $x \quad y$, *where* $x > y$, *and there is no cell directly beneath* x *in* L.

Proof. See [HHL05]. □

Such a triple is called an *inversion triple*.

Proposition 24 is particularly useful, since standardization does not affect the statistics.

Proposition 25. *We have*

$$\operatorname{inv}(\sigma, L) = \operatorname{inv}(std(\sigma), L)$$

and

$$\operatorname{maj}(\sigma, L) = \operatorname{maj}(std(\sigma), L)$$

for all (σ, L).

Proof. It is easy to verify that neither the Inversion set nor the Descent set is affected. □

Example 16.

$$\begin{array}{|c|c|c|c} \hline 1 & 5 & 3 & \\ \hline 3 & 2 & 4 & \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & \multicolumn{1}{c|}{1} \\ \cline{2-4} \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c} \hline 1 & 9 & 6 & \\ \hline 7 & 4 & 8 & \\ \hline \multicolumn{1}{c|}{} & 2 & 5 & \multicolumn{1}{c|}{3} \\ \cline{2-4} \end{array}$$

Note that the inversion triples can be computed by counting the inversion triples $\left(\begin{smallmatrix} 1 & \quad & 9 \\ 7 & & \end{smallmatrix}\right), \left(\begin{smallmatrix} 9 & \quad & 6 \\ 4 & & \end{smallmatrix}\right)$ and the inversions $(7 \quad 4), (5 \quad 3)$.

In what follows it will be useful to have the following notational shorthand:

$$h(\sigma, L) = q^{\mathrm{inv}(\sigma,L)} t^{\mathrm{maj}(\sigma,L)}$$

For example:

$$h\left(\begin{array}{|c|c|c|c} \hline 1 & 5 & 3 & \multicolumn{1}{c}{} \\ \hline 3 & 2 & 4 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & \multicolumn{1}{c|}{1} \\ \cline{2-4} \end{array}\right) = q^4 t^5$$

4.2 Introduction of L-multinomial coefficients.

We begin by recalling the following definitions of q-analogs:

- $[n]_q = 1 + q + \cdots + q^{n-1}$
- $[n]_q! = [n]_q \cdots [1]_q$
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$
- For $\alpha = (\alpha_1, \cdots, \alpha_k) \models n$,

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \frac{[n]_q!}{[\alpha_1]_q! \cdots [\alpha_k]_q!} = \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_q \begin{bmatrix} n-\alpha_1 \\ \alpha_2 \end{bmatrix}_q \cdots \begin{bmatrix} n-\cdots-\alpha_{k-1} \\ \alpha_k \end{bmatrix}_q$$

The following proposition is well known.

Proposition 26.

$$\begin{aligned} \begin{bmatrix} n \\ \alpha \end{bmatrix}_q &= \sum_{w:ct(w)=\alpha} q^{\mathrm{inv}(w)} \\ &= \sum_{w:ct(w)=\alpha} q^{\mathrm{maj}(w)} \end{aligned}$$

We now introduce a family of q,t-analogs of multinomial coefficients indexed by skew diagrams. The motivation will be to allow us to compute more efficiently with the condition (3.1).

Definition 26. For $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ we set

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_L = \sum_{w:ct(w)=\alpha} h(w, L)$$

It will also be convenient to set, for any word w,

$$\begin{bmatrix} n \\ w \end{bmatrix}_L = \sum_{v \in \mathcal{R}(w)} h(v, L)$$

The next proposition states that these are indeed analogs of multinomial coefficients.

Proposition 27. *We have*

$$\left.\begin{bmatrix} n \\ \alpha \end{bmatrix}_L\right|_{q=t=1} = \begin{bmatrix} n \\ \alpha \end{bmatrix}$$

Proof. This is because, for any word w of content α, $|\mathcal{R}(w)| = \left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$. □

These multinomial coefficients interpolate between the q and t single variable multinomial coefficients.

Proposition 28. *We have*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{(n)} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q \text{ and } \begin{bmatrix} n \\ \alpha \end{bmatrix}_{(1^n)} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_t$$

Proof. These are immediate consequences of the previous proposition and the definition of Haglund's statistic. □

4.3 A Combinatorial Description of $C_L(x; q, t)$

Definition 27. For L a diagram of size n

$$C_L(x; q, t) = \sum_{\sigma \in (\mathbb{Z}_+)^n} h(\sigma, L) x_\sigma$$

where x_σ means $\prod_{i=1}^n x_{\sigma(i)}$.

The following results are due to Haglund, Haiman, and Loehr [HHL05]:

Theorem 3. *For L any diagram, $C_L(x; q, t)$ is a symmetric polynomial.*

This has an immediate corollary with respect to the L-multinomial coefficients.

Corollary 1. *For α a rearrangement of $\lambda \vdash n$, $\begin{bmatrix} n \\ \alpha \end{bmatrix}_L$ is the coefficient of m_λ in $C_L[X;q,t]$. In particular, $\begin{bmatrix} n \\ \alpha \end{bmatrix}_L = \begin{bmatrix} n \\ \beta \end{bmatrix}_L$ for any β which is a rearrangement of α.*

Proof. This follows immediately from the definition and Theorem 3. □

Another corollary is the fact that we can determine $C_L[X;q,t]$ by considering only permutational fillings of L.

Corollary 2.

$$C_L(x;q,t) = \sum_{\sigma \in S_n} h(\sigma, L) Q_{\mathrm{co}(\sigma^{-1})}$$

where Q_α is the fundamental quasisymmetric function defined in Section 1.2.5.

Proof. This is a consequence of Proposition 2 and the fact that standardization does not affect the statistics. □

The following property of the $C_L[X;q,t]$ will be also be useful to us:

Proposition 29. *If $L = L_1 \times L_2$ with no rows or columns in common, then $C_L[X;q,t] = C_{L_1}[X;q,t]C_{L_2}[X;q,t]$.*

Proof. The following multiplication formula for fundamental quasisymmetric functions is proven, for example, in [Sta99]:

$$Q_{\mathrm{co}(\sigma)} Q_{\mathrm{co}(\tau)} = \sum_{\rho \in sh(\sigma,\tau)} Q_{\mathrm{co}(\rho)}$$

where we assume $\sigma \in S_n$, $\tau \in S_{n+1,\ldots,n+m}$ and the sum is over all shuffle products of σ and τ. Thus for disjoint diagrams L_1 and L_2 we have

$$C_{L_1}[X;q,t]C_{L_2}[X;q,t] = \sum_{\sigma \in S_n, \tau \in S_{n+1,\ldots,n+m}} h(\sigma, L_1) h(\tau, L_2) \sum_{\rho \in sh(\sigma^{-1},\tau^{-1})} Q_{\mathrm{co}(\rho)}$$

Now, by Theorem 2 we have

$$C_{L_1 L_2}[X;q,t] = \sum_{\nu \in S_{n+m}} h(\nu, L_1 L_2) Q_{\mathrm{co}(\nu^{-1})}$$

Thus we need to show a bijection from triples (σ, τ, ρ) to permutations ν such that $\mathrm{co}(\rho) = \mathrm{co}(\nu^{-1})$ and $h(\sigma, L_1)h(\tau, L_2) = h(\nu, L_1 L2)$. It is tedious but not difficult to verify that choosing $\nu = \rho^{-1}$ provides such a bijection. □

The most important property of the $C_L[X;q,t]$ is the following, due to Haiman, Haglund and Loehr.

Theorem 4 ([HHL05])**.** *For μ a partition,*

$$C_\mu[X;q,t] = \widetilde{H}_\mu[X;q,t]$$

The proof of this fact is by verifying, in a completely combinatorial manner, the properties in Proposition 21. Thus, we now have a combinatorial description (and proof of existence) of the Macdonald polynomials. However, this definition is not entirely satisfactory, for at least two reasons. First, this has provided very little insight into the combinatorial nature of the $\widetilde{K}_{\lambda,\mu}(q,t)$. Some progress has recently been made in this direction by Sami Assaf. In her recent dissertation at the University of California-Berkeley, she conjectures a reason why Theorem 4 implies $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$, and proves this conjecture for the case where μ has at most three columns. However, even should this approach succeed, it still does little to explicitly describe the q,t-Kostka polynomials. A second defect of the current theory is the somewhat surprising fact that no one has been able to prove combinatorially that $C_\mu[X;q,t] = C_{\mu'}[X;t,q]$. This symmetry result is among the most important unsolved combinatorial problems in the theory. This said, the $C_L[X;q,t]$ are to this point the most effective combinatorial tool we have for exploring Macdonald polynomials. And so, with them in hand, we proceed to the next chapter.

5 Extension of Macdonald Polynomials

5.1 The conjectured polynomials of Garsia and Haiman

We recall the defining conditions of the polynomials $G_L[X;q,t]$ here:

1. For μ a Young diagram, the polynomial $G_\mu[X;q,t]$ is the modified Macdonald polynomial $\widetilde{H}_\mu[X;q,t]$.

2. Polynomials indexed by equivalent diagrams are equal.

3. If a diagram $L = L_1 \times L_2$ then $G_L[X;q,t] = G_{L_1}[X;q,t]G_{L_2}[X;q,t]$.

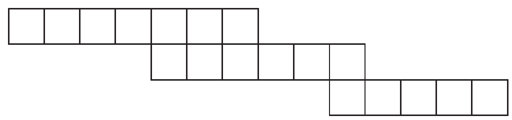
4. $G_L[X;q,t] = G_{L'}[X;t,q]$.

5. The polynomials G_L satisfy the following equation:

$$\partial_{p_1} G_L[X;q,t] = \sum_{c \in L} q^{a'(c)} t^{l(c)} G_{L \setminus c}[X;q,t] \qquad (3.1)$$

A natural question raised by this definition is whether the polynomials $C_L[X;q,t]$ satisfy the properties of the $G_L[X;q,t]$. The immediate answer is no. It is easy to construct shapes L_1 and L_2 which differ by only row and column swaps, for which $C_{L_1}[X;q,t] \neq C_{L_2}[X;q,t]$. There also exist shapes L such that $C_L[X;q,t] \neq C_{L'}[X;t,q]$. A more sophisticated strategy is possible, however. We consider two diagrams equivalent if they differ only by a series of row and column swaps. For a given equivalence class of diagrams, $\mathcal{L}$, we define a unique representative L^*. Finally, we set $G_L[X;q,t] = C_{L^*}[X;q,t]$ for all $L \in \mathcal{L}$. The main result is that this approach successfully defines the polynomials $G_L[X;q,t]$ for the family of diagrams known as general pistols. Involved in the proof are new properties of the "Haglund statistics" on diagrams.

Here and after, the only diagrams under consideration will be those which can be rearranged to general pistols. A general *pistol* is a skew-shape with no column of height greater than 2. For example, the diagram below is a general pistol:



For any diagram L which can be rearranged to a general pistol, we will write L^* for the general pistol to which it can be arranged. This definition brings us to the following theorem.

Theorem 5. *For any diagram L which can be rearranged to a general pistol, the polynomial* $G_L[X;q,t]$ *is given by* $C_{L^*}[X;q,t]$.

The rest of this chapter is devoted to establishing this theorem.

5.2 Reducing to L-multinomial coefficients

We begin the process of establishing Theorem 5 by showing the theorem is equivalent to certain conditions on L-multinomial coefficients.

Proposition 30. *The conditions*

$$\begin{bmatrix} n \\ 1, \alpha \end{bmatrix}_{L^*} = \sum_{c \in L^*} q^{a'(c)} t^{l(c)} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{(L \backslash c)^*} \tag{5.1}$$

respectively,

$$\begin{bmatrix} n \\ 1, \alpha \end{bmatrix}_{L^*} = \sum_{c \in L^*} q^{a(c)} t^{l'(c)} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{(L \backslash c)^*} \tag{5.2}$$

for all $\alpha \models n-1$ *imply*

$$\partial_{p_1} C_{L^*}[X;q,t] = \sum_{c \in L^*} q^{a'(c)} t^{l(c)} C_{(L\backslash c)^*}[X;q,t] \tag{5.3}$$

respectively,

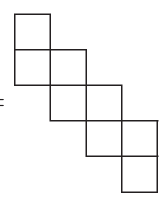
$$\partial_{p_1} C_{L^*}[X;q,t] = \sum_{c \in L^*} q^{a(c)} t^{l'(c)} C_{(L\backslash c)^*}[X;q,t], \tag{5.4}$$

and this implies Theorem 5.

Proof. We begin by establishing the second implication. For this, we must show that the $C_{L^*}[X;q,t]$ satisfy properties (1) through (4).

Condition 2 is satisfied by the definition of L^*. By Theorem 4, Condition 1 is satisfied. By Proposition 29, Condition 3 is satisfied. We first consider Condition 4 when both L^* and L'^* are ribbons with no column of size greater than or equal to two. This is when L^* is a ribbon shape with no row or column of length greater than 2. There are two classes of such diagrams, which we deal with in turn.

The first such class is when L is not symmetrical about the main diagonal. A typical example of this is

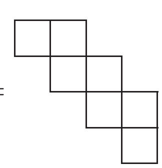
$L =$ 

In this case it is easy to verify that for every filling w of L we have

$$q^{inv(w,L)} t^{maj(w,L)} = q^{maj(w,L')} t^{inv(w,L')}$$

which gives Condition 4.

The other class of ribbons with no rows or columns of length greater 2 are those diagrams L which are symmetrical about the main diagonal. A typical example of this is

$L =$ 

For any filling w of L we let w' be the word w in reverse order, with every letter complemented (*i.e.*, replace i by $|L|-i$). We then have

$$q^{inv(w,L)}t^{maj(w,L)} = q^{maj(w',L)}t^{inv(w',L)}$$

Since the vector $wt(w)$ is, up to rearrangement, the same as the vector $wt(w')$, Condition 4 will hold.

We now must show the first implication in the statement of the proposition. We note here that there is an obvious bijection between the sets of words of content α, and the set of standardizations of words of content α. This allows us to perform the following calculation of the coefficient of $Q_{\mathrm{co}_n(S)}$ in $C_L[X;q,t]$ in terms of L multinomial coefficients:

Proposition 31. *For any $S \subseteq \{1,\ldots,n-1\}$ we have*

$$C_L[X;q,t]\Big|_{Q_{\mathrm{co}_n(S)}} = \sum_{R\subseteq S}(-1)^{|S-R|}\begin{bmatrix} n \\ \mathrm{co}_n(R)\end{bmatrix}_L$$

Proof. This is a straightforward inclusion-exclusion computation:

$$\begin{aligned} C_L[X;q,t]\Big|_{Q_{\mathrm{co}_n(S)}} &= \sum_{d(\sigma^{-1})=S} h(\sigma,L) \\ &= \sum_{R\subseteq S}\sum_{d(\sigma^{-1})\subseteq R}(-1)^{|S-R|}h(\sigma,L). \end{aligned}$$

Applying Proposition 2 gives

$$C_L[X;q,t]\Big|_{Q_{\mathrm{co}_n(S)}} = \sum_{R\subseteq S}\sum_{std(w):ct(w)=\mathrm{co}_n(R)}(-1)^{|S-R|}h(std(w),L)$$

and now we replace $std(w)$ with w, giving

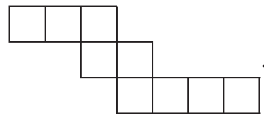
$$\begin{aligned} C_L[X;q,t]\Big|_{Q_{\mathrm{co}_n(S)}} &= \sum_{R\subseteq S}\sum_{w:ct(w)=\mathrm{co}_n(R)}(-1)^{|S-R|}h(w,L) \\ &= \sum_{R\subseteq S}(-1)^{|S-R|}\begin{bmatrix} n \\ \mathrm{co}_n(R)\end{bmatrix}_L. \end{aligned}$$

□

Before completing the proof of Proposition 30, we must introduce one more tool from the machinery of symmetric functions.

Definition 28. For $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$, we set Z_α to be the Schur function indexed by the ribbon shape $\alpha_1, \alpha_2, \ldots, \alpha_k$.

For example, $Z_{(3,2,4)}$ is the skew-Schur function indexed by



.

These functions have some useful properties which can be found, for example, in [Mac]:

- For f a symmetric function, $\langle f, Z_\alpha \rangle$ is the coefficient of Q_α in f.
- $p_1 Z_\alpha = Z_{(1,\alpha_1,\ldots,\alpha_k)} + Z_{(1+\alpha_1,\alpha_2,\ldots,\alpha_k)}$.

We denote the terms in this last sum as $Z_{(1,\alpha)}, Z_{(1+\alpha)}$, respectively. We can now complete the proof of Proposition 30. We show 5.1 and note that the proof of 5.2 is completely analogous. We begin by considering the coefficient of Q_α on both sides of (3.1). The calculation begins by using the fact that, with respect to the Hall scalar product, the adjoint of differentiation by p_1 is multiplication by p_1.

$$\begin{aligned}\langle \partial_{p_1} C_{L^*}(x;q,t), Z_\alpha \rangle =& \langle C_{L^*}(x;q,t), p_1 Z_\alpha \rangle \\ =& \langle C_L(x;q,t), Z_{(1,\alpha)} \rangle + \langle C_{L^*}(x;q,t), Z_{(1+\alpha)} \rangle.\end{aligned}$$

We now apply Proposition 31 to obtain

$$\begin{aligned}\langle \partial_{p_1} C_{L^*}(x;q,t), Z_\alpha \rangle =& \sum_{R\subseteq \mathbb{S}(1,\alpha)} \begin{bmatrix} n \\ \mathrm{co}_n(R) \end{bmatrix}_{L^*} (-1)^{|\mathbb{S}(\alpha)-R|+1} \\ &+ \sum_{S\subseteq \mathbb{S}(1+\alpha)} \begin{bmatrix} n \\ \mathrm{co}_n(S) \end{bmatrix}_{L^*} (-1)^{|\mathbb{S}(\alpha)-S|} \\ =& \sum_{R\subseteq \mathbb{S}(\alpha)} \begin{bmatrix} n \\ \mathrm{co}_n(1,R) \end{bmatrix}_{L^*} (-1)^{|\mathbb{S}(\alpha)-R|}\end{aligned}$$

and similarly,

$$\langle C_{(L\backslash c)^*}(x;q,t), Z_\alpha \rangle = \sum_{R\subseteq \mathbb{S}(\alpha)} \begin{bmatrix} n-1 \\ \mathrm{co}_{n-1}(R) \end{bmatrix}_{(L\backslash c)^*} (-1)^{|\mathbb{S}(\alpha)-R|}$$

Since the Q_α are linearly independent, (3.1) will hold if and only if

$$\sum_{R\subseteq \mathbb{S}(\alpha)} \begin{bmatrix} n \\ \mathrm{co}_n(1,R)\end{bmatrix}_L (-1)^{|\mathbb{S}(\alpha)-R|} = \sum_{c\in L^*} q^{a'(c)} t^{l(c)} \sum_{R\subseteq \mathbb{S}(\alpha)} \begin{bmatrix} n-1 \\ \mathrm{co}_{n-1}(R)\end{bmatrix}_{(L\backslash c)^*} (-1)^{|\mathbb{S}(\alpha)-R|}$$

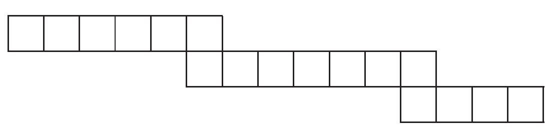
for all α. This is the case if and only if

$$\begin{bmatrix} n \\ 1,\alpha\end{bmatrix}_{L^*} = \sum_{c\in L^*} q^{a'(c)} t^{l(c)} \begin{bmatrix} n \\ \alpha\end{bmatrix}_{(L\backslash c)^*}$$

holds for all $\alpha \models n-1$, which completes the proof that (5.1) implies 5.3. At first it would appear that this is all that is necessary to establish Theorem 5. However, there is one subtlety which we have not considered. For the case when L' is not in $\mathcal{L}$, we must establish the ∂_{p_1} condition on L'. We have no way of doing this directly, so we use Proposition 22. This shows that we have the desired recursion on these L' if we can show 5.4. The proof that 5.2 implies 5.4 is completely analogous to what we have done above. □

5.3 Ribbon Pistol Diagrams

In this section, we show the existence of the $G_L[X;q,t]$ in what will prove to be an informative warm-up case. Namely, we will show the existence of $G_L[X;q,t]$ for the class of diagrams that can be rearranged to *ribbon pistols*. These are shapes which are both ribbons and pistols. For example,



is a ribbon pistol.

We first not that if L^* is a ribbon pistol, and c a cell in L^*, then $(L^*\backslash c)$ is either the disjoint union of two ribbon pistols, or rearranges to a ribbon pistol simply by swapping out the missing cell. So the class of ribbon pistols is closed with respect to removing cells (up to to row and column rearrangements).

Theorem 6. *For L a ribbon pistol, the polynomials $G_L[X;q,t]$ are given by the polynomials $C_L[X;q,t]$.*

Proof. Our basic strategy for showing (5.1) will be to give a bijection between words w with content $(1, \alpha)$ and pairs (c, w') where c is a cell of L^* and w' has content α satisfying

$$h(w, L^*) = q^{a'(c)} t^{l(c)} h(w', (L \setminus c)^*) \tag{5.5}$$

This bijection is established by choosing c to be the cell containing 1, and setting $w' = w \setminus 1$. For example,

$$h\left(\begin{array}{|c|c|c|c|c|cccc} \hline 3&4&1&2&5&&&& \\ \hline &&&&3&2&2&4&3 \end{array}\right) = q^2 t^0 h\left(\begin{array}{cccccccc} 3&4&2&5&&&& \\ &&&3&2&2&4&3 \end{array}\right)$$

and

$$h\left(\begin{array}{ccccccccc} 3&4&2&2&5&&&& \\ &&&&1&3&2&4&3 \end{array}\right) = q^0 t^1 h\left(\begin{array}{ccccccccc} 3&4&2&2&5&&&& \\ &&&&&3&2&4&3 \end{array}\right)$$

From the definition of the statistics, we get that the inversion statistic for ribbon pistols is the sum of the inversions in each of the rows. Furthermore, the major index is simply the number of descents. From this, it is easy to see that the contribution of 1 to the statistic is $q^{a'(c)} t^{l(c)}$ where c is the cell which contains 1. We can establish 5.2 in a completely analogous way, by choosing c to be the cell containing n. In this case, the contribution to the statistic is $q^{a(c)} t^{l'(c)}$. □

This proof is a model for how we would, ideally, prove the general case. In fact, to establish 5.1 attempt to find a bijection satisfying 5.5, and we will always choose c to be the cell containing 1. Similarly, in establishing 5.2, we will always choose c to be the cell containing n. The difficulty will arise from the fact that we cannot always choose $w' = w \setminus 1$.

5.4 Reducing to 2-row rectangles

In this section, we reduce the problem to that of two row rectangles. Essentially, the reason this is possible, is that the bijection of the previous section "works" when the cell c is not in a column of height 2.

Lemma 1. *Let L be a general pistol, w a word of length $n = |L|$ with a unique 1. Let c be the cell of L which contains 1, when L is filled with w, and suppose c is not in a column of height 2. Let w' be w with the 1 removed. Then*

$$h(w, L) = q^{a'(c)} h(w', (L \setminus c)^*)$$

Similarly, suppose w contains a unique n and c, the cell containing this n, has no other cells in the same column. Setting w' to be w with the n removed gives

$$h(w, L) = q^{a(c)} h(w', (L \setminus c)^*)$$

Proof. The first statement follows immediately from the definition of the statistics: Since c is in a column of height one, it cannot be part of a descent. Since L is a skew Young diagram, c cannot be part of an inversion triple. Finally, since c contains a unique 1, every element in the coarm of c forms an inversion with 1. The proof of the second statement is analogous. □

In order to establish 5.5 we must decide what to do when the 1 is in a column of height 2. Unfortunately, this is not as straightforward. Consider the following example:

$$h\left(\begin{array}{ccccccccc} 3 & 4 & 3 & 4 & 2 & 5 & & & \\ & & & 2 & 1 & 4 & 4 & 3 & 2 \end{array}\right) \longrightarrow$$
$$q^1 t^1 h\left(\begin{array}{ccccccccc} 3 & 4 & 3 & ? & ? & ? & & & \\ & & & & ? & ? & 4 & 3 & 2 \end{array}\right).$$

It is not clear how to fill in the question marks to obtain the correct statistic. We continue our convention of letting w be a word with a unique 1, and c be the cell of L that contains w. We think of L as being made up of *blocks* and *strips* where a block is a rectangle of height 2 and a strip is a horizontal row of blocks all of which are unique in their column. Note that the shape $(L \setminus c)^*$ depends only on which block or strip that c is in, and, if c is in a block, which row of that block contains c.

Thus, we can prove 5.1 by establishing the following lemma.

Lemma 2. *Let L be the two row rectangle (l, l). Let r, s be weakly increasing words of length l; i.e., $r_i \leq r_{i+1}$ and $s_i \leq s_{i+1}$ for $1 \leq i < l$. If there is no 1 in s, and* *exactly one* 1 *in* r *then*

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline\end{array}\right) = t[l]_q \sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho'\in\mathcal{R}(r')}} h\left(\begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\sigma} \\ \hline & \multicolumn{1}{|c|}{\rho'} \\ \cline{2-2}\end{array}\right). \tag{5.6}$$

If, on the other hand, we have no 1 *in* r*, and exactly one* 1 *in* s*, then*

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline\end{array}\right) = [l]_q \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|cc|}\cline{1-1} \multicolumn{1}{|c|}{\sigma'} & \\ \hline \multicolumn{2}{|c|}{\rho} \\ \hline\end{array}\right). \tag{5.7}$$

As an aid to readability, the notation in all that follows will be consistent with this lemma:

- Roman letters (*e.g.*, s, r, w) will denote a word whose entries are weakly increasing.
- Greek letters (*e.g.*, σ, ρ) will denote arbitrary words.
- "Primed" letters (*e.g.*, w') will denote the word w with a distinguished element (usually 1) removed.

5.5 Distinguished cell in bottom row

This section is devoted to establishing 5.6. To do so, we first establish a kind of "fermionic formula", in order to more easily manipulate the statistics.

Proposition 32. *Let* L *be the two row partition shape* (l, l)*, and* s, r *be weakly increasing words of length* l*. Then*

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline\end{array}\right) = \begin{bmatrix} l \\ r \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} h\left(\begin{array}{|c|}\hline \sigma \\ \hline r \\ \hline\end{array}\right)$$

To prove this, we first prove a certain "swap" lemma. For ρ a word of length l and $i < l$, we let $s_i(\rho)$ denote ρ with positions i and $i+1$ swapped. It is easy to see that $\mathrm{inv}(\rho) = \epsilon + \mathrm{inv}(s_i(\rho))$, where $\epsilon \in \{-1, 0, 1\}$. The content of the following lemma is that there is an action of the s_i on filled two-row shapes which completely agrees with the action on the bottom row.

Lemma 3. *Given words σ, ρ of the same length, let $\epsilon \in \{-1, 0, 1\}$ be such that*

$$\operatorname{inv}(\rho) = \epsilon + \operatorname{inv}(s_i(\rho)).$$

Then there exists a unique $\pi \in \{\sigma, s_i(\sigma)\}$ such that

$$h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right) = q^{\epsilon} h\left(\begin{array}{|c|}\hline \pi \\ \hline s_i(\rho) \\ \hline \end{array}\right).$$

Proof. We first note that in the case where $\epsilon = 0$ (*i.e.*, when $\rho_i = \rho_{i+1}$) we must choose $\pi = \sigma$. We then note that it is enough to consider the 2×2 square consisting of columns i and $i+1$, because swapping elements within these columns will not affect inversions or descents outside of these columns. Finally, because standardization does not affect the statistics, we can assume that the contents of the top and bottom row are disjoint. Then we consider every possible case, as follows:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\,(1) \leftrightarrow \begin{array}{|c|c|}\hline 1&1\\\hline 3&2\\\hline\end{array}\,(q), \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\,(1) \leftrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array}\,(q), \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\,(q) \leftrightarrow \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\,(q^2),$$

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline\end{array}\,(qt) \leftrightarrow \begin{array}{|c|c|}\hline 1&4\\\hline 3&2\\\hline\end{array}\,(q^2t), \qquad \begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array}\,(q) \leftrightarrow \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array}\,(q^2), \qquad \begin{array}{|c|c|}\hline 2&2\\\hline 1&3\\\hline\end{array}\,(t) \leftrightarrow \begin{array}{|c|c|}\hline 2&2\\\hline 3&1\\\hline\end{array}\,(qt),$$

$$\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}\,(t) \leftrightarrow \begin{array}{|c|c|}\hline 2&3\\\hline 4&1\\\hline\end{array}\,(qt), \qquad \begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array}\,(t) \leftrightarrow \begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline\end{array}\,(qt), \qquad \begin{array}{|c|c|}\hline 3&1\\\hline 2&4\\\hline\end{array}\,(t) \leftrightarrow \begin{array}{|c|c|}\hline 1&3\\\hline 4&2\\\hline\end{array}\,(qt),$$

$$\begin{array}{|c|c|}\hline 3&2\\\hline 1&4\\\hline\end{array}\,(qt) \leftrightarrow \begin{array}{|c|c|}\hline 3&2\\\hline 4&1\\\hline\end{array}\,(q^2t), \qquad \begin{array}{|c|c|}\hline 3&3\\\hline 1&2\\\hline\end{array}\,(t^2) \leftrightarrow \begin{array}{|c|c|}\hline 3&3\\\hline 2&1\\\hline\end{array}\,(qt^2), \qquad \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array}\,(t^2) \leftrightarrow \begin{array}{|c|c|}\hline 3&4\\\hline 2&1\\\hline\end{array}\,(qt^2),$$

$$\begin{array}{|c|c|}\hline 4&1\\\hline 2&3\\\hline\end{array}\,(t) \leftrightarrow \begin{array}{|c|c|}\hline 4&1\\\hline 3&2\\\hline\end{array}\,(qt), \qquad \begin{array}{|c|c|}\hline 4&2\\\hline 1&3\\\hline\end{array}\,(qt) \leftrightarrow \begin{array}{|c|c|}\hline 2&4\\\hline 3&1\\\hline\end{array}\,(q^2t), \qquad \begin{array}{|c|c|}\hline 4&3\\\hline 1&2\\\hline\end{array}\,(qt^2) \leftrightarrow \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline\end{array}\,(q^2t^2)$$

□

We can now prove Proposition 32.

Proof. We begin with the fact that

$$\sum_{\rho \in \mathcal{R}(r)} q^{\operatorname{inv}(\rho)} = \begin{bmatrix} l \\ r \end{bmatrix}_q.$$

Thus the proposition can be shown by giving a family of bijections

$$\sigma \in \mathcal{R}(s) \leftrightarrow \pi \in \mathcal{R}(s)$$

depending on $\rho \in \mathcal{R}(r)$ with the weight-preserving property

$$h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right) = q^{\operatorname{inv}(\rho)} h\left(\begin{array}{|c|}\hline \pi \\ \hline r \\ \hline \end{array}\right).$$

This bijection is given by finding a sequence of s_i which transform ρ into r, applying this sequence to $\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline\end{array}$, and taking the top row of the result. The weight-preserving property is an immediate consequence of Lemma 3. $\square$

This proposition has an interesting corollary, which will not be used in this dissertation, but which we state for its general interest.

Corollary 3. *Let $L = (l^k)$ be a rectangular partition shape, and $s_1, s_2, \cdots, s_k$ be weakly increasing words of length l. Then*

$$\sum_{\substack{\sigma_1 \in \mathcal{R}(s_1)\\ \sigma_2 \in \mathcal{R}(s_2)\\ \cdots \\ \sigma_k \in \mathcal{R}(s_k)}} h(\sigma_k \sigma_{k-1} \cdots \sigma_1, L) = \begin{bmatrix} l \\ s_1 \end{bmatrix}_q \left(\sum_{\sigma_2 \in \mathcal{R}(s_2)} h\left(\begin{array}{|c|}\hline \sigma_2 \\ \hline s_1 \\ \hline\end{array} \right) \Big|_{t \to t^{k-1}} \right)$$
$$\left(\sum_{\sigma_3 \in \mathcal{R}(s_3)} h\left(\begin{array}{|c|}\hline \sigma_3 \\ \hline s_2 \\ \hline\end{array} \right) \Big|_{t \to t^{k-2}} \right) \cdots$$
$$\left(\sum_{\sigma_k \in \mathcal{R}(s_k)} h\left(\begin{array}{|c|}\hline \sigma_k \\ \hline s_{k-1} \\ \hline\end{array} \right) \right)$$

Proof. Following the proof of the proposition, we give a bijection between tuples of words $(\sigma_2 \in \mathcal{R}(s_2), \ldots, \sigma_k \in \mathcal{R}(s_k))$ and tuples of words $(\pi_2 \in \mathcal{R}(s_2), \ldots, \pi_k \in \mathcal{R}(s_k))$, depending on σ_1, with the weight preserving property

$$h((\sigma_k \ldots \sigma_1), L) =$$
$$q^{\mathrm{inv}(\sigma_1)} \left(h\left(\begin{array}{|c|}\hline \pi_2 \\ \hline s_1 \\ \hline\end{array} \right) \Big|_{t \to t^{k-1}} \right) \cdots \left(h\left(\begin{array}{|c|}\hline \pi_k \\ \hline s_{k-1} \\ \hline\end{array} \right) \right)$$

Note that the inversion statistic in $h((\sigma_k \ldots \sigma_1), L)$ can be given by $\mathrm{inv}(\sigma_1)$ plus the number of inversion triples involving σ_1 and σ_2 , plus the number of inversion triples involving σ_2 and σ_3, *etc.*. This fact allows us to construct the bijection with the following procedure.

For each $i \geq 2$, we apply Proposition 32 to the two-row shape $\begin{array}{|c|}\hline \sigma_i \\ \hline \sigma_{i-1} \\ \hline\end{array}$.

This gives us π_i so that the number of inversions in $\begin{array}{|c|}\hline \pi_i \\ \hline s_{i-1} \\ \hline\end{array}$ is the same as the number of inversion triples involving σ_i and σ_{i-1}, without changing the number of descents. $\square$

There is another version of this result, which we will have occasion to use, involving "almost rectangular" two-row shapes of the form $(l-1, l)$:

Corollary 4. *Let s, r be weakly increasing words, with $l(s) = l$, $l(r) = l-1$. Then*

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \;\;\boxed{\quad\rho\quad}\end{matrix}\right) = \begin{bmatrix} l-1 \\ r \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \;\;\boxed{\quad r\quad}\end{matrix}\right).$$

Proof. From the definition of the statistics we have

$$\begin{aligned}\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \;\;\boxed{\quad\rho\quad}\end{matrix}\right) &= \sum_{x\in s}\sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{matrix}\boxed{\quad x\sigma\quad}\\ \;\;\boxed{\quad\rho\quad}\end{matrix}\right)\\ &= \sum_{x\in r} q^{|\{y\in s: x>y\}|}\sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \boxed{\quad\rho\quad}\end{matrix}\right)\end{aligned}$$

where we have set s' to be the word s with x removed. Applying Proposition 32 to this, we have

$$\begin{aligned}\sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \;\;\boxed{\quad\rho\quad}\end{matrix}\right) &= \sum_{x\in r} q^{|\{y\in s: x>y\}|}\begin{bmatrix} l-1 \\ r \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s')} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \boxed{\quad r\quad}\end{matrix}\right)\\ &= \begin{bmatrix} l-1 \\ r \end{bmatrix}_q \sum_{x\in r}\sum_{\sigma\in\mathcal{R}(s')} h\left(\begin{matrix}\boxed{\quad x\sigma\quad}\\ \;\;\boxed{\quad r\quad}\end{matrix}\right)\\ &= \begin{bmatrix} l-1 \\ r \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} h\left(\begin{matrix}\boxed{\quad\sigma\quad}\\ \;\;\boxed{\quad r\quad}\end{matrix}\right).\end{aligned}$$
$\square$

We now have the tools to prove 5.6.

Proof. Let s, r be weakly increasing words of length l, with no 1 in s and a unique

1 in r. Let r' denote r with this 1 removed. Then

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad\rho\quad \\ \hline\end{array}\right) = \begin{bmatrix} l \\ r \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad r\quad \\ \hline\end{array}\right)$$

$$= [l]_q \begin{bmatrix} l-1 \\ r' \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad r\quad \\ \hline\end{array}\right)$$

Since any inversion triple $\begin{matrix} x & & y \\ 1 & & \end{matrix}$ must have $x > y$, we can remove the 1 to get

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad\rho\quad \\ \hline\end{array}\right) = [l]_q \begin{bmatrix} l-1 \\ r' \end{bmatrix}_q \sum_{\sigma\in\mathcal{R}(s)} t\, h\left(\begin{array}{l}\boxed{\quad\quad\sigma\quad\quad} \\ \quad\boxed{\quad\quad r'\quad\quad}\end{array}\right)$$

and Corollary 4 gives

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad\rho\quad \\ \hline\end{array}\right) = t[l]_q \sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho'\in\mathcal{R}(r')}} h\left(\begin{array}{l}\boxed{\quad\quad\sigma\quad\quad} \\ \quad\boxed{\quad\quad \rho'\quad\quad}\end{array}\right)$$

which is 5.6. □

5.6 Distinguished cell in top row

This section is devoted to proving 5.7. In the previous section, the proof of 5.6 hinged on reducing to fillings where the bottom row was weakly increasing. We develop another reduction in this section; that of reducing to diagrams with no descents.

We begin this reduction by first showing that we can assume all descents of a two row shape are located on the left. In what follows, D_k will be the condition that the diagram in question contains exactly k descents, located in the first k columns.

Proposition 33. *Let s, r be two weakly increasing words of length l. Then*

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad\rho\quad \\ \hline\end{array}\right) = \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad\rho\quad \\ \hline\end{array}\right) \chi(D_k).$$

If s' is a word of length $l-1$, we also have

$$\sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{l}\boxed{\qquad\sigma'\qquad}\\ \boxed{\qquad\quad\rho\qquad\quad}\end{array}\right) = \sum_{k=0}^{l-1} \begin{bmatrix} l-1 \\ k \end{bmatrix}_q \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{l}\boxed{\qquad\sigma'\qquad}\\ \boxed{\qquad\quad\rho\qquad\quad}\end{array}\right)\chi(D_k)$$

This is proved by means of another "swap" Lemma. In essence, it says that we can move descents to the right at the cost of introducing an inversion.

Lemma 4. *Let $F = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ satisfy D_1; that is, $a > c$ and $b \leq d$. Then there exists a unique*

$$F' \in \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & a \\ \hline c & d \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & a \\ \hline d & c \\ \hline\end{array} \right\}$$

such that

- *The second column of F' is a descent, and the first column is not, and*
- $\mathrm{inv}(F') = \mathrm{inv}\,(F) + 1$.

Proof. By standardization, we can assume that the two rows have distinct contents. We now check every case:

$$\begin{array}{lll}
\begin{array}{|c|c|}\hline 2&2\\\hline 1&3\\\hline\end{array}(t)\leftrightarrow\begin{array}{|c|c|}\hline 2&2\\\hline 3&1\\\hline\end{array}(qt), &
\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}(t)\leftrightarrow\begin{array}{|c|c|}\hline 2&3\\\hline 4&1\\\hline\end{array}(qt), &
\begin{array}{|c|c|}\hline 3&2\\\hline 1&4\\\hline\end{array}(qt)\leftrightarrow\begin{array}{|c|c|}\hline 3&2\\\hline 4&1\\\hline\end{array}(q^2t), \\
\begin{array}{|c|c|}\hline 4&2\\\hline 1&3\\\hline\end{array}(qt)\leftrightarrow\begin{array}{|c|c|}\hline 2&4\\\hline 3&1\\\hline\end{array}(q^2t), &
\begin{array}{|c|c|}\hline 3&1\\\hline 2&2\\\hline\end{array}(qt)\leftrightarrow\begin{array}{|c|c|}\hline 1&3\\\hline 2&2\\\hline\end{array}(q^2t), &
\begin{array}{|c|c|}\hline 3&1\\\hline 2&4\\\hline\end{array}(t)\leftrightarrow\begin{array}{|c|c|}\hline 1&3\\\hline 4&2\\\hline\end{array}(qt), \\
\begin{array}{|c|c|}\hline 4&1\\\hline 2&3\\\hline\end{array}(t)\leftrightarrow\begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline\end{array}(qt), &
\begin{array}{|c|c|}\hline 4&1\\\hline 3&2\\\hline\end{array}(qt)\leftrightarrow\begin{array}{|c|c|}\hline 1&4\\\hline 3&2\\\hline\end{array}(q^2t) &
\end{array}$$

□

We now prove Proposition 33.

Proof. The lemma allows us to define an action of s_i on fillings of two-row shapes as follows: If columns i and $i+1$ are either both descents or both non-descents, we set $s_i((\sigma, L)) = (\sigma, L)$. If only column i (respectively $i+1$) contains a descent, then we move the descent to column $i+1$ (respectively i) according to Lemma 4.

Now we map an arbitrary filling (σ, L) to a "descent word" $d \in \mathcal{R}(1^k\ 2^{l-k})$ by labeling each column with a 1 (resp. 2) if it is a descent (resp. non-descent). There exists a series of s_i which will transform d to the word $1^k 2^{l-k}$, with the length of this series equal to the number of inversions of d. Applying the same series of s_i to (σ, L) will result in a diagram (σ', L) with all descents on the left and $h(\sigma, L) = q^{\mathrm{inv}(d)} h(\sigma', L)$. This operation is invertible; applying the reverse sequence of s_i will give the original diagram.

Since $\sum_{d \in \mathcal{R}(1^k 2^{l-k})} q^{inv(d)} = \begin{bmatrix} l \\ k \end{bmatrix}_q$, we have

$$\sum_{\substack{\sigma \in \mathcal{R}(s) \\ \rho \in \mathcal{R}(r)}} h\left(\begin{array}{|c|} \hline \sigma \\ \hline \rho \\ \hline \end{array} \right) = \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \sum_{\substack{\sigma \in \mathcal{R}(s) \\ \rho \in \mathcal{R}(r)}} h\left(\begin{array}{|c|} \hline \sigma \\ \hline \rho \\ \hline \end{array} \right) \chi(D_k)$$

The second statement in the Proposition follows immediately; we simply leave the last element of ρ unchanged throughout the whole process. $\square$

We now give an identity which will allow us to reduce to the case where there are no descents whatsoever. Recall that the general situation we are concerned with is that of a filling of the shape (l, l) where the unique 1 is in the top row.

Proposition 34. *If (σ_1, σ_2) is a diagram consisting of only descent columns, (τ_1, τ_2) a diagram consisting of no descent columns with $1 \in \tau_1$, and (ν_1, ν_2) a diagram with no descents and $\nu_1 \in \mathcal{R}(\tau_1 \setminus 1), \nu_2 \in \mathcal{R}(\tau_2)$, then we have*

$$h\left(\begin{array}{|cc|} \hline \sigma_1 & \tau_1 \\ \hline \sigma_2 & \tau_2 \\ \hline \end{array} \right) = q^c h\left(\begin{array}{|cc|c} \hline \sigma_1 & \nu_1 & \multicolumn{1}{|c}{} \\ \hline \sigma_2 & \nu_2 & \multicolumn{1}{c|}{} \\ \hline \end{array} \right)$$

if and only if

$$h\left(\begin{array}{|c|} \hline \tau_1 \\ \hline \tau_2 \\ \hline \end{array} \right) = q^c h\left(\begin{array}{|c|c} \hline \nu_1 & \\ \hline \nu_2 & \multicolumn{1}{c|}{} \\ \hline \end{array} \right)$$

Proof. It is clear that the power of t is correct; we must only check the power of q. The inversions of the LHS (respectively RHS) of the above can be divided into 3 categories:

1. Inversions within $\begin{array}{|c|} \hline \sigma_1 \\ \hline \sigma_2 \\ \hline \end{array}$.

2. Inversions within $\begin{array}{|c|} \hline \tau_1 \\ \hline \tau_2 \\ \hline \end{array} \left(\text{resp. } \begin{array}{|c|c} \hline \nu_1 & \\ \hline \nu_2 & \multicolumn{1}{c|}{} \\ \hline \end{array} \right)$.

3. Inversions between $\begin{array}{|c|}\hline \sigma_1 \\ \hline \sigma_2 \\ \hline \end{array}$ and $\begin{array}{|c|}\hline \tau_1 \\ \hline \tau_2 \\ \hline \end{array}$ $\left(\text{resp. } \begin{array}{|c|c|}\cline{1-1} \nu_1 & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{\nu_2} \\ \hline \end{array}\right)$.

It is clear that we only need to check for equality in (3). The number of such inversions depends only on the content of τ (resp. ν). The only difference in the contents is that τ_1 contains a 1. Thus there is an extra inversion with every element of σ_1 on the LHS, but since the arm of every descent is exactly one shorter on the RHS, this balances out exactly. □

We review the situation at this point, in order to make it clear that we have indeed reduced to the case where there are no descents. Suppose s, r are weakly increasing words of length l, with no 1 in r and exactly one 1 in s. We wish to show

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right) = [l]_q \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|c|}\cline{1-1} \sigma' & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{\rho} \\ \hline \end{array}\right). \tag{5.7}$$

Suppose we can prove this in the case there are no descents. In symbols, assume

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right)\chi(D_0) = [l]_q \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|c|}\cline{1-1} \sigma' & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{\rho} \\ \hline \end{array}\right)\chi(D_0) \tag{5.8}$$

Proposition 35. *The condition in 5.8 implies 5.7.*

Proof. Noting that there can be at most $l-1$ descents, we apply Proposition 33 to the left hand side of 5.7 to get

$$\sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right) = \sum_{k=0}^{l-1} \sum_{\substack{\sigma\in\mathcal{R}(s)\\ \rho\in\mathcal{R}(r)}} \begin{bmatrix} l \\ k \end{bmatrix}_q h\left(\begin{array}{|c|}\hline \sigma \\ \hline \rho \\ \hline \end{array}\right)\chi(D_k)$$

by the assumption and the previous proposition this is

$$= \sum_{k=0}^{l-1} \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} \begin{bmatrix} l \\ k \end{bmatrix}_q [l-k]_q h\left(\begin{array}{|c|c|}\cline{1-1} \sigma' & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{\rho} \\ \hline \end{array}\right)\chi(D_k)$$

$$= \sum_{k=0}^{l-1} \sum_{\substack{\sigma'\in\mathcal{R}(s')\\ \rho\in\mathcal{R}(r)}} [l]_q \begin{bmatrix} l-1 \\ k \end{bmatrix}_q h\left(\begin{array}{|c|c|}\cline{1-1} \sigma' & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{\rho} \\ \hline \end{array}\right)\chi(D_k)$$

and another application of Proposition 33 gives

$$= [l]_q \sum_{\substack{\sigma' \in \mathcal{R}(s') \\ \rho \in \mathcal{R}(r)}} h \left(\begin{array}{|c|} \hline \sigma' \\ \hline \rho \\ \hline \end{array} \right)$$

□

So it remains to prove 5.8. We can simplify this condition even further.

Proposition 36. *The condition 5.8 is equivalent to*

$$\sum_{\sigma \in \mathcal{R}(s)} h \left(\begin{array}{|c|} \hline \sigma \\ \hline r \\ \hline \end{array} \right) \chi(D_0) = \sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} [m_x]_q h \left(\begin{array}{|c|} \hline \sigma' \\ \hline r'x \\ \hline \end{array} \right) \chi(D_0) \tag{5.9}$$

Proof. We begin by taking the coefficient of t^0 on both sides of the equation in Proposition 32, which gives

$$\sum_{\substack{\sigma \in \mathcal{R}(s) \\ \rho \in \mathcal{R}(r)}} h \left(\begin{array}{|c|} \hline \sigma \\ \hline \rho \\ \hline \end{array} \right) \chi(D_0) = \begin{bmatrix} l \\ r \end{bmatrix}_q \sum_{\sigma \in \mathcal{R}(s)} h \left(\begin{array}{|c|} \hline \sigma \\ \hline r \\ \hline \end{array} \right) \chi(D_0). \tag{5.10}$$

This is a simplification of the left hand side of (5.8). We can simplify the right hand side as well. We begin by expanding from the definition of the statistics to get

$$[l]_q \sum_{\substack{\sigma' \in \mathcal{R}(s') \\ \rho \in \mathcal{R}(r)}} h \left(\begin{array}{|c|} \hline \sigma' \\ \hline \rho \\ \hline \end{array} \right) \chi(D_0)$$

$$= [l]_q \sum_{x \in r} \sum_{\substack{\sigma' \in \mathcal{R}(s') \\ \rho' \in \mathcal{R}(r')}} h \left(\begin{array}{|c|} \hline \sigma' \\ \hline \rho' x \\ \hline \end{array} \right) \chi(D_0)$$

$$= [l]_q \sum_{x \in r} \sum_{\substack{\sigma' \in \mathcal{R}(s') \\ \rho' \in \mathcal{R}(r')}} h \left(\begin{array}{|c|} \hline \sigma' \\ \hline \rho' \\ \hline \end{array} \right) q^{|\{y \in r : y > x\}|} \chi(D_0)$$

Applying Proposition 32 to this, we have

$$
\begin{aligned}
&[l]_q \sum_{\substack{\sigma' \in \mathcal{R}(s') \\ \rho \in \mathcal{R}(r)}} h\left(\begin{array}{|c|}\hline \sigma' \\ \hline \rho \\ \hline \end{array}\right)\chi(D_0) \\
&\qquad = [l]_q \sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} \begin{bmatrix} l-1 \\ r' \end{bmatrix}_q h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r' \\ \hline \end{array}\right) q^{|\{y \in r : y > x\}|}\chi(D_0) \\
&\qquad = \sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} [l]_q \begin{bmatrix} l-1 \\ r' \end{bmatrix}_q h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r'x \\ \hline \end{array}\right)\chi(D_0)
\end{aligned}
$$

Denoting the multiplicity of x in r by m_x, this reduces to

$$
\begin{aligned}
&\sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} [l]_q \begin{bmatrix} l-1 \\ m_x - 1, ct(r') \end{bmatrix}_q h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r'x \\ \hline \end{array}\right)\chi(D_0) \\
&= \sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} \begin{bmatrix} l \\ r \end{bmatrix}_q [m_x]_q h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r'x \\ \hline \end{array}\right)\chi(D_0)
\end{aligned}
$$

Thus 5.8 is equivalent to

$$
\begin{aligned}
&\sum_{\sigma \in \mathcal{R}(s)} h\left(\begin{array}{|c|}\hline \sigma \\ \hline r \\ \hline \end{array}\right)\chi(D_0) \\
&\qquad = \sum_{x \in r} \sum_{\sigma' \in \mathcal{R}(s')} [m_x]_q h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r'x \\ \hline \end{array}\right)\chi(D_0)
\end{aligned}
\tag{5.11}
$$

□

We will establish 5.11 bijectively, as follows.

Proposition 37. *Given* $\sigma \in \mathcal{R}(s)$ *with* $\begin{array}{|c|}\hline \sigma \\ \hline r \\ \hline \end{array}$ *having no descents, there exists an* $x \in r$, $p \in \{0, \ldots, m_x - 1\}$ *and* $\sigma' \in \mathcal{R}(s')$ *such that* $\begin{array}{|c|}\hline \sigma' \\ \hline r' \\ \hline \end{array}$ *has no descents and*

$$
h\left(\begin{array}{|c|}\hline \sigma \\ \hline r \\ \hline \end{array}\right) = q^p h\left(\begin{array}{|c|}\hline \sigma' \\ \hline r'x \\ \hline \end{array}\right).
$$

Furthermore the map $\beta : \sigma \to (x, p, \sigma')$ is a bijection.

Here and in the following, s' will mean s with 1 removed, and r' will mean r with x removed.

Before giving this bijection, we must establish some further notation. We begin with a function that decomposes a given a word into the shuffle product of a "small subword" and a "large subword".

Specifically, for σ a word and $x \in \mathbb{N}$, we define the following four functions.

- $\tau_x(\sigma)$ is the word consisting of the letters of σ which are weakly less than x, in the same order as in σ.

- $\tau^x(\sigma)$ is the word consisting of the letters of σ which are strictly greater than x, in the same order as in σ.

- $\omega^x(\sigma) \in \mathcal{R}(1^{|\tau_x(\sigma)|}, 2^{|\tau^x(\sigma)|})$ and gives the positions of the elements of $\tau_x(\sigma)$ and $\tau^x(\sigma)$ in σ.

- $\phi_x(\sigma)$ is the triple $(\tau_x(\sigma), \tau^x(\sigma), \omega^x(\sigma))$.

Example 17. $\phi_3(5234134) = (2313, 544, 2112112)$

It is clear that we can restruct the original word from such a triple. Thus we denote by $\phi^{-1}(\tau_x, \tau^x, \omega^x)$ the word constructed by replacing the 1's in ω^x with the elements of τ_x, and the 2s by the elements of τ^x.

Example 18. $\phi^{-1}(2313, 544, 2112112) = (5234134)$

We note that the number of inversions of a word can be read from the image under ϕ_x:

Proposition 38. *We have*

$$\text{inv}(\sigma) = \text{inv}(\tau_x(\sigma)) + \text{inv}(\tau^x(\sigma)) + \text{inv}(\omega^x(\sigma))$$

Proof. This is immediate from the definition of ϕ_x. □

In what follows, if σ is clear from the context, we will write τ_x for $\tau_x(\sigma)$, τ^x for $\tau^x(\sigma)$ and ω^x for $\omega^x(\sigma)$.

The *left circular shift* ($\hookleftarrow$) of a word of length n is constructed by moving the element in position i to position $i-1$, and the element in position 1 to position n.

Example 19. $\hookleftarrow(2112112) = 1121122$

We now define a map which shifts the "small subword" relative to the "large subword", but leaves each subword unchanged. Precisely, we define Φ_x from words to words by

$$\Phi_x : \sigma \xmapsto{\phi_x} (\tau_x, \tau^x, \omega^x) \xmapsto{\omega' = \hookleftarrow(\omega^x)} (\tau_x, \tau^x, \omega') \xmapsto{\phi^{-1}} \Phi_x(\sigma)$$

Example 20. $\Phi_3(5234134) = (2351344)$

The following property of $\hookleftarrow(\omega^x)$ will be useful:

Proposition 39. *If* ${\omega^x}_1 = 1$*, then the positions of the* 1*'s in* $\hookleftarrow(\omega^x)$ *are weakly greater than the positions of the* 1*'s in* ω^x*, and the positions of the* 2*'s in* $\hookleftarrow(\omega^x)$ *are exactly one less than the positions of the* 2*'s in* ω^x*. The analogous statement holds when* ${\omega^x}_1 = 2$*.*

Proof. This follows from the definition of $\hookleftarrow$. ☐

The significance of the somewhat mysterious map Φ_x is hinted at by its relation to the inversion statistic. This is explained in the following proposition.

Proposition 40. *If* $\sigma_1 > x$*, then*

$$\operatorname{inv}(\Phi_x(\sigma)) = \operatorname{inv}(\sigma) - |\tau_x|.$$

On the other hand, if $\sigma_1 \leq x$*, then*

$$\operatorname{inv}(\Phi_x(\sigma)) = \operatorname{inv}(\sigma) + |\tau^x|.$$

Proof. Based on Proposition 38, we must only consider the relationship between the inversions of ω^x and $\hookleftarrow(\omega^x)$. If ω^x begins with a 1, we are adding $|\tau^x|$ inversions when we move that 1 to the end, and all other inversions remain unchanged. Similarly, if ω^x begins with a 2, we are losing $|\tau_x|$ inversions when we move that 2 to the end, and all other inversions will remain unchanged. ☐

We now give the bijection β from σ to (x, p, σ') subject to the conditions in Proposition 37. We begin by setting i to be the position of 1 in σ. We consider three cases, and provide an example of each case after describing the procedure. The three cases are as follows

1. The number i satisfies $i \leq l - i + 1$ and there is no $j \geq i$ with $\sigma_{j+1} > r_j$.

2. The number i satisfies $i > l - i + 1$ and there is no $j \geq i$ with $\sigma_{j+1} > r_j$.

3. There exists a $j \geq i$ with $\sigma_{j+1} > r_j$.

In each case, we must show the following three properties:

1. The condition D_0 is preserved: *i.e.,* the diagram $\begin{array}{|c|}\hline \quad\sigma'\quad \\ \hline r' \\ \hline\end{array}$ has no descents.

2. The q statistic is properly affected: *i.e.,*

$$h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline r \\ \hline\end{array}\right) = q^p h\left(\begin{array}{|c|c|}\hline \quad\sigma'\quad & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{|c|}{r'x} \\ \hline\end{array}\right)$$

Case 1: If $l - i + 1 \geq i$ and there is no $j \geq i$ with $\sigma_{j+1} > r_j$, we proceed by the following algorithm.

`Step 1:` Let $k = l - i + 1$, and $x = r_k$.

`Step 2:` Let p be the number of x's to the right of r_k. That is, we should have $r_{k+p} = x$ and $r_{k+p+1} \neq x$.

`Step 3:` Split σ into two words μ, ν at position k. Precisely, $\mu = (\sigma_1, \ldots, \sigma_k)$, and $\nu = (\sigma_{k+1}, \ldots, \sigma_l)$.

`Step 4:` Set $\sigma' = (\mu \setminus 1)\Phi_x(\nu)$.

Example 21.

$$\begin{array}{|c|c|c|c|c|c|}\hline 2&3&1&3&2&6\\\hline 2&3&4&5&6&7\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|}\cline{1-5} 2&3&3&6&2&\multicolumn{1}{c}{}\\\hline 2&3&4&6&7&5\\\hline\end{array}$$

with $p = 0$.

We have $i = 3, k = l - i + 1 = 4$. Hence $x = r_4 = 5$, and $p = 0$ since $r_5 \neq 5$. We set $\mu = (231)$, and $\nu = (326)$. Since $\Phi_5(326) = 362$, we have the situation depicted above.

Lemma 5. *In Case 1, the property D_0 is preserved.*

Proof. In symbols, we show that for all $1 \leq j \leq n-1$, we have $\sigma'_j \leq r'_j$. We consider three possibilities for j.

$j < i$: In this case, $\sigma'_j = \mu_j = \sigma_j \leq r_j = r'_j$, as desired.

$i \leq j < k$: In this case, $\sigma'_j = \mu_j = \sigma_{j+1}$ since the 1 has been removed. We have $\sigma_{j+1} \leq r_j$ by the condition of Case 1, and $r_j = r'_j$. Combining these gives the desired inequality.

$j \geq k$: We note that $r'_j = r_{j+1} \geq x$. If $x \geq \sigma'_j$, we are done. If $\sigma'_j > x$, then $\sigma'_j = \sigma_{j+2}$ by Proposition 39 (σ_{j+2} is moved one space to the left by the removal of 1, and one more space to the left by Φ_x). Now, $\sigma_{j+2} \leq r_{j+1}$ by the condition of Case 1, and since $r_{j+1} = r'(j)$ we are done.

□

Lemma 6. *In Case 1, we have*

$$h\left(\begin{array}{|c|}\hline \quad\quad\sigma\quad\quad \\ \hline \quad\quad r\quad\quad \\ \hline\end{array}\right) = q^p h\left(\begin{array}{|c|c}\hline \quad\quad\sigma'\quad\quad & \multicolumn{1}{|c}{} \\ \hline \multicolumn{2}{|c|}{r'x} \\ \hline\end{array}\right)$$

Proof. For a two-row diagram $d = (\sigma, \rho)$ with no descents, we have $\mathrm{inv}(d) = \mathrm{inv}(\rho) + \mathrm{inv}(\sigma) + \mathrm{diag}(\sigma, \rho)$. Here $\mathrm{diag}(\sigma, \rho)$ means the number of pairs (σ_i, ρ_j) with $i > j$ and $\sigma_i > \rho_j$. In what follows we will also use the notation $\mathrm{diag}(\sigma, \rho_k)$ for the number of indices i such that $i > k$ and $\sigma_i > \rho_k$. Furthermore, for general words w, v, we write $\mathrm{inv}(w, v)$ for the number of pairs (i, j) with $w_i > v_j$.

We have the following facts:

$$\begin{aligned}
\mathrm{inv}(\mu) &= \mathrm{inv}(\mu') + (i-1) \\
-\mathrm{diag}(\sigma, r_k) + \mathrm{inv}(\nu, x) &= 0 \\
\mathrm{inv}(\nu, x) &= |\tau^x(\nu)| \\
\mathrm{inv}(\Phi_x(\nu)) &= \mathrm{inv}(\nu) + |\tau^x(\nu)| \\
\mathrm{inv}\left(\mu', \Phi_x(\nu)\right) &= \mathrm{inv}(\mu, \nu) \\
\mathrm{inv}(r', x) &= (l - (k+p)) \\
\mathrm{diag}(\sigma, r) - \mathrm{diag}(\sigma, r_k) &= \mathrm{diag}(\sigma', r').
\end{aligned}$$

Of these, the only one which is not immediate is the last; this follows from the fact that there are no descents in either (σ, r) or in $\sigma', r')$. Thus we may compute

$$\begin{aligned}\operatorname{inv}(\sigma r, (l,l)) &= \operatorname{inv}(\mu) + \operatorname{inv}(\nu) + \operatorname{inv}(\mu,\nu) + \operatorname{diag}(\sigma, r)\\ &= \operatorname{inv}(\mu) + \operatorname{inv}(\nu) + \operatorname{inv}(\mu,\nu) + \operatorname{diag}(\sigma, r) - \operatorname{diag}(\sigma, r_k) + \operatorname{inv}(\nu, x)\\ &= \operatorname{inv}(\mu') + (i-1) + \operatorname{inv}(\nu) + \operatorname{inv}(\mu,\nu) + \operatorname{diag}(\sigma', r') + |\tau^x(\nu)|.\end{aligned}$$

We also have

$$\begin{aligned}\operatorname{inv}(\sigma' r', (l,l)) &= \operatorname{inv}(\mu') + \operatorname{inv}(\Phi_x(\nu)) + \operatorname{inv}(\mu', \Phi_x(\nu)) + \operatorname{diag}(\sigma', r') + \operatorname{inv}(r', x)\\ &= \operatorname{inv}(\mu') + \operatorname{inv}(\nu) + |\tau^x(\nu)| + \operatorname{inv}(\mu,\nu) + \operatorname{diag}(\sigma', r') + (l-(k+p)).\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{inv}(\sigma r, (l,l)) - \operatorname{inv}(\sigma' r', (l,l)) &= (i-1) - (l-(k+p))\\ &= (i-1) - l + (l-i+1) + p\\ &= p\end{aligned}$$

as required. □

We now describe the algoritm in Case 2. If $l-i+1 < i$ and there is no $j > i$ with $\sigma_j > r_{j-1}$, we proceed as follows:

`Step 1:` Let $k = n-i+1$, and $x = r_k$.

`Step 2:` Let p the number of x's to the right of r_k. That is, choose p so that $r_{k+p} = x$ and $r_{k+p+1} \neq x$.

`Step 3:` Split σ into two word μ, ν at position $k-1$. Precisely, set $\mu = (\sigma_1, \ldots, \sigma_{k-1})$, $\nu = (\sigma_k, \ldots, \sigma_l)$.

`Step 4:` Let $\sigma' = \mu\Phi_x(\nu \setminus 1)$.

Example 22.

$$\begin{array}{|c|c|c|c|c|c|}\hline 2&3&4&1&5&2\\\hline 2&3&4&5&6&7\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|}\hline 2&3&5&4&2&\\\hline 2&3&5&6&7&4\\\hline\end{array}$$

with $p = 0$.

We have $i = 4, k = l-i+1 = 3$, $x = r_3 = 4$, and $p = 0$. So $\mu = (23)$, and $\nu = (4152)$. $\Phi_4(452) = (542)$, so we have the situation depicted above.

Lemma 7. *In Case 2, the property D_0 is preserved.*

Proof. In symbols, we show that for all $1 \leq j \leq n-1$, we have $\sigma'_j \leq r'_j$. We consider two possibilities for j.

$j < k$: In this case $\sigma_j = \sigma'_j$ and $r_j = r'_j$, so we have $\sigma'_j = \sigma_j \leq r_j = r'_j$.

$j \geq k$: Note that $r'_j = r_{j+1} \geq x$. If $x \geq \sigma'_j$, we are done. Suppose instead that $\sigma'_j > x$. Then σ'_j is part of $\tau^x(\nu)$. Now, $\omega^x(\nu)$ begins with a 1, since $\nu_1 = \sigma_k < r_k = x$. Therefore, by Proposition 39 we have either $\sigma'_j = \sigma_{j+1}$ (if $j < i$) or $\sigma'_j = \sigma_{j+2}$ (if $j \geq i$). In either case we have $\sigma'_j \leq r_{j+1} = j'_j$, as desired.

□

Lemma 8. *In Case 2, we have*

$$h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline \quad r\quad \\ \hline\end{array}\right) = q^p h\left(\begin{array}{|c|c|}\hline \quad\sigma'\quad & \multicolumn{1}{|c}{} \\ \hline \multicolumn{2}{|c|}{r'x} \\ \hline\end{array}\right)$$

Proof. Similar to Case 1, we have

$$\begin{aligned}
\operatorname{inv}(\sigma r,(l,l)) &= \operatorname{inv}(\mu)+\operatorname{inv}(\nu)+\operatorname{inv}(\mu,\nu)+\operatorname{diag}(\sigma,r)\\
&= \operatorname{inv}(\mu)+\operatorname{inv}(\nu)+\operatorname{inv}(\mu,\nu)+\operatorname{diag}(\sigma,r)-\operatorname{diag}(\sigma,r_k)+\operatorname{inv}(\nu,x)\\
&= \operatorname{inv}(\mu)+(i-1)+\operatorname{inv}(\nu')+\operatorname{inv}(\mu,\nu')+\operatorname{diag}(\sigma',r')+|\tau^x(\nu)|.
\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{inv}(\sigma' r' x,(l,l-1)) &= \operatorname{inv}(\mu)+\operatorname{inv}\left(\Phi_x(\nu')\right)+\operatorname{inv}\left(\mu,\Phi_x(\nu')\right)+\operatorname{diag}(\sigma',r')\\
&\quad +\operatorname{inv}(r',x)\\
&= \operatorname{inv}(\mu)+\operatorname{inv}(\nu')+|\tau^x(\nu')|+\operatorname{inv}(\mu,\nu')\\
&\quad +\operatorname{diag}(\sigma',r')+(l-(k+p)).
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{inv}(\sigma r,(l,l))-\operatorname{inv}(\sigma' r',(l,l)) &= (i-1)-(l-(k+p))\\
&= (i-1)-l+(l-i+1)+p\\
&= p
\end{aligned}$$

as required. □

We now describe the algorithm in Case 3. If there does exist some $j > i$ such that $\sigma_j > r_{j-1}$, we let j' be minimal such that $j' > i$ and $\sigma_{j'} > r_{j'-1}$. We then proceed as follows:

`Step 1:` Let $k = j' - i$ and set $x = r_k$.

`Step 2:` As before, let p be the number of x's to the right of r_k. So again we have $r_{k+p} = x$ and $r_{k+p+1} \neq x$.

`Step 3:` In this case, we must divide σ into three subwords. Precisely, we let $\mu = (\sigma_1, \ldots, \sigma_k)$, $\nu = (\sigma_{k+1}, \ldots, \sigma_{j'-1})$, and $\xi = (\sigma_{j'}, \ldots, \sigma_l)$. Note that ν may be empty, and we don't know if 1 is in μ or ν.

`Step 4:` We set $\sigma' = (\mu \setminus 1)\Phi_x(\nu \setminus 1)\Phi_x(\xi)$.

Example 23.

2	3	1	3	6	2
2	3	4	5	6	7

$\longrightarrow$

2	3	3	2	6	
2	4	5	6	7	3

with $p = 0$.

We have $i = 3$, and $\sigma_5 > r_4$, so $j' = 5$. Thus $k = 5 - 3 = 2$, and $x = r_2 = 3$. Since $r_3 \neq 3$, $p = 0$. We divide σ into $\mu = (23), \nu = (13)$ and $\xi = (62)$. We have $\Phi_3(3) = (3), \Phi_3(62) = (26)$, so $\sigma' = (23326)$.

Lemma 9. *In Case 3, the property D_0 is preserved.*

Proof. We wish to show that $\sigma'_j \leq r_j$ for all j. We first consider what happens to the letters in μ. Suppose that $j < k$ and also $j < i$. In this case, $\sigma'_j = \sigma_j$ and $r'_j = r_j$, and since $\sigma_j \leq r_j$, there is nothing else to show. If instead, $j < k$ but $j \geq i$, we have $j < k < j'$. In particular, $\sigma_{j+1} \leq r_j$, by the condition of Case 3. Since $\sigma'_j = \sigma_{j+1}$ (because $\sigma_{j+1} \in \mu$) and $r_j \leq r'_j$, we have $\sigma'_j \leq r'_j$ in this case.

Now we consider the letters in ν. If $k \geq j < j' - 1$, we note that $r'_j = r_{j+1} \geq x$ and consider three subcases:

$\sigma'_j \leq x$: Since $x \leq r'_j$, we are done.

$\sigma'_j > x$: If $j < i-1$, we have $\sigma'_j = \sigma_{j+1}$ since $\omega^x(\nu)$ begins with a 1. Since $\sigma_{j+1} \leq r_{j+1} = r'_j$, we are done. On the other hand, if $j \geq i-1$, we have $\sigma'_j = \sigma_{j+2}$, since we must also account for the removal of 1. By the condition of Case 3, $\sigma_{j+2} \leq r_{j+1} = r'_j$.

The final case to consider is that of the letters in ξ. This is done by examining the case $j \geq j'-1$. We note that $r'_j = r_{j+1} \geq r_{j'} \geq x$, and consider two subcases:

$\sigma_j \leq x$: Here we must have $\sigma'_j \leq x < r'_j$.

$\sigma'_j > x$: Since $\omega^x(\xi)$ begins with a 2, by Proposition 39, we have $\sigma'_j = \sigma_m$ for some $m \leq j$. Thus $\sigma'_j = \sigma_m \leq r_m \leq r_{j+1} = r'_j$.

□

Lemma 10. *In Case 3, we have*

$$h\left(\begin{array}{|c|}\hline \quad\sigma\quad \\ \hline r \\ \hline\end{array}\right) = q^p h\left(\begin{array}{|l|}\cline{1-1} \quad\sigma'\quad \\ \hline \quad r'x \qquad \\ \hline\end{array}\right)$$

Proof. We have the following calculation:

$$\begin{aligned}
\operatorname{inv}(\sigma r,(l,l)) &= \operatorname{inv}(\mu)+\operatorname{inv}(\nu)+\operatorname{inv}(\mu,\nu)+\operatorname{inv}(\xi)+\operatorname{inv}(\mu,\xi)\\
&\quad +\operatorname{inv}(\nu,\xi)+\operatorname{diag}(\sigma,r)\\
&= \operatorname{inv}(\mu)+\operatorname{inv}(\nu)+\operatorname{inv}(\mu,\nu)+\operatorname{inv}(\xi)+\operatorname{inv}(\mu,\xi)\\
&\quad +\operatorname{inv}(\nu,\xi)+\operatorname{diag}(\sigma,r)-\operatorname{diag}(\sigma,r_k)+\operatorname{inv}(\nu,x)+\operatorname{inv}(\xi,x)\\
&= (i-1)+\operatorname{inv}(\mu')+\operatorname{inv}(\nu')+\operatorname{inv}(\mu',\nu')+\operatorname{inv}(\xi)\\
&\quad +\operatorname{inv}(\mu',\xi)+\operatorname{inv}(\nu',\xi)+\operatorname{diag}(\sigma',r')+|\tau^x(\nu)|+|\tau^x(\xi)|
\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{inv}(\sigma' r' x,(l,l-1)) &= \operatorname{inv}(\mu')+\operatorname{inv}\left(\Phi_x(\nu')\right)+\operatorname{inv}\left(\Phi_x(\xi)\right)+\operatorname{inv}\left(\mu',\Phi_x(\nu')\right)\\
&\quad +\operatorname{inv}\left(\mu',\Phi_x(\xi)\right)+\operatorname{inv}\left(\Phi_x(\nu'),\Phi_x(\xi)\right)+\operatorname{diag}(\sigma',r')+\operatorname{inv}(r',x)\\
&= \operatorname{inv}(\mu')+\operatorname{inv}(\nu')+|\tau^x(\nu')|+\operatorname{inv}(\xi)-|\tau_x(\xi)|+\operatorname{inv}(\mu',\nu')\\
&\quad +\operatorname{inv}(\mu',\xi)+\operatorname{inv}(\nu',\xi)+\operatorname{diag}(\sigma',r')+(l-(k+p))
\end{aligned}$$

Taking the difference gives

$$\begin{aligned}\operatorname{inv}(\sigma r,(l,l)) - \operatorname{inv}(\sigma' r' x) &= (i-1) + |\tau^x(\xi)| + |\tau_x(\xi)| - l + k + p \\ &= i - 1 + |\xi| - l + j' - i + p \\ &= |\xi| - (l - (j'-1)) + p \\ &= p\end{aligned}$$

as required. □

We are now able to prove Proposition 37.

Proof. The weight preserving property of β is established by Lemmas 5, 6, 7, 8, 9, and 10. Thus we must only verify that β is invertible. Given $(\sigma', r'x, p)$ we first find k by requiring that $k + p$ is the position of the last x in r. We then observe whether or not the last element of σ' is greater than x. If so, we must be in case 3. If not, we are in either case 1 or 2. We can determine which of these two cases we are in by setting $i = l - k + 1$ and determining whether or not $i > k$. Once we know k and the case we are in, it is straightforward to invert the map. □

This establishes 5.8, which by Proposition 35 implies 5.7. We now have all the tools necessary to show Theorem 5.

Proof. In Section 5.5 we proved 5.6, and in Section 5.6 we established 5.7. This completes the proof of Lemma 2, which in turn establishes 5.1. Replacing 1 by n in this result establishes 5.2, and this completes the proof. □

6 Further Research

As we have seen, there are many open combinatorial problems in this area. One of the most obvious continuations of the results here would to be to attempt the establish the existence of the $G_L[X;q,t]$ on a larger class of shapes; particularly skew shapes and shapes consisting of a Young diagram with a single cell removed. It has been verified that the statistics of Haglund do not give the correct polynomials, so another approach is necessary. The hope is that a combinatorial description of the $G_L[X;q,t]$ will satisfy some of the deficiencies of the $C_L[X;q,t]$.

Another continuation of this work would be to further explore the implications of the relationships of the L-multinomial coefficients. For example, if a "column version" of Corollary 3 could be found, this could be used to establish the "transpose symmetry" of the $C_\mu[X;q,t]$ in the case where the diagram of μ is a rectangle.

Bibliography

[BBG$^+$99] Franois Bergeron, Nantel Bergeron, Adriano M. Garsia, Mark Haiman, and Glenn Tesler. Lattice diagram polynomials and extended Pieri rules. *Adv. Math.*, 142(2):244–334, 1999.

[Ful91] William Fulton. *Young tableaux. With applications to representation theory and geometry.* Cambridge University Press, Cambridge, United Kingdom, 1991.

[GH93] Adriano M. Garsia and Mark Haiman. A graded representation model for Macdonald's polynomials. *Proc. Nat. Acad. Sci. USA*, 90(8):3607–3610, 1993.

[GH95] Adriano M. Garsia and Mark Haiman. Factorizations of Pieri rules for Macdonald polynomials. *Discrete Math*, 139(1-3):219–256, 1995.

[GH96a] Adriano M. Garsia and Mark Haiman. A remarkable q, t-Catalan sequence and q-Lagrange inversion. *J. Alg. Combin.*, 5(3):191–244, 1996.

[GH96b] Adriano M. Garsia and Mark Haiman. Some natural bigraded S_n-modules and q, t-Kostka coefficients. *Electron. J. Combin.*, 3(2):60 pp, 1996.

[GH98] Adriano M. Garsia and Mark Haiman. A random q, t-hook walk and a sum of Pieri coefficients. *J. Combin. Theory Ser. A*, 82(1):74–111, 1998.

[Hag04] Jim Haglund. A combinatorial model for the Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 101(46):16127–16131 (electronic), 2004.

[Hai99] Mark Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[Hai01] Mark Haiman. Hilbert schemes, polygraphs, and the Macdonald positivity conjecture. *Jour. Amer. Math. Soc.*, 14:941–1006, 2001.

[HHL05] Jim Haglund, Mark Haiman, and Nick Loehr. A combinatorial formula for Macdonald polynomials. *Jour. Amer. Math. Soc.*, 18:735–761, 2005.

[Mac] Ian G. Macdonald. *Symmetric Functions and Hall Polynomials*. Second edition edition.

[Sta97] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, Cambridge, United Kingdom, 1997.

[Sta99] Richard P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, United Kingdom, 1999.