Title
Consequences of Weyl consistency conditions

Journal
Journal of High Energy Physics, 2013(11)

ISSN
1126-6708

Authors
Grinstein, Benjamín
Stergiou, Andreas
Stone, David

Publication Date
2013-11-01

DOI
10.1007/jhep11(2013)195

Peer reviewed

UCSD-PTH-13-12

Consequences of Weyl Consistency Conditions

Benjamín Grinstein, Andreas Stergiou and David Stone

Department of Physics, University of California, San Diego, La Jolla, CA 92093, USA

The running of quantum field theories can be studied in detail with the use of a local renormalization group equation. The usual beta-function effects are easy to include, but by introducing spacetime-dependence of the various parameters of the theory one can efficiently incorporate renormalization effects of composite operators as well. An illustration of the power of these methods was presented by Osborn in the early 90s, who used consistency conditions following from the Abelian nature of the Weyl group to rederive Zamolodchikov's c-theorem in $d = 2$ spacetime dimensions, and also to obtain a perturbative a-theorem in $d = 4$. In this work we present an extension of Osborn's work to $d = 6$ and to general even d. We compute the full set of Weyl consistency conditions, and we discover among them a candidate for an a-theorem in $d = 6$, similar to the $d = 2, 4$ cases studied by Osborn. Additionally, we show that in any even spacetime dimension one finds a consistency condition that may serve as a generalization of the c-theorem, and that the associated candidate c-function involves the coefficient of the Euler term in the trace anomaly. Such a generalization hinges on proving the positivity of a certain "metric" in the space of couplings.

August 2013

bgrinstein@ucsd.edu, (stergiou, dcstone)@physics.ucsd.edu

1. Introduction

When a symmetry of a quantum field theory (QFT) is broken by quantum corrections, then the corresponding anomaly can be reproduced by a contribution to the generating functional of the theory [1, 2]. The algebra of the symmetry that is violated constrains the symmetry-breaking parameters that appear in these anomalous contributions, which are thus forced to satisfy the so-called Wess–Zumino consistency conditions [1].

The study of the Wess–Zumino consistency conditions for the Weyl anomaly was undertaken by Osborn in the early 90s and produced remarkable results [3]. In $d = 2$ spacetime dimensions, for example, Osborn obtained an independent proof of Zamolodchikov's c-theorem [4]. Furthermore, an extension of the c-theorem to 4d, commonly referred to as the a-theorem, was demonstrated perturbatively [3,5], establishing in perturbation theory the intuition that the number of massless degrees of freedom of a QFT decreases under renormalization-group (RG) flow.

This perturbative 4d result was based on previous work by Jack and Osborn [5], who computed the local RG equation for a general renormalizable QFT in a curved background using dimensional regularization. To account for effects of renormalization of composite operators, Jack and Osborn used spacetime-dependent coupling constants, a trick that allows for straightforward computations of Green functions of composite operators (at least of those that appear in the Lagrangian) and the stress-energy tensor. Their candidate c-function agrees with Cardy's suggestion [6]: it is equal to the coefficient a of the Euler term in the trace anomaly at fixed points of the RG-flows. Nevertheless, it differs from a if the corresponding flat-space theory is not a conformal field theory (CFT). In subsequent work Osborn reproduced the main results of [5] by requiring that two successive Weyl variations of the effective action commute (since the Weyl group is Abelian), and also showed that the main results of the analysis are scheme-independent [3].

Invariance under the Weyl group in the flat background limit is *a priori* a stronger requirement than that of scale invariance of a theory on a flat background. The former imposes the vanishing of the trace of the stress-energy tensor, while the latter requires that the trace be the divergence of a suitable vector operator [7]. This suggests that the study of Weyl deformations may elucidate the relation between scale and conformally invariant theories in flat backgrounds. Indeed, these methods have recently been used to show, in perturbation theory, that a unitary 4d QFT invariant under the Poincaré group extended by the generator of scale transformations is automatically invariant under the four generators of special conformal transformations [8], even though the parameters of the theory may display cyclic behavior [9].[1]

In this paper we undertake an investigation of the response of QFTs to Weyl transformations

[1] The same result was also obtained using different methods in [10]. In renormalizable theories with $\mathcal{N} = 1$ supersymmetry it was shown perturbatively that cyclic behavior of parameters does not arise [11]. The situation in $d = 4 - \epsilon$ needs further investigation [12].

in $d = 6$.[2] In particular, we determine the Weyl consistency conditions and general RG properties of six-dimensional QFTs. It is evident that this line of inquiry is interesting if it leads to results similar to those already obtained in $d = 4$, say, a perturbative extension of the 2d c-theorem and a proof that scale implies conformal invariance in 6d. But the investigation is also interesting in light of the advent of strongly coupled conformal CFTs that lack a Lagrangian description in $d = 6$, like the famous $(2,0)$ theory.[3] This suggests the existence of flows, i.e. families of non-conformal QFTs, between such theories, with an associated flow of a presumed c-function. The class of perturbative, renormalizable, $d = 6$ models is restricted to scalar fields with cubic interactions for which the general analysis is of limited interest. They could, however, be put to use as a check of results using standard methods of perturbation theory.

As we will see, the consistency conditions bring us very close to an extension of the c-theorem to $d = 6$. More specifically, one can define a quantity $\tilde{a}$, which is a function of the dimensionless coupling constants g^i, that satisfies the equation

$$\frac{d\tilde{a}}{dt} = -\mathcal{H}_{ij}\beta^i\beta^j, \tag{1.1}$$

where the RG time is $t = -\ln(\mu/\mu_0)$, taken here to increase as we flow to the IR, and $\beta^i = -dg^i/dt$, as usual. The quantity $\tilde{a}$ agrees with the coefficient of the Euler term in the 6d trace anomaly at fixed points. The symmetric tensor $\mathcal{H}_{ij}$ can be viewed as a metric in the space of couplings. A proof of the a-theorem would be immediate if $\mathcal{H}_{ij}$ were shown to be positive-definite. This is analogous to the situation in $d = 4$ where perturbative positivity of the analogous "metric" has been shown by explicit computation in a generic QFT—it is here and only here that perturbation theory is used in proving the a-theorem and that scale implies conformal invariance in $d = 4$.

The analysis of the Weyl consistency conditions in $d = 6$ is significantly more complicated than in $d = 4$. This analysis reveals generic features that were not apparent in Osborn's treatment, and actually allows us to demonstrate the validity of (1.1) for QFTs in any even-dimensional spacetime. The only ingredient missing for a generalization of Zamolodchikov's c-theorem to any even dimension is a demonstration that the "metric" $\mathcal{H}_{ij}$ is positive-definite. As already mentioned, $\mathcal{H}_{ij}$ is positive-definite in $d = 4$ at lowest order in perturbation theory. It would be interesting to extend this result to higher even dimensions. Of course, a non-perturbative proof of the positivity of $\mathcal{H}_{ij}$ in even $d \geq 4$ is the ultimate goal of this line of research.

To address questions similar to those that motivated this work, Komargodski and Schwimmer have put forward an argument that gives a non-perturbative physicist's proof of the weak version[4]

[2] Weyl consistency conditions in $d = 3$ were studied recently in [13].

[3] The consistency conditions we derive can be seen as relations among correlation functions involving composite operators.

[4] For a discussion of the various versions of the a-theorem see [14].

of the a-theorem in $d = 4$ [15]. More specifically, they provided a compelling argument that in the flow from a UV CFT to an IR CFT the inequality $a_{\rm UV} > a_{\rm IR}$ is satisfied, without however providing a monotonically-decreasing c-function. Attempts to extend that line of reasoning to the $d = 6$ case have been unsuccessful, although in explicit examples the validity of $a_{\rm UV} > a_{\rm IR}$ was demonstrated [16]. The method we use in this work is very different from that used in [15, 16], and allows us to obtain results local in the RG scale.

The organization of the paper is as follows. In the next section we summarize the results of Osborn in 2d and 4d. We continue in Section 3 by describing the 6d case in detail, and we then illustrate in Section 4 the ingredients that allow us to genaralize our analysis regarding the a-theorem to all even spacetime dimensions. In Appendices A and B we present details regarding our conventions as well as the terms that participate in the consistency conditions in 6d. A Mathematica file that contains all the consistency conditions in 6d is included with our submission.

2. Summary of the 2d and 4d cases

To begin, let us introduce the basic setting. More details can be found in [3, 17]. We are working in Euclidean space and we define the generating functional W of connected Green functions via

$$e^W = \int [d\phi]\, e^{-S},$$

where S is the Euclidean action with all required counterterms. S contains a potential of the form $g^i\mathcal{O}_i$, where g^i are parameters which can be taken to be dimensionless, and $\mathcal{O}_i$ are scaling-dimension-d operators where d is the spacetime dimension.[5] W is a function of the renormalized couplings g^i and the metric $\gamma_{\mu\nu}$.

Consider now the RG flow as a flow in the space of theories as parametrized by their couplings g^i. The arbitrary RG parameter μ has to be introduced, and the flow is then generated by

$$\mathscr{D} = \mu\frac{\partial}{\partial\mu} + \beta^i\frac{\partial}{\partial g^i},$$

where $\beta^i = \mu\, dg^i/d\mu$ is the beta function. W is a finite scalar function, since it is derived from S which includes all necessary counterterms, and it is thus invariant under the RG flow:

$$\mathscr{D}W = 0. \tag{2.1}$$

This is simply the Callan–Symanzik equation.

To define a local RG equation we let the parameters of S as well as the spacetime metric be arbitrary functions of spacetime. New counterterms involving derivatives on the metric and the

[5] Non-marginal operators can also be included, see [3].

couplings are then necessary for finiteness. With their inclusion in S functional differentiations of W are guaranteed to produce finite operator-insertions in Green functions. Local rescalings of length are described by

$$\gamma^{\mu\nu}(x) \to e^{2\sigma(x)}\gamma^{\mu\nu}(x)$$

and they form the Weyl group.

We define the quantum stress-energy tensor and finite composite operators using

$$T_{\mu\nu}(x) = 2\frac{\delta S}{\delta\gamma^{\mu\nu}(x)}, \qquad [\mathcal{O}_i(x)] = \frac{\delta S}{\delta g^i(x)},$$

where functional derivatives are defined in d spacetime dimensions by

$$\frac{\delta}{\delta\gamma^{\mu\nu}(x)}\gamma^{\kappa\lambda}(y) = \tfrac{1}{2}\delta_{(\mu}{}^{\kappa}\delta_{\nu)}{}^{\lambda}\,\delta^d(x,y), \qquad \frac{\delta}{\delta g^i(x)}g^j(y) = \delta_i^{\,j}\,\delta^d(x,y),$$

with $X_{(i}Y_{j)} \equiv X_iY_j + X_jY_i$, $\delta^d(x,y) = \delta^d(x-y)/\sqrt{\gamma(x)}$, γ is the determinant of the metric, and $\delta^d(x)$ the usual delta function in d dimensions. At the level of the generating functional we implement infinitesimal local Weyl transformations with the generators

$$\Delta_\sigma^{\mathrm{W}} = 2\int d^dx\sqrt{\gamma}\,\sigma\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}, \qquad \Delta_\sigma^{\beta} = \int d^dx\sqrt{\gamma}\,\sigma\beta^i\frac{\delta}{\delta g^i}.$$

With these definitions it is obvious that

$$\Delta_\sigma^{\mathrm{W}}W = -\int d^dx\sqrt{\gamma}\,\sigma\gamma^{\mu\nu}\langle T_{\mu\nu}\rangle, \qquad \Delta_\sigma^{\beta}W = -\int d^dx\sqrt{\gamma}\,\sigma\langle\beta^i[\mathcal{O}_i]\rangle.$$

It is known that the Weyl variation of W, $\Delta_\sigma^{\mathrm{W}}W$, is anomalous in curved space [18], even when the flat-space theory is a CFT.

In general, one can write

$$\Delta_\sigma^{\mathrm{W}}W = \Delta_\sigma^{\beta}W + \int d^dx\sqrt{\gamma}\,(\text{terms with derivatives on } \gamma_{\mu\nu}, g^i, \sigma). \tag{2.2}$$

For a classically scale invariant theory we also have

$$\left(\mu\frac{\partial}{\partial\mu} + 2\int d^dx\sqrt{\gamma}\,\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}\right)W = 0,$$

and so if the integral in (2.2) is neglected, then (2.2) reduces to the Callan–Symanzik equation (2.1). Therefore, (2.2) serves as a local version of the Callan–Symanzik equation. It is straightforward to see that (2.2) is equivalent to

$$\gamma^{\mu\nu}T_{\mu\nu} = \beta^i[\mathcal{O}_i] + (\text{curvature, } \partial_\mu g)\text{-terms}. \tag{2.3}$$

This is the most general form of the trace anomaly.[6] Consistency conditions follow from requiring that $[\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^{\beta}, \Delta_{\sigma'}^{\mathrm{W}} - \Delta_{\sigma'}^{\beta}]W = 0$, as imposed by the fact that the Weyl group is Abelian.

[6] Actually there is a more general form that includes renormalization effects of a specific vector operator of classical scaling dimension $d-1$. In $d=4$ such operators were considered by Osborn [3], and were also found in dimensional regularization in [5]. For the significance of such contributions the reader is referred to [8].

2.1. The 2d case

In two dimensions, the number of terms that are diffeomorphism and scale invariant that can contribute to the trace anomaly is small, and the elegance of the c-theorem is manifest. There is one curvature term and two terms with derivatives on spacetime-dependent couplings one can write down. The trace anomaly is reproduced by

$$\Delta_\sigma^{\mathrm{W}} W = \Delta_\sigma^\beta W - \int d^2x\sqrt{\gamma}\,\sigma(\tfrac{1}{2}\beta^\Phi R - \tfrac{1}{2}\chi_{ij}\partial_\mu g^i \partial^\mu g^j) + \int d^2x\sqrt{\gamma}\,\partial_\mu\sigma\, w_i\partial^\mu g^i, \tag{2.4}$$

where R is the Ricci scalar and β^Φ, χ_{ij}, and w_i are functions of the couplings. From $[\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^\beta, \Delta_{\sigma'}^{\mathrm{W}} - \Delta_{\sigma'}^\beta]W = 0$ with (2.4) we obtain a single consistency condition, namely

$$\partial_\mu\beta^\Phi + w_i\,\partial_\mu\beta^i + \beta^i\partial_i w_j\,\partial_\mu g^j - \chi_{ij}\beta^i\,\partial_\mu g^j = 0,$$

where $\partial_i \equiv \partial/\partial g^i$. Since this has to be true for arbitrary $\partial_\mu g^i$, we conclude that

$$\partial_i\tilde{\beta}^\Phi = \chi_{ij}\beta^j + \partial_{[i}w_{j]}\beta^j, \qquad \tilde{\beta}^\Phi = \beta^\Phi + w_i\beta^i, \tag{2.5}$$

where $X_{[i}Y_{j]} \equiv X_iY_j - X_jY_i$. By multiplying (2.5) by β^i we get,

$$\frac{d\tilde{\beta}^\Phi}{dt} = -\chi_{ij}\beta^i\beta^j,$$

which is equivalent to Zamolodchikov's c-theorem if χ_{ij} is positive-definite.

One should note here that there is a degree of arbitrariness in the definition of the various coefficients in (2.4), corresponding to the addition of allowed terms in the generating functional of the original 2d theory. Indeed, if we shift $W \to W + \delta W$, where

$$\delta W = \int d^2x\sqrt{\gamma}\,(\tfrac{1}{2}bR - \tfrac{1}{2}b_{ij}\partial_\mu g^i\partial^\mu g^j),$$

with arbitrary functions b, b_{ij}, then

$$\delta\beta^\Phi = \beta^i\partial_i b = \mathscr{L}_\beta b, \qquad \delta w_i = -\partial_i b + b_{ij}\beta^j$$
$$\delta\chi_{ij} = \beta^k\partial_k b_{ij} + \partial_i\beta^k\,b_{jk} + \partial_j\beta^k\,b_{ik} = \mathscr{L}_\beta b_{ij},$$

where $\mathscr{L}_\beta$ is the Lie derivative along the beta-function vector. Nevertheless, the consistency condition is invariant under this arbitrariness.[7] Osborn then establishes that there is a choice of the arbitrariness so that the corresponding χ_{ij} is positive-definite, essentially equal to Zamolodchikov's metric $G_{ij} = (x^2)^2\,\langle[\mathcal{O}_i(x)][\mathcal{O}_j(0)]\rangle$. With that choice $\tilde{\beta}^\phi$ becomes Zamolodchikov's c-function C. As a final remark let us point out here that possible dimension-one vector operators are neglected in the treatment of Osborn—such operators were considered in [19].

[7]The arbitrariness we are discussing here is analogous to the arbitrariness that affects the coefficient of $\Box R$ in the 4d trace anomaly at the fixed point. In 2d we see that outside the fixed point β^Φ has a degree of arbitrariness. Of course when $\beta^i = 0$ the well-defined β^Φ is the central charge of the corresponding CFT (up to normalization).

2.2. *The 4d case*

In four dimensions the elegance of the two-dimensional case is obfuscated by the fact that there exist four curvature invariants (that conserve parity) and quite a few terms that involve derivatives on the couplings. The terms that account for the trace anomaly may be written as

$$\Delta_\sigma^{\mathrm{W}} W = \Delta_\sigma^{\beta} W + \int d^4x \sqrt{\gamma}\, \sigma \mathscr{T} + \int d^4x \sqrt{\gamma}\, \partial_\mu \sigma\, \mathscr{Z}^\mu , \tag{2.6}$$

where

$$\begin{aligned} \mathscr{T} = \beta_a I + \beta_b E_4 + \tfrac{1}{9}\beta_c R^2 + \tfrac{1}{3}\chi_i^e \partial_\mu g^i \partial^\mu R + \tfrac{1}{6}\chi_{ij}^f \partial_\mu g^i \partial^\mu g^j R + \tfrac{1}{2}\chi_{ij}^g \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} \\ + \tfrac{1}{2}\chi_{ij}^a \nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\chi_{ijk}^b \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\chi_{ijkl}^c \partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^l , \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \mathscr{Z}^\mu = G^{\mu\nu} w_i \partial_\nu g^i + \tfrac{1}{3}\partial^\mu (b' R) + \tfrac{1}{3} R Y_i \partial^\mu g^i \\ + \partial^\mu (U_i \nabla^2 g^i + \tfrac{1}{2} V_{ij} \partial_\nu g^i \partial^\nu g^j) + S_{ij} \partial^\mu g^i \nabla^2 g^j + \tfrac{1}{2} T_{ijk} \partial_\nu g^i \partial^\nu g^j \partial^\mu g^k , \end{aligned} \tag{2.8}$$

up to terms with vanishing divergence. Definitions of the various curvatures can be found in (A.3). $G_{\mu\nu}$ is the Einstein tensor and the various coefficients are functions of the couplings.

Here one finds six consistency conditions (which can be further decomposed). Two of them are particularly interesting. First, there is a consistency condition like (2.5) involving β_b:

$$\partial_i \tilde{\beta}_b = \tfrac{1}{8}\chi_{ij}^g \beta^j + \tfrac{1}{8}\partial_{[i} w_{j]} \beta^j , \qquad \tilde{\beta}_b = \beta_b + \tfrac{1}{8} w_i \beta^i . \tag{2.9}$$

The consistency condition (2.9) can lead to an extension of Zamolodchikov's result to 4d if the metric χ_{ij}^g can be shown to be positive-definite. Of course, just like in 2d, there is an arbitrariness in the definition of χ_{ij}^g as well as in other coefficients in (2.7) and (2.8). To get an a-theorem it suffices to show that there is a choice of the arbitrariness so that χ_{ij}^g is positive-definite. This relies on the fact that (2.9) is invariant under the arbitrariness.

The other consistency condition we would like to draw attention to is

$$\beta_c = \tfrac{1}{4}(\partial_i b' - \chi_i^e)\beta^i .$$

This shows that the coefficient of R^2 in the trace anomaly is generally non-zero outside the fixed point. It also motivates the use of the term "vanishing anomalies" for contributions to the trace anomaly like R^2 in $d = 4$: these are anomalies that are present along the flow but vanish at the fixed point.

In our treatment so far we have neglected relevant operators with classical scaling dimension three or two that may be present in a four-dimensional theory. Osborn has considered such operators in [3], and has shown that the condition (2.9) is actually unaffected by their presence, except for a shift of β^i due to the presence of dimension-three vector operators. This shift played an important role in [8], where it was calculated at three loops in the most general renormalizable 4d QFT, and was used to show that at the perturbative level scale implies conformal invariance in unitary renormalizable 4d QFTs.

3. The 6d case

As we saw in the previous section the elegance of the consistency conditions rapidly disappears in the jump from 2d to 4d. Nevertheless, a consistency condition similar to (2.5) remains, and it is interesting to see if this is an accident or if such a consistency condition can be obtained in higher (even) dimensions. This is the main motivation behind this work, and the treatment of the highly nontrivial 6d case gives us valuable intuition that actually applies to all even dimensions. We postpone the discussion of the general even-d case until the next section, and we turn now to the consistency conditions in $d = 6$. Appendices A and B contain information on conventions, basis choices, as well as the terms that appear in the trace anomaly in 6d away from the fixed point.

3.1. Basis of curvature tensors

It is clear from the complexity of the 4d case that the situation in 6d is significantly more challenging. As a first step we have to classify the curvature tensors that can be used in the anomaly terms. Of course terms without curvatures also need to be considered.

To begin, note that for the various contributions to $(\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^{\beta})W$ we are only constrained by diffeomorphism invariance and power counting. Let us look at a consequence of this in 4d: in anomaly terms with one power of curvature one cannot involve the Riemann tensor (without contracting its indices). Indeed, the Riemann tensor has four free indices, for which we would need four derivatives on one or more couplings. This would result in a term with mass dimension six. Therefore, in 4d, one can only include curvature tensors with up to two free indices, and those are R, $\gamma_{\mu\nu}R$, and $R_{\mu\nu}$.[8] Since the variation of the 4d Euler density in $d = 4$ is $\delta_\sigma(\sqrt{\gamma}E_4) = -8\sqrt{\gamma}\,G^{\mu\nu}\nabla_\mu\partial_\nu\sigma$, it is preferable to include the Einstein tensor instead of the Ricci tensor. This choice produces the consistency conditions in a convenient form, but it is not essential. Indeed, the consistency conditions in a specific basis can be recast to the form obtained in any other basis by a redefinition of the coefficients of the various anomaly terms.

In 6d a similar choice is dictated by the fact that the Weyl variation of the 6d Euler density is

$$\delta_\sigma(\sqrt{\gamma}E_6) = 12\sqrt{\gamma}\,(3E_4\gamma^{\mu\nu} - 2RR^{\mu\nu} + 4R^{\mu}{}_{\kappa}R^{\kappa\nu} + 4R_{\kappa\lambda}R^{\kappa\mu\lambda\nu} - 2R_{\kappa\lambda\rho}{}^{\mu}R^{\kappa\lambda\rho\nu})\nabla_\mu\partial_\nu\sigma,$$

where E_4 is given in even $d > 2$ by $E_4 = \frac{2}{(d-2)(d-3)}(R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} - 4R^{\kappa\lambda}R_{\kappa\lambda} + R^2)$. The tensors quadratic in curvature that we have to consider can be found in (A.3); the tensor $H_1^{\mu\nu}$ is chosen so that in $d = 6$ the variation of the Euler density is $\delta_\sigma(\sqrt{\gamma}E_6) = 6\sqrt{\gamma}\,H_1^{\mu\nu}\nabla_\mu\partial_\nu\sigma$. As far as terms quadratic in curvature are concerned, we also have to include the terms (A.4), which are basically derivatives of the terms in (A.3). Terms linear in curvature include (A.1) and (A.2). In

[8] Incidentally, using the same argument one sees that $\nabla_\lambda G_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor, can also not be included in the anomaly terms.

writing down the various curvature tensors one has to identify a complete but not over-complete basis, a problem complicated by the symmetries of the Riemann tensor and the Bianchi identities.

As far as scalar terms cubic in curvature are concerned [20, 21], the situation is slightly more subtle. We have to include the terms in (A.5), but among them there are trivial anomalies, i.e. the terms $J_{1,\ldots,6}$ whose coefficient can be varied at will by a choice of local counterterms. These are not genuine anomalies, but they nevertheless appear in the trace anomaly, even at the fixed point. The well-known example is the term $\Box R$ in 4d. In (A.5) there are also vanishing anomalies, i.e. curvature terms that have to be included outside the fixed point, but that do not satisfy the consistency conditions at the fixed point and thus their coefficient has to be set to zero there. These are the terms $L_{1,\ldots,7}$ in (A.5). As we already mentioned there is only one such term in 4d, namely R^2. Here, the form of $L_{1,\ldots,6}$ is chosen based on the fact that these are the terms that shift the coefficients of the trivial anomalies at the fixed point, i.e. $\delta_\sigma \int d^6x\sqrt{\gamma}\, L_{1,\ldots,6} = \int d^6x\sqrt{\gamma}\,\sigma J_{1,\ldots,6}$.

While $J_{1,\ldots,6}$ can be included in the basis of terms cubic in curvature, there is a more convenient choice based on the fact that in order to show that $\delta_\sigma \int d^6x\sqrt{\gamma}\, L_{1,\ldots,6} = \int d^6x\sqrt{\gamma}\,\sigma J_{1,\ldots,6}$ one has to integrate by parts. But since total derivatives can be neglected in our considerations (for σ can be taken to have local support), this implies that we don't have to include the trivial anomalies $J_{1,\ldots,6}$ in $(\Delta^{\rm W}_\sigma - \Delta^\beta_\sigma)W$, so long as we include terms arising from $\delta_\sigma \int d^6x\sqrt{\gamma}\, z_{1,\ldots,6} L_{1,\ldots,6}$ before any integrations by parts. (Here $z_{1,\ldots,6}$ are arbitrary functions of the couplings.) Consequently, terms cubic in curvature that we need to consider are the three terms $I_{1,2,3}$ that lead to Weyl-invariant densities, the 6d Euler term E_6, and the seven vanishing anomalies $L_{1,\ldots,7}$. As we explained, this relies on the ability to discard total derivatives.

3.2. Contributions to the anomaly

Now that we have a complete basis of curvature tensors we are ready to write down the most general anomaly functional $(\Delta^{\rm W}_\sigma - \Delta^\beta_\sigma)W$. It takes the form

$$\Delta^{\rm W}_\sigma W = \Delta^\beta_\sigma W + \sum_{p=1}^{65} \int d^6x\sqrt{\gamma}\,\sigma\,\mathscr{T}_p + \sum_{q=1}^{30} \int d^6x\sqrt{\gamma}\,\partial_\mu\sigma\,\mathscr{Z}^\mu_q,$$

where the $\mathscr{T}_p$ and $\mathscr{Z}^\mu_q$ are dimension-six and dimension-five terms respectively, that can involve curvatures as well as derivatives on the couplings g^i (see Appendix B).

Much like Osborn did in $d = 2, 4$ we split the anomaly contributions into terms with σ and $\partial_\mu\sigma$. This splitting may seem mysterious—as we could also introduce terms of the form $\int d^6x\sqrt{\gamma}\,\Box\sigma\mathscr{V}$, for example—but it is used here in order to get the consistency conditions in a most convenient form. We can obtain the desired form of the consistency conditions even without the splitting, if we carefully choose the coefficients of the various terms in the anomaly. This can be seen by integrating by parts to rewrite the $\mathscr{Z}^\mu_q$ terms in the form of the $\mathscr{T}_p$ terms, which would lead to some new $\mathscr{T}_p$ terms but also to shifts of coefficients of existing $\mathscr{T}_p$ terms.

Let us illustrate this point more clearly in the 2d case. Suppose that instead of (2.4) we started with the equivalent

$$\Delta_\sigma^{\rm W} W = \Delta_\sigma^\beta W - \int d^2x \sqrt{\gamma}\, \sigma(\tfrac{1}{2}\beta^\Phi R - \tfrac{1}{2}\chi'_{ij}\partial_\mu g^i \partial^\mu g^j + w_i \Box g^i). \tag{3.1}$$

After an integration by parts of the $\Box g^i$ term this amounts simply to the definition $\chi_{ij} = \chi'_{ij} + 2\partial_{(i} w_{j)}$ in (2.4). This can also be seen by computing the Weyl consistency condition from (3.1) directly. We get

$$\partial_i \tilde{\beta}^\Phi = (\chi'_{ij} + 2\partial_i w_j)\beta^j = (\chi'_{ij} + \partial_{(i} w_{j)})\beta^j + \partial_{[i} w_{j]}\beta^j. \tag{3.2}$$

Clearly, (3.2) is equivalent to (2.5) with the proper definition of χ_{ij}.

3.3. Some consistency conditions

Here we include some consistency conditions and we comment on the most interesting ones. A Mathematica file with all the consistency conditions is included with our submission.

Just like in 2d and 4d we obtain consistency conditions simply by the requirement $[\Delta_\sigma^{\rm W} - \Delta_\sigma^\beta, \Delta_{\sigma'}^{\rm W} - \Delta_{\sigma'}^\beta]W = 0$. In our case we find a total of forty one consistency conditions.[9] For example, consistency requires that terms proportional to $\partial_\mu\sigma\,\Box^2\sigma' - \partial_\mu\,\sigma'\Box^2\sigma$ add up to zero, which leads to

$$\partial_\mu(4b_{11} - 3\mathcal{A}_i\beta^i) + (4\mathcal{A}_i + 2\mathcal{G}_i^4 + 5\mathcal{H}_i^5 + 2\mathcal{H}_i^6 - 2\mathcal{I}_i^4)\,\partial_\mu g^i + 6\mathcal{A}_i\,\partial_\mu\beta^i - \mathcal{A}'_{ij}\beta^j\,\partial_\mu g^i = 0,$$

which implies that

$$\partial_i(4b_{11} - 3\mathcal{A}_j\beta^j) + 4\mathcal{A}_i + 2\mathcal{G}_i^4 + 5\mathcal{H}_i^5 + 2\mathcal{H}_i^6 - 2\mathcal{I}_i^4 + 6\mathcal{A}_j\,\partial_i\beta^j = \mathcal{A}'_{ij}\beta^j.$$

Among the forty one consistency conditions in 6d the most interesting is the one similar to (2.5), obtained from terms proportional to $(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma)H_1^{\mu\nu}$. It reads

$$\partial_\mu(6a + b_1 - \tfrac{1}{15}b_3) + \mathcal{H}_i^1\,\partial_\mu\beta^i + \beta^i\partial_i\mathcal{H}_j^1\,\partial_\mu g^j - \mathcal{H}_{ij}^1\beta^i\,\partial_\mu g^j = 0,$$

which can be brought to the form

$$\partial_i\tilde{a} = \tfrac{1}{6}\mathcal{H}_{ij}^1\beta^j + \tfrac{1}{6}\partial_{[i}\mathcal{H}_{j]}^1\beta^j, \qquad \tilde{a} = a + \tfrac{1}{6}b_1 - \tfrac{1}{90}b_3 + \tfrac{1}{6}\mathcal{H}_i^1\beta^i. \tag{3.3}$$

The consistency condition (3.3) has a new feature compared to the 2d and 4d cases, i.e. that the function $\tilde{a}$ contains the coefficients b_1 and b_3 of the vanishing anomalies L_1 and L_3 respectively. This is of no consequence as far as the value of $\tilde{a}$ at the fixed point is concerned: there

---

[9] Some of these consistency conditions can be further decomposed as a result of the variety of ways with which spacetime derivatives can act on couplings.

$\tilde{a} = a$, for $b_1 = b_3 = 0$ at the fixed point. This fact is actually made explicit by three consistency conditions. More specifically, from terms proportional to $(\partial_\kappa\sigma\,\nabla_\lambda\partial_\mu\sigma' - \partial_\kappa\sigma'\,\nabla_\lambda\partial_\mu\sigma)\nabla^\kappa G^{\lambda\mu}$, $(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma)\nabla_\nu H_4^{\mu\nu}$, and $(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma)\nabla_\nu H_3^{\mu\nu}$ we find

$$b_7 = \tfrac{1}{8}\mathcal{F}_i\beta^i, \tag{3.4a}$$

$$3b_1 - 8b_7 = -\tfrac{1}{4}(\partial_i b_{14} + \mathcal{I}_i^7)\beta^i, \tag{3.4b}$$

$$12b_1 - b_3 - 16b_7 = -(\partial_i b_{13} + \mathcal{I}_i^6)\beta^i, \tag{3.4c}$$

respectively. From similar consistency conditions we can verify that b_2, b_4, b_5 and b_6 are also zero at the fixed point, as expected since they are coefficients of vanishing anomalies.

3.4. Possibility for an a-theorem in 6d

The consistency condition (3.3) has the potential to lead to a result similar to that of Zamolodchikov in 2d. Indeed, contracting with the beta function it follows that (3.3) implies

$$\frac{d\tilde{a}}{dt} = -\tfrac{1}{6}\mathcal{H}^1_{ij}\beta^i\beta^j. \tag{3.5}$$

Note that here the conditions (3.4) allow us to absorb the b_1 and b_3 contributions in $\tilde{a}$ to a shift of $\mathcal{H}^1_i$. Of course what is missing is a proof of the positive-definiteness of $\mathcal{H}^1_{ij}$.

It is important to point out that the consistency condition (3.3) is actually stronger than (3.5). Indeed, (3.3) also contains information about the possibility of a gradient flow interpretation of the RG flow. For that, it has to be that $\partial_{[i}\mathcal{H}^1_{j]} = 0$, in which case $\tilde{a}$ is the "potential" whose gradient produces the RG flow.

Let us now concentrate on a technical but important point. It turns out that the tensor $H_1^{\mu\nu}$, which appears in $\delta_\sigma(\sqrt{\gamma}E_6) = 6\sqrt{\gamma}\,H_1^{\mu\nu}\nabla_\mu\partial_\nu\sigma$, is divergenceless. A similar statements holds in two, $\delta_\sigma(\sqrt{\gamma}R) = 2\sqrt{\gamma}\,\gamma^{\mu\nu}\nabla_\mu\partial_\nu\sigma$, and four dimensions, $\delta_\sigma(\sqrt{\gamma}E_4) = -8\sqrt{\gamma}\,G^{\mu\nu}\nabla_\mu\partial_\nu\sigma$. This is actually crucial for the coefficient of the Euler term to be involved in a consistency condition like (3.3), which has the chance to lead to an a-theorem. This is not so easy to see in 2d and 4d, but it is clear in 6d.

Indeed, consider, for example, the consistency condition arising from terms proportional to $(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma)H_4^{\mu\nu}$. It reads

$$\partial_i\tilde{b}_1 = \tfrac{1}{12}(\mathcal{H}^4_{ij} + \tfrac{1}{2}\mathcal{F}_{ij})\beta^j + \tfrac{1}{12}\partial_{[i}\mathcal{H}^4_{j]}\beta^j + \tfrac{1}{6}\mathcal{I}^7_i, \qquad \tilde{b}_1 = -b_1 + \tfrac{2}{3}b_7 + \tfrac{1}{12}\mathcal{H}^4_i\beta^i. \tag{3.6}$$

The contribution $\frac{1}{6}\mathcal{I}^7_i$ does not allow (3.6) as a candidate for the generalization of Zamolodchikov's result.[10] This contribution in fact arises from the term $\mathscr{T}_{18} = \mathcal{I}^7_i\,\partial_\mu g^i\,\nabla_\nu H_4^{\mu\nu}$. Were $\nabla_\nu H_1^{\mu\nu}$ non-vanishing, we would not be able to find a consistency condition like (3.3). It can be verified by

[10] Of course this can also be seen from the fact that $\tilde{b}_1$ becomes zero at fixed points, and so it cannot possibly be monotonically-decreasing along an RG flow.

explicit computations that $H_1^{\mu\nu}$ is the only divergenceless symmetric two-index tensor quadratic in curvature. It is thus a generalization of the Einstein tensor. As we will see in the next section Lovelock has constructed all such generalizations a long time ago [22], something that will allow us to argue for a consistency condition similar to (3.3) in all even d.

3.5. Arbitrariness

Just like the coefficient of $\Box R$ in the four-dimensional trace anomaly, the various coefficients in $\mathscr{T}_p$ and $\mathscr{Z}_q^\mu$ are affected by the choice of additive, quantum-field-independent counterterms. Indeed, calculations in curved space and with x-dependent couplings will result in infinities that will need to be renormalized via counterterms whose finite part is arbitrary. Therefore, different subtraction schemes will result in different coefficients for $\mathscr{T}_p$ and $\mathscr{Z}_q^\mu$.

The most general addition to the generating functional of our theory is

$$\delta W = -\sum_{p=1}^{65} \int d^6x \sqrt{\gamma}\, \mathscr{X}_p,$$

where the $\mathscr{X}_p$ terms have the same form as the $\mathscr{T}_p$ terms but with arbitrary coefficients. There is no arbitrariness introduced by terms $\mathscr{X}_q^\mu$ similar in form to the $\mathscr{Z}_q^\mu$ terms, for those are total derivatives. Now, the consistency conditions are invariant under the shift $W \to W + \delta W$, although the coefficients in the consistency conditions will shift. Let us see how this works for (3.3).

The relevant terms are

$$\int d^6x \sqrt{\gamma}\,(z_a E_6 + z_{b_1} L_1 + z_{b_3} L_3 - \tfrac{1}{2} z_{ij}^{\mathcal{H}^1}\, \partial_\mu g^i \partial_\nu g^j H_1^{\mu\nu}) \subset \delta W \tag{3.7}$$

and one can verify that their inclusion leads to shifts

$$\begin{gathered} \delta a = \mathscr{L}_\beta z_a, \qquad \delta b_1 = \mathscr{L}_\beta z_{b_1}, \qquad \delta b_3 = \mathscr{L}_\beta z_{b_3}, \\ \delta\mathcal{H}_i^1 = -6\partial_i(z_a + \tfrac{1}{6} z_{b_1} - \tfrac{1}{90} z_{b_3}) + z_{ij}^{\mathcal{H}^1} \beta^j, \qquad \delta\mathcal{H}_{ij}^1 = \mathscr{L}_\beta z_{ij}^{\mathcal{H}^1}, \end{gathered} \tag{3.8}$$

under which (3.3) is invariant. Note that a, which is of course well-defined at the fixed point, is arbitrary along the flow, while $\mathcal{H}_i^1$ and $\mathcal{H}_{ij}^1$ have a degree of arbitrariness even at the fixed point. Also note that the shifts (3.8) cannot be used to set the corresponding coefficients to zero, except for $\mathcal{H}_i^1$ if $\mathcal{H}_i^1 = \partial_i X$ for some X.

This observation leads to an important point, which we have already emphasized: regarding the a-theorem, one should not be able to prove that the metric $\mathcal{H}_{ij}^1$ is positive-definite in all generality. Instead, one ought to be able to show that there is a choice for the arbitrariness (3.7) such that $\mathcal{H}_{ij}^1$ is positive-definite. That specific choice then gives us the quantity $\tilde{a}$ whose flow is monotonic, through the dependence of $\delta\mathcal{H}_i^1$ on $z_{ij}^{\mathcal{H}^1}$. Recall that in 2d arbitrariness similar to the one described here was used by Osborn to rederive Zamolodchikov's c-theorem (see [3] for details).

4. Consistency conditions in even spacetime dimensions

In this section we identify the ingredients that allow us to conclude that a consistency condition like (3.3) appears in all even spacetime dimensions. Of course non-trivial CFTs in $d > 6$ are not known, but it is still interesting to consider the generalization of our results.

According to the classification of [23], for a CFT in any even spacetime dimension lifted to curved space the conformal anomaly consists of a unique Euler term (type-A anomaly), a number of terms that lead to locally Weyl invariant densities (type-B anomalies), as well as a number of trivial anomalies. Outside the fixed point we also have a number of vanishing anomalies. As for the trivial anomalies, these can always be accounted for by terms with $d/2 - 1$ powers of curvature.

Now, in any even spacetime dimension, $d = 2n$, it is easy to see that the Weyl variation of the Euler density $\sqrt{\gamma}E_{2n}$, where

$$E_{2n} = \frac{1}{2^n} R_{i_1 j_1 k_1 l_1} \cdots R_{i_n j_n k_n l_n} \epsilon^{i_1 j_1 \ldots i_n j_n} \epsilon^{k_1 l_1 \ldots k_n l_n},$$

gives

$$\delta_\sigma(\sqrt{\gamma}E_{2n}) = \sqrt{\gamma}\, H^{\mu\nu} \nabla_\mu \partial_\nu \sigma,$$

for some symmetric tensor $H^{\mu\nu}$ with $n-1$ powers of the curvature. As Lovelock showed [22], this tensor $H^{\mu\nu}$ is the unique tensor with the properties of the Einstein tensor—in particular, it is the only two-index symmetric tensor with $n-1$ powers of the curvature that is divergenceless:

$$\nabla_\nu H^{\mu\nu} = 0.$$

Regarding the consistency condition similar to (3.3), this observation allows us to conclude that the only relevant terms among the various contributions to the anomaly $(\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^{\beta})W$ are

$$\int d^{2n}x \sqrt{\gamma}\, \sigma \left[(-1)^n a E_{2n} + \sum_p b_p L_p + \tfrac{1}{2} \mathcal{H}_{ij}\, \partial_\mu g^i \partial_\nu g^j\, H^{\mu\nu} \right] + \int d^{2n}x \sqrt{\gamma}\, \partial_\mu \sigma\, \mathcal{H}_i\, \partial_\nu g^i\, H^{\mu\nu},$$

where L_p are some vanishing anomalies. A consistency condition similar to (3.3) is thus easily found, and is of course invariant under arbitrariness generated by contributions similar to (3.7).

A relation of the metric $\mathcal{H}_{ij}$ to a positive-definite metric is currently only known in 2d [3]. A similar relation in higher even d is lacking, but its possible existence would immediately imply the generalization of Zamolodchikov's result. To summarize, in any even spacetime dimension one can find a scalar quantity $\tilde{a}$ such that

$$\partial_i \tilde{a} = \mathcal{H}_{ij} \beta^j + \partial_{[i} \mathcal{H}_{j]} \beta^j. \tag{4.1}$$

The quantity $\tilde{a}$ becomes the coefficient of the Euler term in the trace anomaly at the fixed point, but more generally it includes a linear combination of the b_ps and a term $\mathcal{H}_i \beta^i$. The relation (4.1) immediately implies that

$$\frac{d\tilde{a}}{dt} = -\mathcal{H}_{ij} \beta^i \beta^j,$$

which, if $\mathcal{H}_{ij}$ can be related to a positive-definite metric via the arbitrariness $\delta\mathcal{H}_{ij} = \mathscr{L}_\beta z^{\mathcal{H}}_{ij}$ with $z^{\mathcal{H}}_{ij}$ an arbitrary symmetric tensor, is the generalization of the 2d c-theorem.

Acknowledgments

We have relied heavily on Mathematica and the package `xAct`. We would like to thank Aneesh Manohar, John McGreevy, and especially Ken Intriligator for helpful discussions. This work was supported in part by the US Department of Energy under contract DE-SC0009919.

Appendix A. Conventions and definitions

Throughout this paper we follow the conventions of Misner, Thorne and Wheeler [24] for the Riemann tensor. For the Weyl variation of the metric we choose

$$\gamma_{\mu\nu} \to e^{-2\sigma}\gamma_{\mu\nu}.$$

Infinitesimally, then, $\delta_\sigma\gamma_{\mu\nu} = -2\sigma\gamma_{\mu\nu}$ and so $\delta_\sigma\gamma^{\mu\nu} = 2\sigma\gamma^{\mu\nu}$ (we do not use $\delta\sigma$ for an infinitesimal σ, since no confusion can arise).

It is important to classify the curvature terms of various mass dimensions. These will be used subsequently to construct all possible terms that can appear in $(\Delta^{\mathrm{W}}_\sigma - \Delta^{\beta}_\sigma)W$. In two and four spacetime dimensions this is very easy, but in six it becomes a rather cumbersome problem, plagued by complications due to the large number of monomials and the identities of the Riemann tensor.

A complete basis $\mathfrak{B}_2$ of dimension-two curvature terms that can be used in $\Delta^{\mathrm{W}}_\sigma W$ is given by the Ricci scalar, the Einstein tensor, and the Riemann tensor,

$$\tfrac{1}{d-1}R, \qquad G_{\mu\nu}, \qquad R_{\kappa\lambda\mu\nu}, \tag{A.1}$$

where we define the Einstein tensor as

$$G_{\mu\nu} = \tfrac{2}{d-2}(R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R) \qquad (d \geq 3),$$

where $R_{\mu\nu}$ is the Ricci tensor. Taking a derivative leads to three dimension-three terms, but, by diffeomorphism invariance and simple power counting, only two can be used in $\Delta^{\mathrm{W}}_\sigma W$, namely

$$\tfrac{1}{d-1}\partial_\mu R \qquad \text{and} \qquad \nabla_\kappa G_{\mu\nu}. \tag{A.2}$$

These form the basis $\mathfrak{B}_3$.

At the level of dimension-four curvature terms only terms with up to two free indices are allowed in $\Delta^{\mathrm{W}}_\sigma W$. We consider the basis $\mathfrak{B}_4$ with elements

$$\begin{gathered}
E_4 = \tfrac{2}{(d-2)(d-3)}(R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} - 4R^{\kappa\lambda}R_{\kappa\lambda} + R^2),\\
I = R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} - \tfrac{4}{d-2}R^{\kappa\lambda}R_{\kappa\lambda} + \tfrac{2}{(d-1)(d-2)}R^2, \qquad \tfrac{1}{(d-1)^2}R^2, \qquad \tfrac{1}{d-1}\Box R,\\
H_{1\mu\nu} = \tfrac{(d-2)(d-3)}{2}E_4\gamma_{\mu\nu} - 4(d-1)H_{2\mu\nu} + 8H_{3\mu\nu} + 8H_{4\mu\nu} - 4R^{\kappa\lambda\rho}{}_{\mu}R_{\kappa\lambda\rho\nu},\\
H_{2\mu\nu} = \tfrac{1}{d-1}RR_{\mu\nu}, \qquad H_{3\mu\nu} = R_\mu{}^\kappa R_{\kappa\nu}, \qquad H_{4\mu\nu} = R^{\kappa\lambda}R_{\kappa\mu\lambda\nu},\\
H_{5\mu\nu} = \Box R_{\mu\nu}, \qquad H_{6\mu\nu} = \tfrac{1}{d-1}\nabla_\mu\partial_\nu R.
\end{gathered} \tag{A.3}$$

All $H_{1,\dots,6}$ are symmetric. I is the Weyl tensor squared, $I = W^{\kappa\lambda\mu\nu}W_{\kappa\lambda\mu\nu}$, and $\sqrt{\gamma}E_4$ is the four-dimensional Euler density. In our conventions the Weyl tensor is given by

$$W_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} + \tfrac{2}{d-2}(\gamma_{\kappa[\nu}R_{\mu]\lambda} + \gamma_{\lambda[\mu}R_{\nu]\kappa}) + \tfrac{2}{(d-1)(d-2)}\gamma_{\kappa[\mu}\gamma_{\nu]\lambda}R \qquad (d \geq 3).$$

The dimension-five curvature terms we need to consider are given by

$$\partial_\mu E_4, \qquad \partial_\mu I, \qquad \tfrac{1}{(d-1)^2}R\,\partial_\mu R, \qquad \tfrac{1}{d-1}\partial_\mu\Box R, \qquad \nabla^\nu H_{(2,3,4)\mu\nu}, \tag{A.4}$$

and they form the basis $\mathfrak{B}_5$. Note that we do not need $\nabla^\nu H_{1\mu\nu}$, for $\nabla^\nu H_{1\mu\nu} = 0$.[11] Similarly, $\nabla^\nu H_{(5,6)\mu\nu}$ are not necessary, for

$$\nabla^\nu H_{5\mu\nu} = \nabla^\nu\{(d-1)H_{2\mu\nu} - 2H_{4\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}[\tfrac{1}{8}(d-2)^2E_4 - \tfrac{d-2}{4(d-3)}I + \tfrac{d-2}{4(d-1)}R^2 - \Box R]\}$$

and

$$\nabla^\nu H_{6\mu\nu} = \nabla^\nu[H_{2\mu\nu} - \tfrac{1}{d-1}\gamma_{\mu\nu}(\tfrac{1}{4}R^2 - \Box R)].$$

The corresponding matrix of coefficients of the remaining terms in $\nabla^\mu\mathfrak{B}_4$ has full rank, which shows that $\mathfrak{B}_5$ is a good basis.

Finally, a complete basis of scalar dimension-six curvature terms was constructed in [20]. Its building blocks are $K_{1,\dots,17}$ given by

$$\begin{gathered}
K_1 = R^3, \qquad K_2 = RR^{\kappa\lambda}R_{\kappa\lambda}, \qquad K_3 = RR^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}, \qquad K_4 = R^{\kappa\lambda}R_{\lambda\mu}R^\mu{}_\kappa,\\
K_5 = R^{\kappa\lambda}R_{\kappa\mu\nu\lambda}R^{\mu\nu}, \qquad K_6 = R^{\kappa\lambda}R_{\kappa\mu\nu\rho}R_\lambda{}^{\mu\nu\rho}, \qquad K_7 = R^{\kappa\lambda\mu\nu}R_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\kappa\lambda},\\
K_8 = R^{\kappa\lambda\mu\nu}R_{\rho\lambda\mu\sigma}R_\kappa{}^{\rho\sigma}{}_\nu, \qquad K_9 = R\Box R, \qquad K_{10} = R^{\kappa\lambda}\Box R_{\kappa\lambda}, \qquad K_{11} = R^{\kappa\lambda\mu\nu}\Box R_{\kappa\lambda\mu\nu},\\
K_{12} = R^{\kappa\lambda}\nabla_\kappa\partial_\lambda R, \qquad K_{13} = \nabla^\kappa R^{\lambda\mu}\nabla_\kappa R_{\lambda\mu}, \qquad K_{14} = \nabla^\kappa R^{\lambda\mu}\nabla_\lambda R_{\kappa\mu},\\
K_{15} = \nabla^\kappa R^{\lambda\mu\nu\rho}\nabla_\kappa R_{\lambda\mu\nu\rho}, \qquad K_{16} = \Box R^2, \qquad K_{17} = \Box^2 R.
\end{gathered}$$

[11] In any even dimension d, the Weyl variation of $\sqrt{\gamma}E_d$ is $\sqrt{\gamma}\,H^{\mu\nu}\nabla_\mu\partial_\nu\sigma$ for some symmetric tensor $H^{\mu\nu}$. Since $\delta_\sigma\int d^dx\sqrt{\gamma}E_d = 0$, it follows that $\nabla_\mu\nabla_\nu H^{\mu\nu} = 0$. In [22] it was shown, however, that $H^{\mu\nu}$ is actually divergenceless, i.e. $\nabla_\nu H^{\mu\nu} = 0$.

At the fixed point we can express the trace anomaly in the basis K's, and the consistency condition implies that there are seven combinations of K's whose coefficient has to be set to zero [20, 21]. Thus, we can arrange the K's in the basis of [21],

$$
\begin{aligned}
I_1 &= \tfrac{19}{800}K_1 - \tfrac{57}{160}K_2 + \tfrac{3}{40}K_3 + \tfrac{7}{16}K_4 - \tfrac{9}{8}K_5 - \tfrac{3}{4}K_6 + K_8, \\
I_2 &= \tfrac{9}{200}K_1 - \tfrac{27}{40}K_2 + \tfrac{3}{10}K_3 + \tfrac{5}{4}K_4 - \tfrac{3}{2}K_5 - 3K_6 + K_7, \\
I_3 &= -\tfrac{11}{50}K_1 + \tfrac{27}{10}K_2 - \tfrac{6}{5}K_3 - K_4 + 6K_5 + 2K_7 - 8K_8 \\
&\qquad\qquad + \tfrac{3}{5}K_9 - 6K_{10} + 6K_{11} + 3K_{13} - 6K_{14} + 3K_{15}, \\
E_6 &= K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8, \\
J_1 &= 6K_6 - 3K_7 + 12K_8 + K_{10} - 7K_{11} - 11K_{13} + 12K_{14} - 4K_{15}, \\
J_2 &= -\tfrac{1}{5}K_9 + K_{10} + \tfrac{2}{5}K_{12} + K_{13}, \\
J_3 &= K_4 + K_5 - \tfrac{3}{20}K_9 + \tfrac{4}{5}K_{12} + K_{14}, \\
J_4 &= -\tfrac{1}{5}K_9 + K_{11} + \tfrac{2}{5}K_{12} + K_{15}, \\
J_5 &= K_{16}, \\
J_6 &= K_{17},
\end{aligned}
$$

which makes manifest the splitting of anomalies at the fixed point into type A (E_6) and B ($I_{1,2,3}$) according to the classification of [23], and also trivial ($J_{1,\ldots,6}$). To be more specific, $I_{1,2,3}$, which can be expressed as

$$
\begin{aligned}
I_1 &= W^{\kappa\lambda\mu\nu}W_{\rho\lambda\mu\sigma}W_{\kappa}{}^{\rho\sigma}{}_{\nu}, \\
I_2 &= W^{\kappa\lambda\mu\nu}W_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\kappa\lambda}, \\
I_3 &= W^{\kappa\lambda\mu\nu}(\delta_{\kappa}{}^{\rho}\,\Box + 4R_{\kappa}{}^{\rho} - \tfrac{6}{5}\delta_{\kappa}{}^{\rho}R)W_{\rho\lambda\mu\nu} - \tfrac{2}{3}J_1 - \tfrac{13}{3}J_2 + 2J_3 + \tfrac{1}{3}J_4,
\end{aligned}
$$

lead to locally Weyl-invariant densities, while $J_{1,\ldots,6}$ can be set to zero in the trace anomaly by a choice of local counterterm, just like $\Box R$ in four dimensions. The piece $-\tfrac{2}{3}J_1 - \tfrac{13}{3}J_2 + 2J_3 + \tfrac{1}{3}J_4$ in I_3 is necessary for $\sqrt{\gamma}I_3$ to be locally Weyl-invariant [21].

For our purposes all K's are needed, since we are interested in consistency conditions valid along the RG flow. It is convenient to work in the basis $\mathfrak{B}_6$ given by

$$
\begin{gathered}
I_1, \qquad I_2, \qquad I_3, \qquad E_6, \\
L_1 = -\tfrac{1}{30}K_1 + \tfrac{1}{4}K_2 - K_6, \qquad L_2 = -\tfrac{1}{100}K_1 + \tfrac{1}{20}K_2, \\
L_3 = -\tfrac{37}{6000}K_1 + \tfrac{7}{150}K_2 - \tfrac{1}{75}K_3 + \tfrac{1}{10}K_5 + \tfrac{1}{15}K_6, \qquad L_4 = -\tfrac{1}{150}K_1 + \tfrac{1}{20}K_3, \\
L_5 = \tfrac{1}{30}K_1, \qquad L_6 = -\tfrac{1}{300}K_1 + \tfrac{1}{20}K_9, \qquad L_7 = K_{15}, \\
J_1, \qquad J_2, \qquad J_3, \qquad J_4, \qquad \tfrac{1}{25}J_5, \qquad \tfrac{1}{25}J_6.
\end{gathered}
\tag{A.5}
$$

The form of $L_{1,\ldots,6}$ is chosen based on the fact that these are the terms that shift the coefficients of the trivial anomalies at the fixed point, i.e. $\delta_\sigma \int d^6x\sqrt{\gamma}\, L_{1,\ldots,6} = \int d^6x\sqrt{\gamma}\,\sigma J_{1,\ldots,6}$.

The choice of bases is arbitrary, and the form of the consistency conditions depends on the choice. Nevertheless, the essential conclusions derived from the consistency conditions are basis-independent.

Appendix B. Terms in the anomaly

In six spacetime dimensions there are ninety five independent terms that can contribute to $(\Delta_\sigma^{\rm W} - \Delta_\sigma^{\beta})W$. We include them in this appendix for easy reference.

In general, we can write

$$\Delta_\sigma^{\rm W} W = \Delta_\sigma^{\beta} W + \sum_{p=1}^{65}\int d^6x\sqrt{\gamma}\,\sigma\,\mathscr{T}_p + \sum_{q=1}^{30}\int d^6x\sqrt{\gamma}\,\partial_\mu\sigma\,\mathscr{Z}_q^\mu .$$

Clearly, the $\mathscr{T}_p$ and $\mathscr{Z}_q^\mu$ are dimension-six and dimension-five terms respectively, that can involve curvatures as well as derivatives on the couplings g^i. In writing down the various terms below, we neglect total derivatives, and we keep in mind the convenient form in which we want to obtain the consistency conditions.

If only curvatures are included, then we have the terms

$$\mathscr{T}_1 = -c_1 I_1, \qquad \mathscr{T}_2 = -c_2 I_2, \qquad \mathscr{T}_3 = -c_3 I_3, \qquad \mathscr{T}_4 = -aE_6, \qquad \mathscr{T}_{5,\ldots,11} = -b_{1,\ldots,7} L_{1,\ldots,7}.$$

We call these the $(0,6)$ terms, for only curvatures and derivatives on curvatures contribute to the power counting. We also have $(0,5)^\mu$ terms, given by

$$\mathscr{Z}_1^\mu = -b_8\,\partial^\mu E_4, \qquad \mathscr{Z}_2^\mu = -b_9\,\partial^\mu I, \qquad \mathscr{Z}_3^\mu = -\tfrac{1}{25} b_{10}\, R\,\partial^\mu R,$$
$$\mathscr{Z}_4^\mu = -\tfrac{1}{5} b_{11}\,\partial^\mu \Box R, \qquad \mathscr{Z}_{5,6,7}^\mu = -b_{12,13,14}\,\nabla_\nu H_{2,3,4}^{\mu\nu}.$$

Next, we can allow one power of $\partial_\mu g^i$ to get

$$\mathscr{T}_{12} = \mathcal{I}_i^1\,\partial_\mu g^i\,\partial^\mu E_4, \qquad \mathscr{T}_{13} = \mathcal{I}_i^2\,\partial_\mu g^i\,\partial^\mu I, \qquad \mathscr{T}_{14} = \tfrac{1}{25}\mathcal{I}_i^3\,\partial_\mu g^i\, R\,\partial^\mu R,$$
$$\mathscr{T}_{15} = \tfrac{1}{5}\mathcal{I}_i^4\,\partial_\mu g^i\,\partial^\mu \Box R \qquad \mathscr{T}_{16,17,18} = \mathcal{I}_i^{5,6,7}\,\partial_\mu g^i\,\nabla_\nu H_{2,3,4}^{\mu\nu},$$

which are the $(1,5)$ terms. The $(1,4)^\mu$ terms are

$$\mathscr{Z}_8^\mu = \mathcal{G}_i^1\,\partial^\mu g^i\, E_4, \qquad \mathscr{Z}_9^\mu = \mathcal{G}_i^2\,\partial^\mu g^i\, I, \qquad \mathscr{Z}_{10}^\mu = \tfrac{1}{25}\mathcal{G}_i^3\,\partial^\mu g^i\, R^2,$$
$$\mathscr{Z}_{11}^\mu = \tfrac{1}{5}\mathcal{G}_i^4\,\partial^\mu g^i\,\Box R, \qquad \mathscr{Z}_{12,\ldots,17}^\mu = \mathcal{H}_i^{1,\ldots,6}\,\partial_\nu g^i H_{1,\ldots,6}^{\mu\nu}.$$

The $(2,4)$ terms are given by

$$\mathscr{T}_{19} = \tfrac{1}{2}\mathcal{G}_{ij}^1\,\partial_\mu g^i\partial^\mu g^j\, E_4, \qquad \mathscr{T}_{20} = \tfrac{1}{2}\mathcal{G}_{ij}^2\,\partial_\mu g^i\partial^\mu g^j\, I, \qquad \mathscr{T}_{21} = \tfrac{1}{50}\mathcal{G}_{ij}^3\,\partial_\mu g^i\partial^\mu g^j\, R^2,$$
$$\mathscr{T}_{22} = \tfrac{1}{10}\mathcal{G}_{ij}^4\,\partial_\mu g^i\partial^\mu g^j\,\Box R, \qquad \mathscr{T}_{23,\ldots,28} = \tfrac{1}{2}\mathcal{H}_{ij}^{1,\ldots,6}\,\partial_\mu g^i\partial_\nu g^j\, H_{1,\ldots,6}^{\mu\nu},$$

while the $(2,3)^\mu$ terms are

$$\mathscr{Z}^\mu_{18} = \mathcal{F}_i\,\nabla_\kappa\partial_\lambda g^i\,\nabla^\mu G^{\kappa\lambda}, \qquad \mathscr{Z}^\mu_{19} = \tfrac{1}{5}\mathcal{E}_i\,\Box g^i\,\partial^\mu R, \qquad \mathscr{Z}^\mu_{20} = \tfrac{1}{5}\mathcal{E}_{ij}\,\partial^\mu g^i\partial_\nu g^j\,\partial^\nu R.$$

The $(3,3)$ terms are

$$\mathscr{T}_{29} = \mathcal{F}_{ij}\,\partial_\kappa g^i\nabla_\lambda\partial_\mu g^j\,\nabla^\kappa G^{\lambda\mu}, \qquad \mathscr{T}_{30} = \mathcal{F}'_{ij}\,\partial_\kappa g^i\nabla_\lambda\partial_\mu g^j\,\nabla^\lambda G^{\kappa\mu},$$
$$\mathscr{T}_{31} = \tfrac{1}{2}\mathcal{F}_{ijk}\,\partial_\kappa g^i\partial_\lambda g^j\partial_\mu g^k\,\nabla^\kappa G^{\lambda\mu}, \qquad \mathscr{T}_{32} = \tfrac{1}{5}\hat{\mathcal{E}}_{ij}\,\partial_\mu g^i\Box g^j\,\partial^\mu R,$$
$$\mathscr{T}_{33} = \tfrac{1}{10}\mathcal{E}_{ijk}\,\partial_\mu g^i\partial_\nu g^j\partial^\nu g^k\,\partial^\mu R,$$

and the $(3,2)^\mu$ terms are

$$\mathscr{Z}^\mu_{21} = \mathcal{D}_{ij}\,\partial_\kappa g^i\nabla_\lambda\partial_\nu g^j\,R^{\mu\lambda\kappa\nu}, \qquad \mathscr{Z}^\mu_{22} = \mathcal{C}_i\,\partial_\nu\Box g^i\,G^{\mu\nu}, \qquad \mathscr{Z}^\mu_{23} = \mathcal{C}_{ij}\,\partial_\kappa g^i\nabla_\nu\partial^\kappa g^j\,G^{\mu\nu},$$
$$\mathscr{Z}^\mu_{24} = \mathcal{C}'_{ij}\,\partial_\nu g^i\Box g^j\,G^{\mu\nu}, \qquad \mathscr{Z}^\mu_{25} = \tfrac{1}{5}\mathcal{B}_{ij}\,\partial^\mu g^i\Box g^j\,R.$$

The $(4,2)$ terms are

$$\mathscr{T}_{34} = \mathcal{D}_{ijk}\,\partial_\kappa g^i\partial_\mu g^j\nabla_\lambda\partial_\nu g^k\,R^{\kappa\lambda\mu\nu}, \qquad \mathscr{T}_{35} = \tfrac{1}{4}\mathcal{D}_{ijkl}\,\partial_\kappa g^i\partial_\lambda g^j\partial_\mu g^k\partial_\nu g^l\,R^{\kappa\lambda\mu\nu},$$
$$\mathscr{T}_{36} = \hat{\mathcal{C}}_{ij}\,\nabla_\mu\partial_\nu g^i\Box g^j\,G^{\mu\nu}, \qquad \mathscr{T}_{37} = \tfrac{1}{2}\hat{\mathcal{C}}'_{ij}\,\nabla_\kappa\partial_\mu g^i\nabla^\kappa\partial_\nu g^j\,G^{\mu\nu}, \qquad \mathscr{T}_{38} = \tfrac{1}{2}\mathcal{C}_{ijk}\,\partial_\mu g^i\partial_\nu g^j\Box g^k\,G^{\mu\nu},$$
$$\mathscr{T}_{39} = \mathcal{C}'_{ijk}\,\partial_\mu g^i\partial_\kappa g^j\nabla^\kappa\partial_\nu g^k\,G^{\mu\nu}, \qquad \mathscr{T}_{40} = \tfrac{1}{2}\mathcal{C}''_{ijk}\,\partial_\kappa g^i\partial^\kappa g^j\nabla_\mu\partial_\nu g^k\,G^{\mu\nu},$$
$$\mathscr{T}_{41} = \tfrac{1}{4}\mathcal{C}_{ijkl}\,\partial_\mu g^i\partial_\nu g^j\partial_\kappa g^k\partial^\kappa g^l\,G^{\mu\nu}, \qquad \mathscr{T}_{42} = \tfrac{1}{5}\mathcal{B}_i\,\Box^2 g^i\,R, \qquad \mathscr{T}_{43} = \tfrac{1}{10}\hat{\mathcal{B}}_{ij}\,\Box g^i\Box g^j\,R,$$
$$\mathscr{T}_{44} = \tfrac{1}{10}\hat{\mathcal{B}}'_{ij}\,\nabla_\mu\partial_\nu g^i\nabla^\mu\partial^\nu g^j\,R, \qquad \mathscr{T}_{45} = \tfrac{1}{10}\mathcal{B}_{ijk}\,\partial_\mu g^i\partial^\mu g^j\Box g^k\,R,$$
$$\mathscr{T}_{46} = \tfrac{1}{10}\mathcal{B}'_{ijk}\,\partial_\mu g^i\partial_\nu g^j\nabla^\mu\partial^\nu g^k\,R, \qquad \mathscr{T}_{47} = \tfrac{1}{20}\mathcal{B}_{ijkl}\,\partial_\mu g^i\partial^\mu g^j\partial_\nu g^k\partial^\nu g^l\,R,$$

and the $(5,0)^\mu$ terms are

$$\mathscr{Z}^\mu_{26} = \mathcal{A}_{ij}\,\partial_\nu\Box g^i\nabla^\mu\partial^\nu g^j, \qquad \mathscr{Z}^\mu_{27} = \mathcal{A}'_{ij}\,\partial^\mu g^i\Box^2 g^j, \qquad \mathscr{Z}^\mu_{28} = \mathcal{A}_{ijk}\,\partial_\nu g^i\nabla^\mu\partial^\nu g^j\Box g^k,$$
$$\mathscr{Z}^\mu_{29} = \mathcal{A}'_{ijk}\,\partial_\kappa g^i\nabla^\mu\partial_\lambda g^j\nabla^\kappa\partial^\lambda g^k, \qquad \mathscr{Z}^\mu_{30} = \tfrac{1}{2}\mathcal{A}_{ijkl}\,\partial_\nu g^i\partial^\nu g^j\partial^\mu g^k\Box g^l.$$

Finally, the $(6,0)$ terms are

$$\mathscr{T}_{48} = \mathcal{A}_i\,\Box^3 g^i, \qquad \mathscr{T}_{49} = \hat{\mathcal{A}}_{ij}\,\Box^2 g^i\Box g^j, \qquad \mathscr{T}_{50} = \tfrac{1}{2}\hat{\mathcal{A}}'_{ij}\,\partial_\mu\Box g^i\partial^\mu\Box g^j,$$
$$\mathscr{T}_{51} = \tfrac{1}{2}\hat{\mathcal{A}}''_{ij}\,\nabla_\kappa\nabla_\lambda\partial_\mu g^i\nabla^\kappa\nabla^\lambda\partial^\mu g^j, \qquad \mathscr{T}_{52} = \tfrac{1}{8}\hat{\mathcal{A}}_{ijk}\,\Box g^i\Box g^j\Box g^k,$$
$$\mathscr{T}_{53} = \tfrac{1}{2}\hat{\mathcal{A}}'_{ijk}\,\nabla_\kappa\partial_\mu g^i\nabla^\kappa\partial_\nu g^j\nabla^\mu\partial^\nu g^k, \qquad \mathscr{T}_{54} = \hat{\mathcal{A}}''_{ijk}\,\partial_\mu g^i\Box g^j\partial^\mu\Box g^k,$$
$$\mathscr{T}_{55} = \check{\mathcal{A}}_{ijk}\,\partial_\mu g^i\nabla^\mu\partial_\nu g^j\partial^\nu\Box g^k, \qquad \mathscr{T}_{56} = \tfrac{1}{2}\check{\mathcal{A}}'_{ijk}\,\partial_\mu g^i\partial^\mu g^j\Box^2 g^k,$$
$$\mathscr{T}_{57} = \tfrac{1}{2}\check{\mathcal{A}}''_{ijk}\,\partial_\mu g^i\partial_\nu g^j\nabla^\mu\partial^\nu\Box g^k, \qquad \mathscr{T}_{58} = \tfrac{1}{4}\hat{\mathcal{A}}_{ijkl}\,\partial_\mu g^i\partial^\mu g^j\Box g^k\Box g^l,$$
$$\mathscr{T}_{59} = \tfrac{1}{4}\hat{\mathcal{A}}'_{ijkl}\,\partial_\kappa g^i\partial^\kappa g^j\nabla_\mu\partial_\nu g^k\nabla^\mu\partial^\nu g^l, \qquad \mathscr{T}_{60} = \tfrac{1}{2}\hat{\mathcal{A}}''_{ijkl}\,\partial_\kappa g^i\partial_\lambda g^j\nabla^\kappa\partial_\mu g^k\nabla^\lambda\partial^\mu g^l,$$
$$\mathscr{T}_{61} = \tfrac{1}{2}\check{\mathcal{A}}_{ijkl}\,\partial_\mu g^i\partial_\nu g^j\nabla^\mu\partial^\nu g^k\Box g^l, \qquad \mathscr{T}_{62} = \tfrac{1}{2}\check{\mathcal{A}}'_{ijkl}\,\partial_\kappa g^i\partial_\lambda g^j\partial_\mu g^k\nabla^\kappa\nabla^\lambda\partial^\mu g^l,$$
$$\mathscr{T}_{63} = \tfrac{1}{4}\mathcal{A}_{ijklm}\,\partial_\mu g^i\partial^\mu g^j\partial_\nu g^k\partial^\nu g^l\Box g^m, \qquad \mathscr{T}_{64} = \tfrac{1}{4}\mathcal{A}'_{ijklm}\,\partial_\kappa g^i\partial^\kappa g^j\partial_\lambda g^k\partial_\mu g^l\nabla^\lambda\partial^\mu g^m,$$
$$\mathscr{T}_{65} = \tfrac{1}{8}\mathcal{A}_{ijklmn}\,\partial_\kappa g^i\partial^\kappa g^j\partial_\lambda g^k\partial^\lambda g^l\partial_\mu g^m\partial^\mu g^n.$$

Note that when we have more than two derivatives on a coupling only one ordering of the derivatives is independent. That is because all other orderings can be produced by commuting covariant derivatives, a process which introduces Riemann tensors or its contractions. This leads to terms with curvature tensors that we have already included.

The scalar quantities $a, b_{1,\ldots,14}$ and $c_{1,2,3}$ are functions of the couplings g^i. All $\mathcal{A}, \ldots, \mathcal{I}$ are also functions of the couplings, but not all of them are tensors under reparametrizations in the space of couplings, owing to the fact that $\Box g^i$ transforms inhomogeneously under $g^i \to \bar{g}^i(g)$; more specifically, $\Box \bar{g}^i = \partial_j \bar{g}^i \, \Box g^j + \partial_j \partial_k \bar{g}^i \, \partial_\mu g^j \partial^\mu g^k$. $\mathcal{E}_{ijk}$ in $\mathscr{T}_{33}$ is an example of this, because of the $\Box g^j$ in $\mathscr{T}_{32}$.

References

[1] J. Wess & B. Zumino, *"Consequences of anomalous Ward identities"*, Phys.Lett. **B37**, 95 (1971).

[2] E. Witten, *"Global Aspects of Current Algebra"*, Nucl.Phys. **B223**, 422 (1983).

[3] H. Osborn, *"Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories"*, Nucl.Phys. **B363**, 486 (1991).

[4] A. Zamolodchikov, *"Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory"*, JETP Lett. **43**, 730 (1986).

[5] I. Jack & H. Osborn, *"Analogs for the c theorem for four-dimensional renormalizable field theories"*, Nucl.Phys. **B343**, 647 (1990).

[6] J.L. Cardy, *"Is There a c Theorem in Four-Dimensions?"*, Phys.Lett. **B215**, 749 (1988).

[7] J. Polchinski, *"Scale and conformal invariance in quantum field theory"*, Nucl.Phys. **B303**, 226 (1988).

[8] J.F. Fortin, B. Grinstein & A. Stergiou, *"Limit Cycles and Conformal Invariance"*, JHEP **1301**, 184 (2013), `arXiv:1208.3674`.

[9] J.F. Fortin, B. Grinstein & A. Stergiou, *"Limit Cycles in Four Dimensions"*, JHEP **1212**, 112 (2012), `arXiv:1206.2921`.

[10] M.A. Luty, J. Polchinski & R. Rattazzi, *"The a-theorem and the Asymptotics of 4D Quantum Field Theory"*, JHEP **1301**, 152 (2013), `arXiv:1204.5221`.

[11] J.F. Fortin, B. Grinstein, C.W. Murphy & A. Stergiou, *"On Limit Cycles in Supersymmetric Theories"*, Phys.Lett. **B719**, 170 (2013), `arXiv:1210.2718`. ✥ Y. Nakayama, *"Supercurrent, Supervirial and Superimprovement"*, `arXiv:1208.4726`.

[12] J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance: An Example"*, Phys.Lett. **B704**, 74 (2011), `arXiv:1106.2540`. ✥ J.F. Fortin, B. Grinstein & A. Stergiou,

"Scale without Conformal Invariance: Theoretical Foundations", JHEP **1207**, 025 (2012), `arXiv:1107.3840`. ✦ J.F. Fortin, B. Grinstein & A. Stergiou, *"Scale without Conformal Invariance at Three Loops"*, JHEP **1208**, 085 (2012), `arXiv:1202.4757`.

[13] Y. Nakayama, *"Consistency of local renormalization group in d=3"*, `arXiv:1307.8048`.

[14] E. Barnes, K.A. Intriligator, B. Wecht & J. Wright, *"Evidence for the strongest version of the 4d a-theorem, via a-maximization along RG flows"*, Nucl.Phys. **B702**, 131 (2004), `hep-th/0408156`.

[15] Z. Komargodski & A. Schwimmer, *"On Renormalization Group Flows in Four Dimensions"*, JHEP **1112**, 099 (2011), `arXiv:1107.3987`.

[16] H. Elvang, D.Z. Freedman, L.Y. Hung, M. Kiermaier, R.C. Myers & S. Theisen, *"On renormalization group flows and the a-theorem in 6d"*, JHEP **1210**, 011 (2012), `arXiv:1205.3994`.

[17] H. Osborn, *"Local renormalization group equations in quantum field theory"*, in *"Dubna 1991 Proceedings, Renormalization group '91"*, p. 128.

[18] D. Capper & M. Duff, *"Trace anomalies in dimensional regularization"*, Nuovo Cim. **A23**, 173 (1974).

[19] D. Friedan & A. Konechny, *"Gradient formula for the beta-function of 2d quantum field theory"*, J.Phys. **A43**, 215401 (2010), `arXiv:0910.3109`.

[20] L. Bonora, P. Pasti & M. Bregola, *"Weyl Cocycles"*, Class.Quant.Grav. **3**, 635 (1986).

[21] F. Bastianelli, G. Cuoghi & L. Nocetti, *"Consistency conditions and trace anomalies in six-dimensions"*, Class.Quant.Grav. **18**, 793 (2001), `hep-th/0007222`.

[22] D. Lovelock, *"The Einstein tensor and its generalizations"*, J.Math.Phys. **12**, 498 (1971).

[23] S. Deser & A. Schwimmer, *"Geometric classification of conformal anomalies in arbitrary dimensions"*, Phys.Lett. **B309**, 279 (1993), `hep-th/9302047`.

[24] C.W. Misner, K. Thorne & J. Wheeler, *"Gravitation"*, W.H. Freeman and Co. (1973).