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ORIGINAL PAPER

# Stochastic stability of viscoelastic systems under Gaussian and Poisson white noise excitations

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**Abstract** As the use of viscoelastic materials becomes increasingly popular, stability of viscoelastic structures under random loads becomes increasingly important. This paper aims at studying the asymptotic stability of viscoelastic systems under Gaussian and Poisson white noise excitations with Lyapunov functions. The viscoelastic force is approximated as equivalent stiffness and damping terms. A stochastic differential equation is set up to represent randomly excited viscoelastic systems, from which a Lyapunov function is determined by intuition. The time derivative of this Lyapunov function is then obtained by stochastic averaging. Approximate conditions are derived for asymptotic Lyapunov stability with probability one of the viscoelastic system. Validity and utility of this approach are illustrated by a Duffing-type oscillator possessing viscoelastic forces, and the influence of different parameters on the stability region is delineated.

**Keywords** Stochastic stability · Gaussian and Poisson noise · Lyapunov function · Stochastic averaging · Viscoelastic system

## 1 Introduction

Viscoelasticity occurs in modern materials such as polymers, composites and alloys. A viscoelastic material can store energy due to its elasticity, and it can also dissipate energy through damping. In order to use these new materials at elevated temperatures with confidence, it is necessary to develop techniques for the analysis of viscoelastic structures under random loads [1,2]. Zhu and Cai [3] studied the response of a viscoelastic system under broad-band random excitations using the quasi-conservative averaging. Zhao et al. [4] investigated the stochastic stationary responses of a viscoelastic system with impacts under additive Gaussian white noise excitation using stochastic averaging method. Later, Wang et al. [5] and Zhao et al. [6] extended this method to explore responses of viscoelastic systems under additive and multiplicative random excitations.

Due to the existence of cyclic loads associated with earthquakes, high winds, or sea waves, the stability of viscoelastic systems has also been gained much attention in the literature [7–9]. The central issue in stability is the deviation of system response from an equilibrium or stationary state over time. There are many definitions of stochastic stability in the classical literature, for

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example Lyapunov stability with probability one, stability in probability, and stability in $p$th mean [10,11]. This paper is concerned with Lyapunov stability with probability one.

There are two common methods for investigating Lyapunov stability with probability one. They are, respectively, the method of largest Lyapunov exponent and the method of Lyapunov functions. The largest Lyapunov exponent, evaluated from an *Itô* stochastic differential equation, was first described by Khasminskii [12]. Using the largest Lyapunov exponents, the stability of certain two-dimensional systems under white noise or wideband excitations were obtained [13–15]. Later, Ariaratnam et al. [16], Zhu [17], and Liu et al. [18,19] extended this study to obtain the largest Lyapunov exponents of higher-dimensional systems. Using a combination of the method of averaging and the technique of Khas'minskii, Ariaratnam et al. [16] studied the almost-sure asymptotic stability of a class of coupled multi-degree-of-freedom systems subject to parametric excitation of small intensity. Zhu [17] examined the largest Lyapunov exponent of quasi-non-integrable Hamiltonian systems under Gaussian white noise. Subsequently, Liu et al. [18,19] used this method successfully to study stochastic stability of different Hamiltonian systems under Gaussian and Poisson white noise excitations.

A second major technique for studying Lyapunov stability with probability one is the Lyapunov function method. A Lyapunov function is a scalar real function, and it is often necessary to evaluate its derivative to deduce the stability of the equilibrium position. Similar to the largest Lyapunov exponent method, a combination of stochastic averaging and Lyapunov functions is also an efficient procedure to evaluate stochastic stability [20–22]. The asymptotic Lyapunov stability with probability one for quasi-Hamiltonian systems with Gaussian white noise was studied by Huang et al. [20], where the Hamiltonian was taken as the Lyapunov function. Ling et al. [21] adopted a specific linear combination of subsystem energies as the Lyapunov function and then obtained sufficient conditions for stochastic stability with probability one. Liu and Zhu [22] employed Lyapunov functions to study the asymptotic Lyapunov stability with probability one of multi-degree-of-freedom quasi-Hamiltonian systems subject to parametric excitations of combined Gaussian and Poisson white noises. However, stochastic stability of viscoelastic system subject to combined Gaussian and Poisson white noise excitations has not been reported in the open literature.

The objective of this paper is to conduct stability analysis of stochastic viscoelastic systems. The paper is organized as follows. In Sect. 2, the governing equation of viscoelastic systems and a brief description of Poisson white noise are given. A viscoelastic force is replaced as a sum of stiffness and damping forces in Sect. 3, and an *Itô*-type stochastic differential equation is set up to describe viscoelastic systems. A Lyapunov function of viscoelastic systems is constructed and sufficient conditions of stability with probability one are obtained in Sect. 4. In Sect. 5, an example involving a Duffing-type oscillator is given to illustrate the validity and application of the proposed method. Finally, conclusions are summarized in Sect. 6.

## 2 Problem statement

In modeling randomly excited viscoelastic systems, a general equation of motion is

$$\ddot{x} + \varepsilon^2 g(x, \dot{x}) + \varepsilon^2 u(x) + \varepsilon^2 Z = \varepsilon \sum_{k=1}^{n_1} f_{1k}(x, \dot{x})\xi_k(t) + \varepsilon \sum_{l=1}^{n_2} f_{2l}(x, \dot{x})\eta_l(t), \quad (1)$$

where $g(x, \dot{x})$ represents damping force, $u(x)$ nonlinear stiffness force, $Z$ the viscoelastic force, and $\varepsilon$ is a small parameter. The functions $f_{1k}(x, \dot{x})$ $(k = 1, 2, \ldots, n_1)$ are twice differentiable, and $f_{2l}(x, \dot{x})$ $(l = 1, 2, \ldots, n_2)$ are infinitely differentiable, $\xi_k(t)$ $(k = 1, 2, \ldots, n_1)$ are Gaussian white noises with zero mean and correlation functions $E[\xi_k(t)\xi_m(t+\tau)] = 2D_{km}\delta(\tau)(k, m = 1, 2, \ldots, n_1)$ and $\eta_l(t)(l = 1, 2, \ldots, n_2)$ are independent Poisson noises with zero mean. Stability of Eq. (1) is the focus of this paper.

Poisson white noises $\eta_l(t)$ are sequence of impulses arriving at Poisson times:

$$\eta_l(t) = \sum_{p=1}^{N_l(t)} Y_{lp}\delta(t - t_\mathrm{p}), \quad l = 1, 2 \ldots, n_2, \quad (2)$$

where $N_l(t)$ are a homogeneous Poisson counting processes with mean arrival rate $\lambda_l > 0$, which denote the number of pulses in the time interval $[0, t)$. In the above equations, $\delta(\cdot)$ is a delta function, and $Y_{lp}$ are independent and identically distributed random variables representing the impulse amplitudes, which are independent of the pulse occurring time $t_\mathrm{p}$. Poisson white noises $\eta_l(t)$ can be considered as formal derivatives of the following compound Poisson processes [22,23],

$$C_l(t) = \sum_{p=1}^{N_l(t)} Y_{lp} U(t - t_\mathrm{p}), \tag{3}$$

where $U(\cdot)$ is unit step function. The compound Poisson processes in Eq. (3) are defined as [23,26]

$$C_l(t) = \int_0^t \int_{Y_l} Y_l M_l(\mathrm{d}t, \mathrm{d}Y_l), \tag{4}$$

where $M_l(\mathrm{d}t, \mathrm{d}Y_l)$ are the Poisson random measures and $\mathbf{Y_l}$ denote the Poisson mark spaces. The $r$-order correlation functions of Poisson white noises have the following forms

$$\begin{aligned} &R^r \left[\eta_l(t_1), \eta_l(t_2), \ldots, \eta_l(t_r)\right] \\ &\quad = \lambda_l E\left[Y_l^r\right] \delta(t_2 - t_1) \ldots \delta(t_r - t_1), \\ &\quad\quad r = 2, 3, \ldots \infty. \end{aligned} \tag{5}$$

The powers and $j$th moments of $\mathrm{d}C_l(t)$ are

$$\begin{aligned} &(\mathrm{d}C_l(t))^j = \int_{Y_l} Y_l^j M_l(\mathrm{d}t, \mathrm{d}Y_l), \\ &E[(\mathrm{d}C_l(t))^j] = \lambda_l E\left[Y_l^j\right] \mathrm{d}t. \end{aligned} \tag{6}$$

A detailed description of Poisson white noise is given in [23–26].

## 3 Stochastic differential equation of viscoelastic systems

According to Zhu et al. [3–6], the viscoelastic force $Z$ is taken as

$$Z = \int_0^t h(t - \tau) x(\tau) \mathrm{d}\tau, \tag{7}$$

where $h(t)$ is relaxation function and can be obtained from the Maxwell model

$$h(t) = \sum_i \beta_i \exp\left(-\frac{t}{\alpha_i}\right), \quad \alpha_i \geq 0, \tag{8}$$

where $h(t)$ is composed of a number of relaxation domains, $\alpha_i (i = 1, 2, 3, \ldots)$ are the relaxation times, and $\beta_i (i = 1, 2, 3, \ldots)$ denote their magnitudes which can be either positive or negative. From Eqs. (7) and (8), the viscoelastic force can be rewritten as

$$Z = \int_0^t \sum_i \beta_i \exp\left(-\frac{t - \tau}{\alpha_i}\right) x(\tau) \mathrm{d}\tau. \tag{9}$$

Assume that the coefficients of all excitations are proportional to small parameters, on the basis of Liu and Zhu [27], $Z$ can be approximated as

$$x(t - s) \approx x \cos\bar{\omega}s - \frac{\dot{x}}{\bar{\omega}} \sin\bar{\omega}s, \tag{10}$$

Substituting Eq. (10) into the viscoelastic term, it follows that

$$\begin{aligned} &\int_0^t h(t - \tau) x(\tau) \mathrm{d}\tau = \int_0^t h(s) x(t - s) \mathrm{d}s \\ &= \int_0^t \sum_i \beta_i \exp\left(-\frac{s}{\alpha_i}\right) \left(x \cos\bar{\omega}s - \frac{\dot{x}}{\bar{\omega}} \sin\bar{\omega}s\right) \mathrm{d}s \\ &= \sum_i \left\{ \frac{\beta_i \alpha_i}{1 + \bar{\omega}^2 \alpha_i^2} x - \frac{\beta_i \alpha_i^2}{1 + \bar{\omega}^2 \alpha_i^2} \dot{x} \right. \\ &\quad + \frac{\alpha_i \exp(-t/\alpha_i)}{1 + \bar{\omega}^2 \alpha_i^2} \left[\beta_i x (\bar{\omega}\alpha_i \sin\bar{\omega}t - \cos\bar{\omega}t) \right. \\ &\quad \left.\left. + \frac{\beta_i \dot{x}}{\bar{\omega}} (\bar{\omega}\alpha_i \cos\bar{\omega}t + \sin\bar{\omega}t) \right] \right\}. \end{aligned} \tag{11}$$

When the system is exposed to the excitations for a long time, the terms associated with $\exp(-t/\alpha_i)$ can be ignored. Thus the viscoelastic force $Z$ can be approximated as equivalent stiffness and damping terms. Neglect the decaying transient part to obtain

$$Z = \sum_i \frac{\beta_i \alpha_i}{1 + \bar{\omega}^2 \alpha_i^2} (x - \alpha_i \dot{x}). \tag{12}$$

The averaged frequency $\bar{\omega}$ is determined by

$$\frac{\pi}{\bar{\omega}} = 2 \int_0^A \frac{\mathrm{d}x}{\sqrt{2H - 2U(x)}}, \tag{13}$$

where $H$ is the total energy function and $U(x)$ is potential function of system (1). The parameter $A$ is the amplitude for a given $H$, and the positive root is calculated from $H = U(A)$. It is found that the viscoelastic force can be split up into stiffness and damping terms, and the original system (1) can be rewritten as

$$\begin{aligned} &\ddot{x} + \varepsilon^2 g_1(x, \dot{x}) + \varepsilon^2 u_1(x) \\ &\quad = \varepsilon \sum_{k=1}^{n_1} f_{1k}(x, \dot{x}) \xi_k(t) + \varepsilon \sum_{l=1}^{n_2} f_{2l}(x, \dot{x}) \eta_l(t), \end{aligned} \tag{14}$$

where

$$g_1(x, \dot{x}) = g(x, \dot{x}) - \sum_i \frac{\beta_i \alpha_i^2}{1 + \bar{\omega}^2 \alpha_i^2} \dot{x}, \tag{15}$$

$$u_1(x) = u(x) + \sum_i \frac{\beta_i \alpha_i}{1 + \bar{\omega}^2 \alpha_i^2} x. \tag{16}$$

Based on Eqs. (14)–(16), the total energy function $H$ and potential function $U(x)$ can be expressed, respectively, as

$$H = \frac{1}{2}\dot{x}^2 + U(x), \tag{17}$$

$$U(x) = \int_0^x \varepsilon^2 u_1(z)\,\mathrm{d}z. \tag{18}$$

Let $x = x_1$, $\dot{x} = x_2$, and Eq. (14) is equivalent to a pair of first-order equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\varepsilon^2 g_1(x_1, x_2) - \varepsilon^2 u_1(x_1) + \varepsilon \sum_{k=1}^{n_1} f_{1k}(x_1, x_2)\,\xi_k(t) + \varepsilon \sum_{l=1}^{n_2} f_{2l}(x_1, x_2)\,\eta_l(t). \tag{19}$$

The system (19) can be modeled as the following *Itô*-type stochastic differential equation by adding Wong–Zakai correction terms [28],

$$\begin{aligned}\mathrm{d}x_1 &= x_2\mathrm{d}t, \quad \mathrm{d}x_2 = \Bigg(-\varepsilon^2 g_1(x_1, x_2) - \varepsilon^2 u_1(x_1) + \varepsilon^2 \sum_{k,m=1}^{n_1} D_{km} f_{1m}(x_1, x_2)\frac{\partial f_{1k}}{\partial x_2}\Bigg)\mathrm{d}t \\ &+ \varepsilon \sum_{k=1}^{n_1} \sigma_{1k}(x_1, x_2)\,\mathrm{d}B_k(t) + \sum_{l=1}^{n_2}\sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} f_{2l}^{(j)}(x_1, x_2)\left(\mathrm{d}C_l(t)\right)^j,\end{aligned} \tag{20}$$

where $\sigma_{1k} = \left(\sigma\sigma^{\mathrm{T}}\right)_{1k}$ and $\sigma\sigma^{\mathrm{T}} = 2fDf^{\mathrm{T}}$, $f_{2l}^{(j)}(x_1, x_2) = \frac{\partial f_{2l}^{j-1}}{\partial x_2} f_{2l}$, $f_{2l}^{(1)} = f_{2l}\ (l = 1, 2, \ldots, n_2)$. By means of the property of Poisson white noises Eq. (6), Eq. (20) is also equivalent to

$$\begin{aligned}\mathrm{d}x_1 &= x_2\mathrm{d}t, \quad \mathrm{d}x_2 = \Bigg(-\varepsilon^2 g_1(x_1, x_2) - \varepsilon^2 u_1(x_1) + \varepsilon^2 \sum_{k,m=1}^{n_1} D_{km} f_{1m}(x_1, x_2)\frac{\partial f_{1k}}{\partial x_2}\Bigg)\mathrm{d}t \\ &+ \varepsilon \sum_{k=1}^{n_1} \sigma_{1k}(x_1, x_2)\,\mathrm{d}B_k(t) + \sum_{l=1}^{n_2}\int_{Y_l} \gamma_l(x_1, x_2, Y_l) M_l(\mathrm{d}t, \mathrm{d}Y_l),\end{aligned} \tag{21}$$

where $\gamma_l(x_1, x_2, Y_l) = \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} f_{2l}^{(j)}(x_1, x_2)\, Y_l^j$. In the following sections, Eqs. (20) or (21) is used to determine the stochastic stability of the original system (1).

## 4 Lyapunov functions and stability of viscoelastic systems

The trivial solution $x_1 = x_2 = 0$ of system (21) is represented by the state vector $X = 0$. According to [20–22], the stability of the trivial solution can be deduced by a Lyapunov function $V(X, t)$ and its derivative in the vicinity of the equilibrium position. If $V(X, t) > 0$ in a domain except when $X = 0$, it is a positive definite function. In this paper, total energy $H$ in Eq. (17) is chosen as a Lyapunov function.

Apply the stochastic jump-diffusion chain rule [23, 26], the stochastic differential equation of $H$ can be derived from Eq. (21),

$$\begin{aligned}\mathrm{d}H(X, t) &= \varepsilon^2 \Bigg(-x_2 g_1(x_1, x_2) + x_2 \sum_{k,m=1}^{n_1} D_{km} f_{1m}\frac{\partial f_{1k}}{\partial x_2} + \frac{1}{2}\sum_{k=1}^{n_1}\sigma_{1k}^2\Bigg)\mathrm{d}t \\ &+ \varepsilon x_2 \sum_{k=1}^{n_1} \sigma_{1k}\mathrm{d}B_k(t) + \sum_{l=1}^{n_2}\int_{Y_l} \left(H(x_1, x_2 + \gamma_l) - H(x_1, x_2)\right) M_l(\mathrm{d}t, \mathrm{d}Y_l),\end{aligned} \tag{22}$$

where $H(x_1, x_2 + \gamma_l) - H(x_1, x_2)$ is calculated by its Taylor expansion, which results in

$$H(x_1, x_2 + \gamma_l) - H(x_1, x_2) = \sum_{k=1}^{\infty} \varepsilon^k Y_l^k A_{1,k,l}(x_1, x_2), \tag{23}$$

and the $j$th power of $H(x_1, x_2 + \gamma) - H(x_1, x_2)$ is obtained from Eq. (23) as

$$\left(H(x_1, x_2 + \gamma_l) - H(x_1, x_2)\right)^j = \left(\sum_{k=1}^{\infty} \varepsilon^k Y_l^k A_{1,k,l}\right)^j = \sum_{k=j}^{\infty} \varepsilon^k Y_l^k A_{j,k,l}(x_1, x_2). \tag{24}$$

Detailed calculations of Eqs. (23–24) are given in Refs. [23,29].

Using Eq. (24), the stochastic differential equation of $H$ can be rewritten in the following form

$$\begin{aligned}\mathrm{d}H(X, t) &= \varepsilon^2 \Bigg(-x_2 g_1(x_1, x_2) + x_2 \sum_{k,m=1}^{n_1} D_{km} f_{1m}\frac{\partial f_{1k}}{\partial x_2} + \frac{1}{2}\sum_{k=1}^{n_1}\sigma_{1k}^2\Bigg)\mathrm{d}t \\ &+ \varepsilon x_2 \sum_{k=1}^{n_1} \sigma_{1k}\mathrm{d}B_k(t)\end{aligned}$$

$$+\sum_{l=1}^{n_2}\int_{Y_l}\left(\sum_{k=1}^{\infty}\varepsilon^k A_{1,k,l}(x_1,x_2)\,Y_l^k\right)M_l(\mathrm{d}t,\mathrm{d}Y_l)\,, \tag{25}$$

Neglect all terms of power $\varepsilon^{u+1}$ and higher, the averaged *Itô* stochastic differential equation of $H$ has the form [22,23,29]

$$\begin{aligned}\mathrm{d}H(X,t) &= \left(\varepsilon^2\bar{m}(H)+\sum_{i=0}^{u-2}\varepsilon^{i+2}U_i(H)\right)\mathrm{d}t\\ &\quad+\varepsilon\bar{\sigma}(H)\mathrm{d}B(t)+\sum_{l=1}^{n_2}\sum_{k=1}^{u-1}\int_{\bar{Y}_{kl}}\\ &\quad\sum_{s=1}^{k}\varepsilon^s V_{k,s,l}(H)\bar{Y}_{kl}M_{kl}\left(\mathrm{d}t,\mathrm{d}\bar{Y}_{kl}\right)\\ &= \left(\varepsilon^2\bar{m}(H)+\sum_{i=0}^{u-2}\varepsilon^{i+2}U_i(H)\right)\mathrm{d}t\\ &\quad+\varepsilon\bar{\sigma}(H)\mathrm{d}B(t)+\sum_{l=1}^{n_2}\sum_{k=1}^{u-1}\sum_{s=1}^{k}\varepsilon^s V_{k,s,l}(H)\mathrm{d}\bar{C}_{kl},\end{aligned} \tag{26}$$

where $M_{kl}\left(\mathrm{d}t,\mathrm{d}\bar{Y}_{kl}\right)$ are independent Poisson random measures with the property

$$E\left[\left(\mathrm{d}\bar{C}_{kl}\right)^r\right]=\bar{\lambda}_{kl}E\left[\bar{Y}_{kl}^r\right]=\begin{cases}0, & r=1,\\ \bar{M}_{k,r,l}, & r=2,\ldots,u-k+1,\\ \bar{m}_{k,r,l}, & r=u-k+2,u-k+3,\ldots,\end{cases} \tag{27}$$

The parameters $\bar{M}_{k,r,l}\gg\varepsilon$, $\bar{m}_{k,r,l}<\varepsilon^u$. The parameters $\bar{\lambda}_{kl}$ represent the mean numbers of arrivals per unit time of compound Poisson processes $\bar{C}_{kl}$, and $\bar{Y}_{kl}$ are the random magnitudes of compound Poisson processes $\bar{C}_{kl}$. The expressions $\bar{m}(H)$, $U_i(H)$, $\bar{\sigma}(H)$ and $V_{k,s,l}(H)$ of Eq. (26) are

$$\begin{aligned}\bar{m}(H)=\frac{4}{T}\int_0^A&\Bigg(-x_2g_1(x_1,x_2)\\ &+x_2\sum_{k,l=1}^{n_1}D_{kl}f_{1l}\frac{\partial f_{1k}}{\partial x_2}\\ &+\frac{1}{2}\sum_{k=1}^{n_1}\sigma_{1k}^2\Bigg)\Big/x_2\,\mathrm{d}x_1|_{x_2=\sqrt{2H-2U(x_1)}},\end{aligned} \tag{28}$$

$$U_0(H)=\sum_{l=1}^{n_2}\frac{4\lambda_l E\left[Y_l^2\right]}{T}\int_0^A\frac{A_{1,2,l}}{x_2}\mathrm{d}x_1|_{x_2=\sqrt{2H-2U(x_1)}}, \tag{29}$$

$$U_k(H)=\sum_{l=1}^{n_2}\frac{4\lambda_l E\left[Y_l^{k+2}\right]}{T}\int_0^A\frac{A_{1,k+2,l}}{x_2}\mathrm{d}x_1|_{x_2=\sqrt{2H-2U(x_1)}},\ k=1,\ldots,u-2, \tag{30}$$

$$\bar{\sigma}(H)=\frac{4}{T}\int_0^A\sum_{k=1}^{n_1}\sigma_{1k}^2x_2\mathrm{d}x_1|_{x_2=\sqrt{2H-2U(x_1)}}, \tag{31}$$

$$\begin{aligned}&\sum_{k=1}^{u-1}\left(\sum_{s=1}^{k}\varepsilon^s V_{k,s,l}(H)\right)^j\bar{\lambda}_{kl}^*E\left[\bar{Y}_{kl}^{*j}\right]\\ &\quad=\sum_{m=j}^{u}\frac{4\varepsilon^m\lambda_l E\left[Y_l^m\right]}{T}\\ &\quad\int_0^A\frac{A_{j,m,l}}{x_2}\mathrm{d}x_1\Big|_{x_2=\sqrt{2H-2U(x_1)}},\end{aligned} \tag{32}$$

$$T=4\int_0^A\frac{1}{\sqrt{2H-2U(x_1)}}\mathrm{d}x_1, \tag{33}$$

where the parameter $A$ is defined in Eq. (13).

To suppress the stochastically varying terms in Eq. (26), conditional expectation is taken on both sides to yield

$$\left.\frac{\mathrm{d}E(H(X,t))}{\mathrm{d}t}\right|_{X=x}=L(X,t)\,, \tag{34}$$

where $L=\varepsilon^2\bar{m}(H)+\sum_{i=0}^{u-2}\varepsilon^{i+2}U_i(H)$. The Lyapunov stability with probability one of the trivial solution can be determined according to the sign of $H$ and $L$ in the vicinity of the trivial solution. Here are two criteria according to Refs. [20,22].

**Theorem 1** *If there is a positive definite function $H$, which is continuously differentiable with respect to $t$, infinitely differentiable with respect to $x_i$ and $L\le 0$, $t\ge t_0$, $0<||X||\le h$, then the trivial solution $X=0$ of system (20) or (21) is Lyapunov stable with probability one.*

**Theorem 2** *Except that system (20) or (21) satisfies the above condition of Theorem 1, $H$ has the infinitesimal superior limit, and $L<0$, $t\ge t_0$, $0<||X||\le h$. Then the trivial solution $X=0$ of system is asymptotically Lyapunov stable with probability one.*

Based on Theorem 2, the viscoelastic system is asymptotically Lyapunov stable with probability one in the averaged sense if the drift coefficient satisfies the following condition

$$L = \varepsilon^2 \bar{m}(H) + \sum_{i=0}^{u-2} \varepsilon^{i+2} U_i(H) < 0, \quad t \geq 0, 0 < H < h. \tag{35}$$

If $L$ of inequality (35) can be linearized at $H = 0$, the approximately sufficient condition of the asymptotic Lyapunov stability with probability one in the sense of average can be simplified as

$L'(0) < 0.$

Now, if $u = 4$, the truncated averaged *Itô* stochastic differential equation of $H$ is

$$\begin{aligned} \mathrm{d}H = & \left(\varepsilon^2 \bar{m}(H) + \varepsilon^2 U_0(H) + \varepsilon^3 U_1(H) + \varepsilon^4 U_2(H)\right) \mathrm{d}t \\ & + \varepsilon \bar{\sigma}(H) \mathrm{d}B(t) + \sum_{l=1}^{n_2} \varepsilon V_{11l} \mathrm{d}\bar{C}_{1l} \\ & + \sum_{l=1}^{n_2} \left(\varepsilon V_{21l} + \varepsilon^2 V_{22l}\right) \mathrm{d}\bar{C}_{2l} \\ & + \sum_{l=1}^{n_2} \left(\varepsilon V_{31l} + \varepsilon^2 V_{32l} + \varepsilon^3 V_{33l}\right) \mathrm{d}\bar{C}_{3l}. \end{aligned} \tag{36}$$

The approximately sufficient condition of the Lyapunov stability of Eq.(36)

$$L = \varepsilon^2 \bar{m}(H) + \varepsilon^2 U_0(H) + \varepsilon^3 U_1(H) + \varepsilon^4 U_2(H), \quad t \geq 0, \quad 0 < H < h. \tag{37}$$

## 5 Illustrative example

In this section, a Duffing oscillator possessing viscoelastic forces is considered to illustrate the validity and application of the method proposed in this paper. The governing equation of the example system is

$$\begin{aligned} & \ddot{x} + 2\xi \omega_0 \dot{x} + \omega_0^2 x + \gamma x^3 + Z = f_1 x \eta_1(t) + f_2 x \eta_2(t) \\ & Z = \beta_1 \int_0^t \mathrm{e}^{-(t-\tau)/\alpha_1} x(\tau) \mathrm{d}\tau, \end{aligned} \tag{38}$$

where the damping, nonlinear stiffness, and viscoelastic force are assumed to be the order of $\varepsilon^2$, $f_1$ and $f_2$ represent the amplitudes of excitations of order of $\varepsilon$, $\eta_1(t)$ is Gaussian white noise with intensity $2D$ and $\eta_2(t)$ is Poisson white noise with zero mean. According to Eqs. (17, 18), the system equivalent to Eq. (38) with $i = 1$ is

$$\ddot{x} + \alpha \dot{x} + \beta x + \gamma x^3 = f_1 x \eta_1(t) + f_2 x \eta_2(t), \tag{39}$$

where

$$\alpha = 2\xi \omega_0 - \frac{\beta_1 \alpha_1^2}{1 + \bar{\omega}^2 \alpha_1^2}, \quad \beta = \omega_0^2 + \frac{\beta_1 \alpha_1}{1 + \bar{\omega}^2 \alpha_1^2}. \tag{40}$$

Let $x = x_1$, $\dot{x} = x_2$, and Eq. (39) is equivalent to

$$\dot{x}_1 = x_2, \dot{x}_2 = -\alpha x_2 - \beta x_1 - \gamma x_1^3 + f_1 x_1 \eta_1(t) + f_2 x_1 \eta_2(t). \tag{41}$$

The above equation can be converted to the Itô-type stochastic differential equation

$$\begin{aligned} \mathrm{d}x_1 = x_2 \mathrm{d}t, \mathrm{d}x_2 = & \left(-\alpha x_2 - \beta x_1 - \gamma x_1^3\right) \mathrm{d}t \\ & + \sqrt{2D} f_1 x_1 \mathrm{d}B(t) \\ & + \int_Y f_2 x_1 Y M(\mathrm{d}t, \mathrm{d}Y). \end{aligned} \tag{42}$$

The trivial solution is $X = (x_1, x_2) = 0$. The total energy is taken as Lyapunov function of system (42) with

$$H = \frac{1}{2} x_2^2 + \frac{1}{2} \beta x_1^2 + \frac{\gamma}{4} x_1^4. \tag{43}$$

It is found that $H$ is positive in the domain except when $X = 0$. Thus the stochastic differential equation of $H$ is

$$\begin{aligned} \mathrm{d}H = & \left(D f_1^2 x_1^2 - \alpha x_2^2\right) \mathrm{d}t + \sqrt{2D} f_1 x_1 x_2 \mathrm{d}B(t) \\ & + \int_Y \left[H(x_1, x_2 + \kappa) - H(x_1, x_2)\right] M(\mathrm{d}t, \mathrm{d}Y), \end{aligned} \tag{44}$$

where $\kappa = f_2 x_1 Y$ and

$$\begin{aligned} & H(x_1, x_2 + \kappa) - H(x_1, x_2) \\ & = \frac{1}{2} (x_2 + f_2 x_1 Y)^2 - \frac{1}{2} x_2^2 \\ & = f_2 x_1 x_2 Y + \frac{1}{2} f_2^2 x_1^2 Y^2. \end{aligned} \tag{45}$$

The averaged *Itô* stochastic differential equation of $H$ is

$$\begin{aligned} \mathrm{d}H = & (\bar{m}(H) + U(H)) \mathrm{d}t + \bar{\sigma}^2(H) d\mathrm{B}(t) \\ & + \sum_{k=1}^{\mathrm{u}} \int_{\bar{Y}_k} \sum_{s=1}^{k} V_{k,s}(H) \bar{Y}_k \left(\mathrm{d}t, \mathrm{d}\bar{Y}_k\right), \end{aligned} \tag{46}$$

where

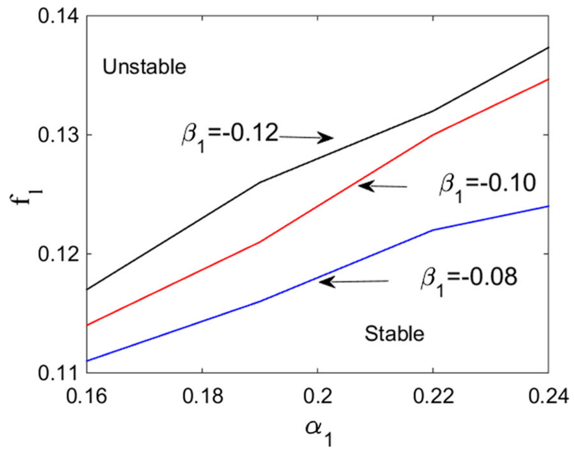


**Fig. 1** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $\beta_1$ by using the Lyapunov function method

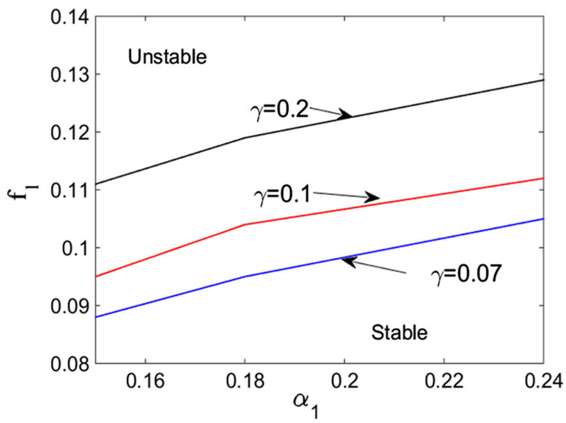


**Fig. 2** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $\gamma$ by using the Lyapunov function method

$$\bar{m}(H) = \frac{4}{T}\int_0^A \frac{Df_1^2x_1^2 - \alpha x_2^2}{x_2}\mathrm{d}x_1\bigg|_{x_2=\sqrt{2H-\beta x_1^2-\frac{\gamma}{2}x_1^4}}, \quad (47)$$

$$U(H) = \frac{2}{T}\int_0^A \frac{\lambda_1 E\left(Y_1^2\right) f_2^2x_1^2}{x_2}\mathrm{d}x_1\bigg|_{x_2=\sqrt{2H-\beta x_1^2-\frac{\gamma}{2}x_1^4}}, \quad (48)$$

$$T = 4\int_0^A \frac{1}{x_2}\mathrm{d}x_1, \quad (49)$$

in which the amplitude $A$ is the positive root of the equation $U(A) = H$. Take conditional expectation on both sides of Eq. (46) to obtain

$$\frac{\mathrm{d}E[H]}{\mathrm{d}t} = \bar{m}(H) + U(H). \quad (50)$$

Based on the results given in Sect. 4, system (42) is asymptotically Lyapunov stable with probability one in the averaged sense if

$$\bar{m}(H) + U(H) < 0, \quad t \geq 0,\ 0 < H < h. \quad (51)$$

If $\bar{m}(H)+U(H)$ in the above inequality is linearized at $H = 0$, an approximate condition for asymptotic Lyapunov stability with probability one can be simplified to

$$\bar{m}'(0) + U'(0) < 0. \quad (52)$$

In the following simulations of system (38), the values of different parameters are chosen as $\lambda_1 = 1$, $D = 0.05$, $EY_1^2 = 0.05$, $\xi = 0.05$, $\omega_0 = 0.5$. According to Eq. (38), $\beta_1$ is the magnitude of the viscoelastic force. And it can be seen in Eq. (40) that the lager magnitude $\beta_1$, the lower damping $\alpha$ and the higher stiffness $\beta$. But it is known that lower damping makes stability region smaller, and higher stiffness makes stability region larger. This means that the magnitude $\beta_1$ has two opposite effects on the stability of system (38). The boundaries of asymptotic Lyapunov stability with probability one of system (38) in plane $(\alpha_1, f_1)$ without Poisson noise are shown in Figs. 1 and 2. Figure 1 shows that the stability region become smaller when $\beta_1$ increases, so it indicates that the damping effect is predominant over that of the stiffness and the effect of the viscoelastic force on the stiffness is insignificant. Figure 2 shows the boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $\gamma$, and it is observed that the area of stability region increases when $\gamma$ is increased; it shows that the effect of the nonlinear stiffness coefficient $\gamma$ on the stability of Duffing system; what is more, the parameter $\gamma$ is even chosen as 0.2, so it means that this method can work for the system with strong nonlinear stiffness. Figure 3 shows the boundaries of asymptotic Lyapunov stability of system (38) with different $f_2$. It is observed that as $f_2$ increases, the stability region decreases. It indicates that Poisson white noise excitation has important influence on the stability of Duffing system. Figures 4 and 5 show the boundaries of asymptotic Lyapunov stability with probability one of system (38) in plane $(\alpha_1, f_1)$ under Poisson noise.

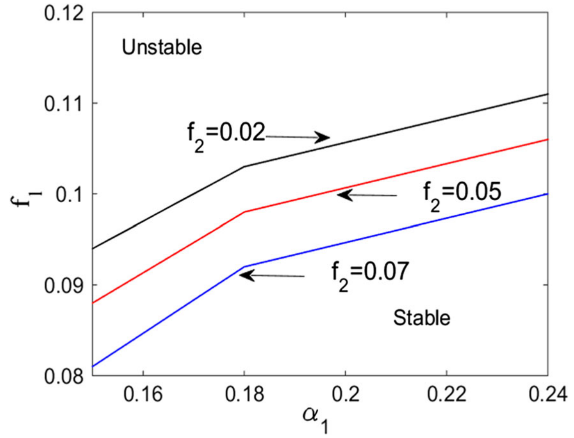


**Fig. 3** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $f_2$ by using the Lyapunov function method

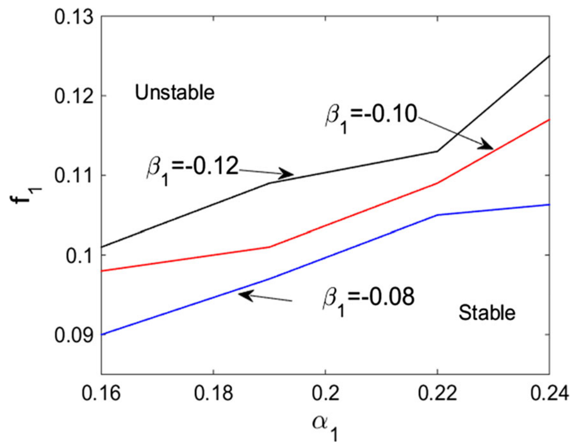


**Fig. 4** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $\beta_1$ and $f_2 = 0.01$ by using the Lyapunov function method

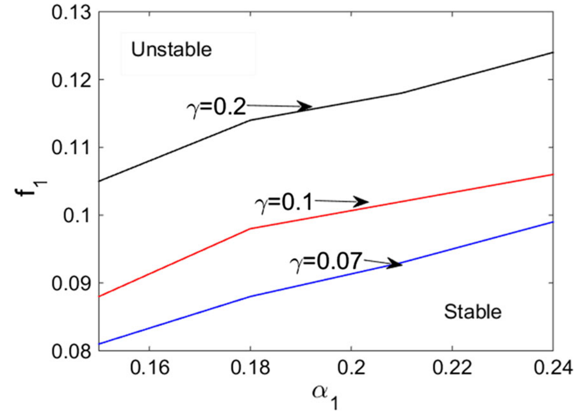


**Fig. 5** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with different $\gamma$ and $f_2 = 0.05$ by using the Lyapunov function method

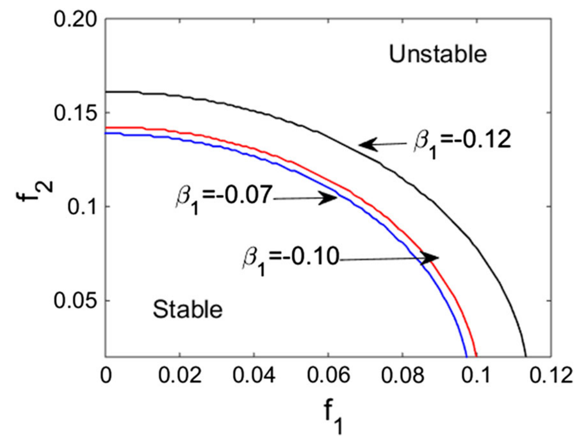


**Fig. 6** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with $\alpha_1 = 0.2, \gamma = 0.1$ by using the Lyapunov function method

With Poisson noise $f_2$, the stability region of system (38) is smaller than that without Poisson noise $f_2$. The stability region of system (38) is also reduced when $\beta_1$ increases and expands when $\gamma$ is increased. These results coincide with those without Poisson noise. Figures 6, 7, and 8 show that the boundaries of asymptotic Lyapunov stability with probability one of system (38) in plane $(f_1, f_2)$. We can find that the boundary value of the intensity of Gaussian noises is smaller than that of Poisson noise in stable region for system (38). It implies that the effect of Gaussian white noise on the stability of system is stronger than that of Poisson white noise.

To verify the effectiveness and correctness of the analytical results, the directly numerical simulations of the largest Lyapunov exponent of system (38) are performed here. The largest Lyapunov exponent is another efficient method to evaluate the stochastic stability [20–22]. Figures 9 and 10 show that the boundaries of asymptotic Lyapunov stability of Duffing system (38) with different parameters by using the Lyapunov function method and the largest Lyapunov exponent method. It is found that the domain of asymptotic stability by using Lyapunov function method is smaller than that by using the largest Lyapunov exponent method. The reason is that the stable condition, obtained by Lyapunov function method, is sufficient, while it is neces-

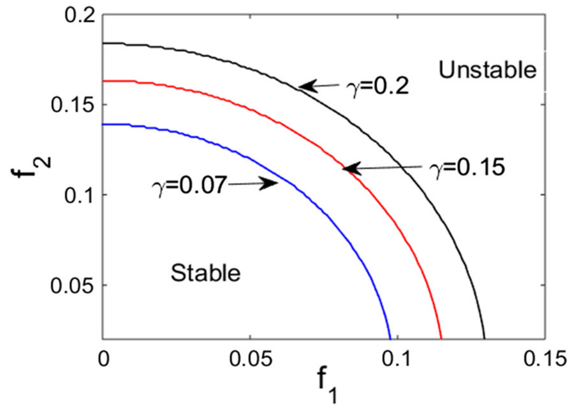


**Fig. 7** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with $\alpha_1 = 0.2$, $\beta_1 = -0.1$ by using the Lyapunov function method

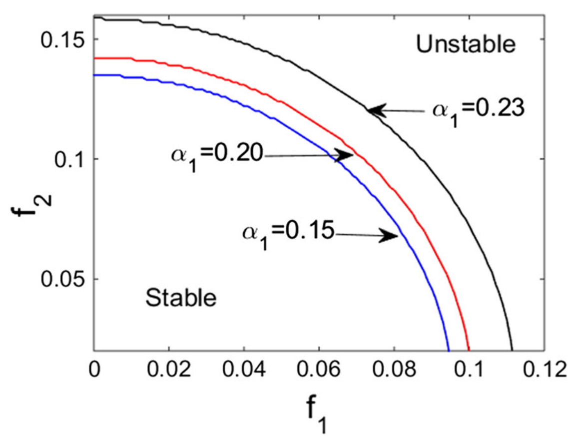


**Fig. 8** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) wit $\beta_1 = -0.1$, $\gamma = 0.1$ by using the Lyapunov function method

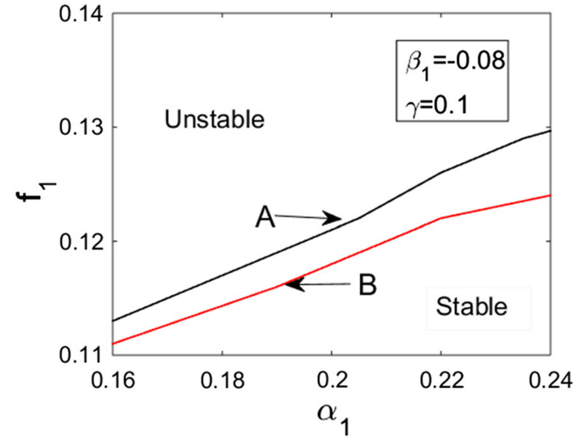


**Fig. 9** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) without Poisson noise by using the Lyapunov function method and the largest Lyapunov exponent method. A: The largest Lyapunov exponent method. B: The Lyapunov function method

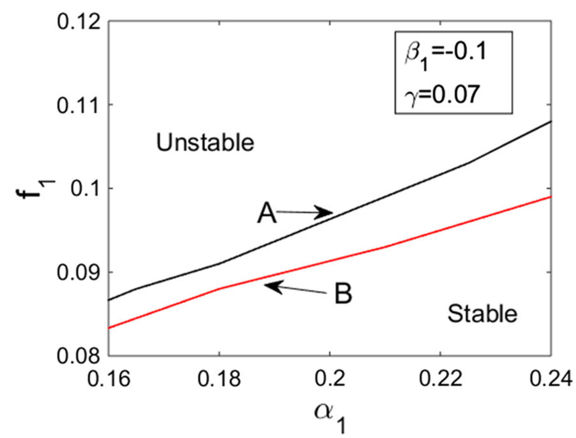


**Fig. 10** Boundaries of asymptotic Lyapunov stability with probability one of Duffing system (38) with $f_2 = 0.05$ by using the Lyapunov function method and the largest Lyapunov exponent method. A: The largest Lyapunov exponent method. B: The Lyapunov function method

sary and sufficient, obtained by the largest Lyapunov exponent method. That implies smaller stability area, obtained by using Lyapunov function method, is reasonable and our proposed method is effective.

## 6 Conclusions

The asymptotic Lyapunov stability with probability one of viscoelastic systems under combined Gaussian and Poisson white noise excitations has been studied. Viscoelastic forces are treated as a sum of stiffness and damping forces, and a stochastic differential equation has been set up to represent randomly excited viscoelastic systems. Total energy of the system is chosen as a Lyapunov function, whose time derivative is obtained by stochastic averaging. Approximate conditions that guarantee asymptotic Lyapunov stability with probability one of viscoelastic systems have been deduced. An example involving a Duffing-type oscillator has been given to illustrate the validity and application of the approach presented herein. As the use of viscoelastic materials becomes increasingly popular, analysis of the behavior of viscoelastic systems becomes increasingly important. This paper contributes to the development of analytical techniques for stability analysis of randomly excited viscoelastic systems.

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## References


1. Xie, W.C.: Moment Lyapunov exponents of a two-dimensional viscoelastic system under bounded noise excitation. ASME J. Appl. Mech. **69**, 346–357 (2002)
2. Deng, J., Xie, W.C., Pandey, M.D.: Stochastic stability of SDOF linear viscoelastic system under wideband noise excitation. Probab. Eng. Mech. **39**, 10–22 (2015)
3. Zhu, W.Q., Cai, G.Q.: Random vibration of viscoelastic system under broad-band excitations. Int. J. Non-Linear Mech. **46**, 720–726 (2011)
4. Zhao, X.R., Xu, W., Gu, X.D., Yang, Y.G.: Stochastic stationary responses of a viscoelastic system with impacts under additive Gaussian white noise excitation. Phys. A **31**, 128–139 (2015)
5. Wang, D.L., Xu, W., Gu, X.D., Yang, Y.G.: Stationary response analysis of vibro-impact system with a unilateral nonzero offset barrier and viscoelastic damping under random excitations. Nonlinear Dyn. **86**, 891–909 (2016)
6. Zhao, X.R., Xu, W., Yang, Y.G., Wang, X.Y.: Stochastic responses of a viscoelastic—impact system under additive and multiplicative random excitations. Commun. Nonlinear Sci. Numer. Simul. **35**, 166–176 (2016)
7. Ling, Q., Jin, X.L., Huang, Z.L.: Response and stability of SDOF viscoelastic system under wideband noise excitations. J. Frankl. Inst. **348**, 2026–2043 (2011)
8. Potapov, V.D.: On almost sure stability of a viscoelastic column under random loading. J. Sound Vib. **173**, 301–308 (1994)
9. Floris, C.: Stochastic stability of a viscoelastic column axially loaded by a white noise force. Mech. Res. Commun. **38**, 57–61 (2011)
10. Khasminskii, R.: Stochastic Stability of Differential Equations. Springer, Berlin (2012)
11. Zhu, W.Q.: Nonlinear stochastic dynamics and control in Hamiltonian formulation. Appl. Mech. Rev. **59**, 230–248 (2006)
12. Khasminskii, R.Z.: Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. Theory Probab. Appl. **12**, 144–147 (1967)
13. Mitchell, R.R., Kozin, F.: Sample stability of second order-linear differential equations with wide band noise coefficients. SIAM J. Appl. Math. **27**, 571–605 (1974)
14. Nishioka, K.: On the stability of two-dimensional linear stochastic systems. Kodai Math. Sem. Rep. **27**, 211–230 (1976)
15. Arnold, L., Papanicolaou, G., Wihstutz, V.: Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator and applications. SIAM J. Appl. Math. **46**, 427–450 (1986)
16. Ariaratnam, S.T., Xie, W.C.: Lyapunov exponents and stochastic stability of coupled linear systems under real noise excitation. ASME J. Appl. Mech. **59**, 664–673 (1992)
17. Zhu, W.Q.: Lyapunov exponent and stochastic stability of quasi-non-integrable Hamiltonian systems. Int. J. Non-Linear Mech. **39**, 569–579 (2004)
18. Liu, W.Y., Zhu, W.Q., Xu, W.: Stochastic stability of quasi non-integrable Hamiltonian systems under parametric excitations of Gaussian and Poisson white noises. Probab. Eng. Mech. **32**, 39–47 (2013)
19. Liu, W.Y., Zhu, W.Q.: Stochastic stability of quasi-integrable and resonant Hamiltonian systems under parametric excitations of combined Gaussian and Poisson white noises. Int. J. Non-Linear Mech. **67**, 52–62 (2014)
20. Huang, Z.L., Jin, X.L., Zhu, W.Q.: Lyapunov functions for quasi-Hamiltonian systems. Probab. Eng. Mech. **24**, 374–381 (2009)
21. Ling, Q., Jin, X.L., Wang, Y., Li, H.F., Huang, Z.L.: Lyapunov function construction for nonlinear stochastic dynamical systems. Nonlinear Dyn. **72**, 853–864 (2013)
22. Liu, W.Y., Zhu, W.Q.: Lyapunov function method for analyzing stability of quasi-Hamiltonian systems under combined Gaussian and Poisson white noise excitations. Nonlinear Dyn. **81**, 1879–1893 (2015)
23. Jia, W.T., Zhu, W.Q., Xu, Y.: Stochastic averaging of quasi-nonintegrable Hamiltonian systems under combined Gaussian and Poisson white noise excitations. Int. J. Non-Linear Mech. **51**, 45–53 (2013)
24. Di Paola, M., Vasta, M.: Stochastic integro-differential and differential equations of non-linear systems excited by parametric Poisson pulses. Int. J. Non-Linear Mech. **32**, 855–862 (1997)
25. Di Paola, M., Falsone, G.: Stochastic dynamics of nonlinear-systems driven by non-normal delta-correlated processes. ASME J. Appl. Mech. **60**, 141–148 (1993)
26. Hanson, F.B.: Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis, and Computation. Society for Industrial and Applied Mathematics, Philadelphia (2007)
27. Liu, Z.H., Zhu, W.Q.: Stochastic averaging of quasi-integrable Hamiltonian systems with delayed feedback control. J. Sound Vib. **299**, 178–195 (2007)
28. Wong, E., Zakai, M.: On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci. **3**, 213–229 (1965)
29. Zeng, Y., Zhu, W.Q.: Stochastic averaging of Quasi-non integrable-Hamiltonian systems under Poisson white noise excitation. ASME J. Appl. Mech. **78**, 021002 (2010)