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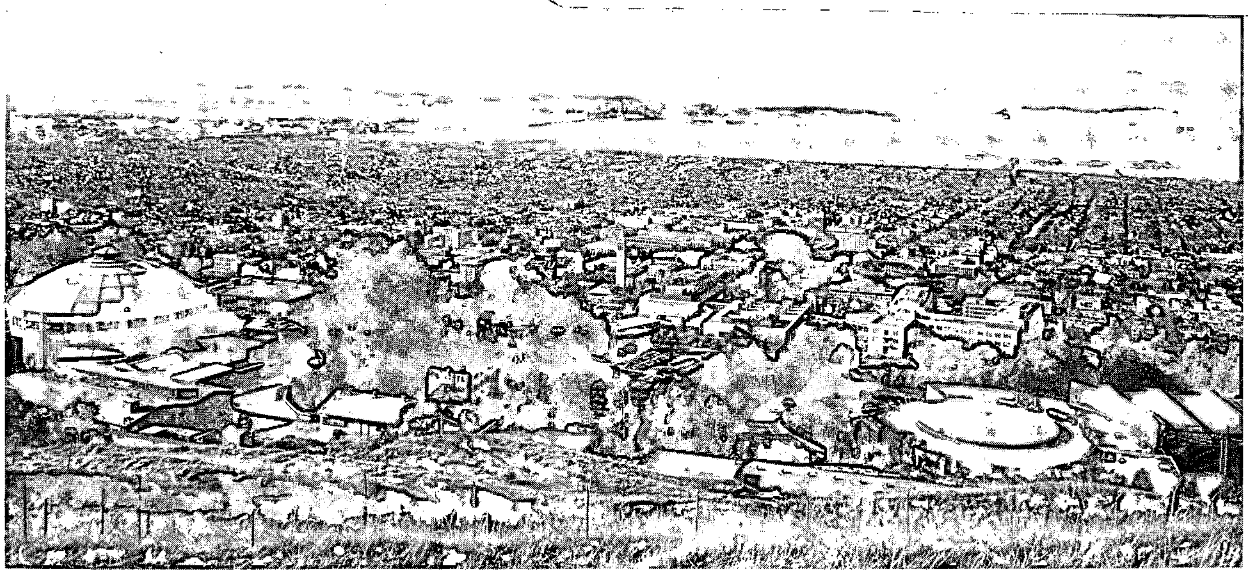

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A NEW FORMULATION FOR ONE-DIMENSIONAL HORIZONTAL IMBIBITION IN UNSATURATED POROUS MEDIA

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ABSTRACT

A formulation for one-dimensional horizontal imbibition in unsaturated porous media is developed using a new concept of a relative imbibition rate function. An exact semi-analytical solution is obtained and analyzed in the Buckley-Leverett style but with the effect of capillary pressure included. In the formulation, the problem is reduced to an integral equation that is easily solved by a rapidly-converging iteration process. This method has some distinct advantages over the existing methods in its generality and simplicity. Examples for the horizontal imbibition of water are discussed using the van Genuchten equations for relative permeability and capillary pressure .

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1. INTRODUCTION

Infiltration of water into an unsaturated porous medium has traditionally been described by Richards' equation. Richards [1931] developed this equation by assuming that, as infiltration takes place, pressure in the air phase remains constant. For the one-dimensional, horizontal case, the Richards equation reduces to a nonlinear diffusion equation. This equation is also applicable to the early stages of vertical infiltration where the gravitational forces are negligible compared to the capillary forces. As Philip [1969] has pointed out, solutions for horizontal imbibition are also important because they provide a basis for developing a solution to the more general problem of infiltration under the influence of gravity.

The nonlinear diffusion equation that describes the process of horizontal imbibition can be solved using the Boltzmann [1894] transformation to obtain a self-similar solution. Klute [1952a, 1952b] has used this approach to obtain numerical solutions, and Philip [1955, 1957a] has extended this process somewhat by converting the resulting ordinary differential equation to obtain an integro-differential equation that can be solved iteratively and numerically. Philip's method is more rapid and yields more accurate answers. In analyzing the mechanism of capillary imbibition in porous media, Rizhik [1959; in Barenblatt et al., 1972] solved the same self-similar problem semi-analytically and numerically. An important contribution of their work was the observation that if the initial saturation of the wetting phase is less than or equal to the irreducible saturation for this phase, then there will be a clearly defined wetting front that travels at a finite velocity. An analytical solution has been proposed by Philip [1960a, 1960b] but with the limitation that the characteristic curves for the imbibition process must be represented by a series of inverse error functions.

In this paper, a new formulation and a semi-analytic exact solution for one-dimensional, horizontal imbibition of water into unsaturated media is obtained by means of an adaptation of a procedure that was first proposed by McWhorter [1971].

This new solution is valid for any uniform initial water saturation and any shape for the characteristic curves. The computational effort only requires numerical integration of definite integrals and some iteration procedures.

2. SELF-SIMILAR PROBLEM OF HORIZONTAL IMBIBITION

Consider a linear porous medium of length L that initially is only partially saturated with water. Water is placed in contact with the porous medium at its inlet, x=0. Under the action of the capillary forces, water will start to imbibe into the unsaturated porous medium, displacing air.

Isothermal flow of water can be described by the generalized Darcy's law [Muskat and Meres, 1936]:

$$u_w = -\frac{k\,k_{rw}(S_w)}{\mu_w}\frac{\partial p_w}{\partial x} \tag{1}$$

in which S_w is the water saturation, u_w is the water velocity, and k_{rw} is the relative permeability of the medium to water. The continuity equation for water is

$$\frac{\partial u_w}{\partial x} + \phi\frac{\partial S_w}{\partial t} = 0 \tag{2}$$

where ϕ is the porosity. At the interface between water and air, there exists a capillary pressure difference that can be expressed as a function of water saturation, S_w, i.e.,

$$p_a - p_w = p_c(S_w) \tag{3}$$

Assuming that the air viscosity is vanishingly small, the pressure in the air phase remains constant and equals p_0, which is the pressure at the inlet where $x=0$. Then:

$$p_w = p_0 - p_c(S_w) \tag{4}$$

Substitution of Equation (4) into Equation (1) gives:

$$u_w = \frac{k}{\mu_w} k_{rw}(S_w) p_c'(S_w) \frac{\partial S_w}{\partial x} \tag{5}$$

where,

$$p_c'(S_w) = \frac{dp_c(S_w)}{dS_w}$$

Substituting Equation (5) into Equation (2), we obtain:

$$\phi \frac{\partial S_w}{\partial t} + \frac{k}{\mu_w} \frac{\partial}{\partial x} \left[k_{rw}(S_w) p_c'(S_w) \frac{\partial S_w}{\partial x} \right] = 0 \tag{6}$$

which is the Richards equation traditionally used to describe the process of imbibition into a one-dimensional, horizontal porous medium.

Equation (6) is a nonlinear diffusion-type equation, and in general, is very difficult to solve in closed-form. However, under certain conditions, some self-similar solutions are obtainable. To develop a self-similar problem, let the length of the flow column L extend to infinity and consider the case where the system is initially saturated with a uniform water saturation, S_{wi}. This initial saturation may be above or below the irreducible water saturation, S_{iw}. Then the initial and boundary conditions become:

$$S_w(x, t=0) = S_{wi} \tag{7a}$$

$$S_w(x \rightarrow \infty, t > 0) = S_{wi} \tag{7b}$$

$$S_w(x = 0, t > 0) = S_w^* \tag{7c}$$

where S_w^* satisfies the condition $p_c(S_w^*) = 0$.

We now introduce the Boltzmann transformation:

$$\xi = \frac{x}{\sqrt{t}} \tag{8}$$

In terms of this new variable, the partial differential equation (6), is transformed into the following ordinary differential equation:

$$\frac{d}{d\xi}\left[k_{rw}(S_w)p_c'(S_w)\frac{ds_w}{d\xi}\right] - \frac{\xi}{2}\frac{dS_w}{d\xi} = 0 \tag{9}$$

and the initial and boundary conditions in Equations (7a, 7b, 7c) transform to:

$$S_w(\xi = 0) = S_w^* \tag{10a}$$

$$S_w(\xi \rightarrow \infty) = S_{wi} \tag{10b}$$

The self-similar problem given by Equations (9), (10a), and (10b) has been solved by Klute [1952a, 1952b], Philip [1955, 1960], Rizhik [1959; in Barenblatt et al., 1972]. In particular, Rizhik solved this problem semi-analytically for the case of $S_w \leq S_{iw}$ using an important finding by Barenblatt [1952]. He showed that for such nonlinear ordinary differential equations, the velocity of the propagating front is finite when $S_w \leq S_{iw}$.

This implies that the self-similar solution for the case of $S_w \leq S_{iw}$ not only applies to a system of infinite length but also to a finite length system as long as the front has not reached the end, $x = L$.

In the next section, we shall formulate the self-similar problem using an integral equation and develop an exact semi-analytical solution in the Buckley-Leverett [1942] style but with capillary pressure effect included. The new formulation and solution are obtained using an adaptation of a procedure that was first proposed by McWhorter [1971].

3. INTEGRAL EQUATION FORMULATION AND EXACT SEMI-ANALYTIC SOLUTION

We shall introduce a relative imbibition rate function, $f(S_w)$, defined by:

$$f(S_w) = \frac{u_w}{u_{w0}} \tag{11}$$

where u_{w0} is the expected water imbibition rate into the porous medium at $x = 0$. Many workers have shown that the imbibition rate is inversely proportional to $\sqrt{t}$, and therefore, we can define u_{w0} as:

$$u_{w0} = \frac{u_0}{\sqrt{t}} \tag{12}$$

where u_0 is some constant to be determined. The function $f(S_w)$ represents the ratio of the water imbibition rate at any given cross-section with saturation S_w to the imbibition rate at the inlet. Combining Equations (5), (11) and (12), we obtain:

$$f(S_w) = \frac{\sqrt{t}k}{u_0\mu_w} k_{rw}(S_w) p_c'(S_w) \frac{\partial S_w}{\partial x} \tag{13}$$

If we use us the Boltzmann transformation given by Equation (8) to change the independent variable from x to ξ, we arrive at

$$f(S_w) = \frac{k}{u_0\mu_w} k_{rw}(S_w) p_c'(S_w) \frac{dS_w}{d\xi} \tag{14}$$

Equation (6) can now be rewritten in terms of the relative imbibition rate, $f(S_w)$, as:

$$\frac{\partial S_w}{\partial t} + \frac{u_0}{\sqrt{t}\phi} \frac{\partial f(S_w)}{\partial x} = 0$$

which can also be written in the form as:

$$\frac{df(S_w)}{d\xi} - \frac{\phi}{2u_0} \xi \frac{dS_w}{d\xi} = 0 \tag{15}$$

Equation (15) gives us an important relationship

$$\xi = \frac{2u_0}{\phi} \frac{df(S_w)}{dS_w} \tag{16}$$

If $f(S_w)$, subject to appropriate boundary conditions, and u_0 can be found, Equation (16) provides the solution to the self-similar problem for S_w given by Equations (9), (10a) and (10b).

Let us now investigate $f(S_w)$ and u_0. Differentiating Equation (16) with respect to S_w gives:

$$\frac{d\xi}{dS_w} = \frac{2u_0}{\phi}\frac{d^2f(S_w)}{dS_w^2} \tag{17}$$

Substituting $dS_w/d\xi$ from Equation (14) into (17), we obtain a non-linear ordinary differential equation of second order for the relative imbibition rate function: $f(S_w)$:

$$\frac{d^2f(S_w)}{dS_w^2} - \frac{\phi k}{2\mu_w u_0^2}\frac{k_{rw}(S_w)p_c'(S_w)}{f(S_w)} = 0 \tag{18}$$

Since Equation (18) is a second-order differential equation, two boundary conditions are needed. From Equation (10b), $S_w \rightarrow S_{wi}$ when $\xi \rightarrow \infty$. Since $dS_w/d\xi \rightarrow 0$ when $\xi \rightarrow \infty$, Equation (14) then shows that:

$$f(S_w)\Big|_{S_w=S_{wi}} = 0 \tag{19}$$

The second boundary condition can be obtained by noting from Equation (11) that $f(S_w)=1$ at the inlet. Then, in view of boundary condition (10a):

$$f(S_w)\Big|_{S_w=S_w^*} = 1 \tag{20}$$

A closed-form solution of Equation (18) subject to (19) and (20) is extremely difficult to obtain because of its nonlinearity. However, this problem can be transformed into an integral equation and then solved numerically by an iterative process.

Direct integration of Equation (18) gives:

$$\frac{df(S_w)}{dS_w} = \frac{\phi k}{2\mu_w u_0^2}\int_{S_w^*}^{S_w}\frac{k_{rw}(S_w)p_c'(S_w)}{f(S_w)}dS_w + C_1 \tag{21}$$

Integrating once again, and reversing the order of the two integrations, results in the following equation for $f(S_w)$:

$$f(S_w) = \frac{\phi k}{2\mu_w u_0^2} \int_{S_w^*}^{S_w} \frac{(S_w - \alpha) k_{rw}(\alpha) p_c'(\alpha)}{f(\alpha)} d\alpha + C_1 (S_w - S_w^*) + C_2 \tag{22}$$

C_1 and C_2 are arbitrary constants of integration that can be determined from boundary conditions (19) and (20) as:

$$C_1 = -\frac{1}{S_{wi} - S_w^*} \left[\frac{\phi k}{2\mu_w u_0^2} \int_{S_w^*}^{S_{wi}} \frac{(S_{wi} - \alpha) k_{rw}(\alpha) p_c'(\alpha)}{f(\alpha)} d\alpha + 1 \right] \tag{23}$$

$$C_2 = 1 \tag{24}$$

Next, the value of u_0 can be determined if we recognize that in the case of imbibition, there is the fundamental condition that $\partial p_c(S_w)/\partial x \big|_{x=0}$ cannot equal zero. Then, according to Equation (16):

$$\left. \frac{df(S_w)}{dS_w} \right|_{S_w = S_w^*} = 0 \tag{25}$$

must be true in order to satisfy boundary condition (10a) because $S_w = S_w^*$ can only occur at $x = 0$. In addition, we see from Equation (21) that when $S_w = S_w^*$:

$$\left. \frac{df(S_w)}{dS_w} \right|_{S_w = S_w^*} = C_1 \tag{26}$$

Comparison of Equations (25) and (26) shows that $C_1=0$, so that Equation (23) leads to the following expression for the imbibition constant u_0:

$$u_0 = \left[\frac{\phi k}{2\mu_w} \int_{S_{wi}}^{S_w^*} \frac{(S_{wi} - \alpha) k_{rw}(\alpha) p_c'(\alpha)}{f(\alpha)} d\alpha \right]^{1/2} \tag{27}$$

Now, Equations (16) and (22) can be written as:

$$\xi = -\frac{k}{\mu_w} u_0 \int_{S_w}^{S_w^*} \frac{k_{rw}(S_w) p_c'(S_w)}{f(S_w)} dS_w \tag{28}$$

$$f(S_w) = 1 - \frac{\displaystyle\int_{S_w}^{S_w^*} \frac{(S_w - \alpha) k_{rw}(\alpha) p_c'(\alpha)}{f(\alpha)} d\alpha}{\displaystyle\int_{S_{wi}}^{S_w^*} \frac{(S_{wi} - \alpha) k_{rw}(\alpha) p_c'(\alpha)}{f(\alpha)} d\alpha} \tag{29}$$

Integral equation (29) can easily be solved using the following numerical iteration process. An initial "guess" for $f(S_w)$ is used used on the rightside of Equation (29), and the integration yields a new estimate of $f(S_w)$. The process can be repeated until the difference between "new" and "old" values for $f(S_w)$ is less than some acceptable value. An obvious choice for the initial guess is $f^{(0)}(S_w)=(S_w-S_{wi})/(S_w^*-S_{wi})$, since this function satisfies the boundary conditions (19) and (20).

The cumulative volume of imbibed water, V_w, can be determined from:

$$V_w = \phi A \int_0^{x_f} S_w dx$$

or

$$V_w = A\int_0^t \frac{u_0}{\sqrt{t}}dt$$

It is easy to verify that both expressions give the same result:

$$V_w = 2Au_0\sqrt{t} \tag{30}$$

Philip [1957b] introduced the concept of sorptivity, S, which he described as, "a measure of the capillary uptake or removal of water." From this work, we can develop the following exact expression for S as:

$$S = \frac{2}{\phi}u_0 = \left[\frac{2k}{\phi\mu_w}\int_{S_{wi}}^{S_w^*}\frac{(S_{wi}-\alpha)k_{rw}(\alpha)p_c'(\alpha)}{f(\alpha)}d\alpha\right]^{1/2} \tag{31}$$

Several important features for this self-similar problem have been clarified by Rizhik [1959; in Barenblatt et al., 1972]. He has shown that the velocity of propagation for the imbibition front is finite when $S_{wi} \leq S_{iw}$, and infinite when $S_{wi} > S_{iw}$. The water saturation decreases monotonically from S_w^* at the inlet to S_{iw} at the front, x_f, and then: (1) if $S_{wi} = S_{iw}$, the saturation remains at the value of S_{iw} for $x_f < x < \infty$, or (2) if $S_{wi} < S_{iw}$, the saturation jumps from S_{iw} to S_{wi} at x_f and then remains at the initial saturation for all $x > x_f$. When $S_{wi} > S_{iw}$, S_w only reaches S_{wi} at ∞. All these important features of imbibition are inherent, implicitly or explicitly, in the solution that has been developed here. In fact, the integral:

$$\int_{S_w^*}^{S_{wi}}\frac{k_{rw}(S_w)p_c'(S_w)}{f(S_w)}dS_w$$

is divergent if $S_{wi} > S_{iw}$ and is convergent if $S_{wi} \leq S_{iw}$. In the former case, Equation (28) shows that $S_w = S_{wi}$ only when $\xi = \infty$; in other words, the location of the front is finite. In the latter case, one recognizes that $k_{rw}(S_w) = 0$ when $S_w \leq S_{iw}$ and no contribution to the integral can exist within the interval $S_{wi} < S_w < S_{iw}$. Equation (28) then shows that ξ will have the same finite value over the range $S_{wi} \leq S_w \leq S_{iw}$. For the case where $S_{wi} \leq S_{iw}$, there is no disturbance ahead of the finite front, and therefore this self-similar solution can also be used for a system of finite length as long as the front has not reached the end, x=L.

4. GRAPHICAL METHOD FOR DETERMINING AVERAGE SATURATION BETWEEN ANY TWO CROSS-SECTIONS

In addition to imbibition rate, saturation profile and location of the front, all of which have been discussed above, the average saturation within any sub-region of the system is also of interest. It can be seen from the results given earlier for the saturation profile, that Equation (16) has the same form as the well known solution for the fluid displacement problem of Buckeley and Leverett [1942]. Therefore, the generalized Welge graphical procedure [Chen and Song, 1963] of determining average saturation between any two cross-sections, where saturation varies from S_{w1} to S_{w2}, also applies to the imbibition case. A typical example for the relative imbibition rate function, $f(S_w)$, is shown in Figure 1. Tangents have been drawn at two given points, $[S_{w1}, f(S_{w1})]$ and $[S_{w2}, f(S_{w2})]$. The intersection of these two tangent lines gives the average saturation, $\bar{S}_{w1,2}$, between the particular cross-sections with saturations S_{w1} and S_{w2} (see Figure 1). If $S_{w1} = S_{wi}$ and $S_{w2} = S_w^*$, this procedure gives the average saturation, $\bar{S}_w$, for the entire zone of imbibition. The cumulative water that has invaded the porous medium is then readily calculated from

$$V_w = A\phi\xi_f\bar{S}_w\sqrt{t} \qquad (32)$$

where ξ_f is the value of ξ at the imbibition front.

5. CALCULATION OF HORIZONTAL IMBIBITION

We now present some examples of horizontal imbibition in which the viscosity of air is assumed to be vanishingly small. To illustrate the nature of imbibition under the assumed conditions, we shall solve Equation (29) for different values of the initial water saturation. The relative permeability to water and capillary pressure functions are taken as:

$$k_{rw}(S_w) = \left\{1 - \left[1 - \left[\frac{S_w - S_{iw}}{S_w^* - S_{iw}}\right]^{1.49}\right]^{0.671}\right\}^2 \left[\frac{S_w - S_{iw}}{S_w^* - S_{iw}}\right]^{0.5} \qquad (33)$$

$$p_c(S_w) = 0.872\left[\left[\frac{S_w - S_{iw}}{S_w^* - S_{iw}}\right]^{-1.49} - 1\right]^{0.329} \text{bar} \qquad (34)$$

These equations are of the form proposed by van Genuchten [1980]. The parameter values are those believed to be appropriate for the Topopah Spring welded tuff from Yucca Mountain, Nevada, a potential site for an underground radioactive waste repository [Rulon et al., 1986]. The maximum value for the water saturation is 0.984, while the irreducible water saturation, S_{iw}, is 0.318. The porosity of the Topopah Spring welded tuff is 0.14, and its absolute permeability is $3.9\times10^{-18}\text{m}^2$. For the water viscosity, we used a value of $\mu_w = 0.001$ Pa·s, which corresponds to a temperature of 20° C.

The results are shown in Figures 2 through 5. An independent check on the proposed method was obtained for an initial saturation of 0.6765 at an elapsed time of 1×10^7 s (116 days). The saturation profile for $S_{wi} = 0.6765$ is shown in Figure 4 where the open circles are results found by Zimmerman and Bodvarsson [1989]. They essentially numerically solved Equation (9) as a two-point boundary-value problem for S_w as a function of ξ. The excellent agreement verifies the correctness of the present formulation and solution. The present formulation has the advantage of requiring only the calculation of definite integrals, as opposed to the integration of a differential equation that is required in other methods. The iteration process was found to converge in about five or six iterations with six-digit accuracy.

6. DISCUSSION OF RESULTS

The curves of the relative imbibition rate function $f(S_w)$, which are plotted in terms of initial saturations on Figure 2, are very similar to those of the well-known fractional flow function, $f_w(S_w)$, from two phase flow theory. Both the physical meaning and the shape of the curves are much the same. In fact, both the relative imbibition rate function and the fractional flow function can be considered as the ratio of the water flow rate at any cross-section with a known S_w to the water flow rate at the inlet. When capillary pressure effects are included, both $f(S_w)$ from this formulation and $f_w(S_w)$ from fractional flow theory are convex upward curves, whereas, in the traditional Buckeley-Leverett treatment in which capillary pressure is ignored, the curves are S-shaped. Inasmuch as $df(S_w)/dS_w$ is a monotonically decreasing function of saturation, the solution is continuous over the whole interval of saturation $[S_{wi}, S_w^*]$. As the initial saturation increases, the $f(S_w)$ curve becomes steeper and in the case where $S_{wi} > S_{iw}$, $f(S_w)$ has an infinite slope at the initial saturation, i.e. where $f(S_w) = 0$. This is difficult to discern in the figure, because the slope becomes large in an extremely small neighborhood near $S_w = S_{wi}$. When $S_{wi} \leq S_{iw}$, the slope is finite at the initial saturation, in addition, when $S_{wi} < S_{iw}$, the slope of the curve is constant over the interval $S_{wi} \leq S_w \leq S_{iw}$. As discussed above in Section 4, the plot of the relative imbibition rate function can be used to determine the average water saturation between any two cross-sections in that part of the system where imbibition has occurred, and the cumulative imbibed water volume can then be calculated using Equation (32).

The saturation profiles for various values of initial saturation are shown in Figure 3. The water saturation decreases monotonically from $S_w = S_w^*$ at the inlet to $S_w = S_{wi}$ at the front. When $S_{wi} \leq S_{iw}$, the saturation reaches S_{wi} at a finite value of ξ, whereas in the case where $S_{wi} > S_{iw}$, the saturation reaches S_{wi} only at infinity. However, in either case, one can still speak of a "front" beyond which the saturation is effectively at its initial value. Note that when $S_{wi} < S_{iw}$, the range of saturations $S_{wi} \leq S_w \leq S_{iw}$

collapses into a single value of ξ, forming an infinitely sharp front. The saturation profile becomes more and more extended as S_{wi} increases, and each saturation propagates with a velocity equal to $\xi(S_w)/(2\sqrt{t})$.

From Equation (12), the water imbibition rate is inversely proportional to $\sqrt{t}$ and directly proportional to u_0, which depends on various parameters of the system as indicated by Equation (27). The imbibition constant, u_0, is related to the sorptivity S by $S=2u_0/\phi$. The effect of the initial saturation on the sorptivity is shown in Figure 5. The imbibition rate reaches its maximum value when $S_{wi}=0$, and then decreases monotonically with increasing initial water saturation, reaching zero when $S_{wi}=S_w^*$. The imbibition rate is a continuous function of S_{wi} over the range $0 \leq S_{wi} \leq S_w^*$.

The exact solution presented here has some advantages over other solutions in the literature. Only numerical integration of definite integrals combined with an iterative procedure is required, and the solution is applicable to systems with arbitrary characteristic curves, as well as any value of the initial water saturation. Most of the existing exact solutions require numerical integration of ordinary differential equations of first order [Philip, 1955] or of second order [Klute, 1952; Rizhik, 1959; Rizhik, in Barenblatt et al., 1972] and an iterative or trial and error procedure. Philip's [1960a, 1960b] analytical solution is in closed form but requires that the characteristic curves be developed in terms of inverse error functions requiring considerable computational effort, and is not especially well adapted to fitting empirical data on the characteristic curves. It should be noted that the solutions of Rizhik [1959; in Barenblatt et al., 1972] have a special theoretical significance because they describe the finite nature of the velocity of propagation of the imbibition front. Since the function that is calculated in the solution procedure, $f(S_w)$, is fairly well-behaved, the integrations are easy to carry out numerically. This is in contrast to methods which directly calculate $S_w(\xi)$. These methods typically run into numerical difficulties when S_{wi} is near S_{iw}, due to the increasingly sharp saturation front. Finally, this solution provides a direct method of calculating the cumulative volume of water that has imbibed into the system, whereas, this is usually estimated from the area beneath the saturation profile with other methods.

7. CONCLUSIONS

1. By introducing the relative imbibition rate function, a new formulation and an exact semi-analytical, self-similar solution for the case of linear horizontal imbibition is obtained which has the same form as the well-known solution for the Buckeley-Leverett problem, but with the effects of capillary pressure included. The solution applies to any system with an initially uniform water saturation. Evaluation of the solution requires only numerical integration of definite integrals, and a rapidly-converging iterative process. This new method offers some distinct advantages over existing exact solutions for the imbibition problem.

2. The saturation profile propagates with a speed of $\xi/(2\sqrt{t})$ for each saturation. The velocity of the imbibition front is finite for the case where $S_{wi} \leq S_{iw}$, whereas it is infinite when $S_{wi} > S_{iw}$. This agrees with the important results that were developed by Rizhik [1959; in Barenblatt et al., 1972]. Since the velocity of the imbibition front is finite, and there is no disturbance ahead of the front when $S_{wi} \leq S_{iw}$, this self-similar solution is also valid for a system of finite length as long as the front has not reached the end of the system.

3. The water imbibition rate, that is initially infinite and then varies at a rate inversely proportional to $\sqrt{t}$, depends on various parameters of the system in a complex way. The rate decreases with increasing initial water saturation in the porous medium.

4. A graphical technique, patterned after the Welge method, can be applied to curves of the relative imbibition rate function to determine the average water saturation within any sub-region of imbibition zone, as well as the average value over the total zone of imbibition.

5. The solution has been used to investigate the behavior of the unsaturated zone in the vicinity of a potential high-level radioactive waste repository at Yucca Mountain. Saturation profiles determined by the procedures developed in this work are

in excellent agreement with those obtained numerically by Zimmerman and Bodvarsson [1989], demonstrating the validity of this new approach.

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NOMENCLATURE

A	=	cross-sectional area $[L^2]$
f	=	relative water infiltration rate function, defined in Equation (37)
f_w	=	fractional flow function in two-phase flow problem
g	=	gravitational constant $[L/T^2]$
k	=	absolute permeability $[L^2]$
k_r	=	relative permeability
p	=	pressure $[F/L^2]$
p_c	=	capillary pressure $[F/L^2]$
S	=	sorptivity $[L/T^{½}]$
S_w	=	water saturation
S_{iw}	=	irreducible water saturation
S_{wi}	=	initial water saturation
S_w^*	=	maximum obtainable water saturation
$\bar{S}_w$	=	average water saturation within the entire imbibed region
$\bar{S}_{w1,2}$	=	average water saturation between two cross-sectional areas with water saturations S_{w1} and S_{w2}, respectively.
t	=	time $[T]$
u	=	flow rate $[L/T]$
u_0	=	imbibition constant introduced in Equation (12) $[L/T^{½}]$
V	=	cumulative imbibed volume $[L^3]$
x	=	distance $[L]$

Greek

α	=	dummy (saturation) variable of integration
μ	=	viscosity $[F\text{-}T/L^2]$

ξ = similarity variable, defined in Equation (8) $[L/T^{1/2}]$

ϕ = porosity

Subscripts

f = front

w = water

0 = inlet

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Figure Captions

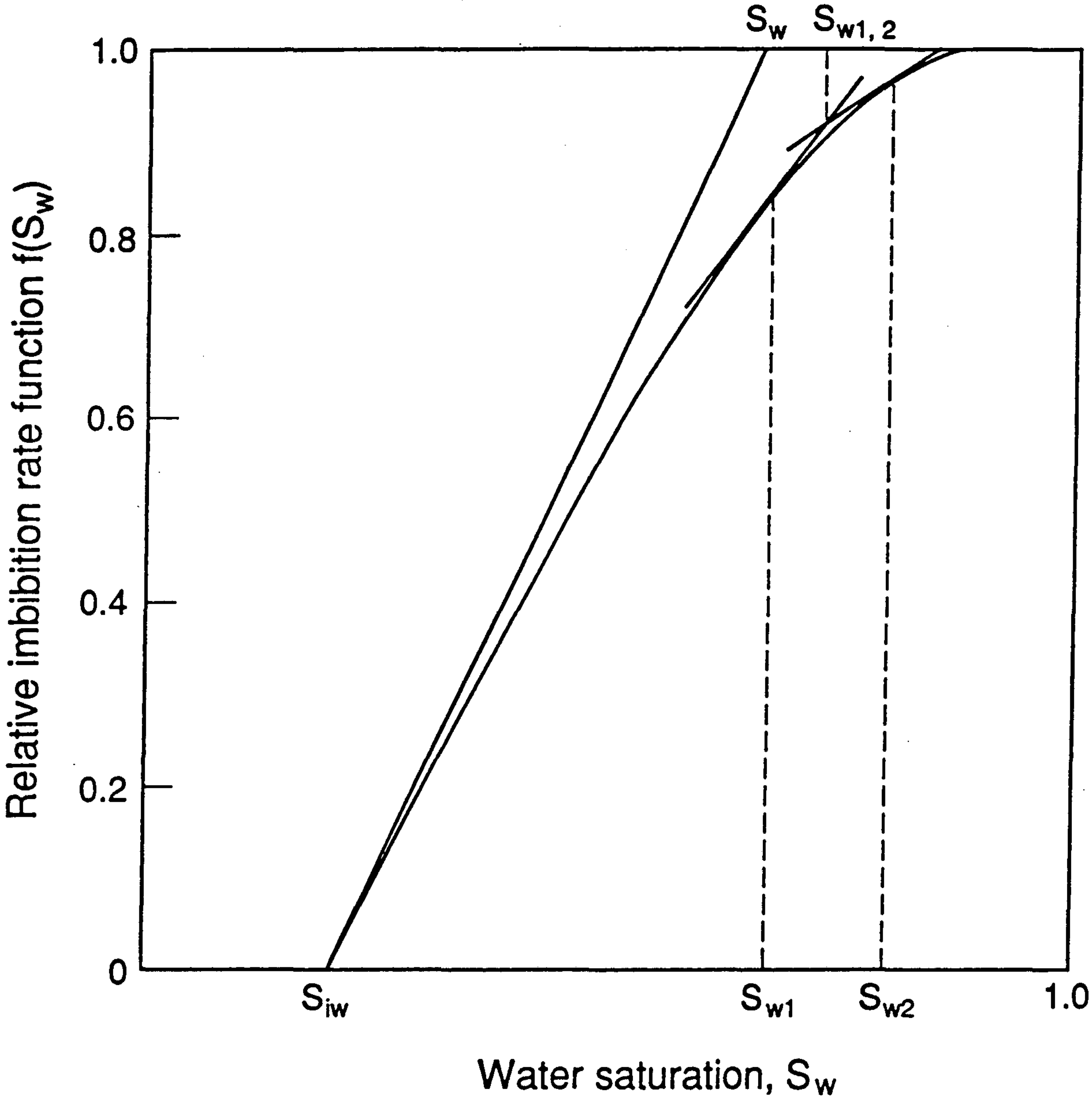
Fig. 1. Graphical method for determining average water saturation between two cross-sections with S_{w1} and S_{w2}.

Fig. 2. Relative imbibition rate function for imbibition into Topopah Spring welded tuff, for different initial water saturations.

Fig. 3. Saturation profiles for imbibition into Topopah Spring welded tuff, for different initial water saturations.

Fig. 4. Comparison of predicted saturation profiles for imbibition into Topopah Spring welded tuff; Solid lines are from the present work, dots are from numerical solution of Zimmerman and Bodvarsson [1989].

Fig. 5. Sorptivity of Topopah Spring welded tuff, as a function of the initial water saturation.



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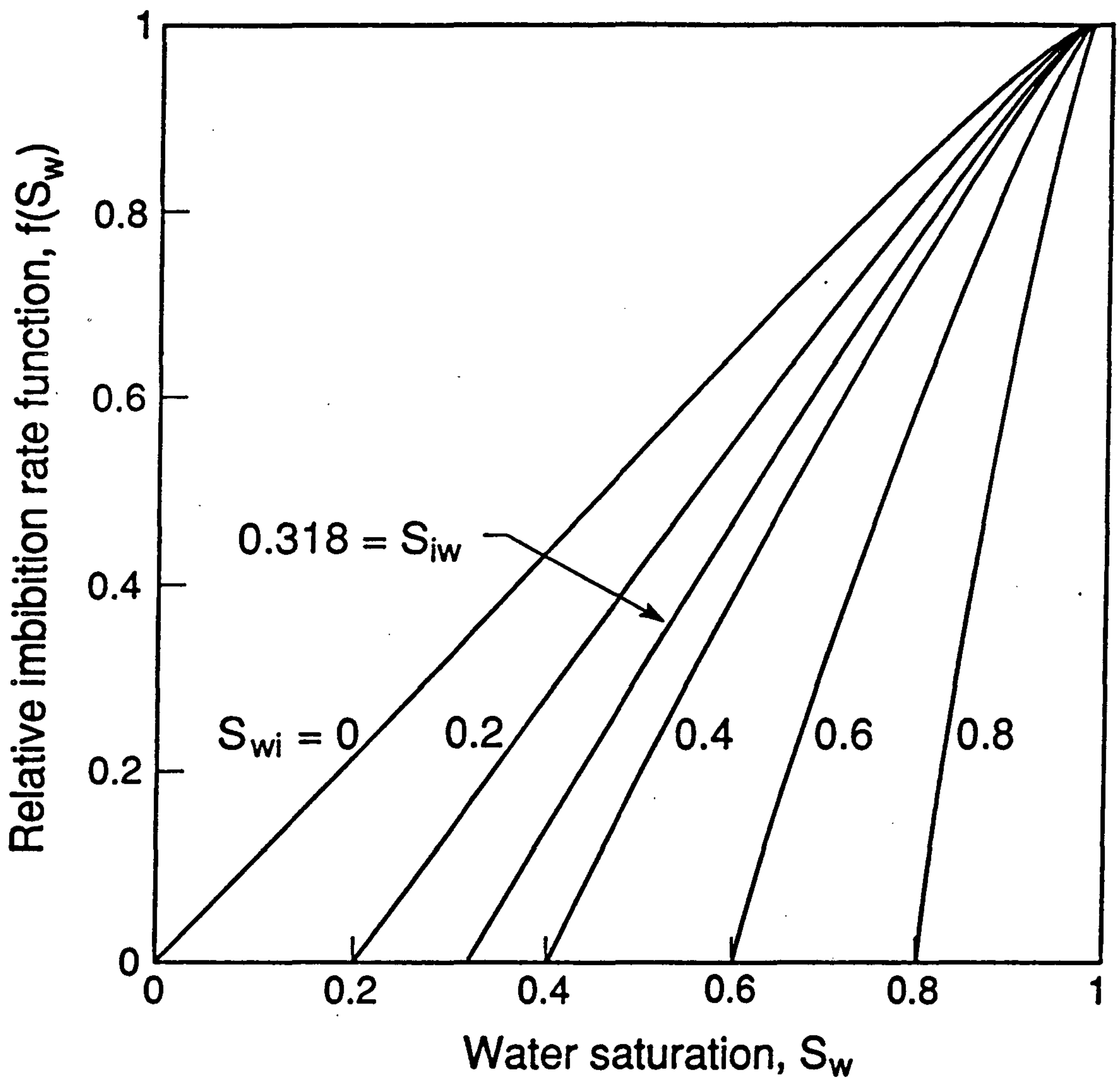
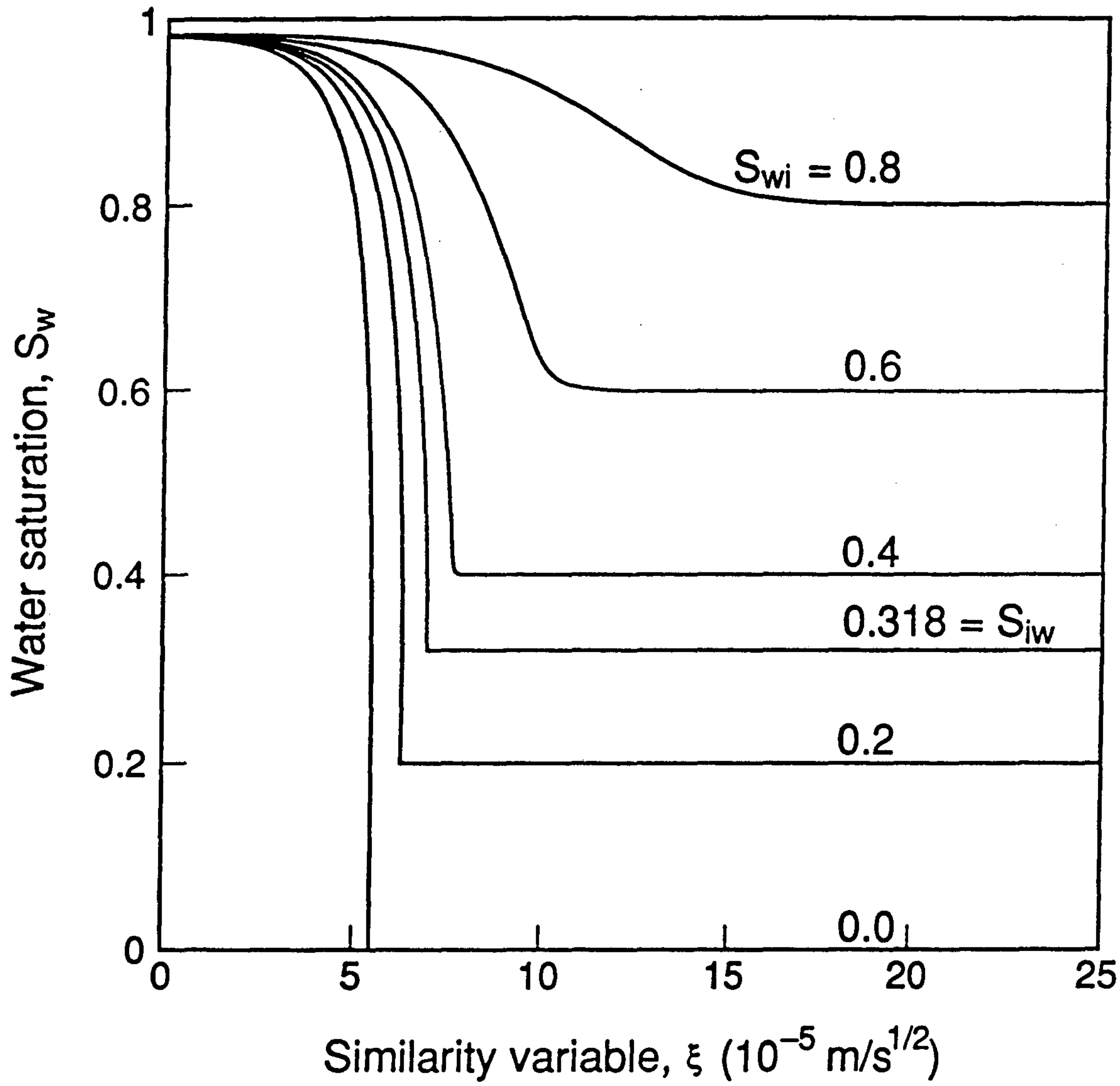


Fig. 2. Relative imbibition rate function for imbibition into Topopah Spring welded tuff, for different initial water saturations.



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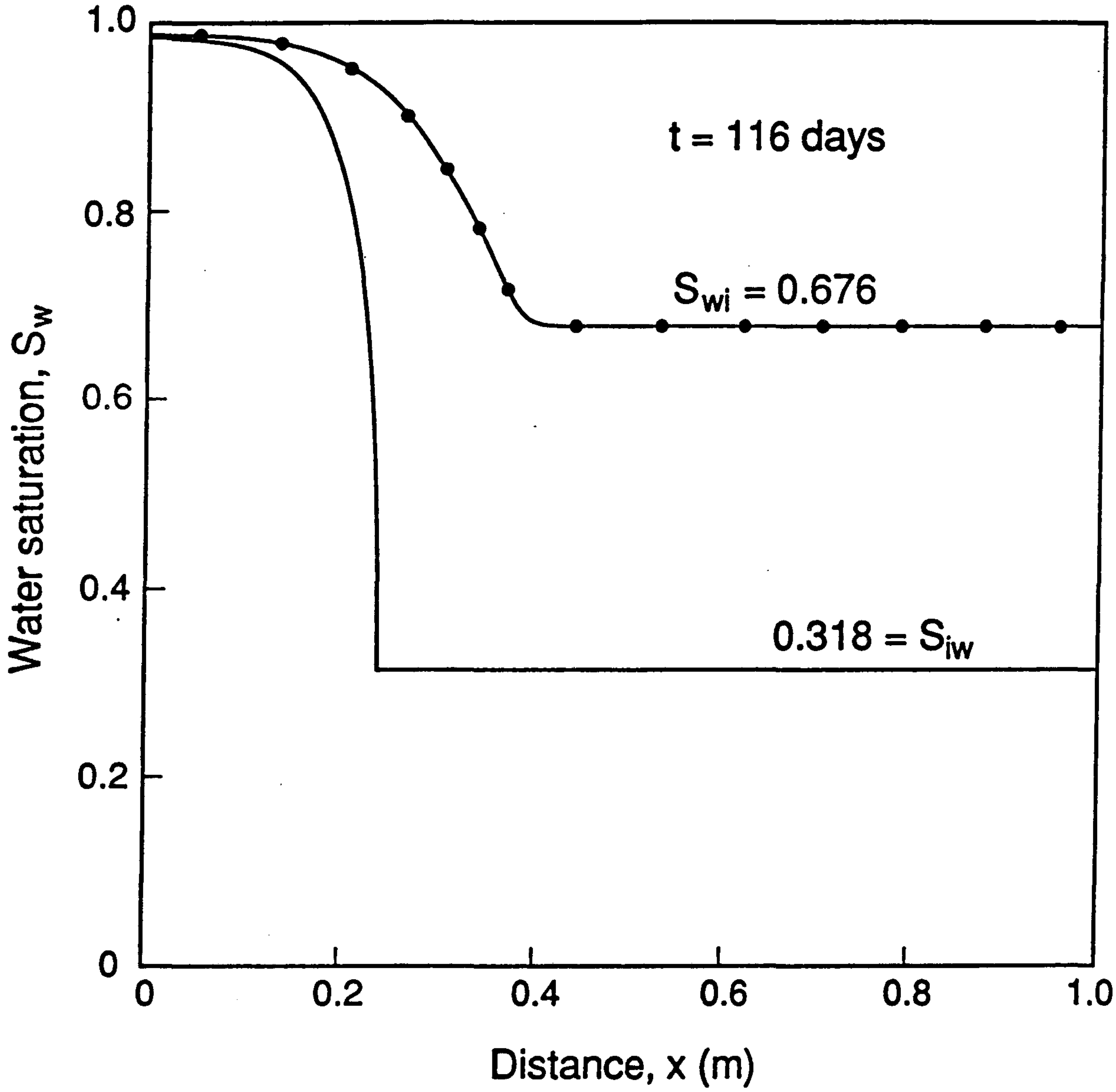

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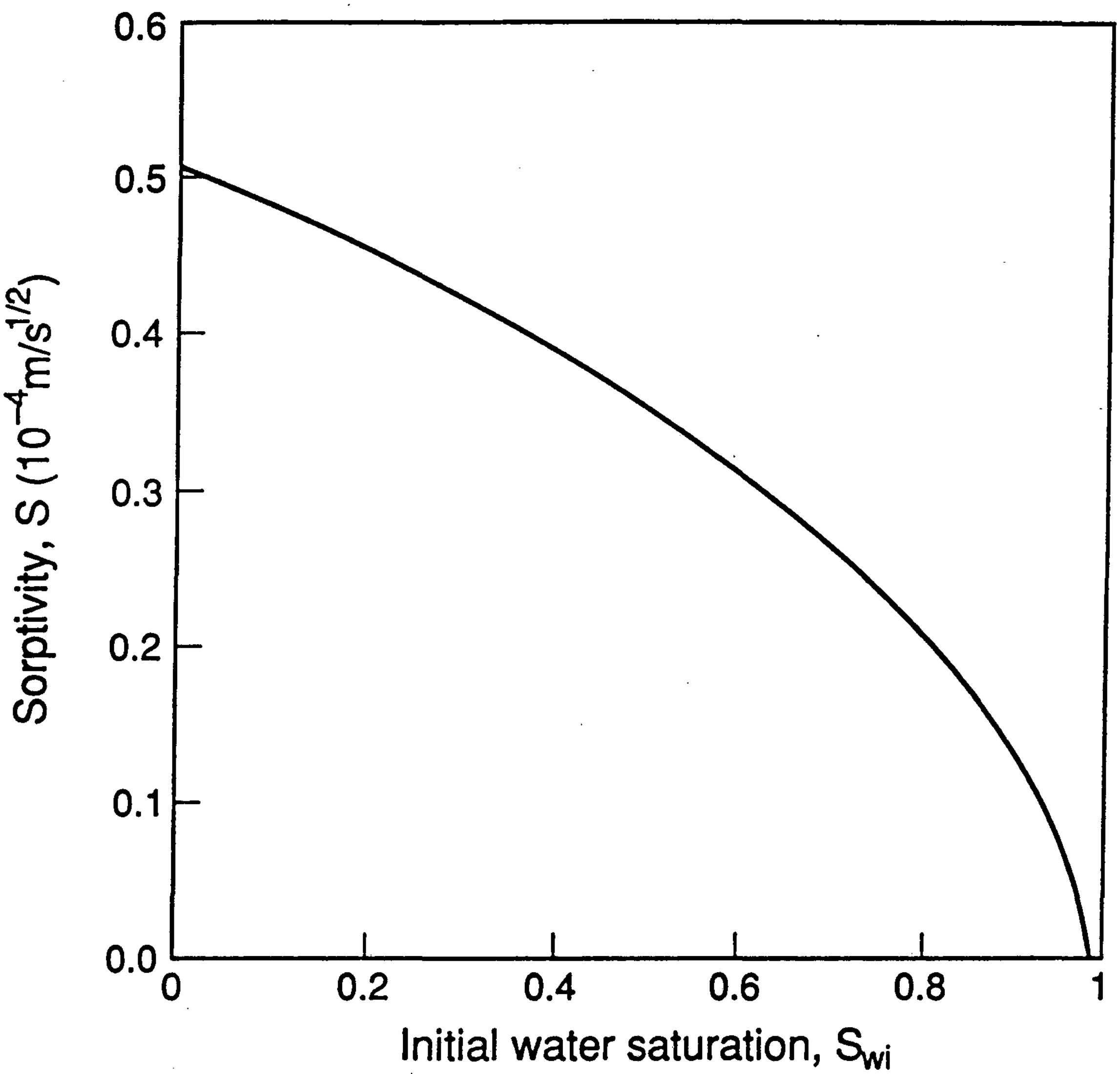

Fig. 5. Sorptivity of Topopah Spring welded tuff, as a function of the initial water saturation.

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