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On Condensation of Anyons and Applications

To my family

who have done so much for me throughout my life

To my teachers

who have made me who I am

To the human project of learning

بناهای آباد گردد خراب ز باران وز تابش آفتاب

پی افکندم نظم کاخی بلند که از باد و باران نابد گزند

- فردوسی

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Abstract

On Condensation of Anyons and Applications

by

Aaron Robert Bagheri

Phase transitions can be understood through the formation of Bose condensates. Anyon condensation is similarly an important tool for transitioning between systems modeled by modular tensor categories. The condensation process can be understood as a functor from one modular tensor category to another fusion category with a modular subcategory.

This dissertation focuses on understanding the condensation functor. After reviewing modular tensor categories, we present and comment on the relationship between two descriptions of the resulting category. We then present general results on the modular data of the resulting category and demonstrate how to explicitly compute the new F- and R-symbols. We finish off with some applications of our work and some speculations about where they might go from here.

Chapter 4 is devoted to implementing the functorial definition of condensation so that it can be carried out by a computer, and a full Mathematica implementation is provided in Supplemental A: Mathematica Code. Supplemental B: $(G_2)_3$ Data provides categorical data for the modular tensor category $(G_2)_3$, which is not otherwise widely available.

Contents

Chapter 1

Introduction

Topological quantum computation was first introduced [32, 21, 20] and further developed [33, 41] as an elegant way to achieve naturally fault-tolerant quantum computation through the braiding of quasi-particles called anyons. Anyons are quasi-particles that exist in two spatial dimensions and are not restricted to Fermi-Dirac or Bose-Einstein statistics. Because of this, they keep track of the topological data of braiding as they interact with each other. The mathematical structure used to describe a system of anyons is the unitary modular tensor category. Modular tensor categories are also important as they provide 3-manifold invariants and TQFTS, and they are useful in the study of braided fusion categories. As such, they are of interest to both mathematicians and physicists.

Phase transitions can be understood through the formation of Bose condensates. Anyon condensation is similarly an important tool for understanding transitions in topologically ordered systems [8]. This condensation process is expressed by a condensation functor from the parent modular tensor category [30, 38]. The target of the functor is a new category in which the vacuum is the algebra formed by the old vacuum and the condensing anyons. At the moment, the data of this new category is not well-understood in general.

Of independent mathematical interest is condensation used in defining Witt equivalence of modular tensor categories. Much is known about the interesting and sometimes surprising structure of modular tensor categories and the Witt group, but a great deal more remains unknown [12, 13]. Any further understanding of the equivalence relation is useful.

Aside from these well-known settings, novel applications are suggesting condensation as a tool with greater general usefulness. Some constructions on local conformal nets and vertex operator algebras can be translated to condensations in modular tensor categories. We also use it in this thesis as a tool to compute categorical data that has otherwise been inaccessible.

In Chapter 2, we attempt to provide a friendly reference for definitions and some light theory of modular tensor categories. Section 2.1 defines modular tensor categories, as well as many relevant structures, properties, and invariants. Section 2.2 introduces the graphical calculus used to work with modular tensor categories and provides graphical counterparts to some of the definitions of Section 2.1. Sections 2.3 and 2.4 briefly provide examples of MTCs, where they come from, and motivations for their study. Section 2.5 mentions the representability of category data in a way that is computer-friendly. Section 2.6 defines algebras in categories. Algebras appear in the general MTC theory of Section 2.7 and are a primary topic of study starting in Chapter 3. We end the chapter with a couple defintions of tensor functors in Section 2.8

In Chapter 3, we introduce two definitions of condensation and demonstrate an explicit equivalence between them. These two definitions are known in the literature, but such a side-by-side presentation is more difficult to find. We then give some results that help determine the condensed category given a parent category and a condensable algebra. Theorem 3.37 is the primary result of this section.

Chapter 4 is perhaps the biggest highlight of this thesis. We use the definitions of [30] to lay out the theory and provide an implementation for explicitly computing all

F- and R-symbols of the condensed category from those of the parent category. The Mathematica code that does this is provided in Supplemental A: Mathematica Code.

Chapter 5 discusses some novel applications of anyon condensation. Section 5.1 is about quantum computing with the minimal model conformal field theories as in [34]. Section 5.2 introduces near-group categories and builds on the data of [19, 7]. It also introduces the modular tensor category $(G_2)_3$, from which condensation gives a near-group category with a maximal fusion coefficient on simple objects of 3. Full F-symbols have never been written down for any such category. Section 5.3 uses the idea of [29] to describe a way to produce condensable algebras from classical error correcting codes.

Finally, Chapter 6 concludes and lists a plethora of continuations of this work that have yet to be explored.

Chapter 2

Modular Tensor Categories

In this chapter, we provide an introduction to the theory and structure of modular tensor categories that is relevant to our work. Some basic knowledge of categories is assumed, though definitions of the relevant structures and properties are presented to establish notation. For the most part, we try to limit the numbering of remarks to those that will be referenced in other parts of the text.

For our purposes, categories are small and base fields are always $\mathbb{C}$. Of course, most instances of the field $\mathbb{C}$ in the general category theory can be replaced by any algebraically closed field $\Bbbk$ with characteristic zero. Many of the definitions in this chapter follow the conventions of [17].

2.1 Definitions

2.1.1 Fusion Categories

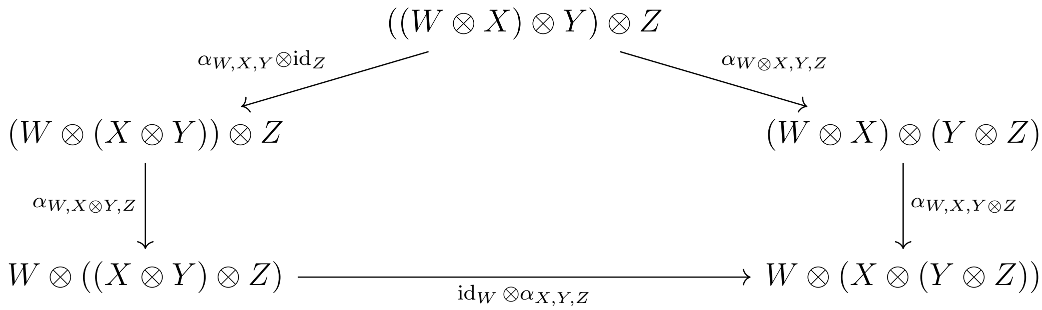
Definition 2.1. A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $\mathbb{1}$ is an object, and α, λ, ρ are natural isomorphisms

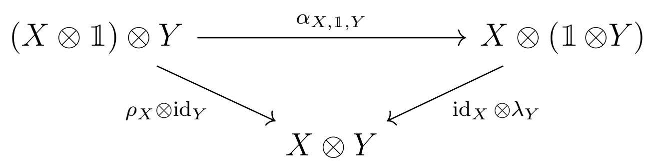
$$\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z),$$

$$\lambda_X\colon \mathbb{1} \otimes X \xrightarrow{\sim} X,$$

$$\rho_X\colon X \otimes \mathbb{1} \xrightarrow{\sim} X,$$

such that the pentagon and triangle axioms hold, i.e. that for any objects W, X, Y, Z, the following diagrams commute.





The bifunctor $\otimes$ is the *tensor product*, the object $\mathbb{1}$ is called the *tensor unit*, and the natural isomorphisms α, λ, ρ are called the *associativity isomorphism*, *left unit isomorphism*, *right unit isomorphism*, respectively.

The pentagon axiom is simply the statement that the way in which parentheses are moved around does not matter, so long as the proper associators are used at each step.

Remark 2.2. A monoidal category can also be defined as a 2-category with a single object. Conversely, every 2-category with a single object is naturally a monoidal 1-category.

Before moving on, we mention two notions that are critically important later in this thesis.

Definition 2.3. A monoidal category is called *strict* if the associativity isomorphism $\alpha_{X,Y,Z}$ is the identity on all triples X, Y, Z.

Definition 2.4. A category is called *skeletal* if each isomorphism class of objects consists of a single object.

Remark 2.5. Every (small) monoidal category is equivalent to a strict monoidal category and (assuming the axiom of choice) to a skeletal monoidal category, but not necessarily a strict and skeletal monoidal category. Later when we work with modular tensor categories, we must choose which adjective is more helpful.

While monoidal categories provide a tensor product, they are far too general for our purposes. We would like to define tensor and fusion categories. These terms do not have standard definitions, and different authors move adjectives around.

To define a tensor category we need definitions of $\mathbb{C}$-linear, abelian, locally finite, and rigid. In the interest of brevity, we only give moral definitions where it seems sufficient for our needs. Proper definitions can be found in [17].

Definition 2.6 (moral)**.** A category $\mathcal{C}$ is called *additive* if

1. $\mathcal{C}$ contains a zero object $0 \in \mathrm{Obj}(\mathcal{C})$ with $\mathrm{Hom}(0,0) = 0$,
2. for every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}(X,Y)$ is an abelian group, and morphism composition is biadditive,
3. for every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, there exists a direct sum $X \oplus Y$.

Thus a category is additive if its objects and morphisms can be added.

Definition 2.7. Let $\mathbb{C}$ be the field of complex numbers. An additive category $\mathcal{C}$ is called $\mathbb{C}$-*linear* if all hom spaces are $\mathbb{C}$-vector spaces and composition of morphisms is $\mathbb{C}$-linear.

Definition 2.8 (moral)**.** An additive category $\mathcal{C}$ is called *abelian* if all kernels and cokernels exist nicely.

Definition 2.9 (moral)**.** A $\mathbb{C}$-linear abelian category $\mathcal{C}$ is called *locally finite* if all hom spaces are finite-dimensional.

Because we will need more from this definition, we devote a longer discussion to rigidity.

Definition 2.10. Consider a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and an object X of $\mathcal{C}$. A *left dual* of X is a triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$, where X^* is an object and $\mathrm{ev}_X \colon X^* \otimes X \to \mathbb{1}$, $\mathrm{coev}_X \colon \mathbb{1} \to X \otimes X^*$ are morphisms such that the compositions

$$X \xrightarrow{(\mathrm{coev}_X \otimes \mathrm{id}_X)\circ\lambda_X^{-1}} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\rho_X \circ(\mathrm{id}_X \otimes \mathrm{ev}_X)} X,$$
$$X^* \xrightarrow{(\mathrm{id}_{X^*} \otimes \mathrm{coev}_X)\circ\rho_{X^*}^{-1}} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\lambda_{X^*} \circ(\mathrm{ev}_X \otimes \mathrm{id}_{X^*})} X^* \tag{2.1}$$

are identity morphisms. The morphism ev_X is called *evaluation*. The morphism coev_X is called *coevaluation*.

Similarly, a *right dual* of X is a triple $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$, where ${}^*X$ is an object and $\mathrm{ev}'_X \colon X \otimes {}^*X \to \mathbb{1}$, $\mathrm{coev}'_X \colon \mathbb{1} \to {}^*X \otimes X$ are morphisms such that the compositions

$$X \xrightarrow{(\mathrm{id}_X \otimes \mathrm{coev}'_X)\circ\rho_X^{-1}} X \otimes ({}^*X \otimes X) \xrightarrow{\alpha^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\lambda_X \circ(\mathrm{ev}'_X \otimes \mathrm{id}_X)} X,$$
$${}^*X \xrightarrow{(\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*X})\circ\lambda_{{}^*X}^{-1}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\rho_{{}^*X} \circ(\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_X)} {}^*X \tag{2.2}$$

are identity morphisms.

Remark 2.11. Lemmas 2.27, 2.29, 2.30 and Definition 2.31 provide a moral definition of a dual object in the setting of Remark 2.15.

Proposition 2.12. *Consider a monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ *and an object* X *of* $\mathcal{C}$.

1. *If* X *has a left dual* X^**, then*

$$\mathrm{Hom}(X^* \otimes Y, Z) \cong \mathrm{Hom}(Y, X \otimes Z), \qquad \mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, Z \otimes X^*).$$

2. *If* X *has a right dual* ${}^*X$*, then*

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(Y, {}^*X \otimes Z), \qquad \mathrm{Hom}(Y \otimes {}^*X, Z) \cong \mathrm{Hom}(Y, Z \otimes X).$$

Proof. Proposition 2.10.8 of [17]. □

The evaluation and coevaluation morphisms allow us to define duals of morphisms as well.

Definition 2.13. Consider a monoidal category $\mathcal{C}$ with objects X, Y that have left duals X^*, Y^*. Let $f \in \mathrm{Hom}(X, Y)$ be a morphism. The *left dual* of f is the morphism $f^* \in \mathrm{Hom}(Y^*, X^*)$ defined by

$$f^* = (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ \alpha^{-1}_{Y^*,X,X^*} \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X)\,.$$

Similarly, suppose $f \in \mathrm{Hom}(X, Y)$ and X, Y have right duals ${}^*X, {}^*Y$. The *right dual* of f is the morphism ${}^*f \in \mathrm{Hom}({}^*Y, {}^*X)$ defined by

$${}^*f = (\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_Y) \circ (\mathrm{id}_{{}^*X} \otimes f \otimes \mathrm{id}_{{}^*Y}) \circ \alpha_{{}^*X,X,{}^*Y} \circ (\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*Y})\,.$$

Definition 2.14. An object in a monoidal category is called *rigid* if it has left and right duals. A monoidal category is called *rigid* if every object has left and right duals.

Remark 2.15. Duals are unique up to isomorphism. Once a braiding of Definition 2.37 is introduced, the left dual is isomorphically a right dual as well. Since all of our categories will be rigid and braided, we will often refer to the *dual* X^* without questioning its existence or specifying left or right. In this case, the isomorphisms of Proposition 2.12 hold through commuting tensor products. Note that the introduction of this sloppiness allows us to forget which isomorphisms are implicitly being used. For an example of braiding appearing in compatibility conditions on duality, see Remark 2.66.

For a graphical presentation of duals, refer to Equations 2.4, 2.5, 2.6. Now we can finally define a tensor category.

Definition 2.16. A monoidal category $\mathcal{C}$ is called a *tensor category* if

- $\mathcal{C}$ is $\mathbb{C}$-linear,
- $\mathcal{C}$ is abelian,

- $\mathcal{C}$ is locally finite,
- $\mathcal{C}$ is rigid,
- $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is bilinear on morphisms,
- $\mathrm{End}_{\mathcal{C}}(\mathbb{1})\cong\mathbb{C}$.

In general, tensor categories are defined over any algebraically closed characteristic zero field, but we always take this field to be $\mathbb{C}$.

Remark 2.17. The condition that $\mathrm{End}_{\mathcal{C}}(\mathbb{1})\cong\mathbb{C}$ is independent of the other conditions and can be omitted. In this case, the category is called a multi-tensor category and does not guarantee the second part of Lemma 2.19 below.

Definition 2.18. An object a is called *simple* if its only subobjects are 0 and a.

Lemma 2.19 (Schur's Lemma)**.** *For simple objects a, b in an abelian category, any nonzero morphism $f\colon a\to b$ is an isomorphism. In particular,* $\mathrm{Hom}(a,b)=0$ *when* $a\not\cong b$*. In a tensor category, we also have* $\mathrm{Hom}(a,b)\cong\mathbb{C}$ *when* $a\cong b$*.*

We now need two more adjectives in order to define a fusion category. Again, refer to [17] for proper definitions.

Definition 2.20 (moral)**.** A $\mathbb{C}$-linear abelian category is called *finite* if it is locally finite and has finitely many isomorphism classes of simple objects.

Definition 2.21. An object in an abelian category is called *semisimple* if it is (isomorphic to) a direct sum of simple objects. An abelian category is called *semisimple* if all objects are semisimple.

Definition 2.22. A tensor category $\mathcal{C}$ is called a *fusion category* if

- $\mathcal{C}$ is finite,
- $\mathcal{C}$ is semisimple.

Remark 2.23. We depend extensively on the fact that every object can be written uniquely (up to isomorphism) as a sum $\bigoplus_i n_i a_i$ for simple objects a_i. Together with Lemma 2.19, Hom spaces of arbitrary object pairs decompose as direct sums of Hom spaces of simple objects, which are each either 0 or $\mathbb{C}$.

We depend so much on the simple objects that it becomes convenient to define some terminology.

Definition 2.24. Consider a fusion category $\mathcal{C}$. The *label set* $\mathcal{L}$ of $\mathcal{C}$ is the set (up to bijection) of simple objects in a skeleton of $\mathcal{C}$.

Explicitly, we choose a single representative a_i from each isomorphism class of simple objects of $\mathcal{C}$. Then the label set is $\mathcal{L} = \{a_i\}$, and each a_i is called a *label.* It is standard to take $a_1 \cong \mathbb{1}$.

The *rank* of $\mathcal{C}$ is the cardinality of $\mathcal{L}$.

Because the simple objects play such a hugely important role, we will often denote labels with lowercase letters $a, b, c, \ldots$ to distinguish them from general objects $X, Y, Z, \ldots$.

Definition 2.25. Consider a fusion category $\mathcal{C}$ with label set $\{a_i\}$. For each i, j, k, the *fusion coefficient* $N_{a_k}^{a_i a_j} = N_k^{ij}$ (note the abuse of notation) is the dimension

$$\dim\Big(\operatorname{Hom}(a_k, a_i \otimes a_j)\Big).$$

That is, $a_i \otimes a_j \cong \bigoplus_k N_k^{ij} a_k$.

In analogy with the names of the F-symbols (Definition 2.72) and R-symbols (Definition 2.74), the fusion coefficients might be called *N-symbols*, but this name is not standard.

Definition 2.26. A fusion category is *multiplicity-free* if it has no fusion coefficients greater than 1.

There has been more work done lately on dropping semisimplicity, but we do not address it here. There has also been more interest in modular tensor categories with multiplicity, which we do address in this thesis.

Since we will need them later, we will also mention a few facts about duals. They are quite foundational and may not be referenced every time they are used. Fix a braided fusion category (Definition 2.38) with an object X and a label set $\{a_i\}$. According to Remark 2.15, we do not bother distinguishing left and right duals.

Lemma 2.27. *If X is simple, then X^* is also simple.*

Lemma 2.28. *If $X \cong \bigoplus n_i a_i$ is an arbitrary object, then $X^* \cong \bigoplus n_i a_i^*$.*

Proof. Lemmas 2.27 and 2.28 follow from $*$ being a (monoidal) equivalence between a rigid (monoidal) category and its opposite. □

Lemma 2.29. *If X is simple, then*

$$\dim\Big(\operatorname{Hom}(X \otimes X^*, 1)\Big) = \dim\Big(\operatorname{Hom}(X, X)\Big) = 1.$$

Lemma 2.30. *If X is simple, then X^* is the only simple object (up to isomorphism) with*

$$\dim\Big(\operatorname{Hom}(X \otimes X^*, 1)\Big) > 0.$$

Proof. Consider an object Y such that

$$\dim\Big(\operatorname{Hom}(X \otimes Y, 1)\Big) \geq 2.$$

Then $\dim(\operatorname{Hom}(X, Y^*)) \geq 2$, so Y^* is not simple, and Y is not simple.

Consider a simple object Y such that

$$\dim\Big(\operatorname{Hom}(X \otimes Y, 1)\Big) = 1.$$

Then $\dim(\operatorname{Hom}(X, Y^*)) = 1$. Since X, Y^* are simple, we must have $X \cong Y^*$ or $Y \cong X^*$. □

Motivated by the above lemmas, the following moral definition is often a convenient characterization of the dual object.

Definition 2.31 (moral)**.** Consider a simple object a in a braided fusion category. The dual a^* is the unique simple object (up to isomorphism) such that $N_{\mathbb{1}}^{aa^*} = 1$.

2.1.2 MTCs as Pivotal Braided Fusion Categories

The first definition of a modular tensor category is as a special type of pivotal braided fusion category.

Definition 2.32. A *pivotal structure* on a fusion category is a natural isomorphism δ from id, the identity functor, to $(-)^{**}$, the double dual functor. On objects, this is a choice of morphisms $\delta_X : X \to X^{**}$ such that $\delta_{X\otimes Y} = \delta_X \otimes \delta_Y$, $\delta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

Remark 2.33. It is unknown whether or not every fusion category admits a pivotal structure.

Definition 2.34. Consider a fusion category with a pivotal structure δ. The *left quantum trace*, or simply the *left trace*, of a morphism $f \in \mathrm{Hom}(X, X)$ is the morphism

$$\mathrm{ev}_{X^*} \circ \Big((\delta_X \circ f) \otimes \mathrm{id}_{X^*} \Big) \circ \mathrm{coev}_X \,.$$

The *right quantum trace*, or simply the *right trace*, of a morphism $f \in \mathrm{Hom}(X, X)$ is the morphism

$$\mathrm{ev}'_{X^*} \circ \Big(\mathrm{id}_{X^*} \otimes (\delta_X \circ f) \Big) \circ \mathrm{coev}'_X \,.$$

Diagrams 2.7 give the graphical representations of these morphisms.

Remark 2.35. Notice that for any morphism f, both traces are members of $\mathrm{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$. Identifying $\mathrm{id}_{\mathbb{1}}$ with $1 \in \mathbb{C}$, the trace morphisms can be identified with complex numbers. When traces are presented in categorical data, they are typically given as numbers in this way.

Definition 2.36. A pivotal structure is called *spherical* if the left trace is equal to the right trace. In this case, we simply refer to the *trace*. Remark 2.71 justifies the term spherical for this property.

Definition 2.37. A *braiding* on a monoidal category is a natural isomorphism $c_{X,Y} \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that the following hexagonal diagrams commute for all objects X, Y, Z.

$$\begin{array}{ccccccc}
 & & X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X & & \\
 & \nearrow^{\alpha_{X,Y,Z}} & & & & \searrow^{\alpha_{Y,Z,X}} & \\
(X \otimes Y) \otimes Z & & & & & & Y \otimes (Z \otimes X) \\
 & \searrow_{c_{X,Y}\otimes \mathrm{id}_Z} & & & & \nearrow_{\mathrm{id}_Y \otimes c_{X,Z}} & \\
 & & (Y \otimes X) \otimes Z & \xrightarrow[\alpha_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & (X \otimes Y) \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) & & \\
 & \nearrow^{\alpha^{-1}_{X,Y,Z}} & & & & \searrow^{\alpha^{-1}_{Z,X,Y}} & \\
X \otimes (Y \otimes Z) & & & & & & (Z \otimes X) \otimes Y \\
 & \searrow_{\mathrm{id}_X \otimes c_{Y,Z}} & & & & \nearrow_{c_{X,Z}\otimes \mathrm{id}_Y} & \\
 & & X \otimes (Z \otimes Y) & \xrightarrow[\alpha^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & &
\end{array}$$

Satisfying the hexagons is simply the condition that braiding is compatible with the associativity of the category.

Definition 2.38. A *braided monoidal category* is a monoidal category along with a choice of braiding. A *braided fusion category* is a fusion category along with a choice of braiding. We will only be considering the fusion case.

Remark 2.39. In similar fashion to Remark 2.2, braided monoidal categories are equivalent to 3-categories with a single object and a single 1-morphism.

Definition 2.40. A braided fusion category is *non-degenerate* if the tensor unit is the only simple object (up to isomorphism) that has a trivial double braiding with every object. That is, given a simple object a, if

$$c_{b,a} \circ c_{a,b} = \mathrm{id}_{a\otimes b} \qquad \text{for all } b, \tag{2.3}$$

then $a \cong \mathbb{1}$.

Remark 2.41. In some sense, a non-degenerate braided fusion category is the opposite of a *symmetric fusion category* (a braided fusion category in which Equation 2.3 holds for all a). A symmetric fusion category is a maximally degenerate braided fusion category.

Definition 2.42. A *modular tensor category* (or *MTC*) is a non-degenerate braided fusion category with a spherical pivotal structure.

Remark 2.43. The term *modular* refers to the non-degeneracy of the braiding. While the modular tensor category we have defined is the most common, modular categories that drop some conditions we have required have been studied as well. The term *modular category* without the adjective *tensor* is sometimes used to refer to such a category.

Note also that a modular tensor category is, in fact, a fusion category, not just a tensor category. For historical reasons, we use the term modular tensor category instead of the term modular fusion category.

Remark 2.44. The definition of a modular tensor category requires the definitions of many structures and properties of those structures. The following table is provided as a reference to help keep in mind which of the above definitions are structures and which are properties of those structures.

adjectives that define structures	adjectives that describe properties of structures
monoidal	tensor, fusion (and subproperties) rigidity*
pivotal	spherical
braided	non-degenerate
	modular

Thus, a modular tensor category is just a category with monoidal, pivotal, and braiding structures that are all sufficiently compatible.

Note that rigidity is the property that all appropriate morphism spaces contain proper ev and coev morphisms. In the settings we work with, the dual of an object is (isomorphically) both a left dual and a right dual. When working at the level of morphisms, we must decide which to use. This is apparent when we draw pictures in the graphical calculus and must make a choice of whether to use ev or ev′. Refer, for example, to the choice of one of the Diagrams 2.7 to denote a quantum trace. If one considers rigidity as making this choice (as opposed to just the property of existence of duals), then it can be thought of as a structure.

2.1.3 Invariants of Modular Tensor Categories

While a modular tensor category can be presented by explicitly providing definitions of structures such as the associativity and braiding isomorphisms (addressed in Section 2.2.1 and the subject of Chapter 4), it is more common to present an MTC in terms of higher-level invariants. We define the important ones here. Some are numbers, while others are matrices whose rows and columns are labeled by representatives of the isomorphism classes of simple objects. In some cases, we diverge from [17].

Definition 2.45. Consider a fusion category $\mathcal{C}$ with a label set $\{a_i\}$ and a spherical pivotal structure. The *quantum dimension*, or simply *dimension*, of an object X is the trace of id_X. We denote the dimension

$$\dim_{\mathcal{C}}(X)$$

or simply $\dim(X)$ if the category is clear from context. Other common notation in the literature is d_X. Diagram 2.8 gives the graphical representation of this definition.

The *global quantum dimension* of the category $\mathcal{C}$ is defined as

$$\dim(\mathcal{C}) = \sqrt{\sum_i \dim(a_i)^2}.$$

Other notation includes $D_{\mathcal{C}}$ or simply D.

Remark 2.46. Note that our global quantum dimension is the square root of the one defined by [17, Definition 7.21.3]. The convention of [17] is quite common in the literature.

Definition 2.47. Consider a braided fusion category $\mathcal{B}$ with a spherical pivotal structure δ. On any object X, we may define a morphism (up to units and associators)

$$\psi_X = X^{**} \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_{X^{**}}} X \otimes X^* \otimes X^{**} \xrightarrow{\mathrm{id}_X \otimes c_{X^*,X^{**}}} X \otimes X^{**} \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_{X^*}} X.$$

The *topological twist* of an object X is defined as the composition

$$\theta_X = \psi_X \circ \delta_X.$$

Diagram 2.9 gives the graphical representation of this definition.

Remark 2.48. Note that $\theta_X \in \mathrm{Hom}(X, X)$, which is one-dimensional when X is simple. For simple objects, the topological twist is typically identified with a complex number as a multiple of the identity morphism. Vafa's theorem says that this complex number is always a root of unity. Diagram 2.10 shows this graphically.

Remark 2.49. The twist is not multiplicative unless the category is symmetric. In general, $\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y)$.

Remark 2.50. In fact, there are two ways to define θ based on whether the braiding in ψ_X is an over-crossing or an under-crossing. For a more in-depth summary of braided pivotal categories, refer to [25, Appendix A.2].

Definition 2.51. Consider a modular tensor category $\mathcal{B}$ with label set $\{a_i\}$. The *modular T-matrix* is defined as

$$T_{i,j} = \delta_{i,j}\theta_{a_i},$$

the diagonal matrix of the topological twists of simple objects.

Definition 2.52. Consider a modular tensor category $\mathcal{B}$ with label set $\{a_i\}$. Define a matrix $\tilde{S}$ by a trace of a full braiding

$$\tilde{S}_{i,j} = \mathrm{tr}\left(c_{a_j,a_i^*} \circ c_{a_i^*,a_j}\right).$$

The *modular S-matrix* is the normalized matrix

$$S = \frac{\tilde{S}}{\dim(\mathcal{B})},$$

where $\dim(\mathcal{B})$ is the global quantum dimension of $\mathcal{B}$ from Definition 2.45. Diagram 2.11 gives the graphical representation of this definition.

The above data satisfy a plethora of useful properties, some of which are listed below.

Proposition 2.53. *Consider a modular tensor category with label set $\{a_i\}$. We have the following.*

1. *If $a_i \otimes a_j \cong \bigoplus_k N_k^{ij} a_k$, then*

$$\dim(a_i)\dim(a_j) = \sum_k N_k^{ij} \dim(a_k)$$

2.

$$\tilde{S}_{i,j} = \theta_{a_i}^{-1}\theta_{a_j}^{-1} \sum_k N_k^{ij} \theta_{a_k} \dim(a_k)$$

3. *The Verlinde formula:*

$$N_k^{ij} = \sum_l \frac{S_{i,l} S_{j,l} \overline{S}_{k,l}}{S_{1,l}^2}$$

4. *The S-matrix is symmetric and unitary.*

5. *If $\det(S) = 0$, then the first column of S is proportional to another column. Thus, degeneracy of S comes from degeneracy of the double braiding in Definition 2.52.*

Other than giving information about the S-matrix, the significance of the last property can be seen in Remark 2.62. For more such relationships and their proofs, refer to, e.g. [51, 2, 18, 5, 52].

Definition 2.54. The *modular data* of a modular tensor category $\mathcal{B}$ is the pair $\{S, T\}$ of the modular S- and T-matrices associated with $\mathcal{B}$.

The term *modular* is chosen because the modular data of a modular tensor category provides a projective representation of the modular group $SL(2, \mathbb{Z})$. To see this, we first define the following additional invariants.

Definition 2.55. Consider a modular tensor category $\mathcal{B}$ with label set $\{a_i\}$. The *charge conjugation matrix* $C_{i,j} = \delta_{i,\hat{j}}$ is the matrix associating each label a_i with its dual. The *central charge* c is a rational number mod 8 defined by

$$\frac{1}{\dim(\mathcal{B})} \sum_i \theta_{a_i} \dim(a_i)^2 = e^{\frac{2c\pi i}{8}}.$$

This whole quantity is called the *multiplicative central charge* and is denoted $\xi = e^{c\pi i/4}$.

Now the following proposition constructs the representation.

Proposition 2.56. *Consider a rank n modular tensor category with label set $\{a_i\}$, modular data $\{S, T\}$, charge conjugation matrix C, and multiplicative central charge ξ. Then*

$$(ST)^3 = \xi S^2, \quad S^2 = C, \quad C^2 = I_n,$$

and the map $\rho\colon PSL(2, \mathbb{Z}) \to U(n)$ defined by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mapsto T$$

is a linear representation to the group of unitary $n \times n$ matrices.

Remark 2.57. These high-level invariants have long been used to uniquely determine the modular tensor category of interest. It was found relatively recently that there exist different modular tensor categories with the same modular data. So-related categories are called *modular isotopes*. Refer to [14] for more on modular isotopes.

2.1.4 MTCs as Ribbon Fusion Categories

For ease of reference in the unlikely event someone reads this thesis, we also mention that there exist other definitions for modular tensor categories. One is by way of ribbon fusion categories.

Definition 2.58. A braiding c and a pivotal structure δ are said to be *compatible* if the morphism θ from Defintion 2.47 satisfies

$$\theta_{X^*} = (\theta_X)^*,$$

where $*$ denotes the dual.

Definition 2.59. A *ribbon fusion category* is a braided fusion category with a compatible pivotal structure. That is, a braided fusion category with a choice of isomorphisms $\delta_X \colon X \to X^{**}$ such that $\delta_{X\otimes Y} = \delta_X \otimes \delta_Y$, $\delta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, $\delta_{X^*} = (\delta_X^*)^{-1}$.

This compatibility is what the spherical property of Definition 2.36 gives.

Proposition 2.60. *A semisimple pivotal braided category is ribbon if and only if it is spherical.*

Note that any ribbon fusion category has the necessary structures to define the modular data of 2.54. Indeed, ribbon fusion categories and spherical braided fusion categories are sometimes called *pre-modular.*

Definition 2.61. A *modular tensor category* is a ribbon fusion category with an invertible S-matrix.

Remark 2.62. A braided fusion category with a spherical pivotal structure is a ribbon fusion category. In this setting, non-degeneracy of the braiding is equivalent to non-degeneracy of the S-matrix. This should be expected from Item 5 of Proposition 2.53.

2.1.5 Unitarity

Finally, we close this section with a brief discussion of unitarity. Unitarity is typically required for physical application of modular tensor categories, such as for use as anyon models. Indeed, all modular tensor categories we work with in this thesis are unitary. While explicit treatment of unitarity is limited in all that follows, it does appear in a few places. The definition of unitary fusion category is due to [36], and the definition of unitary ribbon fusion category is due to [51]. Note that we must now use a base field of $\mathbb{C}$.

Definition 2.63. A *conjugation* $\dagger$ on a fusion category is an assignment of a morphism $f^\dagger \in \mathrm{Hom}(Y, X)$ to each morphism $f \in \mathrm{Hom}(X, Y)$ which is conjugate linear and satisfies

$$\left(f^\dagger\right)^\dagger = f, \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, \qquad (f \circ g)^\dagger = g^\dagger \circ f^\dagger.$$

Definition 2.64. A *unitary fusion category* is a fusion category along with a choice of conjugation such that $f^\dagger \circ f = 0$ implies $f = 0$.

Definition 2.65. A *Hermitian ribbon fusion category* is a ribbon fusion category along with a choice of conjugation such that

- $\mathrm{coev}_X^\dagger = \mathrm{ev}_X \circ c_{X,X^*} \circ (\theta_X \otimes \mathrm{id}_{X^*})$,
- $\mathrm{ev}_X^\dagger = (\mathrm{id}_{X^*} \otimes \theta_X^{-1}) \circ c_{X^*,X}^{-1} \circ \mathrm{coev}_X$,
- $c_{X,Y}^\dagger = c_{X,Y}^{-1}$,
- $\theta_X^\dagger = \theta_X^{-1}$.

Remark 2.66. The first two conditions in Definition 2.65 are compatibility conditions between the evaluation and coevaluation morphisms of left and right duals. Refer to Remark 2.15.

Definition 2.67. A Hermitian ribbon fusion category (or modular tensor category) is called *unitary* if the form $(f, g) \mapsto \mathrm{tr}(f \circ g^\dagger)$ is positive-definite on all Hom spaces.

One helpful feature of unitary categories is that they are also pseudo-unitary. We need a few more definitions.

Definition 2.68. Consider a fusion category $\mathcal{C}$ with label set $\{a_i\}$. The *Grothendieck group* of $\mathcal{C}$ is the free abelian group generated by the formal labels a_i. The *Grothendieck ring* of $\mathcal{C}$ is the Grothendieck group together with the multiplication induced by the tensor product of $\mathcal{C}$.

Roughly speaking, the Grothendieck ring of $\mathcal{C}$ is simply the natural ring obtained from the skeleton of $\mathcal{C}$ together with the operations $\oplus, \otimes$.

Definition 2.69. Consider the Grothendieck ring of a fusion category $\mathcal{C}$ with label set $\{a_i\}$. For each i, the *Frobenius-Perron dimension* $\text{FPdim}(a_i)$ is the maximal non-negative eigenvalue of the matrix of left multiplication by a_i (which is guaranteed to exist).

For an arbitrary object X, define $\text{FPdim}(X)$ additively across $\oplus$.

The *global Frobenius-Perron dimension* is given by

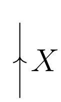
$$\text{FPdim}(\mathcal{C}) = \sum_i \text{FPdim}(a_i)^2.$$

For more on Frobenius-Perron dimensions, refer to [17]. As in Remark 2.46, note that we have defined our global quantum dimension to be the square root of that in [17].

Definition 2.70. A fusion category $\mathcal{C}$ is called *pseudo-unitary* if $\dim(\mathcal{C})^2 = \text{FPdim}(\mathcal{C})$.

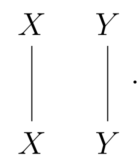
2.2 Graphical Calculus

A standard graphical calculus is often used to conveniently work with modular tensor categories. We adopt the *optimistic* convention, which reads morphisms from bottom to top. The identity morphism on an object X is a single line labeled X, as follows.



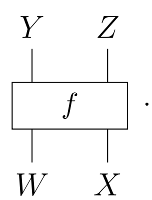
For clarity and reduction of clutter, we will often label the strands of our diagrams at the bottom and top. We will also omit arrows when the optimistic convention leaves little ambiguity.

Tensor products of objects are drawn by placing the objects next to each other. The identity morphism $\mathrm{id}_{X\otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y$ is thus

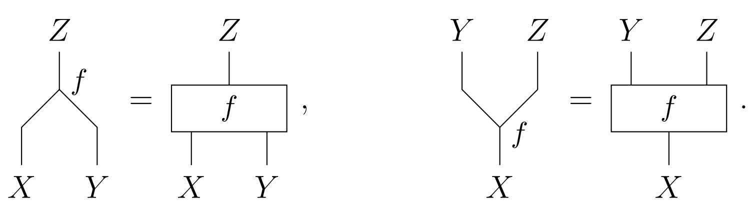


Since $\mathbb{1} \otimes X \cong X$ for tensor unit $\mathbb{1}$ and any object X, we can add and remove $\mathbb{1}$ strands at will. Except when they are the subject of particular focus, $\mathbb{1}$ strands are typically not drawn at all.

In general, morphisms will be represented by boxes. For example, a morphism $f\colon W\otimes X \to Y \otimes Z$ may be drawn

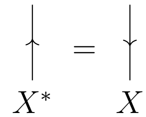


When the meaning is clear, we will often omit boxes. We will do this especially with trivalent vertices so that morphisms $f\colon X\otimes Y \to Z$ and $f\colon X \to Y\otimes Z$ might be drawn

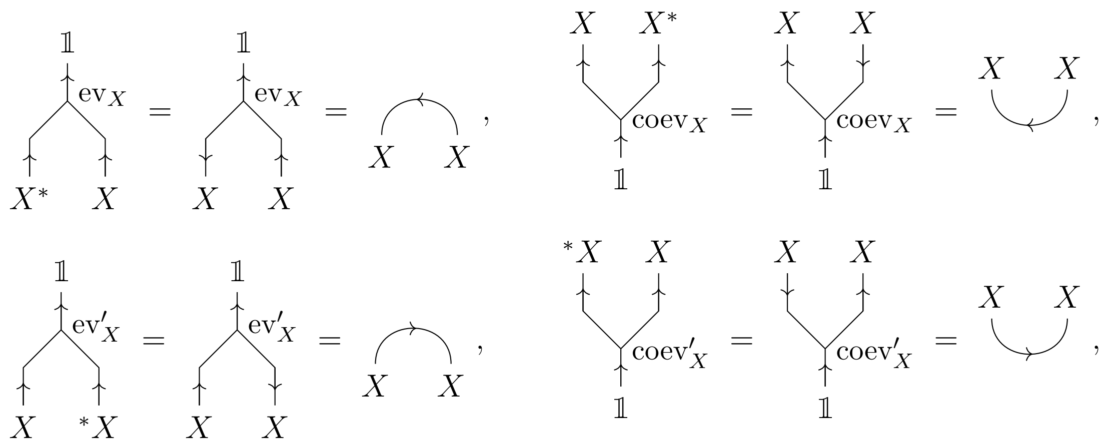


Our pictures will become even cleaner when the morphisms are clear from context and we can leave off the label f entirely.

Recall that modular tensor categories are rigid. If X is an object of a modular tensor category, the identity morphism on the dual X^* is a single line labeled X going the opposite direction, as follows.

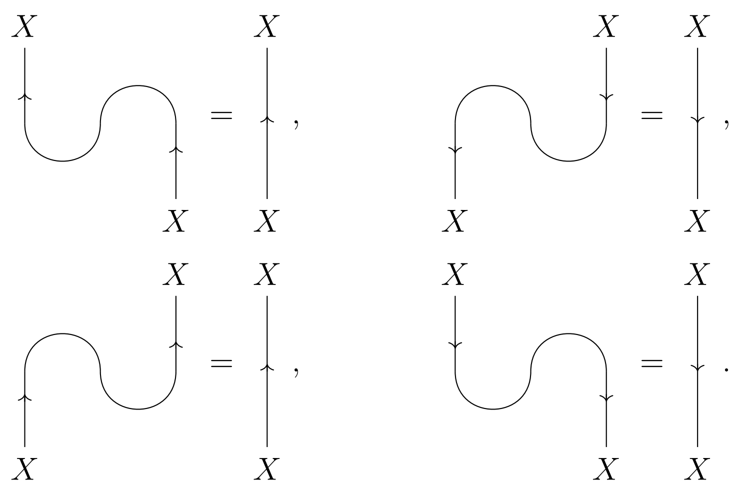


Now, by not drawing $\mathbb{1}$ strands, the evaluation and coevaluation maps of Definition 2.10 can be depicted



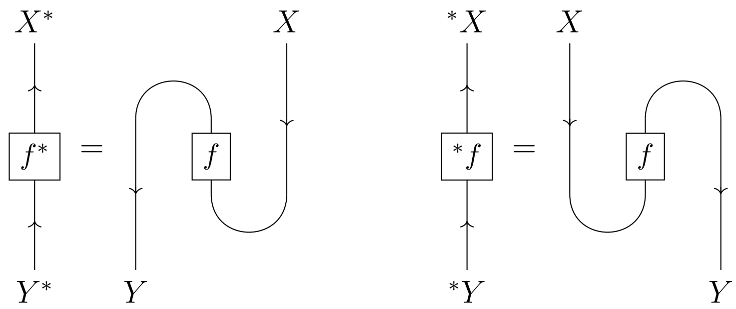
(2.4)

and conditions 2.1,2.2 amount, respectively, to



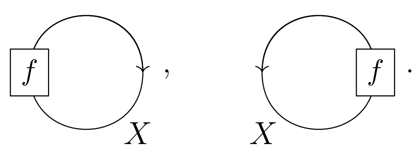
(2.5)

Dual morphisms from Definition 2.13 may be drawn as follows.



(2.6)

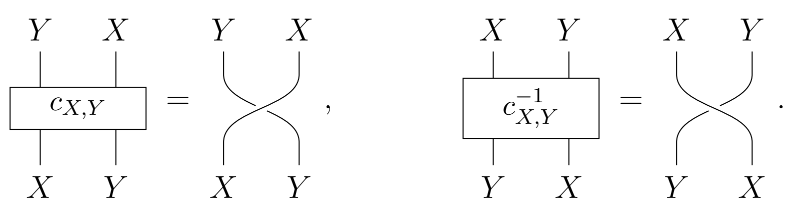
Graphical representations of evaluation and coevaluation in hand, we may present cleaner definitions for the quantum trace and quantum dimension of Definitions 2.34, 2.36, and 2.45. The left and right traces of a morphism $f \in \mathrm{Hom}(X, X)$ are given respectively by the pictures



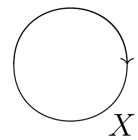
(2.7)

Remark 2.71. The term *spherical* from Definition 2.36 is chosen to suggest that the two diagrams of 2.7 are equal because they live on a sphere and the circle can be pulled around to the other side.

Modular tensor categories are also braided. A braiding is a natural isomorphism $c_{X,Y} \colon X \otimes Y \to Y \otimes X$. These morphisms are often drawn

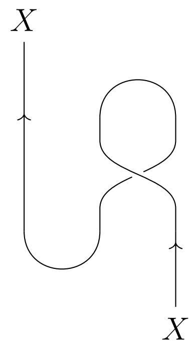


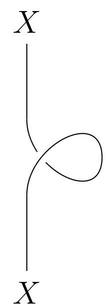
We can now give the graphical representations of the invariants used to discuss modular tensor categories. The quantum dimension of an object X from Definition 2.45 is defined by

$$\dim(X) = $$ 

. (2.8)

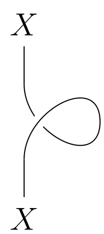
The topological twist of an object X from Definition 2.47 is defined by the morphism

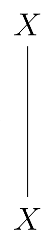
$$\theta_X = $$ 

$=$ 

. (2.9)

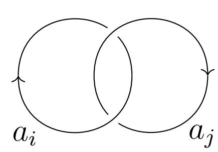
Recall from Remark 2.48 that when X is a simple object, the space $\mathrm{Hom}(X, X)$ is one-dimensional, so the morphism θ_X is a multiple of the identity on X. When X is a simple object, the symbol θ_X can be identified with this complex number, and we can say



$= \theta_X$ 

. (2.10)

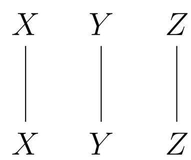
The entries of the $\tilde{S}$-matrix from Definition 2.52 are defined by

$$\tilde{S}_{i,j} = $$ 

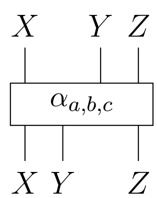
. (2.11)

2.2.1 Skeletalization

A somewhat subtle point is the associativity. Because it will be critically important for us in Chapter 4, we discuss it at length here. The picture



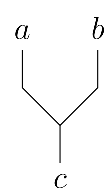
is ambiguous because it could be referring to $(X \otimes Y) \otimes Z$ or $X \otimes (Y \otimes Z)$. That is, we should perhaps make some distinction of the form



,

where proximity indicates the order of the tensor products. In practice, we often ignore this distinction since a modular tensor category can always be taken to be strict without loss of generality. Then the associativity isomorphism is the identity on all X, Y, Z, and we need not worry about parentheses.

This fails when we wish to consider a skeletal MTC. In this case, we need not consider isomorphism classes of objects. Our simple objects are exactly the labels $a, b, c, \ldots$. If we write $a \otimes b = \bigoplus_c N_c^{ab} c$ (which is now a proper equality rather than an isomorphism), then we may choose explicit morphisms

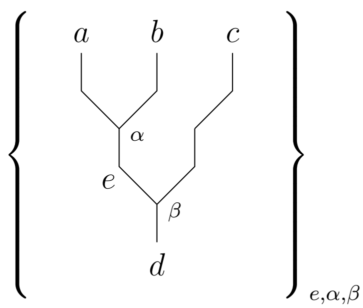


(2.12)

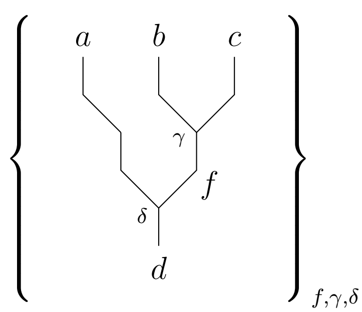
in $\mathrm{Hom}(c, a \otimes b)$. Recall that $\mathrm{Hom}(c, a \otimes b)$ is a vector space of dimension N_c^{ab}. When $N_c^{ab} = 0$, morphism 2.12 is necessarily zero, and we do not bother writing it. When $N_c^{ab} = 1$, morphism 2.12 defines a basis for $\mathrm{Hom}(c, a \otimes b)$. When $N_c^{ab} > 1$, we may label the vertex of picture 2.12 with morphisms $\alpha, \beta, \gamma, \ldots$ to define a basis for $\mathrm{Hom}(c, a \otimes b)$. When we do this, it should be clear from the picture that α is not referring to the associativity isomorphism.

The point here is that pictures of the form of 2.12 set a basis in terms of which the graphical calculus is defined. We may then work with the tensor product $a \otimes b$ using explicit bases of vector spaces. The same trick gives us a concrete way to represent and work with the associativity of the skeletal category in terms of pictures.

To understand the isomorphism from $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$, we consider the spaces $\mathrm{Hom}(d, (a \otimes b) \otimes c)$ and $\mathrm{Hom}(d, a \otimes (b \otimes c))$. Morphisms in $\mathrm{Hom}(d, (a \otimes b) \otimes c)$ are compositions of morphisms in $\mathrm{Hom}(e, a \otimes b)$ and $\mathrm{Hom}(d, e \otimes c)$. Morphisms in $\mathrm{Hom}(d, a \otimes (b \otimes c))$ are compositions of morphisms in $\mathrm{Hom}(f, b \otimes c)$ and $\mathrm{Hom}(d, a \otimes f)$. Thus for each simple object d, we have a basis



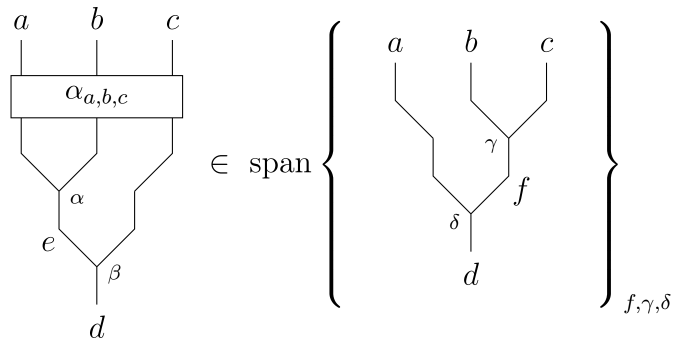
for $\mathrm{Hom}(d, (a \otimes b) \otimes c)$ and a basis



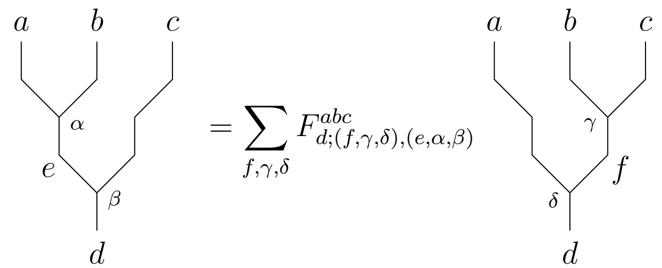
for $\mathrm{Hom}(d, a \otimes (b \otimes c))$. Since $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ and the category is skeletal, the two tensor products are equal as objects, and we have produced two bases for $\mathrm{Hom}(d, (a \otimes b) \otimes c) = \mathrm{Hom}(d, a \otimes (b \otimes c))$. The associativity isomorphism $\alpha_{a,b,c}$ is such that, for each

d, it sends the first basis to the second. That is, for each fixed α, β,

$$\alpha_{a,b,c} \circ (\alpha \otimes \mathrm{id}_c) \circ \beta \;\in\; \mathrm{span}\Big\{ (\mathrm{id}_a \otimes \gamma) \circ \delta \Big\}_{\gamma,\delta}$$



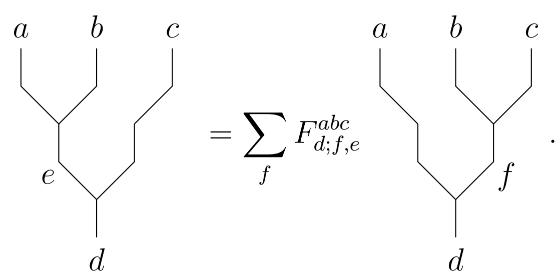
So the associativity isomorphism can be understood as a family of change of basis matrices F_d^{abc} whose entries are the coefficients of



(2.13)

Definition 2.72. The coefficients of Equation 2.13 are sometimes called *F-symbols*.

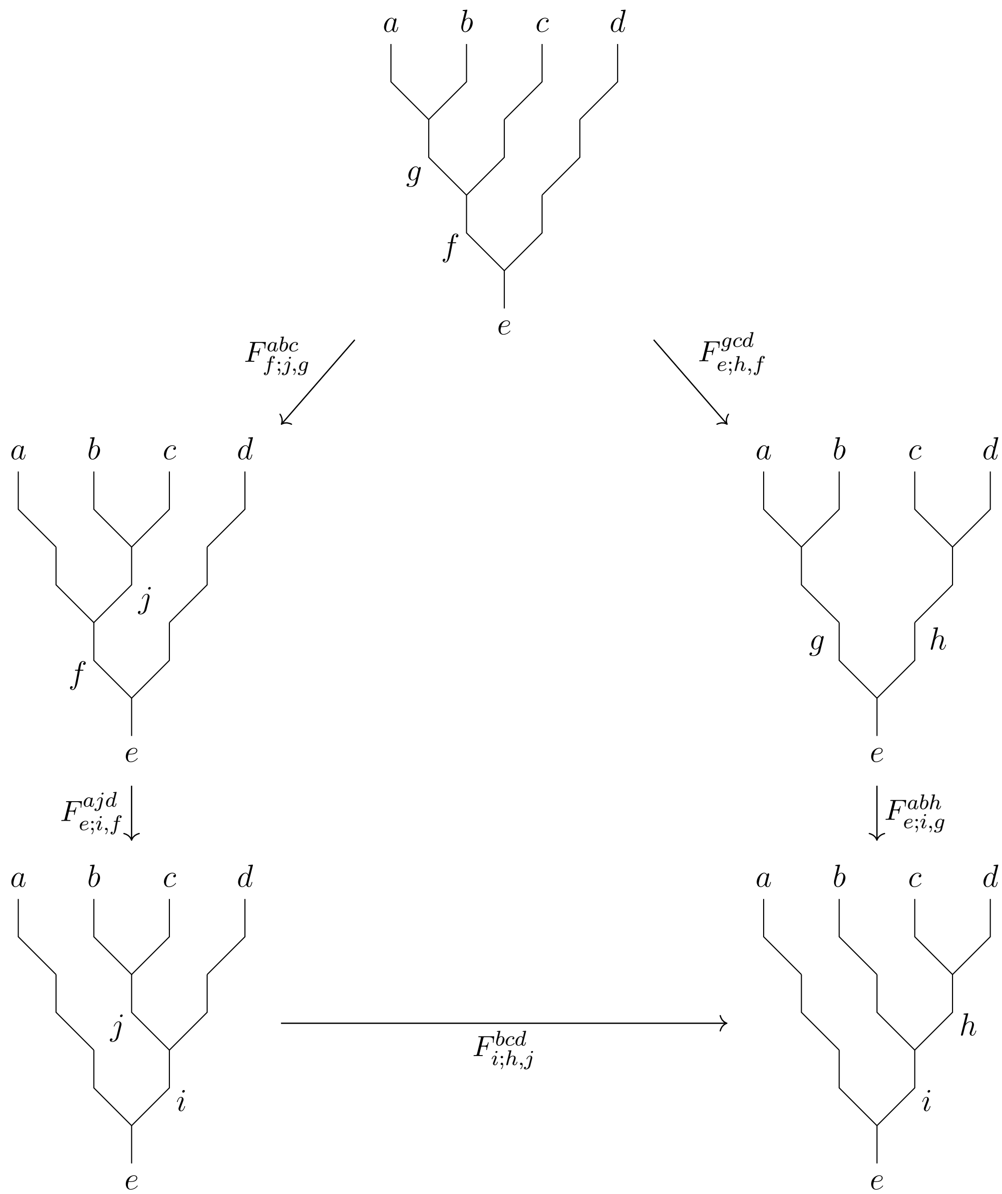
Most of the literature restricts to the simplest case of multiplicity-free categories (Definition 2.26). In this case, we need not distinguish morphisms at the trivalent vertices, and we arrive at the familiar equation



(2.14)

These six-index F-symbols are sometimes known as *6-j symbols*. While we're here, note that our coefficients are matrix entries $(F_d^{abc})_{f,e}$ rather than $(F_d^{abc})_{e,f}$. Defining F-symbols this way makes F_d^{abc} a change-of-basis matrix from the left to the right. The literature is somewhat undecided on which convention to choose.

Since the F-symbols are a presentation of the associativity, they must satisfy the pentagon and triangle axioms of Definition 2.1. In the multiplicity-free case, requiring that the diagram



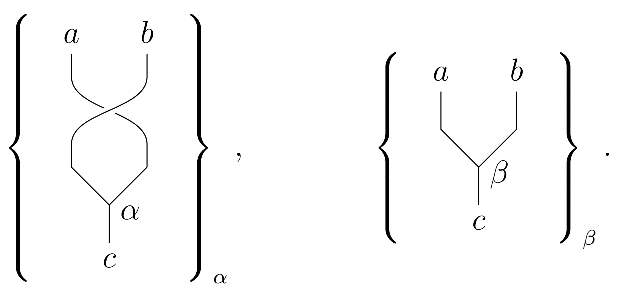
commute gives us the condition

$$F^{abh}_{e;i,g} F^{gcd}_{e;h,f} = \sum_j F^{bcd}_{i;h,j} F^{ajd}_{e;i,f} F^{abc}_{f;j,g}. \tag{2.15}$$

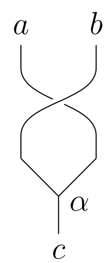
Indeed, a multiplicity-free monoidal category can be defined as a collection of admissible fusion coefficients N_c^{ab} and 6-j symbols $F^{abc}_{d;f,e}$. Refer to [52, Chapter 4] for this definition. By specifying morphisms at each trivalent vertex, we can write down equations of the same form for the F-symbols of a category with multiplicity as well.

Remark 2.73. Explicit monoidal structures (in the form of F-symbols) are generally difficult to compute because their consistency equations look like Equation 2.15. Simultaneously solving many nonlinear equations is computationally challenging.

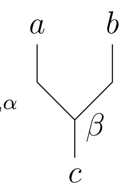
A similar tale can be spun for the braiding. Again if we write $a \otimes b = \bigoplus_c N_c^{ab} c$, we have two bases for $\mathrm{Hom}(c, a \otimes b)$ given by



Then, for each choice of α, we can again write

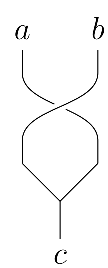


$$= \sum_\beta R^{ab}_{c;\beta,\alpha}$$

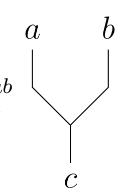


. (2.16)

In the multiplicity-free case, we have the familiar



$$= R_c^{ab}$$



. (2.17)

Definition 2.74. The coefficients of Equation 2.16 are sometimes called *R-symbols*.

These R-symbols define the braiding, and a braided fusion category can be defined as a list of compatible N-, F-, and R-symbols (see Definition 2.25 of fusion coefficients for the N-symbols). A set of t-symbols (not to be confused with the T-symbols of Definition 2.101) called *pivotal coefficients* can also be given so that all structures in Remark 2.44 are defined. Refer to [52, Chapter 4] for these definitions.

2.3 Classification and Examples

Significant effort has gone into the classification of modular tensor categories, e.g. [47, 23]. There is similarly interest in a classification of fusion categories, but this problem is significantly more difficult. The situation is analogous to that of the classification of finite abelian groups and the classification of all finite groups. As an example, a result called Ocneanu rigidity says that for any given set of fusion rules, there exist finitely many fusion categories (up to monoidal equivalence). For modular tensor categories, there is the far stronger result that for any given rank, there exist finitely many modular tensor categories (up to equivalence) [6].

The classification of modular tensor categories that is currently known will be helpful in determining condensations. For examples, refer to Sections 3.3.1 and 3.3.2.

In the remainder of this section, we briefly discuss some sources of modular tensor categories. We also provide a few examples of well-known MTCs to demonstrate the presentation of their data and because we will be using them later.

2.3.1 General Constructions

The theory of modular tensor categories is a confusing array of developments by many people in many different settings. For more information as in this subsection, one might refer to, e.g. [17, 2, 13, 37, 48].

The first construction is just a way to put existing modular tensor categories together, but we must define it since it will be used extensively throughout this thesis.

Definition 2.75 ([17, Definition 1.11.1])**.** Let $\mathcal{C}, \mathcal{D}$ be locally finite abelian categories. The *Deligne tensor product*, or simply *Deligne product*, of $\mathcal{C}$ and $\mathcal{D}$ is a pair $(\mathcal{C} \boxtimes \mathcal{D}, \boxtimes)$, where $\boxtimes$ is a bifunctor $\boxtimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ by $(X, Y) \mapsto X \boxtimes Y$ that is right exact in both variables and is universal with this property. For any bifunctor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ that is right exact in both variables, there exists a unique right exact functor $\overline{F}\colon \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ with $\overline{F} \circ \boxtimes = F$.

Remark 2.76. For us, the categories $\mathcal{C}, \mathcal{D}$ of Definition 2.75 will typically be modular tensor categories, in which case the product $\mathcal{C} \boxtimes \mathcal{D}$ is also a modular tensor category. We will usually take the following as our definitions.

- $\mathrm{Obj}(\mathcal{C} \boxtimes \mathcal{D}) = \{X \boxtimes Y \mid X \in \mathrm{Obj}(\mathcal{C}), Y \in \mathrm{Obj}(\mathcal{D})\}$
- $\mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(X_1 \boxtimes Y_1, X_2 \boxtimes Y_2) = \{f \boxtimes g \mid f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2), g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)\}$

The tensor product of $\mathcal{C} \boxtimes \mathcal{D}$ will be defined as follows.

- $(X_1 \boxtimes Y_1) \otimes_{\mathcal{C} \boxtimes \mathcal{D}} (X_2 \boxtimes Y_2) = (X_1 \otimes_{\mathcal{C}} X_2) \boxtimes (Y_1 \otimes_{\mathcal{D}} Y_2)$
- $(f_1 \boxtimes g_1) \otimes_{\mathcal{C} \boxtimes \mathcal{D}} (f_2 \boxtimes g_2) = (f_1 \otimes_{\mathcal{C}} f_2) \boxtimes (g_1 \otimes_{\mathcal{D}} g_2)$

This agrees with the definition from the proof of [17, Proposition 4.6.1].

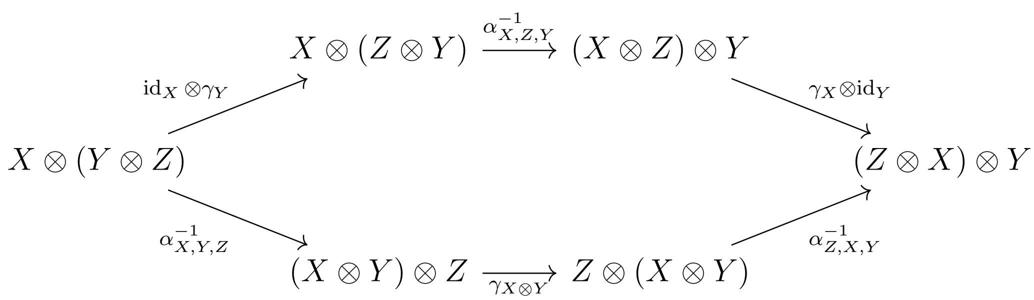
The first standard construction of modular tensor categories is the quantum double. See [2, Section 3.2] for more details.

Construction 2.77. The *quantum double* (or *Drinfeld double*) of a finite group is defined as follows. Let G be a finite group, and let $\Bbbk[G]$ be its group algebra, which has a Hopf algebra structure. The Hopf algebra dual to the group algebra is the function algebra $F(G)$. Then the quantum double $D(G)$ is the semidirect product $F(G) \rtimes \Bbbk[G]$, which is $F(G) \otimes_{\Bbbk} \Bbbk[G]$ as a vector space. Finally, $\mathrm{Rep}_f(D(G))$, the category of finite-dimensional representations of $D(G)$, is a modular tensor category.

This construction can also be given by a Drinfeld center.

Definition 2.78. Let $\mathcal{C}$ be a monoidal category with associativity α. The *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ is the category with objects and morphisms as follows.

- $\operatorname{Obj}(\mathcal{Z}(\mathcal{C}))$ consists of pairs (Z, γ) with $Z \in \operatorname{Obj}(\mathcal{C})$ and $\gamma_X \colon X \otimes Z \to Z \otimes X$ a natural isomorphism such that the following diagram is commutative for all $X, Y \in \operatorname{Obj}(\mathcal{C})$.



- $\operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}((Z, \gamma), (Z', \gamma'))$ is the set of $f \in \operatorname{Hom}_{\mathcal{C}}(Z, Z')$ such that for all $X \in \operatorname{Obj}(\mathcal{C})$, $(f \otimes \mathrm{id}_X) \circ \gamma_X = \gamma'_X \circ (\mathrm{id}_X \otimes f)$.

Thus the Drinfeld center of $\mathcal{C}$ has intertwiner morphisms between objects that are pairs consisting of objects from $\mathcal{C}$ and choices of braidings that satisfy one of the hexagons from Definition 2.37.

Theorem 2.79. *There is a braided equivalence*

$$\operatorname{Rep}_f(D(G)) \cong \mathcal{Z}\left(\operatorname{Rep}_f(G)\right).$$

The Drinfeld center has more general usefulness, as seen in the next two theorems.

Theorem 2.80. *The Drinfeld center of a spherical fusion category is a modular tensor category.*

Recall Definition 2.38 of a braided fusion category. The reverse category is defined as follows.

Definition 2.81. Let $\mathcal{B}$ be a fusion category with a chosen braiding $c_{X,Y}$. The *reverse* $\mathcal{B}^{\text{rev}}$ is the fusion category $\mathcal{B}$ along with the braiding $c^{\text{rev}}_{X,Y} = c^{-1}_{Y,X}$.

Theorem 2.82. *If $\mathcal{B}$ is a modular tensor category (or any non-degenerate braided fusion category), there is a braided equivalence $\mathcal{Z}(\mathcal{B}) \cong \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}$.*

Another common source of modular tensor categories is quantum groups at roots of unity. In this thesis, we will see the categories $SU(2)_k$ (Sections 3.3.3, 5.1.1) and $(G_2)_3$ (Section 5.2.2, Supplemental B: $(G_2)_3$ Data). The construction is somewhat nontrivial. Refer to [2, Section 3.3],[51, Section XI.6] for details or [48] for a summary.

Theorem 2.83. *From a Lie algebra $\mathfrak{g}$ and complex number q with q^2 an ℓ-th root of unity (with properties), a modular tensor category can be constructed from representations of a quantum group $U_q(\mathfrak{g})$.*

Finally, we mention the connection to vertex operator algebras because it is relevant to the motivation for Section 5.3.

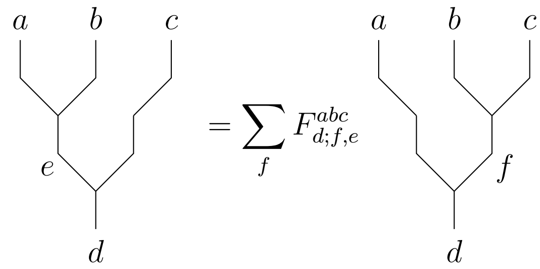
Theorem 2.84 ([26])**.** *The representation category of a vertex operator algebra (with properties) is a modular tensor category.*

2.3.2 Explicit Examples

The data of the following categories are presented in the usual way found in, e.g. [47]. We first establish an order for the isomorphism classes of simple objects when we define a label set. This is then the order in which the rest of the data are given. The i-th entries of the set of quantum dimensions and the set of twists give the quantum dimension and twist of the i-th label in the label set. The i,j-th entry of the S-matrix is the S-matrix entry from Definition 2.52 corresponding to the i-th and j-th labels in the label set.

Finally, the F-matrices are also presented in the usual way. Recall Equation 2.14

from Section 2.2.1 defining the F-symbols in a multiplicity-free category.



$$= \sum_f F^{abc}_{d;f,e}$$

Now each matrix F^{abc}_d has rows labeled by admissible choices for e and columns labeled by admissible choices for f. Then the entry $(F^{abc}_d)_{f,e}$ is the F-symbol $F^{abc}_{d;f,e}$. Presenting the matrices this way requires an order for the choices of e, f. The order we choose is again the order determined by the label set. A single number is a 1×1 matrix and means there is only one admissible choice on each side of the F-symbol equation. Omitted F-symbols are 0 if they are not admissible or 1 if they are admissible.

All examples in this section (and most MTCs anyone works with) are multiplicity-free. Refer to Section 5.2 and Supplemental B: $(G_2)_3$ Data for categories with multiplicity. For more category data, refer to [47, 23].

Example 2.85. The category ***Vec*** of finite-dimensional $\mathbb{C}$-vector spaces with linear transformation morphisms is the trivial modular tensor category with $\mathbb{C}$ itself the only simple object (up to isomorphism).

Example 2.86. The *Fibonacci* modular tensor category has the following data. We use φ to denote the golden ratio $\frac{1+\sqrt{5}}{2}$.

Label set: $\mathcal{L} = \{1, \tau\}$

Fusion rules:

$\otimes$	1	τ
1	1	τ
τ	τ	$1 \oplus \tau$

Quantum dimensions: $\{1, \varphi\}$

Total quantum dimension: $D = \sqrt{\varphi\sqrt{5}}$

Twists: $\left\{1, e^{\frac{4\pi i}{5}}\right\}$

Central charge: $c = \dfrac{14}{5}$

S-Matrix: $\dfrac{1}{\sqrt{2+\varphi}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$

F-Symbols: $F_{\tau}^{\tau\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$

R-Symbols: $R_1^{\tau\tau} = e^{-\frac{4\pi i}{5}}$, $R_{\tau}^{\tau\tau} = e^{\frac{3\pi i}{5}}$

Example 2.87. The term *Ising* refers to any of eight inequivalent categories with the same fusion rules. These eight categories are distinguished by the choice of θ for the simple object σ. The data for the first of these categories (which is the one we will use in this thesis) are as follows.

Label set: $\mathcal{L} = \{1, \sigma, \psi\}$

Fusion rules:

$\otimes$	1	σ	ψ
1	1	σ	ψ
σ	σ	$1 \oplus \psi$	σ
ψ	ψ	σ	1

Quantum dimensions: $\left\{1, \sqrt{2}, 1\right\}$

Total quantum dimension: $D = 2$

Twists: $\left\{1, e^{\frac{\pi i}{8}}, -1\right\}$

Central charge: $c = \dfrac{1}{2}$

S-Matrix: $\dfrac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

F-Symbols: $F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $F_\sigma^{\psi\sigma\psi} = F_\psi^{\sigma\psi\sigma} = (-1)$

R-Symbols: $R_1^{\psi\psi} = -1$, $R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = -i$, $R_1^{\sigma\sigma} = e^{-\frac{\pi i}{8}}$, $R_\psi^{\sigma\sigma} = e^{\frac{3\pi i}{8}}$

Example 2.88. Recall Definition 2.75. The category Ising$\boxtimes$Ising is the Deligne product of two copies of the Ising MTC. Most of its data is multiplicative from Ising.

Label set: $\mathcal{L} = \{1 \boxtimes 1, 1 \boxtimes \sigma, 1 \boxtimes \psi, \sigma \boxtimes 1, \sigma \boxtimes \sigma, \sigma \boxtimes \psi, \psi \boxtimes 1, \psi \boxtimes \sigma, \psi \boxtimes \psi\}$

or, for readability,

$\mathcal{L} = \{11, 1\sigma, 1\psi, \sigma 1, \sigma\sigma, \sigma\psi, \psi 1, \psi\sigma, \psi\psi\}$

Fusion rules: Ising fusion rules on components

$\otimes$	11	1σ	1ψ	$\sigma 1$	$\sigma\sigma$	$\sigma\psi$	$\psi 1$	$\psi\sigma$	$\psi\psi$
11	11	1σ	1ψ	$\sigma 1$	$\sigma\sigma$	$\sigma\psi$	$\psi 1$	$\psi\sigma$	$\psi\psi$
1σ	1σ	$11 \oplus 1\psi$	1σ	$\sigma\sigma$	$\sigma 1 \oplus \sigma\psi$	$\sigma\sigma$	$\psi\sigma$	$\psi 1 \oplus \psi\psi$	$\psi\sigma$
1ψ	1ψ	1σ	11	$\sigma\psi$	$\sigma\sigma$	$\sigma 1$	$\psi\psi$	$\psi\sigma$	$\psi 1$
$\sigma 1$	$\sigma 1$	$\sigma\sigma$	$\sigma\psi$	$11 \oplus \psi 1$	$1\sigma \oplus \psi\sigma$	$1\psi \oplus \psi\psi$	$\sigma 1$	$\sigma\sigma$	$\sigma\psi$
$\sigma\sigma$	$\sigma\sigma$	$\sigma 1 \oplus \sigma\psi$	$\sigma\sigma$	$1\sigma \oplus \psi\sigma$	$11 \oplus 1\psi \oplus \psi 1 \oplus \psi\psi$	$1\sigma \oplus \psi\sigma$	$\sigma\sigma$	$\sigma 1 \oplus \sigma\psi$	$\sigma\sigma$
$\sigma\psi$	$\sigma\psi$	$\sigma\sigma$	$\sigma 1$	$1\psi \oplus \psi\psi$	$1\sigma \oplus \psi\sigma$	$11 \oplus \psi 1$	$\sigma\psi$	$\sigma\sigma$	$\sigma 1$
$\psi 1$	$\psi 1$	$\psi\sigma$	$\psi\psi$	$\sigma 1$	$\sigma\sigma$	$\sigma\psi$	11	1σ	1ψ
$\psi\sigma$	$\psi\sigma$	$\psi 1 \oplus \psi\psi$	$\psi\sigma$	$\sigma\sigma$	$\sigma 1 \oplus \sigma\psi$	$\sigma\sigma$	1σ	$11 \oplus 1\psi$	1σ
$\psi\psi$	$\psi\psi$	$\psi\sigma$	$\psi 1$	$\sigma\psi$	$\sigma\sigma$	$\sigma 1$	1ψ	1σ	11

Quantum dimensions: $\left\{1, \sqrt{2}, 1, \sqrt{2}, 2, \sqrt{2}, 1, \sqrt{2}, 1\right\}$

Total quantum dimension: $D = 4$

Twists: $\left\{1, e^{\frac{\pi i}{8}}, -1, e^{\frac{\pi i}{8}}, e^{\frac{\pi i}{4}}, -e^{\frac{\pi i}{8}}, -1, -e^{\frac{\pi i}{8}}, 1\right\}$

Central charge: $c = \frac{1}{2} + \frac{1}{2} = 1$

S-Matrix: $$\frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & 2 & 0 & -2 & \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & \sqrt{2} & -2 & \sqrt{2} & 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & -2 & -\sqrt{2} \\ 2 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 2 \\ \sqrt{2} & -2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & -\sqrt{2} & -2 & -\sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & -2 & 0 & 2 & \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \end{pmatrix}$$

F-Symbols: $F^{aa',bb',cc'}_{dd';ff',ee'} = F^{abc}_{d;f,e}F^{a'b'c'}_{d';f',e'}$ for all $a, a', b, b', c, c', d, d', e, e', f, f'$ from the label set of Ising

R-Symbols: $R^{aa',bb'}_{cc'} = R^{ab}_{c}R^{a'b'}_{c'}$ for all a, a', b, b', c, c' from the label set of Ising

Example 2.89. The category Ising ⊠ $\overline{\text{Ising}}$ is the Deligne product of the Ising MTC with its complex conjugate. It has the same label set, fusion rules, quantum dimensions, S-matrix, and F-symbols (since they are real) as Ising ⊠ Ising.

Twists: $\left\{1, e^{-\frac{\pi i}{8}}, -1, e^{\frac{\pi i}{8}}, 1, -e^{\frac{\pi i}{8}}, -1, -e^{-\frac{\pi i}{8}}, 1\right\}$

Central charge: $c = \dfrac{1}{2} - \dfrac{1}{2} = 0$

R-Symbols: $R^{aa',bb'}_{cc'} = R^{ab}_{c}\overline{R^{a'b'}_{c'}}$ for all a, a', b, b', c, c' from the label set of Ising

Example 2.90. The category $\mathbb{Z}_4$ is a rank 4 modular tensor category with fusion rules resembling the additive group $\mathbb{Z}_4$.

Label set: $\mathcal{L} = \{1, \epsilon, \sigma, \sigma^*\}$

Fusion rules:

$\otimes$	1	ϵ	σ	σ^*
1	1	ϵ	σ	σ^*
ϵ	ϵ	1	σ^*	σ
σ	σ	σ^*	ϵ	1
σ^*	σ^*	σ	1	ϵ

Quantum dimensions: $\{1, 1, 1, 1\}$

Total quantum dimension: $D = 2$

Twists: $\left\{1, -1, e^{\frac{\pi i}{4}}, e^{\frac{\pi i}{4}}\right\}$

Central charge: $c = 1$

S-Matrix:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$$

F-Symbols: $F_{\sigma}^{\epsilon\sigma\epsilon} = F_{\sigma^*}^{\epsilon\sigma^*\epsilon} = F_{\epsilon}^{\sigma\epsilon\sigma^*} = F_{\epsilon}^{\sigma^*\epsilon\sigma} = F_{\sigma^*}^{\sigma\sigma\sigma} = F_{\sigma}^{\sigma^*\sigma^*\sigma^*} = (-1)$

R-Symbols: $R_1^{\epsilon\epsilon} = -1, \quad R_{\sigma^*}^{\sigma\epsilon} = R_{\sigma^*}^{\epsilon\sigma} = R_{\sigma}^{\sigma^*\epsilon} = R_{\sigma}^{\epsilon\sigma^*} = -i$

$$R_{\epsilon}^{\sigma\sigma} = R_{\epsilon}^{\sigma^*\sigma^*} = e^{\frac{\pi i}{4}}, \quad R_1^{\sigma\sigma^*} = R_1^{\sigma^*\sigma} = e^{-\frac{\pi i}{4}}$$

Example 2.91. The category Toric Code is another rank 4 modular tensor category, but with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules.

Label set: $\mathcal{L} = \{1, e, m, \epsilon\}$

Fusion rules:

$\otimes$	1	e	m	ϵ
1	1	e	m	ϵ
e	e	1	ϵ	m
m	m	ϵ	1	e
ϵ	ϵ	m	e	1

Quantum dimensions: $\{1, 1, 1, 1\}$

Total quantum dimension: $D = 2$

Twists: $\{1, 1, 1, -1\}$

Central charge: $c = 0$

S-Matrix:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

F-Symbols: $F_d^{abc} = (1)$ for all a, b, c, d

R-Symbols: $R_1^{ee} = R_1^{mm} = R_\epsilon^{em} = R_m^{e\epsilon} = R_e^{\epsilon m} = 1$

$$R_1^{\epsilon\epsilon} = R_\epsilon^{me} = R_m^{\epsilon e} = R_e^{m\epsilon} = -1$$

2.4 Motivation

2.4.1 3-Manifold Invariants

Modular tensor categories naturally define invariants of links, something we've almost seen already in the graphical definitions of our category invariants (diagrams 2.7, 2.8, 2.11, etc). Given an oriented 3-manifold M, we also get invariants of M by writing M as a surgery on S^3 along a link L and computing the invariant of L.

The link invariant is defined roughly as follows. Consider a link L and a modular tensor category $\mathcal{B}$. Color the components of the link by simple objects of $\mathcal{B}$. Then the link L becomes a trace as in the diagrams 2.7 and can be identified with a complex number. The sum over all colorings of the components of L is an invariant of L. Refer to [51] for a very complete treatment.

The above is only the beginning of the invariants. Refer also to [51] for a discussion on topological quantum field theories.

2.4.2 Fault-Tolerant Quantum Computing

Quantum information is an exciting field which aims to implement practical quantum computation that offers significant advantages over classical computation. At the moment, one of the biggest hurdles in producing working quantum computers is that the encoded states of the qubits quickly decohere. After any appreciable number of operations, information is mostly lost to noise.

The quantum state of a qubit is said to be topologically protected if a large energy gap protects it from other states. The braiding of anyons is topologically protected, so quantum gates generated by anyon exchange are as well. Algebraically, a sequence of anyon exchanges is an element of a braid group. Extra algebraically, a sequence of anyon exchanges is a matrix given by a representation of a braid group. Skeletal modular tensor categories give anyon models from which these representations can be produced. For an example, refer to Section 5.1.2.

Since quantum gates generated by anyon exchange are topologically protected, there is interest in anyons whose braid representations are dense in the appropriate projective unitary groups. Unitarity of the representation is ensured by unitarity of the modular tensor category as in Definition 2.67. Since quantum computations are unitary operations on sets of qubits, density of the representation allows us to approximate any computation to arbitrary precision by performing a sequence of anyon exchanges. Section 5.1.2 mentions an example.

2.5 Computer Implementation

We devote this section to a brief discussion on how modular tensor categories may be represented on a computer. Thanks to the finiteness and semisimplicity conditions of fusion categories, it can be reduced to linear algebra quite transparently. To see the following discussion in practice, refer to Section 4.3 and specifically Constructions 4.3 and

4.4. For a complete Mathematica implementation, refer to Supplemental A: Mathematica Code or [1]. Sections A.1.1 and A.1.2 may be particularly instructive.

Recall that in a semisimple category (Definition 2.21), all objects may be written as direct sums of simple objects. In a finite category (Definition 2.20), there are finitely many isomorphism classes of these simple objects. Fix a fusion category $\mathcal{C}$ of rank n, and choose and ordered label set $\{a_1, \ldots, a_n\}$. Then every object $X \in \mathrm{Obj}(\mathcal{C})$ can be represented by a list of n integers $c_1, \ldots, c_n$ representing the coefficients in a direct sum decomposition $X \cong \bigoplus c_i a_i$. Note that this assigns the same list to any two isomorphic objects, so our computer implementation necessarily skeletalizes the category (Definition 2.4, Remark 2.5).

Recall that our fusion categories are also $\mathbb{C}$-linear and locally finite, so all morphism spaces are finite-dimensional vector spaces over $\mathbb{C}$. In fact, Lemma 2.19 provides a nice characterization. Let $X \cong \bigoplus b_i a_i$, $Y \cong \bigoplus c_i a_i$ be objects of $\mathcal{C}$ defined above. Then $\mathrm{Hom}(X, Y)$ is (isomorphically) a direct sum $\bigoplus \mathrm{Hom}(b_i a_i, c_i a_i)$. Since each $\mathrm{Hom}(a_i, a_i)$ is a one-dimensional $\mathbb{C}$ vector space, each $f_i \in \mathrm{Hom}(a_i, a_i)$ is a complex multiple of the identity. Then each $f_i \in \mathrm{Hom}(b_i a_i, c_i a_i)$ can be identified with a $c_i \times b_i$ complex matrix, and each $f \in \mathrm{Hom}(X, Y)$ is identified with a graded linear map $\bigoplus f_i$. In Mathematica, this is simply an n-element list of matrices.

For a slightly different implementation, recall from Remark 2.2 that monoidal categories are equivalent to 2-categories with a single object. In [46], a Mathematica package is presented to perform generalized linear algebraic operations on modular tensor categories within the 2-category 2Vec. In light of Remark 2.39, it may also be possible to consider modular tensor categories within 3Vec. We have not explored this.

2.6 Algebras in Categories

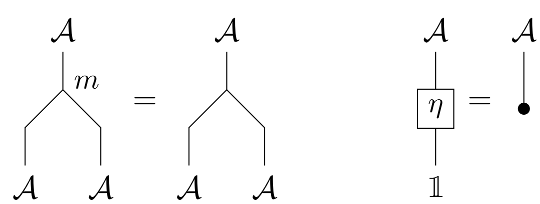
Definition 2.92. An *algebra* in a monoidal category $\mathcal{C}$ is a triple $(\mathcal{A}, m, \eta)$, where $\mathcal{A}$ is an object of $\mathcal{C}$ and $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\eta\colon \mathbb{1} \to \mathcal{A}$ are morphisms in $\mathcal{C}$ such that the following diagrams commute.

$$\begin{array}{ccccc} (\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{A} & & \xrightarrow{\alpha_{\mathcal{A},\mathcal{A},\mathcal{A}}} & & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{A}) \\ \downarrow{\scriptstyle m\otimes\mathrm{id}_{\mathcal{A}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}}\otimes m} \\ \mathcal{A}\otimes\mathcal{A} & & & & \mathcal{A}\otimes\mathcal{A} \\ & \searrow^{m} & & {}^{m}\swarrow & \\ & & \mathcal{A} & & \end{array}$$

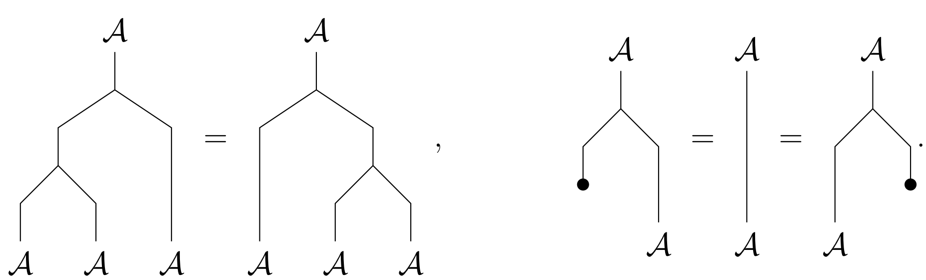
$$\begin{array}{ccc} \mathbb{1}\otimes\mathcal{A} & \xrightarrow{\lambda_{\mathcal{A}}} & \mathcal{A} \\ \downarrow{\scriptstyle \eta\otimes\mathrm{id}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}}} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow[m]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes\mathbb{1} & \xrightarrow{\rho_{\mathcal{A}}} & \mathcal{A} \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}}\otimes\eta} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}}} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow[m]{} & \mathcal{A} \end{array}$$

The morphism m is called the *multiplication* of $\mathcal{A}$, and the morphism η is called the *unit* of $\mathcal{A}$.

We may draw the morphisms m and η in the graphical calculus of Section 2.2. Since these are the only morphisms we use with their respective domains and codomains, we can typically leave off the m and η labels, as follows.

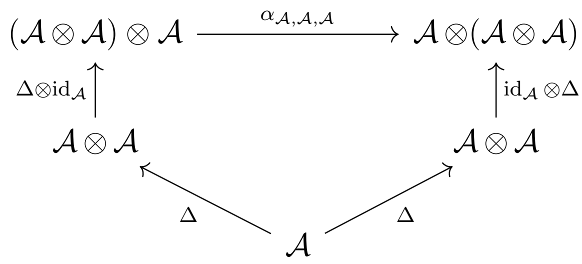


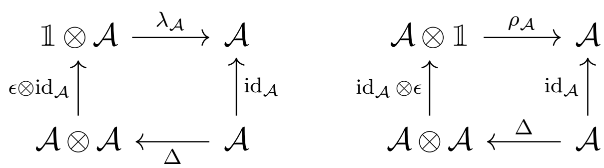
The conditions of Definition 2.92 are then



Reversing all arrows gives us the following definition.

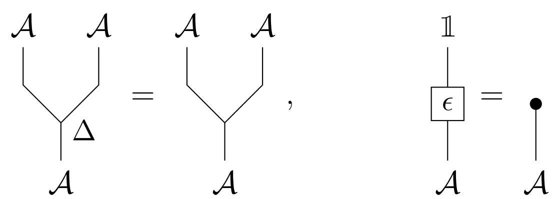
Definition 2.93. A *co-algebra* in a monoidal category $\mathcal{C}$ is a triple $(\mathcal{A}, \Delta, \epsilon)$, where $\mathcal{A}$ is an object of $\mathcal{C}$ and $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\eta\colon \mathcal{A} \to \mathbb{1}$ are morphisms in $\mathcal{C}$ such that the following diagrams commute.



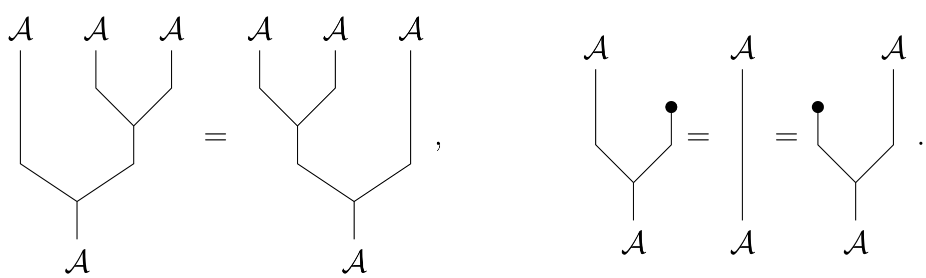


The morphism Δ is called the *co-multiplication* of $\mathcal{A}$, and the morphism ϵ is called the *co-unit* of $\mathcal{A}$.

In the graphical calculus, we draw



so that the conditions of Definition 2.93 are



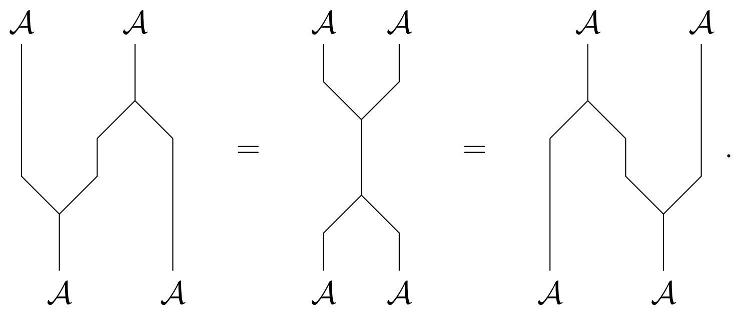
We will be using objects with both of these properties.

Definition 2.94. A *Frobenius algebra* in a monoidal category $\mathcal{C}$ is a quintuple

$(\mathcal{A}, m, \eta, \Delta, \epsilon)$ such that $(\mathcal{A}, m, \eta)$ is an algebra, $(\mathcal{A}, \Delta, \epsilon)$ is a co-algebra, and

$$\mathrm{id}_\mathcal{A} \otimes m \circ \Delta \otimes \mathrm{id}_\mathcal{A} = \Delta \circ m = \Delta \otimes \mathrm{id}_\mathcal{A} \circ \mathrm{id}_\mathcal{A} \otimes m.$$

In the graphical calculus, this property is



2.7 Structure of Modular Tensor Categories

This section provides a glimpse of the interesting structure of modular tensor categories, which is a field in its own right. For more such results, one might refer to, e.g. [39, 12, 13, 49, 44].

In Section 2.3.2, we likened modular tensor categories as a subset of fusion categories to finite abelian groups as a subset of all finite groups. Continuing with this analogy, there is a Witt group of modular tensor categories. Recall Definition 2.75 of the Deligne tensor product and Definition 2.78 of the Drinfeld center.

Definition 2.95. Two modular tensor categories $\mathcal{B}_1, \mathcal{B}_2$ are called *Witt equivalent* if there exist fusion categories $\mathcal{C}_1, \mathcal{C}_2$ such that there is a braided equivalence $\mathcal{B}_1 \boxtimes \mathcal{Z}(\mathcal{C}_1) \cong \mathcal{B}_2 \boxtimes \mathcal{Z}(\mathcal{C}_2)$.

Witt equivalence is an equivalence relation, which allows us to define the following.

Definition 2.96. Witt equivalence classes together with the operation $\boxtimes$ form an abelian group $\mathcal{W}$ called the *Witt group of modular tensor categories*. By Theorem 2.82, the Witt classes satisfy

- $[\mathcal{B}_1][\mathcal{B}_2] = [\mathcal{B}_1 \boxtimes \mathcal{B}_2]$,

- $\mathbb{1}_{\mathcal{W}} = [\mathbf{Vec}]$,
- $[\mathcal{B}]^{-1} = [\mathcal{B}^{\text{rev}}]$,

where $\mathcal{B}^{\text{rev}}$ is the reverse of $\mathcal{B}$ from Definition 2.81.

In fact, we may also consider the Witt group of non-degenerate braided fusion categories (Definition 2.40) and the Witt group of unitary modular tensor categories (Definition 2.67) by changing the choices of $\mathcal{B}_1, \mathcal{B}_2$ in Definition 2.95.

The Witt group $\mathcal{W}$ has interesting and surprising properties, a claim we support by listing a few. Let us begin with some definitions we have seen before.

Proposition 2.97. *If two pseudo-unitary (Definition 2.70) modular tensor categories are Witt equivalent, then they share a central charge (Definition 2.55).*

Mayhap more surprising is the following few properties.

Theorem 2.98.

1. *The group $\mathcal{W}$ is abelian with infinite rank.*
2. *The torsion subgroup of $\mathcal{W}$ is an infinite 2-group with exponent 32 (any element of $\mathcal{W}$ with finite order has order dividing 32).*
3. *The group $\mathcal{W}$ has infinitely-many elements with order 32.*

It is straightforward to see that double constructions like the quantum double of Construction 2.77 are Witt equivalent to **Vec**. Recall the other standard construction of modular tensor categories from quantum groups (Definition 2.83). The following is a conjecture that, in a sense, all modular tensor categories come from quantum groups.

Conjecture 2.99. *The Witt group is generated by quantum group categories.*

The reason we have demonstrated that the Witt group of modular tensor categories is interesting is that equivalent characterizations of Witt equivalence can be given by condensation, the primary subject of study in this thesis. See Section 3.1.3.2 for this relationship.

2.8 Tensor Functors

2.8.1 Definition

Definition 2.100 ([17])**.** A *tensor functor* (T, J) is an exact and faithful $\Bbbk$-linear monoidal functor T with $T(\mathbb{1}) = \mathbb{1}$, along with a functorial isomorphism $J\colon T(-) \otimes T(-) \to T(- \otimes -)$ such that the following diagram commutes for all objects X, Y, Z.

$$
\begin{array}{ccc}
\big(F(X) \otimes F(Y)\big) \otimes F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X) \otimes \big(F(Y) \otimes F(Z)\big) \\
\Big\downarrow {\scriptstyle J_{X,Y} \otimes \mathrm{id}_{F(Z)}} & & \Big\downarrow {\scriptstyle \mathrm{id}_{F(X)} \otimes J_{Y,Z}} \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
\Big\downarrow {\scriptstyle J_{X \otimes Y, Z}} & & \Big\downarrow {\scriptstyle J_{X, Y \otimes Z}} \\
F\big((X \otimes Y) \otimes Z\big) & \xrightarrow{F(\alpha_{X,Y,Z})} & F\big(X \otimes (Y \otimes Z)\big)
\end{array}
$$

A *tensor equivalence* is a tensor functor that is also an equivalence of categories.

The requirement that a tensor functor be exact and faithful is imposed by (and perpetuated by the disciples of) [17], but it is not universal.

2.8.2 Skeletalization

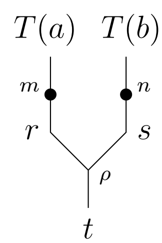
We see later that condensation, the primary subject of study in this thesis is a tensor functor. It will help with computations to have a skeletal description of a tensor functor in the spirit of Section 2.2.1. The description in this section is a slight modification of [11, Chapter 4]. See there for greater detail.

Consider a tensor functor $T\colon \mathcal{C} \to \mathcal{D}$ between skeletal categories. Anticipating the notation of [16] that we will attempt to explain in Section 4.1, take a label set $\{a_i\}$ for $\mathcal{C}$ and a label set $\{r_i\}$ for $\mathcal{D}$. For any two labels $a, b \in \{a_i\}$, the map J is an isomorphism

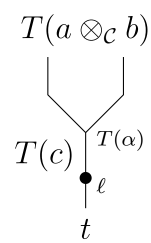
$$J\colon T(a) \otimes_{\mathcal{D}} T(b) \to T(a \otimes_{\mathcal{C}} b).$$

As we did with the associativity and the braiding isomorphisms in Section 2.2.1, we would like to draw trees to fix a basis for each side of the J isomorphism.

On the left, we have $T(a) \otimes_{\mathcal{D}} T(b)$ in $\mathcal{D}$. Write $T(a) = \bigoplus m_j r_j$. For each r_j (with $m_j > 0$), there is an obvious embedding $r_j \hookrightarrow T(a)$. For each integer $1 < m < m_j$, let $I^a_{r_j}(m)$ be the embedding of r_j into the m-th copy of r_j in $T(a)$. Do the same for $T(b) = \bigoplus n_j r_j$. Then for any $r, s, t \in \{r_j\}$ and choice of tensor product morphism $\rho\colon t \to r \otimes_{\mathcal{D}} s$ (see Section 2.2.1), we can draw a picture in $\mathcal{D}$ as follows. In keeping with [11], we use a dot to denote the morphism $I^a_r(m)$, which is the step at which the tensor functor is applied.

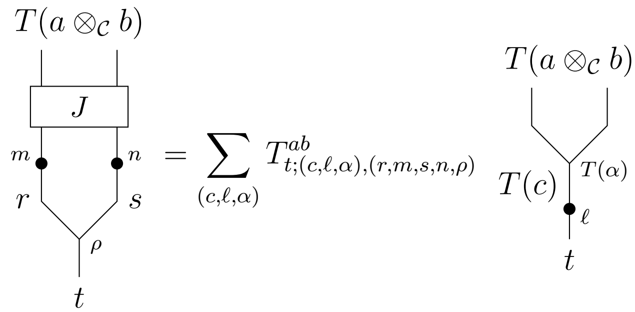


On the right, we have $T(a \otimes_{\mathcal{C}} b)$. As in Section 2.2.1, we may choose tensor product morphisms $\alpha\colon c \to a \otimes_{\mathcal{C}} b$ for each $c \in \{a_i\}$ with $N^{ab}_c > 0$. For each such c, we write $T(c) = \bigoplus \ell_j r_j$ as we did with $T(a)$ and $T(b)$ above. Then for each $t \in \{r_j\}$, this gives us a picture in $\mathcal{D}$ as follows.

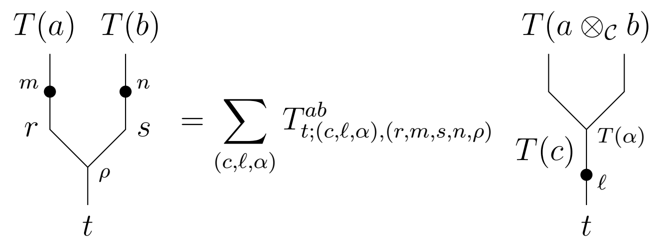


Now for any three labels a, b of $\mathcal{C}$ and t of $\mathcal{D}$, ranging over all admissible choices of r, s, m, n, ρ on the one hand and c, ℓ, α on the other gives bases for the vector spaces $\mathrm{Hom}(t, T(a) \otimes_{\mathcal{D}} T(b)) = \mathrm{Hom}(t, T(a \otimes_{\mathcal{C}} b))$. Composing the former with the isomorphism J from Definition 2.100 gives two bases for $\mathrm{Hom}(t, T(a \otimes_{\mathcal{C}} b))$, and we can present J as a list of change-of-basis matrices as we did with the associator and braiding in Section 2.2.1.

The picture here might be presented as follows.



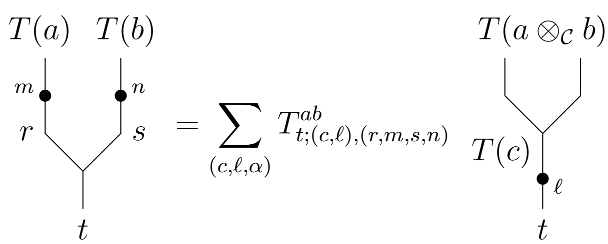
As with the associator in the F-symbols, we may present this in shorthand as the following.



(2.18)

Definition 2.101. In analogy with the F-symbols and R-symbols, the coefficients of Equation 2.18 will be called T-*symbols*.

When working with a multiplicity-free category, we need not specify the morphisms denoted by Greek letters, and these T-symbols reduce to the following.



Notably for us in Chapter 4, these T-symbols are coefficients that convert from a basis in which the tensor product is applied after T to a basis in which T is applied after taking a tensor product.

Chapter 3

Condensation of Algebras

Anyon condensation is really just the process of taking a category of modules over an algebra to produce a sort of categorical quotient by the algebra. This process is expressed by a tensor functor from the parent category to the new *condensed* module category. The interest for those who call this functor *condensation* is in understanding phase transitions in topologically ordered systems, which can be modeled by skeletal unitary modular tensor categories.

This chapter consists of four main sections. In Section 3.1, we provide all of the relevant definitions of condensable algebras and condensation in the setting of modular tensor categories. Of interest may be the side-by-side presentation of two well-known equivalent definitions of condensation with unified notation. Section 3.2 gives this equivalence explicitly and provides some discussion. Section 3.3 provides some useful examples of condensation that continue to appear throughout this thesis. Finally, Section 3.4 presents results determining the data of the condensed category in terms of the data of the parent category.

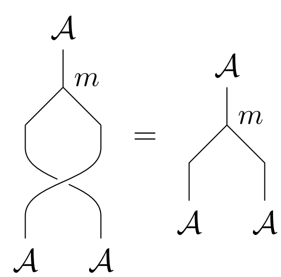
3.1 Premliminaries

3.1.1 Condensable Algebras

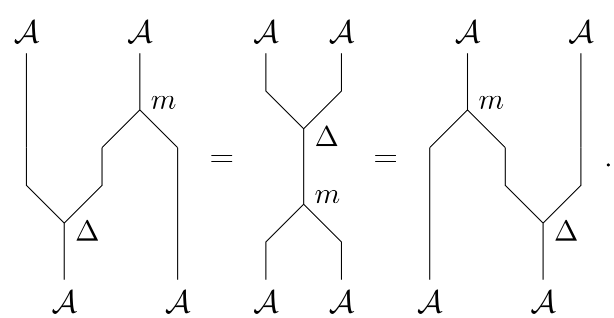
Definition 3.1. [11] Given a modular tensor category $\mathcal{B}$, an algebra $(\mathcal{A}, m, \eta)$ (of Definition 2.92) is called *condensable* if it is

1. *Commutative*: $m \circ c_{\mathcal{A},\mathcal{A}} = m$, where $c_{\mathcal{A},\mathcal{A}}$ is the braiding,
2. *Connected*: $\mathrm{Hom}(1, \mathcal{A}) \cong \mathbb{C}$,
3. *Separable*: m admits a splitting $\Delta\colon\ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ that is a morphism of $(\mathcal{A}, \mathcal{A})$-bimodules.

In the graphical calculus of Section 2.2, we may draw conditions 1 and 3 respectively as

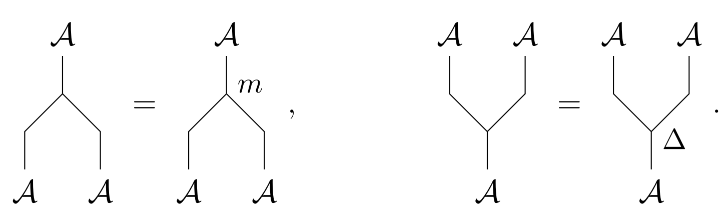


and



(3.1)

In the future, we may not explicitly label m and Δ, and it should be understood that



Remark 3.2. Note the similarity to the Frobenius algebra of Definition 2.94. Algebras related to condensable algebras go by several names in the literature. Commutative separable algebras are called *étale* in [13]. A condensable algebra is called *Lagrangian* if $\mathrm{FPdim}(\mathcal{A})^2 = \mathrm{FPdim}(\mathcal{B})$ (see Definition 2.69 for FPdim). In [35], it is demonstrated that condensable algebras come with a connected commutative symmetric normalized-special Frobenius algebra structure as in [22]. Renormalizing gives the strongly separable Frobenius algebra of [38].

Proposition 3.3. *An algebra* $\mathcal{A} \cong \bigoplus a_i$ *in a modular category is commutative if and only if* $\theta_{a_i} = 1$ *for all* i*.*

Proof. [22, Proposition 2.25] □

3.1.2 Condensation

In all that follows, we typically consider a modular tensor category $\mathcal{B}$ and a condensable algebra $(\mathcal{A}, m, \eta)$. Since we work with both the parent category $\mathcal{B}$ and the new condensed category, it is critical to keep track of which category our morphisms belong to. Where the symbols Hom, $\circ$, $\otimes$, and c are not labelled, they refer to $\mathrm{Hom}_{\mathcal{B}}$, $\circ_{\mathcal{B}}$, $\otimes_{\mathcal{B}}$, and $c_{\mathcal{B}}$, the hom set, composition, tensor product, and braiding of $\mathcal{B}$.

We now provide two equivalent descriptions of the condensed category. The first is $\mathrm{Rep}\,\mathcal{A}$, the category of $\mathcal{A}$-modules over $\mathcal{B}$, developed by [5, 45, 30]. We use many of the conventions of [30]. We present this definition first though we will focus on it more in Chapter 4.

Definition 3.4. Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $(\mathcal{A}, m, \eta)$. The category $\mathrm{Rep}\,\mathcal{A}$ is defined with

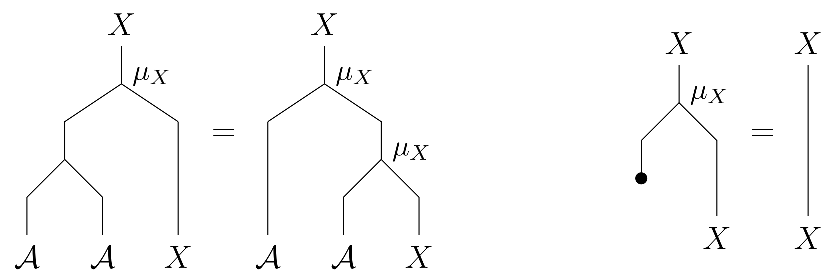
- objects (X, μ_X), where $X \in \mathrm{Obj}(\mathcal{B})$ and $\mu_X \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{A} \otimes X, X)$ such that
    1. $\mu_X \circ (m \otimes \mathrm{id}_X) = \mu_X \circ (\mathrm{id}_{\mathcal{A}} \otimes \mu_X)$,

2. $\mu_X \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X$,

- intertwiner morphisms

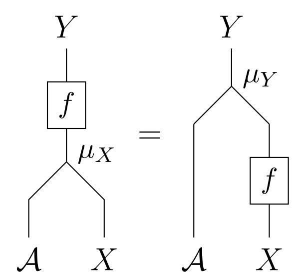
$$\mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}\Big((X,\mu_X),(Y,\mu_Y)\Big) = \Big\{ f \in \mathrm{Hom}_{\mathcal{B}}(X,Y) \;\Big|\; f \circ \mu_X = \mu_Y \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \Big\}.$$

In the graphical calculus of $\mathcal{B}$, an object of $\mathrm{Rep}\,\mathcal{A}$ is a pair (X,μ_X) such that



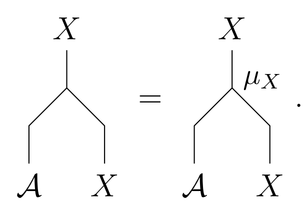
(3.2)

and a morphism $f\colon (X,\mu_X) \to (Y,\mu_Y)$ is a morphism of $\mathcal{B}$ such that



(3.3)

When it is unlikely to cause confusion, we may write $\mathrm{Hom}_{\mathcal{A}}((X,\mu_X),(Y,\mu_Y))$ or even $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ instead of $\mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}((X,\mu_X),(Y,\mu_Y))$. Morphisms μ_X will not be labeled, and it should be understood that



Definition 3.5. Consider the category $\mathrm{Rep}\,\mathcal{A}$ as defined in Definition 3.4. The subcategory consisting of objects (X,μ_X) with $\mu_X \circ c_{X,\mathcal{A}} \circ c_{\mathcal{A},X} = \mu_X$ is called $\mathrm{Rep}^0\,\mathcal{A}$. These modules are referred to as *dyslectic* by [45] and *local* by [30]. They are called *deconfined* objects in the physics literature.

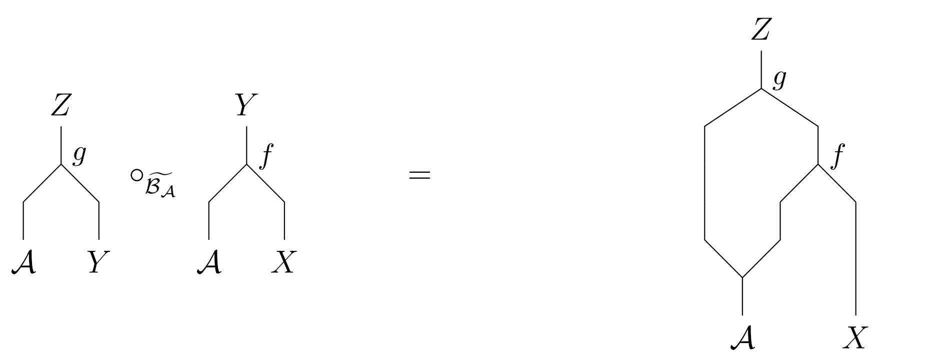
Remark 3.6. [30, Theorem 4.5] shows that when the algebra $\mathcal{A}$ is condensable, the category $\operatorname{Rep}^0 \mathcal{A}$ is a modular tensor category.

The second definition of condensation, which we will be relying on for much of this chapter, is due to [38]. Section 3.2 discusses the equivalence of these two definitions.

Definition 3.7. Consider a (strict) modular tensor category $\mathcal{B}$ with condensable algebra $(\mathcal{A}, m, \eta)$. Define a co-monoid structure so that $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ is a strongly separable Frobenius algebra (see Remark 3.2). Then consider a new tensor category $\widetilde{\mathcal{B}_{\mathcal{A}}}$ defined as follows.

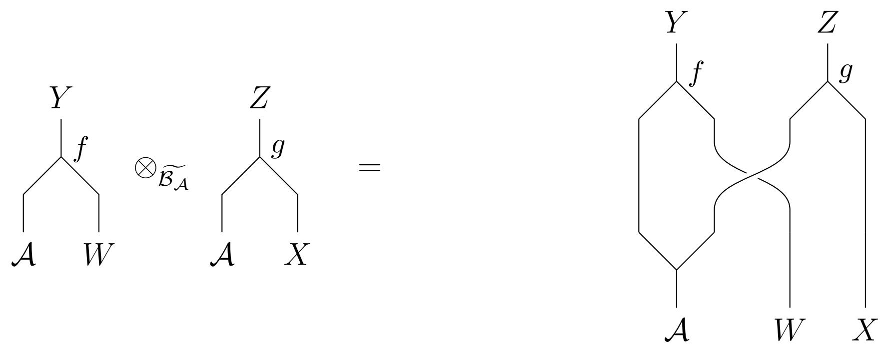
- $\operatorname{Obj}\left(\widetilde{\mathcal{B}_{\mathcal{A}}}\right) = \operatorname{Obj}(\mathcal{B})$
- $X \otimes_{\widetilde{\mathcal{B}_{\mathcal{A}}}} Y = X \otimes Y$
- $\operatorname{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X, Y) = \operatorname{Hom}(\mathcal{A} \otimes X, Y)$
- For $f \in \operatorname{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X, Y), g \in \operatorname{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(Y, Z)$, the composition

$$g \circ_{\widetilde{\mathcal{B}_{\mathcal{A}}}} f \qquad = \qquad g \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\Delta \otimes \mathrm{id}_X)$$



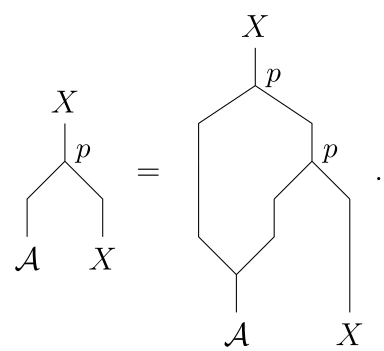
- If $f \in \operatorname{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(W, Y), g \in \operatorname{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X, Z)$, the tensor product

$$f \otimes_{\widetilde{\mathcal{B}_{\mathcal{A}}}} g \qquad = \qquad (f \otimes g) \circ (\mathrm{id}_{\mathcal{A}} \otimes c_{\mathcal{A},W} \otimes \mathrm{id}_X) \circ (\Delta \otimes \mathrm{id}_W \otimes \mathrm{id}_X)$$



The condensed category $\mathcal{B}_{\mathcal{A}}$ is the idempotent completion of $\widetilde{\mathcal{B}_{\mathcal{A}}}$, presented explicitly below.

- $\text{Obj}\,(\mathcal{B}_{\mathcal{A}}) = \left\{(X,p) \,\middle|\, X \in \text{Obj}(\mathcal{B}), p = p^2 \in \text{End}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X)\right\}$. In the graphical calculus of the parent category, this is pairs (X,p) such that



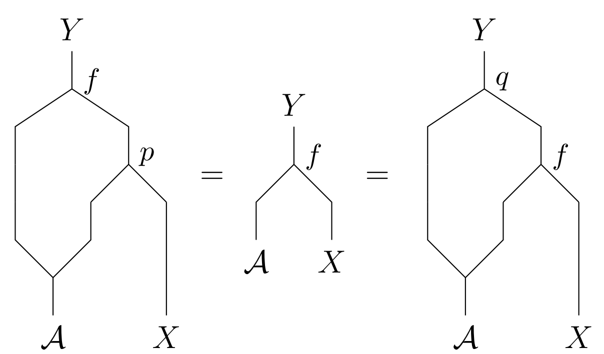
(3.4)

  When $p = (\text{id}_X)_{\widetilde{\mathcal{B}_{\mathcal{A}}}} = (\epsilon \otimes \text{id}_X)_{\mathcal{B}}$, we may write X instead of (X, id_X).

- intertwiner morphisms

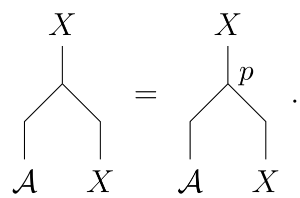
$$\text{Hom}_{\mathcal{B}_{\mathcal{A}}}\Big((X,p),(Y,q)\Big) = \left\{f \in \text{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X,Y) \,\middle|\, f \circ_{\widetilde{\mathcal{B}_{\mathcal{A}}}} p = f = q \circ_{\widetilde{\mathcal{B}_{\mathcal{A}}}} f\right\}$$

  These are morphisms from the parent category with



(3.5)

Again where there is little chance of confusion, we will write $\mathrm{Hom}_{\mathcal{A}}$ instead of $\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}$, and we will not explicitly label p morphisms so that



Definition 3.8. Following the convention for $\mathrm{Rep}\,\mathcal{A}$, we write $\mathcal{B}^0_{\mathcal{A}}$ for the deconfined part of $\mathcal{B}_{\mathcal{A}}$ consisting of objects with consistent θ morphisms.

Remark 3.9. Notice that Definition 3.4 uses only the algebra structure on $\mathcal{A}$ while Definition 3.7 also uses the co-algebra structure. Indeed the co-algebra structure is not utilized at all by [30]. Section 3.2 explores the equivalence of these two definitions and why the algebra structure by itself is not missing information in, e.g. Lemma 3.14, Remark 3.19. Section 3.2.1 is devoted to further discussion of the relationship between the two definitions of condensation.

In both definitions of condensation (since they are equivalent), the resulting category is a fusion category. If $\mathcal{A}$ is condensable, the subcategory $\mathrm{Rep}^0\,\mathcal{A}$ or $\mathcal{B}^0_{\mathcal{A}}$ is modular. Often, we are interested only in this modular part of the result, and we discard the other objects.

Also in both definitions of condensation, we have described a construction of a new category. Condensation is then best described as a tensor functor $T\colon \mathcal{B} \to \mathrm{Rep}\,\mathcal{A}$ or $T\colon \mathcal{B} \to \mathcal{B}_{\mathcal{A}}$.

Remark 3.10. We will use this functor more in Chapter 4. In Section 3.4, we will write

$$T(X) \cong \bigoplus_j n_i^j Y_j$$

as a way to refer to images of objects under condensation generally in terms of a label set $\{Y_j\}$ for the condensed category.

Remark 3.11. It is known ([11]) that the forgetful functor is right adjoint to the condensation functor. This is an application of tensor-hom adjunction.

3.1.3 Motivations

3.1.3.1 Topological Phases of Matter and Gauging

By cooling them near absolute zero, it is possible to have many bosons occupy the same lowest quantum state. This forms a phase of matter called a Bose-Einstein condensate. Bose-Einstein condensation can explain some interesting phenomena, such as superconductivity.

Anyon condensation takes its name from Bose-Einstein condensation. In a topological phase of matter, anyons can form a condensate in the same way. This is the condensable algebra $\mathcal{A}$. The work of this thesis is to determine the new anyon system upon formation of this condensate. For more on anyon condensation, refer to [8].

In the physics literature, condensation is also known as an inverse to a process called *gauging*. Gauging is the sequential process of *defectification* followed by *orbifolding*. The reverse process is called *coring* and consists of condensation followed by deconfinement. The schematic for the relationship is as follows.

$$\begin{array}{ccc} \mathcal{B}_0 & \underset{\text{Deconfinement}}{\overset{\text{Defectification}}{\rightleftarrows}} & \mathcal{B}_{\mathcal{A}} = \mathcal{B}_0 \oplus \mathcal{B}_1 \\ & & \text{Orbifolding} \downarrow\uparrow \text{Condensation} \\ & & \mathcal{B} \end{array}$$

Here $\mathcal{B}_0$ is the category $\operatorname{Rep}^0 \mathcal{A}$ from Definition 3.5 or $\mathcal{B}_{\mathcal{A}}^0$ from Definition 3.8. The terms orbifolding and condensation are also known as *equivariantization* and *deequivariantization*, respectively. For this reason, condensation and its adjoint functor are sometimes called D and E. For more on gauging, refer to [10, 3].

3.1.3.2 Witt Equivalence

Interestingly for us, Witt equivalence (Definition 2.95) can be expressed in terms of condensation. In this subsubsection, we do not choose between Definition 3.4 and Definition 3.7, but we do use the notation $\mathcal{B}_\mathcal{A}$ and $\mathcal{B}^0_\mathcal{A}$ for the convenience of specifying the category $\mathcal{B}$ and the algebra $\mathcal{A}$. These results are due to [13].

Proposition 3.12. *Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $\mathcal{A}$. Then $[\mathcal{B}^0_\mathcal{A}] = [\mathcal{B}_\mathcal{A}]$ in $\mathcal{W}$, the Witt group of modular tensor categories.*

Since gauging is the reverse of process of condensation, it is not surprising that Witt class is also preserved by gauging.

It turns out that invariance of Witt class under condensation is quite significant. Witt equivalence can be presented equivalently in terms of condensation. Recall Definition 2.81 of the reverse of a braided fusion category.

Theorem 3.13. *Let $\mathcal{B}_1, \mathcal{B}_2$ be modular tensor categories. The following are equivalent. Equivalences are all braided.*

1. *$\mathcal{B}_1$, $\mathcal{B}_2$ are Witt equivalent.*
2. *There exists a fusion category $\mathcal{C}$ such that $\mathcal{B}_1 \boxtimes \mathcal{B}_2^{\text{rev}} \cong \mathcal{Z}(\mathcal{C})$.*
3. *There is a condensable algebra $\mathcal{A}$ in $\mathcal{B}_1 \boxtimes \mathcal{B}_2^{\text{rev}}$ with $(\mathcal{B}_1 \boxtimes \mathcal{B}_2^{\text{rev}})^0_\mathcal{A} \cong \mathbf{Vec}$ (so that $\mathcal{A}$ is a Lagrangian algebra defined in Remark 3.2).*
4. *There exist a modular tensor category $\mathcal{B}$ and condensable algebras $\mathcal{A}_1, \mathcal{A}_2$ in $\mathcal{B}$ such that $\mathcal{B}^0_{\mathcal{A}_1} \cong \mathcal{B}_1$ and $\mathcal{B}^0_{\mathcal{A}_2} \cong \mathcal{B}_2$.*
5. *There exist condensable algebras $\mathcal{A}_1$ in $\mathcal{B}_1$ and $\mathcal{A}_2$ in $\mathcal{B}_2$ such that ${\mathcal{B}_1}^0_{\mathcal{A}_1} \cong {\mathcal{B}_2}^0_{\mathcal{A}_2}$.*

Clearly, a better understanding of condensation should be helpful in understanding the Witt group and may provide insight toward answering questions like Conjecture 2.99.

3.2 Equivalence of Definitions

The two definitions of condensation given in Section 3.1.2 yield equivalent categories [38], but both descriptions are useful. The category $\mathcal{B}_{\mathcal{A}}$ is easier to compute examples of, but the category $\operatorname{Rep}\mathcal{A}$ has provided a more clear theoretical framework. For some informal thoughts on why, refer to the discussion of Section 3.2.1. To understand the translation between the two presentations, we have produced an explicit functor $F\colon \mathcal{B}_{\mathcal{A}} \to \operatorname{Rep}\mathcal{A}$ that fills in the equivalence proof from [38]. In this section, we construct this functor and briefly discuss the relationship between the categories on either side.

Fix a modular tensor category $\mathcal{B}$ and a condensable algebra $\mathcal{A}$ with a strongly separable Frobenius algebra structure $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ as in [38]. Consider the categories $\mathcal{B}_{\mathcal{A}}$ and $\operatorname{Rep}\mathcal{A}$ as defined in section 3.1.2. Denote by $\overline{\mathcal{C}}$ the idempotent completion of a category $\mathcal{C}$.

Before proceeding, we establish a few facts we will need. The first of these illuminates how facts about an algebra structure carry to a compatible co-algebra structure so that $\operatorname{Rep}\mathcal{A}$ does not carry less information than $\mathcal{B}_{\mathcal{A}}$. The next three are useful properties of idempotent completions.

Throughout this section, we will often consider two objects X, X' and reserve Y for a retract of a split idempotent. In order to keep expressions clean, we will often omit parentheses in favor of an order of operations which performs tensor product before composition. When in doubt, considering the domains of the morphisms in question should clarify the intention. As long strings of tensor products and compositions become particularly unwieldy, we will begin working with the graphical calculus of Section 2.2. If desired, it should be straightforward to turn the pictures back into algebraic equations and verify their validity.

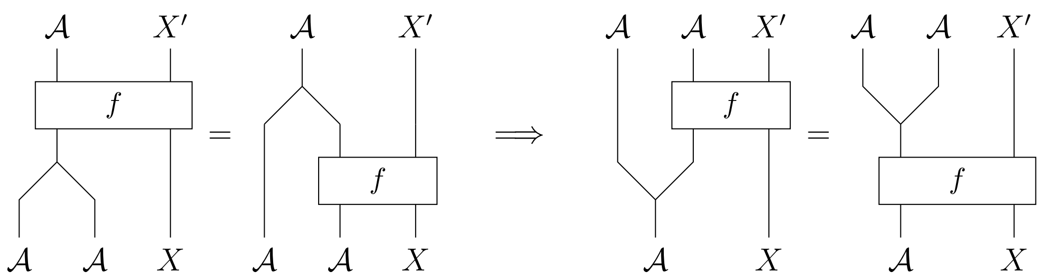
Lemma 3.14. *Consider a modular tensor category $\mathcal{B}$ with condensable algebra $(\mathcal{A}, m, \eta)$. As in Remark 3.2, separability gives a co-algebra structure so that $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ is a* *strongly separable Frobenius algebra of [38]. Suppose* $f \in \mathrm{Hom}(A \otimes X, \mathcal{A} \otimes X')$ *such that*

$$f \circ (m \otimes \mathrm{id}_X) = (m \otimes \mathrm{id}_{X'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f).$$

Then

$$(\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\Delta \otimes \mathrm{id}_X) = (\Delta \otimes \mathrm{id}_{X'}) \circ f.$$

Loosely speaking, if f *can slide past the multiplication, it can also slide past the co-multiplication. Graphically, we have the following implication.*

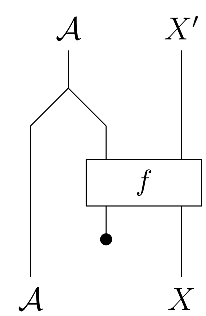


Proof. Consider f as in the statement of the Lemma, and notice that

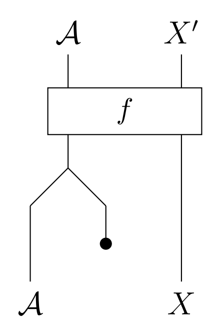
$$f = (m \otimes \mathrm{id}_{X'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\mathrm{id}_{\mathcal{A}} \otimes \eta \otimes \mathrm{id}_X).$$

That is,

$$(m \otimes \mathrm{id}_{X'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\mathrm{id}_{\mathcal{A}} \otimes \eta \otimes \mathrm{id}_X) =$$

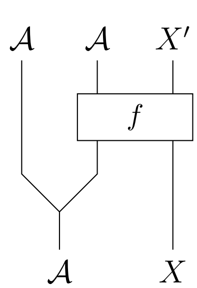


$$=$$



$$= f, \tag{3.6}$$

where we have used the assumption on f and a property of the algebra $\mathcal{A}$ from Definition 2.92. Now



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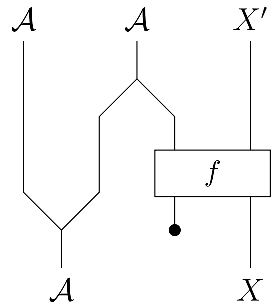
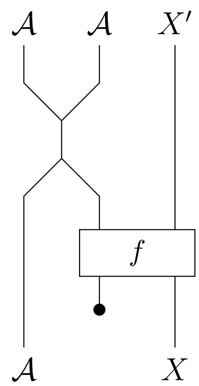


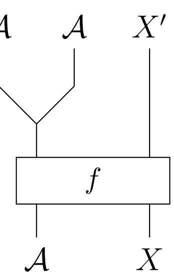
(by 3.6)

$=$



(by 3.1)

$=$



, (by 3.6)

as desired. $\square$

Lemma 3.15. *In any category, given objects X, Y and morphisms $r\colon X \to Y$, $s\colon Y \to X$ with $s \circ r = p$ and $r \circ s = \mathrm{id}_Y$, then*

$$r = \mathrm{id}_Y \circ r = r \circ s \circ r = r \circ p$$

and

$$s = s \circ \mathrm{id}_Y = s \circ r \circ s = p \circ s.$$

Lemma 3.16. *Any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor*

$$\overline{F}\colon \overline{\mathcal{C}} \to \overline{\mathcal{D}}$$
$$(X,p) \mapsto (F(X), F(p))$$
$$f \mapsto F(f).$$

Proof. This is very straightforward.

(i) The pair $(F(X), F(p))$ is an object in $\overline{\mathcal{D}}$ since $F(X) \in \mathrm{Obj}(\mathcal{D})$ and $F(p) \circ F(p) = F(p \circ p) = F(p)$.

(ii) For any $f\colon (X,p) \to (Y,q)$, we have $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ and

$$F(f) \circ F(p) = F(f \circ p) = F(f) = F(q \circ f) = F(q) \circ F(f).$$

(iii) Since $\mathrm{id}_{(X,p)} = p$, we have

$$\overline{F}(\mathrm{id}_{(X,p)}) = F(p) = \mathrm{id}_{(F(X),F(p)}.$$

(iv) For morphisms f, g,

$$\overline{F}(g \circ f) = F(g \circ f) = F(g) \circ F(f).$$

□

Lemma 3.17. *Given any idempotent complete category $\mathcal{C}$, there is a category equivalence $F\colon \overline{\mathcal{C}} \to \mathcal{C}$ from the idempotent completion of $\mathcal{C}$ to $\mathcal{C}$.*

Proof. We first define F. Consider an object $(X,p) \in \mathrm{Obj}(\overline{\mathcal{C}})$. Since $\mathcal{C}$ is idempotent complete, there exists an object $Y \in \mathrm{Obj}(\mathcal{C})$ and morphisms r, s such that $s \circ r = p$ and $r \circ s = \mathrm{id}_Y$. For two objects $(X,p), (X',p')$ with morphism $f\colon (X,p) \to (X',p')$, consider corresponding Y, r, s and Y', r', s'. We define

$$\begin{aligned} F\colon (X,p) &\mapsto Y \\ f &\mapsto r' \circ f \circ s. \end{aligned}$$

This choice is well-defined since Y, r, s are unique up to isomorphism. Given any (X, p), Y, r, s and Y', r', s' with $s \circ r = s' \circ r' = p$, $r \circ s = \mathrm{id}_Y$, and $r' \circ s' = \mathrm{id}_{Y'}$, we have a morphism $r' \circ s\colon Y \to Y'$. The morphism $r \circ s'\colon Y' \to Y$ is an inverse since

$$r \circ s' \circ r' \circ s \;=\; r \circ p \circ s \;=\; \mathrm{id}_Y$$

and

$$r' \circ s \circ r \circ s' \;=\; r' \circ p \circ s' \;=\; \mathrm{id}_{Y'},$$

so $Y \cong Y'$.

To show that F is a category equivalence, we may show that it is both fully faithful and essentially surjective.

(i) Given two objects $(X, p), (X', p')$ with images Y, Y', we need to verify that the function

$$\begin{aligned} \{f \in \mathrm{Hom}_{\mathcal{C}}(X, X') \mid f \circ p = f = p' \circ f\} &\to \mathrm{Hom}_{\mathcal{C}}(Y, Y') \\ f &\mapsto r' \circ f \circ s \end{aligned}$$

on hom sets is bijective. The function

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Y, Y') &\to \{f \in \mathrm{Hom}_{\mathcal{C}}(X, X') \mid f \circ p = f = p' \circ f\} \\ f &\mapsto s' \circ f \circ r \end{aligned}$$

is clearly well-defined and is an inverse since

$$s' \circ r' \circ f \circ s \circ r = p' \circ f \circ p = f$$

and

$$r' \circ s' \circ f \circ r \circ s = \mathrm{id}_{Y'} \circ f \circ \mathrm{id}_Y = f.$$

(ii) Any object $Y \in \mathrm{Obj}(\mathcal{C})$ is the image of the object $(Y, \mathrm{id}_Y) \in \mathrm{Obj}(\overline{\mathcal{C}})$, so F is essentially surjective.

□

As the lemmas suggest, the picture for this equivalence is as follows.

$$\mathcal{B}_{\mathcal{A}} \overset{F}{\underset{G}{\rightleftarrows}} \overline{\mathrm{Rep}\,\mathcal{A}} \rightleftarrows \mathrm{Rep}\,\mathcal{A}$$

It will be helpful to deal with some of these arrows separately. We will first define functors $F_1\colon\ \mathcal{B}_{\mathcal{A}} \to \overline{\mathrm{Rep}\,\mathcal{A}}$, $F_2\colon \overline{\mathrm{Rep}\,\mathcal{A}} \to \mathrm{Rep}\,\mathcal{A}$ and then let $F = F_2 \circ F_1$. We will then define G and show that it is adjoint to F.

Begin by considering a modification of the tensor functor in [30], $F_0\colon \widetilde{\mathcal{B}_{\mathcal{A}}} \to \mathrm{Rep}\,\mathcal{A}$, defined by $F_0(X) = (\mathcal{A} \otimes X, m \otimes \mathrm{id}_X)$ and $F_0(f) = (\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\Delta \otimes \mathrm{id}_X)$. By Lemma 3.16, F_0 induces a functor

$$\begin{aligned} F_1 = \overline{F_0}\colon\ \mathcal{B}_{\mathcal{A}} &\to \overline{\mathrm{Rep}\,\mathcal{A}} \\ (X, p) &\mapsto ((\mathcal{A} \otimes X, m \otimes \mathrm{id}_X), \mathrm{id}_{\mathcal{A}} \otimes p \circ \Delta \otimes \mathrm{id}_X) \\ f &\mapsto \mathrm{id}_{\mathcal{A}} \otimes f \circ \Delta \otimes \mathrm{id}_X, \end{aligned}$$

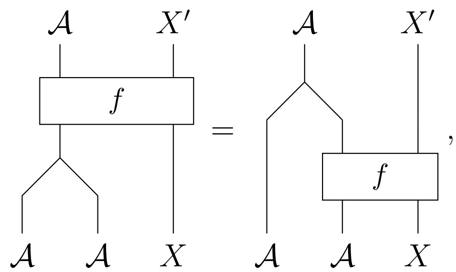
which we can show is fully faithful and not quite essentially surjective.

(i) Given two objects (X, p) and (X', p') with images $((\mathcal{A} \otimes X, m \otimes \mathrm{id}_X), \mathrm{id}_{\mathcal{A}} \otimes p \circ \Delta \otimes$

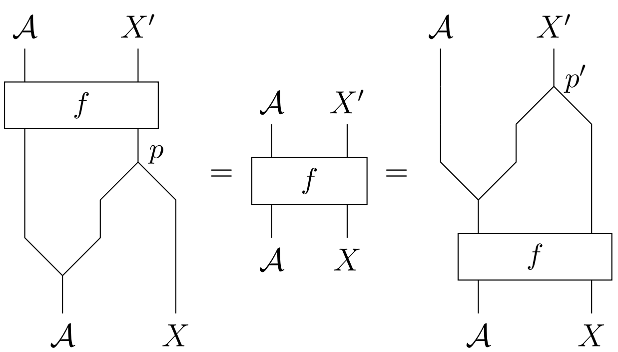
id_X) and $((\mathcal{A} \otimes X', m \otimes \mathrm{id}_{X'}), \mathrm{id}_{\mathcal{A}} \otimes p' \circ \Delta \otimes \mathrm{id}_{X'})$, we need to verify that the function

$$\begin{aligned} \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}\Big((X,p),(X',p')\Big) \to \mathrm{Hom}_{\overline{\mathrm{Rep}\,\mathcal{A}}}\Big(&((\mathcal{A} \otimes X, m \otimes \mathrm{id}_X), \mathrm{id}_{\mathcal{A}} \otimes p \circ \Delta \otimes \mathrm{id}_X), \\ &((\mathcal{A} \otimes X', m \otimes \mathrm{id}_{X'}), \mathrm{id}_{\mathcal{A}} \otimes p' \circ \Delta \otimes \mathrm{id}_{X'})\Big) \\ f \mapsto \mathrm{id}_{\mathcal{A}} \otimes f \circ \Delta \otimes \mathrm{id}_X \end{aligned} \tag{3.7}$$

on hom sets is bijective. Recall that a morphism f on the left hand side satisfies equation 3.5, while a morphism f on the right hand side satisfies both equation 3.3 and the idempotent completion condition. So a morphism f on the right hand side of 3.7 is a morphism $f\colon A \otimes X \to A \otimes X'$ such that



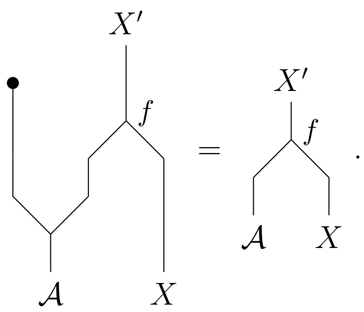
(3.8)



(3.9)

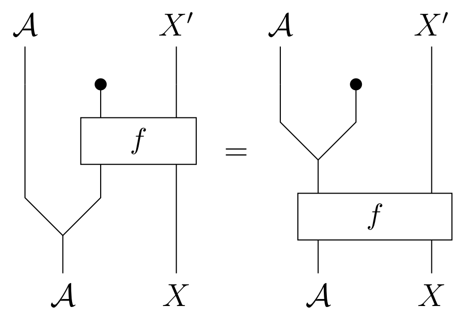
To show the desired function is bijective, we propose an inverse function $f \mapsto \varepsilon \otimes \mathrm{id}_{X'} \circ f$ as in [38]. We can verify that this is an inverse by composing

$$\varepsilon \otimes \mathrm{id}_{X'} \circ (\mathrm{id}_{\mathcal{A}} \otimes f \circ \Delta \otimes \mathrm{id}_X) = f \circ \varepsilon \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_X \circ \Delta \otimes \mathrm{id}_X = f$$



For the other composition, Lemma 3.14 gives us

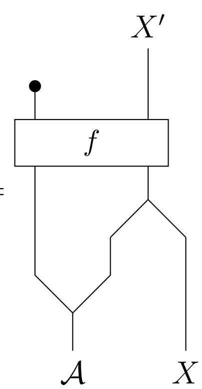
$$\mathrm{id}_{\mathcal{A}} \otimes (\varepsilon \otimes \mathrm{id}_{X'} \circ f) \circ \Delta \otimes \mathrm{id}_X =$$



$$= f. \qquad (3.10)$$

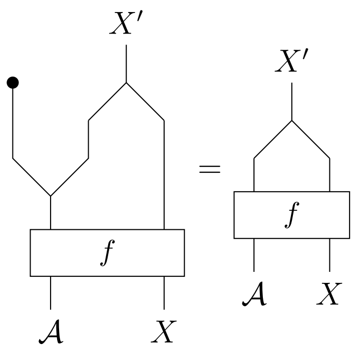
Finally, the image of the proposed inverse function lives in $\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}((X,p),(X',p'))$ because

$$(\varepsilon \otimes \mathrm{id}_{X'} \circ f) \circ_{\mathcal{B}_{\mathcal{A}}} p =$$

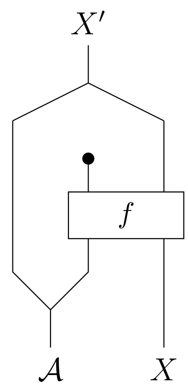


$$= \varepsilon \otimes \mathrm{id}_{X'} \circ f \qquad \text{(by 3.9)}$$

$$=$$



(by 3.9)

$$= \quad$$  $$= p' \circ_{\mathcal{B}_\mathcal{A}} (\varepsilon \otimes \mathrm{id}_{X'} \circ f). \qquad \text{(by 3.10)}$$

(ii) We see here that F_1 is not quite essentially surjective (as far as we know). Let $((X,\mu),p)$ be any object of $\overline{\operatorname{Rep}\mathcal{A}}$. Consider the object (X,μ) in $\mathcal{B}_\mathcal{A}$. Then

$$F_1\Big((X,\mu)\Big) = ((\mathcal{A}\otimes X, m\otimes \mathrm{id}_X), \mathrm{id}_\mathcal{A}\otimes\mu\circ\Delta\otimes\mathrm{id}_X).$$

Now, consider the morphisms $\mu\colon \mathcal{A}\otimes X\to X$ and $\mathrm{id}_\mathcal{A}\otimes\mu\circ\Delta\otimes\mathrm{id}_X\circ\eta\otimes\mathrm{id}_X\colon X\to \mathcal{A}\otimes X$, which give us compositions

$$\mu\circ(\mathrm{id}_\mathcal{A}\otimes\mu\circ\Delta\otimes\mathrm{id}_X\circ\eta\otimes\mathrm{id}_X) = \mu\circ\eta\otimes\mathrm{id}_X = \mathrm{id}_X$$

and

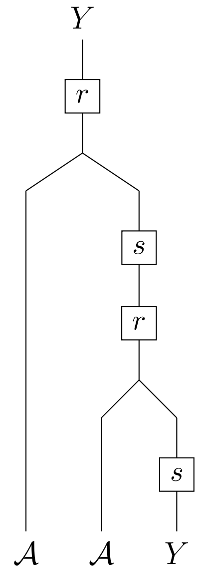
$$\begin{aligned}(\mathrm{id}_\mathcal{A}\otimes\mu\circ\Delta\otimes\mathrm{id}_X\circ\eta\otimes\mathrm{id}_X)\circ\mu &= \mathrm{id}_\mathcal{A}\otimes(\mu\circ\mathrm{id}_\mathcal{A}\otimes\mu)\circ\Delta\otimes\mathrm{id}_{\mathcal{A}\otimes X}\circ\eta\otimes\mathrm{id}_{\mathcal{A}\otimes X}\\ &= \mathrm{id}_\mathcal{A}\otimes(\mu\circ m\otimes\mathrm{id}_X)\circ\Delta\otimes\mathrm{id}_{\mathcal{A}\otimes X}\circ\eta\otimes\mathrm{id}_{\mathcal{A}\otimes X}\\ &= \mathrm{id}_\mathcal{A}\otimes\mu\circ(\Delta\circ m)\otimes\mathrm{id}_X\circ\eta\otimes\mathrm{id}_{\mathcal{A}\otimes X}\\ &= \mathrm{id}_\mathcal{A}\otimes\mu\circ\Delta\otimes\mathrm{id}_X\,.\end{aligned}$$

These morphisms are not properly inverses, so F_1 is not essentially surjective, but we will later show that F is.

Next, since $\operatorname{Rep}\mathcal{A}$ is already idempotent complete, we may use Lemma 3.17 to produce an equivalence F_2. Consider an object $((X,\mu),p)\in\operatorname{Obj}(\overline{\operatorname{Rep}\mathcal{A}})$ and choose $Y\in\operatorname{Obj}(\mathcal{B})$, $r\colon X\to Y$, $s\colon Y\to X$ so that $s\circ r = p$ and $r\circ s = \mathrm{id}_Y$. Consider the object $(Y, r\circ\mu\circ$

$\mathrm{id}_{\mathcal{A}} \otimes s) \in \mathrm{Obj}(\mathrm{Rep}\,\mathcal{A})$. This is an object as stated since it satisfies conditions 3.2.

$$(r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \circ \mathrm{id}_{\mathcal{A}} \otimes (r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) =$$



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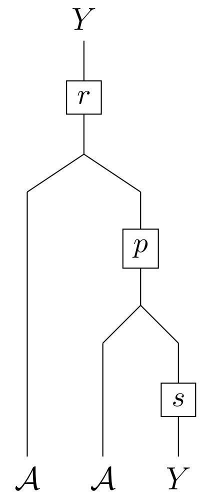

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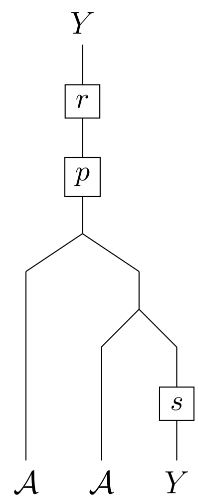

(by 3.3)

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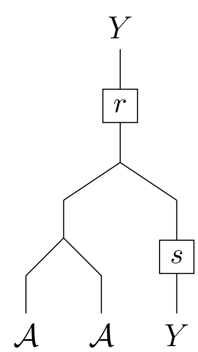
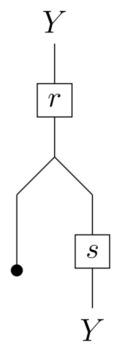
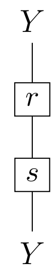


(by 3.2, Lemma 3.15)

$$= (r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \circ m \otimes \mathrm{id}_Y$$

$(r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \circ \eta \otimes \mathrm{id}_Y =$ 

$=$ 

$= \mathrm{id}_Y$ (by 3.2)

We also have $r \in \mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}((X,\mu),(Y, r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s))$ since

$$\begin{aligned} r \circ \mu &= r \circ p \circ \mu \\ &= r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes p \\ &= (r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \circ \mathrm{id}_{\mathcal{A}} \otimes r, \end{aligned}$$

and $s \in \mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}((Y, r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s),(X,\mu))$ since

$$\begin{aligned} s \circ (r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) &= p \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s \\ &= \mu \circ \mathrm{id}_{\mathcal{A}} \otimes p \circ \mathrm{id}_{\mathcal{A}} \otimes s \\ &= \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s. \end{aligned}$$

Thus, $(Y, r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s)$ witnesses the splitting of p, and Lemma 3.17 says there is an equivalence $F_2 \colon \overline{\mathrm{Rep}\,\mathcal{A}} \to \mathrm{Rep}\,\mathcal{A}$ defined by

$$\begin{aligned} F_2 \colon \Big((X,\mu),p\Big) &\mapsto (Y, r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \\ f &\mapsto r' \circ f \circ s, \end{aligned}$$

where $r' \circ f \circ s \colon (Y, r \circ \mu \circ \mathrm{id}_{\mathcal{A}} \otimes s) \to (Y', r' \circ \mu' \circ \mathrm{id}_{\mathcal{A}} \otimes s')$ arises from the splitting of $((X,\mu),p)$ and $((X',\mu'),p')$.

Finally, we define $F = F_2 \circ F_1$ to get a functor

$$\begin{aligned} F \colon \mathcal{B}_{\mathcal{A}} &\to \mathrm{Rep}\,\mathcal{A} \\ (X,p) &\mapsto (Y, r \circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}} \otimes s) \end{aligned}$$

$$f \mapsto r' \circ \mathrm{id}_{\mathcal{A}} \otimes f \circ \Delta \otimes \mathrm{id}_X \circ s.$$

Here we have taken f to be a morphism in $\mathcal{B}_{\mathcal{A}}$ from (X,p) to (X',p'). We are defining $r\colon \mathcal{A}\otimes X \to Y$, $s\colon Y \to \mathcal{A}\otimes X$ to satisfy $s\circ r = \mathrm{id}_{\mathcal{A}}\otimes p \circ \Delta \otimes \mathrm{id}_X$ and $r \circ s = \mathrm{id}_Y$ and $r'\colon \mathcal{A}\otimes X' \to Y'$, $s'\colon Y' \to \mathcal{A}\otimes X'$ to satisfy $s'\circ r' = \mathrm{id}_{\mathcal{A}}\otimes p' \circ \Delta \otimes \mathrm{id}_{X'}$ and $r'\circ s' = \mathrm{id}_{Y'}$. Since F_1, F_2 are fully faithful, so is F.

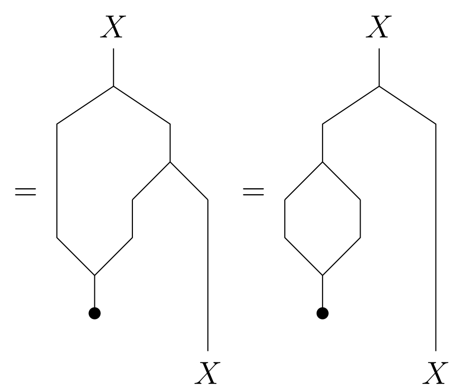
Claim: The functor F is also essentially surjective.

Proof: Consider any object (X,μ) in $\mathrm{Rep}\,\mathcal{A}$. We must find an isomorphic image under F. Since $\mu\colon \mathcal{A}\otimes X \to X$ satisfies equation 3.4, we may consider (X,μ) as an object in $\mathcal{B}_{\mathcal{A}}$ and then consider

$$F\Big((X,\mu)\Big) = (Y, r\circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}}\otimes s).$$

Now, we consider morphisms $\mu\circ s\colon Y \to X$ and $r\circ\eta\otimes\mathrm{id}_X\colon X\to Y$, which are compositions of the morphisms we considered in considering the surjectivity of F_1. We see these are now isomorphisms because

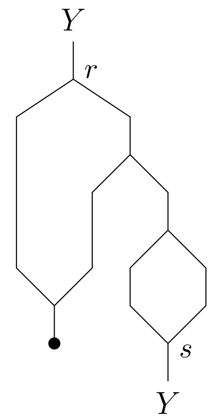
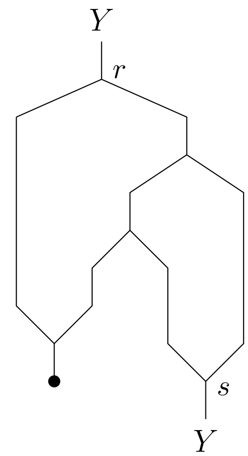
$$(\mu\circ s)\circ(r\circ\eta\otimes\mathrm{id}_X) = \mu\circ(\mathrm{id}_{\mathcal{A}}\otimes\mu\circ\Delta\otimes\mathrm{id}_X)\circ\eta\otimes\mathrm{id}_X$$

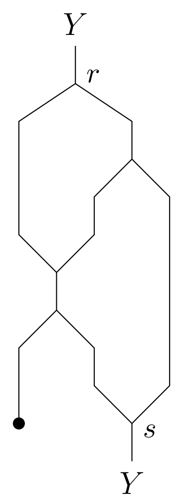
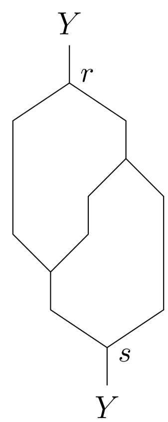


(by 3.2)

and

$$(r\circ\eta\otimes\mathrm{id}_X)\circ(\mu\circ s) = \Big(r\circ(\mathrm{id}_{\mathcal{A}}\otimes\mu\circ\Delta\otimes\mathrm{id}_X)\circ\eta\otimes\mathrm{id}_X\Big)\circ(\mu\circ s) \quad \text{(by Lemma 3.15)}$$

$=$  $=$  (by 3.2)

$=$  $=$  (by 3.1)

$= r \circ s$ (by Lemma 3.15)

$= \mathrm{id}_Y \, .$

■

Since F is both fully faithful and essentially surjective, it is a category equivalence.

Now consider the following definition for G with $(X, \mu) \in \mathrm{Obj}(\mathrm{Rep}\, \mathcal{A})$ and $f \in \mathrm{Hom}_{\mathrm{Rep}\, \mathcal{A}}((X, \mu), (Y, \lambda))$.

$$
\begin{aligned}
G\colon\ \mathrm{Rep}\, \mathcal{A} &\to \mathcal{B}_{\mathcal{A}} \\
(X, \mu) &\mapsto (X, \mu) \\
f &\mapsto f \circ \mu
\end{aligned}
$$

Claim: G is a functor.

Proof:

(i) We have $(X, \mu) \in \mathrm{Obj}(\mathrm{Rep}\,\mathcal{A})$. To have $(X, \mu) \in \mathrm{Obj}(\mathcal{B}_{\mathcal{A}})$, we need $\mu = \mu^2$ in $\mathcal{B}_{\mathcal{A}}$. Observe that

$$\begin{aligned}
\mu \circ_{\mathcal{B}_{\mathcal{A}}} \mu &= \mu \circ (\mathrm{id}_{\mathcal{A}} \otimes \mu) \circ (\Delta \otimes \mathrm{id}_X) \\
&= \mu \circ (m \otimes \mathrm{id}_X) \circ (\Delta \otimes \mathrm{id}_X) && \text{(by 3.2)} \\
&= \mu,
\end{aligned}$$

where we have normalized to have $m \circ \Delta = \mathrm{id}_{\mathcal{A}}$.

(ii) For each $f \in \mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}((X, \mu), (Y, \lambda))$, we have

$$G(f) = f \circ \mu \colon \mathcal{A} \otimes X \to Y$$

with

$$\begin{aligned}
G(f) \circ_{\mathcal{B}_{\mathcal{A}}} \mu &= f \circ \mu \circ_{\mathcal{B}_{\mathcal{A}}} \mu \\
&= f \circ \mu \circ (\mathrm{id}_{\mathcal{A}} \otimes \mu) \circ (\Delta \otimes \mathrm{id}_X) \qquad = f \circ \mu = G(f) && \text{(by part (i))} \\
&= \lambda \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \circ (\mathrm{id}_{\mathcal{A}} \otimes \mu) \circ (\Delta \otimes \mathrm{id}_X) && \text{(by 3.3)} \\
&= \lambda \circ_{\mathcal{B}_{\mathcal{A}}} (f \circ \mu) \\
&= \lambda \circ_{\mathcal{B}_{\mathcal{A}}} G(f),
\end{aligned}$$

which satisfies condition 3.5 and makes $G(f)$ a morphism of $\mathcal{B}_{\mathcal{A}}$.

(iii) Identity morphisms are preserved since

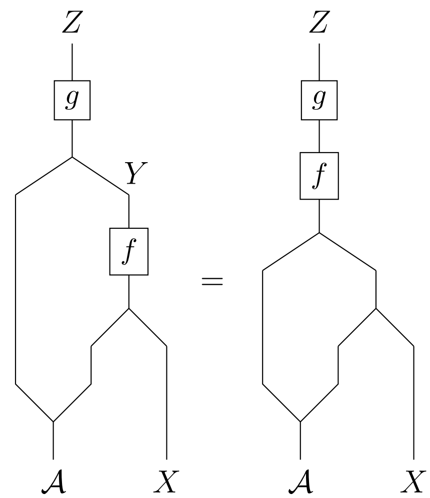
$$\begin{aligned}
G\left(\mathrm{id}_{(X,\mu)}\right) &= G\left(\mathrm{id}_X\right) \\
&= \mathrm{id}_X \circ \mu \\
&= \mu \\
&= \mathrm{id}_{(X,\mu)},
\end{aligned}$$

where the first $\mathrm{id}_{(X,\mu)}$ is in $\operatorname{Rep}\mathcal{A}$ and the last $\mathrm{id}_{(X,\mu)}$ is in $\mathcal{B}_\mathcal{A}$.

(iv) Morphism composition is preserved. For $f \in \mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}((X,\mu),(Y,\lambda))$ and $g \in \mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}((Y,\lambda),(Z,\nu))$,

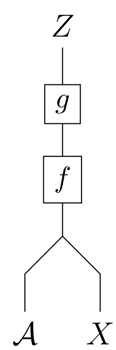
$$G(g)\circ_{\mathcal{B}_\mathcal{A}} G(f) = (g\circ\lambda)\circ_{\mathcal{B}_\mathcal{A}} (f\circ\mu)$$

$=$



(by 3.3)

$=$



(by part (i))

$$= G(g\circ f).$$

■

It remains only to show that G is adjoint to F. Consider objects $(X,p)\in \mathrm{Obj}(\mathcal{B}_\mathcal{A})$, $(X',p')\in\mathrm{Obj}(\operatorname{Rep}\mathcal{A})$. We need a natural bijection

$$\mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}\Big((Y, r\circ m\otimes \mathrm{id}_X\circ \mathrm{id}_\mathcal{A}\otimes s),(X',p')\Big)\longleftrightarrow \mathrm{Hom}_{\mathcal{B}_\mathcal{A}}\Big((X,p),(X',p')\Big)$$

or

$$\left\{f\colon Y \to X' \mid f \circ (r \circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}} \otimes s) = p' \circ (\mathrm{id}_{\mathcal{A}} \otimes f)\right\} \longleftrightarrow$$

$$\left\{g\colon \mathcal{A} \otimes X \to X' \mid g \circ (\mathrm{id}_{\mathcal{A}} \otimes p) \circ (\Delta \otimes \mathrm{id}_X) = g = p' \circ (\mathrm{id}_{\mathcal{A}} \otimes g) \circ (\Delta \otimes \mathrm{id}_X)\right\}. \quad (3.11)$$

Consider the map

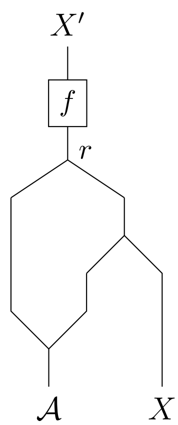
$$\varphi(f) = f \circ r$$

with inverse

$$\varphi^{-1}(g) = g \circ s.$$

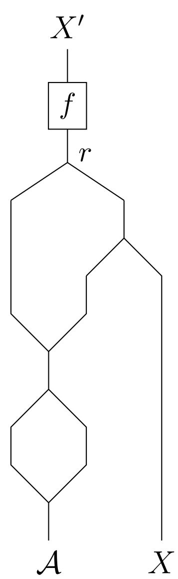
These images live in the correct sets. To see $\varphi(f)$ lives in $\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}((X,p),(X',p'))$, we must verify that $f \circ r$ satisfies the condition on g in the correspondence of 3.11. Observe that

$(f \circ r) \circ (\mathrm{id}_{\mathcal{A}} \otimes p) \circ (\Delta \otimes \mathrm{id}_X) =$



$= f \circ r$ (Lemma 3.15)

$=$



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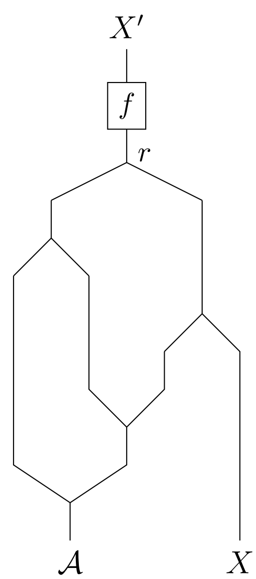
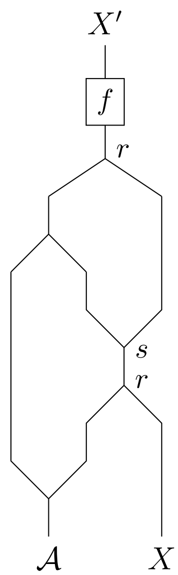
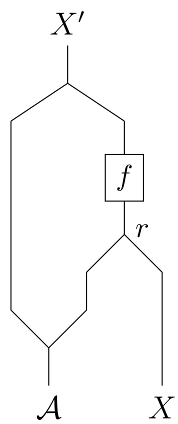


(by 3.1)

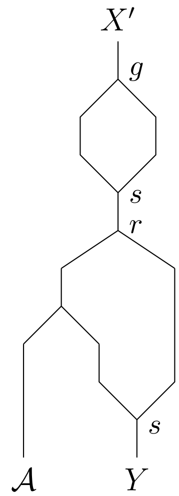
$=$  (by definition of r, s)

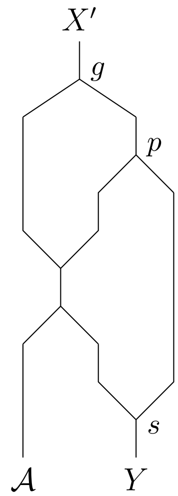
$=$  (by condition in 3.11)

$$= p' \circ (\mathrm{id}_{\mathcal{A}} \otimes (f \circ r)) \circ (\Delta \otimes \mathrm{id}_X).$$

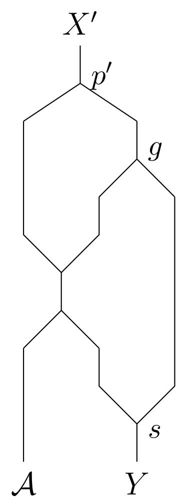
To see $\varphi^{-1}(g)$ lives in $\mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}}((Y, r \circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}} \otimes s), (X', p'))$, we must verify that

$g \circ s$ satisfies the condition on f in the correspondence of 3.11. Observe that

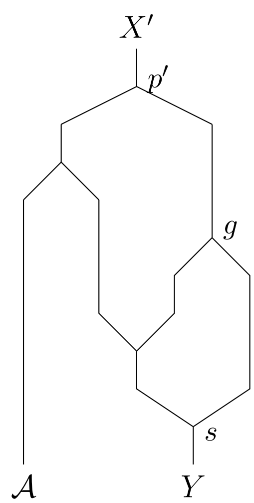
$(g \circ s) \circ (r \circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}} \otimes s) =$ 

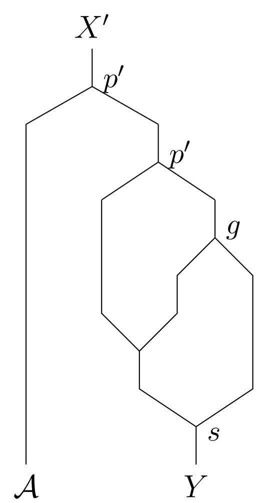
$=$ 

(by definition of r, s)

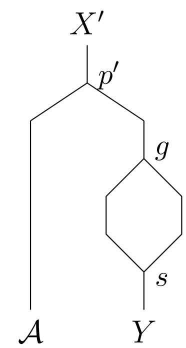
$=$ 

(by condition in 3.11)

$=$ 

$=$ 

(by 3.1,3.2)

$=$ 

(by condition in 3.11)

$$= p' \circ (\mathrm{id}_{\mathcal{A}} \otimes (g \circ s)).$$

These maps are also inverses since

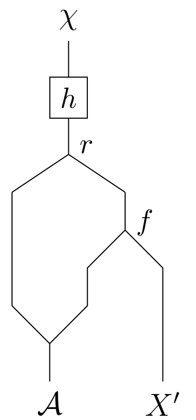
$$\begin{aligned}
\varphi^{-1}(\varphi(f)) &= f \circ r \circ s \\
&= f, \\
\\
\varphi(\varphi^{-1}(g)) &= g \circ s \circ r \\
&= g \circ \mathrm{id}_{\mathcal{A}} \otimes p \circ \Delta \otimes \mathrm{id}_X \\
&= g. \qquad \text{(by condition in 3.11)}
\end{aligned}$$

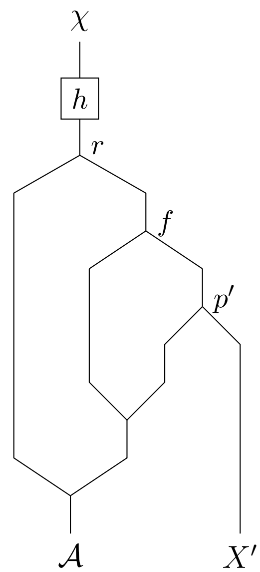
Finally, we show φ is natural. Consider morphisms $f\colon (X', p') \to (X, p)$ in $\mathcal{B}_{\mathcal{A}}$ and $g\colon (\chi, \mu) \to (\chi', \mu')$ in $\operatorname{Rep}\mathcal{A}$. We must show that the following diagrams both commute.

$$\begin{array}{ccc}
\mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}(FX, \chi) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, G\chi) \\
{\scriptstyle (Ff)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}(FX', \chi) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X', G\chi)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}(FX, \chi) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, G\chi) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (Gg)_*} \\
\mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}(FX, \chi') & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, G\chi')
\end{array}$$

Given $h \in \mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}(F(X, p), (\chi, \mu)) = \mathrm{Hom}_{\operatorname{Rep}\mathcal{A}}((Y, r \circ m \otimes \mathrm{id}_X \circ \mathrm{id}_{\mathcal{A}} \otimes s), (\chi, \mu))$, we have

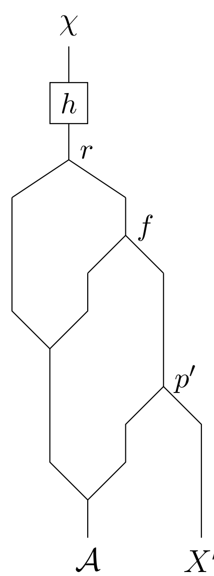
$$f^* \circ \varphi(h) = f^*(h \circ r) = h \circ r \circ_{\mathcal{B}_{\mathcal{A}}} f$$

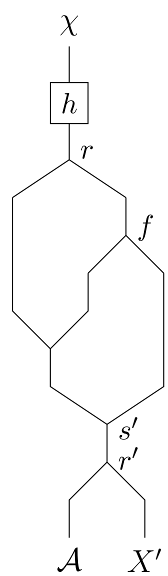
$=$ 

$=$ 

(BA or 3.11)

$$= h \circ \Big(r \circ_{\mathcal{B}_{\mathcal{A}}} (f \circ_{\mathcal{B}_{\mathcal{A}}} p')\Big) = h \circ \Big((r \circ_{\mathcal{B}_{\mathcal{A}}} f) \circ_{\mathcal{B}_{\mathcal{A}}} p'\Big)$$

$=$ 

$=$ 

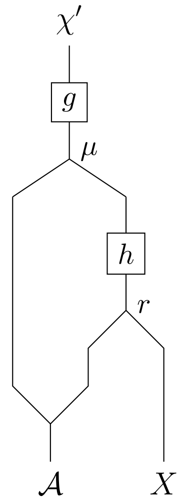
(by definition of r', s')

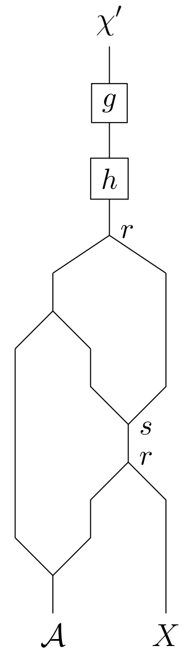
$$= \varphi(h \circ r \circ \mathrm{id}_{\mathcal{A}} \otimes f \circ \Delta \otimes \mathrm{id}_{X'} \circ s')$$

$$= \varphi \circ (Ff)^*(h),$$

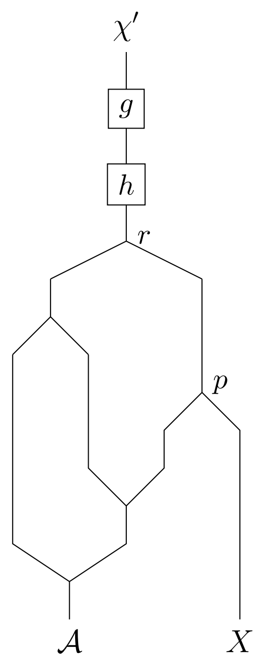
and

$$(Gg)_* \circ \varphi(h) = (Gg)_*(h \circ r)$$
$$= (g \circ \mu) \circ_{\mathcal{B}_{\mathcal{A}}} (h \circ r)$$

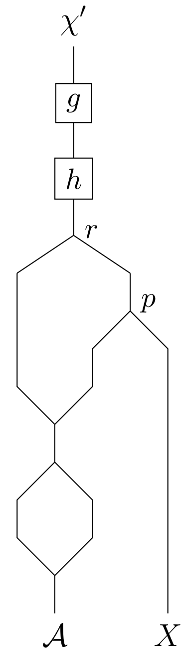
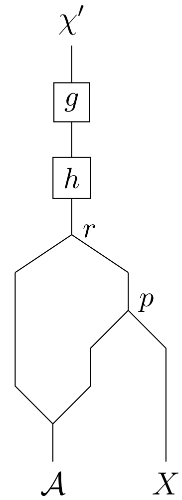
$=$ 

$=$ 

$(h \in \mathrm{Hom}_{\mathrm{Rep}\,\mathcal{A}})$

$=$ 

(by definition of r, s)

$=$  $=$  (by 3.1)

$$= g \circ h \circ r \qquad \text{(by 3.15)}$$

$$= \varphi(g \circ h)$$

$$= \varphi \circ g_*(h).$$

3.2.1 Discussion

It may be helpful to review an example condensation from Section 3.3 for context before reading this discussion.

Fix a modular tensor category $\mathcal{B}$ and condensable algebra $\mathcal{A}$. We may notice that the functor F is built on top of the functor F_0, which is equal on objects to the condensation tensor functor of [30]. That is, the equivalence between two definitions of condensation is built on the condensation functor from the parent category to the condensed category. In the diagram

$$\begin{array}{ccc} & \mathcal{B} & \\ \swarrow & & \searrow^{T \approx F_0} \\ \mathcal{B}_{\mathcal{A}} & \xrightarrow[F]{} & \operatorname{Rep} \mathcal{A} \end{array},$$

the categories $\mathcal{B}_{\mathcal{A}}$ and $\operatorname{Rep} \mathcal{A}$ are the equivalent condensations of $\mathcal{B}$, but the functor F is naturally induced by the functor F_0.

This illustrates an important difference between $\mathcal{B}_\mathcal{A}$ and $\operatorname{Rep}\mathcal{A}$ that we will need to navigate while trying to port results from one to the other (e.g. Lemma 3.28). Fix an object X in the parent category $\mathcal{B}$. Condensation as defined by $\operatorname{Rep}\mathcal{A}$ is a tensor functor taking X to $\mathcal{A}\otimes X$. Morally (up to idempotent completion), condensation as defined by $\mathcal{B}_\mathcal{A}$ alters Hom sets and leaves objects as in $\mathcal{B}$ so that $(X,p)\cong(Y,q)\cong\cdots$ are separate isomorphic objects.

To make this difference more concrete, consider $X=\mathbb{1}$. The unit object in $\operatorname{Rep}\mathcal{A}$ is $\mathcal{A}$. The unit object in $\mathcal{B}_\mathcal{A}$ is $(\mathbb{1},\mathrm{id}_\mathbb{1})\cong(X,\mathrm{id}_X)\cong\cdots$ for each simple X in the decomposition $\mathcal{A}\cong\mathbb{1}\oplus\bigoplus_X n_X X$.

Since $\mathcal{B}_\mathcal{A}$ roughly preserves the objects of $\mathcal{B}$, we still need the functor F_0 to give the new objects of $\operatorname{Rep}\mathcal{A}$ that all have proper module structures. The functor F is constructed to do this exactly when necessary and keep morphisms straight.

A next reasonable question may be "When is this necessary?" Consider a simple object X of $\mathcal{B}$ for which $p=(\mathrm{id}_X)_{\widetilde{\mathcal{B}_\mathcal{A}}}=\epsilon\otimes\mathrm{id}_X$ is a projection in $\operatorname{End}_{\widetilde{\mathcal{B}_\mathcal{A}}}(X)$ so that (X,id_X) is simple in $\mathcal{B}_\mathcal{A}$. Such an object is called *nonsplitting* since it remains simple through the condensation. In this case, when considering the image of (X,id_X) under F, we can (or must, since the choice is unique up to isomorphism) take $Y=\mathcal{A}\otimes X$, $r,s=\mathrm{id}_{A\otimes X}$, and indeed F reduces exactly to the definition of F_0. In order to refer to it later, we immortalize this observation in a remark.

Remark 3.18. Consider a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$, and let X be an object that does not split during condensation. The functor F reduces to the functor F_0 on X.

It is when X does split (or admits multiple idempotents) that the full definition of the functor F is needed. The idea is that the image of each idempotent gives a subobject of X that does not exist in $\mathcal{B}$. In $\mathcal{B}_\mathcal{A}$, these objects are denoted by pairing X with each idempotent. Since $\operatorname{Rep}\mathcal{A}$ is idempotent complete, the subobject should exist, and we call it Y. In either case, these subobjects that admit only a single idempotent or module structure, respectively, are the new simple objects.

On a different note, recall the observation in Remark 3.9 that the algebra structure of [30] and the Frobenius algebra structure of [38] somehow both get us to the same place. The discussion preceding Remark 3.18 suggests the following elucidation of Remark 3.9.

Remark 3.19. The condensation functor of [30] acts on objects by $X \mapsto (\mathcal{A} \otimes X, m \otimes \mathrm{id}_X)$. The point of this functor is to create module structures, especially when there is no such structure on X alone. The definition of $\mathcal{B}_\mathcal{A}$ circumvents the need to modify objects by using the co-algebra structure of $\mathcal{A}$ to define idempotents where the algebra structure of $\mathcal{A}$ alone fails.

3.3 Condensation Examples

We present a few insightful examples to provide a more complete picture of the process of computing condensations. The process follows the construction of the condensed category in Definition 3.7. All of these examples are condensations of self-dual bosons, which are helpful for seeing how the process often runs and for understanding the results of Section 3.4, but are also relatively well-behaved. For a somewhat less nice example, see Section 5.2.2. For more examples, one may refer to, e.g. [11, Section 5.2].

In what follows, the symbol $\boxtimes$ is the Deligne tensor product of Definition 2.75. Data for the categories being used can be found in Section 2.3.2.

3.3.1 Ising $\boxtimes$ Ising to $\mathbb{Z}_4$

Let $\mathcal{B}$ be the Ising MTC, and consider $\mathcal{B} \boxtimes \mathcal{B}$, the Deligne product of two copies of the Ising MTC with simple object representatives $\{1 \boxtimes 1, 1 \boxtimes \sigma, 1 \boxtimes \psi, \sigma \boxtimes 1, \sigma \boxtimes \sigma, \sigma \boxtimes \psi, \psi \boxtimes 1, \psi \boxtimes \sigma, \psi \boxtimes \psi\}$. For brevity, we will write these as $\{11, 1\sigma, 1\psi, \sigma 1, \sigma\sigma, \sigma\psi, \psi 1, \psi\sigma, \psi\psi\}$. The object $\mathcal{A} = 11 \oplus \psi\psi$ has a condensable algebra structure. Following the construction of $\mathcal{B}_\mathcal{A}$ in Definition 3.7, the category $(\mathcal{B} \boxtimes \mathcal{B})_\mathcal{A}$ has the same objects as the parent category

$\mathcal{B} \boxtimes \mathcal{B}$, but with new morphism spaces. Notably,

$$\mathrm{Hom}_{\mathcal{A}}(11, \psi\psi) = \mathrm{Hom}_{\mathcal{B}\boxtimes\mathcal{B}}(11 \oplus \psi\psi, \psi\psi) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(1\sigma, \psi\sigma) = \mathrm{Hom}_{\mathcal{B}\boxtimes\mathcal{B}}(1\sigma \oplus \psi\sigma, \psi\sigma) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(1\psi, \psi 1) = \mathrm{Hom}_{\mathcal{B}\boxtimes\mathcal{B}}(1\psi \oplus \psi 1, \psi 1) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(\sigma 1, \sigma\psi) = \mathrm{Hom}_{\mathcal{B}\boxtimes\mathcal{B}}(\sigma 1 \oplus \sigma\psi, \sigma\psi) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(\sigma\sigma, \sigma\sigma) = \mathrm{Hom}_{\mathcal{B}\boxtimes\mathcal{B}}(\sigma\sigma \oplus \sigma\sigma, \sigma\sigma) \cong \mathbb{C}^2 .$$

So in the category $(\widetilde{\mathcal{B} \boxtimes \mathcal{B}})_{\mathcal{A}}$, we have $11 \cong \psi\psi$, $1\sigma \cong \psi\sigma$, $1\psi \cong \psi 1$, $\sigma 1 \cong \sigma\psi$, and $\sigma\sigma$ not simple. Since $\mathrm{Hom}_{\mathcal{A}}(\sigma\sigma, X) = \{0\}$ for each of the simple objects $X = 11, 1\sigma, 1\psi, \sigma 1$, we realize $\sigma\sigma$ will, in the idempotent completion, be the sum of two simple objects which were not simple (or even in existence) before. Taking the idempotent completion of $(\widetilde{\mathcal{B} \boxtimes \mathcal{B}})_{\mathcal{A}}$ gives us a category $(\mathcal{B} \boxtimes \mathcal{B})_{\mathcal{A}}$ with six simple objects $(11, \mathrm{id}), (1\sigma, \mathrm{id}), (1\psi, \mathrm{id}), (\sigma 1, \mathrm{id}), (\sigma\sigma, p), (\sigma\sigma, q)$.

To determine the modular subcategory $(\mathcal{B} \boxtimes \mathcal{B})^0_{\mathcal{A}}$, we observe

$$\theta_{11} = 1 = 1 = \theta_{\psi\psi},$$

$$\theta_{1\sigma} = e^{\pi i/8} \neq -e^{\pi i/8} = \theta_{\psi\sigma},$$

$$\theta_{1\psi} = -1 = -1 = \theta_{\psi 1},$$

$$\theta_{\sigma 1} = e^{\pi i/8} \neq -e^{\pi i/8} = \theta_{\sigma\psi},$$

$$\theta_{\sigma\sigma} = e^{\pi i/4} = e^{\pi i/4} = \theta_{\sigma\sigma}.$$

So, the modular tensor category resulting from the condensation of $11 \oplus \psi\psi$ in Ising$\boxtimes$Ising has four simple objects $(11, \mathrm{id}), (1\psi, \mathrm{id}), (\sigma\sigma, p), (\sigma\sigma, q)$ and a corresponding T-matrix $\mathrm{Diag}[1, -1, e^{\pi i/4}, e^{\pi i/4}]$. By the classification of modular tensor categories [47], this must be the $\mathbb{Z}_4$ MTC with fusion rules

$$(1\psi, \mathrm{id}) \otimes (\sigma\sigma, p) \cong (\sigma\sigma, q),$$

$$(1\psi, \mathrm{id}) \otimes (\sigma\sigma, q) \cong (\sigma\sigma, p),$$

$$(\sigma\sigma, p) \otimes (\sigma\sigma, q) \cong (11, \mathrm{id}),$$

$$(\sigma\sigma, p) \otimes (\sigma\sigma, p) \cong (\sigma\sigma, q) \otimes (\sigma\sigma, q) \cong (1\psi, \mathrm{id}).$$

3.3.2 Ising $\boxtimes$ $\overline{\text{Ising}}$ to Toric Code

Let $\mathcal{B}$ be the Ising MTC, and consider $\mathcal{B} \boxtimes \overline{\mathcal{B}}$, the Deligne product of one copy of the Ising MTC with one copy of the Ising MTC with complex conjugate modular data. We can again condense the object $\mathcal{A} = 11 \oplus \psi\psi$ and follow the same process as in Section 3.3.1. The process is independent of modular data up to complex conjugation until we must find the θ morphisms inherited from $\mathcal{B} \boxtimes \overline{\mathcal{B}}$.

To determine the modular subcategory $(\mathcal{B} \boxtimes \overline{\mathcal{B}})^0_{\mathcal{A}}$, we observe

$$\theta_{11} = 1 = 1 = \theta_{\psi\psi},$$

$$\theta_{1\sigma} = e^{-\pi i/8} \neq -e^{-\pi i/8} = \theta_{\psi\sigma},$$

$$\theta_{1\psi} = -1 = -1 = \theta_{\psi 1},$$

$$\theta_{\sigma 1} = e^{-\pi i/8} \neq -e^{-\pi i/8} = \theta_{\sigma\psi},$$

$$\theta_{\sigma\sigma} = 1 = 1 = \theta_{\sigma\sigma}.$$

So, the modular tensor category resulting from the condensation of $11 \oplus \psi\psi$ in Ising$\boxtimes\overline{\text{Ising}}$ has four simple objects $(11, \mathrm{id}), (1\psi, \mathrm{id}), (\sigma\sigma, p), (\sigma\sigma, q)$ with T-matrix Diag$[1, -1, 1, 1]$. By the classification of modular tensor categories [47], this must be the Toric Code MTC with fusion rules

$$(1\psi, \mathrm{id}) \otimes (\sigma\sigma, p) \cong (\sigma\sigma, q),$$

$$(1\psi, \mathrm{id}) \otimes (\sigma\sigma, q) \cong (\sigma\sigma, p),$$

$$(\sigma\sigma, p) \otimes (\sigma\sigma, q) \cong (1\psi, \mathrm{id}),$$

$$(\sigma\sigma, p) \otimes (\sigma\sigma, p) \cong (\sigma\sigma, q) \otimes (\sigma\sigma, q) \cong (11, \mathrm{id}).$$

3.3.3 $SU(2)_k$ to Minimal Models

Here we demonstrate the first in a family of examples. A more general treatment of the relationship between $SU(2)_k$ and the minimal model conformal field theories can be found in Corollary 3.35 and in Section 5.1.1.

Consider just the modular tensor categories $SU(2)_1$ and $SU(2)_2$. It is common to name the simple objects in these categories $\{1, s\}$ and $\{1, \sigma, \psi\}$, respectively. The data we will need is given below.

$$s \otimes s = 1 \qquad\qquad \sigma \otimes \psi = \sigma \qquad \psi \otimes \psi = 1$$

$$\sigma \otimes \sigma = 1 \oplus \psi$$

$$\theta_1 = 1 \qquad \theta_s = i \qquad\qquad \theta_1 = 1 \qquad \theta_\sigma = e^{\frac{3\pi i}{8}} \qquad \theta_\psi = -1$$

Now, let $\mathcal{B}$ be the modular tensor category $SU(2)_1 \boxtimes SU(2)_1 \boxtimes \overline{SU(2)_2}$ with condensable algebra $\mathcal{A} = 111 \oplus ss\psi$. Note that

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{A}}(111, ss\psi) \cong \mathbb{C}, && \mathrm{Hom}_{\mathcal{A}}(11\sigma, ss\sigma) \cong \mathbb{C}, \\ &\mathrm{Hom}_{\mathcal{A}}(11\psi, ss1) \cong \mathbb{C}, && \mathrm{Hom}_{\mathcal{A}}(1s1, s1\psi) \cong \mathbb{C}, \\ &\mathrm{Hom}_{\mathcal{A}}(1s\sigma, s1\sigma) \cong \mathbb{C}, && \mathrm{Hom}_{\mathcal{A}}(1s\psi, s11) \cong \mathbb{C}. \end{aligned}$$

The θ values of the pairings in the right column do not match, so those objects are relegated to the confined part of the condensed category. The θ values of the pairings in the right column do match, so $\mathcal{B}_{\mathcal{A}}^0$ consists of three simple objects classes $(111, \mathrm{id})$, $(11\psi, \mathrm{id})$, $(1s\sigma, \mathrm{id})$ with θ values $1, -1, e^{\pi i/8}$, respectively. With fusion rules

$$(11\psi, \mathrm{id}) \otimes (11\psi, \mathrm{id}) \cong (111, \mathrm{id}),$$

$$(1s\sigma, \mathrm{id}) \otimes (11\psi, \mathrm{id}) \cong (1s\sigma, \mathrm{id}),$$

$$(1s\sigma, \mathrm{id}) \otimes (1s\sigma, \mathrm{id}) \cong (111, \mathrm{id}) \oplus (11\psi, \mathrm{id}),$$

we have recovered the Ising MTC, which is also the minimal model $\mathcal{M}(4, 3)$.

3.4 Determining the Condensed Category

Given a modular tensor category $\mathcal{B}$ and a condensable algebra $\mathcal{A}$, we would like to determine the data of the condensed category from the data of the original. To do so, we establish some terminology motivated by the examples of Section 3.3.

In all of the examples, we find that some distinct simple objects become isomorphic after condensation. For example, in the Ising$\boxtimes$Ising $\to \mathbb{Z}_4$ condensation of Section 3.3.1, the objects 11 and $\psi\psi$ are sent to the objects $(11, \mathrm{id})$ and $(\psi\psi, \mathrm{id})$, respectively. While 11 and $\psi\psi$ are nonisomorphic simple objects of Ising $\boxtimes$ Ising, their images in the new category are isomorphic. In this case, we may say that 11 and $\psi\psi$ have been *identified* and write $[11]$ or $[\psi\psi]$ to represent both of the two isomorphic images.

In the two Ising examples (Sections 3.3.1 and 3.3.2), we also saw the object $\sigma\sigma$ *split* into two simple objects $(\sigma\sigma, p), (\sigma\sigma, q)$. This is a common phenomenon that complicates the condensation process. The results in this section aim to explore what happens when objects split.

As usual, we will mostly be working in a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$. We also fix a label set $\{a_i\}_{i \in I}$ for $\mathcal{B}$.

3.4.1 Condensing a Boson

We focus on the simplest case (and the only one we have seen so far) where $\mathcal{A} = \mathbb{1} \oplus B$ for a boson B.

Definition 3.20. A simple object B in a modular category is called *bosonic* if $\theta_B = 1$.

Remark 3.21. Proposition 3.3 implies that if $\mathcal{A} = \bigoplus_i n_i a_i$ is a condensable algebra, then a_i is bosonic for all i with $n_i \neq 0$.

Definition 3.22. A bosonic object B in a modular category is called a *boson* if $\dim(B) = 1$.

Lemma 3.23. *If a boson B is self-dual, then $B \otimes B \cong \mathbb{1}$.*

Proof. Since B is self-dual, we have $B \otimes B \cong \mathbb{1} \oplus \bigoplus_i n_i a_i$. Since $\dim(B) = 1$, we know $\dim(B \otimes B) = 1$. Since $\dim(\mathbb{1}) = 1$, we must have $n_i = 0$ for all i. □

Proposition 3.24. *If B is a self-dual boson, then $\mathcal{A} = \mathbb{1} \oplus B$ is condensable.*

Proof. Consider the object $\mathcal{A} = \mathbb{1} \oplus B$ for a self-dual boson B. By Lemma 3.23, the object $\mathcal{A}$ is a direct sum of the simple objects of a subcategory equivalent to $\mathrm{Rep}(\mathbb{Z}_2)$ and has a $\mathbb{Z}_2$ action. Then we may use, e.g. [31, Theorem 4.2]. □

Remark 3.25. The object $\mathcal{A} = \mathbb{1} \oplus B$ need not be condensable if B is a boson that is not self-dual. In fact, there exist conditions ensuring it is not, as in the no-go theorem of [43]. One of the objects of work on condensation is to find necessary and sufficient conditions for condensability that are easier to decide in practice than the separability condition of Definition 3.1.

Lemma 3.26. *If B is a self-dual boson and X is simple, then $B \otimes X$ is also simple.*

Proof. Let B be a self-dual boson and X be a simple object. Consider $B \otimes X \cong \bigoplus n_i a_i$. Then

$$X \cong B \otimes B \otimes X \cong \bigoplus n_i (B \otimes a_i).$$

But X is simple, so we must have $n_i = 1$ for some i and $n_j = 0$ for all $j \neq i$. Then $B \otimes X$ is simple. □

Lemma 3.27. *Let $\mathcal{B}$ be a modular tensor category with self-dual boson B, condensable algebra $\mathcal{A} = \mathbb{1} \oplus B$, and simple object X. If X splits, then it splits into exactly two objects $(X, p), (X, q)$.*

Proof. Consider the situation of the statement. If X splits, then the dimension of $\mathrm{Hom}_{\mathcal{B}_\mathcal{A}}(X, X) = \mathrm{Hom}_\mathcal{B}(\mathcal{A} \otimes X, X)$ is greater than one. Lemma 3.26 guarantees $\mathcal{A} \otimes X \cong X \oplus X$ so that X splits into two objects $(X, p), (X, q)$. □

We now present two facts about dimensions that will be useful. The first is (at least implicitly) known to [30, 38] and can be stated in greater generality than how it appears here. The specific formulation presented here will be helpful to us for reasons articulated in the discussion of Section 3.2.1. It will also help us resist the confusion wrought by the conventional difference mentioned in Remark 2.46. Our global dimension agrees with that of [30], while the global dimension of [38] agrees with that of [17]. Refer to [30] for a minimally confusing presentation of the same results in $\operatorname{Rep}\mathcal{A}$ instead of $\mathcal{B}_{\mathcal{A}}$.

Lemma 3.28. *Let $\mathcal{B}$ be a modular tensor category with self-dual boson B, condensable algebra $\mathcal{A} = \mathbb{1} \oplus B$, and simple object X. Then*

$$\dim_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X) = \dim_{\mathcal{B}}(X).$$

If X does not split, then

$$\dim_{\mathcal{B}_{\mathcal{A}}}([X]) = \dim_{\mathcal{B}}(X).$$

If X does split, then

$$\dim_{\mathcal{B}_{\mathcal{A}}}\Big((X,p)\Big) + \dim_{\mathcal{B}_{\mathcal{A}}}\Big((X,q)\Big) = \dim_{\mathcal{B}}(X).$$

Proof. In Section 3.2, we used a functor $F_0 \colon \widetilde{\mathcal{B}_{\mathcal{A}}} \to \operatorname{Rep}\mathcal{A}$. Since F_0 is a tensor functor,

$$\dim_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X) = \dim_{\operatorname{Rep}\mathcal{A}}(F_0(X)) = \dim_{\mathcal{B}}(X),$$

where the second equality comes from [30, Theorem 1.18].

If X does not split, then Remark 3.18 completes the proof, and

$$\dim_{\mathcal{B}_{\mathcal{A}}}([X]) = \dim_{\operatorname{Rep}\mathcal{A}}(\mathcal{A} \otimes X) = \dim_{\mathcal{B}}(X).$$

If X does split, then Lemma 3.27 says X splits into two objects $(X,p),(X,q)$. Again we can say

$$\dim_{\mathcal{B}_{\mathcal{A}}}\Big((X,p) \oplus (X,q)\Big) = \dim_{\mathcal{B}_{\mathcal{A}}}\Big((X,\mathrm{id}_X)\Big) = \dim_{\operatorname{Rep}\mathcal{A}}(F_0(X)) = \dim_{\mathcal{B}}(X).$$

□

Lemma 3.29. *Let $\mathcal{B}$ be a modular tensor category with condensable algebra $\mathcal{A}$. Then*

$$\dim(\mathcal{B}_{\mathcal{A}}^0) = \frac{\dim(\mathcal{B})}{\dim_{\mathcal{B}}(\mathcal{A})}.$$

Proof. [30, Theorem 4.5] by way of the tensor equivalence of Section 3.2. □

Another known result that we will use later is the following.

Lemma 3.30. *In any modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$, condensation preserves the central charge from Definition 2.55.*

Proof. [30, Theorem 4.5] □

We present one last lemma before diving in.

Lemma 3.31. *Consider a modular tensor category with simple objects X, Y and a self-dual boson B. Then*

$$S_{X,Y} = S_{B\otimes X,Y}.$$

Proof. We first note that $(B \otimes X)^* \cong B \otimes X^*$ since B is self-dual. Then

$$\begin{aligned} N_Z^{(B\otimes X)^*,Y} &= \dim\Big(\operatorname{Hom}\big(Z, (B\otimes X)^* \otimes Y\big)\Big) \\ &= \dim\Big(\operatorname{Hom}\big(B\otimes Z, X^* \otimes Y\big)\Big) \\ &= N_{B\otimes Z}^{X^*,Y}. \end{aligned}$$

Now since $\theta_{B\otimes Z} = \theta_Z$ and $\dim(B \otimes Z) = \dim(Z)$ for all simple objects Z,

$$\begin{aligned} \tilde{S}_{B\otimes X,Y} &= \theta_{B\otimes X}^{-1}\theta_Y^{-1}\sum_Z N_Z^{(B\otimes X)^*,Y}\theta_Z \dim(Z) \qquad \text{(Proposition 2.53)} \\ &= \theta_X^{-1}\theta_Y^{-1}\sum_Z N_{B\otimes Z}^{X^*,Y}\theta_{B\otimes Z}\dim(B\otimes Z) \\ &= \tilde{S}_{X,Y}. \end{aligned}$$

Multiplying by the dimension of the category gives the result for the S-matrix. □

3.4.1.1 Modular Data

In this section, we begin with a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$, and we try to determine the modular data of the condensed category $\mathcal{B}_{\mathcal{A}}^0$. The primary result is Theorem 3.37, but other results using different proof techniques are also included. We begin with the following.

Theorem 3.32. *Let $\mathcal{B}$ be a modular tensor category with condensable algebra $\mathcal{A} = \mathbb{1} \oplus B$ for a self-dual boson B. Suppose condensing $\mathcal{A}$ does not cause any simple objects to split. Then for simple objects $[X], [Y] \in \mathrm{Obj}(\mathcal{B}_{\mathcal{A}}^0)$, we have*

$$\tilde{S}_{[X],[Y]} = \tilde{S}_{X,Y}, \qquad T_{[X]} = T_X.$$

Proof. Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $\mathcal{A} = 1 \oplus B$ for a self-dual boson B. Assume no simple objects split in the category $\mathcal{B}_{\mathcal{A}}$. Then $B \otimes X \ncong X$ for any simple objects X of $\mathcal{B}$. Then by Lemma 3.26, for any simple object X,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(\mathcal{A} \otimes X, Y) \cong \mathrm{Hom}_{\mathcal{B}}(X \oplus B \otimes X, Y) \cong \mathbb{C}$$

exactly when $Y \cong X$ or $Y \cong B \otimes X$. Note that no object is unpaired since $B \otimes X \ncong X$. Thus, the deconfined simple objects of $\mathcal{B}_{\mathcal{A}}$ (up to isomorphism) are the pairs $[X] = \{X, B \otimes X\}$ with $\theta_X = \theta_{B\otimes X}$. The T-matrix follows immediately since $T_{[X]} = \theta_X = \theta_{B\otimes X}$.

For the $\tilde{S}$-matrix, we observe

$$\begin{aligned}
\tilde{S}_{[X],[Y]} &= \theta_{[X]}^{-1}\theta_{[Y]}^{-1} \sum_{[Z]} N_{[Z]}^{[X]^*,[Y]} \theta_{[Z]} \dim([Z]) && \text{(Proposition 2.53)} \\
&= \theta_X^{-1}\theta_Y^{-1} \sum_{[Z]} N_{[Z]}^{[X]^*,[Y]} \theta_{[Z]} \dim([Z]),
\end{aligned}$$

which is well-defined since $\theta_X = \theta_{B\otimes X}$. Note that

$$\begin{aligned}
N_{[Z]}^{[X]^*,[Y]} &= \dim\Big(\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}\left((Z, \mathrm{id}_Z), (X^*, \mathrm{id}_{X^*}) \otimes (Y, \mathrm{id}_Y) \right) \Big) \\
&= \dim\Big(\mathrm{Hom}_{\mathcal{B}}\left(Z, X^* \otimes Y \oplus B \otimes X^* \otimes Y \right) \Big)
\end{aligned}$$

$$= N_Z^{X^*,Y} + N_Z^{B\otimes X^*,Y}. \qquad (B \text{ self-dual})$$

We have

$$N_Z^{B\otimes X^*,Y} = \dim\Big(\mathrm{Hom}_{\mathcal{B}}(Z, B\otimes X^*\otimes Y)\Big) = \dim\Big(\mathrm{Hom}_{\mathcal{B}}(B\otimes Z, X^*\otimes Y)\Big) = N_{B\otimes Z}^{X^*,Y}$$

since B is self-dual, so

$$\tilde{S}_{[X],[Y]} = \theta_X^{-1}\theta_Y^{-1}\sum_{[Z]}\big(N_Z^{X^*,Y} + N_{B\otimes Z}^{X^*,Y}\big)\theta_{[Z]}\dim([Z]).$$

Since each $[Z]$ is an equivalence class $\{Z, B\otimes Z\}$, the sum can be expanded. Using Lemma 3.28 and the fact that $\theta_Z = \theta_{B\otimes Z}$, $\dim(Z) = \dim(B\otimes Z)$, we find

$$\begin{aligned}\tilde{S}_{[X],[Y]} &= \theta_X^{-1}\theta_Y^{-1}\sum_{[Z]}\Big(N_Z^{X^*,Y}\theta_Z\dim(Z) + N_{B\otimes Z}^{X^*,Y}\theta_{B\otimes Z}\dim(B\otimes Z)\Big)\\ &= \theta_X^{-1}\theta_Y^{-1}\sum_{Z} N_Z^{X^*,Y}\theta_Z\dim(Z),\end{aligned}$$

and $\tilde{S}_{[X],[Y]} = \tilde{S}_{X,Y}$, as desired. □

Remark 3.33. Theorem 3.32 aims to give full modular data for the condensed category, so it is restricted to cases with no splitting objects. The S- and T-matrix entries still hold for nonsplitting X, Y even when there are other objects that split. If the object Z splits, tensor functoriality of condensation guarantees that either none or all of the split objects from Z will count toward the sum of $\tilde{S}_{[X],[Y]}$. Then Lemma 3.28 guarantees that the sum of dimensions of the splitting objects will leave the overall sum unchanged. This allows us to state Theorem 3.32 in slightly greater generality at the cost of the simple classification of condensed simple objects.

Theorem 3.34. *Let $\mathcal{B}$ be a modular tensor category with condensable algebra $\mathcal{A} = \mathbb{1}\oplus B$ for a self-dual boson B. If X, Y are simple objects of $\mathcal{B}$ that do not split and $[X], [Y] \in$ $\mathrm{Obj}(\mathcal{B}_{\mathcal{A}}^0)$, then*

$$\tilde{S}_{[X],[Y]} = \tilde{S}_{X,Y}, \qquad T_{[X]} = T_X.$$

In calculating fusion coefficients, we found that $N_{[Z]}^{[X],[Y]} = N_Z^{X,Y} + N_{B\otimes Z}^{X,Y}$. Indeed if $X \otimes Y$ (or $X^* \otimes Z$ or $Y^* \otimes Z$) is simple, at least one of the two summands is zero, and $N_{[Z]}^{[X],[Y]}$ is equal to the other. Consider the spaces

$$\mathrm{Hom}(Z, X \otimes Y) \cong \mathrm{Hom}(X^* \otimes Z, Y) \cong \mathrm{Hom}(Y^* \otimes Z, X),$$

$$\mathrm{Hom}(B \otimes Z, X \otimes Y) \cong \mathrm{Hom}(X^* \otimes Z, B \otimes Y) \cong \mathrm{Hom}(Y^* \otimes Z, B \otimes X).$$

Lemma 3.26 says $B \otimes X, B \otimes Y, B \otimes Z$ are simple, so at most one of these Hom spaces can be positive-dimensional. By choosing the right representative of $[Z] = \{Z, B \otimes Z\}$, we can always have $N_{[Z]}^{[X],[Y]} = N_Z^{X,Y}$. This is simply confirming that the tensor product in $\mathcal{B}_\mathcal{A}$ works as we want it to on nice nonsplitting objects.

Sadly, Theorem 3.34 makes no statements about splitting objects. The difficulty in considering splitting objects comes from the somewhat opaque definition of the tensor product as a coequalizer, which complicates the fusion coefficient calculation in the proof (we will make use of the coequalizer in Chapter 4). However, the nonsplitting case is already interesting as it includes the minimal model conformal field theories.

Corollary 3.35 ([34])**.** *If $\mathcal{B} = SU(2)_k \boxtimes SU(2)_1 \boxtimes \overline{SU(2)_{k+1}}$ and $\mathcal{A} = 000 + k1(k+1)$, then the modular data of $\mathcal{B}_\mathcal{A}^0$ are the same as those of the minimal model $\mathcal{M}(k+3, k+2)$.*

Proof. Refer to Section 5.1.1 for the condensation of $SU(2)_k \boxtimes SU(2)_1 \boxtimes \overline{SU(2)_{k+1}}$ to the minimal model $\mathcal{M}(k+3, k+2)$. □

Let us now tackle the more complicated situation in which there exist simple objects X with $B \otimes X \cong X$. We may begin to generalize with the following result of [42] using the notation of Remark 3.10.

Lemma 3.36 ([42])**.** *Consider a (restricted) condensation functor $T\colon \mathcal{B} \to \mathcal{B}_\mathcal{A}^0$ from a rank n modular tensor category $\mathcal{B}$ with label set $\{X_i\}$ to a rank m modular tensor category $\mathcal{B}_\mathcal{A}^0$ with label set $\{Y_j\}$ so that*

$$T(X_i) \cong \bigoplus_j n_i^j Y_j.$$

Define an $m \times n$ condensation matrix n *by*

$$n_{j,i} = n_i^j.$$

Then,

$$S_{\mathcal{A}} n = n S_{\mathcal{B}},$$
$$T_{\mathcal{A}} n = n T_{\mathcal{B}},$$

where $\{S_{\mathcal{B}}, T_{\mathcal{B}}\}$ *is the modular data of the category* $\mathcal{B}$ *and* $\{S_{\mathcal{A}}, T_{\mathcal{A}}\}$ *is the modular data of the category* $\mathcal{B}^0_{\mathcal{A}}$.

The condensation matrix is demonstrated in Example 3.42, but a consequence of this lemma is presented first. Recall from Lemma 3.27 that any splitting objects decompose as the sum of exactly two simple objects in $\mathcal{B}^0_{\mathcal{A}}$.

Theorem 3.37. *Consider a modular tensor category* $\mathcal{B}$ *with a self-dual boson* B *and a condensable algebra* $\mathcal{A} = \mathbb{1} \oplus B$. *If* $\mathcal{B}$ *has some nonsplitting objects* $\mathbb{1}, X, \ldots$ *and some splitting objects* $Y, Z, \ldots$ *with* S*-matrix*

$$S_{\mathcal{B}} = \begin{array}{c} \\ \mathbb{1} \\ B \\ X \\ B\otimes X \\ \vdots \\ Y \\ Z \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \mathbb{1} & B & X & B\otimes X & \cdots & Y & Z & \cdots \end{array} \\ \begin{pmatrix} S_{\mathbb{1},\mathbb{1}} & S_{\mathbb{1},B} & S_{\mathbb{1},X} & S_{\mathbb{1},B\otimes X} & & S_{\mathbb{1},Y} & S_{\mathbb{1},Z} & \\ S_{B,\mathbb{1}} & S_{B,B} & S_{B,X} & S_{B,B\otimes X} & \cdots & S_{B,Y} & S_{B,Z} & \cdots \\ S_{X,\mathbb{1}} & S_{X,B} & S_{X,X} & S_{X,B\otimes X} & & S_{X,Y} & S_{X,Z} & \\ S_{B\otimes X,\mathbb{1}} & S_{B\otimes X,B} & S_{B\otimes X,X} & S_{B\otimes X,B\otimes X} & & S_{B\otimes X,Y} & S_{B\otimes X,Z} & \\ & \vdots & & & \ddots & & \vdots & \\ S_{Y,\mathbb{1}} & S_{Y,B} & S_{Y,X} & S_{Y,B\otimes X} & \cdots & S_{Y,Y} & S_{Y,Z} & \cdots \\ S_{Z,\mathbb{1}} & S_{Z,B} & S_{Z,X} & S_{Z,B\otimes X} & & S_{Z,Y} & S_{Z,Z} & \\ & \vdots & & & & & \vdots & \ddots \end{pmatrix} \end{array},$$

then condensing $\mathcal{A}$ gives a new S-matrix of the form

$$S_{\mathcal{A}} = \begin{array}{c} \\ {\scriptstyle [\mathbb{1}]} \\ {\scriptstyle [X]} \\ \vdots \\ {\scriptstyle (Y,p)} \\ {\scriptstyle (Y,q)} \\ {\scriptstyle (Z,p)} \\ {\scriptstyle (Z,q)} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} {\scriptstyle [\mathbb{1}]} & {\scriptstyle [X]} & {\scriptstyle \cdots} & {\scriptstyle (Y,p)} & {\scriptstyle (Y,q)} & {\scriptstyle (Z,p)} & {\scriptstyle (Z,q)} & {\scriptstyle \cdots} \end{array} \\ \begin{pmatrix} S_{\mathbb{1},\mathbb{1}} + S_{B,\mathbb{1}} & S_{\mathbb{1},X} + S_{B,X} & \dots & S_{\mathbb{1},Y} & S_{\mathbb{1},Y} & S_{\mathbb{1},Z} & S_{\mathbb{1},Z} & \dots \\ S_{X,\mathbb{1}} + S_{B\otimes X,\mathbb{1}} & S_{X,X} + S_{B\otimes X,X} & & S_{X,Y} & S_{X,Y} & S_{X,Z} & S_{X,Z} & \\ & \vdots & \ddots & & & \vdots & & \\ S_{Y,\mathbb{1}} & S_{Y,X} & & a & b & e & f & \\ S_{Y,\mathbb{1}} & S_{Y,X} & \dots & b & a & f & e & \dots \\ S_{Z,\mathbb{1}} & S_{Z,X} & & e & f & c & d & \\ S_{Z,\mathbb{1}} & S_{Z,X} & & f & e & d & c & \\ & \vdots & & & & \vdots & & \ddots \end{pmatrix} \end{array},$$

where

$$a + b = S_{Y,Y},$$

$$c + d = S_{Z,Z},$$

$$e + f = S_{Y,Z} = S_{Z,Y},$$

$$\vdots$$

.

Proof. For simplicity, we disregard objects that are confined after condensation. We begin by defining a condensation matrix

$$n = \begin{array}{c} \\ [\mathbb{1}] \\ [X] \\ \vdots \\ (Y,p) \\ (Y,q) \\ (Z,p) \\ (Z,q) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} \mathbb{1} & B & X & B\otimes X & \cdots & Y & Z & \cdots \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & & 0 & 0 & \\ & \vdots & & & \ddots & & \vdots & \ddots \\ 0 & 0 & 0 & 0 & & 1 & 0 & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & & 0 & 1 & \\ 0 & 0 & 0 & 0 & & 0 & 1 & \\ & \vdots & & & \ddots & & \vdots & \ddots \end{pmatrix} \end{array}$$

and a condensed S matrix $S_{\mathcal{A}}$ given by

$$\begin{array}{c} \\ [\mathbb{1}] \\ [X] \\ \vdots \\ (Y,p) \\ (Y,q) \\ (Z,p) \\ (Z,q) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} [\mathbb{1}] & [X] & \cdots & (Y,p) & (Y,q) & (Z,p) & (Z,q) & \cdots \end{array} \\ \begin{pmatrix} S_{[\mathbb{1}],[\mathbb{1}]} & S_{[\mathbb{1}],[X]} & \dots & S_{[\mathbb{1}],(Y,p)} & S_{[\mathbb{1}],(Y,q)} & S_{[\mathbb{1}],(Z,p)} & S_{[\mathbb{1}],(Z,q)} & \dots \\ S_{[X],[\mathbb{1}]} & S_{[X],[X]} & & S_{[X],(Y,p)} & S_{[X],(Y,q)} & S_{[X],(Z,p)} & S_{[X],(Z,q)} & \\ & \vdots & \ddots & & & \vdots & & \ddots \\ S_{(Y,p),[\mathbb{1}]} & S_{(Y,p),[X]} & & S_{(Y,p),(Y,p)} & S_{(Y,p),(Y,q)} & S_{(Y,p),(Z,p)} & S_{(Y,p),(Z,q)} & \\ S_{(Y,q),[\mathbb{1}]} & S_{(Y,q),[X]} & \dots & S_{(Y,q),(Y,p)} & S_{(Y,q),(Y,q)} & S_{(Y,q),(Z,p)} & S_{(Y,q),(Z,q)} & \dots \\ S_{(Z,p),[\mathbb{1}]} & S_{(Z,p),[X]} & & S_{(Z,p),(Y,p)} & S_{(Z,p),(Y,q)} & S_{(Z,p),(Z,p)} & S_{(Z,p),(Z,q)} & \\ S_{(Z,q),[\mathbb{1}]} & S_{(Z,q),[X]} & & S_{(Z,q),(Y,p)} & S_{(Z,q),(Y,q)} & S_{(Z,q),(Z,p)} & S_{(Z,q),(Z,q)} & \\ & \vdots & \ddots & & & \vdots & & \ddots \end{pmatrix} \end{array}.$$

Now, we may observe the products

$$nS_{\mathcal{B}} = \begin{pmatrix} A_{\mathcal{B}} & B_{\mathcal{B}} \\ C_{\mathcal{B}} & D_{\mathcal{B}} \end{pmatrix}$$

with $A_{\mathcal{B}}$ given by

$$\begin{pmatrix} S_{\mathbb{1},\mathbb{1}} + S_{B,\mathbb{1}} & S_{\mathbb{1},B} + S_{B,B} & S_{\mathbb{1},X} + S_{B,X} & S_{\mathbb{1},B\otimes X} + S_{B,B\otimes X} & \\ & & & & \cdots \\ S_{X,\mathbb{1}} + S_{B\otimes X,\mathbb{1}} & S_{X,B} + S_{B\otimes X,B} & S_{X,X} + S_{B\otimes X,X} & S_{X,B\otimes X} + S_{B\otimes X,B\otimes X} & \\ & & \vdots & & \ddots \end{pmatrix},$$

$$B_{\mathcal{B}} = \begin{pmatrix} S_{\mathbb{1},Y} + S_{B,Y} & S_{\mathbb{1},Z} + S_{B,Z} & \\ & & \cdots \\ S_{X,Y} + S_{B\otimes X,Y} & S_{X,Z} + S_{B\otimes X,Z} & \\ & \vdots & \ddots \end{pmatrix},$$

$$C_{\mathcal{B}} = \begin{pmatrix} S_{Y,\mathbb{1}} & S_{Y,B} & S_{Y,X} & S_{Y,B\otimes X} & \\ S_{Y,\mathbb{1}} & S_{Y,B} & S_{Y,X} & S_{Y,B\otimes X} & \\ & & & & \cdots \\ S_{Z,\mathbb{1}} & S_{Z,B} & S_{Z,X} & S_{Z,B\otimes X} & \\ S_{Z,\mathbb{1}} & S_{Z,B} & S_{Z,X} & S_{Z,B\otimes X} & \\ & & \vdots & & \ddots \end{pmatrix}, \quad D_{\mathcal{B}} = \begin{pmatrix} S_{Y,Y} & S_{Y,Z} & \\ S_{Y,Y} & S_{Y,Z} & \\ & & \cdots \\ S_{Z,Y} & S_{Z,Z} & \\ S_{Z,Y} & S_{Z,Z} & \\ & \vdots & \ddots \end{pmatrix},$$

and

$$S_{\mathcal{A}}n = \begin{pmatrix} A_{\mathcal{B}_{\mathcal{A}}} & B_{\mathcal{B}_{\mathcal{A}}} \\ C_{\mathcal{B}_{\mathcal{A}}} & D_{\mathcal{B}_{\mathcal{A}}} \end{pmatrix}$$

with

$$A_{\mathcal{A}} = \begin{pmatrix} S_{[\mathbb{1}],[\mathbb{1}]} & S_{[\mathbb{1}],[\mathbb{1}]} & S_{[\mathbb{1}],[X]} & S_{[\mathbb{1}],[X]} & \\ & & & & \cdots \\ S_{[X],[\mathbb{1}]} & S_{[X],[\mathbb{1}]} & S_{[X],[X]} & S_{[X],[X]} & \\ & & \vdots & & \ddots \end{pmatrix},$$

$$B_{\mathcal{A}} = \begin{pmatrix} S_{[\mathbb{1}],(Y,p)} + S_{[\mathbb{1}],(Y,q)} & S_{[\mathbb{1}],(Z,p)} + S_{[\mathbb{1}],(Z,q)} & \\ & & \cdots \\ S_{[X],(Y,p)} + S_{[X],(Y,q)} & S_{[X],(Z,p)} + S_{[X],(Z,q)} & \\ & \vdots & \ddots \end{pmatrix},$$

$$C_{\mathcal{A}} = \begin{pmatrix} S_{(Y,p),[\mathbb{1}]} & S_{(Y,p),[\mathbb{1}]} & S_{(Y,p),[X]} & S_{(Y,p),[X]} & \\ S_{(Y,q),[\mathbb{1}]} & S_{(Y,q),[\mathbb{1}]} & S_{(Y,q),[X]} & S_{(Y,q),[X]} & \\ & & & & \cdots \\ S_{(Z,p),[\mathbb{1}]} & S_{(Z,p),[\mathbb{1}]} & S_{(Z,p),[X]} & S_{(Z,p),[X]} & \\ S_{(Z,q),[\mathbb{1}]} & S_{(Z,q),[\mathbb{1}]} & S_{(Z,q),[X]} & S_{(Z,q),[X]} & \\ & & \vdots & & \ddots \end{pmatrix},$$

$$D_{\mathcal{A}} = \begin{pmatrix} S_{(Y,p),(Y,p)} + S_{(Y,p),(Y,q)} & S_{(Y,p),(Z,p)} + S_{(Y,p),(Z,q)} & \\ S_{(Y,q),(Y,p)} + S_{(Y,q),(Y,q)} & S_{(Y,q),(Z,p)} + S_{(Y,q),(Z,q)} & \\ & & \cdots \\ S_{(Z,p),(Y,p)} + S_{(Z,p),(Y,q)} & S_{(Z,p),(Z,p)} + S_{(Z,p),(Z,q)} & \\ S_{(Z,q),(Y,p)} + S_{(Z,q),(Y,q)} & S_{(Z,q),(Z,p)} + S_{(Z,q),(Z,q)} & \\ & \vdots & \ddots \end{pmatrix}.$$

Setting the A and C blocks equal immediately gives the columns of $S_{\mathcal{A}}$ corresponding to nonsplitting objects. Symmetry of the S-matrix then gives block B. For the lower right block, let us begin by noticing that

$$S_{(Y,p),(Y,p)} + S_{(Y,p),(Y,q)} = S_{Y,Y} = S_{(Y,q),(Y,p)} + S_{(Y,q),(Y,q)}.$$

Since $S_{(Y,p),(Y,q)} = S_{(Y,q),(Y,p)}$, we see that $S_{(Y,p),(Y,p)} = S_{(Y,q),(Y,q)}$. For simplicity, let us define

$$a = S_{(Y,p),(Y,p)} = S_{(Y,q),(Y,q)},$$

$$b = S_{(Y,p),(Y,q)} = S_{(Y,q),(Y,p)}.$$

Similarly, we may define

$$c = S_{(Z,p),(Z,p)} = S_{(Z,q),(Z,q)},$$

$$d = S_{(Z,p),(Z,q)} = S_{(Z,q),(Z,p)}.$$

For the other terms, we notice

$$S_{(Y,p),(Z,p)} + S_{(Y,p),(Z,q)} = S_{Y,Z} = S_{(Y,q),(Z,p)} + S_{(Y,q),(Z,q)},$$

$$S_{(Z,p),(Y,p)} + S_{(Z,p),(Y,q)} = S_{Z,Y} = S_{(Z,q),(Y,p)} + S_{(Z,q),(Y,q)},$$

$$S_{Y,Z} = S_{Z,Y}.$$

Since $S_{(Y,p),(Z,p)} = S_{(Z,p),(Y,p)}$, we get $S_{(Y,p),(Z,q)} = S_{(Z,p),(Y,q)}$. Since $S_{(Y,q),(Z,q)} = S_{(Z,q),(Y,q)}$, we get $S_{(Y,q),(Z,p)} = S_{(Z,q),(Y,p)}$. Now define

$$e = S_{(Y,p),(Z,p)} = S_{(Z,p),(Y,p)} = S_{(Y,q),(Z,q)} = S_{(Z,q),(Y,q)},$$

$$f = S_{(Y,p),(Z,q)} = S_{(Z,p),(Y,q)} = S_{(Y,q),(Z,p)} = S_{(Z,q),(Y,p)}.$$

This gives us the claimed $S_{\mathcal{A}}$ matrix. □

Remark 3.38. This theorem has been presented so as to demonstrate the role of the boson B and the old S-matrix. By Lemma 3.31, we know $S_{B\otimes X,Y} = S_{X,Y}$, so we can also write

$$
S_{\mathcal{A}} = \begin{array}{c} \\ [\mathbb{1}] \\ [X] \\ \vdots \\ (Y,p) \\ (Y,q) \\ (Z,p) \\ (Z,q) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccccccc} [\mathbb{1}] & [X] & \cdots & (Y,p) & (Y,q) & (Z,p) & (Z,q) & \cdots \end{array} \\
\begin{pmatrix}
2S_{\mathbb{1},\mathbb{1}} & 2S_{\mathbb{1},X} & \dots & S_{\mathbb{1},Y} & S_{\mathbb{1},Y} & S_{\mathbb{1},Z} & S_{\mathbb{1},Z} & \dots \\
2S_{X,\mathbb{1}} & 2S_{X,X} & & S_{X,Y} & S_{X,Y} & S_{X,Z} & S_{X,Z} & \\
 & \vdots & \ddots & & & \vdots & & \\
S_{Y,\mathbb{1}} & S_{Y,X} & & a & b & e & f & \\
S_{Y,\mathbb{1}} & S_{Y,X} & \dots & b & a & f & e & \dots \\
S_{Z,\mathbb{1}} & S_{Z,X} & & e & f & c & d & \\
S_{Z,\mathbb{1}} & S_{Z,X} & & f & e & d & c & \\
 & \vdots & & & & \vdots & & \ddots
\end{pmatrix} \end{array}
$$

and keep in mind that the matrix is symmetric.

Remark 3.39. Note that Theorem 3.34 gives the new $\tilde{S}$-matrix, which is unchanged by condensation. Theorem 3.37 gives the new S-matrix, which features a factor of 2 where Theorem 3.34 does not.

To see that the two theorems agree, let us restrict our attention to the upper left block of $S_{\mathcal{A}}$ from Theorem 3.37. For nonsplitting objects, we have $S_{[X],[Y]} = 2S_{X,Y}$ or

$$\tilde{S}_{[X],[Y]} = 2\dim(\mathcal{B}^0_{\mathcal{A}})S_{X,Y}.$$

From Lemma 3.29, we know

$$\dim(\mathcal{B}^0_{\mathcal{A}}) = \frac{\dim(\mathcal{B})}{\dim_{\mathcal{B}}(\mathcal{A})} = \frac{\dim(\mathcal{B})}{2}.$$

So

$$\tilde{S}_{[X],[Y]} = \dim(\mathcal{B})S_{X,Y} = \tilde{S}_{X,Y},$$

as we saw in Theorem 3.34.

To see what we can do with the lower right block of the new S-matrix, consider the following corollary.

Corollary 3.40. *Consider a modular tensor category $\mathcal{B}$ with a self-dual boson B and a condensable algebra $\mathcal{A} = \mathbb{1} \oplus B$. If $\mathcal{B}$ has some nonsplitting objects $\mathbb{1}, X, \dots$ and exactly one splitting object Y, then $\mathcal{B}^0_{\mathcal{A}}$ has modular data*

$$S_{\mathcal{A}} = \begin{array}{c} {\scriptstyle [\mathbb{1}]} \\ {\scriptstyle [X]} \\ \vdots \\ {\scriptstyle (Y,p)} \\ {\scriptstyle (Y,q)} \end{array} \overset{\begin{array}{ccccc} {\scriptstyle [\mathbb{1}]} & {\scriptstyle [X]} & {\scriptstyle \cdots} & {\scriptstyle (Y,p)} & {\scriptstyle (Y,q)} \end{array}}{\begin{pmatrix} S_{\mathbb{1},\mathbb{1}} + S_{B,\mathbb{1}} & S_{\mathbb{1},X} + S_{B,X} & \dots & S_{\mathbb{1},Y} & S_{\mathbb{1},Y} \\ S_{X,\mathbb{1}} + S_{B\otimes X,\mathbb{1}} & S_{X,X} + S_{B\otimes X,X} & & S_{X,Y} & S_{X,Y} \\ & \vdots & \ddots & & \vdots \\ S_{Y,\mathbb{1}} & S_{Y,X} & & a & S_{Y,Y} - a \\ S_{Y,\mathbb{1}} & S_{Y,X} & \dots & S_{Y,Y} - a & a \end{pmatrix}},$$

$$T_{\mathcal{A}} = Diag[T_{\mathbb{1},\mathbb{1}}, T_{X,X}, \dots, T_{Y,Y}, T_{Y,Y}],$$

where a is solved as a root of a quadratic polynomial depending on whether $(Y,p),(Y,q)$ are self-dual or dual to each other.

Proof. The claimed $S_{\mathcal{A}}$ comes directly from Theorem 3.37. Since $S^2_{\mathcal{A}} = C_{\mathcal{A}}$, the charge conjugation matrix from Definition 2.55, exactly one of

$$S^2_{1Y} + S^2_{XY} + \cdots + a^2 + (S_{YY} - a)^2, \quad S^2_{1Y} + S^2_{XY} + \cdots + a(S_{YY} - a) + (S_{YY} - a)a$$

is zero, and the other is one, depending on whether $(Y,p),(Y,q)$ are self-dual or dual to each other. □

Remark 3.41. A similar result can be stated about any number of splitting objects. In general, at least one of

$$S^2_{1Y} + S^2_{XY} + \cdots + a^2 + b^2 + e^2 + f^2 + \cdots, \quad S^2_{1Y} + S^2_{XY} + \cdots + ab + ba + ef + fe + \cdots$$

is zero. If exactly one is zero, then the other is one, and Y is self-dual. Both are zero if and only if Y is dual to another splitting object.

Somewhat more difficult is the question of whether introducing T-matrices decides the condensed $S_{\mathcal{A}}$ matrix. In Example 3.42, we see a case where it does.

Example 3.42. Since Ising and $\overline{\text{Ising}}$ share an S matrix, so do Ising ⊠ Ising and Ising ⊠ $\overline{\text{Ising}}$. If we condense $1 \boxtimes 1 \oplus \psi \boxtimes \psi$ in a category with an Ising ⊠ Ising S-matrix (as in Sections 3.3.1 and 3.3.2), the process from the proof of Theorem 3.37 gives us

$$n = \begin{array}{c} \\ [\mathbb{1}\,\mathbb{1}] \\ [\mathbb{1}\,\psi] \\ (\sigma\sigma,p) \\ (\sigma\sigma,q) \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{1}\,\mathbb{1} & \psi\psi & \mathbb{1}\,\psi & \psi\,\mathbb{1} & \sigma\sigma \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

and ultimately

$$S_{\mathcal{A}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & a & -a \\ 1 & -1 & -a & a \end{pmatrix}.$$

If $C_{\mathcal{A}}$ is the condensed charge conjugation matrix (Definition 2.55), the condition $S_{\mathcal{A}}^2 = C_{\mathcal{A}}$ now allows for two possibilities: $a = \pm 1$ or $a = \pm i$. If $a = \pm 1$, then $S_{\mathcal{A}}$ is the Toric Code S-matrix and is the result of condensing Ising ⊠ $\overline{\text{Ising}}$. If $a = \pm i$, then $S_{\mathcal{A}}$ is the $\mathbb{Z}_4$ S-matrix and is the result of condensing Ising ⊠ Ising. Note that the choice of positive or negative a is simply exchanging the interchangeable objects e and m in Toric Code or 1 and 3 in $\mathbb{Z}_4$.

Since Ising and $\overline{\text{Ising}}$ share an S-matrix, it is not surprising that $S_{\mathcal{B}}$ alone does not determine $S_{\mathcal{A}}$. In this case, the T-matrix decides the value of a by the identity

$$(ST)^3 = e^{c\pi i/4} S^2,$$

where c is the central charge of the parent and condensed categories (Definition 2.55 and Lemma 3.30).

If $\mathcal{B} = \text{Ising} \boxtimes \overline{\text{Ising}}$, we get

$$T_{\mathcal{A}} = \text{Diag}[1, -1, 1, 1].$$

In Section 3.3.2, this was already enough for us to conclude that the condensation yields the Toric Code MTC. However, that deduction relied on having a classification of modular tensor categories of a sufficiently high rank. This time, we compute

$$(ST)^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \frac{a^3}{2} & \frac{1}{2} - \frac{a^3}{2} \\ 0 & 0 & \frac{1}{2} - \frac{a^3}{2} & \frac{1}{2} + \frac{a^3}{2} \end{pmatrix}$$

and

$$e^{0\pi i/4} S^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \frac{a^2}{2} & \frac{1}{2} - \frac{a^2}{2} \\ 0 & 0 & \frac{1}{2} - \frac{a^2}{2} & \frac{1}{2} + \frac{a^2}{2} \end{pmatrix},$$

where the central charge $c = 0$ comes from the central charge $c = {}^1\!/_2 - {}^1\!/_2$ of $\text{Ising} \boxtimes \overline{\text{Ising}}$. Equality is achieved when $a = 0, 1$. Since $a = 0$ is not an option from earlier, we must have $a = 1$, and $S_{\mathcal{A}}$ is the Toric Code S-matrix.

If $\mathcal{B} = \text{Ising} \boxtimes \text{Ising}$, we get

$$T_{\mathcal{A}} = \text{Diag}[1, -1, e^{\pi i/4}, e^{\pi i/4}].$$

This is the case of Section 3.3.1. This time, we have central charge $c = {}^1\!/_2 + {}^1\!/_2 = 1$, and a similar computation gives us $a = -i$, which recovers the $\mathbb{Z}_4$ S-matrix.

Remark 3.43. Obviously this approach is only pinning down condensation up to modular data. It is not sensitive to the modular isotopes mentioned in Remark 2.57. Chapter 4 discusses the computation of F- and R-symbols after condensation.

3.4.1.2 Duality

The previous section attempts to pin down the modular data after condensation. We now shift gears and present some results about the relationships between parent and condensed objects. Since some of our S-matrix results determined the new S-matrix up to a choice of splitting objects being dual to each other or not, we give some attention to the question of when objects are dual. Perhaps most obviously, we have the following.

Remark 3.44. Condensation is a tensor functor and preserves duality in the sense that $T(X^*) \cong T(X)^*$. Roughly speaking, dual objects map to dual objects, and dual objects come from dual objects.

Let us now try to make this Remark a little more precise.

Proposition 3.45. *Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $\mathcal{A}$ and any simple object X of $\mathcal{B}$. Let T be the condensation functor and set $T(X) \cong \bigoplus_{i\in I}(X, p_i)$, $T(X^*) \cong \bigoplus_{j\in J}(X^*, \hat{p}_j)$. Without loss of generality, take no (X, p_i) or $(X^*, \hat{p}_j)$ isomorphic to 0 so that I, J are minimal. Then $|I| = |J|$, and $(X, p_i)^* \cong (X^*, \hat{p}_i)$ for all i.*

Proof. This follows from the fact that $T(X^*) \cong T(X)^*$ and Lemma 2.28. The order of the $\hat{p}_j$ is arbitrary, so they can be selected to match the p_i. □

Remark 3.46. This proposition can of course be carried to non-simple objects at the cost of less compact notation.

Notice that Proposition 3.45 is not exactly reversible because the condensed category is an idempotent completion. Some objects do not have pre-images in the parent category. When objects do not split, going backwards is possible.

Proposition 3.47. *Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $\mathcal{A}$ and any nonsplitting simple objects X, Y. Then $(X, \mathrm{id}_X), (Y, \mathrm{id}_Y)$ are dual in $\mathcal{B}_\mathcal{A}$ if and* *only if*

$$\dim\Big(\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}\otimes X\otimes Y,\mathbb{1})\Big)=\dim\Big(\mathrm{Hom}_{\mathcal{A}}\left((X,\mathrm{id}_X)\otimes(Y,\mathrm{id}_Y),(\mathbb{1},\mathrm{id}_{\mathbb{1}})\right)\Big)=1.$$

In the more familiar self-dual object setting, we can be more precise.

Proposition 3.48. *Consider a modular tensor category $\mathcal{B}$ with any self-dual object B and condensable algebra $\mathcal{A}=\mathbb{1}\oplus B$. Let X,Y be simple objects of $\mathcal{B}$ with $N_X^{B,X}=0$ and $N_Y^{B,Y}=0$. Then $(X,\mathrm{id}_X),(Y,\mathrm{id}_Y)$ are dual in $\mathcal{B}_{\mathcal{A}}$ if and only if either $N_{\mathbb{1}}^{X,Y}=1, N_B^{X,Y}=0$ or $N_{\mathbb{1}}^{X,Y}=0, N_B^{X,Y}=1$.*

Proof. Note that

$$\dim\Big(\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}\otimes X\otimes Y,\mathbb{1})\Big)=\dim\Big(\mathrm{Hom}_{\mathcal{B}}(X\otimes Y\oplus B\otimes X\otimes Y,\mathbb{1})\Big).$$

The objects $[X],[Y]$ are dual exactly when this dimension is one, which is clearly not the case if $N_{\mathbb{1}}^{X,Y}>1$.

Suppose $N_{\mathbb{1}}^{X,Y}=1$. Then $\dim(\mathrm{Hom}_{\mathcal{B}}(X\otimes Y,\mathbb{1}))=1$, and the objects $[X],[Y]$ are dual exactly when

$$0=\dim\Big(\mathrm{Hom}_{\mathcal{B}}(B\otimes X\otimes Y,\mathbb{1})\Big)=\dim\Big(\mathrm{Hom}_{\mathcal{B}}(X\otimes Y,B)\Big).$$

Now suppose $N_{\mathbb{1}}^{X,Y}=0$. Then $\dim(\mathrm{Hom}_{\mathcal{B}}(X\otimes Y,\mathbb{1}))=0$, and the objects $[X],[Y]$ are dual exactly when

$$1=\dim\Big(\mathrm{Hom}_{\mathcal{B}}(B\otimes X\otimes Y,\mathbb{1})\Big)=\dim\Big(\mathrm{Hom}_{\mathcal{B}}(X\otimes Y,B)\Big).$$

□

Remark 3.49. This proposition extends to any number of unique self-dual summands of $\mathcal{A}$.

Unsurprisingly, the situation is more complicated when there is splitting. Still, Proposition 3.45 clarifies some situations.

Corollary 3.50. *Consider a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$. If the simple object X is self-dual and splits, then the split simple objects are either self-dual or dual to each other.*

Proof. Proposition 3.45 □

Remark 3.51. Both can happen. When we condense $1 \boxtimes 1 \oplus \psi \boxtimes \psi$ in Ising $\boxtimes$ Ising or Ising $\boxtimes$ $\overline{\text{Ising}}$, the self-dual object $\sigma\sigma$ splits into two objects which are dual to each other or self-dual, respectively. Refer to Sections 3.3.1 and 3.3.2.

Corollary 3.52. *Consider a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$. If the simple objects X, Y are dual to each other, then condensation of $\mathcal{A}$ causes X, Y to split into an equal number (possibly one) of simple objects.*

Proof. Proposition 3.45 □

Corollary 3.53. *Consider a modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$. If the simple objects X, Y split and are dual to each other, then (X, p_i) is dual to (Y, q_i) for all i.*

Proof. Proposition 3.45 □

Proposition 3.54. *Consider a modular tensor category $\mathcal{B}$ with a self-dual boson B and condensable algebra $\mathcal{A} = \mathbb{1} \oplus B$. Let X be a simple object of $\mathcal{B}$ that splits in the condensation of $\mathcal{A}$. Then X^* also splits. That is, splitting objects are dual only to splitting objects.*

Proof. Consider $\mathcal{B}, \mathcal{A}, X$ as in the statement. By Lemma 3.26, we have $B \otimes X \cong X$. Then

$$\begin{aligned} \mathbb{1} \oplus \bigoplus X_i &\cong X \otimes X^* \\ &\cong (B \otimes X) \otimes X^* \\ &\cong X \otimes (B \otimes X^*), \end{aligned}$$

so X is also dual to $B \otimes X^*$. Since duals are unique up to isomorphism, we have $B \otimes X^* \cong X^*$, so X^* splits. $\square$

3.4.2 Condensation over Deligne Products

Recall the Deligne product defined in Definition 2.75 and Remark 2.76. We have already seen this notion of product in the condensation examples of Section 3.3. In all of these examples, the Deligne product being condensed is of some minimal size to be interesting. However, it can also be interesting to consider many copies of known smaller condensations. This appears in Section 5.3.

Proposition 3.55. *Suppose $\mathcal{A}_1$ is a condensable algebra in the MTC $\mathcal{B}_1$ and $\mathcal{A}_2$ is a condensable algebra in the MTC $\mathcal{B}_2$. Then ${\mathcal{B}_1 \boxtimes \mathcal{B}_2}_{\mathcal{A}_1 \boxtimes \mathcal{A}_2}$ and ${\mathcal{B}_1}_{\mathcal{A}_1} \boxtimes {\mathcal{B}_2}_{\mathcal{A}_2}$ are isomorphic categories.*

Proof. We first verify that $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is condensable.

(1) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ each have exactly one copy of their respective unit objects, so does $\mathcal{A}_1 \boxtimes \mathcal{A}_2$. Thus, $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is connected.

(2) For all summands $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, we have $\theta_{A_1} = \theta_{A_2} = 1$. So, every summand of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ has $\theta = 1 \cdot 1 = 1$, and $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is commutative.

(3) Suppose that $m_1\colon \mathcal{A}_1 \otimes \mathcal{A}_1 \to \mathcal{A}_1$ admits a splitting $\Delta_1\colon \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_1$ and that $m_2\colon \mathcal{A}_2 \otimes \mathcal{A}_2 \to \mathcal{A}_2$ admits a splitting $\Delta_2\colon \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$ such that the following diagrammatic representations of Equation 3.1 all commute.

$$
\begin{array}{ccc}
\mathcal{A}_1 \otimes \mathcal{A}_1 & \xrightarrow{\mathrm{id} \otimes \Delta_1} & \mathcal{A}_1 \otimes (\mathcal{A}_1 \otimes \mathcal{A}_1) \\
 & & \wr\Vert \\
\Big\downarrow m_1 & & (\mathcal{A}_1 \otimes \mathcal{A}_1) \otimes \mathcal{A}_1 \\
 & & \Big\downarrow m_1 \otimes \mathrm{id} \\
\mathcal{A}_1 & \xrightarrow[\Delta_1]{} & \mathcal{A}_1 \otimes \mathcal{A}_1
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A}_1 \otimes \mathcal{A}_1 & \xrightarrow{\Delta_1 \otimes \mathrm{id}} & (\mathcal{A}_1 \otimes \mathcal{A}_1) \otimes \mathcal{A}_1 \\
 & & \wr\Vert \\
\Big\downarrow m_1 & & \mathcal{A}_1 \otimes (\mathcal{A}_1 \otimes \mathcal{A}_1) \\
 & & \Big\downarrow \mathrm{id} \otimes m_1 \\
\mathcal{A}_1 & \xrightarrow[\Delta_1]{} & \mathcal{A}_1 \otimes \mathcal{A}_1
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{A}_2 \otimes \mathcal{A}_2 & \xrightarrow{\mathrm{id} \otimes \Delta_2} & \mathcal{A}_2 \otimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \\
 & & \wr\Vert \\
\Big\downarrow m_2 & & (\mathcal{A}_2 \otimes \mathcal{A}_2) \otimes \mathcal{A}_2 \\
 & & \Big\downarrow m_2 \otimes \mathrm{id} \\
\mathcal{A}_2 & \xrightarrow[\Delta_2]{} & \mathcal{A}_2 \otimes \mathcal{A}_2
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A}_2 \otimes \mathcal{A}_2 & \xrightarrow{\Delta_2 \otimes \mathrm{id}} & (\mathcal{A}_2 \otimes \mathcal{A}_2) \otimes \mathcal{A}_2 \\
 & & \wr\Vert \\
\Big\downarrow m_2 & & \mathcal{A}_2 \otimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \\
 & & \Big\downarrow \mathrm{id} \otimes m_2 \\
\mathcal{A}_2 & \xrightarrow[\Delta_2]{} & \mathcal{A}_2 \otimes \mathcal{A}_2
\end{array}
$$

Note the associator in the top right corner of each diagram. Up to equivalence, the isomorphism can be taken to be the identity, but it may not be in general (refer to Remark 2.5). In Chapter 4, we will be interested in skeletal categories specifically.

Now in $\mathcal{B}_1 \boxtimes \mathcal{B}_2$, the object $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ has multiplication $m = m_1 \boxtimes m_2$ with splitting $\Delta = \Delta_1 \boxtimes \Delta_2$. From Remark 2.76, morphisms are applied componentwise and tensor products are defined appropriately so that the following diagrams commute as well.

$$
\begin{array}{ccc}
(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) & \xrightarrow{\mathrm{id} \otimes \Delta} & (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes \Big((\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \Big) \\
\Vert & & \wr\Vert \\
(\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) & & \Big((\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \Big) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \\
 & & \Big\downarrow m \otimes \mathrm{id} \\
\Big\downarrow m & & (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \\
 & & \Vert \\
\mathcal{A}_1 \boxtimes \mathcal{A}_2 & \xrightarrow[\Delta]{} & (\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2)
\end{array}
$$

$$
\begin{array}{ccc}
(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) & \xrightarrow{\Delta \otimes \mathrm{id}} & \Big((\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \Big) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \\
\Vert & & \wr\Vert \\
(\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) & & (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes \Big((\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2) \Big) \\
 & & \Big\downarrow \mathrm{id} \otimes m \\
\Big\downarrow m & & (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \otimes (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \\
 & & \Vert \\
\mathcal{A}_1 \boxtimes \mathcal{A}_2 & \xrightarrow[\Delta]{} & (\mathcal{A}_1 \otimes \mathcal{A}_1) \boxtimes (\mathcal{A}_2 \otimes \mathcal{A}_2)
\end{array}
$$

Thus $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ has a proper splitting and is condensable.

Now we may write the objects and morphisms of all categories in sight (as defined by

Remark 2.76) and observe that they match. First,

$$\text{Obj}(\mathcal{B}_{\mathcal{A}}) = \text{pairs } (X, p), \text{ where } X \in \text{Obj}(\mathcal{B}),\ p = p^2 \in \text{End}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X)$$
$$\text{Hom}_{\mathcal{B}_{\mathcal{A}}}\Big((X_1, p_1), (X_2, p_2)\Big) = \Big\{f \in \text{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X_1, X_2) \mid f \circ p_1 = f = p_2 \circ f\Big\},$$

where $\text{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X, Y) = \text{Hom}_{\mathcal{B}}(\mathcal{A} \otimes X, Y)$. So,

$$\begin{aligned}\text{Obj}(\mathcal{B}_{1\mathcal{A}_1} \boxtimes \mathcal{B}_{2\mathcal{A}_2}) = \text{pairs } &(X_1, p_1) \boxtimes (X_2, p_2),\\ &\text{where } X_1 \in \text{Obj}(\mathcal{B}_1), p_1 = p_1^2 \in \text{End}_{\widetilde{\mathcal{B}_{1\mathcal{A}_1}}}(X_1)\\ &X_2 \in \text{Obj}(\mathcal{B}_2), p_2 = p_2^2 \in \text{End}_{\widetilde{\mathcal{B}_{2\mathcal{A}_2}}}(X_2),\end{aligned}$$

$$\begin{aligned}&\text{Hom}_{\mathcal{B}_{1\mathcal{A}_1} \boxtimes \mathcal{B}_{2\mathcal{A}_2}}\Big((X_1, p_1) \boxtimes (X_2, p_2), (Y_1, q_1) \boxtimes (Y_2, q_2)\Big)\\ &= \Big\{f_1 \boxtimes f_2 \in \text{Hom}_{\widetilde{\mathcal{B}_{1\mathcal{A}_1}}}(X_1, Y_1) \boxtimes \text{Hom}_{\widetilde{\mathcal{B}_{2\mathcal{A}_2}}}(X_2, Y_2) \mid\\ &\qquad (f_1 \boxtimes f_2) \circ (p_1 \boxtimes p_2) = (f_1 \boxtimes f_2) = (q_1 \boxtimes q_2) \circ (f_1 \boxtimes f_2)\Big\}.\end{aligned}$$

Since all objects and morphisms in the Deligne product $\mathcal{B}_1 \boxtimes \mathcal{B}_2$ are pairs of objects and morphisms from $\mathcal{B}_1$ and $\mathcal{B}_2$, we have

$$\begin{aligned}\text{Obj}(\mathcal{B}_1 \boxtimes \mathcal{B}_{2\mathcal{A}_1 \boxtimes \mathcal{A}_2}) = \text{pairs } &(X_1 \boxtimes X_2, p_1 \boxtimes p_2),\\ &\text{where } X_1 \boxtimes X_2 \in \text{Obj}(\mathcal{B}_1 \boxtimes \mathcal{B}_2),\\ &p_1 \boxtimes p_2 = p_1^2 \boxtimes p_2^2 \in \text{End}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X_1 \boxtimes X_2)\end{aligned}$$

$$\begin{aligned}&\text{Hom}_{\mathcal{B}_1 \boxtimes \mathcal{B}_{2\mathcal{A}_1 \boxtimes \mathcal{A}_2}}\Big((X_1 \boxtimes X_2, p_1 \boxtimes p_2), (Y_1 \boxtimes Y_2, q_1 \boxtimes q_2)\Big)\\ &= \Big\{f_1 \boxtimes f_2 \in \text{Hom}_{\widetilde{\mathcal{B}_{\mathcal{A}}}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \mid\\ &\qquad (f_1 \boxtimes f_2) \circ (p_1 \boxtimes p_2) = (f_1 \boxtimes f_2) = (q_1 \boxtimes q_2) \circ (f_1 \boxtimes f_2)\Big\}.\end{aligned}$$

Now the obvious inverse functors demonstrate the category isomorphism. □

Chapter 4

Computing F- and R-Symbols

Chapter 3 gives us some general results about condensation, but they are mostly limited to the simple case of a self-dual boson. This chapter takes a different approach to understanding condensation. Instead of trying to find modular data alone, we attempt to find an entire set of F- and R-symbols for the condensed category by applying the condensation tensor functor to morphisms.

This chapter consists of three discussions. Section 4.1 gives background and the current state of the literature. Section 4.2 discusses the definition of condensation as a tensor functor and uses it to produce condensed objects and morphisms, including the new associator and braiding. Section 4.3 provides an explanation of how to actually implement Section 4.2 on a computer. The chapter concludes in Section 4.4 with a brief discussion of the choices that were made along the way.

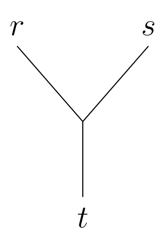
We also provide our full Mathematica implementation of an F- and R-symbol solver in Supplemental A: Mathematica Code and at [1]. This is an exciting new tool for writing down data of new and well-known categories that we do not know explicitly, as well as a proof-of-concept for the feasibility of pinning down the condensed category.

Throughout this chapter, we abuse notation by writing e.g., $(X, \mu_X) \otimes_{\mathcal{A}} (Y, \mu_Y) = (X \otimes_{\mathcal{A}} Y, \mu_{X \otimes_{\mathcal{A}} Y})$. The expression $X \otimes_{\mathcal{A}} Y$ is not meaningful since X, Y are not objects of Rep $\mathcal{A}$, but it is given a definition in diagram 4.5 below.

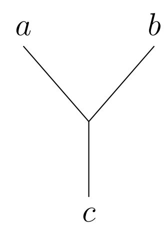
4.1 Graphical Calculus

We discussed the graphical calculus of modular tensor categories in Section 2.2. Given a modular tensor category and a condensable algebra, we would like to understand the condensed category. In particular, we would like to use the graphical calculus we already have for $\mathcal{B}$ after condensation. Part of the difficulty here is that even when we begin with a skeletal category, the condensation functor sends objects that get identified (in the sense of Section 3.4) to isomorphic, but unequal, objects. Producing a graphical calculus for the condensed category is indeed equivalent to skeletalizing the functor in the sense of Section 2.8.2.

This idea appears in the physics literature. Vertex lifting coefficients (VLCs) are introduced by [16] as a way to understand pictures in the graphical calculus of the condensed category as linear combinations of pictures in the graphical calculus of the parent category. VLCs are defined in [16] as the coefficients on the right hand side of the following graphical equation.



$$= \sum_{a,b,c} \left[\begin{matrix} r & s \\ a & b \end{matrix} \middle| \begin{matrix} t \\ c \end{matrix}\right]$$



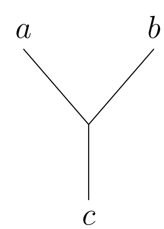
(4.1)

Notice that we are utilizing the graphical language of Section 2.2.1, which is a slight deviation from our conventions in Chapter 3. We often draw tensor products as parallel lines and avoid drawing direct sums of diagrams. In this case, we wish to understand the new tensor product of $\mathcal{A}$-modules in terms of the old tensor product of objects, so the vertices in Equation 4.1 are explicit choices of morphisms $t \to r \otimes s$ and $c \to a \otimes b$. We will discuss this more shortly.

Once all VLCs are determined, any picture of condensed tensor products can be written as a large linear combination of parent tensor products. Notice that VLCs are specifically a way to understand the condensed tensor product and any pictures composed only of tensor products. The framework we build in this chapter understands induced morphisms more generally.

In [16], VLCs are computed by writing down many relationships that would be true in a category where the object $\mathcal{A}$ were the tensor unit. These equations must then be solved simultaneously, a process that experience suggests can be quite difficult. One might suspect that this solving should not be necessary since the definitions of $\operatorname{Rep}\mathcal{A}$ and the condensation tensor functor in [30] directly define the morphisms of $\operatorname{Rep}\mathcal{A}$. The approach outlined in this chapter readily gives access to the categorical data of $\operatorname{Rep}\mathcal{A}$ without the need to simultaneously solve many equations.

Let us now briefly digress to understand VLCs in a categorical context. While the equality in equation 4.1 can be understood as an equality since the morphisms of $\operatorname{Rep}\mathcal{A}$ are formally subsets of morphisms in $\mathcal{B}$, it may be helpful to recognize that the two sides are morphisms in different categories. On the right, we have pictures of the form

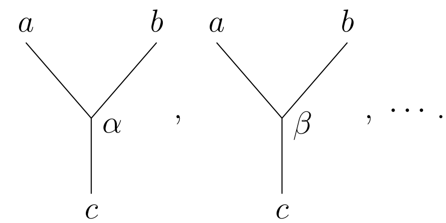


. (4.2)

This is an explicit choice of morphism $B_c^{ab}\colon c \to a \otimes b$ for some c with nonzero N_c^{ab} (if $N_c^{ab} = 0$, then B_c^{ab} is the zero morphism, and we do not bother including it in the sum of Equation 4.1). Recall that

$$\operatorname{Hom}(c, a \otimes b) \cong \mathbb{C}^{N_c^{ab}} .$$

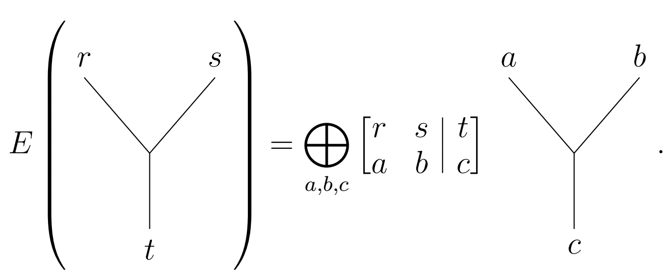
Picture 4.2 establishes a choice of basis vector for $\operatorname{Hom}(c, a\otimes b)$ in the parent category $\mathcal{B}$, which all future pictures will be drawn in terms of. Since [16] assumes multiplicity-free categories, N_c^{ab} can be only 0 or 1. In this case, Picture 4.2 provides a full basis for $\mathrm{Hom}(c, a \otimes b)$. When $\mathrm{Hom}(c, a \otimes b)$ has dimension $n > 1$, we can label a basis of n different morphisms $c \to a \otimes b$



On the left of Equation 4.1, we have a tensor product in the condensed category. This is again a choice of basis vector, but we can no longer choose arbitrarily because we would like these pictures to be consistent with the ones we draw for the parent category $\mathcal{B}$. This requires some minimal amount of solving to make sure the morphisms we choose satisfy condition 3.3 of Rep $\mathcal{A}$. We will typically try to choose these morphisms to maximize the number of 1 coefficients.

For the sake of clarity, note that the sum of Equation 4.1 is slightly suspect. Within each choice of a, b, c, we may take sums of morphisms in $\mathrm{Hom}(c, a \otimes b)$. If $T(a), T(a')$ both contain x, we must consider an element from the direct sum of two separate vector spaces, $\mathrm{Hom}(c, a \otimes b)$ and $\mathrm{Hom}(c, a' \otimes b)$. This distinction is unimportant because in the direct sum of these vector spaces, addition of vectors from different components is a direct sum.

Putting everything together, we can think of VLCs as coefficients determining the condensation adjoint functor (Remark 3.11) on tensor product morphisms. That is,



Since the adjoint to condensation is the forgetful functor, we can get away with being loose about using the E functor. To understand how to compute these coefficients, we must talk about the categorical definition of condensation in the framework of Section 2.8.2

4.2 Condensation as a Functor

From the description of Rep $\mathcal{A}$ (Definition 3.4) given by [30], we are able to work with the condensed category quite concretely. Given a modular tensor category $\mathcal{B}$ and a condensable algebra $\mathcal{A}$ with multiplication m, condensation is a tensor functor defined by

$$\begin{aligned} T\colon\ \mathcal{B} &\to \operatorname{Rep}\mathcal{A} \\ X &\mapsto (\mathcal{A}\otimes X, m\otimes \mathrm{id}_X) \end{aligned} \tag{4.3}$$

The problem with working entirely with this functor is that T provides a non-skeletal description of Rep $\mathcal{A}$.

Example 4.1. Consider the Ising $\boxtimes$ Ising $\to \mathbb{Z}_4$ condensation of Section 3.3.1. Acting by functor 4.3 on the objects 11 and $\psi\psi$ gives us

$$T(11) = (11 \oplus \psi\psi, m \otimes \mathrm{id}_{11}),$$

$$T(\psi\psi) = (11 \oplus \psi\psi, m \otimes \mathrm{id}_{\psi\psi}).$$

These objects are not equal, but they are isomorphic.

This non-skeletal description presents a challenge since the morphisms of the new category are determined by data that is finer than just isomorphism classes of simple objects in the parent category. We wish then to consider a skeletal description of condensation that allows us to use the old graphical calculus.

Consider a modular tensor category $\mathcal{B}$ with a condensable algebra $(\mathcal{A}, m, \eta)$. Let (X, μ_X) and (Y, μ_Y) be objects of Rep $\mathcal{A}$. To understand the condensed tensor product, we observe that there are two obvious module structures on $\mu_1, \mu_2\colon A\otimes X\otimes Y \to X\otimes Y$.

$$\mu_1 = \mu_X \otimes \mathrm{id}_Y \qquad\qquad \mu_2 = (\mathrm{id}_X \otimes \mu_Y) \circ c^{-1}_{X,\mathcal{A}} \tag{4.4}$$

Then the condensed tensor product $(X, \mu_X) \otimes_{\mathcal{A}} (Y, \mu_Y)$ is the cokernel of $\mu_1 - \mu_2$, as in the following diagram.

$$\begin{array}{ccc} \mathcal{A} \otimes X \otimes Y & \xrightarrow{\mu_1 - \mu_2} & X \otimes Y \\ & & \big\downarrow q \\ & & X \otimes_{\mathcal{A}} Y = (X \otimes Y)/\operatorname{im}(\mu_1 - \mu_2) \end{array} \tag{4.5}$$

The morphism $\mu_{X \otimes_{\mathcal{A}} Y}$ is either μ_1 or μ_2 on the quotient $\mathcal{A} \otimes X \otimes_{\mathcal{A}} Y$ (since the two are now equivalent). In an abelian category, the cokernel of $\mu_1 - \mu_2$ is the coequalizer of μ_1 and μ_2, so we may equivalently consider the following diagram.

$$\begin{array}{ccc} \mathcal{A} \otimes X \otimes Y & \overset{\mu_1}{\underset{\mu_2}{\rightrightarrows}} & X \otimes Y \\ & & \big\downarrow q \\ & & X \otimes_{\mathcal{A}} Y \end{array}$$

From here, we would like also to understand the new tensor product on morphisms. Consider morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(V, V')$ and $g \in \mathrm{Hom}_{\mathcal{A}}(W, W')$, which must satisfy

$$f \circ \mu_V = \mu_{V'} \circ (\mathrm{id}_{\mathcal{A}} \otimes f),$$

$$g \circ \mu_W = \mu_{W'} \circ (\mathrm{id}_{\mathcal{A}} \otimes g).$$

To define $f \otimes_{\mathcal{A}} g$, we consider the diagram

$$\begin{array}{ccccc} \mathcal{A} \otimes V \otimes W & \overset{\mu_1}{\underset{\mu_2}{\rightrightarrows}} & V \otimes W & \xrightarrow{f \otimes g} & V' \otimes W' \\ & & \big\downarrow q & & \big\downarrow q' \\ & & V \otimes_{\mathcal{A}} W & & V' \otimes_{\mathcal{A}} W' \end{array}$$

Note that

$$(f \otimes g) \circ \mu_1 = (f \otimes g) \circ (\mu_V \otimes \mathrm{id}_W)$$

$$
\begin{aligned}
&= (\mu_{V'} \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f \otimes g), \\
(f \otimes g) \circ \mu_2 &= (f \otimes g) \circ (\mathrm{id}_V \otimes \mu_W) \circ (c_{\mathcal{A},V} \otimes \mathrm{id}_W) \\
&= (\mathrm{id}_{V'} \otimes \mu_{W'}) \circ (c_{\mathcal{A},V'} \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f \otimes g).
\end{aligned}
$$

Then $q' \circ (f \otimes g) \circ \mu_1 = q' \circ (f \otimes g) \circ \mu_2$ since q' is the coequalizer of μ_1' and μ_2'. By the universal property of coequalizers, there exists a unique map $f \otimes_{\mathcal{A}} g$ which makes the square of the following diagram commute.

$$
\begin{array}{ccccc}
\mathcal{A} \otimes V \otimes W & \underset{\mu_2}{\overset{\mu_1}{\rightrightarrows}} & V \otimes W & \xrightarrow{f \otimes g} & V' \otimes W' \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\
 & & V \otimes_{\mathcal{A}} W & \underset{f \otimes_{\mathcal{A}} g}{\dashrightarrow} & V' \otimes_{\mathcal{A}} W'
\end{array}
\tag{4.6}
$$

This defines the condensed tensor product of f and g.

Motivated by this, we also find the condensed associator and braiding according to the following diagrams.

$$
\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\alpha} & X \otimes (Y \otimes Z) \\
{\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_1'} \\
(X \otimes_{\mathcal{A}} Y) \otimes Z & & X \otimes (Y \otimes_{\mathcal{A}} Z) \\
{\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle q_2'} \\
(X \otimes_{\mathcal{A}} Y) \otimes_{\mathcal{A}} Z & \underset{\alpha_{\mathcal{A}}}{\dashrightarrow} & X \otimes_{\mathcal{A}} (Y \otimes_{\mathcal{A}} Z)
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y & \xrightarrow{c} & Y \otimes X \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\
X \otimes_{\mathcal{A}} Y & \underset{c_{\mathcal{A}}}{\dashrightarrow} & Y \otimes_{\mathcal{A}} X
\end{array}
\tag{4.7}
$$

4.3 Implementation

In this section, we make the previous theory more concrete by describing how it might actually be implemented. Our Mathematica code is included in Supplemental A: Mathematica Code and (at the time of writing) can be accessed at [1]. Refer to Section 2.5 for a brief explanation of how we work with modular tensor categories in Mathematica.

To begin determining the condensed category, we first find its simple objects. The category $\mathcal{B}_A$ of Definition 3.7 makes this fairly straightforward. Once the simple objects of $\mathcal{B}_{\mathcal{A}}$ have been found, they can be converted to simple objects of $\operatorname{Rep} \mathcal{A}$ by way of functor 4.3. Nonsplitting objects can be mapped directly by functor 4.3, and the possible module structures on splitting objects can be found.

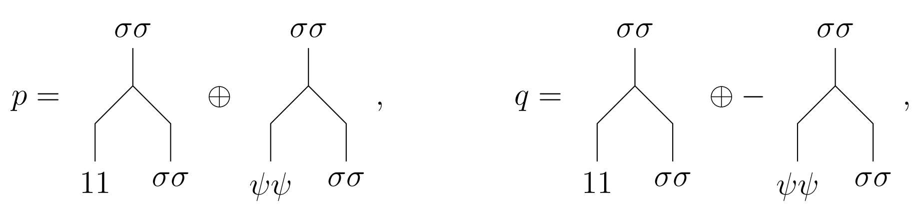
Example 4.2. In Section 3.3.1, we demonstrated the condensation of $11 \oplus \psi\psi$ in Ising $\boxtimes$ Ising. This gave us new simple objects $(11, \mathrm{id}), (1\psi, \mathrm{id}), (\sigma\sigma, p), (\sigma\sigma, q), (1\sigma, \mathrm{id}), (\sigma 1, \mathrm{id})$, with the first four objects forming the $\mathbb{Z}_4$ MTC.

Then the condensation functor 4.3 gives us nonsplitting simple object isomorphism classes

$$11 \mapsto (11 \oplus \psi\psi, m \otimes \mathrm{id}_{11}), \qquad 1\psi \mapsto (1\psi \oplus \psi 1, m \otimes \mathrm{id}_{1\psi}),$$

$$1\sigma \mapsto (1\sigma \oplus \psi\sigma, m \otimes \mathrm{id}_{1\sigma}), \qquad \sigma 1 \mapsto (\sigma 1 \oplus \sigma\psi, m \otimes \mathrm{id}_{\sigma 1}).$$

Directly solving for (nonzero) morphisms $\mathcal{A} \otimes \sigma\sigma \to \sigma\sigma$ gives us the two options



where the trivalent vertices are the tensor product morphisms chosen in $\mathcal{B}$. So the condensation functor 4.3 maps

$$\sigma\sigma \mapsto (\sigma\sigma \oplus \sigma\sigma, m \otimes \mathrm{id}_{\sigma\sigma}),$$

which decomposes as simple objects

$$(\sigma\sigma \oplus \sigma\sigma, m \otimes \mathrm{id}_{\sigma\sigma}) \cong (\sigma\sigma, p) \oplus (\sigma\sigma, q).$$

Having found the new simple objects, we must now figure out how they behave. Since we wish to give a skeletal description of the category, we fix representatives of the simple object isomorphism classes. Let us first look at computing the new tensor product on objects.

Construction 4.3. Consider a skeletal modular tensor category $\mathcal{B}$ with label set $\{X_i\}$

and condensable algebra $\mathcal{A}$. Given objects $X = \bigoplus_i m_i X_i$ and $Y = \bigoplus_i n_i X_i$, a morphism $f\colon X \to Y$ can be represented as a graded linear map $\bigoplus_i f_i\colon \mathbb{C}^{m_i} \to \mathbb{C}^{n_i}$. The morphism $q = \bigoplus_i q_i$ from diagram 4.5 can be computed explicitly in this form.

The object $X \otimes Y = \bigoplus_i \ell_i X_i$ is given by the fusion rules of $\mathcal{B}$. We may also construct graded linear maps representing μ_1, μ_2, and $\mu_1 - \mu_2$ from the definitions 4.4. Set $\mu_1 - \mu_2 = \bigoplus_i f_i$. Then,

$$X \otimes_{\mathcal{A}} Y = \bigoplus_i (\ell_i - \operatorname{rank}(f_i)) X_i, \qquad \mu_{X \otimes_{\mathcal{A}} Y} = q \circ \mu_1 = q \circ \mu_2.$$

Setting $k_i = \ell_i - \operatorname{rank}(f_i)$, we can find $q\colon \bigoplus_i \ell_i X_i \to \bigoplus_i k_i X_i$ as a graded linear map. For each i, we can choose an orthonormal basis for $\operatorname{im}(f_i)^{\perp}$, the orthogonal complement of $\operatorname{im}(f_i)$. Then $q = \bigoplus_i q_i$ is the projection onto the span of this basis, with each q_i given simply by the conjugates of the basis vectors.

We must now also determine the tensor product on morphisms in $\operatorname{Rep}\mathcal{A}$. For this, we refer to diagram 4.6. Morphisms of $\operatorname{Rep}\mathcal{A}$ are formally a subset of morphisms of $\mathcal{B}$, and diagram 4.6 suggests which composition to choose.

Construction 4.4. Consider a skeletal modular tensor category $\mathcal{B}$ with label set $\{X_i\}$ and condensable algebra $\mathcal{A}$. Given morphisms $f = \bigoplus_i f_i$ and $g = \bigoplus_i g_i$, we first define $f \otimes g$ as a tensor product in $\mathcal{B}$. The discussion in Construction 4.3 allows us to find q, q'. In constructing q for each i, choose an orthonormal basis $\{v_1, \dots, v_{\ell_i - k_i}, w_1, \dots, w_{k_i}\}$ for $\mathbb{C}^{\ell_i}$ so that q is a projection onto $\operatorname{span}\{w_1, \dots, w_{k_i}\}$. Then the quotient q has a natural section $\hat{q}$ which, for each i, is simply the inclusion of $\{w_1, \dots, w_{k_i}\}$ into $\mathbb{C}^{\ell_i}$. Then we define $f \otimes_{\mathcal{A}} g = q' \circ (f \otimes g) \circ \hat{q}$. Now if

$$q' \circ (f \otimes g) \circ \hat{q} \circ q = q' \circ (f \otimes g),$$

then we have found the unique map $f \otimes_{\mathcal{A}} g$ from diagram 4.6.

Claim: $q' \circ (f \otimes g) \circ \hat{q} \circ q = q' \circ (f \otimes g)$.

Proof: Write $q = \bigoplus_i q_i$, $q' = \bigoplus_i q'_i$, and $f \otimes g = \bigoplus_i (f \otimes g)_i$. Fix an index i and an orthonormal basis as in the construction of q. Then $\hat{q} \circ q$ is zero on the v_j and the identity on the w_j. The claim follows if $q'_i \circ (f \otimes g)_i$ is zero on the v_j.

By the discussion preceding diagram 4.6, we see that indeed $q' \circ (f \otimes g) \circ (\mu_1 - \mu_2) = 0$ or $q' \circ (f \otimes g)$ is zero on the image of $\mu_1 - \mu_2$. ■

In addition to determining how the new tensor product behaves on objects and morphisms, we must write explicit morphisms for the tensor product $(Z, \mu_Z) \to (X, \mu_X) \otimes_{\mathcal{A}} (Y, \mu_Y)$ in terms of the tensor product morphisms of $\mathcal{B}$. These choices will determine the vertex lifting coefficients of [16].

Suppose $N_Z^{XY} \neq 0$. Then $\dim(\mathrm{Hom}(Z, X \otimes Y)) \neq 0$, and a morphism can be chosen for the tensor product. If $\mathcal{B}$ is multiplicity free, then $N_Z^{XY} = 1$, and we identify the morphism with $1 \in \mathbb{C}$. That is, the chosen tensor product morphism becomes our chosen basis vector for $\mathrm{Hom}(Z, X \otimes Y)$. In general, we can identify the N_Z^{XY} standard basis vectors with the tensor product morphisms from distinguished copies of Z in $X \otimes Y$. In $\mathrm{Rep}\,\mathcal{A}$, we do not have the same freedom to choose whatever convenient morphisms we like since the tensor product of $\mathcal{B}$ has already been fixed.

Construction 4.5. Consider a skeletal modular tensor category $\mathcal{B}$ with condensable algebra $\mathcal{A}$. For a morphism $(Z, \mu_Z) \to (X, \mu_X) \otimes_{\mathcal{A}} (Y, \mu_Y)$ in $\mathrm{Rep}\,\mathcal{A}$, we are only free to choose convenient coefficients for $f \colon Z \to X \otimes_{\mathcal{A}} Y$ up to the constraint $f \circ \mu_Z = \mu_{X \otimes_{\mathcal{A}} Y} \circ (\mathrm{id}_{\mathcal{A}} \otimes f)$. We fix such a choice for each triple $(Z, \mu_Z), (X, \mu_X), (Y, \mu_Y)$ and use these choices in all future computations.

All we need now are a few more induced morphisms that run much like Construction 4.4. We need to know how to define the tensor isomorphism of the condensation tensor functor, the associativity of the new category, and the braiding of the new category.

Recall from Definition 2.100 that a tensor functor T is equipped with an isomorphism $J \colon T(-) \otimes T(-) \to T(- \otimes -)$. The condensation tensor functor of [30] defines a morphism

$f\colon T(a)\otimes_{\mathcal{A}} T(b)\to T(a\otimes b)$ by

$$f\colon (\mathcal{A}\otimes a)\otimes_{\mathcal{A}}(\mathcal{A}\otimes b)\xrightarrow{c^{-1}_{\mathcal{A}a}}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}\otimes a\otimes b\xrightarrow{m}\mathcal{A}\otimes a\otimes b.$$

This definition is somewhat opaque since the objects being acted on are not always in the correct category. We build this function as follows.

Construction 4.6. Let $T\colon\mathcal{B}\to\operatorname{Rep}\mathcal{A}$ be the condensation tensor functor. The J isomorphism is the induced morphism f defined by the following diagram.

$$\begin{array}{ccccc}
(\mathcal{A}\otimes a)\otimes(\mathcal{A}\otimes b) & \xrightarrow{\alpha_{\mathcal{A},a,\mathcal{A}}\circ\alpha^{-1}_{\mathcal{A}\otimes a,\mathcal{A},b}} & (\mathcal{A}\otimes(a\otimes\mathcal{A}))\otimes b & \xrightarrow{\alpha_{\mathcal{A}\otimes\mathcal{A},a,b}\circ\alpha^{-1}_{\mathcal{A},\mathcal{A},a}\circ c^{-1}_{\mathcal{A},a}} & (\mathcal{A}\otimes\mathcal{A})\otimes(a\otimes b)\\
\Big\downarrow q & & & & \Big\downarrow m\\
(\mathcal{A}\otimes a)\otimes_{\mathcal{A}}(\mathcal{A}\otimes b) & & \xrightarrow{\qquad\qquad f\qquad\qquad} & & \mathcal{A}\otimes(a\otimes b)
\end{array}$$

In the style of Construction 4.4, we now define

$$f=(m\otimes\mathrm{id}_{a\otimes b})\circ\alpha_{\mathcal{A}\otimes\mathcal{A},a,b}\circ(\alpha^{-1}_{\mathcal{A},\mathcal{A},a}\otimes\mathrm{id}_b)\circ(\mathrm{id}_{\mathcal{A}}\otimes c^{-1}_{\mathcal{A},a}\otimes\mathrm{id}_b)\circ(\alpha_{\mathcal{A},a,\mathcal{A}}\otimes\mathrm{id}_b)\circ\alpha^{-1}_{\mathcal{A}\otimes a,\mathcal{A},b}\circ\hat{q}.$$

Finally, we would like a new associator and braiding.

Construction 4.7. This time, we apply the process of Construction 4.4 to diagram 4.7. Then

$$\alpha_{\mathcal{A}}=q_2'\circ q_1'\circ\alpha\circ\hat{q}_1\circ\hat{q}_2.$$

Construction 4.8. Now apply the process of Construction 4.4 to diagram 4.7 to get

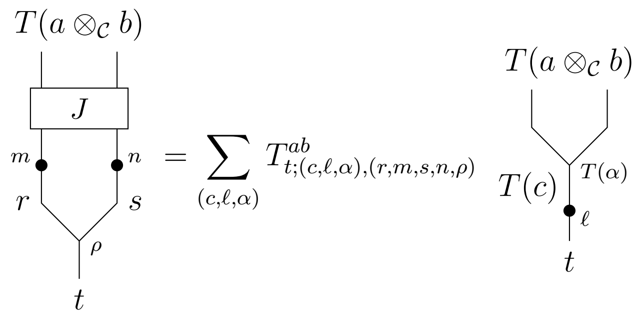
$$c_{\mathcal{A}}=q'\circ c\circ\hat{q}.$$

With all of these building blocks, we have a great deal of power in what we can write down about the condensed category. We can compute the T-symbols of the skeletalized condensation tensor functor, which are equivalent to the VLCs of [16]. We can also compute the F- and R-symbols of the condensed theory.

4.3.1 T-symbols and VLCs

We would like to determine the VLCs of [16]. However, the condensation tensor functor does not directly relate old objects to new objects. Instead, we can compare old vertices to new vertices using the skeletal description of a tensor functor.

Recall from Section 2.8.2 that a skeletal tensor functor may be defined by T-symbols as follows.



(4.8)

Here $J\colon T(-)\otimes T(-)\to T(-\otimes -)$ is the functorial isomorphism we get with any tensor functor, as in Definition 2.100.

Consider the case where the tensor functor is the condensation functor of [30]. Then we have a tensor product $t\to r\otimes_{\mathcal{A}} s$ in $\operatorname{Rep}\mathcal{A}$ on the left of Equation 4.8 and a tensor product $c\to a\otimes b$ in $\mathcal{B}$ on the right. The J isomorphism is what [30] presents as

$$f\colon (\mathcal{A}\otimes a)\otimes_{\mathcal{A}}(\mathcal{A}\otimes b)\xrightarrow{c^{-1}_{\mathcal{A}\,a}}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}\otimes a\otimes b\xrightarrow{m}\mathcal{A}\otimes a\otimes b.$$

This composition is made sense of in Construction 4.6.

Now let us see how to compute T-symbols. We first consider the picture on the left. From bottom to top, this is a morphism

$$t\xrightarrow{\quad\rho\quad} r\otimes_{\mathcal{A}} s\xrightarrow{\quad I^a_r(m)\otimes_{\mathcal{A}} I^b_s(n)\quad} T(a)\otimes_{\mathcal{A}} T(b)\xrightarrow{\quad J\quad} T(a\otimes b)\ , \tag{4.9}$$

where $\rho\in\operatorname{Hom}(t,r\otimes_{\mathcal{A}} s)$ is a tensor product morphism as defined in Construction 4.5 and $I^a_r(m)\colon r\to T(a)$ is an embedding whose image is the m-th distinct copy of r in $T(a)$ (see Section 2.8.2).

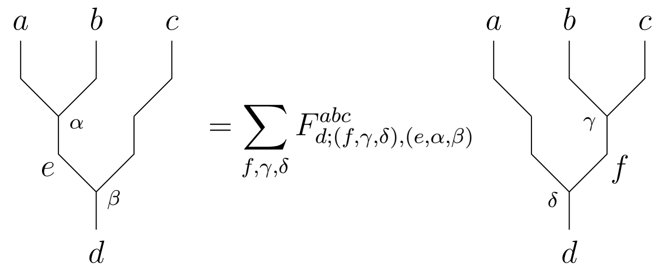
Now let us consider the picture on the right. From bottom to top, this is a morphism

$$t \xrightarrow{I_t^c(\ell)} T(c) \xrightarrow{T(\alpha)} T(a \otimes b) \ . \tag{4.10}$$

By ranging over the choices of $(r, m, s, n, \rho), (c, \ell, \alpha)$, morphisms 4.9 and 4.10 give two different bases for $\mathrm{Hom}_{\mathcal{A}}(t, T(a \otimes b))$. Then T_t^{ab} is the change of basis matrix from the first basis to the second. Writing Equation 4.8 in the form of Equation 2.18 and forgetting the embedding dots gives us coefficients as defined by the original Equation 4.1.

4.3.2 F-symbols

Recall the following picture defining F-symbols.



In the condensed category, the picture on the left is a morphism

$$d \xrightarrow{\beta} e \otimes_{\mathcal{A}} c \xrightarrow{\alpha \otimes_{\mathcal{A}} \mathrm{id}_c} (a \otimes_{\mathcal{A}} b) \otimes_{\mathcal{A}} c \ ,$$

where α, β are chosen according to Construction 4.5. Composing with the associator $\alpha_{\mathcal{A}}$ from Construction 4.7 gives a morphism

$$d \xrightarrow{\beta} e \otimes_{\mathcal{A}} c \xrightarrow{\alpha \otimes_{\mathcal{A}} \mathrm{id}_c} (a \otimes_{\mathcal{A}} b) \otimes_{\mathcal{A}} c \xrightarrow{\alpha_{\mathcal{A}}} a \otimes_{\mathcal{A}} (b \otimes_{\mathcal{A}} c) \ . \tag{4.11}$$

The picture on the right is a morphism

$$d \xrightarrow{\delta} a \otimes_{\mathcal{A}} f \xrightarrow{\mathrm{id}_a \otimes_{\mathcal{A}} \gamma} a \otimes_{\mathcal{A}} (b \otimes_{\mathcal{A}} c) \ , \tag{4.12}$$

where again γ, δ are chosen according to Construction 4.5. By ranging over the choices of $(e, \alpha, \beta), (f, \gamma, \delta)$, morphisms 4.11 and 4.12 give two different bases for $\mathrm{Hom}_{\mathcal{A}}(d, a \otimes_{\mathcal{A}}$

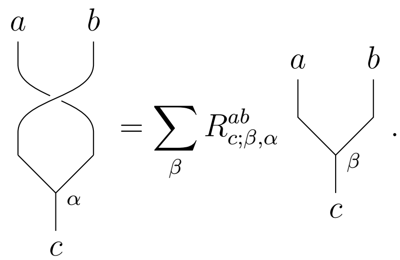
$(b \otimes_{\mathcal{A}} c))$. Then F_d^{abc} is the change of basis matrix from the first basis to the second.

In the case of a multiplicity-free condensed category, there is only a single choice for each $\alpha, \beta, \gamma, \delta$, and our F-matrices reduce to $F_{d;f,e}^{abc}$.

Refer to Supplemental Section A.2.2 for an implementation of this computation.

4.3.3 R-symbols

Recall the following picture defining R-symbols.



In the condensed category, the picture on the left is a morphism

$$c \xrightarrow{\alpha} b \otimes_{\mathcal{A}} a \xrightarrow{c_{\mathcal{A}b,a}} a \otimes_{\mathcal{A}} b \ , \tag{4.13}$$

and the picture on the right is a morphism

$$c \xrightarrow{\beta} a \otimes_{\mathcal{A}} b \ . \tag{4.14}$$

Here α, β are chosen according to Construction 4.5, and $c_{\mathcal{A}}$ comes from Construction 4.8. By ranging over the choices of α, β, morphisms 4.13 and 4.14 give two different bases for $\mathrm{Hom}_{\mathcal{A}}(c, a \otimes_{\mathcal{A}} b)$. Then R_c^{ab} is the change of basis matrix from the first basis to the second.

In the case of a multiplicity-free condensed category, $\alpha = \beta$ and R_c^{ab} is a single number determined by diagram 4.7.

Refer to Supplemental Section A.2.3 for an implementation of this computation.

4.4 Gauge Freedom

Recall from Chapter 2 that modular tensor categories are categories along with choices of various structures (associativity, braiding, pivotal, etc). Obviously categories are equal if all of their structures are equal. However, it is possible for categories with different structures to be equivalent. When presenting the F- and R-symbols of a modular tensor category, we are choosing which, of the many that give equivalent categories, to present. In computing condensation data, the new F- and R-symbols are determined by choices we make along the way.

Perhaps most obviously, we made a choice of basis in Construction 4.5. Note that our approach here is different from that of [16]. In [16], equations of trivalent vertices are used to solve for the vertices. In our approach, the definitions of the trivalent vertices are inconsequential because the following computations are done with respect to whichever bases were chosen for the vertices. Indeed, the selection of an associativity and braiding isomorphism for the new category is independent of choices of morphisms for the vertices. The T-, F- and R-symbols are then given in terms of these choices.

Let us return to where the associativity and braiding of the condensed category are defined. Recall that in the discussion around Diagram 4.6, the morphisms f, g are morphisms of $\operatorname{Rep} \mathcal{A}$. Their intertwiner properties allow for a universal property argument that there exists a unique induced morphism $f \otimes_{\mathcal{A}} g$. Since the associator and braiding are not necessarily morphisms of $\operatorname{Rep} \mathcal{A}$, the same argument fails, and there is not necessarily a natural choice for the morphisms $\alpha_{\mathcal{A}}, c_{\mathcal{A}}$ in Diagrams 4.7.

In Construction 4.7 and Construction 4.8, the condensed associator and braiding are selected as compositions involving the section $\hat{q}$ of the quotient morphism chosen in Construction 4.4. While different choices of $\hat{q}$ could give category data that define equivalent categories, it may be that some choices of $\hat{q}$ do not. This question has not yet been fully investigated. A question that might be of interest is whether modular isotopes of the condensed category can be produced from different choices of $\hat{q}$.

Chapter 5

Applications

There is no shortage of interesting usage for condensation. We have already seen it as a physical process on topological phases of matter (Section 3.1.3.1) and as an equivalence relation for braided fusion categories (Section 3.1.3.2).

In this chapter, we provide applications for our work that are quite novel. We begin with the least novel but perhaps most exciting application. Section 5.1 applies some of our general results from Section 3.4 in providing a connection to topological quantum computing.

We then apply our computational work from Chapter 4 to the study of near-group categories in Section 5.2. We see that condensation may be a way to access category data that would otherwise be computationally out of reach.

Finally, Section 5.3 provides the opposite of an application, a co-application if you will. An interesting and surprising scheme is laid out for producing condensable algebras from classical error-correcting codes. This makes condensation an unexpected application of error-correcting codes. We hope this condensation can, in turn, be applied to the problem of generalizing the phenomenon of monstrous moonshine.

5.1 Quantum Computing with Conformal Field Theories

In Section 2.4.2, we mentioned naturally fault tolerant quantum computation as a motivation for studying modular tensor categories. In [34], a model for universal quantum computation is developed from the minimal model conformal field theories. The minimal models are realized as a coset of $SU(2)_k$ theories, and braiding universality (see Section 2.4.2) is demonstrated. Braid group representations are calculated and shown to agree with monodromy representations from braiding conformal blocks. Refer to [34] for the complete story.

5.1.1 Minimal Models from Condensation

Refer to Section 3.3.3 for a first example of this condensation. We begin this section with a refresher on the data of the minimal models from [15].

The minimal model $\mathcal{M}(p,q)$ has primary fields $N_{m,n}$ with $m = 1, \ldots, q-1$ and $n = 1, \ldots, p-1$ and fusion rules

$$N_{r,s} \otimes N_{m,n} = \sum_{k \overset{2}{=} |m-r|+1}^{\min(m+r-1, 2q-1-m-r)} \sum_{l \overset{2}{=} |n-s|+1}^{\min(n+s-1, 2p-1-n-s)} N_{k,l},$$

where $\overset{2}{=}$ denotes incrementing the summation variables k and l by 2. The primary field $N_{m,n}$ has conformal dimension

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}.$$

If we reindex from 0, then the minimal model $\mathcal{M}(p,q)$ has primary fields $N_{m,n}$ with $m = 0, \ldots, q-2$ and $n = 0, \ldots, p-2$. The fusion rules and conformal dimensions are given by

$$N_{r,s} \otimes N_{m,n} = \sum_{k \overset{2}{=} |m-r|}^{\min(m+r, 2(q-2)-m-r)} \sum_{l \overset{2}{=} |n-s|}^{\min(n+s, 2(p-2)-n-s)} N_{k,l},$$

$$h_{r,s}(m) = \frac{\left[(m+1)(r+1) - m(s+1)\right]^2 - 1}{4m(m+1)}.$$

Finally, we will be interested in the minimal model $\mathcal{M}(k+3, k+2)$, which has primary fields $N_{m,n}$ with $m = 0, \ldots, k$ and $n = 0, \ldots, k+1$ with fusion rules and dimensions

$$N_{r,s} \otimes N_{m,n} = \sum_{k \overset{2}{=} |m-r|}^{\min(m+r, 2k-m-r)} \sum_{l \overset{2}{=} |n-s|}^{\min(n+s, 2(k+1)-n-s)} N_{k,l}, \tag{5.1}$$

$$h_{r,s}(m) = \frac{\left[(k+3)(r+1) - (k+2)(s+1)\right]^2 - 1}{4(k+2)(k+3)}. \tag{5.2}$$

We also review the data of $SU(2)_k$ from [4]. The MTC $SU(2)_k$ has simple objects $0, \frac{1}{2}, \ldots, \frac{k}{2}$ with fusion rules

$$j_1 \otimes j_2 = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} j$$

and twist

$$\theta_j = e^{2\pi i \frac{j(j+1)}{k+2}}.$$

Giving the objects integer labels, the MTC $SU(2)_k$ has simple objects $0, 1, \ldots, k$ with fusion rules

$$j_1 \otimes j_2 = \sum_{j \overset{2}{=} |j_1-j_2|}^{\min(j_1+j_2, 2k-j_1-j_2)} j,$$

where $\overset{2}{=}$ denotes incrementing the summation variable j by 2, and twist

$$\theta_j = e^{2\pi i \frac{\frac{j}{2}\left(\frac{j}{2}+1\right)}{k+2}} = e^{\pi i \frac{j(j+2)}{2(k+2)}}.$$

Lemma 5.1. *In $SU(2)_k$, we have $k \otimes s = r$ if and only if $s = k - r$. Moreover, when $s \neq k - r$, the product $k \otimes s$ contains no r term.*

Proof. (⇐) Observe

$$k \otimes (k-r) = \sum_{j \overset{2}{=} |k-(k-r)|}^{\min(k+(k-r),2k-k-(k-r))} j = \sum_{j \overset{2}{=} r}^{\min(2k-r,r)} j = r$$

since $\min(2k-r, r) = r$ for all $r = 0, \ldots, k$.

(⇒) We have

$$k \otimes s = \sum_{j \overset{2}{=} k-s}^{\min(k+s,k-s)} j.$$

If $s < k-r$, then $k-s > r$, and there is no r term in the product $k \otimes s$. If $s > k-r$, then $k-s < r$, so $\min(k+s, k-s) < r$, and there is no r term in the product $k \otimes s$.

□

Proposition 5.2. *If* $\mathcal{B} = SU(2)_k \boxtimes SU(2)_1 \boxtimes \overline{SU(2)_{k+1}}$ *and* $\mathcal{A} = 000 + k1(k+1)$*, then* $\mathcal{B}_{\mathcal{A}} = \mathcal{B}_0 \oplus \mathcal{B}_1$*, where* $\mathcal{B}_0$ *has the same fusion rules as the Minimal Model* $\mathcal{M}(k+3, k+2)$*.*

Proof. (1) We begin by finding simple objects of $\mathcal{B}_{\mathcal{A}}$ and determining which objects of $\mathcal{B}$ get identified. Let X and Y be objects and consider

$$\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(\mathcal{A} \otimes X, Y).$$

Since $\mathcal{A} \otimes X = X + k1(k+1) \otimes X$, we have $\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, Y) \cong \mathbb{C}$ exactly when $Y \cong X$ or $k1(k+1) \otimes X$ contains exactly one (isomorphic) copy of Y. Note that these cases have no overlap since $1 \otimes X \neq X$ for any X in $SU(2)_1$. Lemma 5.1 gives us the condition for the latter case. So, if $X = rst$, then $\mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, Y) \cong \mathbb{C}$ exactly when $Y \cong rst$ or $Y \cong (k-r)(1-s)(k+1-t)$. Note that every object is paired since $SU(2)_1$ has two simple objects, so $\mathcal{B}$ has an even number of simple objects.

(2) Now, to figure out which object classes compose the modular subcategory $\mathcal{B}_0$, we compare θ_{rst} and $\theta_{(k-r)(1-s)(k+1-t)}$. We have

$$\theta_{rst} = e^{\pi i \left(\frac{r(r+2)}{2(k+2)} + \frac{s(s+2)}{2(3)} - \frac{t(t+2)}{2(k+3)} \right)}$$

and

$$\theta_{(k-r)(1-s)(k+1-t)} = e^{\pi i\left(\frac{(k-r)(k-r+2)}{2(k+2)}+\frac{(1-s)(1-s+2)}{2(3)}-\frac{(k+1-t)(k+1-t+2)}{2(k+3)}\right)}.$$

Notice

$$\begin{aligned}(k-r)(k-r+2) &= k(k-r+2)-r(k-r+2)\\ &= k(k+2)-2rk+r(r-2)\\ &= k(k+2)-2r(k+2)+r(r+2)\\ &= (k-2r)(k+2)+r(r+2),\end{aligned}$$

$$\begin{aligned}(1-s)(1-s+2) &= (1-s+2)-s(1-s+2)\\ &= 3-2s+s(s-2)\\ &= 3-2s(1+2)+s(s+2)\\ &= 3(1-2s)+s(s+2),\end{aligned}$$

and

$$\begin{aligned}(k+1-t)(k+1-t+2) &= (k+1)(k+1-t+2)-t(k+1-t+2)\\ &= (k+1)(k+3)-2t(k+1)+t(t-2)\\ &= (k+1)(k+3)-2t(k+1+2)+t(t+2)\\ &= (k+1-2t)(k+3)+t(t+2).\end{aligned}$$

Now,

$$\begin{aligned}\theta_{(k-r)(1-s)(k+1-t)} &= e^{\pi i\left(\frac{(k-2r)(k+2)+r(r+2)}{2(k+2)}+\frac{3(1-2s)+s(s+2)}{2(3)}-\frac{(k+1-2t)(k+3)+t(t+2)}{2(k+3)}\right)}\\ &= \theta_{rst}e^{\pi i\left(\frac{(k-2r)(k+2)}{2(k+2)}+\frac{3(1-2s)}{2(3)}-\frac{(k+1-2t)(k+3)}{2(k+3)}\right)}\\ &= \theta_{rst}e^{\pi i\frac{k-2r+1-2s-k-1+2t}{2}}\\ &= \theta_{rst}e^{(t-r-s)\pi i},\end{aligned}$$

so $\theta_{rst} = \theta_{(k-r)(1-s)(k+1-t)}$ exactly when $t-r-s$, or equivalently, $r+s+t$ is even.

(3) For each $r = 0, \dots, k$ and $t = 0, \dots, k+1$, we may uniquely choose $s = 0$ or $s = 1$ so that $r+s+t$ is even. Then, we may identify $rst \sim N_{r,t}$ in $\mathcal{M}(k+3, k+2)$. We have

$$rst \otimes mnp = \sum_{j \overset{2}{=} |r-m|}^{\min(r+m, 2k-r-m)} \sum_{l \overset{2}{=} |t-p|}^{\min(t+p, 2(k+1)-t-p)} jsl,$$

where s is chosen to make $j+s+l$ even, and

$$N_{r,t} \otimes N_{m,p} = \sum_{j \overset{2}{=} |m-r|}^{\min(m+r, 2k-m-r)} \sum_{l \overset{2}{=} |p-t|}^{\min(p+t, 2(k+1)-p-t)} N_{j,l}.$$

□

Proposition 5.3. *The twists of the objects of $\mathcal{B}_0$ in Proposition 5.2 agree with those of the corresponding objects in the minimal model $\mathcal{M}(k+3, k+2)$.*

Proof. Using equation 5.2, we find

$$\begin{aligned}
\theta_{N_{r,t}} &= e^{2\pi i h_{r,t}(k+2)} \\
&= e^{2\pi i \left(\frac{[(k+2+1)(r+1)-(k+2)(t+1)]^2 - 1}{4(k+2)(k+2+1)} \right)} \\
&= e^{\pi i \left(\frac{(k+2+1)^2(r+1)^2 - 2(k+2+1)(k+2)(r+1)(t+1) + (k+2)^2(t+1)^2 - 1}{2(k+2)(k+2+1)} \right)}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{(k+2+1)^2(r+1)^2 - 2(k+2+1)(k+2)(r+1)(t+1) + (k+2)^2(t+1)^2 - 1}{2(k+2)(k+2+1)} \\
&= \frac{(k+2+1)^2(r+1)^2 - 2(k+2+1)(k+2)(r+1)(t+1) + (k+2)^2(t+1)^2 - 1}{2(k+2)(k+2+1)} \\
&\quad + \frac{r(r+2)(k+2+1) - r(r+2)(k+2+1) + t(t+2)(k+2) - t(t+2)(k+2)}{2(k+2)(k+2+1)} \\
&= \frac{r(r+2)}{2(k+2)} - \frac{t(t+2)}{2(k+3)} + \frac{(k+2+1)(r+1)^2 - r(r+2)}{2(k+2)} \\
&\quad + \frac{(k+2)(t+1)^2 + t(t+2)}{2(k+2+1)} - \frac{1}{2(k+2)(k+2+1)} - \frac{2(r+1)(t+1)}{2}.
\end{aligned}$$

Considering two of the terms separately, we have

$$\frac{(k+2+1)(r+1)^2 - r(r+2)}{2(k+2)} = \frac{(k+2)(r+1)^2 + r^2 + 2r + 1 - r^2 - 2r}{2(k+2)}$$
$$= \frac{(r+1)^2}{2} + \frac{1}{2(k+2)}$$

and

$$\frac{(k+2)(t+1)^2 + t(t+2)}{2(k+2+1)} = \frac{(k+2)(t^2+2t+1) + t^2 + 2t}{2(k+2+1)}$$
$$= \frac{(k+2+1)t^2 + 2(k+2+1)t + k + 2}{2(k+2+1)}$$
$$= \frac{t^2+2t+1}{2} - \frac{1}{2(k+2+1)}.$$

Now,

$$\frac{(k+2+1)(r+1)^2 - r(r+2)}{2(k+2)} + \frac{(k+2)(t+1)^2 + t(t+2)}{2(k+2+1)} - \frac{1}{2(k+2)(k+2+1)} - \frac{2(r+1)(t+1)}{2}$$
$$= \frac{(r+1)^2}{2} + \frac{t^2+2t+1}{2} - \frac{2(r+1)(t+1)}{2} + \frac{1}{2(k+2)} - \frac{1}{2(k+2+1)} - \frac{1}{2(k+2)(k+2+1)}$$
$$= \frac{[(r+1)-(t+1)]^2}{2} + \frac{k+2+1-k-2-1}{2(k+2)(k+2+1)}$$
$$= \frac{(r-t)^2}{2}.$$

Finally, if we have $r - t = 2n$ for an integer n, then we have $s = 0$ and

$$\frac{(r-t)^2}{2} = 2n^2,$$

so

$$\theta_{N_{r,t}} = e^{\pi i\left(\frac{r(r+2)}{2(k+2)} - \frac{t(t+2)}{2(k+3)} + 2n^2\right)} = e^{\pi i\left(\frac{r(r+2)}{2(k+2)} - \frac{t(t+2)}{2(k+3)}\right)} = \theta_{rst}.$$

If we have $r - t = 2n + 1$ for an integer n, then we have $s = 1$ and

$$\frac{(r-t)^2}{2} = 2n^2 + 2n + \frac{1}{2},$$

so

$$\theta_{N_{r,t}} = e^{\pi i\left(\frac{r(r+2)}{2(k+2)} - \frac{t(t+2)}{2(k+3)} + 2(n^2+n) + \frac{1}{2}\right)} = e^{\pi i\left(\frac{r(r+2)}{2(k+2)} + \frac{1}{2} - \frac{t(t+2)}{2(k+3)}\right)} = \theta_{rst}.$$

In both cases, we have $\theta_{N_{r,t}} = \theta_{rst}$, as desired. □

5.1.2 Braid Group Representations

Anyons that are universal for quantum computing by braiding alone (see Section 2.4.2) can be found in the minimal models. Given such anyons, a computation is a braid and can be written as a unitary transformation as an image of a unitary braid group representation. These braid group representations can be computed in the graphical calculus of modular tensor categories. This computation agrees with the monodromy representations in [34] and provides a far simpler way to compute them.

As an example, consider the tricritical Ising model $\mathcal{M}(5,4) \cong \text{Ising} \boxtimes \overline{\text{Fib}}$. It is known that the Fibonacci anyon τ is universal for quantum computing by braiding alone [21]. Then its complex conjugate $\overline{\tau}$ is also universal. Since ψ is a fermion, the anyon $\psi \boxtimes \overline{\tau}$ in $\mathcal{M}(5,4)$ is also universal for quantum computing by braiding alone, and the braid group representation we get from it is equivalent to that from τ up to phases. A universal quantum computation scheme from representations of the braid groups B_3, B_4, and B_6 is developed in [34]. Representations of these three braid groups are computed below.

To understand the braid group representation, select a braid σ and an orthonormal basis of fusion trees labeled by τ anyons. For each basis element, we compose with the braid σ and then use the Fibonacci F-symbols and R-symbols to write the result as a linear combination of the basis itself. Thus each braid is assigned a matrix which is the change-of-basis matrix from a braided basis to the original unbraided one. Computing this matrix for each of the braid group generators allows us to extend to any braid by simple matrix multiplication. For data of the Fibonacci category, see Example 2.86.

Before we begin, we note a few nontrivial identities that simplify the matrices that follow. Let φ be the golden ratio $\frac{1+\sqrt{5}}{2}$. Then

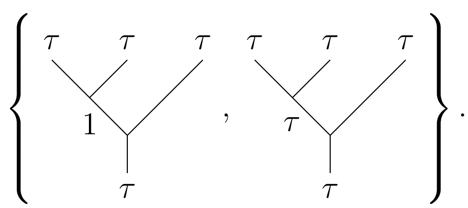
$$\varphi^{-1}e^{-4\pi i/5} + e^{3\pi i/5} = e^{4\pi i/5},$$

$$e^{-4\pi i/5} + \varphi^{-1}e^{3\pi i/5} = -1,$$

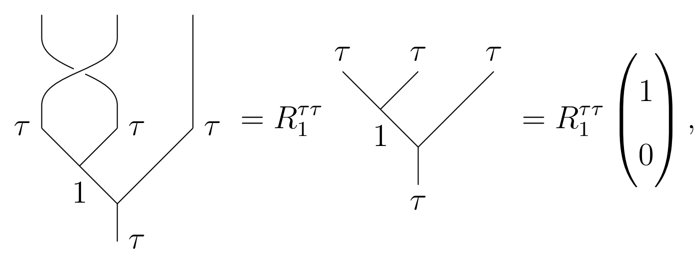
$$\varphi^{-1}e^{-4\pi i/5} - \varphi^{-1}e^{3\pi i/5} = e^{-3\pi i/5}.$$

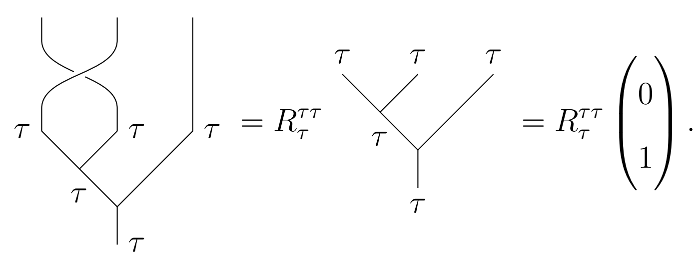
A Representation of the Braid Group B_3

We wish to write the generators $\{\sigma_1, \sigma_2\}$ of B_3 in terms of a basis



To find σ_1, we observe

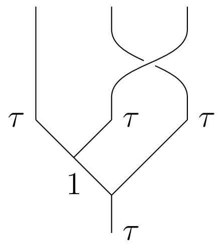


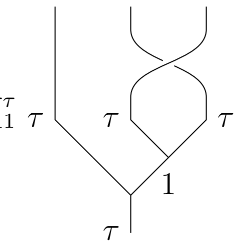


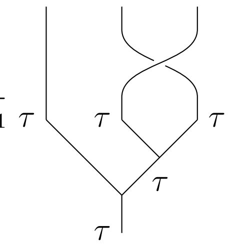
Thus, we have

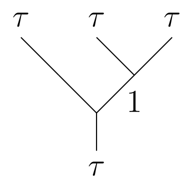
$$\sigma_1 \mapsto \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}.$$

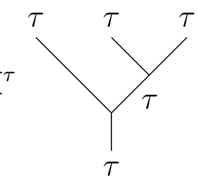
To find σ_2, we observe

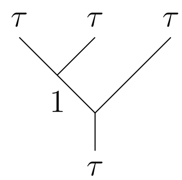


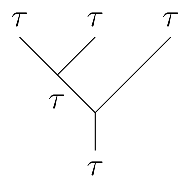
$= F^{\tau\tau\tau}_{\tau;11}$


$+ F^{\tau\tau\tau}_{\tau;\tau 1}$


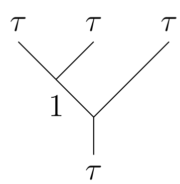
$= F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_1$


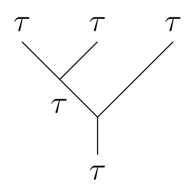
$+ F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_\tau$


$= F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_1 \Bigg((F^{\tau\tau\tau}_\tau)^{-1}_{11}$


$+ (F^{\tau\tau\tau}_\tau)^{-1}_{\tau 1}$


$\Bigg)$

$+ F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_\tau \Bigg((F^{\tau\tau\tau}_\tau)^{-1}_{1\tau}$


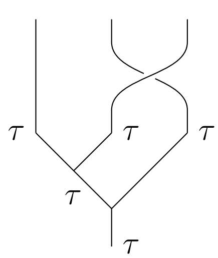
$+ (F^{\tau\tau\tau}_\tau)^{-1}_{\tau\tau}$


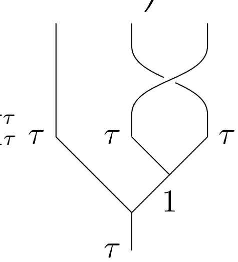
$\Bigg)$

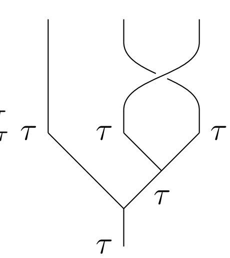
$$= \left(F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_1 \, (F^{\tau\tau\tau}_\tau)^{-1}_{11} + F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_\tau \, (F^{\tau\tau\tau}_\tau)^{-1}_{1\tau} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

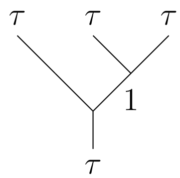
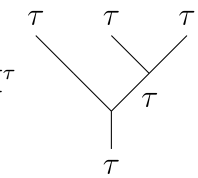
$$+ \left(F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_1 \, (F^{\tau\tau\tau}_\tau)^{-1}_{\tau 1} + F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_\tau \, (F^{\tau\tau\tau}_\tau)^{-1}_{\tau\tau} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

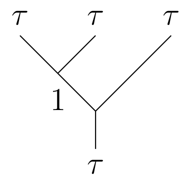
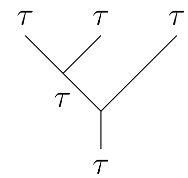
$$= \begin{pmatrix} \varphi^{-1} e^{4\pi i/5} \\ \varphi^{-1/2} e^{-3\pi i/5} \end{pmatrix},$$

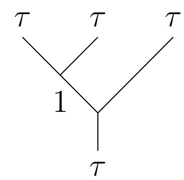
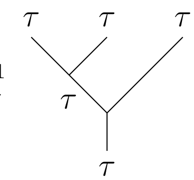


$= F^{\tau\tau\tau}_{\tau;1\tau}$


$+ F^{\tau\tau\tau}_{\tau;\tau\tau}$


$= F^{\tau\tau\tau}_{\tau;1\tau} R^{\tau\tau}_{1}$  $+ F^{\tau\tau\tau}_{\tau;\tau\tau} R^{\tau\tau}_{\tau}$ 

$= F^{\tau\tau\tau}_{\tau;1\tau} R^{\tau\tau}_{1} \Big((F^{\tau\tau\tau}_{\tau})^{-1}_{11}$  $+ (F^{\tau\tau\tau}_{\tau})^{-1}_{\tau 1}$  $\Big)$

$+ F^{\tau\tau\tau}_{\tau;\tau\tau} R^{\tau\tau}_{\tau} \Big((F^{\tau\tau\tau}_{\tau})^{-1}_{1\tau}$  $+ (F^{\tau\tau\tau}_{\tau})^{-1}_{\tau\tau}$  $\Big)$

$$= \left(F^{\tau\tau\tau}_{\tau;1\tau} R^{\tau\tau}_{1} (F^{\tau\tau\tau}_{\tau})^{-1}_{11} + F^{\tau\tau\tau}_{\tau;\tau\tau} R^{\tau\tau}_{\tau} (F^{\tau\tau\tau}_{\tau})^{-1}_{1\tau}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ \left(F^{\tau\tau\tau}_{\tau;1\tau} R^{\tau\tau}_{1} (F^{\tau\tau\tau}_{\tau})^{-1}_{\tau 1} + F^{\tau\tau\tau}_{\tau;\tau\tau} R^{\tau\tau}_{\tau} (F^{\tau\tau\tau}_{\tau})^{-1}_{\tau\tau}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

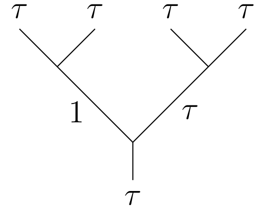
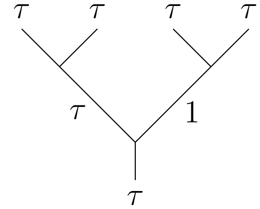
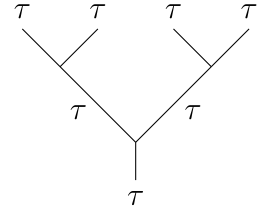
$$= \begin{pmatrix} \varphi^{-1/2} e^{-3\pi i/5} \\ -\varphi^{-1} \end{pmatrix}.$$

Thus, we have

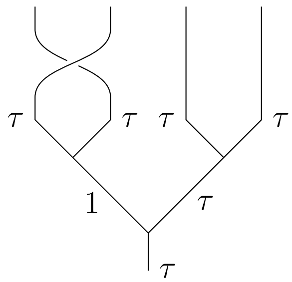
$$\sigma_2 \mapsto \begin{pmatrix} \varphi^{-1} e^{4\pi i/5} & \varphi^{-1/2} e^{-3\pi i/5} \\ \varphi^{-1/2} e^{-3\pi i/5} & -\varphi^{-1} \end{pmatrix}.$$

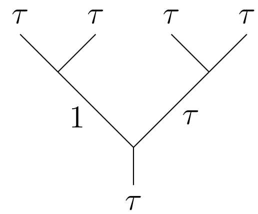
A Representation of the Braid Group B_4

We wish to write the generators $\{\sigma_1, \sigma_2, \sigma_3\}$ of B_4 in terms of the basis

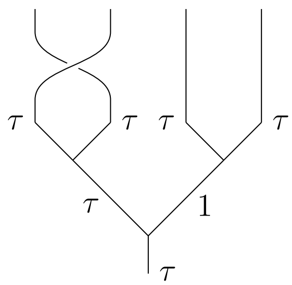
$\Big\{$  ,  ,  $\Big\}$.

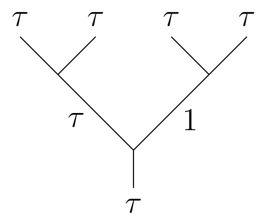
To find σ_1, we observe



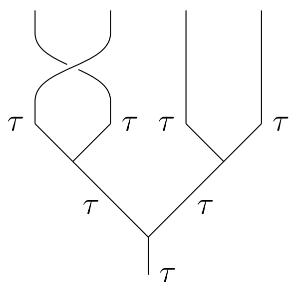
$= R_1^{\tau\tau}$


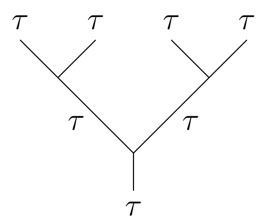
$$= R_1^{\tau\tau}\begin{pmatrix}1\\0\\0\end{pmatrix},$$



$= R_\tau^{\tau\tau}$


$$= R_\tau^{\tau\tau}\begin{pmatrix}0\\1\\0\end{pmatrix},$$



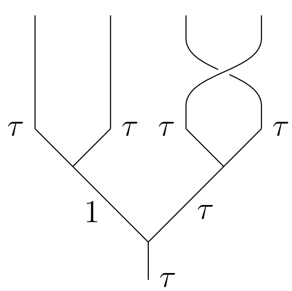
$= R_\tau^{\tau\tau}$


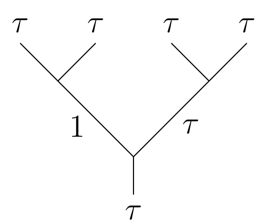
$$= R_\tau^{\tau\tau}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Thus, we have

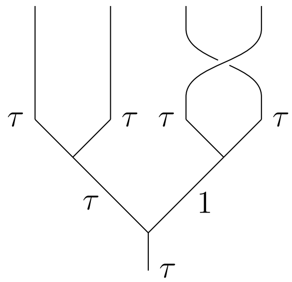
$$\sigma_1 \mapsto \begin{pmatrix} e^{-4\pi i/5} & 0 & 0 \\ 0 & e^{3\pi i/5} & 0 \\ 0 & 0 & e^{3\pi i/5} \end{pmatrix}.$$

Similarly, to find σ_3, we observe

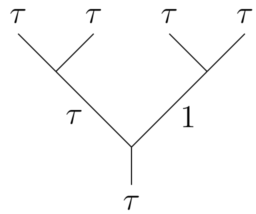


$= R_\tau^{\tau\tau}$


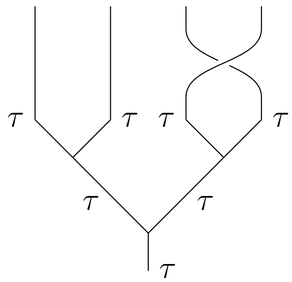
$$= R_\tau^{\tau\tau}\begin{pmatrix}1\\0\\0\end{pmatrix},$$



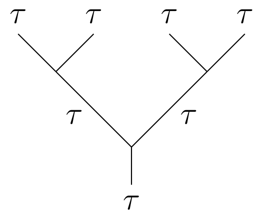
$= R_1^{\tau\tau}$



$$= R_1^{\tau\tau} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$



$= R_\tau^{\tau\tau}$

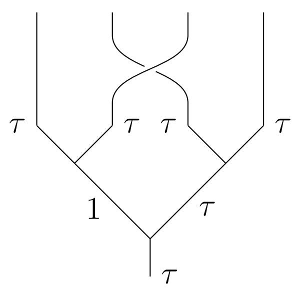


$$= R_\tau^{\tau\tau} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

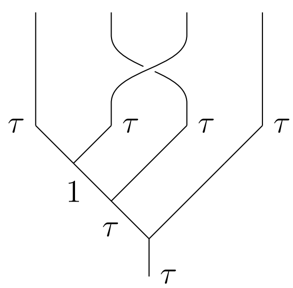
Thus, we have

$$\sigma_3 \mapsto \begin{pmatrix} e^{3\pi i/5} & 0 & 0 \\ 0 & e^{-4\pi i/5} & 0 \\ 0 & 0 & e^{3\pi i/5} \end{pmatrix}.$$

Now an abbreviated version of σ_2:

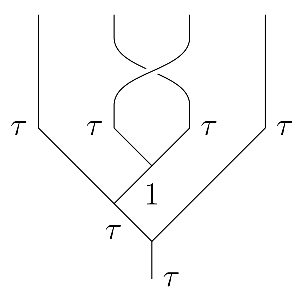


$= F^{1\tau\tau}_{\tau;\tau\tau}$

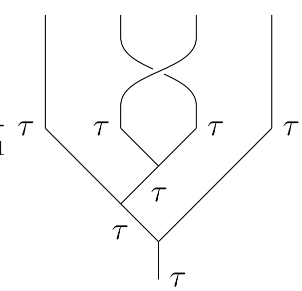


$(F^{1\tau\tau}_{\tau;\tau\tau} = 1)$

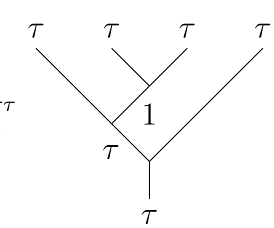
$= F^{\tau\tau\tau}_{\tau;11}$



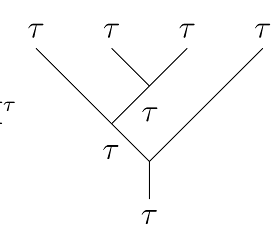
$+ F^{\tau\tau\tau}_{\tau;\tau 1}$



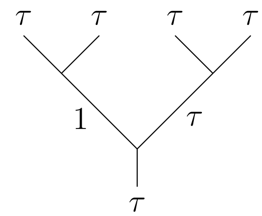
$= F^{\tau\tau\tau}_{\tau;11} R_1^{\tau\tau}$



$+ F^{\tau\tau\tau}_{\tau;\tau 1} R_\tau^{\tau\tau}$

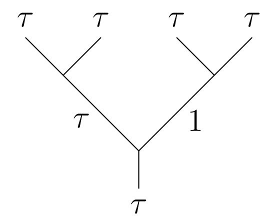


$$= \left(F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_{1} \left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{11} + F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_{\tau} \left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{1\tau}\right)$$

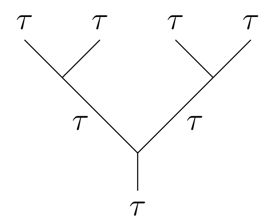


$$+ \left(F^{\tau\tau\tau}_{\tau;11} R^{\tau\tau}_{1} \left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{\tau 1}\right.$$

$$\left. + F^{\tau\tau\tau}_{\tau;\tau 1} R^{\tau\tau}_{\tau} \left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{\tau\tau}\right) \Bigg(\left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{1\tau}$$

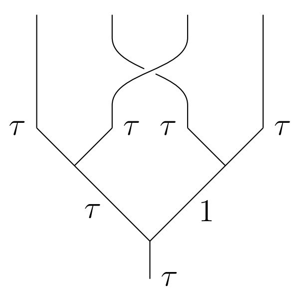


$$+ \left(F^{\tau\tau\tau}_{\tau}\right)^{-1}_{\tau\tau}$$

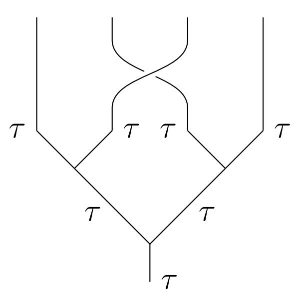


$$\Bigg)$$

$$= \begin{pmatrix} \varphi^{-1} e^{4\pi i/5} \\ \varphi^{-1} e^{-3\pi i/5} \\ -\varphi^{-3/2} e^{-3\pi i/5} \end{pmatrix},$$



$$= \begin{pmatrix} \varphi^{-1} e^{-3\pi i/5} \\ \varphi^{-1} e^{4\pi i/5} \\ -\varphi^{-3/2} e^{-3\pi i/5} \end{pmatrix},$$



$$= \begin{pmatrix} -\varphi^{-3/2} e^{-3\pi i/5} \\ -\varphi^{-3/2} e^{-3\pi i/5} \\ \varphi^{-1} e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}.$$

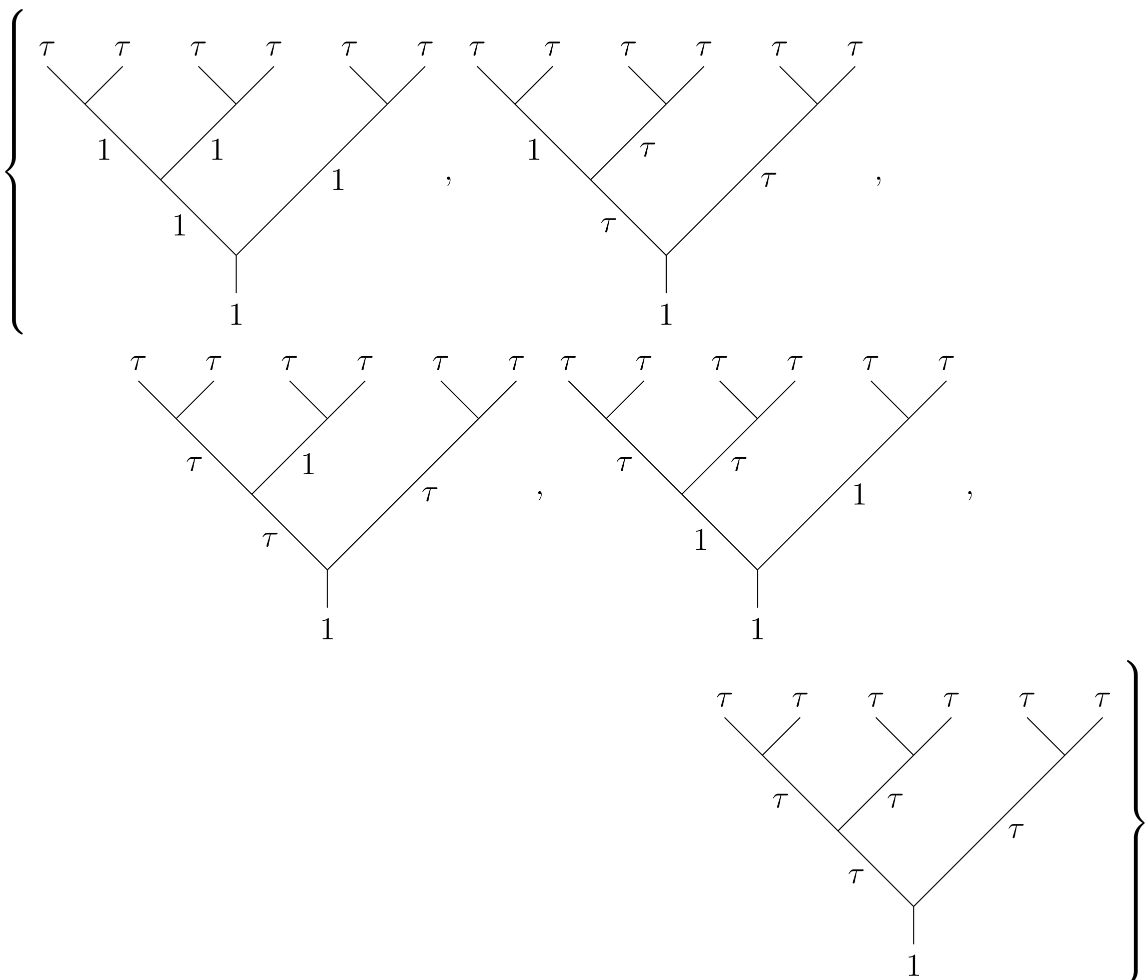
Finally,

$$\sigma_2 \mapsto \begin{pmatrix} \varphi^{-1}e^{4\pi i/5} & \varphi^{-1}e^{-3\pi i/5} & -\varphi^{-3/2}e^{-3\pi i/5} \\ \varphi^{-1}e^{-3\pi i/5} & \varphi^{-1}e^{4\pi i/5} & -\varphi^{-3/2}e^{-3\pi i/5} \\ -\varphi^{-3/2}e^{-3\pi i/5} & -\varphi^{-3/2}e^{-3\pi i/5} & \varphi^{-1}e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}.$$

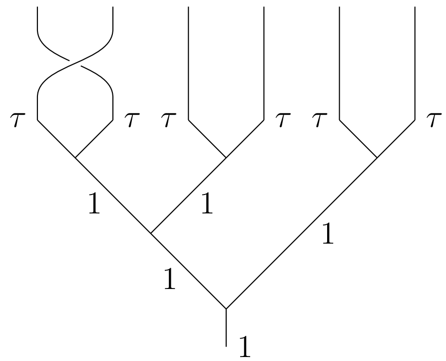
It can be checked that in fact $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$.

A Representation of the Braid Group B_6

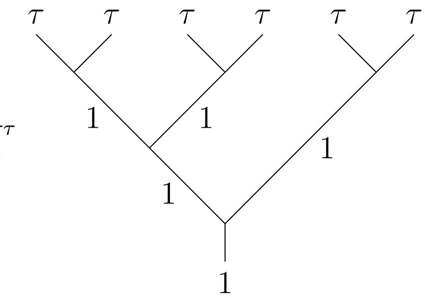
We wish to write the generators $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ of B_6 in terms of the basis



To find σ_1, we observe



$= R_1^{\tau\tau}$



$$= R_1^{\tau\tau} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

The next four give factors of $R_1^{\tau\tau}$, $R_\tau^{\tau\tau}$, $R_\tau^{\tau\tau}$, $R_\tau^{\tau\tau}$, respectively. So

$$\sigma_1 \mapsto \begin{pmatrix} e^{-4\pi i/5} & 0 & 0 & 0 & 0 \\ 0 & e^{-4\pi i/5} & 0 & 0 & 0 \\ 0 & 0 & e^{3\pi i/5} & 0 & 0 \\ 0 & 0 & 0 & e^{3\pi i/5} & 0 \\ 0 & 0 & 0 & 0 & e^{3\pi i/5} \end{pmatrix}.$$

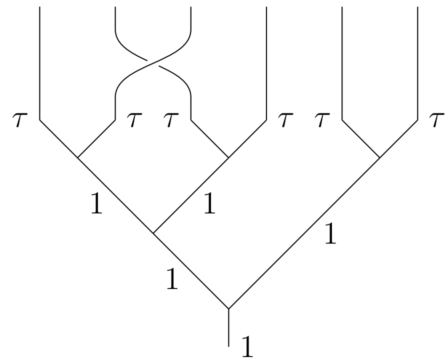
For σ_3, we have factors $R_1^{\tau\tau}$, $R_\tau^{\tau\tau}$, $R_1^{\tau\tau}$, $R_\tau^{\tau\tau}$, and $R_\tau^{\tau\tau}$. So

$$\sigma_3 \mapsto \begin{pmatrix} e^{-4\pi i/5} & 0 & 0 & 0 & 0 \\ 0 & e^{3\pi i/5} & 0 & 0 & 0 \\ 0 & 0 & e^{-4\pi i/5} & 0 & 0 \\ 0 & 0 & 0 & e^{3\pi i/5} & 0 \\ 0 & 0 & 0 & 0 & e^{3\pi i/5} \end{pmatrix}.$$

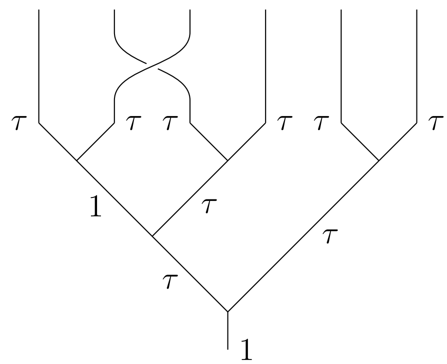
For σ_5, we have factors $R_1^{\tau\tau}$, $R_\tau^{\tau\tau}$, $R_\tau^{\tau\tau}$, $R_1^{\tau\tau}$, and $R_\tau^{\tau\tau}$. So

$$\sigma_5 \mapsto \begin{pmatrix} e^{-4\pi i/5} & 0 & 0 & 0 & 0 \\ 0 & e^{3\pi i/5} & 0 & 0 & 0 \\ 0 & 0 & e^{3\pi i/5} & 0 & 0 \\ 0 & 0 & 0 & e^{-4\pi i/5} & 0 \\ 0 & 0 & 0 & 0 & e^{3\pi i/5} \end{pmatrix}.$$

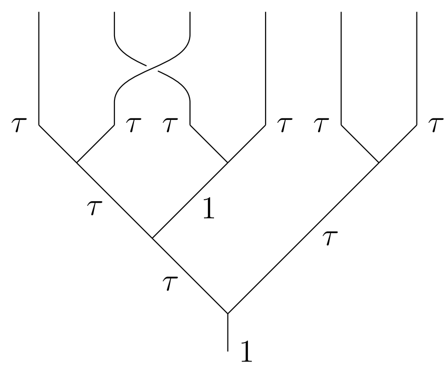
For σ_2, we find



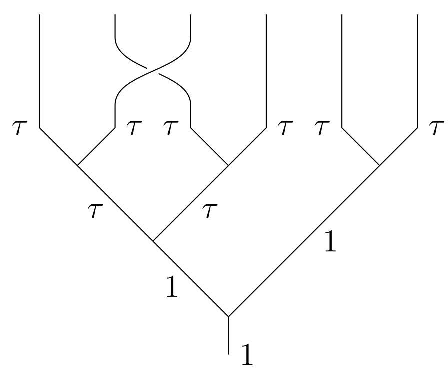
$$= \begin{pmatrix} \varphi^{-1}e^{4\pi i/5} \\ 0 \\ 0 \\ \varphi^{-1/2}e^{-3\pi i/5} \\ 0 \end{pmatrix},$$



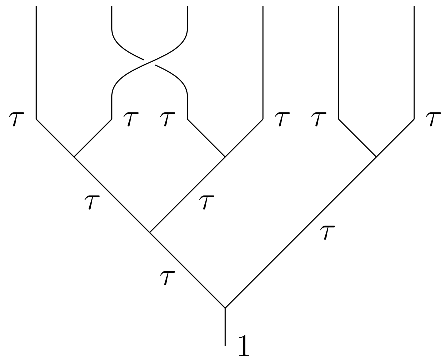
$$= \begin{pmatrix} 0 \\ \varphi^{-1}e^{4\pi i/5} \\ \varphi^{-1}e^{-3\pi i/5} \\ 0 \\ -\varphi^{-3/2}e^{-3\pi i/5} \end{pmatrix}, \qquad \text{(see } B_4\text{)}$$



$$= \begin{pmatrix} 0 \\ \varphi^{-1}e^{-3\pi i/5} \\ \varphi^{-1}e^{4\pi i/5} \\ 0 \\ -\varphi^{-3/2}e^{-3\pi i/5} \end{pmatrix}, \qquad \text{(see } B_4\text{)}$$



$$= \begin{pmatrix} \varphi^{-1/2}e^{-3\pi i/5} \\ 0 \\ 0 \\ -\varphi^{-1} \\ 0 \end{pmatrix},$$

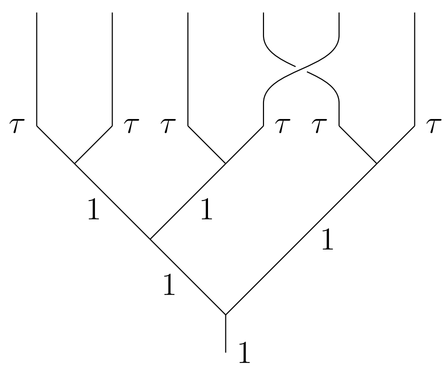


$$= \begin{pmatrix} 0 \\ -\varphi^{-3/2}e^{-3\pi i/5} \\ -\varphi^{-3/2}e^{-3\pi i/5} \\ 0 \\ \varphi^{-1}e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}. \qquad \text{(see } B_4\text{)}$$

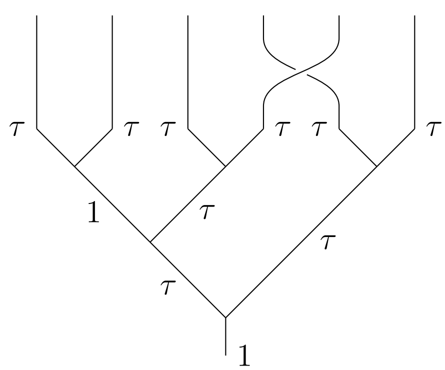
Now,

$$\sigma_2 \mapsto \begin{pmatrix} \varphi^{-1}e^{4\pi i/5} & 0 & 0 & \varphi^{-1/2}e^{-3\pi i/5} & 0 \\ 0 & \varphi^{-1}e^{4\pi i/5} & \varphi^{-1}e^{-3\pi i/5} & 0 & -\varphi^{-3/2}e^{-3\pi i/5} \\ 0 & \varphi^{-1}e^{-3\pi i/5} & \varphi^{-1}e^{4\pi i/5} & 0 & -\varphi^{-3/2}e^{-3\pi i/5} \\ \varphi^{-1/2}e^{-3\pi i/5} & 0 & 0 & -\varphi^{-1} & 0 \\ 0 & -\varphi^{-3/2}e^{-3\pi i/5} & -\varphi^{-3/2}e^{-3\pi i/5} & 0 & \varphi^{-1}e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}.$$

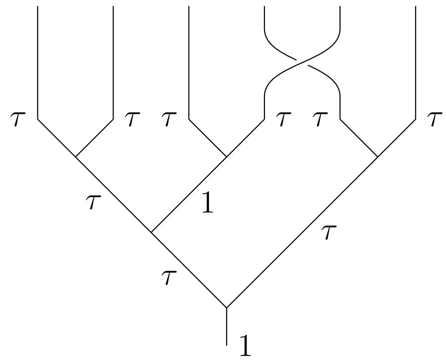
And finally for σ_4,



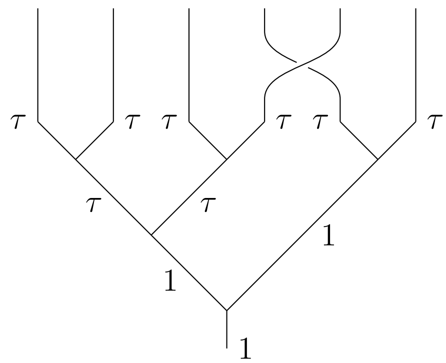
$$= \begin{pmatrix} \varphi^{-1}e^{4\pi i/5} \\ \varphi^{-1/2}e^{-3\pi i/5} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$



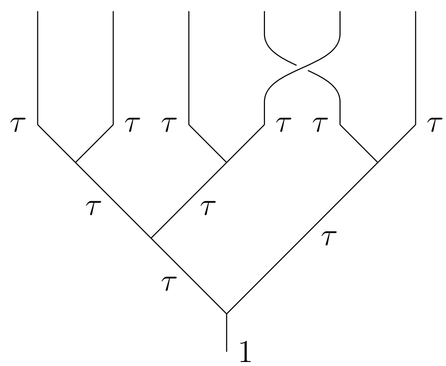
$$= \begin{pmatrix} \varphi^{-1/2}e^{-3\pi i/5} \\ -\varphi^{-1} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$



$$= \begin{pmatrix} 0 \\ 0 \\ \varphi^{-2}e^{3\pi i/5} - \varphi^{-2} \\ \varphi^{-1}e^{-3\pi i/5} \\ \varphi^{-3/2}e^{3\pi i/5} + \varphi^{-5/2} \end{pmatrix},$$



$$= \begin{pmatrix} 0 \\ 0 \\ \varphi^{-1}e^{-3\pi i/5} \\ \varphi^{-1}e^{4\pi i/5} \\ -\varphi^{-3/2}e^{-3\pi i/5} \end{pmatrix},$$



$$= \begin{pmatrix} 0 \\ 0 \\ \varphi^{-3/2}e^{3\pi i/5} + \varphi^{-5/2} \\ -\varphi^{-3/2}e^{-3\pi i/5} \\ \varphi^{-1}e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}.$$

Now, σ_4 maps to

$$\begin{pmatrix} \varphi^{-1}e^{4\pi i/5} & \varphi^{-1/2}e^{-3\pi i/5} & 0 & 0 & 0 \\ \varphi^{-1/2}e^{-3\pi i/5} & -\varphi^{-1} & 0 & 0 & 0 \\ 0 & 0 & \varphi^{-2}e^{3\pi i/5} - \varphi^{-2} & \varphi^{-1}e^{-3\pi i/5} & \varphi^{-3/2}e^{3\pi i/5} + \varphi^{-5/2} \\ 0 & 0 & \varphi^{-1}e^{-3\pi i/5} & \varphi^{-1}e^{4\pi i/5} & -\varphi^{-3/2}e^{-3\pi i/5} \\ 0 & 0 & \varphi^{-3/2}e^{3\pi i/5} + \varphi^{-5/2} & -\varphi^{-3/2}e^{-3\pi i/5} & \varphi^{-1}e^{3\pi i/5} - \varphi^{-3} \end{pmatrix}.$$

It can be checked that both braid relations hold for all pairs of σ_1, σ_2, σ_3, σ_4, and σ_5.

5.2 Near-Group Categories

There is a class of fusion categories constructed from finite abelian groups called near-group categories. While definitions are not well-established, one might find some in, e.g. [50]. We will take the following as our definition.

Definition 5.4. Given a finite abelian group G and integer n, a *near-group category* *NG($G, n|G|$)* is any fusion category with label set $G \cup \{x\}$ and fusion rules

$$g \otimes h \cong gh,$$

$$g \otimes x \cong x \otimes g \cong x,$$

$$x \otimes x \cong \left(\bigoplus_{g \in G} g \right) \oplus n|G|x$$

for all $g, h \in G$.

Note we have only defined the near-group category with N_x^{xx} a multiple of the order of the group. It is known that this is necessary to achieve a fusion category, so we do not consider other cases.

Example 5.5. The Tambara-Yamagami category $\mathrm{TY}(G)$ is the near-group category $\mathrm{NG}(G, 0)$. $\mathrm{TY}(\mathbb{Z}_2)$ is the Ising MTC.

Example 5.6. The near-group category $\mathrm{NG}(\{0\}, 1)$ is the Fibonacci MTC.

Example 5.7. The near-group category $\mathrm{NG}(\mathbb{Z}_2, 2)$ is the category known as $\frac{1}{2}E_6$. It is already complicated as it has multiplicity.

Conjecture 5.8. *A fusion category NG($\mathbb{Z}_p, p$) exists for all primes p.*

Section 5.2.1 presents one approach to resolving Conjecture 5.8. Section 5.2.2 provides another.

5.2.1 Equations Determining Near-Groups

Work has been done on writing and solving polynomial equations which determine different classes of near-group categories. A summary can be found in [27]. A set that we are interested in was first defined in [28] and later worked on by [19]. The equations differ based on who has been working on them but amount to the following. Definitions are given below.

$$a(1) = 1, \qquad a(g) = a(-g), \qquad a(g+h)\,\langle g, h\rangle = a(g)a(h), \qquad \sum_g a(g) = \sqrt{|G|}\,\eta^{-3},$$

$$b(0) = -\frac{1}{d}, \qquad \sum_h \overline{\langle g, h\rangle} b(h) = \sqrt{|G|}\,\eta\overline{b(g)}, \qquad a(g)b(-g) = \overline{b(g)},$$

$$\sum_g b(g+h)\overline{b(g)} = \delta_{h,0} - \frac{1}{d}, \quad \sum_g b(g+h_1)b(g+h_2)\overline{b(g)} = \overline{\langle h_1, h_2\rangle} b(h_1)b(h_2) - \frac{\eta}{d\sqrt{|G|}}$$

Here, G is a group and $g, h, h_i \in G$. The paper [19] finds all solutions for all groups up to order 13, and [7] adds solutions for most groups up to order 30. We restrict our attention to groups of prime order since we are interested in answering Conjecture 5.8. Our numerical results agree with [19, 7], though we add solutions for $|G| = 19$, which [7] has left out.

From here on, we use p to refer to the prime order of G. For $\mathrm{NG}(\mathbb{Z}_p, p)$, we set $d = (p + \sqrt{p^2 + 4p})/2$ to be the dimension of the noninvertible simple object x. We would now like to choose $\eta^3 = e^{c\pi i/4}$ so that η is a third root of the multiplicative central charge from Definition 2.55. We resist this urge and settle for $\overline{\eta}$ being a third root of the multiplicative central charge in order to keep our equations consistent with the literature.

Finally, $a, b, \langle\cdot,\cdot\rangle$ are complex numbers. We choose $a(g) = e^{2\pi i m g^2/p}$ for an integer m and $\langle g, h\rangle = a(g)a(h)\overline{a(g+h)}$. What remains is to solve for $b(g), g \in \mathbb{Z}_p$ so that the equations involving b are all satisfied.

We first provide a summary of numerical results for b. These results are followed by some general statements and conjectures.

5.2.1.1 Numerical Results

For each of the following solutions, we also have a solution given by $b'(g) = b(-g)$. Choosing different m can give permutations of the a function, which gives permutations of the b solutions. The general results of Section 5.2.1.2 address these solutions. No other solutions have been intentionally omitted.

$p = 3$ $(3 \textbf{ mod } 8)$

$m = 1$

No solutions for $\eta = i, ie^{2\pi i/3}$.

$$\eta = ie^{4\pi i/3},$$
$$b(0) \approx -0.26376, b(1) \approx -0.55272 - 0.16683i, b(2) \approx 0.42084 + 0.39526i$$

$m = 2$

No solutions for $\eta = -i, -ie^{4\pi i/3}$.

$$\eta = -ie^{2\pi i/3},$$
$$b(0) \approx -0.26376, b(1) \approx -0.55272 + 0.16683i, b(2) \approx 0.42084 - 0.39526i$$

This is the complex conjugate of $m = 1$, $\eta = ie^{4\pi i/3}$.

$p = 5$ $(5 \textbf{ mod } 8)$

$m = 1$

No solutions for $\eta = 1$.

$$\eta = e^{2\pi i/3},$$
$$b(0) \approx -0.17082, b(1) \approx -0.01268 - 0.44703i, b(2) \approx 0.23894 - 0.37803i$$

$$b(3) \approx -0.28569 + 0.34407i, b(4) \approx 0.42124 + 0.15021i$$

Taking $\eta = e^{4\pi i/3}$ gives the (rearranged) complex conjugate of the above.

$m = 2$

No solutions for $\eta = -e^{2\pi i/3}, -e^{4\pi i/3}$.

$$\eta = -1,$$

$$b(0) \approx -0.17082, b(1) \approx 0.13820 - 0.42533i, b(2) \approx 0.13820 + 0.42533i$$

$$b(3) \approx 0.13820 + 0.42533i, b(4) \approx 0.13820 - 0.42533i$$

The complex conjugate of this solution appears by taking $m = 3$. This should be considered a rearranged complex conjugate even though the symmetry of this case obfuscates it.

$p = 7$ (7 **mod** 8)

$m = 1$

No solutions for $\eta = ie^{2\pi i/3}, ie^{4\pi i/3}$.

$$\eta = i,$$

$$b(0) \approx -0.12678, b(1) \approx 0.35945 - 0.11686i, b(2) \approx -0.31133 - 0.21432i$$

$$b(3) \approx -0.08353 + 0.36862i, b(4) \approx -0.34079 + 0.16346i,$$

$$b(5) \approx 0.18751 - 0.32817i, b(6) \approx 0.31548 - 0.20817i$$

$m = 6$

No solutions for $\eta = -ie^{2\pi i/3}, -ie^{4\pi i/3}$.

The solution for $\eta = -i$ is the complex conjugate of the $m = 1$ solution.

$p = 11$ (3 **mod** 8)

$m = 1$

No solutions for $\eta = i$.

$$\eta = ie^{2\pi i/3},$$

$$b(0) \approx -0.08387, b(1) \approx -0.05339 - 0.29675i, b(2) \approx -0.16303 - 0.25363i$$

$$b(3) \approx 0.19404 - 0.23078i, b(4) \approx -0.11063 + 0.28048i, b(5) \approx 0.28473 + 0.09920i,$$

$$b(6) \approx -0.13871 - 0.26771i, b(7) \approx 0.02712 + 0.30029i, b(8) \approx -0.12932 + 0.27237i,$$

$$b(9) \approx 0.29845 - 0.04288i, b(10) \approx 0.11552 + 0.27850i$$

$$\eta = ie^{4\pi i/3},$$

$$b(0) \approx -0.08387, b(1) \approx 0.19809 - 0.22731i, b(2) \approx -0.24269 + 0.17892i$$

$$b(3) \approx -0.28715 - 0.09194i, b(4) \approx -0.2919 + 0.07552i, b(5) \approx -0.17208 - 0.24759i,$$

$b(6) \approx 0.26956 + 0.13509i, b(7) \approx 0.25880 + 0.1547i, b(8) \approx -0.20292 - 0.22301i,$

$b(9) \approx 0.02371 + 0.30058i, b(10) \approx 0.28954 + 0.08412i$

$m = 10$

No solutions for $\eta = -i$.

The solution for $\eta = -ie^{2\pi i/3}$ is the complex conjugate of the solution for $m = 1$, $\eta = ie^{4\pi i/3}$.

The solution for $\eta = -ie^{4\pi i/3}$ is the complex conjugate of the solution for $m = 1$, $\eta = ie^{2\pi i/3}$.

$p = 13$ $(5 \textbf{ mod } 8)$

$m = 1$

No solutions for $\eta = e^{2\pi i/3}, e^{4\pi i/3}$.

$$\eta = 1,$$

$$b(0) \approx -0.07177, b(1) \approx 0.16523 - 0.22276i, b(2) \approx 0.26626 + 0.07764i$$

$$b(3) \approx 0.12648 - 0.24683i, b(4) \approx -0.27179 + 0.05525i, b(5) \approx -0.21893 - 0.17027i,$$

$$b(6) \approx 0.02205 + 0.27647i, b(7) \approx 0.27711 - 0.01144i, b(8) \approx -0.27298 + 0.04903i,$$

$$b(9) \approx -0.08761 + 0.26315i, b(10) \approx -0.27564 + 0.03073i,$$

$$b(11) \approx -0.16701 - 0.22143i, b(12) \approx 0.24983 + 0.12046i$$

A rearrangement of the complex conjugates is also a solution for $\eta = 1$.

$m = 2$

No solutions for $\eta = -e^{2\pi i/3}, -e^{4\pi i/3}$.

$$\eta = -1,$$

$$b(0) \approx -0.07177, b(1) \approx 0.25259 + 0.11456i, b(2) \approx 0.11571 + 0.25206i$$

$$b(3) \approx -0.21083 + 0.18020i, b(4) \approx 0.25000 - 0.12008i, b(5) \approx -0.11157 + 0.25392i,$$

$$b(6) \approx 0.03203 - 0.27550i, b(7) \approx -0.09703 - 0.25982i, b(8) \approx 0.14560 - 0.23606i,$$

$$b(9) \approx -0.21401 - 0.17642i, b(10) \approx 0.03831 + 0.27469i,$$

$$b(11) \approx 0.08054 + 0.26540i, b(12) \approx 0.04921 - 0.27295i$$

A rearrangement of the complex conjugates is also a solution for $\eta = -1$.

$p = 17$ (1 **mod** 8)

$m = 1$

No solutions for $\eta = 1$.

$\eta = e^{2\pi i/3}$,

$b(0) \approx -0.05572, b(1) \approx 0.08844 - 0.22584i, b(2) \approx -0.15849 + 0.18359i,$

$b(3) \approx -0.24086 - 0.02848i, b(4) \approx 0.22510 - 0.09030i, b(5) \approx 0.11245 - 0.21489i,$

$b(6) \approx -0.09140 - 0.22466i, b(7) \approx -0.05366 - 0.23652i, b(8) \approx 0.20685 - 0.12664i,$

$b(9) \approx -0.10701 + 0.21765i, b(10) \approx -0.19900 + 0.13864i, b(11) \approx 0.08380 + 0.22760i,$

$b(12) \approx -0.07105 - 0.23190i, b(13) \approx 0.17728 + 0.16552i, b(14) \approx 0.23153 - 0.07225i,$

$b(15) \approx -0.19743 + 0.14088i, b(16) \approx 0.16405 + 0.17864i$

The solution for $\eta = e^{4\pi i/3}$ is the (rearranged) complex conjugate of the above.

$m = 3$

No solutions for $\eta = -1$.

$\eta = -e^{2\pi i/3}$,

$b(0) \approx -0.05572, b(1) \approx -0.18355 - 0.15853i, b(2) \approx -0.05371 + 0.23651i,$

$b(3) \approx 0.18821 - 0.15298i, b(4) \approx 0.23525 - 0.05900i, b(5) \approx -0.21586 - 0.11059i,$

$b(6) \approx -0.24132 + 0.02427i, b(7) \approx -0.20412 - 0.13099i, b(8) \approx -0.22741 - 0.08431i,$

$b(9) \approx 0.14332 + 0.19566i, b(10) \approx 0.01847 - 0.24183i, b(11) \approx 0.12606 + 0.20720i,$

$b(12) \approx 0.24174 + 0.01961i, b(13) \approx 0.05204 + 0.23689i, b(14) \approx -0.24055 - 0.03098i,$

$b(15) \approx 0.24218 + 0.01306i, b(16) \approx 0.06009 + 0.23497i$

The solution for $\eta = -e^{4\pi i/3}$ is the (rearranged) complex conjugate of the above.

$p = 19$ (3 **mod** 8)

$m = 1$

No solutions for $\eta = ie^{2\pi i/3}, ie^{4\pi i/3}$.

$$\eta = i,$$

$$b(0) \approx -0.05012, b(1) \approx -0.21986 - 0.06547i, b(2) \approx 0.19622 + 0.11886i,$$

$$b(3) \approx -0.22922 - 0.00947i, b(4) \approx 0.01705 + 0.22878i, b(5) \approx -0.00510 + 0.22936i,$$

$$b(6) \approx -0.01303 - 0.22904i, b(7) \approx -0.19824 + 0.11547i, b(8) \approx 0.11594 - 0.19796i,$$

$$b(9) \approx 0.21655 - 0.07576i, b(10) \approx 0.05762 - 0.22206i, b(11) \approx 0.06712 - 0.21938i,$$

$$b(12) \approx 0.22930 + 0.00720i, b(13) \approx -0.15097 + 0.17274i, b(14) \approx -0.20799 + 0.09681i,$$

$$b(15) \approx 0.20085 - 0.11086i, b(16) \approx 0.22765 + 0.02839i,$$

$$b(17) \approx -0.06706 - 0.21940i, b(18) \approx -0.18670 + 0.13332i$$

$m = 18$

No solutions for $\eta = -ie^{2\pi i/3}, -ie^{4\pi i/3}$.

The solution for $\eta = -i$ is the complex conjugate of the solution above.

$p = 23$ (7 **mod** 8)

$m = 1$

$$\eta = i,$$

$$b(0) \approx -0.04174, b(1) \approx 0.02385 - 0.20715i, b(2) \approx 0.18008 + 0.10512i,$$

$$b(3) \approx 0.15973 - 0.13404i, b(4) \approx -0.17080 + 0.11961i, b(5) \approx 0.18461 + 0.09694i,$$

$$b(6) \approx -0.20839 + 0.00712i, b(7) \approx -0.20709 - 0.02431i, b(8) \approx 0.05372 - 0.20148i,$$

$$b(9) \approx 0.17228 + 0.11747i, b(10) \approx -0.11588 - 0.17335i, b(11) \approx -0.09699 + 0.18458i,$$

$b(12) \approx -0.17754 + 0.10936i, b(13) \approx 0.20845 - 0.00529i, b(14) \approx -0.15468 + 0.13983i,$

$b(15) \approx -0.18633 + 0.09359i, b(16) \approx -0.12358 + 0.16795i, b(17) \approx 0.19398 - 0.07649i,$

$b(18) \approx 0.10736 - 0.17875i, b(19) \approx 0.16990 - 0.12088i, b(20) \approx -0.03931 - 0.20478i,$

$b(21) \approx -0.01048 - 0.20825i, b(22) \approx 0.07885 + 0.19303i$

Solutions for $\eta = ie^{2\pi i/3}$ that we did not find can be found in [7].

$m = 22$

The solution for $\eta = -i$ is the complex conjugate of the solution above.

5.2.1.2 General results

Proposition 5.9. *If a choice of $\eta, a(g), b(g)$ is a solution to the equations at the beginning of Section 5.2.1, then so is the choice of $\eta, a(g), b(-g)$.*

Proof. Trivial since $\sum_{g \in \mathbb{Z}_p} f(g) = \sum_{g \in \mathbb{Z}_p} f(-g)$. For example, suppose that for each $g \in \mathbb{Z}_p$, we have

$$\sum_h \overline{\langle g, h \rangle} b(h) = \sqrt{p}\,\eta \overline{b(g)}.$$

Fixing some $g \in \mathbb{Z}_p$, we also have

$$\sum_h \overline{\langle -g, h \rangle} b(h) = \sqrt{p}\,\eta \overline{b(-g)}.$$

Then

$$\sum_h \overline{\langle g, h \rangle} b(-h) = \sum_h \overline{\langle -g, -h \rangle} b(-h) = \sqrt{p}\,\eta \overline{b(-g)}.$$

□

As noted at the beginning of Section 5.2.1.1, the $b(-g)$ solutions were omitted from that section.

Proposition 5.10. *If a choice of $\eta, a(g), b(g)$ is a solution to the equations at the beginning of Section 5.2.1, then so is $\overline{\eta}, \overline{a(g)}, \overline{b(g)}$.*

Proof. Trivial. For example, suppose

$$\sum_h \overline{\langle g,h\rangle} b(h) = \sqrt{p}\,\eta \overline{b(g)}.$$

It follows that

$$\sum_h \langle g,h\rangle\, \overline{b(h)} = \sqrt{p}\,\overline{\eta} b(g).$$

Since $\langle g,h\rangle = a(g)a(h)\overline{a(g+h)}$, the set $\overline{\eta}, \overline{a}, \overline{b}$ also satisfy the equation. All equations follow similarly. □

Remark 5.11. Note that the rearranged complex conjugate solutions noted in Section 5.2.1.1 are not solutions guaranteed by Proposition 5.10. The complex conjugate solutions of Proposition 5.10 are governed by Proposition 5.12 and Corollary 5.14.

Proposition 5.12. *Up to permutation of outputs, there exist exactly two functions* $a(g) = e^{2\pi i \frac{mg^2}{p}}$ *for each odd prime* p*. These two functions correspond to the Legendre symbols* $(\frac{m}{p}) = \pm 1$*, and the intersection of their images is only* $a(0) = 1$*. They are complex conjugates of each other exactly when* $(\frac{1}{p}) \neq (\frac{-1}{p})$*.*

Proof. Let p be an odd prime. Then $(\frac{m}{p}) = 1$ if and only if there exists an integer x such that $x^2 \equiv m \pmod{p}$, which is true if and only if for each $g \in \mathbb{Z}_p^\times$, we have

$$a(g) = e^{2\pi i \frac{x^2 g^2}{p}} = e^{2\pi i \frac{(xg)^2}{p}}.$$

When $(\frac{m}{p}) = -1$, $a(g)$ cannot be written in the form $e^{2\pi i \frac{(xg)^2}{p}}$ unless $g = 0$. So the image of a when $(\frac{m}{p}) = -1$ is disjoint from the image of a when $(\frac{m}{p}) = 1$ (for nonzero input g).
Claim: For each m, the image of a contains $(p-1)/2$ elements not equal to 1.
Proof: Fix $m \in \{1, \ldots, p-1\}$ and recall that $a(g) = a(-g)$. We wish to show that the values mg^2 are otherwise distinct mod p. Take $g, h \in \{1, \ldots, p-1\}$. We see that $mg^2 \equiv mh^2 \pmod{p}$ if and only if $g^2 \equiv h^2 \pmod{p}$, or $g = \pm h$. Thus $a(g)$ takes on exactly $(p-1)/2$ values for $g = 1, \ldots, p-1$. ■

Now, choices m_1, m_2 with $(\frac{m_1}{p}) = 1$ and $(\frac{m_2}{p}) = -1$ give functions $a_1(g), a_2(g)$ with images of size $(p+1)/2$ intersecting only on $g = 0$.

Since $a(g) = a(-g)$, the domain of a can contain no more than $(p+1)/2$ elements with distinct images. Thus, for each m, the function a is , and its range is determined by $(\frac{2m}{p})$.

For the last claim, note that $e^{2\pi i \frac{g^2}{p}}$ is the complex conjugate of $e^{2\pi i \frac{(p-1)g^2}{p}}$ for all g. Thus, the two distinct a functions are complex conjugates exactly when $m = 1$ and $m = p-1$ belong to different Legendre classes. □

Corollary 5.13. *The two functions $a(g)$ are complex conjugates exactly when $p \equiv 3 \pmod 4$.*

Proof. It is known that, for any integer n,

$$p \equiv 1 \pmod 4 \implies \left(\frac{n}{p}\right) = \left(\frac{-n}{p}\right),$$
$$p \equiv 3 \pmod 4 \implies \left(\frac{n}{p}\right) \neq \left(\frac{-n}{p}\right).$$

□

Corollary 5.14. *Let $p \equiv 3 \pmod 4$ be prime. Consider $m_1, m_2 \in \mathbb{Z}_p^\times$ so that $(\frac{2m_1}{p}) \neq (\frac{2m_2}{p})$. The solutions for η, a, b are exactly those for m_1 together with the complex conjugates $\overline{\eta}, \overline{a}, \overline{b}$ for m_2.*

Proof. Let p, m_1, m_2 be as defined in the statement. Proposition 5.12 says there exist exactly two distinct functions a_1, a_2 defined by m_1, m_2, respectively. The functions a_1, a_2 are complex conjugates by Corollary 5.13. Finally, Proposition 5.10 guarantees that the complex conjugate of any solution corresponding to m_1 also appears as a solution to m_2, and vice versa. □

Lemma 5.15. *When $p \equiv 1 \pmod 4$, the image of each of the two functions $a(g)$ guaranteed by Proposition 5.12 is closed under complex conjugation.*

Proof. Fix a prime $p \equiv 1 \pmod 4$. Then $(\frac{-1}{p}) = 1$, and there exists an integer x such that $x^2 \equiv -1 \pmod p$. Fix such an x. Now fix $g \in \mathbb{Z}_p^\times$. We wish to find $h \in \mathbb{Z}_p$ such that

$$e^{2\pi i \frac{mh^2}{p}} = e^{-2\pi i \frac{mg^2}{p}}.$$

Consider $h \in \mathbb{Z}_p$ with $xg \equiv h \pmod p$. On the one hand, $(xg)^2 \equiv h^2 \pmod p$. On the other hand, $x^2 g^2 \equiv -g^2 \pmod p$. Thus, $h^2 \equiv -g^2 \pmod p$, and there exists an integer n such that $h^2 = -g^2 + np$. Finally,

$$e^{2\pi i \frac{mh^2}{p}} = e^{2\pi i \frac{m(-g^2+np)}{p}} = e^{-2\pi i \frac{mg^2}{p}},$$

and the complex conjugate of each image under a is also an image under a. □

Corollary 5.16. *When $p \equiv 1$ (mod 4), each solution a, η, b gives another solution with a, $\overline{\eta}$, and a permutation of $\overline{b}$ (or equivalently, $\overline{\eta}$, $\overline{b}$, and a permutation of a).*

Proof. Consider a prime $p \equiv 1 \pmod 4$ and a solution a, η, b. By Lemma 5.15, there is a permutation $\sigma \in S_p$ with $a(\sigma(g)) = \overline{a(g)}$. Then $a(\sigma(g)), \overline{\eta}, \overline{b}$ is a solution. □

In addition to these proven results, we also note the following pattern. In all of the numerical results, solving the two equations

$$\sum_h \overline{\langle g, h\rangle} b(h) = \sqrt{p}\,\eta a(g) b(-g), \qquad \sum_g b(g+h) a(g) b(-g) = \delta_{h,0} - d^{-1}$$

was enough to guarantee

$$a(g)b(-g) = \overline{b(g)}, \qquad \sum_g b(g+h_1) b(g+h_2) \overline{b(g)} = \overline{\langle h_1, h_2\rangle} b(h_1) b(h_2) - \frac{\eta}{d\sqrt{p}}.$$

This suggests the following conjecture.

Conjecture 5.17. *In the case of $\mathbb{Z}_p$, the first two equations imply the latter two.*

5.2.2 Condensing $(G_2)_3$

Another approach to answering Conjecture 5.8 is to try to extend Examples 5.6 and 5.7. More examples of near-group categories $\mathrm{NG}(G, |G|)$ for small order groups may provide some insight toward the conjecture.

While the F-symbols for the near-group category $\mathrm{NG}(\mathbb{Z}_2, 2)$ are known, significant effort was required to find them [24]. See Remark 2.73 for a word on why this is generally a difficult problem.

The same data for the near-group category $\mathrm{NG}(\mathbb{Z}_3, 3)$ are unknown. In fact, full F-symbols are not known for any category with any multiplicities greater than 2. However, it is known that $\mathrm{NG}(\mathbb{Z}_3, 3)$ can be realized as a condensation of $(G_2)_3$, a modular tensor category constructed from the exceptional Lie group G_2. The complete data for $(G_2)_3$ can be found in Supplemental B: $(G_2)_3$ Data or at [1]. A summary is provided here.

$(G_2)_3$ **Data**

Label set: $\mathcal{L} = \{a, b, c, d, e, f\}$

Fusion rules:

$\otimes$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	$a \oplus b \oplus c \oplus e$	$b \oplus c \oplus d \oplus e \oplus f$	$c \oplus d \oplus f$	$b \oplus c \oplus f$	$c \oplus d \oplus e \oplus f$
c	c	$b \oplus c \oplus d \oplus e \oplus f$	$a \oplus b \oplus 2c \oplus d \oplus e \oplus 2f$	$b \oplus c \oplus d \oplus e \oplus f$	$b \oplus c \oplus d \oplus e \oplus f$	$b \oplus 2c \oplus d \oplus e \oplus f$
d	d	$c \oplus d \oplus f$	$b \oplus c \oplus d \oplus e \oplus f$	$a \oplus b \oplus c \oplus d$	$c \oplus e \oplus f$	$b \oplus c \oplus e \oplus f$
e	e	$b \oplus c \oplus f$	$b \oplus c \oplus d \oplus e \oplus f$	$c \oplus e \oplus f$	$a \oplus c \oplus d \oplus e$	$b \oplus c \oplus d \oplus f$
f	f	$c \oplus d \oplus e \oplus f$	$b \oplus 2c \oplus d \oplus e \oplus f$	$b \oplus c \oplus e \oplus f$	$b \oplus c \oplus d \oplus f$	$a \oplus b \oplus c \oplus d \oplus e \oplus f$

Quantum dimensions: $\left\{1, \frac{3+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}\right\}$

Total quantum dimension: $D = \sqrt{\frac{21\left(5+\sqrt{21}\right)}{2}}$

Twists: $\left\{1, e^{\frac{4\pi i}{7}}, e^{\frac{4\pi i}{3}}, e^{\frac{16\pi i}{7}}, e^{\frac{8\pi i}{7}}, 1\right\}$

S Matrix:

$$\sqrt{\frac{5-\sqrt{21}}{42}} \begin{pmatrix} 1 & \frac{3+\sqrt{21}}{2} & \frac{7+\sqrt{21}}{2} & \frac{3+\sqrt{21}}{2} & \frac{3+\sqrt{21}}{2} & \frac{5+\sqrt{21}}{2} \\ \frac{3+\sqrt{21}}{2} & \alpha & 0 & \beta & \gamma & \frac{-3-\sqrt{21}}{2} \\ \frac{7+\sqrt{21}}{2} & 0 & \frac{-7-\sqrt{21}}{2} & 0 & 0 & \frac{7+\sqrt{21}}{2} \\ \frac{3+\sqrt{21}}{2} & \beta & 0 & \gamma & \alpha & \frac{-3-\sqrt{21}}{2} \\ \frac{3+\sqrt{21}}{2} & \gamma & 0 & \alpha & \beta & \frac{-3-\sqrt{21}}{2} \\ \frac{5+\sqrt{21}}{2} & \frac{-3-\sqrt{21}}{2} & \frac{7+\sqrt{21}}{2} & \frac{-3-\sqrt{21}}{2} & \frac{-3-\sqrt{21}}{2} & 1 \end{pmatrix},$$

with

$$\alpha = D \cdot \sqrt{\frac{\mathcal{S}}{7}}$$

$$= D \cdot \sqrt{\frac{5}{21} + \frac{\sqrt[3]{\frac{1+3i\sqrt{3}}{2}}}{3\sqrt[3]{49}} + \frac{1}{3\sqrt[3]{\frac{7+21i\sqrt{3}}{2}}}}$$

$$\approx 6.8316646397,$$

$$\beta = D \cdot \left(\frac{\alpha}{\mathcal{S}} - \alpha\right)$$

$$= D \cdot \frac{1-\mathcal{S}}{\sqrt{7\mathcal{S}}}$$

$$\approx -4.7276586176,$$

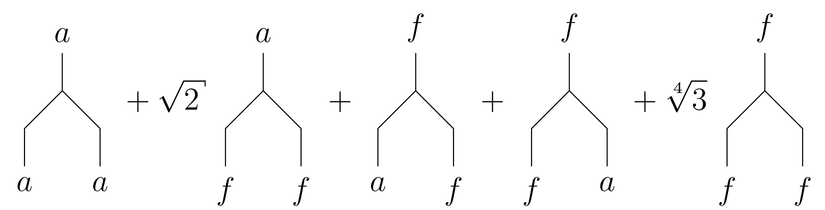
$$\gamma = D \cdot (\mathcal{S} - 3)\,\alpha$$

$$= D \cdot \frac{(\mathcal{S}-3)\sqrt{\mathcal{S}}}{\sqrt{7}}$$

$$\approx 1.6872818254,$$

where $\mathcal{S} = 2 + 2\sin\left(\frac{3\pi}{14}\right)$ is the silver constant.

From the above, we note that f is not a boson, but it is bosonic. The object $\mathcal{A} = a \oplus f$ has a condensable algebra structure with multiplication



Following the examples in Section 3.3, we compute Hom spaces. Some of the relevant ones are

$$\mathrm{Hom}_{\mathcal{A}}(a, a) \cong \mathrm{Hom}_{\mathcal{A}}(a, f) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(b, b) \cong \mathrm{Hom}_{\mathcal{A}}(d, d) \cong \mathrm{Hom}_{\mathcal{A}}(e, e) \cong \mathbb{C},$$

$$\mathrm{Hom}_{\mathcal{A}}(b, X) \cong \mathrm{Hom}_{\mathcal{A}}(d, X) \cong \mathrm{Hom}_{\mathcal{A}}(e, X) \cong \mathrm{Hom}(b \oplus c \oplus d \oplus e \oplus f, X),$$

$$\mathrm{Hom}_{\mathcal{A}}(c, c) \cong \mathbb{C}^3,$$

$$\mathrm{Hom}_{\mathcal{A}}(c, X) \cong \mathrm{Hom}(b \oplus 3c \oplus d \oplus e \oplus f, X),$$

$$\mathrm{Hom}_{\mathcal{A}}(f, f) \cong \mathbb{C}^2,$$

$$\mathrm{Hom}_{\mathcal{A}}(f, X) \cong \mathrm{Hom}(a \oplus b \oplus c \oplus d \oplus e \oplus 2f, X).$$

In condensing, the objects b, d, e remain simple but become isomorphic. The image of the object c is a direct sum of three simple objects, one of which is isomorphic to $b \sim d \sim e$. The object f gives a direct sum of two simple objects, one of which is isomorphic to the tensor unit and one of which is isomorphic to $b \sim d \sim e$. That is, if $\mathrm{NG}(\mathbb{Z}_3, 1)$ has label set $\{0, 1, 2, X\}$, the condensation functor maps

$$T(a) = (a \oplus f, m \otimes \mathrm{id}_a) \cong 0 \text{ simple},$$

$$T(b) = (b \oplus c \oplus d \oplus e \oplus f, m \otimes \mathrm{id}_b) \cong X \text{ simple},$$

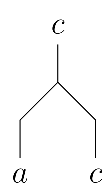
$$T(c) = (b \oplus 3c \oplus d \oplus e \oplus f, m \otimes \mathrm{id}_c) \cong 1 \oplus 2 \oplus X,$$

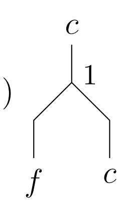
$$T(d) = (b \oplus c \oplus d \oplus e \oplus f, m \otimes \mathrm{id}_d) \cong X \text{ simple},$$

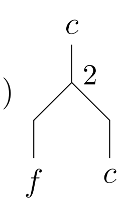
$$T(e) = (b \oplus c \oplus d \oplus e \oplus f, m \otimes \mathrm{id}_e) \cong X \text{ simple},$$

$$T(f) = (a \oplus b \oplus c \oplus d \oplus e \oplus 2f, m \otimes \mathrm{id}_f) \cong 0 \oplus X.$$

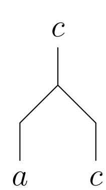
Indeed, there exist two nontrivial module structures on c. The objects 1 and 2 of $\mathrm{NG}(\mathbb{Z}_3, 3)$ can be identified with the simple modules

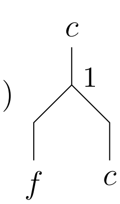
$(c, \mu_1), \quad \mu_1 \approx$ 

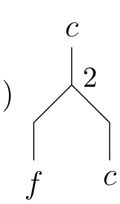
$+ (-0.27944 - 1.09709i)$ 

$+ (0.88766 - 0.34537i)$ 

,

$(c, \mu_2), \quad \mu_2 \approx$ 

$+ (-0.27944 + 1.09709i)$ 

$+ (0.88766 + 0.34537i)$ 

.

At the time of writing, our Mathematica implementation lacked the efficiency to run the full F-symbol computation. As can be seen in Supplemental B: $(G_2)_3$ Data, much of the $(G_2)_3$ data is not pretty. It lends itself very well to producing abysmally conditioned matrices.

Ultimately, we do hope to find the data for $\mathrm{NG}(\mathbb{Z}_3, 3)$ and provide insight to answering the question of Conjecture 5.8.

5.3 Error Correcting Codes

One unexpected source of condensable algebras is classical error correcting codes. In classical computing, an error correcting code is an encoding of data bits in such a way that a receiver of the data may detect, and ideally correct, any errors that may have been introduced in transmission.

Example 5.18. The Hamming(7,4) error correcting code uses three parity bits to encode four data bits. A total of seven bits are used to transmit four bits of data. The Hamming(7,4) generator matrix is

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Encoding a bitstring $d_1d_2d_3d_4$ is achieved by multiplying

$$G\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} d_1+d_2+d_4 \\ d_1+d_3+d_4 \\ d_1 \\ d_2+d_3+d_4 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

When a seven-bit string is received, parity bits 1, 2, and 4 can be checked for consistency. When a bitstring is received, multiplying by the matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

adds each parity bit to the data bits that determine it. If transmission was error-free, the result is the zero vector. If a single error occurred, the resulting vector indicates which bit is in error.

In addition to the encoding algorithm, the term *code* will refer also to the full set of possible encoded bitstrings. That is, the Hamming(7,4) code is the image of the generator matrix G,

$$\{0000000, 0001111, 0010110, 0011001, 0100101, 0101010, 0110011, 0111100,$$
$$1000011, 1001100, 1010101, 1011010, 1100110, 1101001, 1110000, 1111111\}.$$

For reasons that are not yet clear, we now add a parity bit to the end of each element to arrive at a doubly even code.

$C = \{00000000, 00011110, 00101101, 00110011, 01001011, 01010101, 01100110, 01111000,$

$10000111, 10011001, 10101010, 10110100, 11001100, 11010010, 11100001, 11111111\}$

As in [29], we map $00 \mapsto 00$, $11 \mapsto 20$, $10 \mapsto 11$, $01 \mapsto 31$ to get a subset of $\mathbb{Z}_4^8$.

$$\hat{C} = \{00000000, 00312011, 00112031, 00200020, 31001120, 31313131, 31113111, 31201100,$$
$$11003120, 11311131, 11111111, 11203100, 20002000, 20310011, 20110031, 20202020\}$$

Again as in [29], we let $\Sigma_2^4 = \{00, 22\}^4$ and consider the set $\hat{C} + \Sigma_2^4$, a subgroup of $\mathbb{Z}_4^8$ with 256 elements.

Finally, recall our Ising $\boxtimes$ Ising $\to \mathbb{Z}_4$ condensation from Section 3.3.1. Let $\mathcal{B}$ be the Ising modular tensor category, and consider the 16-fold Deligne product $(\mathcal{B} \boxtimes \mathcal{B})^{\boxtimes 8}$. Proposition 3.55 demonstrates that condensation of the algebra $(11 \oplus \psi\psi)^{\boxtimes 8}$ gives the MTC $\mathbb{Z}_4^{\boxtimes 8}$.

Recall the data of the $\mathbb{Z}_4$ modular tensor category.

$$\mathcal{L} = \{0, 1, 2, 3\}$$

$$d_1 = d_2 = d_3 = d_4 = 1$$

$$\theta_0 = 1 \qquad \theta_2 = -1$$
$$\theta_1 = \theta_3 = e^{\pi i/4}$$

Considering the set $\hat{C} + \Sigma_2^4$ as a set of objects of the MTC $\mathbb{Z}_4^{\boxtimes 8}$, we may observe that every element has dimension 1 and $\theta = 1$. Since $\mathbb{Z}_4$ is unitary, it is also pseudo-unitary, and we need not distinguish between the dimension and the Frobenius-Perron dimension of objects (see 2.70). We now proceed treating $\hat{C} + \Sigma_2^4$ as a subgroup of the Grothendieck group (Definition 2.68) of $\mathbb{Z}_4^{\boxtimes 8}$.

Consider the object

$$\mathcal{A} = \bigoplus_{x \in \hat{C} + \Sigma_2^4} x,$$

the direct sum of all elements of this bosonic subgroup.

Claim: It follows from [9, Corollary 3.8] that $\mathcal{A}$ has the structure of a Lagrangian algebra (Remark 3.2) in $\mathbb{Z}_4^{\boxtimes 8}$.

Proof: Certainly $\mathcal{A}$ is commutative by Proposition 3.3 and connected since it contains a single copy of 00000000. Since all objects have dimension 1, we also have

$$\text{FPdim}(\mathcal{A})^2 = 256^2 = 4^8 = \text{FPdim}\left(\mathbb{Z}_4^{\boxtimes 8}\right).$$

Finally, the object $\mathcal{A}$ is written as a direct sum with a coefficient of $n_x = 1$ on each element x of the set $\hat{C} + \Sigma_2^4$. Taking any $x, y \in \hat{C} + \Sigma_2^4$, we have

$$n_x n_y = 1 \leq \sum_{z \in \mathbb{Z}_4^{\boxtimes 8}} N_z^{xy} n_z$$

since the sum includes $z = xy$. ■

This object $\mathcal{A}$ as a Lagrangian algebra is also condensable. Recall Lemma 3.29 governing global dimensions after condensation. Since

$$\dim\left(\left(\mathbb{Z}_4^{\boxtimes 8}\right)_{\mathcal{A}}^0\right) = \frac{\dim\left(\mathbb{Z}_4^{\boxtimes 8}\right)}{\dim_{\mathbb{Z}_4^{\boxtimes 8}}(\mathcal{A})} = 1,$$

the condensation yields **Vec**, the trivial MTC. So we are able to sequentially condense

$$\mathcal{B}^{\boxtimes 16} \xrightarrow{(11 \oplus \psi\psi)^{\boxtimes 8}} \mathbb{Z}_4^{\boxtimes 8} \xrightarrow{\text{Hamming}} \mathbf{Vec}.$$

We might now ask how else we can generate Lagrangian algebras of $\mathbb{Z}_4$ to a power. In the Hamming(7,4) example, we added a parity bit to the end of every word to arrive at a doubly even code. In fact, the parity bit can be added in any of the eight positions to form an eight-bit string. These eight options all give distinct Lagrangian algebras.

A similar process can be run using the Golay code. The Golay code operates similarly to the Hamming code with generator matrix

$$G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}.$$

The image of this matrix is a subgroup of $\mathbb{Z}_2^{24}$, which gets mapped to a subgroup of $\mathbb{Z}_4^{24}$. From here, we produce a subgroup of $\mathbb{Z}_4^{24}$ with order 2^{24} and again have a Lagrangian algebra for the sequential condensation

$$\mathcal{B}^{\boxtimes 48} \xrightarrow{(1\!1 \oplus \psi\psi)^{\boxtimes 24}} \mathbb{Z}_4^{\boxtimes 24} \xrightarrow{\text{Golay}} \mathbf{Vec}.$$

While this is a pretty picture, it is unclear why this is interesting and why we have even introduced Ising when we could have left the picture at a condensation of many copies of $\mathbb{Z}_4$. The hope is to translate this condensation in general to a form of monstrous moonshine. This is touched on briefly in Section 6.6.

Chapter 6

Future Directions

As we have already asserted, there is no shortage of interesting applications of condensation. Some that we have mentioned are the study of the structure theory of modular tensor categories (Section 3.1.3.2), the understanding of topological phases of matter (Section 3.1.3.1) and how they may be used to build quantum computers (Section 5.1), the computation of category data that is otherwise exceedingly challenging to find (Section 5.2), and the study of moonshine by way of error-correcting codes (Section 5.3). Excitingly, all of this that we have presented is merely the beginning. Here we mention a host of problems we are aware of and hope to work on soon.

6.1 Further Understanding Condensation

It is not immediately obvious which algebras in a given modular tensor category are condensable because of the separability condition. We mentioned in Remark 3.25 that [43] gives a condition which is sufficient to conclude that an algebra $\mathcal{A} = \mathbb{1} \oplus B$ is not condensable for a bosonic object B. Going forward, we would like to fully characterize condensable algebras (and not just the simple ones of the form $1 \oplus B$) in a way that is easier to use in practice.

Though we can compute all the F- and R-symbols of the new category, we would like to have closed forms for the data in general.

Finally, there is much optimization work left to be done on the Mathematica code in Supplemental A: Mathematica Code. Once it has been done, we would like to compute the F-symbols of the near group category NG($\mathbb{Z}_3$,3) from those of $(G_2)_3$.

6.2 Modifications to Condensation

There seem to exist several natural modifications to condensation and the way we have defined condensed morphisms in Chapter 4.

Perhaps the natural way to define the condensed tensor product is by

$$\begin{array}{ccccc}
c & \xrightarrow{\quad\otimes\quad} & a \otimes b & & \\
{\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} & & \\
T(c) & \xrightarrow[\mathrm{id}_{\mathcal{A}} \otimes\otimes]{T(\otimes)} & T(a \otimes b) & \xrightarrow[\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_a \otimes \eta \otimes \mathrm{id}_b]{f} & T(a) \otimes_{\mathcal{A}} T(b)
\end{array}$$

The morphism f in [30, Theorem 1.6] uses the unit η of the algebra structure since [30] does not use a co-algebra structure at all. Another possibility might be to use the co-multiplication Δ to define $f = (\mathrm{id}_{\mathcal{A}} \otimes c_{\mathcal{A},a} \otimes \mathrm{id}_b) \circ (\Delta \otimes \mathrm{id}_a \otimes \mathrm{id}_b)$.

Other modifications one might make are switching the braiding in Diagram 4.4 or choosing different quotient sections as in Section 4.4.

It may be interesting to see which of these modifications yield equivalent categories, which do not, and whether any are interesting.

6.3 Property F Conjecture and Quantum Computing

A braided fusion category is said to have *property F* if the associated braid group representations on $\mathrm{End}(X^{\otimes n})$ have finite image for all $n \in \mathcal{N}$ and objects X [40]. There is a conjecture that a braided fusion category has property F if and only if its Frobenius-Perron dimension is an integer. This is significant as it seems that having property F is characteristic of modular categories whose physical realizations would not allow universal quantum computation. Section 5.1.2 and [34] use some work on braid group representations coming from modular tensor categories. We would like to study these more broadly and investigate the property F conjecture.

6.4 Structure of MTCs and the Witt Group

Classification of modular tensor categories has been a longstanding problem of interest [47], and braided fusion categories as a group seem to admit interesting structure (Section 2.7). At this point, there are enough elegant theorems about how these categories arise and behave to indicate that there exist more to be discovered. We would like to further contribute to this picture, perhaps through a better understanding Witt equivalence (Section 3.1.3.2).

6.5 Theory of Near-Group Categories

The numerical results of Section 5.2.1.1 were obtained through the use of Mathematica's equation solvers. More sophisticated approaches should be able to push the results further. We would also like to optimize the algorithms for computing condensed F-symbols so we can find data for the category $\mathrm{NG}(\mathbb{Z}_3, 3)$. With this additional example on hand, we hope to be better equipped to address Conjecture 5.8.

6.6 Moonshine for all Finite Groups

It was seen in Section 5.3 that lattices derived from the Hamming and Golay codes give subgroups of $\mathbb{Z}_4^n$ that are condensable bosonic algebras of modular tensor categories. These are notable, as the Golay lattice is the Leech lattice, which corresponds to the moonshine vertex operator algebra, whose group of automorphisms is the Monster group.

The Hamming lattice is the E_8 lattice. Following the sequential condensation process on MTCs to one on conformal field theories gives a form of monstrous moonshine.

We would like to explore the existence of doubly or triply even codes in general that give sequential condensations from $\text{Ising}^{\boxtimes 8n}$ to Vec, with the hope that this would shine some sunlight on the program of generalizing moonshine to other finite groups.

Bibliography

[1] Aaron Bagheri. *condensation-of-modular-tensor-categories*. URL: https://github.com/aaron-bagheri/condensation-of-modular-tensor-categories.

[2] Bojko Bakalov and Alexander Kirillov. *Lectures on Tensor Categories and Modular Functor*. May 2000. URL: https://www.math.stonybrook.edu/~kirillov/tensor/tensor.html.

[3] Maissam Barkeshli et al. "Symmetry fractionalization, defects, and gauging of topological phases". en. In: *Physical Review B* 100.11 (Sept. 2019), p. 115147. ISSN: 2469-9950, 2469-9969. DOI: 10.1103/PhysRevB.100.115147. URL: https://link.aps.org/doi/10.1103/PhysRevB.100.115147 (visited on 09/04/2023).

[4] Parsa Hassan Bonderson. "Non-Abelian Anyons and Interferometry". en. PhD thesis. California Institute of Technology, June 2007. DOI: 10.7907/5NDZ-W890. URL: https://resolver.caltech.edu/CaltechETD:etd-06042007-101617 (visited on 09/04/2023).

[5] Alain Bruguières. "Catégories prémodulaires, modularisations et invariants des variétés de dimension 3". In: *Mathematische Annalen* 316.2 (Feb. 2000), pp. 215–236. ISSN: 0025-5831, 1432-1807. DOI: 10.1007/s002080050011. URL: http://link.springer.com/10.1007/s002080050011 (visited on 05/23/2023).

[6] Paul Bruillard et al. "Rank-finiteness for modular categories". en. In: *Journal of the American Mathematical Society* 29.3 (July 2015), pp. 857–881. ISSN: 0894-0347,

1088-6834. DOI: 10.1090/jams/842. URL: https://www.ams.org/jams/2016-29-03/S0894-0347-2015-00842-6/ (visited on 08/28/2023).

[7] Paul Garrett Budinski. "Exotic Fusion Categories and Their Modular Data". In: (2021). DOI: 10.7939/R3-7P7K-FG18. URL: https://era.library.ualberta.ca/items/daeeb131-2efc-40b8-aef0-9d0a5f502b47 (visited on 07/09/2023).

[8] F.J. Burnell. "Anyon Condensation and Its Applications". en. In: *Annual Review of Condensed Matter Physics* 9.1 (Mar. 2018), pp. 307–327. ISSN: 1947-5454, 1947-5462. DOI: 10.1146/annurev-conmatphys-033117-054154. URL: https://www.annualreviews.org/doi/10.1146/annurev-conmatphys-033117-054154 (visited on 08/08/2023).

[9] Iris Cong, Meng Cheng, and Zhenghan Wang. "Hamiltonian and Algebraic Theories of Gapped Boundaries in Topological Phases of Matter". en. In: *Communications in Mathematical Physics* 355.2 (Oct. 2017), pp. 645–689. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s00220-017-2960-4. URL: http://link.springer.com/10.1007/s00220-017-2960-4 (visited on 11/08/2022).

[10] Shawn X. Cui et al. "On Gauging Symmetry of Modular Categories". en. In: *Communications in Mathematical Physics* 348.3 (Dec. 2016), pp. 1043–1064. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s00220-016-2633-8. URL: http://link.springer.com/10.1007/s00220-016-2633-8 (visited on 09/04/2023).

[11] Shawn Xingshan Cui, Modjtaba Shokrian Zini, and Zhenghan Wang. "On generalized symmetries and structure of modular categories". en. In: *Science China Mathematics* 62.3 (Mar. 2019), pp. 417–446. ISSN: 1674-7283, 1869-1862. DOI: 10.1007/s11425-018-9455-5. URL: http://link.springer.com/10.1007/s11425-018-9455-5 (visited on 11/08/2022).

[12] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. "On the structure of the Witt group of braided fusion categories". en. In: *Selecta Mathematica* 19.1 (Mar. 2013),

pp. 237–269. ISSN: 1022-1824, 1420-9020. DOI: 10.1007/s00029-012-0093-3. URL: http://link.springer.com/10.1007/s00029-012-0093-3 (visited on 09/04/2023).

[13] Alexei Davydov et al. "The Witt group of non-degenerate braided fusion categories". en. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2013.677 (Jan. 2013). ISSN: 1435-5345, 0075-4102. DOI: 10.1515/crelle.2012.014. URL: https://www.degruyter.com/document/doi/10.1515/crelle.2012.014/html (visited on 04/20/2023).

[14] Colleen Delaney, Sung Kim, and Julia Plavnik. "Zesting produces modular isotopes and explains their topological invariants". In: (2021). DOI: 10.48550/ARXIV.2107.11374. URL: https://arxiv.org/abs/2107.11374 (visited on 08/15/2023).

[15] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. en. Graduate Texts in Contemporary Physics. New York, NY: Springer New York, 1997. ISBN: 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9. URL: https://link.springer.com/10.1007/978-1-4612-2256-9 (visited on 11/08/2022).

[16] I. S. Eliëns, J. C. Romers, and F. A. Bais. "Diagrammatics for Bose condensation in anyon theories". en. In: *Physical Review B* 90.19 (Nov. 2014), p. 195130. ISSN: 1098-0121, 1550-235X. DOI: 10.1103/PhysRevB.90.195130. URL: https://link.aps.org/doi/10.1103/PhysRevB.90.195130 (visited on 09/04/2023).

[17] P. I. Etingof et al., eds. *Tensor categories*. en. Mathematical surveys and monographs volume 205. Providence, Rhode Island: American Mathematical Society, 2015. ISBN: 978-1-4704-2024-6.

[18] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. *On fusion categories*. en. Apr. 2017. DOI: https://doi.org/10.48550/arXiv.math/0203060. URL: http://arxiv.org/abs/math/0203060 (visited on 11/08/2022).

[19] David E. Evans and Terry Gannon. “Near-group fusion categories and their doubles”. en. In: *Advances in Mathematics* 255 (Apr. 2014), pp. 586–640. ISSN: 00018708. DOI: 10.1016/j.aim.2013.12.014. URL: https://linkinghub.elsevier.com/retrieve/pii/S000187081300460X (visited on 09/04/2023).

[20] Michael Freedman et al. “Topological quantum computation”. en. In: *Bulletin of the American Mathematical Society* 40.1 (Oct. 2002), pp. 31–38. ISSN: 0273-0979, 1088-9485. DOI: 10.1090/S0273-0979-02-00964-3. URL: https://www.ams.org/bull/2003-40-01/S0273-0979-02-00964-3/ (visited on 08/20/2023).

[21] Michael H. Freedman, Michael Larsen, and Zhenghan Wang. “A Modular Functor Which is Universal¶for Quantum Computation”. In: *Communications in Mathematical Physics* 227.3 (June 2002), pp. 605–622. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s002200200645. URL: http://link.springer.com/10.1007/s002200200645 (visited on 08/16/2023).

[22] Jürg Fröhlich et al. “Correspondences of ribbon categories”. en. In: *Advances in Mathematics* 199.1 (Jan. 2006), pp. 192–329. ISSN: 00018708. DOI: 10.1016/j.aim.2005.04.007. URL: https://linkinghub.elsevier.com/retrieve/pii/S0001870805001027 (visited on 09/04/2023).

[23] Terry Gannon, Gerald Hohn, and Yamauchi Hiroshi. *The online database of Vertex Operator Algebras and Modular Categories (Version 0.5)*. URL: https://www.math.ksu.edu/~gerald/voas/ (visited on 08/09/2023).

[24] Tobias J. Hagge and Seung-Moon Hong. “Some non-braided fusion categories of rank 3”. en. In: *Communications in Contemporary Mathematics* 11.04 (Aug. 2009), pp. 615–637. ISSN: 0219-1997, 1793-6683. DOI: 10.1142/S0219199709003521. URL: https://www.worldscientific.com/doi/abs/10.1142/S0219199709003521 (visited on 09/04/2023).

[25] André Henriques, David Penneys, and James Tener. "Categorified trace for module tensor categories over braided tensor categories". In: *Documenta Mathematica* 21.2016 (Sept. 2016), pp. 1089–1149. ISSN: 1431-0635.

[26] Yi-Zhi Huang. "Vertex operator algebras, the Verlinde conjecture, and modular tensor categories". en. In: *Proceedings of the National Academy of Sciences* 102.15 (Apr. 2005), pp. 5352–5356. ISSN: 0027-8424, 1091-6490. DOI: 10.1073/pnas.0409901102. URL: https://pnas.org/doi/full/10.1073/pnas.0409901102 (visited on 08/27/2023).

[27] Masaki Izumi. *A Cuntz algebra approach to the classification of near-group categories*. arXiv:1512.04288 [math]. Dec. 2015. URL: http://arxiv.org/abs/1512.04288 (visited on 07/07/2023).

[28] Masaki Izumi. "The Structure of Sectors Associated with Longo-Rehren Inclusions II: Examples". en. In: *Reviews in Mathematical Physics* 13.05 (May 2001), pp. 603–674. ISSN: 0129-055X, 1793-6659. DOI: 10.1142/S0129055X01000818. URL: https://www.worldscientific.com/doi/abs/10.1142/S0129055X01000818 (visited on 07/09/2023).

[29] Yasuyuki Kawahigashi and Roberto Longo. "Local conformal nets arising from framed vertex operator algebras". en. In: *Advances in Mathematics* 206.2 (Nov. 2006), pp. 729–751. ISSN: 00018708. DOI: 10.1016/j.aim.2005.11.003. URL: https://linkinghub.elsevier.com/retrieve/pii/S0001870805004007 (visited on 11/08/2022).

[30] Alexander Kirillov and Viktor Ostrik. "On a q-Analogue of the McKay Correspondence and the ADE Classification of sl2 Conformal Field Theories". en. In: *Advances in Mathematics* 171.2 (Nov. 2002), pp. 183–227. ISSN: 00018708. DOI: 10.1006/aima.2002.2072. URL: https://linkinghub.elsevier.com/retrieve/pii/S0001870802920724 (visited on 11/08/2022).

[31] Alexander Kirillov Jr. *On G–equivariant modular categories*. arXiv:math/0401119. Jan. 2004. URL: http://arxiv.org/abs/math/0401119 (visited on 06/06/2023).

[32] A.Yu. Kitaev. "Fault-tolerant quantum computation by anyons". en. In: *Annals of Physics* 303.1 (Jan. 2003), pp. 2–30. ISSN: 00034916. DOI: 10.1016/S0003-4916(02)00018-0. URL: https://linkinghub.elsevier.com/retrieve/pii/S0003491602000180 (visited on 09/04/2023).

[33] Alexei Kitaev. "Anyons in an exactly solved model and beyond". en. In: *Annals of Physics* 321.1 (Jan. 2006), pp. 2–111. ISSN: 00034916. DOI: 10.1016/j.aop.2005.10.005. URL: https://linkinghub.elsevier.com/retrieve/pii/S0003491605002381 (visited on 08/16/2023).

[34] Elias Kokkas et al. *Quantum Computing with Two-dimensional Conformal Field Theories*. en. arXiv:2112.06144 [quant-ph]. Dec. 2021. URL: http://arxiv.org/abs/2112.06144 (visited on 11/13/2022).

[35] Liang Kong. "Anyon condensation and tensor categories". en. In: *Nuclear Physics B* 973 (Dec. 2021). arXiv:1307.8244 [cond-mat], p. 115607. ISSN: 05503213. DOI: 10.1016/j.nuclphysb.2021.115607. URL: http://arxiv.org/abs/1307.8244 (visited on 04/20/2023).

[36] Michael Müger. "From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories". en. In: *Journal of Pure and Applied Algebra* 180.1-2 (May 2003), pp. 81–157. ISSN: 00224049. DOI: 10.1016/S0022-4049(02)00247-5. URL: https://linkinghub.elsevier.com/retrieve/pii/S0022404902002475 (visited on 08/28/2023).

[37] Michael Müger. "From subfactors to categories and topology II: The quantum double of tensor categories and subfactors". en. In: *Journal of Pure and Applied Algebra* 180.1-2 (May 2003), pp. 159–219. ISSN: 00224049. DOI: 10.1016/S0022-

4049(02)00248-7. URL: https://linkinghub.elsevier.com/retrieve/pii/S0022404902002487 (visited on 08/10/2023).

[38] Michael Müger. "Galois extensions of braided tensor categories and braided crossed G-categories". en. In: *Journal of Algebra* 277.1 (July 2004), pp. 256–281. ISSN: 00218693. DOI: 10.1016/j.jalgebra.2004.02.026. URL: https://linkinghub.elsevier.com/retrieve/pii/S0021869304001383 (visited on 08/20/2023).

[39] Michael Müger. "On the Structure of Modular Categories". en. In: *Proceedings of the London Mathematical Society* 87.02 (Sept. 2003), pp. 291–308. ISSN: 0024-6115, 1460-244X. DOI: 10.1112/S0024611503014187. URL: http://doi.wiley.com/10.1112/S0024611503014187 (visited on 09/04/2023).

[40] Deepak Naidu and Eric C. Rowell. "A Finiteness Property for Braided Fusion Categories". en. In: *Algebras and Representation Theory* 14.5 (Oct. 2011), pp. 837–855. ISSN: 1386-923X, 1572-9079. DOI: 10.1007/s10468-010-9219-5. URL: http://link.springer.com/10.1007/s10468-010-9219-5 (visited on 09/04/2023).

[41] Chetan Nayak et al. "Non-Abelian anyons and topological quantum computation". en. In: *Reviews of Modern Physics* 80.3 (Sept. 2008), pp. 1083–1159. ISSN: 0034-6861, 1539-0756. DOI: 10.1103/RevModPhys.80.1083. URL: https://link.aps.org/doi/10.1103/RevModPhys.80.1083 (visited on 08/16/2023).

[42] Titus Neupert et al. "Boson condensation in topologically ordered quantum liquids". en. In: *Physical Review B* 93.11 (Mar. 2016), p. 115103. ISSN: 2469-9950, 2469-9969. DOI: 10.1103/PhysRevB.93.115103. URL: https://link.aps.org/doi/10.1103/PhysRevB.93.115103 (visited on 09/04/2023).

[43] Titus Neupert et al. "No-go theorem for boson condensation in topologically ordered quantum liquids". In: *New Journal of Physics* 18.12 (Dec. 2016), p. 123009. ISSN: 1367-2630. DOI: 10.1088/1367-2630/18/12/123009. URL: https://iopscience.iop.org/article/10.1088/1367-2630/18/12/123009 (visited on 09/04/2023).

[44] Siu-Hung Ng et al. “Higher central charges and Witt groups”. en. In: *Advances in Mathematics* 404 (Aug. 2022), p. 108388. ISSN: 00018708. DOI: 10.1016/j.aim.2022.108388. URL: https://linkinghub.elsevier.com/retrieve/pii/S0001870822002043 (visited on 08/29/2023).

[45] B. Pareigis. “On Braiding and Dyslexia”. en. In: *Journal of Algebra* 171.2 (Jan. 1995), pp. 413–425. ISSN: 00218693. DOI: 10.1006/jabr.1995.1019. URL: https://linkinghub.elsevier.com/retrieve/pii/S0021869385710198 (visited on 05/23/2023).

[46] D. Roberts. “Representing modular tensor categories: a computer algebra system for topological quantum computing”. English. http://purl.org/dc/dcmitype/Text. Oxford University, UK, 2011. URL: https://ora.ox.ac.uk/objects/uuid:c9b6eaf8-29d4-4637-a576-5a35d3c957bb (visited on 12/18/2022).

[47] Eric Rowell, Richard Stong, and Zhenghan Wang. “On Classification of Modular Tensor Categories”. en. In: *Communications in Mathematical Physics* 292.2 (Dec. 2009), pp. 343–389. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s00220-009-0908-z. URL: http://link.springer.com/10.1007/s00220-009-0908-z (visited on 09/04/2023).

[48] Eric C. Rowell. “From Quantum Groups to Unitary Modular Tensor Categories”. In: (2005). DOI: 10.48550/ARXIV.MATH/0503226. URL: https://arxiv.org/abs/math/0503226 (visited on 08/10/2023).

[49] Eric C. Rowell, Yuze Ruan, and Yilong Wang. “The Witt classes of so(2r)2r”. en. In: *Communications in Algebra* 50.12 (Dec. 2022), pp. 5246–5265. ISSN: 0092-7872, 1532-4125. DOI: 10.1080/00927872.2022.2083630. URL: https://www.tandfonline.com/doi/full/10.1080/00927872.2022.2083630 (visited on 09/04/2023).

[50] Jacob A. Siehler. *Braided Near-group Categories*. arXiv:math/0011037. Nov. 2000. URL: http://arxiv.org/abs/math/0011037 (visited on 06/29/2023).

[51] Vladimir G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. De Gruyter, July 2016. ISBN: 9783110435221. DOI: 10.1515/9783110435221. URL: https://www.degruyter.com/document/doi/10.1515/9783110435221/html (visited on 08/02/2023).

[52] Zhenghan Wang, ed. *Topological quantum computation*. Regional conference series in mathematics / Conference Board of the Mathematical Sciences no. 112. Providence, R.I: Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society with support from the National Science Foundation, 2010. ISBN: 9780821849309.

از آمدنم نبود گردون را سود   وز رفتن من جلال و جاهش نفزود
وز هیچ کسی نیز دو گوشم نشنود   کاین آمدن و رفتنم از بهر چه بود

ـ خیام