UNIVERSITY OF CALIFORNIA, SAN DIEGO

Noncommutative Plurisubharmonic Polynomials

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Jeremy Michael Greene

Committee in charge:

Professor J. William Helton, Chair
Professor Jim Agler
Professor Robert Bitmead
Professor Mauricio de Oliveira
Professor Lei Ni

2011

The dissertation of Jeremy Michael Greene is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2011

DEDICATION

To my parents: Bill and Marilyn

And to theirs: Al, Margie, David, and Rose.

EPIGRAPH

You're wealthy when you're healthy.
Don't trade gold for copper.
Go all the way, baby!
—Al

Don't dream it, be it.
—No Fear

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

To my parents for their unconditional love, support, and belief in me since the beginning: August 16, 1984.

To Bill Helton: You have been the most incredible, understanding, and patient mentor one could ask for. Your constant encouragement was so helpful. You took me under your wing when I was an undergraduate. That was such a rewarding opportunity and experience for me. I feel so lucky to have been granted entrance to the summer NCAlgebra club. I don't know what I would have done without your generous summer funding for all of these years. I thank you for all of your wisdom and mathematical conversations. I will never forget these last 6 years.

Thanks to my committee members: Jim Agler, Robert Bitmead and Lei Ni for listening to my talks and being interested in my work. And to Mauricio de Oliveira, thanks for your Mathematica expertise and your awesome MAE 280B class!

Thanks to Harry Dym, Scott McCullough, and Victor Vinnikov for writing and working with me. It has been a pleasure.

Thanks to NSF grants DMS-0700758 and DMS-0757212 for partial research support.

Chapter 2 of this dissertation is taken from [GHV] that has been submitted for publication with coauthors J. William Helton and Victor Vinnikov as
J. M. Greene, J. W. Helton and V. Vinnikov, Noncommutative Plurisubharmonic Polynomials, Part I: Global Assumptions, preprint, http://arxiv.org/abs/1101.0107 .

Chapter 3 of this dissertation is taken from [G10] that has been submitted for publication as
J. M. Greene, Noncommutative Plurisubharmonic Polynomials, Part II: Local Assumptions, preprint, http://arxiv.org/abs/1101.0111 .

I would like to thank my family: Grandparents Dave and Rose Voss, Al and Margie Greene. Thanks to all of my aunts, uncles, and cousins for their love, care, vacations, holiday get-togethers, and laughter. Uncle John and Aunt Marie:

thank you so much for giving me a home away from home during my nine years of school here in San Diego – I don't know where I would be without that.

Thanks to Daniel Robinson for being the best roommate one could ask for. We sure had a lot of fun and I'll never forget it. You have had a serious impact on my life. Thanks to Ryan Szypowski for helping me through real analysis.

Thanks to the Plush family for their love and support through the end of this process. I feel so lucky to have you all as family and we have a lot of fun times to look forward to.

Finally, Rochelle. Words can not describe my gratitude for your love and support through my final years of this. You keep me sane, you keep me laughing, and you are an inspiration. Always remember and never forget 11-13-11.

VITA

2006	B.A. in Mathematics, *summa cum laude*, University of California, San Diego
2008	M.A. in Mathematics, University of California, San Diego
2009	C.Phil. in Mathematics, University of California, San Diego
2006-2011	Graduate Teaching Assistant, University of California, San Diego
2010	Associate Instructor, University of California, San Diego
2011	Ph.D. in Mathematics, University of California, San Diego

PUBLICATIONS

H. Dym, J. M. Greene, J. W. Helton and S. A. McCullough: Classification of all noncommutative polynomials whose Hessian has negative signature one and a noncommutative second fundamental form, *J. Anal. Math. 108 (2009), 19-59* .

J. M. Greene, Noncommutative Plurisubharmonic Polynomials, Part II: Local Assumptions, preprint, `http://arxiv.org/abs/1101.0111` .

J. M. Greene, J. W. Helton and V. Vinnikov, Noncommutative Plurisubharmonic Polynomials, Part I: Global Assumptions, preprint, `http://arxiv.org/abs/1101.0107` .

ABSTRACT OF THE DISSERTATION

Noncommutative Plurisubharmonic Polynomials

by

Jeremy Michael Greene

Doctor of Philosophy in Mathematics

University of California, San Diego, 2011

Professor J. William Helton, Chair

Many optimization problems and engineering problems connected with linear systems lead to matrix inequalities. Matrix inequalities are constraints in which a polynomial or a matrix of polynomials with matrix variables is required to take a positive semidefinite value. Many of these problems have the property that they are "dimension free" and, in this case, the form of the polynomials remains the same for matrices of all size. In other words, we have noncommutative polynomials. One very much desires these polynomials to be "convex" in the unknown matrix variables, since if they are, then numerical calculations are reliable and local optima are global optima.

In this dissertation, we classify all changes of variables (not containing transposes) from noncommutative non-convex polynomials to noncommutative convex

polynomials. This introduces notions of noncommutative complex Hessians and plurisubharmonicity, classical notions from several complex variables. In addition, we present a theory of noncommutative integration and we prove a “local implies global” result in that we show noncommutative plurisubharmonicity on a noncommutative open set implies noncommutative plurisubharmonicity everywhere.

Chapter 1

Introduction

Many optimization problems and engineering problems connected with linear systems lead to matrix inequalities. Matrix inequalities are constraints in which a polynomial or a matrix of polynomials with matrix variables is required to take a positive semidefinite value. Many of these problems have the property that they are "dimension free" and, in this case, the form of the polynomials remains the same for matrices of all size. In other words, we have noncommutative polynomials. One very much desires these polynomials to be "convex" in the unknown matrix variables, since if they are, then numerical calculations are reliable and local optima are global optima.

Often, one is most interested in the Hessian of a polynomial and its positivity, as this determines convexity. However, in this dissertation, we are concerned with the *complex Hessian*, since it turns out to be related to which problems can be made convex by an nc analytic changes of variables.

In the classical study of complex variables, we have polynomials in z and $\bar{z}$. We can then take derivatives with respect to z and $\bar{z}$; i.e., $\frac{\partial p}{\partial z}(z, \bar{z})$ and $\frac{\partial p}{\partial \bar{z}}(z, \bar{z})$. We can also construct a matrix and fill it with mixed second partial derivatives; i.e., the (i, j)-th entry of the matrix is $\frac{\partial^2 p}{\partial z_i \partial \bar{z}_j}$. In classical several complex variables, this matrix of mixed partial derivatives is called the complex Hessian and if this complex Hessian is positive semidefinite, then the original polynomial is said to be plurisubharmonic (plush).

In this dissertation, we present noncommutative directional derivatives, the noncommutative analogue of the complex Hessian, and we define noncommutative plurisubharmonicity. We classify all noncommutative changes of variables (not containing transposes) from noncommutative non-convex polynomials to noncommutative convex ones. This introduces notions of noncommutative complex Hessians and noncommutative plurisubharmonicity, extending classical notions from several complex variables. In addition, we prove a “local implies global” result in that we show noncommutative plurisubharmonicity on an “nc open set” implies noncommutative plurisubharmonicity everywhere (this is false in classical analysis).

We also present a theory of noncommutative integration. We give necessary and sufficient conditions as to when a given nc polynomial is an nc directional derivative. We also give necessary and sufficient conditions as to when a given nc polynomial is an nc complex Hessian. In addition, we prove a noncommutative version of the Frobenius theorem where we determine when a given nc polynomial is the “nc gradient of a potential”.

This introductory chapter gives the necessary basic definitions and summarizes the main results in this dissertation. Chapter 2 contains the result that classifies all nc plush polynomials and the theory of nc integration. Chapter 3 contains the “local implies global” result.

1.1 NC Polynomials

Many of the the definitions we shall need sit in the context of an elegant theory of noncommutative analytic functions, such as is developed in the articles [K-VV] and [Voi1, Voi2]; see also [Pop6]. Also, related to the results presented in this dissertation are those on various classes of noncommutative functions on balls as in [AK, BGM]. Transformations on nc variables with analytic functions are described in [HKM, Pop6].

1.1.1 NC Variables and Monomials

We consider the free semi-group on the $2g$ noncommuting formal variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$. The variables x_j^T are the formal transposes of the variables x_j. The free semi-group in these $2g$ variables generates monomials in all of these variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$, often called monomials in x, x^T.

If m is a monomial, then m^T denotes the transpose of the monomial m. For example, given the monomial (in the x_j's) $x^w = x_{j_1} x_{j_2} \ldots x_{j_n}$, the involution applied to x^w is $(x^w)^T = x_{j_n}^T \ldots x_{j_2}^T x_{j_1}^T$.

1.1.2 The Ring of NC Polynomials

Let $\mathbb{R}\langle x_1, \ldots, x_g, x_1^T, \ldots, x_g^T \rangle$ denote the ring of noncommutative polynomials over $\mathbb{R}$ in the noncommuting variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$. We often abbreviate

$$\mathbb{R}\langle x_1, \ldots, x_g, x_1^T, \ldots, x_g^T \rangle \quad \text{by} \quad \mathbb{R}\langle x, x^T \rangle.$$

Note that $\mathbb{R}\langle x, x^T \rangle$ maps to itself under the involution ^{T}.

We call a polynomial **nc analytic** if it contains only the variables x_j and none of the transposed variables x_i^T. Similarly, we call a polynomial **nc antianalytic** if it contains only the variables x_j^T and none of the variables x_i.

We call an nc polynomial, p, **symmetric** if $p^T = p$. For example, $p = x_1 x_1^T + x_2^T x_2$ is symmetric. The polynomial $\tilde{p} = x_1 x_2 x_4 + x_3 x_1$ is nc analytic but not symmetric. Finally, the polynomial $\hat{p} = x_2^T x_1^T + 4x_3^T$ is nc antianalytic but not symmetric.

We call an nc polynomial **hereditary** if all $x_1^T, x_2^T, \ldots x_g^T$ variables appear to the *left* of every $x_1, x_2, \ldots, x_g$ variable. Similarly, we call an nc polynomial **antihereditary** if all $x_1^T, x_2^T, \ldots x_g^T$ variables appear to the *right* of every $x_1, x_2, \ldots, x_g$ variable. For example, when $g = 1$, the nc polynomial $p = x^T x^T x x$ is hereditary, $p = x x x^T x^T$ is antihereditary, and $p = x x^T x + x^T x x^T$ is neither hereditary nor antihereditary.

1.1.3 Substituting Matrices for NC Variables

If p is an nc polynomial in the variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$ and

$$X = (X_1, X_2, \ldots, X_g) \in (\mathbb{R}^{n\times n})^g,$$

the evaluation $p(X, X^T)$ is defined by replacing x_j by X_j and x_j^T by X_j^T. Note that, for $Z_n = (0_n, 0_n, \ldots, 0_n) \in (\mathbb{R}^{n\times n})^{2g}$ where each 0_n is the $n \times n$ zero matrix, $p(0_n) = I_n \otimes p(0_1)$. Because of this relationship, we often write $p(0)$ with the size n unspecified. The involution, ^{T}, is compatible with matrix transposition, i.e.,

$$p^T(X, X^T) = p(X, X^T)^T.$$

Matrix Positivity

We say that an nc symmetric polynomial, p, in the $2g$ variables $x_1, \ldots, x_g$, $x_1^T, \ldots, x_g^T$, is **matrix positive** if $p(X, X^T)$ is a positive semidefinite matrix when evaluated on every $X \in (\mathbb{R}^{n\times n})^g$ for every size $n \geq 1$; i.e.,

$$p(X, X^T) \succeq 0$$

for all $X \in (\mathbb{R}^{n\times n})^g$ and all $n \geq 1$.

In [H02], Helton classified all matrix positive nc symmetric polynomials as sums of squares. We recall Theorem 1.1 from [H02]:

Theorem 1.1.1. *Suppose p is a noncommutative symmetric polynomial. If p is a sum of squares, then p is matrix positive. If p is matrix positive, then p is a sum of squares.*

Matrix Convexity

We say that an nc symmetric polynomial, p, is **matrix convex** if

$$tp(X, X^T) + (1-t)p(Y, Y^T) - p(tX + (1-t)Y, tX^T + (1-t)Y^T) \succeq 0$$

for all $0 \leq t \leq 1$ and for all $X, Y \in (\mathbb{R}^{n\times n})^g$ for every $n \geq 1$. This is the usual convexity inequality known from classical analysis. In [HM04], Helton and McCullough classified all nc matrix convex symmetric polynomials as having degree two. More specifically, we now recall Corollary 7.1 in [HM04]:

Theorem 1.1.2. *A noncommutative symmetric matrix convex polynomial p can be written as*

$$p(x, x^T) = c_0 + \Lambda_0(x, x^T) + \sum_{j=1}^{N} \Lambda_j(x, x^T)^T \Lambda_j(x, x^T)$$

where $\Lambda_0, \ldots, \Lambda_N$ are linear in x, x^T and c_0 is a real constant.

1.2 NC Differentiation

Now we make some definitions and state some properties about nc differentiation. In the classical study of complex variables, we have polynomials in z and $\bar{z}$. We can then take derivatives with respect to z and $\bar{z}$; i.e., $\frac{\partial p}{\partial z}(z, \bar{z})$ and $\frac{\partial p}{\partial \bar{z}}(z, \bar{z})$. We can also make a matrix and fill it with mixed second partial derivatives; i.e., the (i, j)-th entry of the matrix is $\frac{\partial^2 p}{\partial z_i \partial \bar{z}_j}$. In classical several complex variables, this matrix of mixed partial derivatives is called the complex Hessian.

The noncommutative differentiation of polynomials in x and x^T defined in this dissertation is analogous to classical differentiation of polynomials in z and $\bar{z}$ from several complex variables.

Definition of Directional Derivative

Let p be an nc polynomial in the nc variables $x = (x_1, \ldots, x_g)$ and $x^T = (x_1^T, \ldots, x_g^T)$. In order to define a directional derivative, we first replace all x_i^T by y_i. Then the **directional derivative of p with respect to x_j in the direction h_j** is

$$p_{x_j}[h_j] := \frac{\partial p}{\partial x_j}(x, x^T)[h_j] = \frac{dp}{dt}(x_1, \ldots, x_j + th_j, \ldots, x_g, y_1, \ldots, y_g)|_{t=0}|_{y_i = x_i^T}. \quad (1.1)$$

The **directional derivative of p with respect to x_j^T in the direction k_j** is

$$p_{x_j^T}[k_j] := \frac{\partial p}{\partial x_j^T}(x, x^T)[k_j] = \frac{dp}{dt}(x_1, \ldots, x_g, y_1, \ldots, y_j + tk_j, \ldots, y_g)|_{t=0}|_{y_i = x_i^T}. \quad (1.2)$$

Often, we take $k_j = h_j^T$ in Equation (1.2) and we define

$$\begin{aligned} p_x[h] &:= \frac{\partial p}{\partial x}(x, x^T)[h] = \frac{dp}{dt}(x+th, y)|_{t=0}|_{y=x^T} = \sum_{i=1}^{g} \frac{\partial p}{\partial x_i}(x, x^T)[h_i] \\ p_{x^T}[h^T] &:= \frac{\partial p}{\partial x^T}(x, x^T)[h^T] = \frac{dp}{dt}(x, y+tk)|_{t=0}|_{y=x^T, k=h^T} = \sum_{i=1}^{g} \frac{\partial p}{\partial x_i^T}(x, x^T)[h_i^T]. \end{aligned}$$

Then, we (abusively[1]) define the ℓ^{th} directional derivative of p in the direction h as

$$p^{(\ell)}(x)[h] := \frac{d^\ell p}{dt^\ell}(x+th, y+tk)|_{t=0}|_{y=x^T, k=h^T}$$

so the first directional derivative of p in the direction h is

$$\begin{aligned} p'(x)]h] &= \frac{\partial p}{\partial x}(x, x^T)[h] + \frac{\partial p}{\partial x^T}(x, x^T)[h^T] && (1.3) \\ &= p_x[h] + p_{x^T}[h^T]. && (1.4) \end{aligned}$$

It is important to note that the directional derivative is an nc polynomial that is homogeneous degree 1 in h, h^T. If p is symmetric, so is p'.

Examples of Differentiation

Here we provide some examples of how to compute directional derivatives.

Example 1.2.1. Let $p = x_1 x_2^T x_1 + x_1^T x_2 x_1^T$. Then we have

$$\begin{aligned} p_{x_1}[h_1] &= \frac{\partial p}{\partial x_1}(x, x^T)[h_1] = h_1 x_2^T x_1 + x_1 x_2^T h_1 \\ p_{x_2^T}[h_2^T] &= \frac{\partial p}{\partial x_2^T}(x, x^T)[h_2^T] = x_1 h_2^T x_1 \\ p_x[h] &= \frac{\partial p}{\partial x}(x, x^T)[h] = h_1 x_2^T x_1 + x_1 x_2^T h_1 + x_1^T h_2 x_1^T \end{aligned}$$

and,

$$p'(x)[h] = h_1 x_2^T x_1 + x_1 h_2^T x_1 + x_1 x_2^T h_1 + h_1^T x_2 x_1^T + x_1^T h_2 x_1^T + x_1^T x_2 h_1^T.$$

[1] For more detail, see [HMV06]. The idea for computing $p^{(\ell)}(x)[h]$ is that we first noncommutatively expand $p(x+th)$. Then, $p^{(\ell)}(x)[h]$ is the coefficient of t^ℓ multiplied by $\ell!$; i.e., $p^{(\ell)}(x)[h] = (\ell!)(\text{coefficient of } t^\ell)$.

Example 1.2.2. Given a general monomial, with $c \in \mathbb{R}$,

$$m = cx_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_n}^{i_n}$$

where each i_k is either 1 or T, we get that

$$m' = ch_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_n}^{i_n} + cx_{j_1}^{i_1} h_{j_2}^{i_2} x_{j_3}^{i_3} \cdots x_{j_n}^{i_n} + \cdots + cx_{j_1}^{i_1} \cdots x_{j_{n-1}}^{i_{n-1}} h_{j_n}^{i_n} \ .$$

1.2.1 NC Hessian and NC Complex Hessian

Often, one is most interested in the Hessian of a polynomial and its positivity; as this determines convexity. However, in this dissertation, we are concerned with the *complex Hessian* since it turns out to be related to nc analytic changes of variables.

We define the **nc complex Hessian** , $q(x, x^T)[h, h^T]$, of an nc polynomial p as the nc polynomial in the $4g$ variables $x = (x_1, \dots, x_g)$, $x^T = (x_1^T, \dots, x_g^T)$, $h = (h_1, \dots, h_g)$, and $h^T = (h_1^T, \dots, h_g^T)$

$$q(x, x^T)[h, h^T] := \frac{\partial^2 p}{\partial s \partial t}(x + th, y + sk)|_{t,s=0}|_{y=x^T, k=h^T}. \tag{1.5}$$

The nc complex Hessian is an iterated nc directional derivative in the sense that we compute it as follows. We first take the nc directional derivative of p with respect to x^T in the direction h^T to get $p_{x^T}[h^T]$. Then, we take the nc directional derivative of that with respect to x in the direction h to get $(p_{x^T}[h^T])_x[h]$. We will see later, in Lemma 2.2.12, that we can switch the order of differentiation to $(p_x[h])_{x^T}[h^T]$ and we still get the same polynomial. Sometimes we denote the nc complex Hessian as $p_{x^T,x}[h^T, h]$. Hence, we have the following equivalent notations for the nc complex Hessian (and we will use each one when context is convenient):

$$q(x, x^T)[h, h^T] = p_{x^T,x}[h^T, h] = (p_{x^T}[h^T])_x[h] = (p_x[h])_{x^T}[h^T]. \tag{1.6}$$

An extremely important fact about $q(x, x^T)[h, h^T]$, which is restated in Theorem 2.2.16 (P1), is that it is quadratic in h, h^T and that **each term contains some h_j and some h_k^T**. The nc complex Hessian is actually a piece of the full **nc**

Hessian which is

$$\begin{aligned} p'' &= \frac{\partial^2 p}{\partial t^2}(x+th, y)|_{t=0}|_{y=x^T} + \frac{\partial^2 p}{\partial t \partial s}(x+th, y+sk)|_{t,s=0}|_{y=x^T,k=h^T} \\ &+ \frac{\partial^2 p}{\partial s \partial t}(x+th, y+sk)|_{t,s=0}|_{y=x^T,k=h^T} + \frac{\partial^2 p}{\partial s^2}(x, y+sk)|_{s=0}|_{y=x^T,k=h^T} \\ &= 2q(x,x^T)[h,h^T] + \frac{\partial^2 p}{\partial t^2}(x+th, y)|_{t=0}|_{y=x^T} + \frac{\partial^2 p}{\partial s^2}(x, y+sk)|_{s=0}|_{y=x^T,k=h^T}. \end{aligned}$$

1.2.2 NC Plurisubharmonicity

We call a symmetric nc polynomial, p, **nc plurisubharmonic** (or **nc plush**) if the nc complex Hessian, q, of p is matrix positive. In other words, we require that q be positive semidefinite when evaluated on all tuples of real $n \times n$ matrices for every size n; i.e.,

$$q(X, X^T)[H, H^T] \succeq 0$$

for all $X, H \in (\mathbb{R}^{n\times n})^g$ for every $n \geq 1$.

Examples of NC Complex Hessians and NC Plurisubharmonicity

Here we provide some examples of how to compute nc complex Hessians.

Example 1.2.3. Let $p = x_1 x_2^T x_1 + x_1^T x_2 x_1^T$ as in Example 1.2.1. Then we have

$$q = h_1 h_2^T x_1 + x_1 h_2^T h_1 + h_1^T h_2 x_1^T + x_1^T h_2 h_1^T.$$

Example 1.2.4. Let $p = x^T x^T x x$. Then, we have

$$\begin{aligned} q(x,x^T)[h,h^T] &= h^T x^T h x + h^T x^T x h + x^T h^T h x + x^T h^T x h \\ &= (h^T x^T + x^T h^T)(hx + xh) \\ &= (hx + xh)^T (hx + xh). \end{aligned}$$

We can see that, for any $X, H \in \mathbb{R}^{n\times n}$ for any size $n \geq 1$, we have that

$$q(X, X^T)[H, H^T] = (HX + XH)^T (HX + XH) \succeq 0.$$

Hence, this nc polynomial, $p = x^T x^T x x$, is nc plush.

Example 1.2.5. The nc complex Hessian of any nc analytic polynomial is 0. The nc complex Hessian of any nc antianalytic polynomial is 0. Hence, both nc analytic and nc antianalytic polynomials are nc plush.

1.3 Main Results of Chapter 2

In Chapter 2 we classify all symmetric nc plush polynomials in g free variables.

Theorem 1.3.1. *An nc symmetric polynomial p in free variables is nc plurisubharmonic if and only if p can be written in the form*

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{1.7}$$

where the sums are finite and each f_j, k_j, F is nc analytic.

Proof. The proof requires all of Chapter 2 and culminates in Section 2.4. □

Chapter 3 strengthens the result of Theorem 1.3.1 by weakening the hypothesis while keeping the same conclusion. Specifically, in Chapter 3, we assume that the nc polynomial is nc plush on an "nc open set" and conclude that it is nc plush everywhere and hence has the form in Equation (1.7). The proof, in Chapter 3, draws on most of the theorems in Chapter 2 together with a very different technique involving representations of noncommutative quadratic functions.

The representation in Equation (1.7) is unique up to the natural transformations.

Theorem 1.3.2. *Let p be an nc symmetric polynomial in free variables that is nc plurisubharmonic and let*

$$\begin{aligned}
\widetilde{N} &:= \min\{N : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\} \\
\widetilde{M} &:= \min\{M : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\}.
\end{aligned}$$

Then, we can represent p as

$$p = \sum_{j=1}^{\widetilde{N}} \tilde{f}_j^T \tilde{f}_j + \sum_{j=1}^{\widetilde{M}} \tilde{k}_j \tilde{k}_j^T + \tilde{F} + \tilde{F}^T$$

and if N and M are integers such that $N \geq \widetilde{N}$, $M \geq \widetilde{M}$ and

$$p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T \ ,$$

then there exist isometries $U_1 : \mathbb{R}^{\widetilde{N}} \longrightarrow \mathbb{R}^N$ *and* $U_2 : \mathbb{R}^{\widetilde{M}} \longrightarrow \mathbb{R}^M$ *such that*

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = U_1 \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_{\widetilde{N}} \end{pmatrix} + \vec{c}_1 \quad \textit{and} \quad \begin{pmatrix} k_1^T \\ \vdots \\ k_M^T \end{pmatrix} = U_2 \begin{pmatrix} \tilde{k}_1^T \\ \vdots \\ \tilde{k}_{\widetilde{M}}^T \end{pmatrix} + \vec{c}_2$$

where $\vec{c}_1 \in \mathbb{R}^N$ *and* $\vec{c}_2 \in \mathbb{R}^M$*.*

Proof. Theorem 1.3.1 gives the desired form of p and nc integration will give the uniqueness. We provide the details of the proof in Section 2.4. □

A byproduct of the proof of Theorem 1.3.1 is noncommutative integration theory of nc polynomials. This includes a Frobenius theorem for nc polynomials and is discussed further in Section 2.2.

1.4 Direct Sums and NC Open Sets

Now we present the additional definitions needed for Chapter 3. We start with direct sums and nc open sets. We then state the main results of Chapter 3.

1.4.1 Direct Sums

Our definition of the direct sum is the usual one, which for two matrices X_1 and X_2, is given by

$$X_1 \oplus X_2 := \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

Given a finite set of matrix tuples $\{X^1, \ldots, X^t\}$ with

$$X^j = \{X_{j1}, X_{j2}, \ldots, X_{jg}\} \in (\mathbb{R}^{n_j \times n_j})^g$$

for $j = 1, \ldots, t$, we define

$$\bigoplus_{j=1}^t X^j := \left\{ \bigoplus_{j=1}^t X_{j1}, \bigoplus_{j=1}^t X_{j2}, \ldots, \bigoplus_{j=1}^t X_{jg} \right\}.$$

For example, if $X^1 = \{X_{11}, \ldots, X_{1g}\}$, $X^2 = \{X_{21}, \ldots, X_{2g}\}$, and $X^3 = \{X_{31}, \ldots, X_{3g}\}$, we get

$$X^1 \oplus X^2 \oplus X^3 = \{X_{11} \oplus X_{21} \oplus X_{31}, \ldots, X_{1g} \oplus X_{2g} \oplus X_{3g}\}.$$

Now let

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$$

where $\mathcal{B}_n \subseteq (\mathbb{R}^{n\times n})^g$ for $n \geq 1$ be given. The graded set $\mathcal{B}$ **respects direct sums** if for each finite set

$$\{X^1, \ldots, X^t\} \quad \text{with} \quad X^j \in \mathcal{B}_{n_j} \quad \text{and} \quad n = \sum_{j=1}^{t} n_j \ ,$$

with repetitions allowed, $\oplus_{j=1}^{t} X^j \in \mathcal{B}_n$.

1.4.2 NC Open Set

A set $\mathcal{G} \subseteq \cup_{n\geq 1}(\mathbb{R}^{n\times n})^g$ is an **nc open set** if $\mathcal{G}$ satisfies the following two conditions:

(i) $\mathcal{G}$ respects direct sums, and

(ii) there exists a positive integer n_0 such that if $n > n_0$, the set $\mathcal{G}_n := \mathcal{G} \cap (\mathbb{R}^{n\times n})^g$ is an open set of matrix tuples.

We say that an nc polynomial, p, is **nc plush on an nc open set**, $\mathcal{G}$, if the nc complex Hessian, q, of p satisfies

$$q(X, X^T)[H, H^T] \succeq 0 \tag{1.8}$$

for all $X \in \mathcal{G}$ and all $H \in (\mathbb{R}^{n\times n})^g$ for all $n \geq 1$.

1.5 Main Results of Chapter 3

As we will see, in Section 3.4, the nc complex Hessian, q, if matrix positive on an nc open set, can be factored as

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T] \tag{1.9}$$

where $D(x, x^T)$ is a diagonal matrix, $L(x, x^T)$ is a lower triangular matrix with ones on the diagonal (we call this a ***unit* lower triangular matrix**), and $V(x, x^T)[h, h^T]$ is a vector of monomials in x, x^T, h, h^T.

When we take the transpose of a matrix (or vector) with monomial or polynomial entries (e.g., $L(x, x^T)^T$ or $V(x, x^T)[h, h^T]^T$), we get the matrix obtained by taking the transpose (as a matrix) *and* applying the transpose (involution) to every entry.

Example 1.5.1. *If*

$$v = \begin{pmatrix} hxx \\ hx \\ h \end{pmatrix},$$

then

$$v^T = \begin{pmatrix} x^T x^T h^T & x^T h^T & h^T \end{pmatrix}.$$

The next theorem shows the surprising result that the diagonal matrix, $D(x, x^T)$, in Equation (1.9) does not depend on x, x^T and that $L(x, x^T)$ has nc polynomial entries.

Theorem 1.5.2. *If p is an nc symmetric polynomial that is nc plurisubharmonic on an nc open set, then q, the nc complex Hessian of p, can be written as*

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $V(x, x^T)[h, h^T]$ is a vector of monomials in x, x^T, h, h^T,

$$D = diag(d_1, d_2, \ldots, d_{\mathcal{N}})$$

is a positive semidefinite constant real matrix, and $L(x, x^T)$ is a unit lower triangular matrix with nc polynomial entries.

Proof. The proof of this theorem requires all of Chapter 3 and culminates in Subsection 3.4.4. □

This gives rise to an extension of Theorem 1.3.1. In Chapter 2, it is shown that an nc polynomial which is nc plush *everywhere* has the specific form given in Equation (1.10) below (same as Equation (1.7) above). In Chapter 3, Theorem 1.5.3, below, is a stronger, "local implies global", result in that an nc polynomial that is nc plush just on an nc open set is actually nc plush everywhere (and has the form in Equation (1.10)).

Theorem 1.5.3. *If an nc symmetric polynomial, p, is nc plurisubharmonic on an nc open set, then p is, in fact, nc plurisubharmonic everywhere and has the form expressed in Equation* (1.7) *from Theorem 1.3.1*

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{1.10}$$

where the sums are finite and each f_j, k_j, and F is nc analytic.

Proof. That $D = D(x, x^T)$, in Theorem 1.5.2, is a positive semidefinite constant real matrix immediately implies

$$q(X, X^T)[H, H^T] \succeq 0$$

for all $X, H \in \cup_{n \geq 1} (\mathbb{R}^{n \times n})^g$; that is, p is nc plush at all $X \in \cup_{n \geq 1} (\mathbb{R}^{n \times n})^g$. Consequently, Theorem 1.3.1 gives that p is of the desired form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T$$

where the sums are finite and f_j, k_j, F are nc analytic. □

Note that with an nc polynomial, p, as in Equation (1.10), the nc complex Hessian, q, of p is

$$q = \sum (f_j^T)_{x^T}[h^T](f_j)_x[h] + \sum (k_j)_x[h](k_j^T)_{x^T}[h^T] \ , \tag{1.11}$$

which is obviously matrix positive as it is a sum of squares. From Equation (1.11), we see that the nc complex Hessian for an nc polynomial that is nc plush on an nc open set has even degree.

Chapter 2

Noncommutative Plurisubharmonic Polynomials, Global Assumptions

In this chapter, we classify all symmetric nc plush polynomials as convex polynomials with an nc analytic change of variables; i.e., an nc symmetric polynomial p is nc plush if and only if it has the form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{2.1}$$

where the sums are finite and f_j, k_j, F are all nc analytic.

We also present a theory of noncommutative integration for nc polynomials and we prove a noncommutative version of the Frobenius theorem.

The next chapter, Chapter 3, proves that if the nc complex Hessian, q, of p takes positive semidefinite values on an "nc open set" then q takes positive semidefinite values on all tuples X, H. Thus, p has the form in Equation (2.1). The proof, in Chapter 3, draws on most of the theorems in Chapter 2 together with a very different technique involving representations of noncommutative quadratic functions.

Now, we recall the main theorems in this chapter.

2.1 Main Results of Chapter 2

In this chapter we classify all symmetric nc plush polynomials in g free variables.

Theorem 2.1.1. *An nc symmetric polynomial p in free variables is nc plurisubharmonic if and only if p can be written in the form*

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{2.2}$$

where the sums are finite and each f_j, k_j, F is nc analytic.

Proof. The proof requires the rest of this chapter and culminates in Section 2.4. □

Chapter 3 strengthens the result of Theorem 2.1.1 by weakening the hypothesis while keeping the same conclusion. Specifically, in Chapter 3, we assume that the nc polynomial is nc plush on an "nc open set" and conclude that it is nc plush everywhere and hence has the form in Equation (2.2). The proof, in Chapter 3, draws on most of the theorems in this chapter together with a very different technique involving representations of noncommutative quadratic functions.

The representation in Equation (2.2) is unique up to the natural transformations.

Theorem 2.1.2. *Let p be an nc symmetric polynomial in free variables that is nc plurisubharmonic and let*

$$\begin{aligned}
\widetilde{N} &:= \min\{N : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\} \\
\widetilde{M} &:= \min\{M : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\}.
\end{aligned}$$

Then, we can represent p as

$$p = \sum_{j=1}^{\widetilde{N}} \tilde{f}_j^T \tilde{f}_j + \sum_{j=1}^{\widetilde{M}} \tilde{k}_j \tilde{k}_j^T + \tilde{F} + \tilde{F}^T$$

and if N and M are integers such that $N \geq \widetilde{N}$, $M \geq \widetilde{M}$ and

$$p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T \ ,$$

then there exist isometries $U_1 : \mathbb{R}^{\widetilde{N}} \longrightarrow \mathbb{R}^N$ *and* $U_2 : \mathbb{R}^{\widetilde{M}} \longrightarrow \mathbb{R}^M$ *such that*

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = U_1 \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_{\widetilde{N}} \end{pmatrix} + \vec{c}_1 \quad \textit{and} \quad \begin{pmatrix} k_1^T \\ \vdots \\ k_M^T \end{pmatrix} = U_2 \begin{pmatrix} \tilde{k}_1^T \\ \vdots \\ \tilde{k}_{\widetilde{M}}^T \end{pmatrix} + \vec{c}_2$$

where $\vec{c}_1 \in \mathbb{R}^N$ *and* $\vec{c}_2 \in \mathbb{R}^M$*.*

Proof. Theorem 2.1.1 gives the desired form of p and nc integration will give the uniqueness. We provide the details of the proof in Section 2.4. □

A byproduct of the proof of Theorem 2.1.1 is noncommutative integration theory of nc polynomials. This includes a Frobenius theorem for nc polynomials and is discussed further in Section 2.2.

2.1.1 Guide to Chapter 2

In Section 2.2, we provide a theory of noncommutative integration for nc polynomials and in Section 2.2.4, we state and prove a noncommutative version of the Frobenius theorem. In Section 2.3, we prove that the nc complex Hessian for an nc plush polynomial is the sum of hereditary and antihereditary squares. Finally, in Section 2.4, we prove the main results. We apply nc integration theory to the sum of squares representation of the nc complex Hessian found in Section 2.3. We also settle the issue of uniqueness of this sum of squares representation.

2.2 NC Integration

In this section, we introduce a natural notion of noncommutative (nc) integration and then give some basic properties. We say that an nc polynomial p in $x = (x_1, \ldots, x_g)$ and h_j is **integrable in** x_j if there exists an nc polynomial $f(x)$ such that $f_{x_j}[h_j] = p$. We say that an nc polynomial p in $x = (x_1, \ldots, x_g)$ and $h = (h_1, \ldots, h_g)$ is **integrable** if there exists an nc polynomial $f(x)$ such that $f'(x)[h] = p$.

2.2.1 Notation

Let m be a monomial containing only the variables x_1, x_2, …, x_g. When we write $m|_{x_i \to h_i}$, we mean the set of monomials that are degree one in h_i where one x_i in m has been replaced by h_i. For example, if $m = x_1x_2x_1x_2$, then

$$m|_{x_1 \to h_1} = \{h_1x_2x_1x_2,\ x_1x_2h_1x_2\}$$

and

$$m|_{x_2 \to h_2} = \{x_1h_2x_1x_2,\ x_1x_2x_1h_2\}.$$

We also define a double substitution as follows. When we write

$$m|_{x_i \to h_i, x_j \to h_j} := (m|_{x_i \to h_i})|_{x_j \to h_j},$$

we mean the set of monomials that are degree one in h_i and degree one in h_j where one x_i in m has been replaced by h_i and one x_j in m has been replaced by h_j. Note that we have

$$m|_{x_i \to h_i, x_j \to h_j} = m|_{x_j \to h_j, x_i \to h_i}. \tag{2.3}$$

Using $m = x_1x_2x_1x_2$, we have that

$$\begin{aligned} m|_{x_1 \to h_1, x_2 \to h_2} &= m|_{x_2 \to h_2, x_1 \to h_1} \\ &= \{h_1h_2x_1x_2,\ h_1x_2x_1h_2,\ x_1h_2h_1x_2,\ x_1x_2h_1h_2\}. \end{aligned}$$

Sometimes we will start with a monomial m that is degree 1 in h_i and we wish to replace this h_i by x_i. When we write $m|_{h_i \to x_i}$, the set we get contains just one monomial so we abuse notation and use $m|_{h_i \to x_i}$ to represent the actual monomial in this set.

2.2.2 Differentially Wed Monomials

For γ either 1 or T, two monomials m and $\tilde{m}$ are called **1-differentially wed with respect to** x_j^γ if both m and $\tilde{m}$ have degree one in h_j^γ and if m has an x_j^γ where $\tilde{m}$ has an h_j^γ and if $\tilde{m}$ has an x_j^γ where m has an h_j^γ. Thus, interchanging h_j^γ and this x_j^γ in m produces $\tilde{m}$; i.e.,

$$m|_{h_j^\gamma \to x_j^\gamma} = \tilde{m}|_{h_j^\gamma \to x_j^\gamma}$$

More generally, two monomials m and $\tilde{m}$ are called **1-differentially wed** if both are degree one in h or h^T and if

$$m|_{h_i^\alpha \to x_i^\alpha} = \tilde{m}|_{h_j^\beta \to x_j^\beta}$$

for some α, β either 1 or T and some i, j.

From these definitions, if m and $\tilde{m}$ are 1-differentially wed with respect to a particular variable then m and $\tilde{m}$ are 1-differentially wed but not the other way around (which we demonstrate below).

Example 2.2.1. The monomials $m = h_1 x_2^T x_1$ and $\tilde{m} = x_1 x_2^T h_1$ are 1-differentially wed with respect to x_1.

Example 2.2.2. The monomials $m = h_1 x_2^T x_1$ and $\tilde{m} = x_1 h_2^T x_1$ are 1-differentially wed (but not with respect to a particular variable).

Example 2.2.3. The monomials $m = x_2 h_2 x_2$ and $\tilde{m} = x_1 x_2 h_2$ are not 1-differentially wed (and, therefore, also not 1-differentially wed with respect to any variable).

Theorem 2.2.4. *An nc polynomial p in $x = (x_1, \ldots, x_g), h = (h_1, \ldots, h_g)$ is integrable if and only if each monomial in p has degree one in h (i.e., contains exactly one h_j for some j) and whenever a monomial m occurs in p, each monomial which is 1-differentially wed to m also occurs in p and has the same coefficient.*

Proof. First suppose the nc polynomial p in x, h is integrable. Then, there exists an nc polynomial, $f(x)$, such that $f'(x)[h] = p$. Write f as

$$f = \sum_{i=1}^{N} c_i m_i$$

where each $c_i \in \mathbb{R}$ and each m_i is a monomial in x. Then, by applying Example 1.2.2 to each term $c_i m_i$, if a monomial $\tilde{m}$ occurs in $p = f'$, then every 1-differentially wed monomial to $\tilde{m}$ also occurs in $p = f'$ with the same coefficient.

Now suppose each monomial in p has degree 1 in h (i.e., contains some h_j) and if m is a monomial in p, then each monomial which is 1-differentially wed also occurs in p with the same coefficient. We will show that p is integrable.

Write

$$p = \sum_{i=1}^{N} c_i m_i$$

where each $c_i \in \mathbb{R}$ and each m_i is a monomial in x and degree 1 in h. Now we will change the order of summation of these terms so that we group together all monomials with the same coefficient that are 1-differentially wed. We do this in the following way.

Let w_1 be the polynomial that contains $c_1 m_1$ and all 1-differentially wed monomials to m_1 with the same coefficient c_1.

Let $1 \leq \alpha_2$ be the smallest integer such that $c_{\alpha_2} m_{\alpha_2}$ is not a term in w_1. Then let w_2 be the polynomial that contains $c_{\alpha_2} m_{\alpha_2}$ and all 1-differentially wed monomials to m_{α_2} with the same coefficient c_{α_2}.

Let $1 \leq \alpha_3$ be the smallest integer such that $c_{\alpha_3} m_{\alpha_3}$ is not a term in w_1 and not a term in w_2. Then let w_3 be the polynomial that contains $c_{\alpha_3} m_{\alpha_3}$ and all 1-differentially wed monomials to m_{α_3} with the same coefficient c_{α_3}.

We continue this process until it stops (it stops since p is a finite sum of monomials). Then we have written p as

$$p = \sum_{i=1}^{\ell} w_i.$$

It is important to note that with this construction, each w_i is a homogeneous polynomial of some fixed degree where each term in w_i is degree 1 in h.

Now define $\alpha_1 = 1$ and

$$f_i(x) := c_{\alpha_i}(m_{\alpha_i}|_{h \to x}), \quad 1 \leq i \leq \ell.$$

Then, by properties of differentiation and construction of w_i, we have that $f_i' = w_i$. Finally, define

$$f(x) := \sum_{i=1}^{\ell} f_i(x)$$

and notice that $f' = p$. $\square$

Corollary 2.2.5. *An nc polynomial p in x, h_j is integrable in x_j if and only if each monomial in p has degree one in h_j and whenever a monomial m occurs in p, each monomial which is 1-differentially wed with respect to x_j also occurs in p and has the same coefficient.*

2.2.3 Uniqueness of Noncommutative Integration

In this subsection, we explore the uniqueness of noncommutative integration. In classical calculus, integrating produces constants of integration. Here, we provide the noncommutative analogue.

Proposition 2.2.6. *Suppose m and $\tilde{m}$ are distinct monomials in the variables $x = (x_1, \ldots, x_g)$. Then, we have that*

1. *$(m)_{x_i}[h_i]$ and $(\tilde{m})_{x_i}[h_i]$ have no terms in common and hence*

$$(m)_{x_i}[h_i] \neq (\tilde{m})_{x_i}[h_i]$$

   provided x_i is contained in either m or $\tilde{m}$; and

2. *we have that*

$$m' = (m)_x[h] \neq (\tilde{m})_x[h] = \tilde{m}'.$$

Moreover, If m and $\tilde{m}$ are distinct monomials in the variables $x = (x_1, \ldots, x_g)$ and $y = (y_1, \ldots, y_s)$, then

$$m_x[h] \neq \tilde{m}_x[h].$$

Proof. If m and $\tilde{m}$ have different degree, then so do their nc directional derivatives and we are done. Suppose m and $\tilde{m}$ have the same degree. If m contains x_i and $\tilde{m}$ does not, then $(\tilde{m})_{x_i}[h_i] = 0$ while $(m)_{x_i}[h_i]$ is a nonzero nc polynomial.

Suppose both m and $\tilde{m}$ contain x_i and are the same degree. Then, write

$$m = x_{j_1} x_{j_2} \cdots x_{j_s} \qquad \text{and} \qquad \tilde{m} = x_{k_1} x_{k_2} \cdots x_{k_s}$$

where the tuple of integers $(j_1, j_2, \ldots, j_s)$ is not the same as the tuple of integers $(k_1, k_2, \ldots, k_s)$. This forces $(m)_{x_i}[h_i]$ and $(\tilde{m})_{x_i}[h_i]$ to have no terms in common. This completes the proof of *(1)*.

To prove *(2)*, note that

$$m' = (m)_x[h] = \sum_{i=1}^{g} (m)_{x_i}[h_i] \qquad \text{and} \qquad \tilde{m}' = (\tilde{m})_x[h] = \sum_{i=1}^{g} (\tilde{m})_{x_i}[h_i]$$

and if $m' = \tilde{m}'$, then we must have $(m)_{x_i}[h_i] = \tilde{m}_{x_i}[h_i]$ for each i. However, *(1)* implies that this is impossible.

If m and $\tilde{m}$ are distinct monomials in the variables $x = (x_1, \ldots, x_g)$ and $y = (y_1, \ldots, y_g)$, the proof follows exactly the way the proof of *(2)* does. $\square$

Lemma 2.2.7. *Suppose p is an nc polynomial in the variables $x = (x_1, \ldots, x_g)$ such that $p_{x_i}[h_i] = 0$. Then, $p(x_1, \ldots, x_g) = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g)$ is an nc polynomial in the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g$.*

Proof. First, if m is a monomial in the variables $x = (x_1, \ldots, x_g)$ that contains x_i, then $m_{x_i}[h_i]$ is a sum of terms where each instance of x_i is replaced by h_i (see Example 1.2.1). Note that each term in $m_{x_i}[h_i]$ has a different number of variables to the left of h_i; hence, the terms can not cancel. Thus, $m_{x_i}[h_i] \neq 0$.

Now suppose p is an nc polynomial in the variables $x = (x_1, \ldots, x_g)$ such that $p_{x_i}[h_i] = 0$. We write the nc polynomial p as

$$p = \sum_{j=1}^{N} \alpha_j \, m_j \tag{2.4}$$

where the α_j are nonzero real constants and the m_j are distinct monomials. Then, we have that

$$0 = p_{x_i}[h_i] = \sum_{j=1}^{N} \alpha_j \, (m_j)_{x_i}[h_i]. \tag{2.5}$$

Since the m_j are distinct monomials, Proposition 2.2.6 implies that no cancellation can occur in Equation (2.5). This implies that

$$(m_j)_{x_i}[h_i] = 0$$

for all $j = 1, \ldots, N$. Then, by the first paragraph in this proof, we get that each m_j is a monomial in the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g$. This implies that p, as in Equation (2.4), is a polynomial in the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g$. $\square$

Proposition 2.2.8. *Suppose p is an nc polynomial in the $g+s$ variables $x = (x_1, \ldots, x_g)$ and $y = (y_1, \ldots, y_s)$. If $p_x[h] = 0$, then*

$$p(x,y) = p(x_1, \ldots, x_g, y_1, \ldots, y_s) = f(y_1, \ldots, y_s)$$

is an nc polynomial in the variables $y = (y_1, \ldots, y_s)$.

Proof. If p is an nc polynomial in the $g+s$ variables $x = (x_1, \ldots, x_g)$ and $y = (y_1, \ldots, y_s)$, then

$$p_x[h] = \sum_{i=1}^{g} p_{x_i}[h_i].$$

It is important to note that $p_{x_i}[h_i]$ is an nc polynomial in x, y, and linear in h_i. Since

$$p_x[h] = \sum_{i=1}^{g} p_{x_i}[h_i] = 0$$

and since each $p_{x_i}[h_i]$ is an nc polynomial that is linear in h_i, it follows that

$$p_{x_i}[h_i] = 0 \quad \text{for all} \quad 1 \leq i \leq g.$$

Then, Lemma 2.2.7 implies that p is an nc polynomial in the variables

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_g, y_1, \ldots, y_s \quad \text{for all} \quad 1 \leq i \leq g$$

This can only happen if p is an nc polynomial in the variables $y = (y_1, \ldots, y_s)$. $\square$

Corollary 2.2.9. *Suppose p is an nc polynomial in $x = (x_1, \ldots, x_g)$. Then, we have that*

1. *if*

$$p'(x)[h] = p_x[h] = \sum_{i=1}^{g} p_{x_i}[h_i] = 0,$$

then p is constant, and

2. *if $\tilde{p}$ is another nc polynomial in the variables $x = (x_1, \ldots, x_g)$ such that $p' = \tilde{p}'$ then $p = \tilde{p} + \alpha$ where α is a real constant.*

Proof. Property *(1)* directly follows from Proposition 2.2.8.

If $p' = \tilde{p}'$, then, since nc differentiation is linear, we get that

$$0 = p' - \tilde{p}' = (p - \tilde{p})'$$

which, by property *(1)*, implies that $p - \tilde{p}$ is constant. $\square$

Noncommutative Complex Differentiation

Here, we specialize from the variables $x = (x_1, \dots, x_g)$ and $y = (y_1, \dots, y_s)$ to the variables $x = (x_1, \dots, x_g)$ and $x^T = (x_1^T, \dots, x_g^T)$. The first corollary below, Corollary 2.2.10, follows directly from Proposition 2.2.8 above.

Corollary 2.2.10. *Suppose p is an nc polynomial in the variables $x = (x_1, \dots, x_g)$ and $x^T = (x_1^T, \dots, x_g^T)$. Then, we have that*

1. *if $p_x[h] = 0$, then p is an nc antianalytic polynomial, and*

2. *if $p_{x^T}[h^T] = 0$, then p is an nc analytic polynomial.*

Lemma 2.2.11. *Let p be an nc polynomial in the nc variables $x = (x_1, \dots, x_g)$, $x^T = (x_1^T, \dots, x_g^T)$ and let q be the nc complex Hessian of p. Then $q = 0$ if and only if $p = F + G^T$ where F and G are nc analytic polynomials.*

If, in addition, p is symmetric, then $q = 0$ if and only if $p = F + F^T$ where F is an nc analytic polynomial.

Proof. Lemma 2.2.12, in Section 2.2.4 below, allows us to switch the order of differentiation to get

$$q = p_{x^T,x}[h^T, h] = (p_{x^T}[h^T])_x[h] = (p_x[h])_{x^T}[h^T].$$

Then, we have that the nc complex Hessian of $p = F + G^T$ is

$$q = (F_{x^T}[h^T])_x[h] + (G_x^T[h])_{x^T}[h^T] = 0.$$

Now suppose p contains a term with both x and x^T. Write p as

$$p = \sum_{j=1}^{N} \alpha_j m_j$$

where α_j are nonzero real constants and m_j are distinct monomials in x and/or x^T. Then, the nc complex Hessian of p is

$$q = \sum_{j=1}^{N} \alpha_j (m_j)_{x^T,x}[h^T, h].$$

Since the m_j are distinct, Proposition 2.2.6 implies that the nc polynomials

$$(m_i)_{x^T}[h^T] \qquad \text{and} \qquad (m_j)_{x^T}[h^T]$$

have no terms in common for all $i \neq j$. Then, we apply Proposition 2.2.6 again to get that the nc polynomials $(m_i)_{x^T,x}[h^T, h]$ and $(m_j)_{x^T,x}[h^T, h]$ have no terms in common for all $i \neq j$. This implies that no cancellation occurs in q so that $q \neq 0$. □

2.2.4 NC "Gradient" of a Potential

In this subsection, we give a noncommutative Frobenius Theorem and present some equivalent tests to determine if a list of nc polynomials is simultaneously integrable.

Lemma 2.2.12. *Suppose $p(x_1, \ldots, x_g)$ is an nc polynomial in $x_1, \ldots, x_g$. Then*

$$(p_{x_i}[h_i])_{x_j}[h_j] = (p_{x_j}[h_j])_{x_i}[h_i].$$

Proof. Write

$$p = \sum_{\alpha=1}^{t} c_\alpha m_\alpha$$

where each $c_\alpha \in \mathbb{R}$ and each m_α is a monomial in $x_1, \ldots, x_g$. Then we have

$$(p_{x_i}[h_i])_{x_j}[h_j] = \sum_{\alpha=1}^{t} c_\alpha((m_\alpha)_{x_i}[h_i])_{x_j}[h_j]$$

and

$$(p_{x_j}[h_j])_{x_i}[h_i] = \sum_{\alpha=1}^{t} c_\alpha((m_\alpha)_{x_j}[h_j])_{x_i}[h_i].$$

Note that the nc directional derivative $(m_\alpha)_{x_i}[h_i]$ is the sum of all monomials in the set $m_\alpha|_{x_i \to h_i}$ and the nc directional derivative $((m_\alpha)_{x_i}[h_i])_{x_j}[h_j]$ is the sum of all monomials in the set $m_\alpha|_{x_i \to h_i, x_j \to h_j}$. Equation (2.3) implies that

$$m_\alpha|_{x_i \to h_i, x_j \to h_j} = m_\alpha|_{x_j \to h_j, x_i \to h_i}$$

which implies that

$$c_\alpha((m_\alpha)_{x_i}[h_i])_{x_j}[h_j] = c_\alpha((m_\alpha)_{x_j}[h_j])_{x_i}[h_i].$$

Hence, $(p_{x_i}[h_i])_{x_j}[h_j] = (p_{x_j}[h_j])_{x_i}[h_i]$. □

The following theorem is the noncommutative analogue of the Frobenius Theorem in that the classical specialization of $(a) \Leftrightarrow (b)$ to $x \in \mathbb{R}^g$ in Theorem 2.2.13 below says that

$$\begin{pmatrix} f_1, & f_2, & \ldots, & f_g \end{pmatrix}$$

is the gradient of a function if and only if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

Theorem 2.2.13. *Suppose δ is an nc polynomial such that*

$$\delta(x_1, \ldots, x_g, h_1, \ldots, h_g) = \sum_{i=1}^{g} f_i(x_1, \ldots, x_g, h_i)$$

where each $f_i(x_1, \ldots, x_g, h_i)$ is homogeneous of degree 1 in h_i. Then, the following are equivalent:

(a) δ is integrable.

(b) Each $f_i(x_1, \ldots, x_g, h_i)$ is integrable in x_i and $(f_i)_{x_j}[h_j] = (f_j)_{x_i}[h_i]$ for any i, j.

(c) For each monomial, m, in δ, every 1-differentially wed monomial to m also occurs in δ with the same coefficient.

Proof. Theorem 2.2.4 gives the equivalence of (a) and (c).

Now we show (a) and (b) are equivalent. First, suppose (a) holds. Then there exists an nc polynomial $\mathcal{P}(x_1, \ldots, x_g)$ such that

$$\mathcal{P}' = \delta \qquad \Longrightarrow \qquad \sum_{i=1}^{g} \mathcal{P}_{x_i}[h_i] = \sum_{i=1}^{g} f_i(x_1, \ldots, x_g, h_i).$$

This forces $\mathcal{P}_{x_i}[h_i] = f_i(x_1, \ldots, x_g, h_i)$ and then Lemma 2.2.12 gives that

$$(f_i)_{x_j}[h_j] = (\mathcal{P}_{x_i}[h_i])_{x_j}[h_j] = \big(\mathcal{P}_{x_j}[h_j]\big)_{x_i}[h_i] = (f_j)_{x_i}[h_i].$$

Now suppose (a) is false; i.e., δ is not integrable. Then, there exists some term αm ($\alpha \in \mathbb{R}$) in δ such that not all 1-differentially wed monomials to m with the same coefficient α occur in δ. Without loss of generality, suppose αm is a term in $f_1(x, h_1)$. Recall, this implies m is degree one in h_1.

If δ does not contain a monomial that is 1-differentially wed to m with respect to x_1 with the same coefficient α, then $f_1(x, h_1)$ is not integrable in x_1.

Suppose αm contains the variable x_k and that δ (more specifically, $f_k(x, h_k)$) does not contain the term $\alpha\tilde{m}$, where $\tilde{m}$ is a specific monomial in the set

$$(m|_{h_1 \to x_1})|_{x_k \to h_k} = (m|_{x_k \to h_k})|_{h_1 \to x_1} \quad .$$

Note that $\tilde{m}$ is 1-differentially wed to m and this implies that the sets $m|_{x_k \to h_k}$ and $\tilde{m}|_{x_1 \to h_1}$ are equal. If $(f_1)_{x_k}[h_k] = (f_k)_{x_1}[h_1]$, then $\alpha\hat{\tilde{m}}$, where $\hat{\tilde{m}}$ is a specific monomial in the set $m|_{x_k \to h_k} = \tilde{m}|_{x_1 \to h_1}$, is a term in $(f_1)_{x_k}[h_k] = (f_k)_{x_1}[h_1]$. This implies that $\alpha\tilde{m}$, where $\tilde{m} = \hat{\tilde{m}}|_{h_1 \to x_1}$, is a term in $f_k(x, h_k)$ which is contained in δ.

Thus, we have shown that if δ is not integrable then either some $f_i(x, h_i)$ is not integrable with respect to x_i or $(f_i)_{x_j}[h_j] \neq (f_j)_{x_i}[h_i]$ for some $i \neq j$. $\square$

2.2.5 Levi-differentially Wed Monomials

Now we turn to properties of the nc complex Hessian q, as q is just a second nc directional derivative.

Two monomials m and $\tilde{m}$ are called **Levi-differentially wed** if m and $\tilde{m}$ are both degree 2 in h, h^T, m contains some h_i, h_j^T, $\tilde{m}$ contains some h_k, h_s^T and

$$m|_{h_i \to x_i, h_j^T \to x_j^T} = \tilde{m}|_{h_k \to x_k, h_s^T \to x_s^T}$$

Indeed, Levi-differentially wed is an equivalence relation on the monomials in the nc complex Hessian, q, with the coefficients of all Levi-differentially wed monomials in q being the same.

Example 2.2.14. The monomials $h^T h x^T x$, $h^T x x^T h$, $x^T h h^T x$, and $x^T x h^T h$ are all Levi-differentially wed to each other.

Example 2.2.15. None of the monomials $h^T h x^T x$, $h^T x h^T x$, $x^T h x^T h$ are Levi-differentially wed to each other.

The next theorem provides necessary and sufficient conditions as to when a given nc polynomial is actually an nc complex Hessian.

Theorem 2.2.16. *An nc polynomial q in x, x^T, h, h^T is an nc complex Hessian if and only if the following two conditions hold:*

(P1) Each monomial in q contains exactly one h_j and one h_k^T for some j, k.

(P2) If a certain monomial m is contained in q, any monomial $\tilde{m}$ that is Levi-differentially wed to m is also contained in q with the same coefficient.

Proof. First, suppose q is an nc complex Hessian. Equation (1.5) shows that q is an nc directional derivative of an nc directional derivative. Then, properties of nc directional derivatives imply that (P1) and (P2) hold.

Now suppose (P1) and (P2) hold. Write q as

$$q = \sum_{i=1}^{N} c_i m_i$$

where each $c_i \in \mathbb{R}$ and each m_i is a monomial that contains some h_j and h_k^T. Now we will change the order of summation of these terms so that we group together all monomials with the same coefficient that are Levi-differentially wed to each other. We do this in the following way.

Let w_1 be the nc polynomial that contains $c_1 m_1$ and all Levi-differentially wed monomials to m_1 with the same coefficient c_1.

Let $1 \leq \alpha_2$ be the smallest integer such that $c_{\alpha_2} m_{\alpha_2}$ is not a term in w_1. Then let w_2 be the nc polynomial that contains $c_{\alpha_2} m_{\alpha_2}$ and all Levi-differentially wed monomials to m_{α_2} with the same coefficient c_{α_2}.

Let $1 \leq \alpha_3$ be the smallest integer such that $c_{\alpha_3} m_{\alpha_3}$ is not a term in w_1 and not a term in w_2. Then let w_3 be the nc polynomial that contains $c_{\alpha_3} m_{\alpha_3}$ and all Levi-differentially wed monomials to m_{α_3} with the same coefficient c_{α_3}.

We continue this process until it stops (it stops since q is a finite sum of monomials). Then we have written q as

$$q = \sum_{i=1}^{\ell} w_i.$$

It is important to note that with this construction, each w_i is a homogeneous polynomial of some fixed degree where each term in w_i contains some h_j and some h_k^T.

Now define $\alpha_1 = 1$ and

$$f_i(x, x^T) := c_{\alpha_i}(m_{\alpha_i}|_{h \to x, h^T \to x^T}), \quad 1 \le i \le \ell.$$

Then we have, by properties of differentiation and construction of w_i, that the nc complex Hessian of each f_i is w_i. Finally, define

$$f(x, x^T) := \sum_{i=1}^{\ell} f_i(x, x^T)$$

and notice that the nc complex Hessian of f is q. □

Lemma 2.2.17. *Let m, m', n, n' be nc analytic monomials with degree 1 in h (or all nc antianalytic monomials with degree 1 in h^T). Then m, m' are 1-differentially wed and n, n' are 1-differentially wed if and only if $n^T m$ and $n'^T m'$ are Levi differentially wed.*

Proof. Without loss of generality, suppose m, m', n, n' are all nc analytic monomials with degree 1 in h.

m, m' are 1-differentially wed and n, n' are 1-differentially wed if and only if

$$\begin{aligned} m|_{h_i \to x_i} &= m'|_{h_j \to x_j} \\ n|_{h_k \to x_k} &= n'|_{h_s \to x_s} \quad . \end{aligned}$$

This happens if and only if

$$\begin{aligned} (n^T m)|_{h_k^T \to x_k^T, h_i \to x_i} &= (n|_{h_k \to x_k})^T \, (m|_{h_i \to x_i}) \\ &= (n'|_{h_s \to x_s})^T \left(m'|_{h_j \to x_j}\right) = (n'^T m')|_{h_s^T \to x_s^T, h_j \to x_j} \end{aligned}$$

since m, m', n, n' are all nc analytic and each is degree 1 in h. □

2.3 Complex Hessian as a Sum of Squares

Assuming nc plurisubharmonicity means we have a matrix positive nc complex Hessian. By Theorem 1.1.1, this leads to a sum of squares representation for the nc complex Hessian.

The next lemma follows the proof of Proposition 4.1 in [HM04] with the nc Hessian now replaced by the nc complex Hessian.

Lemma 2.3.1. *If p is an nc symmetric plush polynomial then the nc complex Hessian, q, of p can be written as*

$$q(x, x^T)[h, h^T] = \sum_{j=1}^{m} r_j^T r_j$$

where each r_j is an nc polynomial that is homogeneous of degree 1 in h (or h^T).

Proof. Since p is nc plush, $q(X, X^T)[H, H^T] \succeq 0$ for all $X, H \in (\mathbb{R}^{n\times n})^g$ for all $n \geq 1$. By Theorem 1.1.1, $q(x, x^T)[h, h^T]$ is a sum of squares. Hence, we can write q as

$$q(x, x^T)[h, h^T] = \sum_{j=1}^{m} r_j^T r_j$$

where each r_j is a polynomial in x, x^T, h, and h^T. Write

$$r_j = \sum_{w\in Mon(x,x^T,h,h^T)} r_j(w)w$$

where $Mon(x, x^T, h, h^T)$ is the set of monomials in the given variables and where all but finitely many of the $r_j(w) \in \mathbb{R}$ are 0. Let $deg_h(r)$ denote the degree of r in h (and h^T) and let $deg_x(r)$ denote the degree of r in x (and x^T). Let

$$\begin{aligned} d_h &= \max\{deg_h(r_j) : j\} \\ d_x &= \max\{deg_x(w) : \exists j \text{ s.t. } r_j \text{ contains } w \text{ and } deg_h(w) = d_h\} \\ S_{d_x,d_h} &= \{w : r_j \text{ contains } w \text{ for some } j, deg_h(w) = d_h, deg_x(w) = d_x\}. \end{aligned}$$

The portion of q homogeneous of degree $2d_h$ in h and $2d_x$ in x is

$$\mathcal{Q} = \sum_{\{j=1,\ldots,m,v,w\in S_{d_x,d_h}\}} r_j(v)r_j(w)v^T w.$$

Since for $v_j, w_j \in S_{d_x,d_h}$, $v_1^T w_1 = v_2^T w_2$ can occur if and only if $v_1 = v_2$ and $w_1 = w_2$, we see that $Q \neq 0$ and thus $deg_h(q) = 2d_h$. Since q has degree 2 in h and h^T, we obtain $2d_h = 2$ which implies $d_h = 1$. □

Since we know q is positive, Theorem 1.1.1 allows us to represent q as a sum of squares, $q = \sum r_j^T r_j$. We now wish to show that these r_j are either nc analytic or nc antianalytic.

Theorem 2.3.2. *If p is an nc symmetric plush polynomial, then the nc complex Hessian, q, of p can be written as in Lemma 2.3.1,*

$$q(x, x^T)[h, h^T] = \sum_{j=1}^{m} r_j^T r_j$$

where each r_j is either nc analytic or nc antianalytic.

Proof. Since p is assumed nc plush we get that $q(X, X^T)[H, H^T] \succeq 0$ for all $X, H \in (\mathbb{R}^{n\times n})^g$ for all $n \geq 1$. Again, by Theorem 1.1.1, we get that q is a finite sum of squares,

$$q = \sum_{j=1}^{m} r_j^T r_j.$$

By Lemma 2.3.1, each r_j is homogeneous of degree 1 in h or h^T. We wish to show that each r_j is either nc analytic or nc antianalytic. Consider all monomials in the r_i's of the form

$$Lh_j^T M x_k N \tag{2.6}$$

or of the form

$$Lx_j^T M h_k N \tag{2.7}$$

$$Lx_k M h_j^T N \tag{2.8}$$

or of the form

$$Lh_k M x_j^T N. \tag{2.9}$$

Here, L, M, N are monomials in x and x^T. The theorem being false is equivalent to some such monomial existing and we say these are monomials of the *offending form*. This is easy to check just by comparing the form of each offending monomial to r_j being nc analytic or nc antianalytic.

We now focus on offending monomials of the highest degree (over all offending monomials).

Case 1: Suppose that the offending monomial of highest degree is of the form $Lh_j^T Mx_k N$. Without loss of generality, say this monomial occurs in r_1. Then $r_1^T r_1$ contains the monomial

$$m := N^T x_k^T M^T h_j L^T L h_j^T M x_k N.$$

We claim that this monomial, m, appears in q. To be cancelled, $\tilde{L} h_j^T M x_k N$ must appear in some r_ℓ where $\tilde{L}$ factors L or L factors $\tilde{L}$. This implies that either $r_\ell^T r_\ell$ contains a monomial of the form $w^T w$, where w is of the offending form and w has higher degree than m, or we must have $\tilde{L} = L$. The first option would contradict the highest degree assumption of m so we must have $\tilde{L} = L$. In this case, the coefficient of m arising from $r_\ell^T r_\ell$ is positive so no cancellation occurs.

Observe that q contains many Levi-differentially wed monomials to m. For example,

$$N^T x_k^T M^T x_j L^T L h_j^T M h_k N$$

is contained in q, so it appears in some square, say $r_k^T r_k$. Thus, r_k^T contains $N^T x_k^T M^T x_j L^T L h_j^T$ (or $N^T x_k^T M^T x_j L^T L h_j^T M$) which is of the offending form (2.8). But this monomial is longer than the longest offending monomial we selected; namely, m. This is a contradiction.

Case 2: Suppose that the offending monomial of highest degree is of the form $Lx_k M h_j^T N$. Without loss of generality, say this monomial occurs in r_1. Then $r_1^T r_1$ contains the monomial

$$m := N^T h_j M^T x_k^T L^T L x_k M h_j^T N.$$

We claim that this monomial, m, appears in q. To be cancelled, $\tilde{K} h_j^T N$ must appear in some r_ℓ where $\tilde{K}$ factors $Lx_k M$ or $Lx_k M$ factors $\tilde{K}$. This implies that either $r_\ell^T r_\ell$ contains a monomial of the form $w^T w$, where w is of the offending form and w has higher degree than m, or we must have $\tilde{K} = Lx_k M$. The first option would contradict the highest degree assumption of m so we must have $\tilde{K} = Lx_k M$.

In this case, the coefficient of m arising from $r_\ell^T r_\ell$ is positive so no cancellation occurs.

Observe that q contains many Levi-differentially wed monomials to m. For example,

$$N^T x_j M^T x_k^T L^T L h_k M h_j^T N$$

is contained in q, so it appears in some square, say $r_k^T r_k$. Thus, r_k^T contains $N^T x_j M^T x_k^T L^T L h_k$ (or $N^T x_j M^T x_k^T L^T L h_k M$) which is of the offending form (2.7). But this monomial is longer than the longest offending monomial we selected; namely, m. This is a contradiction.

Case 3: This case concerns $L h_k M x_j^T N$ and the argument is parallel to that in Case 1.

Case 4: This case concerns $L x_j^T M h_k N$ and the argument is parallel to that in Case 2. □

2.4 Proof of Main Results of Chapter 2

We now prove the main theorem which we now recall from Section 2.1.

Theorem 2.4.1. *An nc symmetric polynomial p in free variables is nc plurisubharmonic if and only if p can be written in the form*

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{2.10}$$

where the sums are finite and each f_j, k_j, F is nc analytic.

Proof. If p has the form given in Equation (2.10) then $q(x, x^T)[h, h^T]$, the nc complex Hessian of p, is

$$\begin{aligned} q &= \sum (f_j^T)_{x^T}[h^T](f_j)_x[h] + \sum (k_j)_x[h](k_j^T)_{x^T}[h^T] \\ &= \sum (f_j)_x[h]^T (f_j)_x[h] + \sum (k_j)_x[h](k_j)_x[h]^T, \end{aligned}$$

which is a finite sum of squares. Hence, $q(X, X^T)[H, H^T] \succeq 0$ for all $X, H \in (\mathbb{R}^{n\times n})^g$ for all $n \geq 1$ and thus p is nc plush.

Now, suppose p is nc plush. By Theorem 2.3.2, write

$$q = \sum r_j^T r_j$$

where each r_j is nc analytic or nc antianalytic and homogeneous of degree one in h or h^T. In view of Theorem 2.2.4, we now show that each r_j is integrable. Suppose m is a monomial in r_j and that m' is any monomial 1-differentially wed to m (other than m). We shall now show that m' occurs in r_j with the same coefficient as m.

To do this, suppose r_j contains $C_j m + C_j' m'$ for some j's. Note that C_j' may certainly be 0. Then $r_j^T r_j$ must contain the terms $C_j^2 m^T m$, $C_j'^2 m'^T m'$, $C_j C_j' m^T m'$, $C_j' C_j m'^T m$. By summing over all j such that r_j contains $C_j m + C_j' m'$ we get that q must have the terms

$$(\sum_j C_j^2) m^T m, \quad (\sum_j C_j C_j') m^T m', \quad (\sum_j C_j'^2) m'^T m'.$$

By Lemma 2.2.17, the monomials $m^T m$, $m^T m'$, and $m'^T m'$ are Levi-differentially wed. Thus, all 3 coefficients are equal. This means we have

$$\sum_j C_j^2 = \sum_j C_j C_j' = \sum_j C_j'^2. \tag{2.11}$$

The Cauchy Schwartz inequality gives

$$(\sum_j C_j C_j')^2 \leq (\sum_j C_j^2)(\sum_j C_j'^2) \tag{2.12}$$

and Equation (2.11) implies we have equality in Equation (2.12). This means we have $C_j = \alpha C_j'$ for all j. Then we get

$$\sum_j C_j C_j' = \sum_j \alpha C_j'^2 = \alpha \sum_j C_j'^2$$

and by Equation (2.11), we get $\alpha = 1$. Hence $C_j = C_j'$ for all j. This means that r_j contains $C_j m$ if and only if it contains $C_j m'$ where m and m' are any two 1-differentially wed monomials.

Since m and m' are arbitrary 1-differentially wed monomials, we get that, by Theorem 2.2.4, r_j is integrable. We integrate it to get f_j in Equation (2.10) if r_j is nc analytic and k_j^T in Equation (2.10) if r_j is nc antianalytic. We note that there are other antiderivatives for the r_j (for example, $f_j + x^T$) but when r_j is nc analytic (resp. nc antianalytic) we only care about the nc analytic (resp. nc antianalytic) ones.

Define

$$\tilde{p} := \sum f_j^T f_j + \sum k_j k_j^T.$$

By construction, $\tilde{p}$ is a sum of hereditary and antihereditary squares. Also note that the nc complex Hessian of $\tilde{p}$ is equal to the nc complex Hessian of p. Apply Lemma 2.2.11 to finish the proof. □

Now we prove the uniqueness of the representation of an nc symmetric plush polynomial. We recall Theorem 2.1.2 from Section 2.1:

Theorem 2.4.2. *Let p be an nc symmetric polynomial in free variables that is nc plurisubharmonic and let*

$$\begin{aligned} \widetilde{N} &:= \min\{N : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\} \\ \widetilde{M} &:= \min\{M : p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T\}. \end{aligned}$$

Then, we can represent p as

$$p = \sum_{j=1}^{\widetilde{N}} \tilde{f}_j^T \tilde{f}_j + \sum_{j=1}^{\widetilde{M}} \tilde{k}_j \tilde{k}_j^T + \tilde{F} + \tilde{F}^T$$

and if N and M are integers such that $N \geq \widetilde{N}$, $M \geq \widetilde{M}$ and

$$p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T \ ,$$

then there exist isometries $U_1 : \mathbb{R}^{\widetilde{N}} \longrightarrow \mathbb{R}^N$ and $U_2 : \mathbb{R}^{\widetilde{M}} \longrightarrow \mathbb{R}^M$ such that

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = U_1 \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_{\widetilde{N}} \end{pmatrix} + \vec{c}_1 \quad \text{and} \quad \begin{pmatrix} k_1^T \\ \vdots \\ k_M^T \end{pmatrix} = U_2 \begin{pmatrix} \tilde{k}_1^T \\ \vdots \\ \tilde{k}_{\widetilde{M}}^T \end{pmatrix} + \vec{c}_2$$

where $\vec{c}_1 \in \mathbb{R}^N$ and $\vec{c}_2 \in \mathbb{R}^M$.

Proof. Suppose N and M are integers such that $N \geq \widetilde{N}, M \geq \widetilde{M}$ where p can be written as

$$p = \sum_{j=1}^{N} f_j^T f_j + \sum_{j=1}^{M} k_j k_j^T + F + F^T \tag{2.13}$$

and suppose $\widehat{M} \geq \widetilde{M}$ is such that we have

$$p = \sum_{j=1}^{\widetilde{N}} \tilde{f}_j^T \tilde{f}_j + \sum_{j=1}^{\widehat{M}} \hat{k}_j \hat{k}_j^T + F + F^T. \tag{2.14}$$

Then, the nc complex Hessian, q, of p based on the representations in Equations (2.14) and (2.13) is

$$q = \sum_{j=1}^{\widetilde{N}} (\tilde{f}_j)_x[h]^T (\tilde{f}_j)_x[h] + \sum_{j=1}^{\widehat{M}} (\hat{k}_j)_x[h] (\hat{k}_j)_x[h]^T \tag{2.15}$$

$$= \sum_{j=1}^{N} (f_j)_x[h]^T (f_j)_x[h] + \sum_{j=1}^{M} (k_j)_x[h] (k_j)_x[h]^T.$$

We define $\mathcal{H}_{hered}(q)$ as the **purely hereditary** part of q to be all of the terms that contain h^T to the left of h and we define $\mathcal{H}_{antihered}(q)$ as the **purely antihereditary** part of q to be all of the terms that contain h to the left of h^T.

First, from Equation (2.15), consider the purely hereditary part of the nc complex Hessian,

$$\mathcal{H}_{hered}(q) = \sum_{j=1}^{\widetilde{N}} (\tilde{f}_j)_x[h]^T (\tilde{f}_j)_x[h] = \sum_{j=1}^{N} (f_j)_x[h]^T (f_j)_x[h].$$

Since $\mathcal{H}_{hered}(q)$ is a sum of squares, it is matrix positive. Hence, the Gram representations[1] for this purely hereditary part of q contain $\ell \times \ell$ unique positive semidefinite matrices, $\widetilde{G}$ and G, that are both of rank $\widetilde{N}$ such that

$$\sum_{j=1}^{\widetilde{N}} (\tilde{f}_j)_x[h]^T (\tilde{f}_j)_x[h] = y^T \widetilde{G} y \quad \text{and} \quad \sum_{j=1}^{N} (f_j)_x[h]^T (f_j)_x[h] = y^T G y \ ,$$

where y is an $\ell \times 1$ vector of monomials in x and h. The purely hereditary nature of $\mathcal{H}_{hered}(q)$ forces $\widetilde{G}$ and G to be unique (so, in fact, $\widetilde{G} = G$).

[1]See [PW] as a reference for the Gram representation.

Since $\widetilde{G}$ is positive semidefinite, we can write $\widetilde{G}$ as $\widetilde{G} = \widetilde{W}^T\widetilde{W}$, where $\widetilde{W} : \mathbb{R}^\ell \longrightarrow \mathbb{R}^{\widetilde{N}}$ is an $\widetilde{N} \times \ell$ matrix with $\operatorname{rank}(\widetilde{W}) = \widetilde{N}$ such that

$$\widetilde{W}y = \begin{pmatrix} (\tilde{f}_1)_x[h] \\ \vdots \\ (\tilde{f}_{\widetilde{N}})_x[h] \end{pmatrix}.$$

Similarly, we can write G as $G = W^TW$, where $W : \mathbb{R}^\ell \longrightarrow \mathbb{R}^N$ is an $N \times \ell$ matrix with $\operatorname{rank}(W) = \widetilde{N}$ such that

$$Wy = \begin{pmatrix} (f_1)_x[h] \\ \vdots \\ (f_N)_x[h] \end{pmatrix}.$$

Note that the range of W is an $\widetilde{N}$-dimensional subspace sitting inside of $\mathbb{R}^N$.

Let $\mathcal{R}^{\widetilde{N}}$ denote the subspace of $\mathbb{R}^N$ spanned by the first $\widetilde{N}$ coordinates of $\mathbb{R}^N$; i.e.,

$$\mathcal{R}^{\widetilde{N}} = \operatorname{span}\{e_1, e_2, \ldots, e_{\widetilde{N}}\}$$

where e_i is the i^{th} standard basis vector in $\mathbb{R}^N$. Then define the $N \times \widetilde{N}$ matrix $E : \mathbb{R}^{\widetilde{N}} \longrightarrow \mathbb{R}^N$ as $E = \begin{pmatrix} I_{\widetilde{N}} \\ 0 \end{pmatrix}$ so that $E\widetilde{W}y = \begin{pmatrix} \widetilde{W}y \\ 0 \end{pmatrix} \in \mathbb{R}^N$. We note that if $N = \widetilde{N}$, then $E = I_{\widetilde{N}}$.

Let $V : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be an $N \times N$ unitary matrix that maps $\mathcal{R}^{\widetilde{N}}$ onto the range of W such that

$$V\begin{pmatrix} \widetilde{W}y \\ 0 \end{pmatrix} = Wy.$$

This implies that $Wy = VE\widetilde{W}y$ and note that the $N \times \widetilde{N}$ matrix $U_1 = VE : \mathbb{R}^{\widetilde{N}} \longrightarrow \mathbb{R}^N$ is an isometry and that

$$\begin{pmatrix} (f_1)_x[h] \\ \vdots \\ (f_N)_x[h] \end{pmatrix} = U_1 \begin{pmatrix} (\tilde{f}_1)_x[h] \\ \vdots \\ (\tilde{f}_{\widetilde{N}})_x[h] \end{pmatrix}. \tag{2.16}$$

Now, we perform nc integration to each nc polynomial in the vectors on both sides of Equation (2.16). We do this according to Corollary 2.2.9 to get

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = U_1 \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_{\widetilde{N}} \end{pmatrix} + \vec{c}_1 \tag{2.17}$$

where $\vec{c}_1 \in \mathbb{R}^N$.

Similarly, if, at the start of the proof, we assumed $\widehat{N} \geq \widetilde{N}$ is such that we have

$$p = \sum_{j=1}^{\widehat{N}} \hat{f}_j^T \hat{f}_j + \sum_{j=1}^{\widetilde{M}} \tilde{k}_j \tilde{k}_j^T + F + F^T \ ,$$

then, we would have constructed an isometry $U_2 : \mathbb{R}^{\widetilde{M}} \longrightarrow \mathbb{R}^M$ such that

$$\begin{pmatrix} k_1 \\ \vdots \\ k_M \end{pmatrix} = U_2 \begin{pmatrix} \tilde{k}_1 \\ \vdots \\ \tilde{k}_{\widetilde{M}} \end{pmatrix} + \vec{c}_2 \tag{2.18}$$

where $\vec{c}_2 \in \mathbb{R}^M$ and M is as in Equation (2.13).

Combining Equation (2.17) and Equation (2.18), we can then write p with the minimal number of hereditary and antihereditary squares as

$$p = \sum_{j=1}^{\widetilde{N}} \tilde{f}_j^T \tilde{f}_j + \sum_{j=1}^{\widetilde{M}} \tilde{k}_j \tilde{k}_j^T + \tilde{F} + \tilde{F}^T.$$

□

Chapter 2 of this dissertation is taken from [GHV] that has been submitted for publication with coauthors J. William Helton and Victor Vinnikov as
J. M. Greene, J. W. Helton and V. Vinnikov, Noncommutative Plurisubharmonic Polynomials, Part I: Global Assumptions, preprint, ***http://arxiv.org/abs/1101.0107*** .

Chapter 3

Noncommutative Plurisubharmonic Polynomials, Local Assumptions

In this chapter, we show that if an nc polynomial is nc plurisubharmonic on an nc open set then the polynomial is actually nc plurisubharmonic everywhere and has the form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{3.1}$$

where the sums are finite and f_j, k_j, F are all nc analytic.

In Chapter 2, it was shown that if p is nc plurisubharmonic everywhere then p has the form in Equation (3.1). In other words, Chapter 2 makes a global assumption while the current chapter makes a local assumption, but both reach the same conclusion.

This chapter requires a technique that is not used in Chapter 2. We use a Gram-like vector and matrix representation (called the border vector and middle matrix) for homogeneous degree 2 nc polynomials. We then analyze this representation for the nc complex Hessian on an nc open set and positive semidefiniteness forces a very rigid structure on the border vector and middle matrix. This rigid structure plus the theorems in Chapter 2 ultimately force the form in Equation (3.1).

Now, we recall the main theorems of Chapter 3.

3.1 Main Results of Chapter 3

As we will see, in Section 3.4, the nc complex Hessian, q, if matrix positive on an nc open set, can be factored as

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T] \tag{3.2}$$

where $D(x, x^T)$ is a diagonal matrix, $L(x, x^T)$ is a lower triangular matrix with ones on the diagonal (we call this a ***unit* lower triangular matrix**), and $V(x, x^T)[h, h^T]$ is a vector of monomials in x, x^T, h, h^T.

When we take the transpose of a matrix with monomial or polynomial entries (e.g., $L(x, x^T)^T$ or $V(x, x^T)[h, h^T]^T$), we get the matrix obtained by taking the transpose (as a matrix) *and* applying the transpose (involution) to every entry.

Example 3.1.1. *If*

$$v = \begin{pmatrix} hxx \\ hx \\ h \end{pmatrix},$$

then

$$v^T = \begin{pmatrix} x^T x^T h^T & x^T h^T & h^T \end{pmatrix}.$$

The next theorem shows the surprising result that the diagonal matrix, $D(x, x^T)$ in Equation (3.2), does not depend on x, x^T and that $L(x, x^T)$ has nc polynomial entries.

Theorem 3.1.2. *If p is an nc symmetric polynomial that is nc plurisubharmonic on an nc open set, then q, the nc complex Hessian of p, can be written as*

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $V(x, x^T)[h, h^T]$ is a vector of monomials in x, x^T, h, h^T,

$$D = diag(d_1, d_2, \ldots, d_{\mathcal{N}})$$

is a positive semidefinite constant real matrix, and $L(x, x^T)$ is a unit lower triangular matrix with nc polynomial entries.

Proof. The proof of this theorem requires the rest of this chapter and culminates in Subsection 3.4.4. □

This gives rise to an extension of the main theorem from Chapter 2. In Chapter 2, it is shown that an nc polynomial which is nc plush *everywhere* has the specific form given in Equation (3.3) below. In this chapter, Theorem 3.1.3, below, is a stronger, "local implies global", result in that an nc polynomial that is nc plush just on an nc open set is actually nc plush everywhere (and has the form in Equation (3.3)).

Theorem 3.1.3. *If an nc symmetric polynomial, p, is nc plurisubharmonic on an nc open set, then p is, in fact, nc plurisubharmonic everywhere and has the form expressed in Chapter 2*

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{3.3}$$

where the sums are finite and each f_j, k_j, and F is nc analytic.

Proof. That $D = D(x, x^T)$, in Theorem 3.1.2, is a positive semidefinite constant real matrix immediately implies

$$q(X, X^T)[H, H^T] \succeq 0$$

for all $X, H \in \cup_{n\geq 1}(\mathbb{R}^{n\times n})^g$; that is, p is nc plush at all $X \in \cup_{n\geq 1}(\mathbb{R}^{n\times n})^g$. Consequently, Theorem 2.1.1 in Chapter 2 gives that p is of the desired form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T$$

where the sums are finite and f_j, k_j, F are nc analytic. □

Note that with an nc polynomial, p, as in Equation (3.3), the nc complex Hessian, q, of p is

$$q = \sum (f_j^T)_{x^T}[h^T](f_j)_x[h] + \sum (k_j)_x[h](k_j^T)_{x^T}[h^T] \ , \tag{3.4}$$

which is matrix positive since it is a sum of squares. From Equation (3.4), we see that the nc complex Hessian for an nc polynomial that is nc plush on an nc open set has even degree.

3.1.1 Guide to Chapter 3

In Section 3.2, we introduce a Gram-like representation of nc quadratics. In Section 3.3, we study this Gram-like representation for the nc complex Hessian and prove some properties for this representation. In Section 3.4, we introduce the LDL^T decomposition of the nc complex Hessian and conclude that D is constant.

3.2 Middle Matrix Representation For A General NC Quadratic

In this section, we turn to a special representation for nc symmetric quadratic polynomials called the **middle matrix representation** (MMR). We represent nc quadratics in a factored form, $v^T Mv$. This representation greatly facilitates the study of the positivity of nc quadratics by letting us study the positivity of M. Now we give details.

Any noncommutative symmetric polynomial, $f(x, x^T, h, h^T)$, in the variables $x = (x_1, \ldots, x_g)$, $x^T = (x_1^T, \ldots, x_g^T)$, $h = (h_1, \ldots, h_g)$, and $h^T = (h_1^T, \ldots, h_g^T)$ that is degree s in x, x^T and homogeneous of degree two in h, h^T admits a representation of the form

$$f(x, x^T, h, h^T) = V(x, x^T)[h, h^T]^T M(x, x^T)V(x, x^T)[h, h^T] \tag{3.5}$$

where $M(x, x^T)$, called the **middle matrix**, is a symmetric matrix of nc polynomials in x, x^T and $V(x, x^T)[h, h^T]$, called the **border vector**, is given by

$$V(x, x^T)[h, h^T] = \begin{pmatrix} V_s(x, x^T)[h] \\ \vdots \\ V_0(x, x^T)[h] \\ V_s(x, x^T)[h^T] \\ \vdots \\ V_0(x, x^T)[h^T] \end{pmatrix}. \tag{3.6}$$

The $V_k(x, x^T)[h]$ (resp. $V_k(x, x^T)[h^T]$) are vectors of nc monomials of the form $h_j m(x, x^T)$ (resp. $h_j^T m(x, x^T)$) where $m(x, x^T)$ runs through the set of $(2g)^k$ monomials in x, x^T of length k for $j = 1, \ldots, g$. Note that the degree of the monomials in V_k is $k+1$.

We note that the vector of monomials, $V(x, x^T)[h, h^T]$, might contain monomials that are not required in the representation of the nc quadratic, f. Therefore, we can omit all monomials from the border vector that are not required. This gives us a *minimal length* border vector and prevents extraneous zeros from occurring in the middle matrix. The next lemma, Lemma 3.2.1, says that a minimal length border vector contains distinct monomials.

Lemma 3.2.1. *If $f(x, x^T, h, h^T)$ is an nc symmetric polynomial that has a middle matrix representation, then there is a middle matrix representation for f such that the border vector contains distinct monomials. Here, distinct precludes one monomial being a scalar multiple of another.*

Proof. Suppose we have f with the representation

$$f(x, x^T, h, h^T) = \begin{pmatrix} m \\ \alpha m \\ n \end{pmatrix}^T \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} m \\ \alpha m \\ n \end{pmatrix}$$

with α a real number and m and n distinct monomials. Write f as

$$\begin{aligned} f &= m^T(p_{11} + \alpha^2 p_{22} + \alpha p_{21} + \alpha p_{12})m \\ &+ m^T(p_{13} + \alpha p_{23})n + n^T(p_{31} + \alpha p_{32})m + n^T p_{33} n \end{aligned}$$

which leads to the representation

$$f(x, x^T, h, h^T) = \begin{pmatrix} m \\ n \end{pmatrix} \begin{pmatrix} p_{11} + \alpha^2 p_{22} + \alpha p_{21} + \alpha p_{12} & p_{13} + \alpha p_{23} \\ p_{31} + \alpha p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

that has distinct monomials in the border vector. $\square$

To aid us in the following sections, we cite a theorem (Theorem 8.3 in [CHSY03] and Theorem 6.1 in [HM04]). Note that in [CHSY03], the following theorem is stated for a positivity domain but the proof only uses the fact that positivity domains are nc open sets (satisfy the two conditions in Subsection 1.4.2). Hence, we slightly generalize the statement of the theorem to work on a general nc open set as defined in Subsection 1.4.2.

Theorem 3.2.2. *Consider a noncommutative polynomial $\mathcal{Q}(x,x^T)[h,h^T]$ which is quadratic in the variables h,h^T that is defined on $\mathcal{G} \subseteq \cup_{n\geq 1}(\mathbb{R}^{n\times n})^g$. Write $\mathcal{Q}(x,x^T)[h,h^T]$ in the form*

$$\mathcal{Q}(x,x^T)[h,h^T] = V(x,x^T)[h,h^T]^T M(x,x^T) V(x,x^T)[h,h^T]$$

and suppose that the following two conditions hold:

(i) the set $\mathcal{G}$ is an nc open set as defined in Subsection 1.4.2;

(ii) the border vector $V(x,x^T)[h,h^T]$ of the quadratic function $\mathcal{Q}(x,x^T)[h,h^T]$ has distinct monomials.

Then, the following statements are equivalent:

(a) $\mathcal{Q}(X,X^T)[H,H^T]$ is a positive semidefinite matrix for each pair of tuples of matrices X and H for which $X \in \mathcal{G}$;

(b) $M(X,X^T) \succeq 0$ for all $X \in \mathcal{G}$.

We will also need the following well known lemma (c.f. [HM04]). Just for notational purposes of stating the lemma, let $\mathcal{B}(\mathcal{H})^g$ denote all g-tuples of operators on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space.

Lemma 3.2.3. *Given d, there exists a Hilbert space $\mathcal{K}$ of dimension $\sum_0^{2d}(2g)^j$ such that if G is an open subset of $\mathcal{B}(\mathcal{K})^g$, if p has degree at most d, and if $p(X,X^T) = 0$ for all $X \in G$, then $p = 0$.*

Next, we proceed to study this middle matrix representation for the nc complex Hessian.

3.3 Middle Matrix Representation For The NC Complex Hessian

In Section 3.2, we introduced the middle matrix representation for a general nc quadratic polynomial, and this section specializes it to the nc complex Hessian. The requirement that the nc complex Hessian be positive on an nc open set forces rigid structure to the border vector and middle matrix.

3.3.1 Border Vector for a Complex Hessian: Choosing an Order for Monomials

Let p be an nc symmetric polynomial in g free variables such that the degree of its nc complex Hessian is d. Then the complex Hessian will be homogeneous of degree two in h, h^T.

For a fixed degree k, there are g^k nc analytic monomials and g^k nc antianalytic monomials in x, x^T. That means there are $(2g)^k - g^k - g^k = (2g)^k - 2g^k$ 'mixed' monomials of degree k (i.e., monomials that are not nc analytic nor nc antianalytic).

Analytic Border Vector

For $0 \le k \le d-2$, let $A_k = A_k(x)[h]$ be the vector of nc analytic monomials with entries $h_j m(x)$ where $m(x)$ runs through the set of g^k nc analytic monomials of length k for $j = 1, \ldots, g$. The order we impose on the monomials in this vector is lexicographic order. Thus, the length of $A_k = A_k(x)[h]$ is g^{k+1} and the vector

$$A(x)[h] = col(A_{d-2}, \ldots, A_1, A_0) \tag{3.7}$$

has length $g^{d-1} + \cdots + g^2 + g = g\nu$ where $\nu = g^{d-2} + \cdots + g^2 + g + 1$.

Antianalytic Border Vector

Let $A_k^t = A_k(x^T)[h^T]$ be the same as $A_k = A_k(x)[h]$ except replace each h_j with h_j^T and replace each x_i by x_i^T. So A_k^t is the vector of nc antianalytic monomials with entries $h_j^T m(x^T)$ where $m(x^T)$ runs through the set of g^k nc antianalytic monomials of length k for $j = 1, \ldots, g$ (again, the order is lexicographic). Thus, the length of $A_k^t = A_k(x^T)[h^T]$ is g^{k+1} and the vector

$$A(x^T)[h^T] = col(A_{d-2}^t, \ldots, A_1^t, A_0^t) \tag{3.8}$$

also has length $g\nu$.

Mixed Term Border Vector

Next, we define notation to handle all nonanalytic and nonantianalytic monomials. Let $B_1 = B_1(x, x^T)[h]$ be the vector of monomials with entries $h_j x_i^T$ for $i = 1, \ldots, g$ and $j = 1, \ldots, g$. The length of B_1 is g^2. For $2 \leq k \leq d-2$, let $B_k = B_k(x, x^T)[h]$ be the vector of monomials with entries $h_j m(x, x^T)$ where $m(x, x^T)$ runs through the set of $(2g)^k - 2g^k$ monomials of length k that are not nc analytic nor nc antianalytic for $j = 1, \ldots, g$. Again, we put the same lexicographic order on the monomials. Thus, the length of $B_k = B_k(x, x^T)[h]$ is $g((2g)^k - 2g^k)$ and the vector

$$B(x, x^T)[h] = col(B_{d-2}, \ldots, B_2, B_1)$$

has length $g^2 + \sum_{k=2}^{d-2} g((2g)^k - 2g^k)$. Then we can also define $B_1^t = B_1^t(x, x^T)[h^T]$ to be the vector of monomials with entries $h_j^T x_i$ for $i = 1, \ldots, g$ and $j = 1, \ldots, g$. This also has length g^2. Then we define, for $2 \leq k \leq d-2$, the vector $B_k^t = B_k(x, x^T)[h^T]$ to be the same as B_k except h_j is replaced by h_j^T. In other words, each entry looks like $h_j^T m(x, x^T)$. Then the vector

$$B(x, x^T)[h^T] = col(B_{d-2}^t, \ldots, B_2^t, B_1^t)$$

has the same length as $B(x, x^T)[h]$.

Note that the degree of the monomials in A_k, A_k^t, B_k, B_k^t is $k+1$.

3.3.2 The Middle Matrix of a Complex Hessian

Now we can represent the nc complex Hessian, q, of a symmetric nc polynomial p as

$$q(x, x^T)[h, h^T] = \begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \\ A(x^T)[h^T] \\ B(x, x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_2 & 0 & 0 \\ Q_2^T & Q_4 & 0 & 0 \\ 0 & 0 & Q_5 & Q_6 \\ 0 & 0 & Q_6^T & Q_8 \end{pmatrix} \begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \\ A(x^T)[h^T] \\ B(x, x^T)[h^T] \end{pmatrix} \tag{3.9}$$

where $Q_i = Q_i(x, x^T)$ are matrices with nc polynomial entries in the variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$.

Again, we wish to stress that the vectors $A(x)[h]$, $A(x^T)[h^T]$, $B(x,x^T)[h]$, and $B(x,x^T)[h^T]$ may contain monomials that are not required in the representation of the nc complex Hessian, q. Therefore, we omit all monomials from the border vector that are not required. This gives us a *minimal length* border vector and prevents extraneous zeros from occurring in the middle matrix. Lemma 3.2.1 says that a minimal length border vector contains only distinct monomials.

We also note that Theorem 2.2.16 (P1) shows that every term in the nc complex Hessian, q, contains exactly one h_j and one h_k^T for some j and k. This structure forces the zeros in the middle matrix in Equation (3.9) above.

3.3.3 Structure of the Middle Matrix

In this subsection, we prove some properties about the structure of the middle matrix in the MMR for a matrix positive nc complex Hessian.

Lemma 3.3.1. *Let p be an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Then, the MMR in Equation* (3.9) *for its nc complex Hessian, q, of p has $Q_2 = Q_4 = Q_6 = Q_8 = 0$. Thus,*

$$q = \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1(x,x^T) & 0 \\ 0 & Q_5(x,x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}. \tag{3.10}$$

Proof. We consider the upper left block of the middle matrix in Equation (3.9)

$$\begin{pmatrix} A(x)[h] \\ B(x,x^T)[h] \end{pmatrix}^T \begin{pmatrix} Q_1(x,x^T) & Q_2(x,x^T) \\ Q_2(x,x^T)^T & Q_4(x,x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ B(x,x^T)[h] \end{pmatrix}$$

with the goal of showing $Q_2 = 0$ and $Q_4 = 0$. Thus, suppose the border vector contains a nonzero monomial which is an entry in the vector of mixed monomials, $B(x,x^T)[h]$; i.e., the border vector contains a term

$$h_k m_1(x,x^T) x_j^T m_2(x,x^T) \tag{3.11}$$

for some monomials m_1 and m_2 in the variables $x_1,\ldots,x_g,x_1^T,\ldots,x_g^T$.

Soon we shall look at the diagonal entry, $\mathcal{P}^{(0)}$, in the middle matrix corresponding to this border vector monomial in (3.11) and show it is 0. By Theorem

3.2.2, we have the middle matrix positive semidefinite for every X in the nc open set, $\mathcal{G}$. By Lemma 3.2.3, if an nc polynomial is zero on an open set of matrix tuples with sufficiently large dimension, then the nc polynomial is identically zero. Hence, if there is ever a diagonal entry in the middle matrix that is zero on an open set of matrix tuples of large enough dimension, then that diagonal entry is identically zero. Hence, to force matrix positivity, the corresponding row and column in the middle matrix must be zero. This implies that the particular monomial in the border vector is not needed in the representation, thereby contradicting the border vector being of minimal length. Thus, showing $\mathcal{P}^{(0)}$ is 0, a contradiction.

The term(s) in the nc complex Hessian corresponding to the diagonal entry $\mathcal{P}^{(0)}$ of the middle matrix and monomial (3.11) in the border vector are

$$m_2^T x_j m_1^T h_k^T \mathcal{P}^{(0)} h_k m_1 x_j^T m_2$$

where $\mathcal{P}^{(0)}$ is some matrix positive polynomial in $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$. By Theorem 2.2.16 (P2), q must also contain the Levi-differentially wed term(s)

$$m_2^T h_j m_1^T h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2.$$

This means the border vector must contain the monomial(s)

$$\{h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2\}_{mon} \tag{3.12}$$

where $\{h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2\}_{mon}$ is the list of the monomials that appear as terms in the nc polynomial $h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2$.

Again, we shall look at the term(s) in q corresponding to the diagonal in the middle matrix corresponding to any one of the border vector monomial(s) in (3.12). Pick $h_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$ as a specific border vector monomial in the list in (3.12). Then, the term(s) in q look like

$$m_2^T x_j m_1^T x_k^T (\widehat{\mathcal{P}}^{(0)})^T h_k \mathcal{P}^{(1)} h_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$$

where $\mathcal{P}^{(1)}$ is a matrix positive polynomial in $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$, which is a diagonal entry of the middle matrix. Theorem 2.2.16 (P2) implies q must also contain the Levi-differentially wed term(s)

$$m_2^T h_j m_1^T h_k^T (\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$$

which means the border vector must contain the monomial(s)

$$\{h_k^T(\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2\}_{mon} \tag{3.13}$$

where $\{h_k^T(\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2\}_{mon}$ is the list of the monomials that appear as terms in the nc polynomial $h_k^T(\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$.

Note that the border vector monomial in (3.13) has degree at least 2 more than the degree of the border vector monomial in (3.12) which has degree at least 2 more than the degree of the border vector monomial in (3.11). We can continue this process and the degree of the successive border vector monomials will keep increasing by at least 2 at each step. At some step, the degree of the border vector monomial will exceed $d-1$. This contradicts the fact that the border vector monomials must have degree at most $d-1$. Thus, we have shown that $Q_4 = 0$. A similar argument shows that $Q_8 = 0$. Since the middle matrix is positive semidefinite, by the argument at the beginning of this proof, we also get that $Q_2 = 0$ and $Q_6 = 0$. Hence, the nc complex Hessian has the representation in Equation (3.10), as claimed by the theorem. □

Theorem 3.3.2. *The nc complex Hessian, q, of an nc symmetric polynomial that is nc plush on an nc open set can be written as in Equation* (3.10)

$$q(x,x^T)[h,h^T] = \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1(x,x^T) & 0 \\ 0 & Q_5(x,x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}$$

where every nc polynomial entry in $Q_1(x,x^T)$ is hereditary and every nc polynomial entry in $Q_5(x,x^T)$ is antihereditary.

Proof. Suppose, for the sake of contradiction, Q_1 contains an nc polynomial entry which is not hereditary. Without loss of generality, this nc polynomial contains a term of the form

$$m_1(x^T)x_j x_k^T m_2(x,x^T) \tag{3.14}$$

where m_1 is a monomial in x^T and m_2 is a monomial in x and x^T. Since this is part of an entry in the middle matrix, this means that the nc complex Hessian must contain a term of the form

$$m_3(x^T)h_\ell^T m_1(x^T)x_j x_k^T m_2(x,x^T)h_s m_4(x)$$

where $m_3(x^T)h_\ell^T$ is a specific monomial entry from the vector $A(x)[h]^T$ and $h_s m_4(x)$ is a specific monomial entry from the vector $A(x)[h]$. Then, Theorem 2.2.16 (P2) implies that the nc complex Hessian must also contain the Levi-differentially wed term

$$m_3(x^T)h_\ell^T m_1(x^T)h_j x_k^T m_2(x, x^T)x_s m_4(x).$$

This implies that the border vector must contain the monomial

$$h_j x_k^T m_2(x, x^T)x_s m_4(x)$$

which contradicts having an nc analytic or nc antianalytic border vector, as required by Lemma 3.3.1. The proof that Q_5 contains antihereditary nc polynomial entries is similar. □

For a real number, r, we define $\lfloor r \rfloor$ as the largest integer less than or equal to r and we define $\lceil r \rceil$ as the smallest integer greater than or equal to r. The next theorem puts an upper bound on the degree of the monomials in the border vector for q.

Lemma 3.3.3. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set. If the degree of its nc complex Hessian, q, is d, then the degree of the border vector monomials is at most $\lfloor \frac{d}{2} \rfloor$.*

Proof. Write the MMR for $q(x, x^T)[h, h^T]$ as

$$q = V^T M V = \begin{pmatrix} V_1^T & V_2^T \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

with the following property. If d is odd, V_1 contains monomials of degree $1, \ldots, \lfloor \frac{d}{2} \rfloor$ and V_2 contains monomials of degree $\lceil \frac{d}{2} \rceil, \ldots, d-1$. If d is even, V_1 contains monomials of degree $1, \ldots, \frac{d}{2}$ and V_2 contains monomials of degree $\frac{d}{2}+1, \ldots, d-1$. In either case, polynomials in M_4 correspond to terms in q having degree strictly greater than d. Hence $M_4 = 0$. By Theorem 3.2.2, $M(X, X^T) \succeq 0$ for all X in an nc open set. This forces $M_2(X, X^T) = 0$ for all X in an nc open set. Then, by taking X to have large enough size, Lemma 3.2.3 implies $M_2 = 0$. □

Consequences of Positivity of the Complex Hessian

Now we turn from a description of the middle matrix to describing the structure of the nc complex Hessian of an nc polynomial that is nc plush on an nc open set.

Proposition 3.3.4. *The nc complex Hessian, q, of an nc symmetric polynomial that is nc plush on an nc open set is a sum of hereditary and antihereditary polynomials.*

Proof. This follows immediately from Lemma 3.3.1 and Theorem 3.3.2. □

Finally, we show that the degree of q must be even when p is nc plush on an nc open set. This fact is obvious if p is assumed nc plush everywhere because then the nc complex Hessian is a sum of squares, as provided by Theorem 1.1.1.

Theorem 3.3.5. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set. Then, the degree of its nc complex Hessian, q, is even.*

Proof. Suppose the degree of q is $2N + 1$. Without loss of generality, Proposition 3.3.4 and Theorem 2.2.16, requiring the presence of Levi-differentially wed monomials, imply that q must contain a hereditary term of the form

$$x_{i_1}^T x_{i_2}^T \cdots h_{i_s}^T h_{j_1} x_{j_2} \cdots x_{j_\ell}$$

where $s, \ell > 0$, $s + \ell = 2N + 1$, and $i_1, \dots, i_s, j_1, \dots, j_\ell \in \{1, \dots, g\}$. This means that in the middle matrix representation for q, the border vector must contain $h_{j_1} x_{j_2} \cdots x_{j_\ell}$ and $h_{i_s} x_{i_{s-1}} \cdots x_{i_1}$ which have degree ℓ and s, respectively. But since $s+\ell = 2N+1$ and $s, \ell > 0$, one of either s or ℓ is at least $\lceil \frac{2N+1}{2} \rceil$. This contradicts Lemma 3.3.3. □

3.4 LDL^T Decomposition Has Constant D

This section concerns the "algebraic Cholesky" factorization, LDL^T, of the middle matrix. We will show that for an nc polynomial that is nc plush on an nc open set, this D is a positive semidefinite matrix whose diagonal entries are all nonnegative real constants, and L is unit lower triangular with entries which are nc polynomials. This is a stronger conclusion than one would expect because, typically, such factorizations have nc rational entries, see [CHSY03, HMV06]. In our approach, the LDL^T factorization of a symmetric matrix with noncommutative entries will be the key tool for the determination of the matrix positivity of an nc quadratic function.

3.4.1 The LDL^T Decomposition

Begin by considering the block 2×2 matrix

$$M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

where A is a constant real symmetric invertible matrix and B and C are matrices with nc polynomial entries with C symmetric. Then, M has the following decomposition

$$M = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - BA^{-1}B^T \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & I \end{pmatrix}, \qquad (3.15)$$

where all matrices in this decomposition contain nc polynomial entries. If $C - BA^{-1}B^T$ contains a constant real symmetric invertible matrix somewhere on the diagonal, then we can apply a permutation, Π, on the left of M and its transpose, Π^T, on the right of M to move this constant real symmetric invertible matrix to the first (block) diagonal position of $C - BA^{-1}B^T$. We then pivot off this constant real symmetric invertible matrix, factor $C - BA^{-1}B^T$ as $\hat{L}\hat{D}\hat{L}^T$, and we get

$$\Pi M \Pi^T = \begin{pmatrix} I & 0 \\ BA^{-1} & \hat{L} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \hat{D} \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & \hat{L}^T \end{pmatrix}.$$

This can be continued, provided at each step, a constant real symmetric invertible matrix appears somewhere on the diagonal to obtain $\Pi M \Pi^T = LDL^T$ where L is a unit lower triangular matrix with nc polynomial entries and D is a (block) diagonal matrix with real constant blocks. This special situation is the one which turns out to hold in the derivation which follows.

Indeed, we shall only care about the case where A is a constant real symmetric invertible matrix. For the case where A contains nc polynomial entries and is considered to be "noncommutative invertible", see [CHSY03]. In this case, we also have the notion of "noncommutative rational" functions (see [HMV06]). However, as we soon shall see, while nc rationals are mentioned, they never actually appear in any calculations in this dissertation.

We recall an immediate consequence of Theorem 3.3 in [CHSY03]:

Theorem 3.4.1. *Suppose $M(x, x^T)$ is a symmetric $r \times r$ matrix with noncommutative rational function entries and that $M(X, X^T) \succeq 0$ for all X in some nc open set. Then, there exists a permutation matrix, Π, a diagonal matrix, $D(x, x^T)$, with nc rational entries, and a unit lower triangular matrix, $L(x, x^T)$, with nc rational entries such that*

$$\Pi M(x, x^T)\Pi^T = L(x, x^T)D(x, x^T)L(x, x^T)^T.$$

Remark 3.4.2. *In this chapter, we care about the positivity of the middle matrix, $M(x, x^T)$. If Π is a permutation matrix, it is clear that*

$$\Pi M(X, X^T)\Pi^T \succeq 0 \qquad \Longleftrightarrow \qquad M(X, X^T) \succeq 0$$

for any $X \in \mathbb{R}^{n\times n}$ and any $n \geq 1$. As a result, for ease of exposition, we will often, without loss of generality, omit the permutation matrix, Π.

Also, there will be some instances where we will, without loss of generality, assume a specific order in the border vector, $V(x, x^T)[h, h^T]$. For example, we may assume a given monomial, say, $hm(x, x^T)$, is the first monomial in $V(x, x^T)[h, h^T]$. This assumption also amounts to a permutation of $V(x, x^T)[h, h^T]$ which, again, does not affect positivity of $M(x, x^T)$ so we omit it from the discussion.

We now proceed to apply the LDL^T factorization to the middle matrix of the nc complex Hessian. Let p be an nc symmetric polynomial and let q denote the nc complex Hessian of p. Since q is homogeneous of degree 2 in h, h^T, q admits the MMR

$$q = V(x, x^T)[h, h^T]^T M(x, x^T)V(x, x^T)[h, h^T]. \tag{3.16}$$

If p is nc plush on an nc open set, then $M(x, x^T)$ is symmetric and matrix positive on an nc open set and we can factor $M(x, x^T)$ following the process underlying Equation (3.15) and Theorem 3.4.1, thus converting Equation (3.16) to

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T] \tag{3.17}$$

up to a harmless rearrangement of the border vector.

In Section 3.4.4, we prove one of the main theorems of this dissertation, Theorem 3.4.9, which was stated in Section 3.1 as Theorem 3.1.2. We recall that this theorem says that $D(x, x^T)$ in Equation (3.17) does not depend on x, x^T and is a positive semidefinite constant real diagonal matrix for an nc polynomial that is nc plush on an nc open set. In addition, we will prove that $L(x, x^T)$ contains nc polynomials instead of nc rationals. Now we start the build up to Section 3.4.4.

3.4.2 Properties of LDL^T for NC Polynomials that are NC Plush on an NC Open Set

In this subsection, we present properties of the LDL^T factorization of the nc complex Hessian for an nc polynomial that is nc plush on an nc open set.

Recall from Section 1.4.2 that a set $\mathcal{G} \subseteq \cup_{n \geq 1} (\mathbb{R}^{n \times n})^g$ is an nc open set if:

(i) $\mathcal{G}$ respects direct sums, and

(ii) there exists a positive integer n_0 such that if $n > n_0$, the set $\mathcal{G}_n := \mathcal{G} \cap (\mathbb{R}^{n \times n})^g$ is an open set of matrix tuples;

and an nc symmetric polynomial, p, is nc plush on an nc open set, $\mathcal{G}$, if p has an nc complex Hessian, q, such that $q(X, X^T)[H, H^T]$ is positive semidefinite for all $X \in \mathcal{G}$ and for all $H \in (\mathbb{R}^{n \times n})^g$ for every $n \geq 1$.

For an nc symmetric polynomial that is nc plush on an nc open set, Theorem 3.3.5 shows that the nc complex Hessian has even degree; denote it $2N$. We will use this fact throughout the duration of the chapter. The next lemma is a stepping stone for Lemma 3.4.4.

Lemma 3.4.3. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex Hessian, q. Then, q must contain a term of the form*

$$\alpha m^T h^T hm \quad (or \quad \alpha mhh^T m^T)$$

where m is an nc analytic monomial of degree $N-1$ and α is a positive real constant.

Proof. Proposition 3.3.4 implies q is a sum of hereditary and antihereditary polynomials. Let w be a term of degree $2N$ in q. Without loss of generality, suppose w is hereditary; i.e., w has the form

$$w = \alpha m_1^T h^T m_2^T m_3 h m_4$$

where $\alpha \in \mathbb{R}$, m_1, m_2, m_3, m_4 are nc analytic monomials in x, and

$$\deg(m_1) + \deg(m_2) + \deg(m_3) + \deg(m_4) = 2N - 2.$$

By Theorem 2.2.16 (P2), q must contain the Levi-differentially wed term

$$\tilde{w} = \alpha \tilde{m}_1^T h^T h \tilde{m}_2$$

where $\tilde{m}_1, \tilde{m}_2$ are nc analytic monomials in x and $\deg(\tilde{m}_1) = \deg(\tilde{m}_2) = N - 1$.

If $\tilde{m}_1 = \tilde{m}_2$, we are done (except for showing $\alpha > 0$). If the conclusion of the lemma is false, so that q contains no term of the form $\alpha m^T h^T hm$, then this implies $\tilde{m}_1 \neq \tilde{m}_2$. Since q is symmetric, q must also contain the term

$$\tilde{w}^T = \alpha \tilde{m}_2^T h^T h \tilde{m}_1.$$

If we partition the border vector so that $e_1^T V = h\tilde{m}_1$ and $e_2^T V = h\tilde{m}_2$, then we get that

$$q = \begin{pmatrix} h\tilde{m}_1 \\ h\tilde{m}_2 \\ \vdots \end{pmatrix}^T \begin{pmatrix} 0 & \alpha & \cdots \\ \alpha & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} h\tilde{m}_1 \\ h\tilde{m}_2 \\ \vdots \end{pmatrix}.$$

This middle matrix is not positive semidefinite for any $X \in \mathcal{G}$. Hence, Theorem 3.2.2 implies that q is not positive semidefinite for all $X \in \mathcal{G}$. This contradicts the positivity of q on the nc open set, $\mathcal{G}$. Hence, q must contain some term of the form $\alpha m^T h^T h m$.

We now show $\alpha > 0$. Since we know that q contains a term of the form $\alpha m^T h^T h m$ with m an nc analytic or nc antianalytic monomial of degree $N-1$, the real constant α will appear on the diagonal in the middle matrix. Then, Theorem 3.2.2 implies that this α must be positive. $\square$

When we write e_i, we mean the vector whose i^{th} entry is 1 and every other entry is 0. From Equation (3.17), we can write q as a sum of outer products

$$\begin{aligned} q &= V(x,x^T)[h,h^T]^T \left(\sum_{i=1}^{\mathcal{N}} (Le_i) d_i (Le_i)^T \right) V(x,x^T)[h,h^T] \\ &= \sum_{i=1}^{\mathcal{N}} V(x,x^T)[h,h^T]^T (Le_i) d_i (Le_i)^T V(x,x^T)[h,h^T]. \end{aligned} \tag{3.18}$$

We stress that in Equation (3.18), each Le_i and d_i depend on x and x^T. However, the next lemma shows that one element of D is constant and one column of L contains nc polynomials rather than nc rationals.

Lemma 3.4.4. *Let p be an nc symmetric polynomial that is nc plush on an nc open set. Let $2N$ denote the degree of its nc complex Hessian, q. Then, we can write the nc complex Hessian, q, as in Equations (3.17) and (3.18) where $L(x,x^T)$ is unit lower triangular and $D(x,x^T) = diag(d_1,\ldots,d_{\mathcal{N}})$ with d_1 a positive real constant.*

Hence, each entry in Le_1, the first column of $L(x,x^T)$, is an nc polynomial rather than an nc rational.

Proof. Theorem 3.4.1 implies $D(x,x^T)$ is a diagonal matrix. Without loss of generality, Lemma 3.4.3 implies that q contains a term of the form

$$\alpha m^T h^T h m$$

where $\alpha > 0$ is a positive real constant and m is an nc analytic monomial of degree $N-1$. The MMR of q can be written as

$$q = \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} \alpha & \ell^T \\ \ell & \widehat{M} \end{pmatrix} \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}.$$

Since $\alpha > 0$, we can first pivot off α in computing the LDL^T factorization of the middle matrix to get

$$q = \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ \frac{1}{\alpha}\ell & I \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \widehat{M} - \frac{1}{\alpha}\ell\ell^T \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\alpha}\ell^T \\ 0 & I \end{pmatrix} \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}.$$

Now we see that $d_1 = e_1^T D e_1 = \alpha > 0$ and that

$$Le_1 = \begin{pmatrix} 1 \\ \frac{1}{\alpha}\ell \end{pmatrix} \qquad \text{and} \qquad \widehat{M} - \frac{1}{\alpha}\ell\ell^T$$

contain only nc polynomials as entries. □

The next lemma provides even more specific structure to Le_1 and maintains the nc polynomial structure.

Lemma 3.4.5. *Under the same hypotheses of Lemma 3.4.4, either:*

(i) every entry of Le_1 (the 1st column of $L(x, x^T)$) is an nc antianalytic polynomial, d_1 (the 1st diagonal entry of $D(x, x^T)$) is a positive real constant, and the corresponding monomials in $V(x, x^T)[h, h^T]$ are nc analytic; or

(ii) every entry of Le_1 (the 1st column of $L(x, x^T)$) is an nc analytic polynomial, d_1 (the 1st diagonal entry of $D(x, x^T)$) is a positive real constant, and the corresponding monomials in $V(x, x^T)[h, h^T]$ are nc antianalytic.

Proof. Lemma 3.3.1 implies that q can be written as

$$q = A(x)[h]^T Q_1(x, x^T) A(x)[h] + A(x^T)[h^T]^T Q_5(x, x^T) A(x^T)[h^T]$$

where each entry of $A(x)[h]$ is an nc analytic monomial and each entry of $A(x^T)[h^T]$ is an nc antianalytic monomial. Also, Q_1 contains hereditary nc polynomials and Q_5 contains antihereditary nc polynomials. Then, we have that

$$q = A(x)[h]^T L_1 D_1 L_1^T A(x)[h] + A(x^T)[h^T]^T L_2 D_2 L_2^T A(x^T)[h^T]$$

$$= \underbrace{\begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T}_{V(x,x^T)[h,h^T]^T} \underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}^T}_{L^T} \underbrace{\begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}}_{V(x,x^T)[h,h^T]}. \tag{3.19}$$

Without loss of generality, Lemma 3.4.3 allows us to assume that q contains a term of the form

$$d_1 m^T h^T hm$$

where m is an nc analytic monomial in x (so that hm is an entry in $A(x)[h]$) of degree $N-1$ and d_1 is a positive real constant. Lemma 3.4.4 implies that $e_1^T D_1 e_1 = d_1$ and that each entry of Le_1 is an nc polynomial. From Equation (3.19), we have that

$$Le_1 = \begin{pmatrix} L_1 e_1 \\ 0 \end{pmatrix}$$

and $(Le_1)^T V = (L_1 e_1)^T A(x)[h]$.

Next, write q as in Equation (3.18) and see that the first term in this sum becomes

$$V^T (Le_1) d_1 (Le_1)^T V = d_1((L_1 e_1)^T A(x)[h])^T((L_1 e_1)^T A(x)[h])$$

Proposition 3.3.4 implies that q is a sum of hereditary and antihereditary polynomials. Therefore, since $A(x)[h]$ contains only nc analytic monomials, this forces $(L_1 e_1)^T$ to contain only nc analytic polynomials (which means that $L_1 e_1$ contains only nc antianalytic polynomials). This completes the proof of Case (i).

The proof of Case (ii) works the same way, from Lemma 3.4.3, whenever we assume that q contains a term of the form

$$d_1 m h h^T m^T$$

where m is an nc analytic monomial in x of degree $N-1$ and d_1 is a positive real constant. $\square$

The next lemma is a technical lemma that is used as a stepping stone to help prove Proposition 3.4.7.

Lemma 3.4.6. *Let p be an nc symmetric polynomial that is nc plush on an nc open set. Let $2N$ denote the degree of its nc complex Hessian, q. Then, we can write q as in Equation* (3.18)

$$q = \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i) d_i (Le_i)^T V(x, x^T)[h, h^T]$$

with

$$V(x,x^T)[h,h^T]^T e_1 = x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \qquad (\textit{resp.} \quad V(x,x^T)[h,h^T]^T e_1 = x_{i_N} \cdots x_{i_2} h_{i_1})$$

in which case, any term in q that has the form

$$d_1 \gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h) \qquad (\textit{resp.} \quad d_1 \gamma x_{i_N} \cdots x_{i_2} h_{i_1} m(x^T, h^T))$$

where γ is a real constant and $m(x,h)$ is some nc analytic monomial in x, h of degree 1 in h (resp. $m(x^T,h^T)$ is some nc antianalytic monomial in x^T, h^T of degree 1 in h^T), is a term in the nc polynomial

$$d_1 V(x,x^T)[h,h^T]^T (Le_1)(Le_1)^T V(x,x^T)[h,h^T].$$

Moreover, $\gamma m(x,h)$ (resp. $\gamma m(x^T,h^T)$) is a term in the nc analytic (resp. nc antianalytic) polynomial

$$(Le_1)^T V(x,x^T)[h,h^T].$$

Proof. Proposition 3.3.4 implies q is a sum of hereditary and antihereditary polynomials. Since the degree of q is $2N$, there exists a term, w, in q of degree $2N$. Without loss of generality, Lemma 3.4.3 allows us to assume that w looks like

$$w = d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N}.$$

with $d_1 \in \mathbb{R}_+$. We partition the border vector $V(x,x^T)[h,h^T]$ as

$$V(x,x^T)[h,h^T] = \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix}$$

where $h_{i_1} x_{i_2} \cdots x_{i_N}$ is not a monomial entry in the vector $\widehat{V}$. Then, q becomes

$$q = \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ \ell & \widehat{L} \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & \widehat{D} \end{pmatrix} \begin{pmatrix} 1 & \ell^T \\ 0 & \widehat{L}^T \end{pmatrix} \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix}$$

$$= d_1 \overbrace{(x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N} + x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \ell^T \widehat{V} + \widehat{V}^T \ell h_{i_1} x_{i_2} \cdots x_{i_N} + \widehat{V} \ell \ell^T \widehat{V})}^{V(x,x^T)[h,h^T]^T (Le_1)(Le_1)^T V(x,x^T)[h,h^T] = (x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T + \widehat{V}^T \ell)(h_{i_1} x_{i_2} \cdots x_{i_N} + \ell^T \widehat{V})} \tag{3.20}$$

$$+\widehat{V}^T\widehat{L}\widehat{D}\widehat{L}^T\widehat{V}.$$

Since $x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T$ is not a monomial entry in the vector $\widehat{V}^T$, this shows that any term in q of the form $d_1\gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h)$, where γ is a real constant and $m(x,h)$ is an nc analytic monomial of degree 1 in h, is a term in the nc polynomial

$$d_1 V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^T V(x,x^T)[h,h^T].$$

Equation (3.20) implies that either

$$d_1\gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h) = d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N}$$

or that $d_1\gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h)$ is a term in the nc polynomial

$$d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \ell^T \widehat{V}.$$

This implies that either $\gamma = 1$ and $m(x,h) = h_{i_1} x_{i_2} \cdots x_{i_N}$ or that $\gamma m(x,h)$ is a term in the nc polynomial $\ell^T\widehat{V}$. Hence, $\gamma m(x,h)$ is a term in the nc polynomial

$$h_{i_1} x_{i_2} \cdots x_{i_N} + \ell^T\widehat{V} = (Le_1)^T V(x,x^T)[h,h^T]$$

and Lemma 3.4.5 implies that $(Le_1)^T V(x,x^T)[h,h^T]$ is nc analytic. □

3.4.3 Part of the NC Complex Hessian is an NC Complex Hessian

In this subsection, we focus on writing the nc complex Hessian, q, as in Equation (3.18)

$$q = \sum_{i=1}^{\mathcal{N}} V(x,x^T)[h,h^T]^T(Le_i)d_i(Le_i)^T V(x,x^T)[h,h^T].$$

This subsection culminates with the result that the nc polynomial

$$d_1 V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^T V(x,x^T)[h,h^T],$$

is the nc complex Hessian for some nc polynomial that is nc plush on an nc open set. In order to do this, we first show that the nc polynomial

$$(Le_1)^T V(x,x^T)[h,h^T]$$

is the nc directional derivative of some nc analytic or nc antianalytic polynomial.

Proposition 3.4.7. *Let p be an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex Hessian, q. If we write q as in Equation* (3.18) *and d_1 is constant, then the nc polynomial*

$$(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc directional derivative of an nc analytic polynomial or an nc antianalytic polynomial.

In addition, the nc polynomial

$$d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc complex Hessian of some nc polynomial that is nc plush on $\mathcal{G}$.

Proof. Without loss of generality, we can assume, by Lemma 3.4.5, that

$$(Le_1)^T V(x, x^T)[h, h^T]$$

is an nc analytic polynomial where $V(x, x^T)[h, h^T]$ and Le_1 are partitioned as

$$V(x, x^T)[h, h^T] = \begin{pmatrix} V_N \\ V_{N-1} \\ \vdots \\ V_1 \end{pmatrix}, \quad Le_1 = \begin{pmatrix} \ell_0 \\ \ell_1 \\ \vdots \\ \ell_{N-1} \end{pmatrix}, \quad \ell_0 = \begin{pmatrix} 1 \\ \star \\ \vdots \\ \star \end{pmatrix}, \tag{3.21}$$

where $\star$ is any nc polynomial and V_j is a vector that contains only nc analytic monomials of the form $h_{i_1} x_{i_2} \cdots x_{i_j}$ having total degree j. Each ℓ_j is a vector with the same length as V_j and, by Lemma 3.4.5, ℓ_j contains only nc antianalytic polynomials (ℓ_j^T contains only nc analytic polynomials). With this setup, we have that

$$\mathcal{F}(x, h) := (Le_1)^T V(x, x^T)[h, h^T] = \sum_{j=0}^{N-1} \ell_j^T V_{N-j}$$

is an nc analytic polynomial in x and h. We define this as $\mathcal{F}(x, h)$ for convenience.

Lemma 3.4.4 implies $d_1 \in \mathbb{R}_+$ is a constant and Equation (3.18) implies that q contains the terms

$$d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T] = d_1 \left(\sum_{j=0}^{N-1} \ell_j^T V_{N-j} \right)^T \left(\sum_{j=0}^{N-1} \ell_j^T V_{N-j} \right) \tag{3.22}$$

Then, since the degree of q is $2N$ and the degree of each border vector monomial in V_{N-j} is $N-j$, it follows that the degree of each nc analytic polynomial in ℓ_j^T is at most j.

Lemma 3.4.3 implies that q contains some term of the form

$$\alpha^2 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N} \tag{3.23}$$

with α a nonzero real constant. This implies that the vector V_N contains the monomial $h_{i_1} x_{i_2} \cdots x_{i_N}$ as an entry. Without loss of generality, assume this monomial is first in lexicographic order. Then,

$$e_1^T V(x, x^T)[h, h^T] = e_1^T V_N = h_{i_1} x_{i_2} \cdots x_{i_N}.$$

As in the proof of Lemma 3.4.4, if M represents the middle matrix of q, then $e_1^T M e_1 = \alpha^2$ and, after one step in the LDL^T algorithm, we see that $\alpha^2 = d_1$. Then, by Theorem 2.2.16 (P2), q also contains the Levi-differentially wed terms

$$\begin{gathered} d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T x_{i_1} h_{i_2} x_{i_3} \cdots x_{i_N} \\ d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T x_{i_1} x_{i_2} h_{i_3} \cdots x_{i_N} \\ \vdots \\ d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T x_{i_1} x_{i_2} \cdots x_{i_{N-1}} h_{i_N}. \end{gathered}$$

Since q contains these terms and the term in (3.23), Lemma 3.4.6 implies that $\mathcal{F}(x, h)$ contains the term

$$h_{i_1} x_{i_2} x_{i_3} \cdots x_{i_N} \tag{3.24}$$

and the terms

$$\begin{gathered} x_{i_1} h_{i_2} x_{i_3} \cdots x_{i_N} \\ x_{i_1} x_{i_2} h_{i_3} \cdots x_{i_N} \\ \vdots \\ x_{i_1} x_{i_2} \cdots x_{i_{N-1}} h_{i_N}. \end{gathered}$$

Hence, $\mathcal{F}(x, h)$ contains all 1-differentially wed monomials to $h_{i_1} x_{i_2} x_{i_3} \cdots x_{i_N}$ as terms. Theorem 2.2.4 implies that $\mathcal{F}(x, h)$ contains the nc directional derivative of $x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_N}$.

Now we pick any other term in $\mathcal{F}(x,h)$ and show that $\mathcal{F}(x,h)$ contains all other 1-differentially wed monomials to it and that they all occur with the same coefficient. Suppose $\mathcal{F}(x,h)$ contains the term

$$\gamma x_{s_1}\cdots x_{s_k}h_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}.$$

We already showed that $\mathcal{F}(x,h)$ contains the monomial in (3.24), $h_{i_1}x_{i_2}\cdots x_{i_N}$, as a term so $\mathcal{F}(x,h)^T$ must contain the monomial $x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T$ as a term. This implies that $d_1\mathcal{F}(x,h)^T\mathcal{F}(x,h)$ contains the terms

$$d_1 x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T(h_{i_1}x_{i_2}\cdots x_{i_N}+\gamma x_{s_1}\cdots x_{s_k}h_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}).$$

Hence, q contains the term

$$d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}\cdots x_{s_k}h_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}$$

and Theorem 2.2.16 (P2) implies that q contains the Levi-differentially wed terms

$$\begin{array}{c}
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T h_{s_1}x_{s_2}\cdots x_{s_k}x_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}h_{s_2}\cdots x_{s_k}x_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}x_{s_2}\cdots h_{s_k}x_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}x_{s_2}\cdots x_{s_k}x_{\beta_1}h_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}x_{s_2}\cdots x_{s_k}x_{\beta_1}x_{\beta_2}\cdots h_{\beta_{N-j}}.
\end{array}$$

Since q contains all of these terms with $x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T$ on the left, Lemma 3.4.6

implies $\mathcal{F}(x,h)$ must contain the terms

$$\begin{gathered}
\gamma h_{s_1} x_{s_2} \cdots x_{s_k} x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{N-j}} \\
\gamma x_{s_1} h_{s_2} \cdots x_{s_k} x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{N-j}} \\
\vdots \\
\gamma x_{s_1} x_{s_2} \cdots h_{s_k} x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{N-j}} \\
\gamma x_{s_1} x_{s_2} \cdots x_{s_k} h_{\beta_1} x_{\beta_2} \cdots x_{\beta_{N-j}} \\
\gamma x_{s_1} x_{s_2} \cdots x_{s_k} x_{\beta_1} h_{\beta_2} \cdots x_{\beta_{N-j}} \\
\vdots \\
\gamma x_{s_1} x_{s_2} \cdots x_{s_k} x_{\beta_1} x_{\beta_2} \cdots h_{\beta_{N-j}}.
\end{gathered}$$

All of these terms in $\mathcal{F}(x,h)$ have the same coefficient, γ, and the monomials are 1-differentially wed to each other. Thus, Theorem 2.2.4 implies that they sum to the nc directional derivative of

$$\gamma x_{s_1} x_{s_2} \cdots x_{s_k} x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{N-j}}.$$

Hence, we have shown that $\mathcal{F}(x,h) = (Le_1)^T V(x,x^T)[h,h^T]$ is an nc directional derivative, where, without loss of generality, we assumed that $\mathcal{F}(x,h)$ was nc analytic.

Now we have that

$$\mathcal{F}(x,h) := (Le_1)^T V(x,x^T)[h,h^T]$$

is the nc directional derivative of some nc analytic or nc antianalytic polynomial. Suppose, without loss of generality, that $\mathcal{F}(x,h)$ is the nc directional derivative of some nc analytic polynomial, $\mathcal{F}(x)$. Then, $\mathcal{F}(x,h)$ is nc analytic and

$$d_1\mathcal{F}(x,h)^T\mathcal{F}(x,h) = d_1V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^TV(x,x^T)[h,h^T]$$

is the nc complex Hessian of the nc polynomial

$$d_1\mathcal{F}(x)^T\mathcal{F}(x).$$

Hence, for any $n \geq 1$, any $X \in \mathcal{G}$, and any $H \in (\mathbb{R}^{n\times n})^g$, we have

$$d_1\mathcal{F}(X,H)^T\mathcal{F}(X,H) \succeq 0.$$

$\square$

3.4.4 Constant D Result

In this subsection, we show that for an nc symmetric polynomial, p, that is nc plush on an nc open set, the matrix $D(x, x^T)$ in Equation (3.17) has no dependence on x or x^T and is actually a positive semidefinite constant real matrix. First, we require a helpful lemma.

Lemma 3.4.8. *If p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$, then its nc complex Hessian, q, can be written as in Equation* (3.17)

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $D(x, x^T)$ is a diagonal matrix of nc rationals and $D(X, X^T) \succeq 0$ for all $X \in \mathcal{G}$.

Proof. This follows immediately from Theorem 3.4.1. □

Theorem 3.4.9. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex Hessian, q. Then q can be written as in Equation* (3.17)

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $D(x, x^T) = diag(d_1, d_2, \ldots, d_{\mathcal{N}})$ is a positive semidefinite constant real matrix (i.e., $d_i \in \mathbb{R}_{\geq 0}$ for all $i = 1, \ldots, \mathcal{N}$) and $L(x, x^T)$ is a unit lower triangular matrix of nc polynomials.

Proof. Lemma 3.4.8 implies $D(X, X^T) \succeq 0$ for every $X \in \mathcal{G}$. This means

$$d_i(X, X^T) \succeq 0$$

for every $X \in \mathcal{G}$ and every $i = 1, \ldots, \mathcal{N}$. It remains to show that each d_i is a nonnegative constant real number.

First, write the nc complex Hessian, q, as in Equation (3.18)

$$q = \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i) d_i(x, x^T) (Le_i)^T V(x, x^T)[h, h^T].$$

Lemma 3.4.4 shows $d_1 \in \mathbb{R}_+$ is a constant, Le_1 contains nc polynomial entries, and Proposition 3.4.7 shows that

$$d_1 V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^T V(x,x^T)[h,h^T]$$

is the nc complex Hessian for some nc polynomial that is nc plush on $\mathcal{G}$. Since nc differentiation is linear, we know that the difference of two nc complex Hessians is an nc complex Hessian. This implies that

$$\begin{aligned} \widetilde{q} &:= q - d_1 V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^T V(x,x^T)[h,h^T] \\ &= \sum_{i=2}^{\mathcal{N}} V(x,x^T)[h,h^T]^T(Le_i)d_i(x,x^T)(Le_i)^T V(x,x^T)[h,h^T] \end{aligned}$$

is an nc complex Hessian. Since $d_i(X,X^T) \succeq 0$ for all $X \in \mathcal{G}$ and for all i, we have that $\widetilde{q}$ is the nc complex Hessian for an nc symmetric polynomial that is nc plush on $\mathcal{G}$. $\square$

Chapter 3 of this dissertation is taken from [G10] that has been submitted for publication as

J. M. Greene, Noncommutative Plurisubharmonic Polynomials, Part II: Local Assumptions, preprint, `http://arxiv.org/abs/1101.0111` .

Bibliography

[AK] D. Alpay, D. Kalyuzhnyi-Verbovetzkiĭ: Matrix-J-unitary non-commutative rational formal power series, In: *The state space method generalizations and applications*, Oper. Theory Adv. Appl. **161** (2006) 49–113.

[BGM] J.A. Ball, G. Groenewald, T. Malakorn: Conservative structured noncommutative multidimensional linear systems, In: *The state space method generalizations and applications*, Oper. Theory Adv. Appl. **161** (2006) 179–223.

[CHSY03] J. F. Camino, J. W. Helton, R. E. Skelton, J. Ye: Matrix inequalities: A Symbolic Procedure to Determine Convexity Automatically, Integral Equations and Operator Theory Vol 46, issue 4, August 2003 on pp. 399-454.

[G10] J. M. Greene: Noncommutative Plurisubharmonic Polynomials Part II: Local Assumptions, preprint, `http://arxiv.org/abs/1101.0111`.

[GHV] J. M. Greene, J. W. Helton, V. Vinnikov: Noncommutative Plurisubharmonic Polynomials Part I: Global Assumptions, preprint, `http://arxiv.org/abs/1101.0107`.

[H02] J. W. Helton: "Positive" Noncommutative Polynomials are a Sum of Squares, Annals of Mathematics vol 56 num 2, Sept 2002 pp.675-694.

[HKM] J. W. Helton, I. Klep, S. McCullough: Proper Analytic Free Maps, preprint. `http://arxiv.org/abs/1004.1381`.

[HM04] J. W. Helton, S. A. McCullough: Convex polynomials Degree Two or Less, SIAM Journal of Matrix Analysis, **25**(2004) no. 4, pp. 1124-1139.

[HMV06] J. W. Helton, S. A. McCullough, V. Vinnikov: Noncommutative convexity arises from linear matrix inequalities, J. Funct. Anal., **240** (2006), no. 1, 105–191.

[K-VV] D. Kalyuzhnyi-Verbovetskiĭ, V. Vinnikov: Foundations of noncommutative function theory, in preparation.

[Pop6] G. Popescu: Free holomorphic automorphisms of the unit ball of $B(H)^n$, J. reine angew. Math. **638** (2010) 119–168.

[PW] V. Powers, T. Wormann: An algorithm for sums of squares of real polynomials, J. Pure and Appl. Alg. 127 (1998), 99-104.

[Voi1] D.-V. Voiculescu: Free analysis questions I: Duality transform for the coalgebra of $\partial_{X:B}$, Int. Math. Res. Not. **16** (2004) 793–822.

[Voi2] D.-V. Voiculescu: Free Analysis Questions II: The Grassmannian Completion and The Series Expansions at the Origin, preprint, `http://arxiv.org/abs/0806.0361`.