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Multicast Networks with Variable-Length Limited Feedback

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***Abstract*—We investigate the channel quantization problem for two-user multicast networks where the transmitter is equipped with multiple antennas and either receiver is equipped with only a single antenna. Our goal is to design a global quantizer to minimize the outage probability. It is known that any fixed-length quantizer with a finite-cardinality codebook cannot obtain the same minimum outage probability as the case where all nodes in the network know perfect channel state information (CSI). To achieve the minimum outage probability, we propose a variable-length global quantizer that knows perfect CSI and sends quantized CSI to the transmitter and receivers. With a random infinite-cardinality codebook, we prove that the proposed quantizer is able to achieve the minimum outage probability with a low average feedback rate. We also extend the proposed quantizer to the multicast networks with more than two users. Numerical simulations validate our theoretical analysis.**

***Index Terms*—Multicasting, limited feedback, variable-length feedback, outage probability.**

I. Introduction

It is known that using more than one antenna at the transmitters can greatly improve the performance of communication systems. However, the performance depends on the availability of channel state information (CSI) at the transmitters and receivers [1]–[3]. Receivers can obtain CSI through training sequences; however, the transmitters must rely on the feedback information from receivers to do so. Additionally, perfect CSI at the transmitters requires an "infinite" number of feedback bits, which is unrealistic due to the limitations of feedback links. Therefore, it is more practical to employ quantized CSI to design efficient transmission schemes for wireless networks.

There has been a lot of work on channel quantization in point-to-point multiple antenna systems. An overview of research on limited feedback can be found in [4]. In multiple-input single-output (MISO) systems, a fixed-length quantizer (FLQ) is proposed in [1] to maximize the capacity by applying the beamforming vector at the transmitter. In FLQs, the number of feedback bits per channel state is a fixed positive integer. Compared to the case that all the nodes know CSI perfectly, fixed-length quantization always suffers from some performance loss. On the other hand, [5] proposes a variable-length quantizer (VLQ) to achieve the full-CSI outage probability with a low average feedback rate. VLQs allow binary codewords of different lengths to represent different channel states. It has been shown in [5] that variable-length quantization does not suffer from performance loss in MISO systems.

In this paper, we study the channel quantization problem in multicast networks with two receivers. We use transmit beamforming and consider the outage probability gap between the proposed quantizer and the full-CSI case. For a FLQ, the standard encoding rule is to choose the codeword "closest" to the channel state. For any finite-cardinality codebook, the outage probability of a FLQ is strictly worse than that of the full-CSI case [5]. To achieve the full-CSI outage probability with a finite average feedback rate, we propose a VLQ with a codebook of infinite cardinality. We incorporate the idea of variable-length coding and expect that in such a VLQ, the codeword covering a larger partition of channel space can be represented by a fewer number of bits. In this way, the average feedback rate can be made finite.

Based on the above analysis, we propose a VLQ in multicast networks that has access to full CSI and sends quantized CSI to the transmitter and receivers via error-free and delay-free feedback links. We consider a random codebook with infinite cardinality that is tractable for analysis [6]. Also, if a random codebook can provide a certain level of performance, then one codebook that will surpass this performance can be found. We first prove that the outage probability for the VLQ is the same as that of the full-CSI case. Afterwards, through a derived upper bound on the average feedback rate, we will show that: (i) the average feedback rate is finite and small in the entire range of transmit power; (ii) the average feedback rate will converge to zero when the transmit power approaches infinity or zero. Moreover, we extend the proposed VLQ to the multicast networks with more than two users. In addition to theoretical analysis, numerical simulations are presented to verify the effectiveness of the proposed VLQ.

Our contributions in this paper are threefold:

1) A novel VLQ is proposed for the multicast networks with two users. It can be extended to the multicast networks with more than two users. The performance of the proposed quantizer is the same as that of a system with full CSI.
2) For the first time, we provide a framework for analyzing the performance of random codebooks using variable-length limited feedback. The derivations based on random codebooks in our paper can be applied to many other scenarios.
3) Our work is an important necessary first-step towards the

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goal of designing VLQs for multicast networks using only local CSI. The availability of a global quantizer that achieves the full-CSI performance, as shown in this paper, opens the door for designing distributed quantizers.

The rest of this paper is organized as follows. In Section II, we describe the system model. In Section III, we introduce the proposed VLQ, including its encoding rule and the infinite-cardinality random codebook. In Section IV, we prove that the proposed VLQ achieves the minimum outage probability. An upper bound on the average feedback rate is given in Section V. Numerical simulations are provided in Section VI to validate our theoretical analysis. In Section VII, we extend the proposed VLQ to the multicast networks with more than two users. We draw our main conclusions and discuss future work in Section VIII. Technical proofs are provided in the appendices.

Notation: For a vector or matrix, T represents its transpose and $\dagger$ represents its conjugate transpose. $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{C}^{m\times n}$ denotes the set of complex vectors or matrices. $\mathbb{CN}(a,b)$ represents a circularly-symmetric complex Gaussian random variable (r.v.) with mean a and covariance b. $\mathrm{E}[\cdot]$ denotes the expectation and $\mathrm{Prob}\{\cdot\}$ denotes the probability. $\mathbb{N}$ is the set consisting of all natural numbers. For any real number x, $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. $\mathbf{1}_{\mathrm{ST}} = \mathbf{1}$ when the logical statement ST is true, and 0 otherwise. Finally, $f_X(\cdot)$ is the probability density function (PDF) for r.v. X.

II. System Model

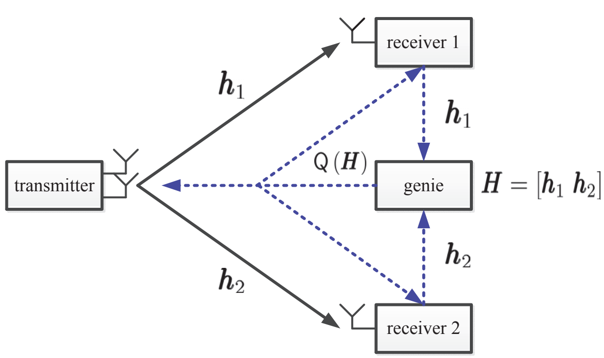

Fig. 1. System block diagram (solid and dash lines represent signal transmission and feedback links, respectively. The "genie" stands for a global channel quantizer Q).

Consider the multicast network in Fig. 1, where a transmitter with t antennas ($t \geq 2$) is sending common information to two singe-antenna receivers. The channel vector from the transmitter to receiver m is denoted by $\boldsymbol{h}_m = [h_{m1}\cdots h_{mt}]^{\mathsf{T}} \in \mathbb{C}^{t\times 1}$, where $h_{mn} \simeq \mathbb{CN}(0,1)$ for $m = 1,2$, $n = 1,\ldots,t$. Let $\chi_m = ||\boldsymbol{h}_m||^2$ for $m = 1,2$. Then the entire channel state is represented by $\boldsymbol{H} = [\boldsymbol{h}_1\, \boldsymbol{h}_2] \in \mathbb{C}^{t\times 2}$. We assume $\boldsymbol{h}_m$ is perfectly estimated at receiver m and consider a quasi-static block fading channel model in which the channel realizations vary independently from one block to another while remain constant within each block [2], [7].

At the transmitter, $\boldsymbol{x} \in \mathcal{X} \triangleq \left\{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{C}^{t\times 1}, ||\boldsymbol{x}||^2 = 1\right\}$ is employed as the beamforming vector and a scalar symbol $s \in \mathbb{C}$ is sent through t antennas. The received signal at receiver m is

$$y_m = \sqrt{P}\boldsymbol{x}^{\dagger}\boldsymbol{h}_m s + g_m,$$

where P denotes the transmit power and $g_m \simeq \mathbb{CN}(0,1)$ is the additive white Gaussian noise term. We assume $\mathrm{E}\left[|s|^2\right] = 1$. For the multicast network, the maximum achievable rate is $\log_2(1 + P\min_{m=1,2}|\boldsymbol{x}^{\dagger}\boldsymbol{h}_m|^2)$ [8].[1] Let $\gamma(\boldsymbol{x},\boldsymbol{H}) = \min_{m=1,2}\left|\boldsymbol{x}^{\dagger}\boldsymbol{h}_m\right|^2$, then for the target data transmission rate ρ, an outage event will occur if $\log_2(1 + P\gamma(\boldsymbol{x},\boldsymbol{H})) < \rho$, or equivalently, if $\gamma(\boldsymbol{x},\boldsymbol{H}) < \frac{2^{\rho}-1}{P}$. Without loss of generality, we assume $\rho = 1$ throughout this paper. Thus, $\frac{2^{\rho}-1}{P} = \frac{1}{P}$. Results for other values of ρ can be obtained similarly.

The full-CSI case where perfect CSI is known by all nodes in the multicast network is studied in [8], and the optimal beamforming vector is computed as $\mathtt{Full}(\boldsymbol{H}) = \mathrm{argmax}_{\boldsymbol{x}\in\mathcal{X}}\,\gamma(\boldsymbol{x},\boldsymbol{H})$. [2] Then the full-CSI outage probability is

$$\begin{aligned}\mathtt{Out}(\mathtt{Full}) &= \mathrm{Prob}\left\{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) < \frac{1}{P}\right\}\\ &= \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})<\frac{1}{P}}.\end{aligned} \tag{1}$$

In contrast to the full-CSI case where the perfect CSI needs to be fed back to all nodes, we consider a global quantizer denoted by Q which only requires perfect CSI to be available at a "genie" in the multicast network. As depicted in Fig. 1, the "genie" first gathers $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ from receivers 1 and 2 via error-free and delay-free feedback links. Then it quantizes $\boldsymbol{H} = [\boldsymbol{h}_1\ \boldsymbol{h}_2]$ and sends limited feedback information $\mathsf{Q}(\mathbf{H})$ to both receivers and the transmitter. The "genie" does not have to be a specific node outside the network and it can be either receiver or the transmitter. For example, if receiver 1 plays the role of "genie", it only needs to collect $\boldsymbol{h}_2$ from receiver 2.

For an arbitrary global quantizer Q, the distortion with respect to the outage probability is defined as $\mathsf{Dist} = \mathtt{Out}(\mathsf{Q}) - \mathtt{Out}(\mathtt{Full})$. Since $\mathtt{Out}(\mathtt{Full})$ is invariant for fixed P, minimizing Dist is equivalent to designing a quantizer to minimize $\mathtt{Out}(\mathsf{Q})$. In the subsequent sections, we are going to propose a VLQ and show that even if perfect CSI is no longer available at all nodes, the full-CSI outage probability or zero distortion can still be achieved.

III. Channel Quantization and Encoding Rule

In the multicast network, we consider a global VLQ associated with a random codebook $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ where $\boldsymbol{x}_i \in \mathcal{X}$ is independent and identically distributed with a uniform distribution on $\mathcal{X}$ for $i \in \mathbb{N}$ [10]. The random codebook is

[1] In this paper, we only consider the channel quantization problem for transmit beamforming. Although the precoding matrix can have higher rank than the beamforming vector, it can be inferred from [8, Theorem 1] and [8, Theorem 2] that optimal beamforming vector actually achieves the same maximum achievable rate as the optimal precoding matrix in multicast networks with two users. This also holds in the three-user case [9].

[2] For any $\boldsymbol{H}$, $\mathtt{Full}(\boldsymbol{H})$ exists because $\gamma(\boldsymbol{x},\boldsymbol{H})$ is a continuous function on $\boldsymbol{x}$ and $\mathcal{X}$ is a bounded and closed set. There might exist more than one unit-normal vector that can achieve maximum value of $\gamma(\boldsymbol{x},\boldsymbol{H})$ and $\mathtt{Full}(\boldsymbol{H})$ can be any one of them.

generated each time the channel state changes and revealed to all nodes in the network. It provides a performance benchmark since if a random codebook can achieve certain performance, one deterministic codebook can be found to surpass this performance. For any realization of $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$, the proposed VLQ is represented by

$$\mathsf{Q}_{\mathrm{VLQ}} = \{\boldsymbol{x}_i, \mathcal{R}_i, \mathtt{b}_i\}, \qquad (2)$$

where $\mathcal{R}_i$ denotes the partition channel region of $\boldsymbol{x}_i$ for $i \in \mathbb{N}$. In other words, $\boldsymbol{x}_i$ is used as the transmit beamforming vector when $\boldsymbol{H} \in \mathcal{R}_i$. Also, $\mathtt{b}_i$ is the feedback binary string that represents the index $\boldsymbol{x}_i$. We shall later specify $\mathtt{b}_i$ explicitly for every $i \in \mathbb{N}$.

Let us now specify the partition regions $\mathcal{R}_i$. In this context, our main observation is that for a given H, it is not necessary to always choose the best codeword $\boldsymbol{x}^\star$ that maximizes $\gamma(\boldsymbol{x}, \boldsymbol{H})$ among $\boldsymbol{x} \in \{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$. Any codeword $\boldsymbol{x}$ that enables $\gamma(\boldsymbol{x}, \boldsymbol{H}) \geq \frac{1}{P}$ can be applied. Hence, different from channel-partition regions of FLQs which consist of channel states that achieve the best performance with the "centroid" codeword, $\mathcal{R}_0$ in $\mathsf{Q}_{\mathrm{VLQ}}$ is set as

$$\mathcal{R}_0 = \left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_0, \boldsymbol{H}) \geq \frac{1}{P}\right\} \cup \bigcap_{i\in\mathbb{N}} \left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_i, \boldsymbol{H}) < \frac{1}{P}\right\}, \qquad (3)$$

and $\mathcal{R}_i$ for $i \in \mathbb{N} - \{0\}$ is set as

$$\mathcal{R}_i = \left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_i, \boldsymbol{H}) \geq \frac{1}{P}\right\} \cap \bigcap_{k=0}^{i-1} \left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_k, \boldsymbol{H}) < \frac{1}{P}\right\}. \qquad (4)$$

For any $\boldsymbol{x} \in \mathcal{X}$, $\left\{\boldsymbol{H} : \gamma(\boldsymbol{x}, \boldsymbol{H}) < \frac{1}{P}\right\}$ includes all channel states for which an outage incident will happen if $\boldsymbol{x}$ is employed as the beamforming vector, and $\left\{\boldsymbol{H} : \gamma(\boldsymbol{x}, \boldsymbol{H}) \geq \frac{1}{P}\right\}$ is the complement set. Thus $\mathcal{R}_0$ is the union set of channel states for which using any codeword in the codebook as the transmit beamforming vector cannot prevent outage and channel states for which using $\boldsymbol{x}_0$ will not result in outage.[3] For any $i \in \mathbb{N} - \{0\}$, $\mathcal{R}_i$ consists of channel states for which using $\boldsymbol{x}_i$ can prevent outage while using $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{i-1}$ cannot. It can be easily inferred that $\{\mathcal{R}_i\}$ is a collection of disjoint sets and $\cup_{i\in\mathbb{N}}\mathcal{R}_i$ is equal to the entire channel space.

We apply variable-length coding to encode $\boldsymbol{x}_i$ for $i \in \mathbb{N}$. To be specific, we set $\mathtt{b}_0 = \epsilon$, which is an empty codeword, [4] $\mathtt{b}_1 = \{0\}$, $\mathtt{b}_2 = \{1\}$, $\mathtt{b}_3 = \{00\}$, $\mathtt{b}_4 = \{01\}$ and sequentially so on for all codewords in the set $\{\epsilon, 0, 1, 00, 01, 10, 11, \ldots\}$. The length of $\mathtt{b}_i$ is $\lfloor \log_2(i+1) \rfloor$.

With perfect CSI and any realization of $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$, $\mathsf{Q}_{\mathrm{VLQ}}$ first determines the partition channel region $\mathcal{R}_i$ in which the current channel state $\boldsymbol{H}$ falls according to (3) and (4). Then the corresponding codeword $\boldsymbol{x}_i$ is chosen and $\lfloor \log_2(i+1) \rfloor$ bits are fed back to notify the index of $\boldsymbol{x}_i$.[5] After decoding the feedback information, $\boldsymbol{x}_i$ is employed by the transmitter as the beamforming vector. Therefore, the average feedback rate of $\mathsf{Q}_{\mathrm{VLQ}}$ is

$$\begin{aligned} \mathtt{R}(\mathsf{Q}_{\mathrm{VLQ}}) &= \sum_{i=1}^{\infty} \lfloor \log_2(i+1) \rfloor \mathrm{Prob}\{\boldsymbol{H} \in \mathcal{R}_i\} \\ &= \sum_{i=1}^{\infty} \lfloor \log_2(i+1) \rfloor \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{R}_i}. \end{aligned} \qquad (5)$$

The outage probability is given by

$$\begin{aligned} \mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}}) &= \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathrm{Prob}\left\{\gamma(\boldsymbol{x}_i, \boldsymbol{H}) < \frac{1}{P}, \forall i \in \mathbb{N}\right\} \\ &= \mathrm{E}_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\gamma(\boldsymbol{x}_i, \boldsymbol{H}) < \frac{1}{P}, \forall i\in\mathbb{N}}. \end{aligned} \qquad (6)$$

IV. Outage Optimality

In this section, we show that the proposed VLQ in (2) will achieve the full-CSI outage probability.

Intuitively, to attain the full-CSI outage probability means for any channel state $\boldsymbol{H}$ where strict non-outage achieved by the optimal beamforming vector $\mathtt{Full}(\boldsymbol{H})$ (i.e., $\gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) > \frac{1}{P}$), the proposed VLQ should return a unit-normal vector $\boldsymbol{x}$ that is "close" enough to $\mathtt{Full}(\boldsymbol{H})$ so that $\boldsymbol{x}$ also succeeds in $\gamma(\boldsymbol{x}, \boldsymbol{H}) > \frac{1}{P}$.[6] For such $\boldsymbol{H}$, there exists a certain region in the unit sphere of beamforming vectors with non-zero probability, where all the unit-normal vectors also result in strict non-outage. In order to "closely" represent $\mathtt{Full}(\boldsymbol{H})$ for any $\boldsymbol{H} \in \mathbb{C}^{t\times 2}$, we need infinitely many codewords in the codebook for the proposed VLQ, so that these infinite vectors ensure at least one efficient vector in that region will eventually be chosen to make $\boldsymbol{H}$ non-outage. This also tells why a FLQ with a finite feedback rate cannot achieve the full-CSI outage probability.

The following theorem says the outage probability of the proposed VLQ is equal to that of the full-CSI case, the proof of which is given in Appendix A.

Theorem 1. *For any $P > 0$, we have*

$$\boxed{\mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathtt{Out}(\mathtt{Full}).} \qquad (7)$$

V. Average Feedback Rate

In Section IV, we have shown that the infinite codebook cardinality is the key to achieve the full-CSI outage probability. In this section, we will show that when variable-length design in Section III is applied to encode these infinite codewords, a finite average feedback rate is attainable.

Define

$$\begin{aligned} \mathcal{H}_0 &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \chi_1 \geq \tfrac{1}{P}, \chi_2 \geq \tfrac{1}{P}\right\}, \\ \mathcal{H}_1 &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{H}_0, \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) < \tfrac{1}{P}\right\}, \\ \mathcal{H}_2 &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{H}_0, \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) = \tfrac{1}{P}\right\}, \\ \mathcal{H}_3 &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{H}_0, \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) > \tfrac{1}{P}\right\}. \end{aligned}$$

[3] It will be shown in Appendix A that $\mathcal{R}_0$ is equal to the expectation of $\left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_0, \boldsymbol{H}) \geq \frac{1}{P}\right\} \bigcup \left\{\boldsymbol{H} : \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) < \frac{1}{P}\right\}$ with regard to the random codebook $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ with probability one. Therefore, $\mathsf{Q}_{\mathrm{VLQ}}$ can determine whether $\boldsymbol{H}$ belongs to this region or not based on the expression of the optimal beamforming vector given by [8, Theorem 2], rather than checking all codewords in $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$.

[4] An empty codeword is used here for illustration. Adding 1 bit to each codeword to avoid an empty codeword only increases the average feedback rate by 1 bit per channel realization, thereby not impacting the result of the average feedback rate being finite.

[5] We reemphasize that $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ refers to the infinite-cardinality codebook while $\boldsymbol{x}_i$ represents any beamforming vector selected from $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$.

[6] We will show the channel state $\boldsymbol{H}$ satisfying $\gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) = \frac{1}{P}$ has probability zero.

As defined in Section II, $\chi_m = ||\boldsymbol{h}_m||^2$ for $m = 1, 2$. Based on the encoding rules in (3), (4) and the random codebook $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$, the feedback rate in (5) can be rewritten as

$$\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) = \sum_{l=1}^{3} \int_{\boldsymbol{H}\in\mathcal{H}_l} \Phi f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{d}\boldsymbol{H}, \tag{8}$$

where

$$\Phi = \sum_{i=1}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor,$$
$$p = \mathrm{Prob}\left\{\gamma(\boldsymbol{x}_i, \boldsymbol{H}) < \frac{1}{P}\right\}.$$

From the proof of Theorem 1 in Appendix A, it is directly obtained that $p = 1$ and $\Phi = 0$ for any $\boldsymbol{H} \in \mathcal{H}_1 \cup \mathcal{H}_2$. Hence, $\int_{\boldsymbol{H}\in\mathcal{H}_1} \Phi f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H} = \int_{\boldsymbol{H}\in\mathcal{H}_2} \Phi f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H} = 0$. Then $\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right)$ in (8) is equivalent to

$$\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) = \int_{\boldsymbol{H}\in\mathcal{H}_3} \Phi f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{d}\boldsymbol{H}. \tag{9}$$

The following lemma exhibits an upper bound on Φ, the proof of which is presented in Appendix B.

Lemma 1. *For any* $0 \le p < 1$*, we have*

$$\Phi \le p(1-p) + \left(\frac{6}{\log 2} + 2\right) p^2 + \frac{2}{\log 2} p^2 \log \frac{1}{1-p}. \tag{10}$$

Substituting (10) into (9), it follows that

$$\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) \le I_1 + I_2 + I_3, \tag{11}$$

where

$$I_1 = C_1 \int_{\boldsymbol{H}\in\mathcal{H}_3} p(1-p) f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{d}\boldsymbol{H},$$
$$I_2 = C_2 \int_{\boldsymbol{H}\in\mathcal{H}_3} p^2 f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{d}\boldsymbol{H},$$
$$I_3 = C_3 \int_{\boldsymbol{H}\in\mathcal{H}_3} p^2 \left(\log \frac{1}{1-p}\right) f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{d}\boldsymbol{H},$$

and $C_1 = 1$, $C_2 = \frac{6}{\log 2} + 2$, $C_3 = \frac{2}{\log 2}$. To further proceed, we also need useful bounds on p. For an upper bound on p, using [11, Lemma 2] and [12], we obtain

$$p \le \sum_{m=1}^{2} \mathrm{Prob}\left\{\left|\boldsymbol{x}_i^\dagger \boldsymbol{h}_m\right|^2 < \frac{1}{P}\right\}$$
$$= \sum_{m=1}^{2} \left[1 - \left(1 - \frac{1}{P\chi_m}\right)^{t-1}\right],$$

where the last equality arises from $\mathrm{Prob}\left\{\left|\boldsymbol{x}_i^\dagger \boldsymbol{h}_m\right|^2 < x\right\} = 1 - \left(1 - \frac{x}{\chi_m}\right)^{t-1}$ [12]. Since $(1-a)^b \ge 1 - ab$ for $0 < a < 1$ and $b \ge 1$, $\left(1 - \frac{1}{P\chi_m}\right)^{t-1} \ge 1 - \frac{t-1}{P\chi_m}$. Therefore, p is upper-bounded by

$$p \le \frac{t-1}{P} \sum_{m=1}^{2} \frac{1}{\chi_m}. \tag{12}$$

Another upper bound on p obtained from Lemma 2 in Appendices A and D is given as

$$p \le 1 - (1 - \Pi)^{t-1}, \tag{13}$$

where

$$\Pi = 1 - \min_{m=1,2} \left[\frac{\left|\left[\mathtt{Full}\left(\boldsymbol{H}\right)\right]^\dagger \boldsymbol{h}_m\right|^2 - \frac{1}{P}}{\chi_m}\right]^2.$$

In addition, a lower bound on p (or equivalently, the upper bound on $1-p$) is given by

$$1 - p \le \mathrm{Prob}\left\{\left|\boldsymbol{x}_i^\dagger \boldsymbol{h}_1\right|^2 \ge \frac{1}{P}\right\} = \left(1 - \frac{1}{P\chi_1}\right)^{t-1}. \tag{14}$$

With bounds on p in (12), (13), (14) and based on (11), we deduce an upper bound on $\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right)$ and present it in the following theorem, the detailed proof of which is shown in Appendix C.

Theorem 2. *For any* $P > 0$*, we have*

$$\boxed{\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) \le C_0 e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right]}, \tag{15}$$

where $C_0 > 0$ *is a constant that is independent of* P*.*

Remark 1: We mainly focus on showing how the number of average feedback bits for $\mathsf{Q}_{\mathrm{VLQ}}$ changes with P. Therefore, it is beyond the scope of this paper to find the tightest bound, i.e., the smallest value for C_0.

Remark 2: From (15), it can be seen that in the medium and high regions for P, the derived upper bound on average feedback rate is dominated by $e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{\log(1+P)}{P}\right]$; in the low region for P, it is dominated by $\frac{e^{-\frac{1}{P}}}{P^{2t}}$. Moreover, the upper bound will approach zero when $P \to \infty$ and $P \to 0$. The average feedback rate also behaves like this. This can be intuitively interpreted as follows: when $P \to \infty$, any vector in the codebook will not cause an outage event, while when $P \to 0$, any vector will result in outage. According to the encoding rule of $\mathsf{Q}_{\mathrm{VLQ}}$, only empty codewords will be fed back in both situations. Thus the average feedback rate approaches zero.

VI. Numerical Simulations

In this section, we perform numerical simulations to verify the theoretical results for the outage probability and the average feedback rate.

In the pseudo-code, a sufficiently large number of channel realizations will be generated in order to observe 1000 outage events for each t and P. Moreover, `Out` stands for the simulated outage probability, `R` refers to the simulated average feedback rate and `Loop` records the number of channel realizations. For each channel realization, whether the full-CSI case could prevent outage will be checked in line 6. If not, an outage event is declared in line 7; otherwise, in lines 9 to 14, a random unit-normal vector will be generated repeatedly until one that allows the current channel realization to prevent outage is found, and the index of the selected codeword is `Index`. Together with line 16, the simulated feedback rate is

Simulation Procedure:

```
1: Initialization: Given t, P. Set Out = 0, R = 0, Loop = 0;
2: while Out < 1000
3:     Index = 0;
4:     Loop = Loop + 1;
5:     Generate a realization of H;
6:     if γ(Full(H), H) < 1/P
7:         Out = Out + 1;
8:     else
9:         Randomly generate x ∈ 𝒳;
10:        while γ(x, H) < 1/P
11:            Randomly generate y ∈ 𝒳;
12:            x = y;
13:            Index = Index + 1;
14:        end
15:    end
16:    R = R + ⌊log_2(1 + Index)⌋;
17: end
18: return R = R/Loop, Out = Out/Loop.
```

the average number of feedback bits calculated in line 18, where the simulated outage probability is computed as 1000 divided by the number of all channel realizations. In all the simulations, no endless iteration has been detected, which is equivalent to say that as long as the channel state realization is able to avoid outage in the full-CSI case, a randomly-generated codeword that also prevents outage will eventually be found.

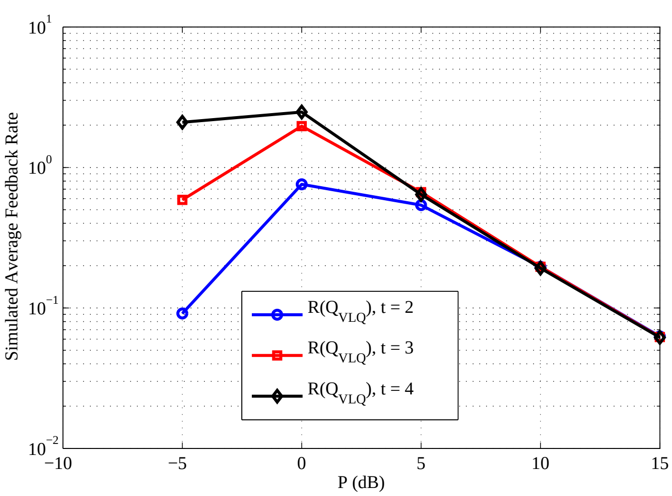

Fig. 2. Simulated average feedback rates when $t = 2, 3, 4$ (t is the number of transmit antennas).

Fig. 2 shows the simulated average feedback rates for $t = 2, 3, 4$. The horizontal axis represents P in decibels. It can be observed that: (i) all the average feedback rates will decrease towards zero when P increases towards infinity or decreases to zero; (ii) the average feedback rate is finite and small for any P; (iii) the average feedback rates for $t = 2, 3, 4$ coincide in the high-P region. These observations correspond to the upper bound derived in Theorem 2.[7]

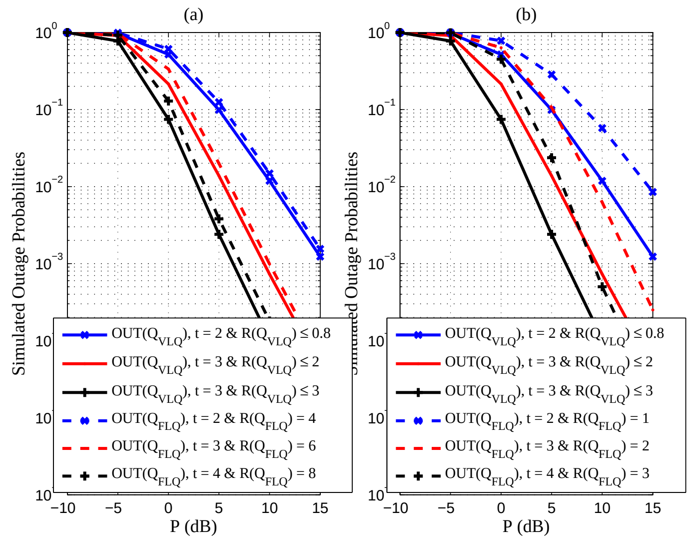

Fig. 3. Simulated outage probabilities of $\mathsf{Q}_{\rm VLQ}$ and $\mathsf{Q}_{\rm FLQ}$ when $t = 2, 3, 4$ (t is the number of transmit antennas).

In Figs. 3(a) and 3(b), we compare the outage probabilities of $\mathsf{Q}_{\rm VLQ}$ and a traditional FLQ denoted by $\mathsf{Q}_{\rm FLQ}$. For any given $\boldsymbol{H}$, $\mathsf{Q}_{\rm FLQ}$ employs B bits to quantize $\boldsymbol{H}$ based on the random codebook $\left\{\boldsymbol{x}_i, i = 0, \ldots, 2^B - 1\right\}$ according to $\mathsf{Q}_{\rm FLQ}(\boldsymbol{H}) = \operatorname{argmax}_{\boldsymbol{x} \in \{\boldsymbol{x}_i, i=0,\ldots,2^B-1\}} \gamma(\boldsymbol{x}, \boldsymbol{H})$. Then the outage probability is $\mathtt{Out}(\mathsf{Q}_{\rm FLQ}) = \mathrm{E}_{\{\boldsymbol{x}_i, i=0,\ldots,2^B-1\}} \mathrm{Prob}\left\{\gamma(\mathsf{Q}_{\rm FLQ}(\boldsymbol{H}), \boldsymbol{H}) < \frac{1}{P}\right\}$, and the average feedback rate is $\mathtt{R}(\mathsf{Q}_{\rm FLQ}) = B$. It is observed from Fig. 2 that $\mathtt{R}(\mathsf{Q}_{\rm VLQ})$ is no larger than $0.8, 2$ or 3 bits per channel state when $t = 2, 3$ or 4, respectively. Thus in Fig. 3 (a), we choose the number of feedback bits assigned to $\mathsf{Q}_{\rm FLQ}$ to be $B = 4, 6$ and 8 when $t = 2, 3, 4$, respectively. Curves in Fig. 3(a) demonstrate that $\mathsf{Q}_{\rm VLQ}$ outperforms $\mathsf{Q}_{\rm FLQ}$ even when the latter one has a much larger feedback rate. In Fig. 3(b), we let $B = 1, 2, 3$ for $t = 2, 3, 4$, which are close to (but still larger than) $\mathtt{R}(\mathsf{Q}_{\rm VLQ})$. It can be seen that the outage probabilities of $\mathsf{Q}_{\rm FLQ}$ are much worse than those of $\mathsf{Q}_{\rm VLQ}$. Therefore, it is revealed from Figs. 3 (a) and 3(b) that $\mathsf{Q}_{\rm VLQ}$ is superior to $\mathsf{Q}_{\rm FLQ}$.

VII. Generalization to Multicast Networks with More than Two Users

The quantizer proposed for multicast networks with two users can be applied to the multicast networks with more than two users after slight modifications. We still name it $\mathsf{Q}_{\rm VLQ}$ for simplicity. Denote the number of receivers by M. When $M \geq 3$, $\boldsymbol{h}_m = [h_{m1} \cdots h_{mt}]^{\intercal}$ stands for the channel vector from the transmitter to receiver m with $h_{mn} \simeq \mathbb{CN}(0, 1)$ for $m = 1, \ldots, M$ and $n = 1, \ldots, t$. Then $\boldsymbol{H} = [\boldsymbol{h}_1 \cdots \boldsymbol{h}_M] \in \mathbb{C}^{t \times M}$ represents the entire channel state. Let $\gamma(\boldsymbol{x}, \boldsymbol{H}) = \min_{m=1,\ldots,M} \left|\boldsymbol{x}^{\dagger}\boldsymbol{h}_m\right|^2$ for any $\boldsymbol{x} \in \mathcal{X}$. With such modifications, we can apply the proposed quantizer

[7] For (iii), the upper bound in Theorem 2 shows the average feedback rate is dominated by $e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{\log(1+P)}{P}\right]$ in the high-P region, which is independent of t.

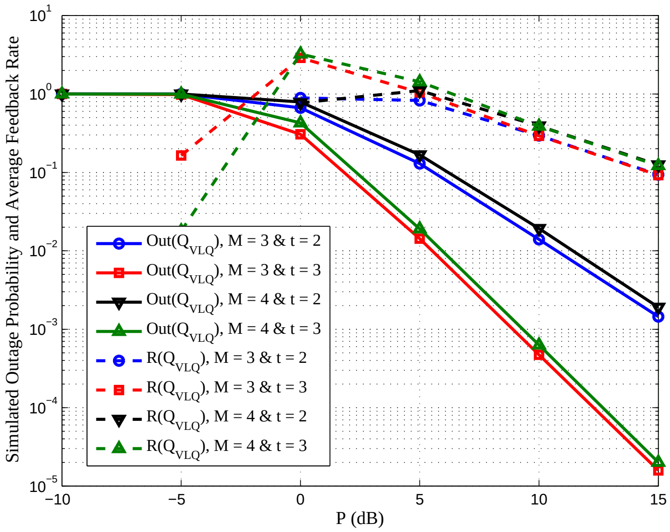

Fig. 4. Simulated outage probabilities and average feedback rates when $M = 3, 4$ and $t = 2, 3$ (M is the number of receivers and t is the number of transmit antennas).

$\mathsf{Q}_{\mathrm{VLQ}}$ in (2) with encoding rules in (3), (4) to the multicast networks with more than two users.

For the two-user case, we have rigorously proved $\mathsf{Q}_{\mathrm{VLQ}}$ could achieve the optimal outage probability with a finite average feedback rate, and the proofs rely on the closed-form expression of $\mathtt{Full}\,(\boldsymbol{H})$ given in [8]. But when the multicast network has more than two users, there is no optimal solution for $\mathtt{Full}\,(\boldsymbol{H})$ in the literature. Thus, we cannot apply the same method to prove $\mathsf{Q}_{\mathrm{VLQ}}$ could achieve the optimal outage probability with a finite average feedback rate in the general case with an arbitrary number of users.

Nevertheless, our proposed quantizer $\mathsf{Q}_{\mathrm{VLQ}}$ is still effective in the multicast networks with more than two users. The "closest" solution we have found for $\mathtt{Full}\,(\boldsymbol{H})$ is in [13], which uses approximation but generates optimal solutions in many scenarios. In Fig. 4, we simulate the outage probabilities and average feedback rates according to the simulation procedure in Section VI when the numbers of users are $M = 3, 4$ and the numbers of transmit antennas are $t = 2, 3$, respectively. We use the solution in [13] as the base for the simulation procedure, thus its outage probability is treated as the full-CSI performance. We believe that if the exactly optimal solution for $\mathtt{Full}\,(\boldsymbol{H})$ is found, our proposed quantizer will also yield the optimal outage probability. Fig. 4 shows that $\mathsf{Q}_{\mathrm{VLQ}}$ could attain the full-CSI outage probability using finite average feedback rates when there are more than two users.

VIII. Conclusions and Future Work

In this paper, we have proved that in the two-user multicast network, the proposed VLQ can achieve the full-CSI outage probability with a low average feedback rate. We have also extended the proposed VLQ to the multicast networks with more than two users. In the future, we intend to work on a distributed quantizer for the multicast network by localizing the proposed VLQ. In this scenario, each receiver only feedbacks its local channel information and no node can acquire the full CSI. We aim to approach or even achieve the full-CSI outage probability at the cost of a finite average feedback rate.

Appendix A - Proof of Theorem 1

Proof. Before showing the detailed proof, let us summarize the main idea behind the proof first. Based on (1) and (6), to show $\mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathtt{Out}(\mathtt{Full})$, it is equivalent to prove:

1) For any $\boldsymbol{H}$ satisfying $\gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) < \frac{1}{P}$, $\mathbf{1}_{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})<\frac{1}{P}} = \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathbf{1}_{\gamma(\boldsymbol{x}_i,\boldsymbol{H})<\frac{1}{P},\forall i\in\mathbb{N}} = 1$;
2) For any $\boldsymbol{H}$ satisfying $\gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) = \frac{1}{P}$,
$$\begin{aligned}&\int_{\boldsymbol{H}} \mathbf{1}_{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})<\frac{1}{P}} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right)\mathrm{d}\boldsymbol{H}\\ &= \int_{\boldsymbol{H}} \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathbf{1}_{\gamma(\boldsymbol{x}_i,\boldsymbol{H})<\frac{1}{P},\forall i\in\mathbb{N}} f_{\boldsymbol{H}}\left(\boldsymbol{H}\right)\mathrm{d}\boldsymbol{H} = 0;\end{aligned}$$
3) For any $\boldsymbol{H}$ satisfying $\gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) > \frac{1}{P}$, $\mathbf{1}_{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})>\frac{1}{P}} = \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathbf{1}_{\gamma(\boldsymbol{x}_i,\boldsymbol{H})>\frac{1}{P},\forall i\in\mathbb{N}} = 0$.

Define
$$\mathcal{S}_1 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) < \frac{1}{P}\right\}.$$
For any realization of $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$, define
$$\mathcal{S}_2\left(\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}\right) = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\boldsymbol{x}_i,\boldsymbol{H}\right) < \frac{1}{P}, \forall i \in \mathbb{N}\right\}.$$
For brevity, we omit the dependency of $\mathcal{S}_2\left(\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}\right)$ on $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ and simply use $\mathcal{S}_2$. From (1) and (6), $\mathtt{Out}\left(\mathtt{Full}\right)$ and $\mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}})$ can be rewritten as
$$\mathtt{Out}\left(\mathtt{Full}\right) = \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1}, \quad (16)$$
$$\mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_2}. \quad (17)$$
For convenience, we define
$$\begin{aligned}\mathcal{S}_{21} &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{S}_2, \gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) < \tfrac{1}{P}\right\},\\ \mathcal{S}_{22} &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{S}_2, \gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) = \tfrac{1}{P}\right\},\\ \mathcal{S}_{23} &= \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{S}_2, \gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) > \tfrac{1}{P}\right\}.\end{aligned}$$
Since $\mathcal{S}_2 = \mathcal{S}_{21} \cup \mathcal{S}_{22} \cup \mathcal{S}_{23}$ and $\mathcal{S}_{21}$, $\mathcal{S}_{22}$, $\mathcal{S}_{23}$ are mutually exclusive, $\mathtt{Out}\left(\mathsf{Q}_{\mathrm{VLQ}}\right)$ in (17) is rewritten as
$$\mathtt{Out}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) = \sum_{l=1}^{3} \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{2l}}. \quad (18)$$
In order to prove $\mathtt{Out}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathtt{Out}(\mathtt{Full})$, according to (16) and (18), we will show $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1}$, $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{22}} = 0$ and $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{23}} = 0$.

First, to prove $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1}$, it is sufficient to prove $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1} = \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}}$ for any given $\boldsymbol{H}$ and $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$. When $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1} = 0$, based on the definition of $\mathcal{S}_1$, it means $\boldsymbol{H} \notin \mathcal{S}_1$ and $\gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) \geq \frac{1}{P}$. From the definition of $\mathcal{S}_{21}$, we have $\boldsymbol{H} \notin \mathcal{S}_{21}$ and $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = 0$. When $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1} = 1$, $\boldsymbol{H} \in \mathcal{S}_1$. By the optimality of $\mathtt{Full}\left(\boldsymbol{H}\right)$, it must have $\boldsymbol{H} \in \mathcal{S}_2$. Since $\mathcal{S}_{21} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\boldsymbol{H} \in \mathcal{S}_{21}$ and $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = 1$. Therefore, $\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1}$ and $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_1}$.

Second, we will prove $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{22}} = 0$. Define $\mathcal{S}_3 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) = \frac{1}{P}\right\}$. By definition, $\mathcal{S}_{22} = \mathcal{S}_2 \cap \mathcal{S}_3 \subseteq \mathcal{S}_3$. Then $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}}\mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{22}} \leq \mathrm{E}_{\boldsymbol{H}}\mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_3} = \mathrm{Prob}\left\{\gamma\left(\mathtt{Full}\left(\boldsymbol{H}\right),\boldsymbol{H}\right) = \frac{1}{P}\right\}$. Let $m_{\min} =$

$\operatorname*{argmin}_{m=1,2} \chi_m$, $m_{\max} = \operatorname*{argmax}_{m=1,2} \chi_m$ and $\theta = \frac{|\boldsymbol{h}_1^\dagger \boldsymbol{h}_2|^2}{\chi_1 \chi_2}$. According to [8, Theorem 2],

$$\gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) = \begin{cases} \chi_{m_{\min}}, & \theta \geq \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \\ \frac{\chi_{m_{\min}}}{1+\beta^2}, & \theta < \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \end{cases} \tag{19}$$

where $\beta = \frac{\sqrt{\chi_{m_{\min}}} - \sqrt{\chi_{m_{\max}}\theta}}{\sqrt{\chi_{m_{\max}} - \chi_{m_{\max}}\theta}}$. Since θ, χ_1, and χ_2 are mutually independent, θ and $\chi_{m_{\min}}$ and $\chi_{m_{\max}}$ are also mutually independent [12]. With (19), it is straightforward to show that $\text{Prob}\left\{\gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) = \frac{1}{P}\right\} = 0$ by fixing $\chi_{m_{\min}}, \chi_{m_{\max}}$ or θ as well as using the fact that the probability of a continuous r.v. assuming a specific value is zero. Therefore, $\mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathtt{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{22}} \leq 0$. Since the probability is non-negative, $\mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathtt{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{22}} = 0$.

Finally, we will prove $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathrm{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{23}} = 0$. Define $\mathcal{S}_4 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) > \frac{1}{P}\right\}$. Since $\mathcal{S}_{23} = \mathcal{S}_2 \cap \mathcal{S}_4$, $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathrm{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{23}} = \int_{\boldsymbol{H}\in\mathcal{S}_4} f_{\boldsymbol{H}}(\boldsymbol{H}) \mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_2} \mathrm{d}\boldsymbol{H}$. To prove $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathrm{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{23}} = 0$, it is sufficicent to show $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_2} = 0$ for any $\boldsymbol{H} \in \mathcal{S}_4$. By contradiction, assume $\exists \tilde{\boldsymbol{H}} \in \mathcal{S}_4$, s.t. $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} = \varepsilon > 0$. Then

$$\begin{aligned} &\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} \\ &= \text{Prob}\left\{\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right) < \frac{1}{P}, \forall i \in \mathbb{N}\right\} \\ &\leq \text{Prob}\left\{\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right) < \frac{1}{P}, \forall i \in \{0, 1, \ldots, K-1\}\right\} \\ &= \left[\text{Prob}\left\{\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right) < \frac{1}{P}\right\}\right]^K, \end{aligned} \tag{20}$$

where $K \geq 1$ can be any finite natural number, and the last equality is because for a given $\tilde{\boldsymbol{H}}$, $\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right)$ for $i = 1, \ldots, K$ are mutually independent due to the independence of $\boldsymbol{x}_i$ for $i = 1, \ldots, K$. We shall use the following lemma, the proof of which is in Appendix D.

Lemma 2. *If $\gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) > \frac{1}{P}$, there exists $\Pi \in (0,1)$ such that for any $\boldsymbol{x} \in \mathcal{X}$ with $\left|\boldsymbol{x}^\dagger \mathtt{Full}(\boldsymbol{H})\right|^2 \geq \Pi$, $\gamma(\boldsymbol{x}, \boldsymbol{H}) \geq \frac{1}{P}$ holds.*

From Lemma 1, for a given $\tilde{\boldsymbol{H}}$, we have $\text{Prob}\left\{\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right) \geq \frac{1}{P}\right\} \geq \text{Prob}\left\{\left|\boldsymbol{x}_i^\dagger \mathtt{Full}\left(\tilde{\boldsymbol{H}}\right)\right|^2 \geq \Pi\right\} = (1-\Pi)^{t-1} > 0$. Therefore, $\text{Prob}\left\{\gamma\left(\boldsymbol{x}_i, \tilde{\boldsymbol{H}}\right) < \frac{1}{P}\right\} \leq 1-(1-\Pi)^{t-1} < 1$. By (20), it can be derived that $\mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} \leq \left[1-(1-\Pi)^{t-1}\right]^K$. Let $K = \left\lceil \log_{(1-(1-\Pi)^{t-1})} \varepsilon \right\rceil + 1$, then $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} \leq \left[1-(1-\Pi)^{t-1}\right]^{\left\lceil \log_{(1-(1-\Pi)^{t-1})} \varepsilon \right\rceil + 1} < \left[1-(1-\Pi)^{t-1}\right]^{\log_{(1-(1-\Pi)^{t-1})} \varepsilon} = \varepsilon$, which contradicts the assumption that $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} = \varepsilon$. Thus $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\tilde{\boldsymbol{H}}\in\mathcal{S}_2} = 0$ and $\mathrm{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathrm{E}_{\boldsymbol{H}} \mathbf{1}_{\boldsymbol{H}\in\mathcal{S}_{23}} = 0$, which completes the proof. □

Remark 3: It follows from (7) and (18) that $\mathtt{Out}(\mathsf{Q}_{\text{VLQ}}) = \mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathtt{E}_{\boldsymbol{H}} \mathbf{1}_{\mathcal{S}_2} = \mathtt{Out}(\mathtt{Full}) = \mathtt{E}_{\boldsymbol{H}} \mathbf{1}_{\mathcal{S}_1}$, thus $\mathtt{E}_{\boldsymbol{H}}\left[\mathbf{1}_{\mathcal{S}_1} - \mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\mathcal{S}_2}\right] = 0$. Based on the definitions of $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathbf{1}_{\mathcal{S}_1} - \mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\mathcal{S}_2}$ is always non-positive for any $\boldsymbol{H}$. Therefore, $\mathbf{1}_{\mathcal{S}_1} - \mathtt{E}_{\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}} \mathbf{1}_{\mathcal{S}_2} = 0$ for any $\boldsymbol{H}$ with probability one. In other words, $\mathcal{R}_0$ in (3) is equal to the expectation of $\left\{\boldsymbol{H} : \gamma(\boldsymbol{x}_0, \boldsymbol{H}) \geq \frac{1}{P}\right\} \bigcup \left\{\boldsymbol{H} : \gamma(\mathtt{Full}(\boldsymbol{H}), \boldsymbol{H}) < \frac{1}{P}\right\}$ with regard to $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}}$ with probability one.

Appendix B - Proof of Lemma 1

Proof. For $p = 0$, $\Phi = 0$, then the upper bound in (10) holds. Hence, suppose that $0 < p < 1$. Then

$$\begin{aligned} \Phi &= \sum_{i=1}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor \\ &= p(1-p) + \sum_{i=2}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor \\ &\leq p(1-p) + \sum_{i=2}^{\infty} p^i (1-p) \log_2(i+1) \\ &= p(1-p) + p(1-p) \sum_{i=1}^{\infty} p^i \log_2(i+2) \\ &= p(1-p) + p(1-p) \left[p \log_2 3 + \sum_{i=2}^{\infty} p^i \log_2(i+2)\right] \\ &\leq p(1-p) + p(1-p) \left[p \log_2 3 + \frac{2}{\log 2} \sum_{i=1}^{\infty} p^i \log i\right]. \end{aligned} \tag{21}$$

We estimate the sum $\sum_{i=1}^{\infty} p^i \log i$ via the integral of the function $f(x) = e^{-\beta x} \log x$, where $0 < \beta \triangleq -\log p < \infty$. We calculate $f'(x) = e^{-\beta x}\left(\frac{1}{x} - \beta \log x\right)$, where f' represents the derivative of f. For $y \log y = \frac{1}{\beta}$, $f'(x) > 0$ for $1 \leq x < y$, $f'(x) = 0$ for $x = y$, and $f'(x) < 0$ for $x > y$. The global maximum of f is thus $f(y)$. Since $y \log y = \frac{1}{\beta} > 0$, $y \geq 1$ must hold, which implies $f(y) = e^{-\beta y} \log y \leq e^{-\beta} \log y \leq e^{-\beta} y \log y = \frac{e^{-\beta}}{\beta}$. Let $j = \lfloor y \rfloor$. Then $1 \leq j \leq y < j+1$, and

$$\begin{aligned} &\sum_{i=1}^{\infty} f(i) \\ &= \mathbf{1}_{j\geq 2} \sum_{i=1}^{j-1} f(i) + f(j) + f(j+1) + \sum_{i=j+2}^{\infty} f(i) \\ &= \mathbf{1}_{j\geq 2} \sum_{i=1}^{j-1} \int_i^{i+1} f(i)\mathrm{d}x + f(j) + f(j+1) + \sum_{i=j+2}^{\infty} \int_{i-1}^{i} f(i)\mathrm{d}x \\ &\leq \mathbf{1}_{j\geq 2} \sum_{i=1}^{j-1} \int_i^{i+1} f(x)\mathrm{d}x + f(y) + f(y) + \sum_{i=j+2}^{\infty} \int_{i-1}^{i} f(x)\mathrm{d}x \\ &= \mathbf{1}_{j\geq 2} \int_1^j f(x)\mathrm{d}x + 2f(y) + \int_{j+1}^{\infty} f(x)\mathrm{d}x \\ &< 2f(y) + \int_1^{\infty} f(x)\mathrm{d}x \leq \frac{2e^{-\beta}}{\beta} + \int_1^{\infty} f(x)\mathrm{d}x, \end{aligned} \tag{22}$$

where the first inequality follows since f is increasing on $(1, j)$ and decreasing on $(j+1, \infty)$. We now estimate the integral. With a change of variables $u = \log x$, $\mathrm{d}v = e^{-\beta x}\mathrm{d}x$, we obtain

$$\begin{aligned} \int_1^{\infty} f(x)\mathrm{d}x &= \left(-\frac{1}{\beta} \log x e^{-\beta x}\right)\Big|_1^{\infty} + \frac{1}{\beta} \int_1^{\infty} \frac{1}{x} e^{-\beta x} dx \\ &= \frac{1}{\beta} \mathtt{E}_1(\beta) < \frac{e^{-\beta}}{\beta} \log\left(1 + \frac{1}{\beta}\right). \end{aligned}$$

Combining with (22) and subsituting $\beta = -\log p$, it follows that

$$\sum_{i=1}^{\infty} f(i) < \frac{p}{-\log p}\left[2+\log\left(1+\frac{1}{-\log p}\right)\right] < \frac{p}{1-p}\left[2+\log\left(1+\frac{1}{1-p}\right)\right] < \frac{p}{1-p}\left[2+\log\frac{2}{1-p}\right] < \frac{p}{1-p}\left[3+\log\frac{1}{1-p}\right], \tag{23}$$

where the second inequality is because $-\log p > 1-p$ for $0<p<1$. Substituting (23) into (21) yields that

$$\Phi \le p(1-p) + p^2(1-p)\log_2 3 + \frac{2p^2}{\log 2}\left(3+\log\frac{1}{1-p}\right) \le p(1-p) + 2p^2 + \frac{6p^2}{\log 2} + \frac{2p^2}{\log 2}\log\frac{1}{1-p} = p(1-p) + \left(\frac{6}{\log 2}+2\right)p^2 + \frac{2}{\log 2}p^2\log\frac{1}{1-p}.$$

This concludes the proof. □

Appendix C - Proof of Theorem 2

Proof. Based on (11), we will derive upper bounds on I_1, I_2 and I_3, separately. First, since $\mathcal{H}_3 \subseteq \mathcal{H}_0$, we get

$$I_1 \le C_1 \int_{\boldsymbol{H}\in\mathcal{H}_0} p(1-p) f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H}.$$

Substituting the upper bounds in (12) and (14) into I_1, it is derived that

$$I_1 \le \frac{C_4}{P}\sum_{m=1}^{2}\int_{\boldsymbol{H}\in\mathcal{H}_0}\frac{1}{\chi_m}\left(1-\frac{1}{P\chi_1}\right)^{t-1} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H},$$

where $C_4 = (t-1)C_1$. Since χ_m is chi-squared distributed, the PDF of χ_m is $f_{\chi_m}(\chi_m) = \frac{\chi_m^{t-1}e^{-\chi_m}}{(t-1)!}$ for $m=1,2$ [14]. Then we obtain

$$\begin{aligned} I_1 &\le \frac{C_4}{P}\sum_{m=1}^{2}\int_{\frac{1}{P}}^{\infty}\int_{\frac{1}{P}}^{\infty}\frac{1}{\chi_m}\left(1-\frac{1}{P\chi_1}\right)^{t-1} \\ &\times\left[\frac{\chi_1^{t-1}e^{-\chi_1}}{(t-1)!}\right]\left[\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\right]\mathrm{d}\chi_1\mathrm{d}\chi_2 \\ &= \frac{C_5}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1 \\ &\times\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &+ \frac{C_5}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-1}e^{-\chi_1}\mathrm{d}\chi_1 \\ &\times\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2, \end{aligned}$$

where $C_5 = \frac{C_4}{(t-1)!}$. Noting that $\int_0^{\infty} x^{n-1}e^{-x}\mathrm{d}x = (n-1)!$ for $n \ge 1$ and $n \in \mathbb{N}$ [14], I_1 is bounded by

$$\begin{aligned} I_1 &\le \frac{C_5}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1 \\ &\times\int_0^{\infty}\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &+ \frac{C_5}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-1}e^{-\chi_1}\mathrm{d}\chi_1 \\ &\times\int_0^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &\le \frac{C_5}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1 \\ &+ \frac{C_6}{P}\int_{\frac{1}{P}}^{\infty}\left(1-\frac{1}{P\chi_1}\right)^{t-1}\chi_1^{t-1}e^{-\chi_1}\mathrm{d}\chi_1, \end{aligned}$$

where $C_6 = \frac{C_5}{t-1}$. Letting $\chi_1 - \frac{1}{P} = \lambda_1$, the bound is derived as

$$\begin{aligned} I_1 &\le C_5\frac{e^{-\frac{1}{P}}}{P}\int_0^{\infty}\frac{\lambda_1}{\lambda_1+\frac{1}{P}}\lambda_1^{t-2}e^{-\lambda_1}\mathrm{d}\lambda_1 \\ &+ C_6\frac{e^{-\frac{1}{P}}}{P}\int_0^{\infty}\lambda_1^{t-1}e^{-\lambda_1}\mathrm{d}\lambda_1 \\ &\le C_5\frac{e^{-\frac{1}{P}}}{P}\int_0^{\infty}\lambda_1^{t-2}e^{-\lambda_1}\mathrm{d}\lambda_1 + C_6(t-1)!\frac{e^{-\frac{1}{P}}}{P} \\ &= C_5(t-2)!\frac{e^{-\frac{1}{P}}}{P} + C_6(t-1)!\frac{e^{-\frac{1}{P}}}{P} = C_7\frac{e^{-\frac{1}{P}}}{P}, \end{aligned} \tag{24}$$

where $C_7 = (t-2)!C_5 + (t-1)!C_6$.

To derive I_2, applying the upper bound in (12) and based on the fact that $\mathcal{H}_1 \subseteq \mathcal{H}_0$, we obtain

$$\begin{aligned} I_2 &\le \frac{C_8}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_0}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H} \\ &= \frac{C_8}{P^2}\int_{\frac{1}{P}}^{\infty}\int_{\frac{1}{P}}^{\infty}\left[\frac{1}{\chi_1^2}+\frac{1}{\chi_2^2}+\frac{2}{\chi_1\chi_2}\right] \\ &\times\left[\frac{\chi_1^{t-1}e^{-\chi_1}}{(t-1)!}\right]\left[\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\right]\mathrm{d}\chi_1\mathrm{d}\chi_2 \\ &= \frac{2C_8}{(t-1)!P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &+ \frac{2C_8}{(t-1)!P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &\le \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1\int_0^{\infty}\frac{e^{-\chi_2}\chi_2^{t-1}}{(t-1)!}\mathrm{d}\chi_2 \\ &+ \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1\int_0^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\ &= \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1 + \frac{C_{10}}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1, \end{aligned}$$

where $C_8 = (t-1)^2C_2$, $C_9 = \frac{2C_8}{(t-1)!}$ and $C_{10} = \frac{C_9}{t-1}$. When $t \ge 3$, I_2 is upper-bounded by

$$I_2 \le \frac{C_9}{P^2}\Gamma\left(t-2,\frac{1}{P}\right) + \frac{C_{10}}{P^2}\Gamma\left(t-1,\frac{1}{P}\right), \tag{25}$$

where $\Gamma(n,a) = \int_a^{\infty} x^{n-1}e^{-x}\mathrm{d}x$ is the incomplete gamma function for $n > 0, a > 0$. The following lemma shows an upper bound on the incomplete gamma function, the proof of which is in Appendix E.

Lemma 3. *For $n > 0, n \in \mathbb{N}$ and $a > 0$, we have*

$$\Gamma(n,a) \le n! e^{-a}\left(1+\alpha^{n-1}\right). \tag{26}$$

Applying (26) to (25) yields

$$\begin{aligned} I_2 &\le \frac{C_9}{P^2}(t-2)! e^{-\frac{1}{P}}\left(1+\frac{1}{P^{t-3}}\right) \\ &+ \frac{C_{10}}{P^2}(t-1)! e^{-\frac{1}{P}}\left(1+\frac{1}{P^{t-2}}\right) \\ &= C_{11}\frac{e^{-\frac{1}{P}}}{P^2} + C_{12}\frac{e^{-\frac{1}{P}}}{P^{t-1}} + C_{13}\frac{e^{-\frac{1}{P}}}{P^t}, \end{aligned} \tag{27}$$

where $C_{11} = C_9(t-2)! + C_{10}(t-1)!$, $C_{12} = C_9(t-2)!$ and $C_{13} = C_{10}(t-1)!$. When $t = 2$, the upper bound on I_2 is

$$\begin{aligned} I_2 &\le \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\frac{e^{-\chi_1}}{\chi_1}\mathrm{d}\chi_1 + \frac{C_{10}}{P^2}\int_{\frac{1}{P}}^{\infty}e^{-\chi_1}\mathrm{d}\chi_1 \\ &= \frac{C_9}{P^2}\mathtt{E}_1\left(\frac{1}{P}\right) + C_{10}\frac{e^{-\frac{1}{P}}}{P^2} \\ &\le C_9\frac{e^{-\frac{1}{P}}}{P^2}\log(1+P) + C_{10}\frac{e^{-\frac{1}{P}}}{P^2}, \end{aligned} \tag{28}$$

where $\mathtt{E}_1(z) = \int_z^{\infty}\frac{e^{-z}}{z}\mathrm{d}z$ is the exponential integral with an upper bound as $\mathtt{E}_1(z) \le e^{-z}\log\left(1+\frac{1}{z}\right)$ [14]. From (27) and (28), the upper bound on I_2 for any $t \ge 2$ can be obtained as

$$\begin{aligned} I_2 &\le \left[C_{11}\frac{e^{-\frac{1}{P}}}{P^2} + C_{12}\frac{e^{-\frac{1}{P}}}{P^{t-1}} + C_{13}\frac{e^{-\frac{1}{P}}}{P^t}\right]\times \mathbf{1}_{t\ge 3} \\ &+ \left[C_9\frac{e^{-\frac{1}{P}}}{P^2}\log(1+P) + C_{10}\frac{e^{-\frac{1}{P}}}{P^2}\right]\times \mathbf{1}_{t=2} \\ &\le C_{14}e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right], \end{aligned} \tag{29}$$

where $C_{14} = [C_{11}+C_{12}+C_{13}]\times \mathbf{1}_{t\ge 3} + [C_9 + C_{10}]\times \mathbf{1}_{t=2}$. The last inequality is obtained by comparing both cases where $0 < P \le 1$ and $P > 1$.

To derive I_3, we need an upper bound on $\log\frac{1}{1-p}$ first. By applying (13), we obtain

$$\log\frac{1}{1-p} \le 2(t-1)\log\frac{1}{\min\limits_{m=1,2}\frac{\left|[\mathtt{Full}(\boldsymbol{H})]^{\dagger}\boldsymbol{h}_m\right|^2 - \frac{1}{P}}{\chi_m}}.$$

From (19), it is found when $\theta \ge \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}$, $\min\limits_{m=1,2}\frac{\left|[\mathtt{Full}(\boldsymbol{H})]^{\dagger}\boldsymbol{h}_m\right|^2 - \frac{1}{P}}{\chi_m} \ge \frac{\chi_{m_{\min}} - \frac{1}{P}}{\chi_{m_{\max}}} = \frac{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}}{\chi_{m_{\max}}}$; when $\theta < \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}$, $\min\limits_{m=1,2}\frac{\left|[\mathtt{Full}(\boldsymbol{H})]^{\dagger}\boldsymbol{h}_m\right|^2 - \frac{1}{P}}{\chi_m} = \frac{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}}{\chi_{m_{\max}}}$. Therefore,

$$\min_{m=1,2}\frac{\left|[\mathtt{Full}(\boldsymbol{H})]^{\dagger}\boldsymbol{h}_m\right|^2 - \frac{1}{P}}{\chi_m} \ge \frac{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}}{\chi_{m_{\max}}}.$$

and

$$\log\frac{1}{1-p} \le 2(t-1)\log\frac{\max\limits_{m=1,2}\chi_m}{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}}. \tag{30}$$

Define $\mathcal{H}_4 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \chi_1 \ge \chi_2\right\}$ and $\mathcal{H}_5 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathbb{C}^{t\times 2}, \chi_1 < \chi_2\right\}$. Substituting (30) and (12) into I_3 yields

$$\begin{aligned} I_3 &\le \frac{C_{15}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 \\ &\times \log\frac{\chi_1}{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &+ \frac{C_{15}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_1\cap\mathcal{H}_5}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 \\ &\times \log\frac{\chi_2}{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H} \\ &= \frac{2C_{15}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 \\ &\times \log\frac{\chi_1}{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}, \end{aligned}$$

where $C_{15} = 2(t-1)^3 C_3$. For any $\boldsymbol{H} \in \mathcal{H}_1 \cap \mathcal{H}_4$, $\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 \le \left[\frac{1}{\chi_2}+\frac{1}{\chi_2}\right]^2 = \frac{4}{\chi_2^2}$. Therefore, it follows that

$$I_3 \le \frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\frac{1}{\chi_2^2}\log\frac{\chi_1}{\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\,\mathrm{d}\boldsymbol{H}, \tag{31}$$

where $C_{16} = 8C_{15}$.

Define $\mathcal{H}_6 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{H}_1 \bigcap \mathcal{H}_4, \chi_2 \le \left|\boldsymbol{h}_1^{\dagger}\boldsymbol{h}_2\right|\right\}$ and $\mathcal{H}_7 = \left\{\boldsymbol{H} : \boldsymbol{H} \in \mathcal{H}_1 \bigcap \mathcal{H}_4, \chi_2 > \left|\boldsymbol{h}_1^{\dagger}\boldsymbol{h}_2\right|\right\}$. With such notations, $\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})$ in [8, Theorem 2] can be rewritten as

$$\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H}) = \begin{cases} \chi_2, & \boldsymbol{H} \in \mathcal{H}_6, \\ \frac{\chi_2}{1+\beta^2}, & \boldsymbol{H} \in \mathcal{H}_7, \end{cases}$$

where $\beta = \frac{\sqrt{\chi_2} - \sqrt{\chi_1\theta}}{\sqrt{\chi_1 - \chi_1\theta}}$ and $\theta = \frac{\left|\boldsymbol{h}_1^{\dagger}\boldsymbol{h}_2\right|^2}{\chi_1\chi_2}$. Then the upper bound on I_3 in (31) can be further deduced as

$$\begin{aligned} I_3 &\le \frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_6}\frac{1}{\chi_2^2}\log\frac{\chi_1}{\chi_2 - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H} \\ &+ \frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_7}\frac{1}{\chi_2^2}\log\frac{\chi_1}{\frac{\chi_2}{1+\beta^2} - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H} \\ &= \underbrace{\frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_6\cup\mathcal{H}_7}\frac{1}{\chi_2^2}\overbrace{(\log\chi_1)}^{\le\chi_1} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H}}_{=I_{3,1}} \\ &+ \underbrace{\frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_6}\frac{1}{\chi_2^2}\log\frac{1}{\chi_2 - \frac{1}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H}}_{=I_{3,2}} \\ &+ \underbrace{\frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_7}\frac{1}{\chi_2^2}\log\left(1+\beta^2\right) f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H}}_{=I_{3,3}} \\ &+ \underbrace{\frac{C_{16}}{P^2}\int_{\boldsymbol{H}\in\mathcal{H}_7}\frac{1}{\chi_2^2}\log\frac{1}{\chi_2 - \frac{1+\beta^2}{P}} f_{\boldsymbol{H}}(\boldsymbol{H})\mathrm{d}\boldsymbol{H}}_{=I_{3,4}}. \end{aligned}$$

Based on the fact that $\{\mathcal{H}_6 \cup \mathcal{H}_7\} = \{\mathcal{H}_1 \cap \mathcal{H}_4\} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_0$ and using a similar mathematical derivation for the upper bounds on I_1, I_2, the upper bound on $I_{3,1}$ is derived as

$$I_{3,1} \leq C_{17} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \quad (32)$$

where $C_{17} = \frac{2t}{t-1} C_{16} \times \mathbf{1}_{t\geq 3} + 2C_{16} \times \mathbf{1}_{t=2}$.

For $I_{3,2}$, since $\mathcal{H}_6 \subseteq \mathcal{H}_0$, its upper bound can be

$$\begin{aligned} I_{3,2} \leq& \frac{C_{16}}{P^2} \int_{\frac{1}{P}}^{\infty} \left[\frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!} \right] \mathrm{d}\chi_1 \\ &\times \int_{\frac{1}{P}}^{\infty} \frac{1}{\chi_2^2} \log \frac{1}{\chi_2 - \frac{1}{P}} \left[\frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!} \right] \mathrm{d}\chi_2 \\ \leq& \frac{C_{16}}{(t-1)! P^2} \int_0^{\infty} \frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!} \mathrm{d}\chi_1 \\ &\times \int_{\frac{1}{P}}^{\infty} \chi_2^{t-3} e^{-\chi_2} \log \frac{1}{\chi_2 - \frac{1}{P}} \mathrm{d}\chi_2 \\ =& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\infty} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2, \end{aligned} \quad (33)$$

where $C_{18} = \frac{C_{16}}{(t-1)!}$ and the last equality is obtained by replacing $\chi_2 - \frac{1}{P}$ with λ_2. When $t \geq 4$, with the help of (26), we obtain

$$\begin{aligned} I_{3,2} \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ &+ C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_{\frac{1}{P}}^{\infty} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \left(\log \frac{1}{\lambda_2} \right) \left(\frac{1}{P} + \frac{1}{P} \right)^{t-3} \mathrm{d}\lambda_2 \\ &+ C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_{\frac{1}{P}}^{\infty} \frac{1}{\lambda_2} \left(\lambda_2 + \lambda_2 \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ \leq& \frac{2^{t-3} C_{18} e^{-\frac{1}{P}}}{P^{t-1}} \int_0^{\frac{1}{P}} \log \frac{1}{\lambda_2} \mathrm{d}\lambda_2 \\ &+ \frac{2^{t-3} C_{18}}{P^2} \int_{\frac{1}{P}}^{\infty} \lambda_2^{t-4} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ =& \frac{C_{19} e^{-\frac{1}{P}}}{P^{t-1}} \left[\frac{1}{P} + \frac{\log P}{P} \right] + \frac{C_{19}}{P^2} \Gamma \left(t-3, \frac{1}{P} \right) \\ \leq& \frac{C_{19} e^{-\frac{1}{P}}}{P^{t-1}} \left[\frac{1}{P} + 1 \right] + \frac{(t-3)! C_{19} e^{-\frac{1}{P}}}{P^2} \left[1 + \frac{1}{P^{t-4}} \right] \\ \leq& C_{20} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} \right], \end{aligned} \quad (34)$$

where $C_{19} = 2^{t-3} C_{18}$ and $C_{20} = 2 \times (t-3)! C_{19} + 2C_{19}$. When $t = 3$, (33) becomes

$$\begin{aligned} I_{3,2} \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \left(\log \frac{1}{\lambda_2} \right) e^{-\lambda_2} \mathrm{d}\lambda_2 \\ &+ C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_{\frac{1}{P}}^{\infty} \left(\log \frac{1}{\lambda_2} \right) e^{-\lambda_2} \mathrm{d}\lambda_2 \\ \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \log \frac{1}{\lambda_2} \mathrm{d}\lambda_2 + \frac{C_{18}}{P^2} \int_{\frac{1}{P}}^{\infty} \frac{e^{-\lambda_2}}{\lambda_2} \mathrm{d}\lambda_2 \\ \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \left[\frac{1}{P} + \frac{\log P}{P} \right] + \frac{C_{18}}{P^2} \mathtt{E}_1 \left(\frac{1}{P} \right) \\ \leq& C_{18} \frac{e^{-\frac{1}{P}}}{P^2} \left[\frac{1}{P} + 1 \right] + \frac{C_{18}}{P^2} e^{-\frac{1}{P}} \log(1+P) \\ \leq& C_{21} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \end{aligned} \quad (35)$$

where $C_{21} = 3C_{18}$. When $t = 2$, since $\frac{1}{\lambda_2 + \frac{1}{P}} \leq P$, $I_{3,2} \leq C_{18} \frac{e^{-\frac{1}{P}}}{P} \int_0^{\infty} \log \frac{1}{\lambda_2} e^{-\lambda_2} \mathrm{d}\lambda_2$. Following the same steps in (35), $I_{3,2}$ can be bounded by

$$I_{3,2} \leq C_{22} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \quad (36)$$

where $C_{22} = 3C_{18}$. Based on (34), (35) and (36), the upper bound on $I_{3,2}$ for any $t \geq 2$ is

$$I_{3,2} \leq C_{23} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \quad (37)$$

where $C_{23} = C_{20} \times \mathbf{1}_{t\geq 4} + C_{21} \times \mathbf{1}_{t=3} + C_{22} \times \mathbf{1}_{t=2}$.

In $I_{3,3}$, since $0 \leq \beta \leq 1$, $\log\left(1+\beta^2\right) \leq \log 2 < 1$. Similarly, the bound on $I_{3,3}$ is obtained as

$$I_{3,3} \leq C_{24} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \quad (38)$$

where $C_{24} = \frac{2C_{16}}{t-1} \times \mathbf{1}_{t\geq 3} + \frac{C_{16}}{(t-1)!} \times \mathbf{1}_{t=2}$.

For $I_{3,4}$, since χ_1, χ_2 and θ are mutually independent, its upper bound can be derived as

$$\begin{aligned} I_{3,4} \leq& \frac{C_{16}}{P^2} \int_{\{\chi_1,\chi_2,\theta\}\in\mathcal{H}_7'} \frac{1}{\chi_2^2} \left(\log \frac{1}{\chi_2 - \frac{1+\beta^2}{P}} \right) \\ &\times f_{\chi_1}(\chi_1) f_{\chi_2}(\chi_2) f_\theta(\theta) \mathrm{d}\chi_1 \mathrm{d}\chi_2 \mathrm{d}\theta \\ =& \frac{C_{25}}{P^2} \int_{\{\chi_1,\chi_2,\theta\}\in\mathcal{H}_7'} \left(\log \frac{1}{\chi_2 - \frac{1+\beta^2}{P}} \right) \\ &\times \chi_1^{t-1} e^{-\chi_1} \chi_2^{t-3} e^{-\chi_2} (1-\theta)^{t-2} \mathrm{d}\chi_1 \mathrm{d}\chi_2 \mathrm{d}\theta, \end{aligned}$$

where $C_{25} = \frac{C_{16}}{(t-1)!(t-2)!}$ and $\mathcal{H}_7'$ is a transformed version of the pre-defined $\mathcal{H}_7$ with respect to χ_1, χ_2 and θ. The PDF of θ is given by $f_\theta(\theta) = (t-1)(1-\theta)^{t-2}$ for $0 \leq \theta \leq 1$ [12]. By changing the integration variables from (χ_1, χ_2, θ) into (β, χ_2, θ), we obtain the Jacobian of the transformation as $\left| \frac{\partial(\chi_1,\chi_2,\theta)}{\partial(\beta,\chi_2,\theta)} \right| = \left| \frac{\partial \chi_1}{\partial \beta} \right|$. For any $\boldsymbol{H} \in \mathcal{H}_7'$, $\beta = \frac{\sqrt{\chi_2} - \sqrt{\chi_1 \theta}}{\sqrt{\chi_1 - \chi_1 \theta}}$, $\chi_1 = \frac{\chi_2}{\left(\sqrt{\theta} + \beta\sqrt{1-\theta}\right)^2}$ and $\left| \frac{\partial \chi_1}{\partial \beta} \right| = \frac{2\sqrt{1-\theta}\chi_2}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^3}$. Therefore, $I_{3,4}$ can be bounded by

$$\begin{aligned} I_{3,4} \leq& \frac{C_{26}}{P^2} \int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''} \left(\log \frac{1}{\chi_2 - \frac{1+\beta^2}{P}} \right) \\ &\times \left[\frac{\chi_2}{(\sqrt{\theta} + \beta\sqrt{1-\theta})^2} \right]^{t-1} e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}} \chi_2^{t-3} e^{-\chi_2} \\ &\times (1-\theta)^{t-2} \frac{\sqrt{1-\theta}\chi_2}{\left(\sqrt{\theta} + \beta\sqrt{1-\theta}\right)^3} \mathrm{d}\chi_2 \mathrm{d}\beta \mathrm{d}\theta \\ =& \frac{C_{26}}{P^2} \int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''} \left(\log \frac{1}{\chi_2 - \frac{1+\beta^2}{P}} \right) \\ &\times \frac{\chi_2^{2t-3} e^{-\chi_2} (1-\theta)^{t-\frac{3}{2}}}{\left(\sqrt{\theta} + \beta\sqrt{1-\theta}\right)^{2t+1}} e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}} \mathrm{d}\chi_2 \mathrm{d}\beta \mathrm{d}\theta \end{aligned}$$

$$\leq \frac{C_{26}}{P^2}\int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''}\left(\log\frac{1}{\chi_2-\frac{1+\beta^2}{P}}\right) \times \frac{\chi_2^{2t-3}e^{-\chi_2}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}}e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}\mathrm{d}\chi_2\mathrm{d}\beta\mathrm{d}\theta,$$

where $C_{26}=2C_{25}$ and $\mathcal{H}_7''$ is a transformed version of $\mathcal{H}_7'$ with respect to β,χ_2 and θ. By replacing $\chi_2-\frac{1+\beta^2}{P}$ by χ, $\left|\frac{\partial(\beta,\chi_2,\theta)}{\partial(\beta,\chi,\theta)}\right|=\left|\frac{\partial\chi_2}{\partial\chi}\right|=1$, then $I_{3,4}$ is further bounded by

$$I_{3,4}\leq\frac{C_{26}}{P^2}\int_{\{\beta,\chi,\theta\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)\frac{e^{-\frac{\chi+\frac{1+\beta^2}{P}}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}} \times\left[\chi+\frac{1+\beta^2}{P}\right]^{2t-3}e^{-\chi-\frac{1+\beta^2}{P}}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\theta$$
$$\leq\frac{C_{26}}{P^2}\int_{\{\beta,\chi,\theta\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)\frac{e^{-\frac{\chi+\frac{1+\beta^2}{P}}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}} \times\left[\chi+\frac{1+\beta^2}{P}\right]^{2t-3}e^{-\chi-\frac{1}{P}}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\theta, \quad (39)$$

where $\mathcal{H}_7'''$ is a transformed version of $\boldsymbol{H}_7''$ with respect to β,χ and θ. Letting $\phi=\frac{\chi+\frac{1+\beta^2}{P}}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}$, $\left|\frac{\partial(\chi,\beta,\theta)}{\partial(\chi,\beta,\phi)}\right|=\left|\frac{\partial\theta}{\partial\phi}\right|$. Since $\frac{\sqrt{\chi+\frac{1+\beta^2}{P}}}{\sqrt{\phi}}=\sqrt{\theta}+\beta\sqrt{1-\theta}$, $\left|\frac{\partial\theta}{\partial\phi}\right|=\frac{\phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}}{\left|\frac{1}{\sqrt{\theta}}-\frac{\beta}{\sqrt{1-\theta}}\right|}$. For any $\boldsymbol{H}\in\mathcal{H}_7'''$, $\chi_1\geq\chi_2$, thus $\phi=\frac{\chi_1}{\chi_2}\geq 1$ and $0\leq\sqrt{\theta}+\beta\sqrt{1-\theta}\leq 1$. Then $0\leq\beta\leq\frac{1-\sqrt{\theta}}{\sqrt{1-\theta}}$. Hence, $\frac{1}{\sqrt{\theta}}-\frac{\beta}{\sqrt{1-\theta}}\geq\frac{1}{\sqrt{\theta}}-\frac{1}{\sqrt{1-\theta}}\times\frac{1-\sqrt{\theta}}{\sqrt{1-\theta}}=\frac{1}{(1+\sqrt{\theta})\sqrt{\theta}}>0$. Therefore, $\left|\frac{\partial\theta}{\partial\phi}\right|\leq\phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}(1+\sqrt{\theta})\sqrt{\theta}\leq 2\phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}$ due to $0\leq\theta\leq 1$. Moreover, since $\mathcal{H}_7'''\subseteq\{(\beta,\chi,\phi):0\leq\beta\leq 1,\chi>0,\phi>0\}$, the upper bound in (39) becomes

$$I_{3,1,4}\leq 2C_{26}\frac{e^{-\frac{1}{P}}}{P^2}\int_{\{\beta,\chi,\phi\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)e^{-\phi} \times\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}\phi^{t-1}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\phi$$
$$\leq 2C_{26}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\int_0^1\int_0^\infty\left(\log\frac{1}{\chi}\right)e^{-\phi} \times\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}\phi^{t-1}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\phi$$
$$=2C_{26}\frac{e^{-\frac{1}{P}}}{P^2}\left[\int_0^\infty\phi^{t-1}e^{-\phi}\mathrm{d}\phi\right] \times\int_0^\infty\int_0^1\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta$$
$$\leq C_{27}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\int_0^1\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta,$$

where $C_{27}=2(t-1)!C_{26}$. When $t\geq 4$, $\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}\leq\left[\chi+\frac{2}{P}\right]^{t-3}$ due to $0\leq\beta\leq 1$. Similar to (34), an upper bound on $I_{3,1,4}$ is derived as

$$I_{3,4}\leq C_{28}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}\right], \quad (40)$$

where $C_{28}=2^{2t-5}C_{27}+(t-4)!2^{t-2}C_{27}$. When $t=3$, similar to (35), the upper bound on $I_{3,4}$ is

$$I_{3,4}\leq C_{29}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right], \quad (41)$$

where $C_{29}=3C_{27}$. When $t=2$, since $\frac{1}{\chi+\frac{1+\beta^2}{P}}\leq\frac{P}{1+\beta^2}\leq P$, $I_{3,4}\leq C_{27}\frac{e^{-\frac{1}{P}}}{P}\int_0^\infty\left(\log\frac{1}{\chi}\right)e^{-\chi}\mathrm{d}\chi$. Similar to (35), we obtain

$$I_{3,4}\leq C_{30}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right], \quad (42)$$

where $C_{30}=3C_{27}$. Combining bounds in (40), (41) and (42), the upper bound on $I_{3,4}$ for any $t\geq 2$ is

$$I_{3,4}\leq C_{31}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right], \quad (43)$$

where $C_{31}=C_{28}\times\mathbf{1}_{t\geq 4}+C_{29}\times\mathbf{1}_{t=3}+C_{30}\times\mathbf{1}_{t=2}$. Based on (32), (37), (38) and (43), I_3 is upper-bounded by

$$I_3\leq C_{32}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right], \quad (44)$$

where $C_{32}=C_{19}+C_{23}+C_{26}+C_{31}$. From (24), (29) and (44), we finally get the upper bound in (15), where $C_0=C_7+C_{14}+C_{32}$. □

APPENDIX D - PROOF OF LEMMA 2

Proof. We use the following lemma, the proof of which is given in Appendix F.

Lemma 4. *For unit-normal complex vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{C}^{t\times 1}$, we have*

$$\left||\boldsymbol{u}^\dagger\boldsymbol{v}|^2-|\boldsymbol{u}^\dagger\boldsymbol{w}|^2\right|\leq\sqrt{1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2}. \quad (45)$$

For any $\boldsymbol{H}$ satisfying $\gamma(\mathtt{Full}(\boldsymbol{H}),\boldsymbol{H})>\frac{1}{P}$, let $\Delta_m=\left|[\mathtt{Full}(\boldsymbol{H})]^\dagger\boldsymbol{h}_m\right|^2-\frac{1}{P}$, where $0<\frac{\Delta_m}{\chi_m}<1$ for $m=1,2$. If $|\boldsymbol{x}^\dagger\mathtt{Full}(\boldsymbol{H})|^2\geq\Pi=1-\min_{m=1,2}\left[\frac{\Delta_m}{\chi_m}\right]^2$, by applying (45) and letting $\boldsymbol{u}=\frac{\boldsymbol{h}_m}{|\boldsymbol{h}_m|}$, $\boldsymbol{v}=\boldsymbol{x}$, $\boldsymbol{w}=\mathtt{Full}(\boldsymbol{H})$, we derive that

$$\left|\left|\frac{\boldsymbol{h}_m^\dagger}{|\boldsymbol{h}_m|}\boldsymbol{x}\right|^2-\left|\frac{\boldsymbol{h}_m^\dagger}{|\boldsymbol{h}_m|}\mathtt{Full}(\boldsymbol{H})\right|^2\right|\leq\sqrt{1-|\boldsymbol{x}^\dagger\mathtt{Full}(\boldsymbol{H})|^2}$$
$$\Longrightarrow\left|\frac{\boldsymbol{h}_m^\dagger}{|\boldsymbol{h}_m|}\boldsymbol{x}\right|^2\geq\left|\frac{\boldsymbol{h}_m^\dagger}{|\boldsymbol{h}_m|}\mathtt{Full}(\boldsymbol{H})\right|^2-\sqrt{1-|\boldsymbol{x}^\dagger\mathtt{Full}(\boldsymbol{H})|^2}$$
$$\Longrightarrow\left|\frac{\boldsymbol{h}_m^\dagger}{|\boldsymbol{h}_m|}\boldsymbol{x}\right|^2\geq\frac{1}{P\chi_m}+\frac{\Delta_m}{\chi_m}-\sqrt{1-\Pi}\geq\frac{1}{P\chi_m}$$
$$\Longrightarrow\left|\boldsymbol{h}_m^\dagger\boldsymbol{x}\right|^2\geq\frac{1}{P},$$

where "$\Longrightarrow$" represents "it follows that". Since $0<\Pi<1$, the proof is complete. □

APPENDIX E - PROOF OF LEMMA 3

Proof. $\Gamma(n,a)$ can be expanded as $\Gamma(n,a) = (n-1)!e^{-a}\sum_{k=0}^{n-1}\frac{a^k}{k!}$ [14]. When $0 < a \leq 1$, $\Gamma(n,a) \leq (n-1)!e^{-a}\sum_{k=0}^{n-1}\frac{1}{k!} \leq n!e^{-a}$; when $\alpha > 1$, $\Gamma(n,a) \leq (n-1)!e^{-a}\sum_{k=0}^{n-1}\alpha^k \leq (n-1)!e^{-a}\sum_{k=0}^{n-1}\alpha^{n-1} = n!e^{-a}\alpha^{n-1}$. Therefore, $\Gamma(n,a) \leq \max\left\{n!e^{-a}, n!e^{-a}\alpha^{n-1}\right\} \leq n!e^{-a} + n!e^{-a}\alpha^{n-1} = n!e^{-a}\left(1+\alpha^{n-1}\right)$. □

APPENDIX F - PROOF OF LEMMA 4

Proof. The left hand side of (45) can be rewritten as $\left|\boldsymbol{u}^\dagger\boldsymbol{G}\boldsymbol{u}\right|$, where $\boldsymbol{G} = \boldsymbol{v}\boldsymbol{v}^\dagger - \boldsymbol{w}\boldsymbol{w}^\dagger$. Therefore, it is upper-bounded by the maximum value of $\left||\boldsymbol{u}^\dagger\boldsymbol{v}|^2 - |\boldsymbol{u}^\dagger\boldsymbol{w}|^2\right|$ with respect to $\boldsymbol{u}$, which is the maximum absolute value for the singular value of $\boldsymbol{G}$. Using Gram-Schmidt orthogonalization, we obtain $\boldsymbol{v}_\perp = \frac{\boldsymbol{w}-\boldsymbol{v}\boldsymbol{v}^\dagger\boldsymbol{w}}{\sqrt{1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2}}$, which satisfies $|\boldsymbol{v}_\perp^\dagger\boldsymbol{v}_\perp|^2 = 1$ and $\boldsymbol{v}^\dagger\boldsymbol{v}_\perp = 0$. Then $\boldsymbol{w}$ can be rewritten as $\boldsymbol{w} = \boldsymbol{v}\boldsymbol{v}^\dagger\boldsymbol{w} + \sqrt{1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2}\boldsymbol{v}_\perp$. Therefore, $\boldsymbol{G} = \left(1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2\right)\boldsymbol{v}\boldsymbol{v}^\dagger + \left(|\boldsymbol{v}^\dagger\boldsymbol{w}|^2-1\right)\boldsymbol{v}_\perp\boldsymbol{v}_\perp^\dagger$ and $\boldsymbol{G}\boldsymbol{G}^\dagger = \left(1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2\right)\boldsymbol{v}\boldsymbol{v}^\dagger + \left(1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2\right)\boldsymbol{v}_\perp\boldsymbol{v}_\perp^\dagger$. Since $1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2 \geq 0$, the maximum absolute singular value of $\boldsymbol{G}$ is $\sqrt{1-|\boldsymbol{v}^\dagger\boldsymbol{w}|^2}$. □

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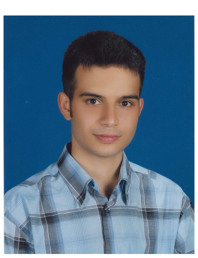

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