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Measurement Errors and Outliers in Seasonal Unit Root Testing

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Abstract. Frequently, seasonal and non-seasonal data (especially macro time series) are observed with noise. For instance, the time series can have irregular abrupt changes and interruptions following as a result of additive or temporary change outliers caused by external circumstances which are irrelevant for the series of interest. Equally, the time series can have measurement errors. In this paper we analyse the above types of data irregularities on the behaviour of seasonal unit root tests. It occurs that in most cases outliers and measurement errors can seriously affect inference towards the rejection of seasonal unit roots. It is shown how the distortion of the tests will depend upon the frequency, magnitude, and persistence of the outliers as well as on the signal to noise ratio associated with measurement errors. Some solutions to the implied inference problems are suggested.

Keywords: Seasonal unit roots, HEGY tests, additive outliers, measurement errors, Brownian motion.

JEL Classification: C12, C2, C22,

1. Introduction

It is often the case that observed time series are measured with noise. The noise can take many different forms. For instance, data measurement can be inaccurate due to inappropriate availability of data or because the definitions used by statistical offices do not correspond to the desired series intended for statistical or economic analysis. Data series like personal taxable income and unemployment rate series are typical series that are likely to be measured with a significant measurement error component. However, there might be other sources which contaminate the data. For

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instance, outliers may occur as a result of computer breakdown in the registration of data, union strikes, governmental interventions, and other reasons for some non-repetitive events that may flaw data quality. Also, such events may tend to persistent for some time but eventually will die out. In this paper we study the implications of measurement errors and different types of additive and temporary change outliers on tests for seasonal unit roots.

The implications of additive outliers in the levels of economic time series (as opposed to the differences) on unit root tests has previously been examined by Franses and Haldrup (1994) and Vogelsang (1999), see also Cati, Garcia and Perron (1999) for a slightly different class of models. These studies indicate that zero frequency unit roots may too often be rejected in the presence of outliers of the aforementioned type. The implications are the opposite of those analyzed by Perron (1989, 1990) and Perron and Vogelsang (1992), amongst others, who find that level shifts and trend breaks may bias unit root tests towards the acceptance of the unit root hypothesis. Focusing on seasonal data, Franses and Vogelsang (1998) examine the implications of additive and innovational outliers in the seasonally differenced series. Their type of outliers are fundamentally different from those examined in the present exposition because additive outliers in the seasonally differenced series will produce shifting seasonal means. In this situation the seasonal unit roots are too frequently accepted and hence the Franses and Vogelsang study is much more in line with the setup of Perron and his work with Vogelsang for non-seasonal data.

Here we focus our attention particularly on the behavior of seasonal unit root test statistics (but also non-seasonal) of the Hylleberg, Engle Granger, and Yoo (1990) (HEGY) type when the observed series contains measurement errors as well as additive and temporary change outliers in the levels of the series. We allow for data being non-seasonal, biannual, quarterly, or monthly. The design of our statistical model is such that we can control the impact on the various tests when the frequency, the magnitude, and the persistence of outliers changes. Also, the noise resulting from measurement errors can be controlled via an appropriately defined signal to noise ratio. Generally, the outliers and measurement errors imply that the regression errors will have an autocorrelation type component that is similar to a moving average process with a negative coefficient, and hence the size distortions that are known to arise in unit root testing for exactly this case, see e.g. Schwert (1989), are not surprisingly seen to apply to the present kind of problems as well. However, because of the irregularity of outliers, for instance, these kind of problems needs to be dealt with differently (compared to usual autocorrelation problems) because we may be able to identify the location of the outliers. It appears that the HEGY tests will be affected differently depending upon the frequency being tested for a unit root, and also, the way the tests are distorted will generally depend upon the type of outliers that occurs. In most cases, however, the biases of the tests turn out to be towards rejection of seasonal unit roots. We suggest different routes to correct for outliers and measurement errors. One route is to extend the suggestion of Franses and Haldrup (1991) to seasonal data and hence the idea is to add dummy variables in an augmented HEGY regression which will yield asymptotic distributions that are the same as when no outliers are present in the data. It remains to identify possible outliers and in so doing a new procedure building on Vogelsang (1999) is suggested for seasonal data. It appears that a combination of augmentation of the HEGY regression and outlier removal (where possible) capture much of the distortion resulting from the outliers although the remedy is far from being perfect. The second route one can pursue is to use modified Phillips-Perron type of tests to remove the influence from the outliers. However, such a generalization turns out not to be straightforward when applied to seasonal data.

The paper proceeds as follows. In the next section we present a representation of a seasonal random walk process which is due to Osborn and Rodrigues (1999) that is useful for deriving our analytical results. Section 4 presents the HEGY class of tests for any (even) value of s and the limiting distributions of the relevant test statistics are subsequently derived and discussed under the maintained assumption of noisy data. In section 5 a Monte Carlo study is presented to quantify the implications of our analytical findings. The subsequent section suggests some ways of solving the problems with outliers and measurement errors and in particular we suggest a new outlier detection procedure for seasonal data. An empirical illustration is given in section 7, and finally we conclude.

2. THE STATISTICAL MODEL

Consider the univariate (seasonal) process

$$\Delta_s y_t = \varepsilon_t, \qquad t = 1, 2, ...T \tag{1}$$

where ε_t is as $i.i.d.(0, \sigma_\varepsilon^2)$, s is the sampling frequency of the data, and Δ_s is the seasonal differencing filter. Hence, in practical situations we have $s = 1, 2, 4$, and 12 corresponding respectively to annually, biannually, quarterly, and monthly sampled data. In order to telescope on particular problems associated with measurement errors and outliers, the observed series is[1]

$$z_t = y_t + v_t \tag{2}$$

where v_t is an error term contaminating y_t. In particular, we consider the noise mechanism

$$v_t = \frac{\theta}{(1 - \alpha B)} \delta_t + \eta_t \tag{3}$$

[1]We abstract from the possibility of deterministic components such as seasonal dummy variables and trends. Such components will have no influence on our qualitative conclusions as long as we allow for these components in the relevant regression models.

where $\eta_t \sim i.i.d(0, \sigma_\eta^2)$ is a measurement error. The first term in (3) is a general outlier component where we assume $|\alpha| < 1$, with B being the lag-operator. If $\alpha = 0$, $\theta\delta_t$ is a noise term generated by irregularly observed additive outliers (AO). The parameter θ is the magnitude of the outliers, whilst δ_t is an indicator (Bernoulli) variable which can take either of the values 1 or -1 with a specified probability $p/2$. Otherwise, the value of δ_t equals zero. Hence AO's are characterized by some non repetitive events which occur irregularly and are unaffected by the dynamics of the y_t process. A different kind of outliers occur when $\alpha \neq 0$. In this situation the outliers also appear irregularly but tend to persistent although eventually their effect will die out given the assumption $|\alpha| < 1$. We will refer to such outliers as temporary change (TC) outliers. The above classification of outliers follows the terminology of Chen and Liu (1993). Note that neither of the noise components are such that their presence is dependent upon the season[2].

As can be seen, the design of the model is such that we can control the impact on the various tests when the frequency, the (relative) magnitude, and the persistence of the outliers changes. Also, the noise resulting from measurement errors can be controlled via the variance signal to noise ratio $(\sigma_\eta/\sigma_\varepsilon)^2$.

In Figures 1 and 2 examples are given of how the various kinds of data irregularities may appear in actual series. The series y_t (following a seasonal random walk with $s = 4$) has been contaminated with the three different kinds of outliers, one at a time. For the measurement error case $(\sigma_\eta/\sigma_\varepsilon)^2 = \{2, 4\}$ whilst for the AO and TC cases $p, \theta = \{(p = .05, \theta = 4), (p = .01, \theta = 20)\}$ and $\alpha = \{.75\}$. These cases correspond to situations with moderate and strong noise, respectively. Observe that the TC outliers tend to persist for some time compared to the additive outliers.

Insert Figure 1 about here

Insert Figure 2 about here

Before presenting the HEGY test procedure and the various test statistics of interest, we will present a representation of a seasonal random walk that is useful for our analytical derivations.

3. Preliminaries on the representation of a seasonal random walk

In a recent paper, Osborn and Rodrigues (1999) have developed an appealing and very elegant approach for deriving asymptotic results for estimators and test statistics in seasonal models with unit roots. In the sequel we will follow their exposition closely,

[2]Such a generalization could be considered of course, but it is not clear why in practical situations data irregularities should be seasonally dependent.

although alternative approaches are available, see e.g. Smith and Taylor (1999)[3]. The idea of Osborn and Rodrigues is to use a particular parameterization of the model such that for seasonally integrated data of frequency s, the model can be written as a sequence of s mutually independent random walks $\{x_{1n}\}, ..., \{x_{sn}\}$, that is, the series y_t defined in (1) can be written in terms of

$$x_{in} = x_{i,n-1} + \varepsilon_{in}, \qquad n = 1, 2, ..., N = T/s \tag{4}$$

where we make the simplifying assumption that $x_{i0} = 0, i = 1, ..., s$, while $\varepsilon_{in} \sim i.i.d.(0, \sigma_\varepsilon^2)$. Independency is across both indices i and n. More explicitly, (4) and (1) are related through

$$y_t = x_{i,\left[\frac{t-1}{s}\right]+1}$$

where $i = t \bmod s$, with the exception that $i = s$ when $t \bmod s = 0$. We have used the notation $n = \left[\frac{t-1}{s}\right] + 1$, where [.] takes the integer value of its argument.

A convenient way of writing the process in compact form is

$$\mathbf{X}_n = \mathbf{X}_{n-1} + \varepsilon_n, \qquad n = 1, ..., N$$

where

$$\begin{aligned} \mathbf{X}_n &= (x_{1n}, x_{2n}, ..., x_{sn})' \\ \varepsilon_n &= (\varepsilon_{1n}, \varepsilon_{2n}, ..., \varepsilon_{sn})' \sim i.i.d.(0, \sigma_\varepsilon^2 \mathbf{I}_s). \end{aligned}$$

Also, we will define the partial sum vector

$$\mathbf{S}_n = (S_{1n}, S_{2n}, ..., S_{sn})' = (\sum_{n=1}^{N} \varepsilon_{1n}, \sum_{n=1}^{N} \varepsilon_{2n}, ..., \sum_{n=1}^{N} \varepsilon_{sn})'. \tag{5}$$

It occurs that the vector representation of the seasonally integrated process is especially useful in deriving asymptotic results. The multivariate invariance principle, see e.g. Phillips (1986) and Phillips and Durlauf (1986), implies that

$$N^{-1/2}\mathbf{S}_n \Rightarrow \sigma \mathbf{W}(\mathbf{r}) \tag{6}$$

for $N \to \infty$ where "$\Rightarrow$ " signifies weak convergence, $\mathbf{W}(\mathbf{r}) = (W_1(r), W_2(r), ..., W_s(r))'$, and $W_i(r)$ for $i = 1, 2, ..., s$ are standard Wiener processes defined on [0,1]. As can be seen, the seasonal random walk model can be appropriately formulated such that much of the theory applying for unit root processes (at the zero frequency) can readily be applied to this extended class of processes.

[3]Smith and Taylor (1999) develop an integrated framework based on a spectral representation upon which asymptotic theory for seasonal data can be described for any frequency of observations.

To relate the process y_t to the seasonal random walks (5), it can be seen that

$$y_t = x_{in} = e_k'\mathbf{S}_n = e_k'(\mathbf{S}_{n-1} + \varepsilon_n) \tag{7}$$

where e_k is an $(s \times 1)$ selection vector which picks out the k'th row of a matrix, (i.e. the season in which y_t occurs) and lagged values can be expressed as

$$y_{t-i} = e_k'\mathbf{A}_i(\mathbf{S}_{n-1} + \mathbf{D}_i\varepsilon_n), \quad \text{for } i = 1, 2, ..., s. \tag{8}$$

The $s \times s$ matrix $\mathbf{D}_i$ takes the form $\mathbf{D}_i = diag(\mathbf{I}_{s-i}, \mathbf{0}_i)$ whilst the $\mathbf{A}_i$ matrices are "circulant" or permutation matrices, see also Barnett (1990) and Kunst (1997), which appear to play an important role for the results to follow. The matrices allocate the various lags of y_t to the relevant Wiener processes associated with a particular season. In particular, $\mathbf{A}_i\mathbf{X}$ will move the last i rows of the matrix $\mathbf{X}$ to the top with the remaining rows correspondingly moved down. The $\mathbf{A}_i$ matrices have a number of algebraic properties which are well described in Osborn and Rodrigues' paper. The interested reader is advised to consult their paper for further details.

4. THE HEGY TEST WITH NOISY DATA

The HEGY test procedure, see Hylleberg *et al.* (1990) and Engle *et al.* (1993), is widely used in seasonal unit root testing and references to applied papers seem to be unnecessary. Although originally developed for quarterly data, extensions to biannual and monthly observations, say, are relatively straightforward, see e.g. Beaulieu and Miron (1993), Taylor (1998), and Smith and Taylor (1999). Basically, the HEGY test is based upon an auxiliary regression where the regressors are transformed in such a way that they are orthogonal which equally implies that the frequency specific roots of the autoregressive process can be examined separately. In addition to providing a convenient interpretation of model parameters, the orthogonalization of the regressors greatly simplifies analytical derivations. In the present context, given that the observations are measured with noise as indicated in (2), the auxiliary regression using the z_t series can be written as[4]

$$\Delta_s z_t = \sum_{j=1}^{s} \pi_j^s z_{j,t-1}^s + u_t^s = \mathbf{z}_t^{s\prime}\boldsymbol{\pi}^s + u_t \tag{9}$$

where z_{jt}^s, $j = 1, 2, ..., s$ are the filtered z_t series, $\mathbf{z}_t^s = (z_{1t}^s, z_{2t}^s, ..., z_{st}^s)'$ and $\boldsymbol{\pi}^s = (\pi_1^s, \pi_2^s, ..., \pi_s^s)'$. If a (real) unit root at a particular frequency is present, the associated π_j^s coefficient will be zero; imaginary unit roots implies pairs of π_j^s coefficients to be zero,

[4]In the following we let a superscript "s" indicate the sampling frequency of the data.

in particular $\pi_j^s = \pi_{j+1}^s$ for j odd and larger than one. The least squares estimate of the $\boldsymbol{\pi}^s$ coefficients is given by

$$\begin{aligned}\widehat{\boldsymbol{\pi}}^s &= \left(\sum_{t=1}^{T} \mathbf{z}_t^s \mathbf{z}_t^{s\prime}\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{z}_t^{s\prime} \Delta_s z_t\right) \\ &= \operatorname{diag}\left\{\sum_{t=1}^{T} (z_{1t}^s)^2, \sum_{t=1}^{T} (z_{2t}^s)^2, ..., \sum_{t=1}^{T} (z_{st}^s)^2\right\}^{-1} \left(\sum_{t=1}^{T} \mathbf{z}_t^{s\prime} \Delta_s z_t\right)\end{aligned} \tag{10}$$

due to the orthogonality of the regressors. The $t-$statistic of a zero coefficient null associated with a particular parameter reads

$$t_{\pi_i^s} = \frac{\widehat{\pi}_i^s}{\widehat{\sigma}_u \left(\sum_{t=1}^{T} z_{it}^{s2}\right)^{1/2}}. \tag{11}$$

where

$$\widehat{\sigma}_u^2 = \frac{1}{T} \sum_{t=1}^{T} \widehat{u}_t^2 \tag{12}$$

is an estimate of the regression standard error. Also, because of the orthogonality of the regressors, joint hypothesis testing is especially simple. If a joint hypothesis of the form $\mathrm{H}_0 : \pi_i^s = \pi_j^s = ... = \pi_k^s = 0$ is considered with a total of m coefficients being tested, then the F-test statistic of the joint hypothesis can be written as a simple average of the associated squared $t-$ratios:

$$F_{\pi_i^s, \pi_j^s, ..., \pi_k^s} = \frac{1}{m}\left(t_{\pi_i^s}^2 + t_{\pi_j^s}^2 + ... + t_{\pi_k^s}^2\right). \tag{13}$$

In particular, it is of interest to consider joint tests of the form $F_{\pi_j^s, \pi_{j+1}^s}$ for j odd and larger than one which entails a test of complex pairs of the unit roots.

The following Lemma applies:

Lemma 1. *Assume that $z_t = y_t + v_t$ where $\Delta_s y_t = \varepsilon_t$ with $\varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2)$ and the noise term $v_t = \frac{\theta}{1-\alpha B}\delta_t + \eta_t$ satisfies , $\eta_t \sim i.i.d.(0, \sigma_\eta^2), |\alpha| < 1$, and δ_t is a Bernoulli indicator variable taking the values ± 1 with probability $p/2$, $(0 < p < 1)$. Let initial observations be zero, i.e. $z_0 = z_{-1} = z_{-2} = ... = z_{-s+1} = 0$. Then, as $T \to \infty$*

a) $\frac{1}{T^{3/2}} \sum_{t=1}^{T} z_{t-k} \Rightarrow \frac{\sigma_\varepsilon}{s^{3/2}} \int_0^1 \mathbf{1}' \mathbf{A}_k \mathbf{W}(r) dr$ $\quad k = 1, 2, ..., s$

b) $\frac{1}{T^2} \sum_{t=1}^{T} z_{t-k}^2 \Rightarrow \frac{\sigma_\varepsilon^2}{s^2} \int_0^1 \mathbf{W}(r)' \mathbf{W}(r) dr$ $\quad k = 1, 2, ..., s$

c) $\frac{1}{T^2} \sum_{t=1}^{T} z_{t-k} z_{t-j} \Rightarrow \frac{\sigma_\varepsilon^2}{s^2} \int_0^1 \mathbf{W}(r)' \mathbf{A}_k' \mathbf{A}_j \mathbf{W}(r) dr$ $\quad k \neq j$

d) $\frac{1}{T} \sum_{t=1}^{T} z_{t-k} \Delta_s z_t \Rightarrow \frac{\sigma_\varepsilon^2}{s} \int_0^1 \mathbf{W}(r)' \mathbf{A}_k' d\mathbf{W}(r) + \frac{\theta^2 p(\alpha^k - \alpha^{s-k})}{1-\alpha^2}$ $\quad k = 1, 2, ..., s-1$

$\frac{1}{T} \sum_{t=1}^{T} z_{t-k} \Delta_s z_t \Rightarrow \frac{\sigma_\varepsilon^2}{s} \int_0^1 \mathbf{W}(r)' \mathbf{A}_k' d\mathbf{W}(r) + (\frac{\theta^2 p(\alpha^s - 1)}{1-\alpha^2} - \sigma_\eta^2)$ $\quad k = s$

Proof: See appendix.

The regression (9) and the form of the regressors will depend upon the frequency of the data. However, we need some further notation to derive our main results. Let $\phi_j(B)$ be the polynomial filter adopted to define the z^s_{jt-1} series:

$$z^s_{jt-1} = \phi_j(B) z_t. \tag{14}$$

Then we define $\mathbf{B}_j$ to be the matrix consisting of an appropriate sum of $\mathbf{A}_i$ matrices, each of which is associated with a particular power B^i of the $\phi_j(B)$ filter. In other words, if

$$\phi_j(B) = \sum_{i=0}^{s} b_{ij} B^i \tag{15}$$

then

$$\mathbf{B}_j = \sum_{i=0}^{s} b_{ij} \mathbf{A}_i.$$

As clarified by Osborn and Rodrigues, it appears that the $\mathbf{B}_j$ matrices have a number of algebraic properties that make them easy to manipulate, see Appendix B.2 of their paper. A complete list of the $\mathbf{B}_j$ matrices for $s = 1, 2, 4$, and 12 are given in the appendix of the present paper. Let us consider some examples which also serve to clarify the particular forms the auxiliary regressions will take for given frequencies of the data observed.

4.1. Examples.

Biannual data. With $s = 2$ the filter $(1 - B^2)$ factors as

$$(1 - B^2) = (1 - B)(1 + B)$$

which corresponds to a zero frequency unit root, +1, and a seasonal unit root, -1, with two cycles per year. The associated frequencies are 0 and π. The regressors in the HEGY auxiliary regression are respectively $z^2_{1t-1} = (1 + B)z_t$ and $z^2_{2t-1} = -(1 - B)z_t$ and hence it follows that $\mathbf{B}_1 = \mathbf{A}_1 + \mathbf{A}_2$, $\mathbf{B}_2 = -\mathbf{A}_1 + \mathbf{A}_2$.

Quarterly data. With $s = 4$ the differencing filter $(1 - B^4)$ decomposes as

$$(1 - B^4) = (1 - B)(1 + B)(1 + B^2)$$

and thus (compared to biannual data) gives the additional unit roots $\pm i$ which forms a complex pair at one cycle per year and correspond to the frequencies $\pm\pi/2$. In this case the HEGY regression has the regressand $\Delta_4 z_t$ and the regressors

$$\begin{aligned} z^4_{1t-1} &= (B + B^2 + B^3 + B^4)z_t \\ z^4_{2t-1} &= (-B + B^2 - B^3 + B^4)z_t \\ z^4_{3t-1} &= (-B^2 + B^4)z_t \\ z^4_{4t-1} &= (-B + B^3)z_t \end{aligned}$$

Note that $z_{3t-1} = z_{4t-2}$ and hence there is a phase shift between these two series. The variable z_{1t} filters away the seasonal roots associated with $(1+B)$ and $(1+B^2)$, z_{2t} is adjusted for $(1-B)$ and $(1+B^2)$, whilst z_{3t-1} and z_{4t-1} jointly filter away the roots concerning $(1-B)$ and $(1+B)$. From these transformations we can define the 4×4 matrices

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 \\ \mathbf{B}_2 &= -\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_3 + \mathbf{A}_4 \\ \mathbf{B}_3 &= -\mathbf{A}_2 + \mathbf{A}_4 \\ \mathbf{B}_4 &= -\mathbf{A}_1 + \mathbf{A}_3. \end{aligned}$$

Monthly data. Finally, for monthly data, $s = 12$, the differencing filter $(1 - B^{12})$ can be decomposed as

$$\begin{aligned} (1 - B^{12}) &= (1-B)(1+B)(1+B^2)(1-B+B^2) \\ &\quad \times (1+B+B^2)(1-\sqrt{3}B+B^2)(1+\sqrt{3}B+B^2) \end{aligned}$$

which corresponds to a non-seasonal unit root $+1$, and the seasonal unit roots, -1; $\pm i$; $-\frac{1}{2}(1 \pm \sqrt{3}i)$; $\frac{1}{2}(1 \pm \sqrt{3}i)$; $-\frac{1}{2}(\sqrt{3} \pm i)$; $\frac{1}{2}(\sqrt{3} \pm i)$ which respectively correspond to 6,3,9,8,4,2,10,7,5,1, and 11 cycles per year[5]. Note that the last ten unit roots form complex conjugate harmonic frequency pairs.

The necessary data transformations required to orthogonalize the regressors in (9) of the HEGY regression are rather involved in the monthly case and are well described in Beaulieu and Miron[6] (1993). From the pattern given above the associated $\mathbf{B}_j$ 12×12 matrices follow naturally, see Osborn and Rodrigues for details.

4.2. Limiting distributions of HEGY test statistics. We now have the required results to obtain the relevant distributions of the various estimators and test statistics for an arbitrary frequency of data s. First, using Lemma 1, the following second Lemma can be derived.

[5]The associated frequencies of the roots are respectively: $0, \pi, \pm\pi/2, \pm 2\pi/3, \pm\pi/3, \pm 5\pi/6$, and $\pm\pi/6$.

[6]Note that in Beaulieu and Miron (1993) there is a misprint in the expressions for y_{6t} and y_{10t}. The filters defining these processes should both be multiplied by minus one.

Lemma 2. *Given the assumptions of Lemma 1*

$$
\begin{array}{lll}
a) & \frac{1}{T}\sum_{t=1}^{T} z_{jt-1}^{s}\Delta_s z_t \Rightarrow \frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_j d\mathbf{W}(r) + \mathcal{K}_j - \sigma_\eta^2 & for\ j=1,2,3,5,7,...,s-1 \\
 & \frac{1}{T}\sum_{t=1}^{T} z_{jt-1}^{s}\Delta_s z_t \Rightarrow \frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_j d\mathbf{W}(r) + \mathcal{K}_j & for\ j=4,6,8,...,s \\
b) & \frac{1}{T^2}\sum_{t=1}^{T}\left(z_{jt-1}^{s}\right)^2 \Rightarrow \frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_j \mathbf{W}(r)dr & for\ j=1,2 \\
 & \frac{1}{T^2}\sum_{t=1}^{T}\left(z_{jt-1}^{s}\right)^2 \Rightarrow \frac{1}{2}\frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_j \mathbf{W}(r)dr & for\ j=3,5,7,...,s-1 \\
 & \frac{1}{T^2}\sum_{t=1}^{T}\left(z_{jt-1}^{s}\right)^2 \Rightarrow \frac{1}{2}\frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_{j-1} \mathbf{W}(r)dr & for\ j=4,6,8,...,s \\
c) & \widehat{\sigma}_u^2 \Rightarrow \sigma_\varepsilon^2 + p\theta^2\frac{2-\alpha^s}{1-\alpha^2} + 2\sigma_\eta^2 &
\end{array}
$$

where

$$
\mathcal{K}_j = \begin{cases}
-\theta^2 p\left(\frac{1-\alpha^s}{1-\alpha^2}\right) & for\ j=1,2,3,5,7,....\ ,s-1 \\
-2\theta^2 p\alpha\left(\frac{1-\alpha^s}{1-\alpha^4}\right) & for\ j=4 \\
-\sqrt{3}\theta^2 p\alpha\left(\alpha^8-\alpha^7+\alpha^6+\alpha^2-\alpha+1\right) & for\ j=6 \\
-\sqrt{3}\theta^2 p\alpha\left(\alpha^8+\alpha^7+\alpha^6+\alpha^2+\alpha+1\right) & for\ j=8 \\
-\sqrt{3}\theta^2 p\alpha\left(\alpha^8-\sqrt{3}\alpha^7+3\alpha^6-2\sqrt{3}\alpha^5+4\alpha^4-2\sqrt{3}\alpha^3+3\alpha^2-\sqrt{3}\alpha+1\right) & for\ j=10 \\
-\sqrt{3}\theta^2 p\alpha\left(\alpha^8+\sqrt{3}\alpha^7+3\alpha^6+2\sqrt{3}\alpha^5+4\alpha^4+2\sqrt{3}\alpha^3+3\alpha^2+\sqrt{3}\alpha+1\right) & for\ j=12
\end{cases}
$$

Proof. See appendix.

Using Lemma 2 it is now straightforward to show the following main theorem.

Theorem 3. *Given the assumptions of Lemma 1 the t-ratios from the auxiliary HEGY regression* $\Delta_s z_t = \sum_{i=1}^{s}\pi_i^s z_{i,t-1}^s + u_t^s = \mathbf{z}_t^{s\prime}\boldsymbol{\pi}^s + u_t^s$ *have the distributions*

$$
\begin{array}{lll}
a) & t_{\pi_j^s} \Rightarrow \dfrac{\int_0^1 \mathbf{W}(r)'\mathbf{B}_j d\mathbf{W}(r) + \sqrt{s}\frac{\mathcal{K}_j-\sigma_\eta^2}{\sigma_\varepsilon^2}}{\sqrt{s}\left(1+p\left(\frac{\theta}{\sigma_\varepsilon}\right)^2\frac{2-\alpha^s}{1-\alpha^2}+2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2\right)^{1/2}\left(\int_0^1 \mathbf{W}(r)'\mathbf{B}_j\mathbf{W}(r)dr\right)^{1/2}} & for\ j=1,2 \\
b) & t_{\pi_j^s} \Rightarrow \dfrac{\int_0^1 \mathbf{W}(r)'\mathbf{B}_j d\mathbf{W}(r) + \sqrt{s}\frac{\mathcal{K}_j-\sigma_\eta^2}{\sigma_\varepsilon^2}}{\sqrt{s/2}\left(1+p\left(\frac{\theta}{\sigma_\varepsilon}\right)^2\frac{2-\alpha^s}{1-\alpha^2}+2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2\right)^{1/2}\left(\int_0^1 \mathbf{W}(r)'\mathbf{B}_j\mathbf{W}(r)dr\right)^{1/2}} & for\ j=3,5,7,...,s-1 \\
c) & t_{\pi_j^s} \Rightarrow \dfrac{\int_0^1 \mathbf{W}(r)'\mathbf{B}_j d\mathbf{W}(r) + \sqrt{s}\frac{\mathcal{K}_j}{\sigma_\varepsilon^2}}{\sqrt{s/2}\left(1+p\left(\frac{\theta}{\sigma_\varepsilon}\right)^2\frac{2-\alpha^s}{1-\alpha^2}+2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2\right)^{1/2}\left(\int_0^1 \mathbf{W}(r)'\mathbf{B}_{j-1}\mathbf{W}(r)dr\right)^{1/2}} & for\ j=4,6,8,...,s \\
d) & F_{\pi_j^s,\pi_{j+1}^s} = \frac{1}{2}\left(t_{\pi_j^s}^2 + t_{\pi_{j+1}^s}^2\right) & for\ j=3,5,7,....,s-1
\end{array}
$$

Proof. See appendix.

Note that all the statistics presented in Theorem 3 simplify to the distributions reported in Dickey and Fuller (1979), Phillips (1987), Engle, Granger, Hylleberg, and Lee (1993), Beaulieu and Miron (1993), Osborn and Rodrigues (1999), and Smith and Taylor (1999) for $s = 1, 2, 4, 12$ when $\mathcal{K}_j = \sigma_\eta^2 = 0$ for all j, that is, when there is no outliers or measurement errors in the model[7].

Generally, with noisy data the distributions are seen to be affected through a scale factor, $\left(1 + p\left(\frac{\theta}{\sigma_\varepsilon}\right)^2 \frac{2-\alpha^s}{1-\alpha^2} + 2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2\right)^{1/2}$, and a location factor, $\sqrt{s}\frac{\mathcal{K}_j - \sigma_\eta^2}{\sigma_\varepsilon^2}$ (or $\sqrt{s}\frac{\mathcal{K}_j}{\sigma_\varepsilon^2}$). To discuss the implications we separate the discussion to special cases.

4.3. Measurement errors. In this situation we assume that $\theta = 0$, $\sigma_\eta^2 \neq 0$, so the focus is entirely on measurement errors. As seen, the distribution of $t_{\pi_j^s}$ is shifted to the left for $j = 1, 2, 3, 5, ..., s-1$ whilst for even values of j exceeding two, there is no location shift. On the other hand, the scale effect tends to narrow the distributions of the t statistics. The larger the noise to signal ratio, $\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2$, the narrower the distribution becomes. For j even and exceeding two, the results therefore tend to indicate that for these values of j, the single $t-$tests will reject the null hypothesis less often than indicated by the significance level. For $j = 1, 2, 3, 5, ..., s-1$ the location and shift effects are of opposite sign and hence it is difficult to predict the summary effect for a particular value of the noise-signal ratio. However, as this ratio increases, the location effect is clearly going to dominate and hence leading to inflated test size compared to the nominal significance level. Too frequently we will thus reject the seasonal unit root associated with these particular statistics.

It is frequently suggested to test the complex paired unit root via the joint F-test, $F_{\pi_j^s, \pi_{j+1}^s} = \frac{1}{2}\left(t_{\pi_j^s}^2 + t_{\pi_{j+1}^s}^2\right)$, for $j = 3, 5, 7, ..., 11$. The implications for this test follow those of $t_{\pi_j^s}$ with j odd which have just been discussed. However, because $t_{\pi_j^s}$ is expected to have actual size in excess of the nominal size when j is even, the overall size distortion of $F_{\pi_j^s, \pi_{j+1}^s}$ is anticipated to lie between the corresponding sizes of the single t tests.

It will be instructive to explain the nature of measurement errors in the present context. In fact, the population error from the HEGY regression will take the form

$$\Delta_s z_t = \varepsilon_t + \eta_t - \eta_{t-s}$$

which is equivalent to an MA(s) process[8] $\Delta_s z_t = (1 + \lambda_s B^s)$. For instance, when the

[7]Although it is not obvious from the present exposition, it occurs that $t_{\pi_j^s}$ will have the same distribution for $j = 3, 5, 7, ..., s-1$, and similarly $t_{\pi_j^s}$ will have identical distributions for $j = 4, 6, 8, ..., s$.

[8]In terms of the present notation the MA parameter reads

$$\lambda_s = \frac{1}{2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2}\left(-1 - 2\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2 + \sqrt{\left(1 + 4\left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2\right)}\right).$$

noise-signal ratio ranges from .5 to 1 the associated MA parameter is in the range .38 to .46 and in the limit as the noise-signal ratio tends to infinity, the parameter approaches minus one. Traditionally in unit root testing, error autocorrelation is accounted for either parametrically or semi-parametrically. However, both types of corrections appear to have especially bad properties when the errors are negatively autocorrelated[9].

4.4. Additive outliers. Turning to additive outliers we assume $\sigma_\eta^2 = \alpha = 0, \theta \neq 0$. In this case the distribution results, (both scale and location effects), of the various statistics are qualitatively similar to those applying for measurement errors. Hence size will be affected: For $j = 1, 2$, and all odd values of j, the test size will tend to be larger than the significance level, while the size will be smaller for even values of j that exceed two. Concerning the interpretation, it is also apparent that some kind of MA errors arise. In particular, we have

$$\Delta_s z_t = \varepsilon_t + \theta \Delta_s \delta_t$$

where the MA parameter is as in the measurement error case with $\theta^2 p$ in place of σ_η^2. However, because additive outliers are irregularly observed and often of a magnitude that makes them easily identifiable, simple correction via extended autoregressive schemes or semi-parametric correction is a less attractive way of proceeding. Instead, we would prefer to identify the (few) outliers that might exist and subsequently adjust the testing procedure. We return to this topic in section 6.

4.5. Temporary change outliers. TC outliers occur irregularly, like additive outliers, but tend to persist for some time. Here we assume $\sigma_\eta^2 = 0$ and hence leaving α, θ non-zero, with $|\alpha| < 1$. Because dynamics is now introduced in the outliers, it occurs that all statistics will have both scale and location affected. Qualitatively, all statistics are shifted to the left whilst, quantitatively, the actual shift will depend upon which statistic is considered. As opposed to measurement errors and additive outliers we thus have that for TC outliers the size of all tests will be in excess of the nominal level if the outliers are not being properly dealt with. Hence we will tend to reject seasonal unit roots too frequently. Again, a moving average interpretation can be given although the irregular nature of the outliers clearly makes the comparison less straightforward for empirical considerations.

5. A MONTE CARLO STUDY

In order to analyze the quantitative implications of our analytical findings discussed in the previous section a small Monte Carlo study was conducted. The purpose of this

[9] In the limiting case where $\lambda_s \approx -1$ this is hardly surprising because any test will have difficulties in identifying a unit root when there is near cancellation of the AR and MA roots of the model.

study is two fold. First, we want to get an idea of the magnitude of the asymptotic size-distortions that we know will exist according to the theoretical analysis, and second, the distortions for finite stretches of data will be of separate interest. To simplify the presentation of the results, we have limited our analysis to quarterly data, $s = 4$. Essentially, the model with quarterly data covers all of the qualitative cases that follow more generally from the theoretical analysis. The experimental design is given as follows:

$$\begin{aligned} \Delta_4 y_t &= \varepsilon_t \\ z_t &= y_t + \frac{\theta}{(1-\alpha B)}\delta_t + \eta_t \end{aligned}$$

where we assume zero initial values and $\varepsilon_t \sim N(0,1)$. Each of the AO, TC and measurement error models are analyzed separately to ease the comparisons and in order to discriminate between the implications of the various types of data irregularities. From our theoretical findings, AO's and measurement errors have identical asymptotic size for a fixed value of $\varkappa = \left(\frac{\sigma_\eta}{\sigma_\varepsilon}\right)^2 = p\left(\frac{\theta}{\sigma_\varepsilon}\right)^2$. However, because time series contaminated with either measurement errors or AO's appear to be much different in nature, c.f. section 2, we decided to conduct the simulations for both cases. For the additive AO model parameter values were chosen such that $p\theta^2 = \{.4, .8\}$. It occured that different values of p and θ, yielding a fixed value of $p\theta^2$, would entail quite similar distributions even in small samples. Hence we only report the values for fixed values of $p\theta^2$. For comparison with the measurement error model we considered the noise generating mechanism $\eta_t \sim N(0, \sigma_\eta^2 = p\theta^2)$. The parameters of the additive outlier model were further extended to allow for TC outliers. In particular, we chose the following parameter values of the autoregressive parameter: $\alpha = \{.25, .75\}$. All experiments were conducted for sample sizes of $T = \{48, 100, 200, 400\}$ and the number of Monte Carlo repetitions was 1000. The RNDN and RNDU routines of the Gauss programming language were used to generate the data series.

For each Monte Carlo repetition the various HEGY test statistics $t_{\pi_1}, t_{\pi_2}, t_{\pi_3}, t_{\pi_4}$, F_{π_3,π_4}, F_{π_2,π_3,π_4}, and $F_{\pi_1,\pi_2,\pi_3,\pi_4}$ were calculated and the rejection frequencies were subsequently calculated. Table 1, panels A and B, respectively display the rejection frequencies for the measurement error and AO models. It appears that although measurement errors and AO's of the specified types are rather different in their appearance, the implications concerning finite sample distributions are rather similar as we also know would be the case asymptotically. As seen, if no proper account is made to adjust for the outliers or noisy errors, then the size distortions can be huge in some cases. The finite sample distortions appear similar for the $t_{\pi_1^4}$ and $t_{\pi_2^4}$ statistics which is suggested by the asymptotic results as well. Also, according to the asymptotic formulae, the $t_{\pi_4^4}$ statistic will have an actual size which is smaller than the nominal size. Because the $F_{\pi_3^4,\pi_4^4}$ test is constructed from a combination of an over-sized and an under-sized test statistic, it is not surprising that the overall size distortion is a convex combination of the simple $t-$test sizes.

The results for the TC model are displayed in Table 2. For moderate values of the outlier persistence parameter, ($\alpha = .25$ more particularly), it is seen that the size distortions are much similar to when $\alpha = 0$. However, the distortions of the $t_{\pi_4^4}$ test appears to be somewhat moderated and as a result size distortions for $F_{\pi_3^4,\pi_4^4}$ tend to worsen compared to the AO case. Generally, the actual sizes are seen to become far larger when the persistency parameter increases.

Insert Table 1 about here

Insert Table 2 about here

6. Identification and Correction for outliers

6.1. Correction for outliers. As discussed in the previous section both measurement errors and outliers give rise to (negative) moving average errors, and hence any procedure attempting to adjust for autocorrelation may be considered in the correction for noisy data. However, as is well known from the unit roots literature, the cure is especially problematic when autocorrelation exhibits negative moving average behavior. Parametric corrections via long autoregressive approximations, as suggested by Said and Dickey (1984), may cause practical problems because a large number of lags may be necessary to effectively whiten the errors[10]. Similarly, with respect to the semi-parametric tests of the Phillips-Perron type, the results of Schwert (1989) generalize to seasonal unit root testing, (Breitung and Franses (1999)), and hence, although negative autocorrelation can be allowed for in theory, serious size inflation of the tests may still occur in finite samples.

For (zero frequency) unit root models some successful attempts have been made to further adjust the Phillips-Perron statistics to account for the difficulties with MA processes, see Stock (1990), Perron and Ng (1996), and Ng and Perron (1996). In particular, Ng and Perron (1997) use a combination of GLS detrending and semi-parametric correction for nuisance parameters. Vogelsang (1999) shows how their approach can be used in the context of additive outliers; in fact, he shows that the Ng and Perron modifications of the Phillips-Perron tests are rather robust to the presence of additive outliers.

In principle, the Ng-Perron idea could be generalized to seasonal data. However, such an extension is not straightforwardly conducted. Also, one might expect that

[10] Said and Dickey (1984) show that if the order of the autoregression is $o_p(T^{1/2})$, the influence from MA errors can be effectively annihilated asymptotically.

the potential allowance for seasonal dummy variables for instance, may considerably complicate the analysis. It is beyond the scope of the present paper to solve these problems and we leave it for future research to develop modified semi-parametric tests for seasonal unit roots that are robust to negative moving average errors, measurement errors, and various types of outliers.

As has been demonstrated, the most serious problems with noisy data occur when the measurement error component has a relatively large variance or if the outliers are sufficiently large or frequent. In applied work, it might be hard to identify when there is a problem. What we might expect, however, is that if the outliers are sufficiently large, we can hope to identify them and correspondingly can take some action towards reduction of their effect. In Franses and Haldrup (1994) it is suggested to augment the Dickey-Fuller regression with an appropriate number of dummy variables associated with the outlying observations. In so doing, the resulting test statistics can be shown to follow the distributions that would apply if no outliers were present. Of course, this presupposes that the outliers are satisfactorily identified.

A similar approach can be pursued when the focus is on seasonal unit root testing. Assume that q additive outliers have been identified for the time periods $T_0^1, T_0^2, ..., T_0^q$. An appropriate way of reformulating the HEGY auxiliary regression is given by:

$$\Delta_s z_t = \mu_s + \sum_{j=1}^{s-1} \mu_j D_{jt} + \beta t + \sum_{j=1}^{s} \pi_j^s z_{j,t-1}^s + \sum_{j=1}^{k} \alpha_j \Delta_s z_{t-j} + \sum_{i=0}^{s+k} \sum_{j=1}^{q} \gamma_{ij} D(T_0^j)_{t-i} + u_t. \quad (16)$$

This regression is somewhat more general than previously assumed. In particular, as opposed to (9) we allow for deterministic regressors such as a constant and seasonal dummy variables (and hence allowing for seasonally varying intercepts), as well as for a time trend[11]. Moreover, since the order of the autoregressive process for z_t might be larger than s, the auxiliary regression is augmented by k lags of $\Delta_s z_t$. The presence of outliers is accounted for via the dummy variables $D(T_0^j)_t$, for $j = 1, 2, ...q$, and lags thereof; the lags are included due to the presence of autoregressive lags in the model[12]. In a similar fashion (16) can be augmented with dummy variables accounting for TC outliers. Compared to (16) this will require further lags of (or clusters of) the outlier dummies to capture the temporal dependence.

6.2. Identification of outliers. It remains to suggest how outliers can be identified to conduct HEGY tests based on a regression such as (16). A large literature exists which focuses on this problem for univariate time series models. In much of

[11]This does not affect the qualitative conclusions of the previous discussion. In practice, a range of deterministic regressors should be considered in the auxiliary HEGY regression.

[12]In case the outliers overlap, one should naturally exclude the superflous regressors to avoid perfect collinearity of the regressors.

this literature the approach is to estimate a fully parameterized ARMA model and constructing supremum t type of tests for the presence of an outlier across all possible dates in the sample, see e.g. Tsay (1986), Chen and Liu (1993), and Franses and Haldrup (1994).

In this paper we take a different route of departure in the detection of additive or temporary change outliers, and in so doing extend the analysis of Vogelsang (1999). The suggested outlier detection procedure is fairly simple to implement and it does not require any knowledge about the parametric form of the model. Neither is it required to estimate unknown serial correlation parameters. The idea is to test for additive outliers, one at a time, and sequentially to remove these from the sample as they are identified. The test relies on an auxiliary ordinary least squares regression which in the most general case takes the form

$$z_t = \widehat{\mu} + \sum_{j=1}^{s-1} \widehat{\delta}_j D_{jt} + \widehat{\beta} t + \widehat{\theta} D(T_0)_t + \widehat{\varepsilon}_t \tag{17}$$

where $D(T_0)$ is a dummy variable with the value one in period T_0, and is zero otherwise. Three subcases will be considered: a) model with constant, b) model with constant and seasonal dummies, and finally c) model with constant, seasonal dummies and trend. Under the null hypothesis $\theta = 0$ and hence, abstracting from measurement errors, $y_t = z_t$. Define $\lambda = \frac{T_0}{T}$ to be the fixed proportion of the data, where a possible outlier is considered. For the asymptotic theory to work we assume this number to be a fixed constant[13]. The test statistic τ is based on the supremum value of $t_\theta(T_0)$ from the regression (17) for all possible dates T_0 in the sample. We have the following result:

Theorem 4. *Assuming data is generated according to the process $\Delta_s y_t = \varepsilon_t$ and the auxiliary regression (17) is conducted with a particular choice of deterministic regressors (constant, seasonal dummies, and trend). Then for $T = Ns \to \infty$*

$$\tau = \sup_{T_0} |t_\theta(T_0)| \Rightarrow \sup_{\lambda} \left| \frac{s^{1/2} \mathbf{e}_k' \mathbf{W}^*(\lambda)}{\int_0^1 \mathbf{W}^*(r)' \mathbf{W}^*(r) dr} \right|$$

where $\mathbf{e}_k$, k=1,2,..s, is a vector selecting (any) element of $\mathbf{W}^*(\lambda) = (W_1^*(r), W_2^*(r), ..., W_s^*(r))'$ which consists of independent Brownian motion processes appropriately corrected for deterministic regressors.

Proof. See appendix.

[13] This is a standard assumption in the unit root literature allowing for structural breaks and outliers.

A number of observations are worth mentioning in relation to the asymptotic distribution reported in Theorem 4. First, note that in the distribution above $\mathbf{e}_k'\mathbf{W}^*(\lambda)$ selects the k'th row of $\mathbf{W}^*(\lambda)$ where k indicates the season in which the outlier might occur. However, because the single elements of $\mathbf{W}^*(\lambda)$ are uncorrelated, any row can be picked and hence the limiting distribution will be unaffected by the season in which the outlier is assumed[14]. This is an intuitively reasonable property. The asymptotic distribution will depend upon the frequency of the observations, s. Also, the distribution is affected by the deterministic regressors included in the auxiliary regression. The precise way the Brownian motion expressions will be affected in the presence of constant, seasonal intercepts, and trend can be found in e.g. Smith and Taylor (1999).

Tables 3-5 report the asymptotic and finite sample critical values for $s = 2, 4$, and 12 and for the three cases with additional deterministic regressors.

Insert Table 3 about here

Insert Table 4 about here

Insert Table 5 about here

The outlier detection procedure can be accomplished according to the following steps.

1. First the $\tau-$statistic is calculated for the entire sample. If τ exceeds a particular critical value an outlier has been detected at time $\text{argmax}_{T_0} \left| t_\theta(T_0) \right|$. The outlier is subsequently dropped from the sample.

2. The regression (17) is re-conducted but now on the reduced sample. If a second outlier is detected, it is removed from the sample as well, and yet another iteration is performed.

3. The procedure continues until no outliers occur to be left.

4. Given the outliers that have been detected, the auxiliary HEGY regression (16) is conducted upon which inference about unit roots is subsequently drawn.

The procedure outlined above has a number of caveats as Vogelsang (1999) rightly argue. First, the iterative nature of the algorithm will obviously affect the overall size in the outlier detection phase of the procedure. And secondly, the maintained assumption of a seasonal unit root is likely to affect the properties of the outlier

[14]Hence, in the expression for the distribution of the τ statistic, we can replace $\mathbf{e}_k'\mathbf{W}^*(\lambda)$ by $W_s^*(\lambda)$, for instance.

detection procedure when, in fact, one or several of the unit roots are absent. Also, depending upon the sign of the outliers and the interaction with the season in which they occur, it might be difficult to identify some of the outliers. In order to examine these potential problems, a small Monte Carlo experiment was conducted for the situation with quarterly data, $s = 4$.

Insert Table 6 about here

Insert Table 7 about here

Insert Table 8 about here

Insert Table 9 about here

Insert Table 10 about here

Table 6 demonstrates the properties of the HEGY testing procedure when a single additive outlier of the magnitudes $\theta = \{5, 10, 15\}$ occur for sample sizes of $T = \{48, 100, 200\}$. It is assumed that the outlier appears in the middle of the sample and that the supremum τ statistic is used to identify the outlier. As seen, the largest size distortions occur when the outlier happens to be large in magnitude. However, when the outlier is identified and subsequently corrected for, the inflated size is effectively reduced to something near the nominal significance level.

In Table 7 a similar experiment is conducted, but now the additive outliers occur systematically in accordance with the setup in section 2 (and elsewhere in the paper). Outliers of magnitude $\pm\theta = \{5, 15\}$ occur with probabilities $p = \{.01, .05\}$ and for the same sample sizes as given above. Overall, the HEGY-tests appear to be seriously size distorted when no correction is made with respect to the presence of the outliers. When the outliers are corrected for, (given that they are identified by use of the supremum τ statistic), some improvement of the sizes will occur for large outliers ($\theta = 15$), i.e. for $\theta = 5$ there is hardly any improvement. This could reflect that when the outliers are relatively small, then they might be hard to identify and hence the size distortions tend to persist after the data has been through the outlier detection filter. In other words, we might expect the supremum τ test to have low power when the additive outliers are small in magnitude.

Finally, Table 8 shows the experiment similar to the setup of the previous exercise, but now the outliers are allowed to persist with $\alpha = .75$. For this case there is only very minor improvements when correcting for outliers. In fact, there is indication that in many cases the size distortion actually deteriorates. This is especially so when addressing the $t_{\pi_1^4}$ statistic. The reasons for this could be that when the outliers are actually identified, then the subsequent reduction in available observations is reduced so much that the finite sample distribution of the test statistic will change. Hence, when TC outliers are present or otherwise many outliers appear to be present because their probability of occurence is high, then our simulations seem to indicate that the suggested procedure will suffer from degrees of freedom problems that introduce new size difficulties. Hence outlier correction might be infeasible if too many outliers appear to exist.

In the detection of outliers via the supremum τ test one might fear that too few outliers are identified because the seasonal fluctuations and the sign of the outliers coincidentally interfere. One way to adjust for this is to identify outliers by use of the τ statistic applied to the filtered series $S(B)z_t$ which equals $(1+B+B^2+B^3)z_t$ in the quarterly case. The resulting $\tau-$statistic will thus follow the distribution reported in Vogelsang (1999).

In Tables 7 and 8 we have deliberately excluded lags of $\Delta_4 z_t$ in the auxiliary HEGY regression because we know the associated coefficients are zero (provided the outliers are correctly identified and removed). However, as we have previously argued, additive outliers (and TC outliers, in particular) introduce dynamics in the model and hence, to the extent that the outlier detection appears insufficient, a combination of outlier correction and augmentation og the HEGY test might show useful. The augmentation counterparts[15] of Tables 7 and 8 are displayed in Tables 9 and 10. Also, the identification of outliers is now based on the transformed series $S(B)z_t$. As it can be seen, the combined effect of augmentation and outlier removal helps improving the size properties of the tests remarkably. In comparing Tables 7 and 9 (respectively 8 and 10), it can also be seen that the augmentation captures relatively more of the size-distortion compared to the case with outlier correction. This applies to both the AO and TC cases.

7. EMPIRICAL APPLICATION

In order to illustrate the suggested procedure for outlier detection and correction, we analyze the time properties of two economic variables: the Brazilian inflation rate and the Production Index of the US Transportation industry. For the Brazilian inflation rate the data is sampled monthly and covers the period 1974:1 to 1993:5. The sample corresponds with the data used in Cati et al. (1998), where an exact description of the data sources can be found. For the Production Index of the US Transportation industry, the quarterly data covers the period 1934:1 to 1991:2. A description of this Index is given in Cooper (1998), although note that the author uses the monthly version of the index. We have graphed the evolution of the two variables in Figures

[15]See the note of Table 9 for details on how the augmented HEGY regression was truncated.

3 and 4. It is quite remarkable that a simple visual inspection of these figures allow us to conclude that the presence of some outliers seems to be a sensible hypothesis. Thus, a priori we could expect that the use of standard methods to determine the time series properties, the HEGY tests for example, to be distorted by their existence.

Let us begin the analysis by using standard methods, that is to say, those which do not account for the removal of the possible outliers. Given that the two variables are seasonal, the use of the HEGY-type tests is advisable. In Table 11 we report the results for the Production Index of the US Transportation industry, obtained by using the quarterly version of the HEGY test. Table 12 presents the monthly version of these statistics when applied to the Brazilian inflation rate. In both cases, an intercept, a deterministic trend and the corresponding seasonal dummy variables have been included. Finally, we also had to include some lags of the seasonally differenced variable in order to remove the possible autocorrelation pattern. Although a number of methods for the selection of the lag truncation parameter are available, we have chosen to use the *k(t)* procedure recommended in Ng and Perron (1995). This method involves a general-to-specific strategy, starting with a predetermined value of the lag truncation parameter *(kmax)* and then testing the significance of the single coefficient associated with the last lag until a significant statistic is encountered. The single significance of the lags is analyzed by comparing their t-ratios with the value 1.65.

When we analyze the first row of Table 11, we can clearly reject the presence of a unit root at the non-seasonal frequency for the production series, whilst more evidence is found in favor of the seasonal unit root hypothesis. However, we should take into account that these results may be influenced by the presence of some outliers and, therefore, it is possible that the rejection of the non-seasonal unit root null hypothesis to have a spurious nature. To confirm this point, we have used the statistic τ proposed in the previous section in order to determine whether some outliers are contaminating the evolution of this variable. Thus, when using the liberal 10% significance level for the determination of the number of outliers, then the statistic τ finds 11 outliers, all of which being related to the 1942:4-1945:2 period. The economic interpretation of this result is easy to understand, in that they are clearly related to the increase of the economic activity of the Transportation industry during the US participation in the World War II. Therefore, we should remove the presence of these outliers in order to carry out a proper inference on the time properties of this variable. To that end, we include impulse dummy variables, each of which is associated with the data points where the 11 outliers were found; compare (16). Once the presence of these outliers is removed, the conclusions on the time series properties of the variable are quite different in that now we cannot reject the non-seasonal unit root hypothesis. The presence of a unit root in the seasonal frequency is still accepted.

When analyzing the Brazilian inflation rate, qualitatively similar results are obtained. First, the application of the monthly version of the HEGY tests (without accounting for the presence of outliers) lead us to conclude that there is a unit root at the non-seasonal frequency[16], whilst the presence of unit roots in all the seasonal frequencies is not so clear. For example, the statistics $F_{5,6}$, $F_{9,10}$ and $F_{11,12}$ can reject the seasonal unit root hypothesis for their corresponding frequencies. By contrast, we accept the presence of a unit root for the rest of the frequencies. Once again, we should interpret these results with caution, as they may have a spurious nature due to the influence of some outliers which have not been born in mind. To determine the possible existence of these outliers, we again apply the τ statistic. When using the liberal 10% significance level, the τ statistic finds 5 outliers, related to the 1989:9-1990:1 period. This coincides with the results presented in Cati et al. (1999), where the seasonal component is not taken into account. As it is explained in greater detail in their paper, these outliers are related to the period of very rapid inflation growth previous to the implementation of an inflation stabilization plan; see their paper for details. The removal of the outlier influence clearly changes the results obtained from the use of the HEGY tests. As we can now observe in the second row of Table 12, the HEGY tests do not provide any evidence against the seasonal/nonseasonal unit root hypothesis for any of the different frequencies.

In summary, the analysis of the two previous variables alerts us on the distortionary effect on seasonal/nonseasonal unit root inference caused by the presence of outliers.

Insert Table 11 about here

Insert Table 12 about here

Insert Figure 3 about here

Insert Figure 4 about here

8. CONCLUSION

Data irregularities in economic time series are probably more the rule than the exception. As has been demonstrated in this paper, measurement errors and outliers of various types introduce a particular form of dynamics - negative moving average errors - that appears to be particularly difficult to deal with when using standard

[16]This contrasts the finding by Cati *et al.* (1999) who rejected a zero frequency unit root for the full sample when no outlier correction was made. A likely explanation behind this difference could be the fact that the dynamics (especially the seasonal properties of the data) is dealt with differently.

procedures in testing for unit roots at both the seasonal and the non-seasonal frequencies. We have given some suggestions to deal with noise in the form of large additive or temporary change outlying observations. In particular, a combination of augmented HEGY tests and outlier removal, where it is possible, will help improving the size-distortions that otherwise will be present. However, this does not change the fact, that minor outliers and measurement errors, that we cannot generally identify, will bias standard tests and existing procedures are not well designed to account for this caveat. On the other hand, if (large) outliers appear too frequently the correction for the outliers can lead to serious "degrees of freedom"-difficulties which will make outlier correction infeasible in practice.

At the zero frequency, unit root tests that are very little sensitive to negative moving average autocorrelation, measurement errors, and outliers have been developed by Ng and Perron (1997). One suggestive route for future research is to generalize these tests to the seasonal frequencies to avoid, or at least to reduce, the problems described in the present paper.

9. APPENDIX A
DEFINITION OF B MATRICES USED IN SECTION 4.

The **B** matrices displayed below correspond to the relevant linear filters of the $z^s_{j,t-1}$ regressors and whose exact form can be found in Hylleberg, Engle, Granger and Yoo (1990) and in Beaulieu and Miron (1993).

$$\mathbf{B}_1 = \textstyle\sum_{i=1}^{s} \mathbf{A}_i \qquad \text{for } s = 1,2,4,12$$
$$\mathbf{B}_2 = \textstyle\sum_{i=1}^{s} (-1)^i \mathbf{A}_i \qquad \text{for } s = 2,4,12$$
$$\mathbf{B}_3 = \textstyle\sum_{i=0}^{s/2-1} (-1)^{i+1} \mathbf{A}_{2i+2} \qquad \text{for } s = 4,12$$
$$\mathbf{B}_4 = \textstyle\sum_{i=0}^{s/2-1} (-1)^{i+1} \mathbf{A}_{2i+1} \qquad \text{for } s = 4,12$$
$$\mathbf{B}_5 = -\tfrac{1}{2}(\textstyle\sum_{i=1}^{12} \mathbf{A}_i - 3\sum_{i=1}^{4} \mathbf{A}_{3i})$$
$$\mathbf{B}_6 = -\tfrac{\sqrt{3}}{2}(\mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_4 - \mathbf{A}_5 + \mathbf{A}_7 - \mathbf{A}_8 + \mathbf{A}_{10} - \mathbf{A}_{11})$$
$$\mathbf{B}_7 = \tfrac{1}{2}(\textstyle\sum_{i=0}^{5} \mathbf{A}_{2i+1} - \sum_{i=0}^{5} \mathbf{A}_{2i+2} + 3\sum_{i=1}^{4} (-1)^i \mathbf{A}_{3i})$$
$$\mathbf{B}_8 = -\tfrac{\sqrt{3}}{2}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_4 - \mathbf{A}_5 + \mathbf{A}_7 + \mathbf{A}_8 - \mathbf{A}_{10} - \mathbf{A}_{11})$$
$$\mathbf{B}_9 = -\tfrac{1}{2}(\sqrt{3}\mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_4 - \sqrt{3}\mathbf{A}_5 + \mathbf{2A}_6 - \sqrt{3}\mathbf{A}_7 + \mathbf{A}_8 - \mathbf{A}_{10} + \sqrt{3}\mathbf{A}_{11} - 2\mathbf{A}_{12})$$
$$\mathbf{B}_{10} = -\tfrac{1}{2}(\mathbf{A}_1 - \sqrt{3}\mathbf{A}_2 + 2\mathbf{A}_3 - \sqrt{3}\mathbf{A}_4 + \mathbf{A}_5 - \mathbf{A}_7 + \sqrt{3}\mathbf{A}_8 - 2\mathbf{A}_9 + \sqrt{3}\mathbf{A}_{10} - \mathbf{A}_{11})$$
$$\mathbf{B}_{11} = \tfrac{1}{2}(\sqrt{3}\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_4 - \sqrt{3}\mathbf{A}_5 - \mathbf{2A}_6 - \sqrt{3}\mathbf{A}_7 - \mathbf{A}_8 + \mathbf{A}_{10} + \sqrt{3}\mathbf{A}_{11} + 2\mathbf{A}_{12})$$
$$\mathbf{B}_{12} = \tfrac{1}{2}(\mathbf{A}_1 + \sqrt{3}\mathbf{A}_2 + 2\mathbf{A}_3 + \sqrt{3}\mathbf{A}_4 + \mathbf{A}_5 - \mathbf{A}_7 - \sqrt{3}\mathbf{A}_8 - 2\mathbf{A}_9 - \sqrt{3}\mathbf{A}_{10} - \mathbf{A}_{11})$$

10. APPENDIX B
PROOFS OF LEMMAS AND THEOREMS.

The proofs are essentially based on a combination of Lemma A.1 of Osborn and Rodrigues (1999) and an extension of the Lemma provided in Franses and Haldrup (1994). First, we need the following Lemma:

Lemma 5. *For the process* $v_t = \frac{\theta}{1-\alpha B}\delta_t + \eta_t$ *which satisfies the conditions exposed in Lemma 1, it follows as* $T \to \infty$

a)	$T^{-1}\sum_{t=1}^{T}\delta_{t-k}^2 \Rightarrow p$	$k \geq 0$
b)	$T^{-1}\sum_{t=1}^{T}\delta_{t-k}\delta_{t-j} \Rightarrow 0$	$k \neq j$
c)	$T^{-1}\sum_{t=1}^{T}\delta_{t-k}\Delta_s\delta_t \Rightarrow -p$	$k = s$
d)	$T^{-1}\sum_{t=1}^{T}\delta_{t-k}\Delta_s y_t = o_p(1)$	$k = 1, 2, ...$
e)	$T^{-3/2}\sum_{t=1}^{T}\delta_{t-j}y_{t-k} = o_p(1)$	$j, k = 1, 2, ...$
f)	$T^{-1}\sum_{t=1}^{T}y_{t-k}\Delta_s\delta_t = o_p(1)$	$k = 1, 2, ...$
g)	$T^{-1}\sum_{t=1}^{T}v_{t-k}\Delta_s v_t \Rightarrow \frac{\theta^2 p(\alpha^k - \alpha^{s-k})}{1-\alpha^2}$	$k = 1, 2, ..., s-1$
h)	$T^{-1}\sum_{t=1}^{T}v_{t-k}\Delta_s v_t \Rightarrow \frac{\theta^2 p(\alpha^s - 1)}{1-\alpha^2} - \sigma_\eta^2$	$k = s$

Proof. a) We let $\delta_t = \delta_t^+ + \delta_t^-$ where $\delta_t^+ = 1$ with probability $p/2$ and $\delta_t^- = -1$ with probability $p/2$. Hence it follows that $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\delta_{t-k}^2 = \mathrm{E}(\delta_t^2) = \mathrm{E}\left(\left(\delta_t^+ + \delta_t^-\right)^2\right) = E\left(\left(\delta_t^+\right)^2 + \left(\delta_t^-\right)^2 + 2\delta_t^+\delta_t^-\right) = 2p/2 = p$. The result *b*) follows a similar line of proof: $\mathrm{E}(\delta_{t-k}\delta_{t-j}) = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\delta_{t-k}\delta_{t-j} = \mathrm{E}\left(\left(\delta_{t-k}^+ + \delta_{t-k}^-\right)\left(\delta_{t-j}^+ + \delta_{t-j}^-\right)\right) = E\left(\left(\delta_{t-k}^+\delta_{t-j}^+\right) + \left(\delta_{t-k}^+\delta_{t-j}^-\right) + \left(\delta_{t-k}^-\delta_{t-j}^+\right) + \left(\delta_{t-k}^-\delta_{t-j}^-\right)\right) = p/2 - p/2 + p/2 - p/2 = 0.$

The limit *c*) follows trivially given *b*). Because the single ε_t' s are independently distributed, the result *d*) follows because $\sum_{t=1}^{T}\delta_{t-k}\Delta_s y_t = \sum_{t=1}^{T}\delta_{t-k}\varepsilon_t$ asymptotically will follow a Gaussian distribution with mean zero and variance $p\sigma^2 T$, hence

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\delta_{t-k}\Delta_s y_t \Rightarrow N(0, p\sigma^2).$$

Next, we see that *e*) applies since $\sum_{t=1}^{T}\delta_{t-j}y_{t-k}$ is also a Gaussian process in the limit. It is straightforward to see that the mean is zero and the variance is given by $p\sigma^2\frac{T(T+1)(2T+1)}{6}$ and hence as $T \to \infty$

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\delta_{t-j}y_{t-k} \Rightarrow N(0, \frac{1}{3}p\sigma^2).$$

f) follows because $T^{-1}\sum_{t=1}^{T}y_{t-k}\Delta_s\delta_t = T^{-1}\sum_{t=1}^{T}\varepsilon_{t-k}\delta_t = o_p(1)$ according to *d*).

g) and *h*) are given as follows.

$$E(v_{t-k}\Delta_s v_t) = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}v_{t-k}\Delta_s v_t =$$
$$E\left(\left(\frac{\theta}{1-\alpha B}\delta_{t-k} + \eta_{t-k}\right)\left(\frac{\theta}{1-\alpha B}\Delta_s\delta_t + \Delta_s\eta_t\right)\right).$$

Since δ_{t-i} and η_{t-j} are uncorrelated for any value of i, j we can focus on each term constituting v_t in separation. First we note that $E(\eta_{t-k}\Delta_s\eta_t) = -\sigma_\eta^2$ for $k = s$ and is zero otherwise. With respect to the δ_t terms we get by using $a) - c)$ of Lemma 5

$$E\left(\left(\frac{\theta}{1-\alpha B}\delta_{t-k}\right)\left(\frac{\theta}{1-\alpha B}\Delta_s\delta_t\right)\right) =$$

$$\theta^2 E\left(\left(\sum_{j=0}^{\infty}\alpha^j\delta_{t-k-j}\right)\left(\sum_{j=0}^{\infty}\alpha^j\delta_{t-j} - \sum_{j=0}^{\infty}\alpha^j\delta_{t-s-j}\right)\right)$$

$$= \frac{\theta^2 p\alpha^k}{1-\alpha^2} - \frac{\theta^2 p\alpha^{s-k}}{1-\alpha^2} = \frac{\theta^2 p(\alpha^k - \alpha^{s-k})}{1-\alpha^2}$$

This yields the desired results ■

Proof of Lemma 1.

We now return to the proof of Corolloray 1. In fact, results $a) - c)$ follow straightforwardly from the analysis of Osborn and Rodrigues (1999). Using the notation of section 3, we see that

$$\begin{aligned}\frac{1}{T^{3/2}}\sum_{t=1}^{T} z_{t-k}^s &= \frac{1}{(sN)^{3/2}}\sum_{t=1}^{N}(y_{t-k} + v_{t-k}) \\ &= \frac{1}{(sN)^{3/2}}\sum_{t=1}^{N} y_{t-k} + o_p(1) \Rightarrow \frac{\sigma_\varepsilon}{s^{3/2}}\int_0^1 \mathbf{1}'\mathbf{A}_k\mathbf{W}(r)dr\end{aligned}$$

and hence the type of outliers considered are of an order in probability that makes them have no impact on the limiting results. The results for $b)$ and $c)$ follow accordingly. See Osborn and Rodrigues for a full proof.

With respect to $d)$ we use the Lemma proven above jointly with the results of Osborn and Rodrigues (1999), Lemma A.1. We have that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} z_{t-k}^s\Delta_s z_t &= \frac{1}{T}\sum_{t=1}^{T}(y_{t-k} + v_{t-k})(\varepsilon_t + \Delta_s v_t) \\ &= \frac{1}{T}\sum_{t=1}^{T} y_{t-k}\varepsilon_t + \frac{1}{T}\sum_{t=1}^{T} v_{t-k}\Delta_s v_t + o_p(1) \\ &= \frac{1}{T}\sum_{t=1}^{T} y_{t-k}\varepsilon_t + \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\theta\delta_{t-k}}{1-\alpha B}\right)\left(\frac{\theta\Delta\delta_t}{1-\alpha B}\right) + \frac{1}{T}\sum_{t=1}^{T}\eta_{t-k}\Delta_s\eta_t + o_p(1) \\ &\Rightarrow \frac{\sigma_\varepsilon^2}{s}\int_0^1 \mathbf{W}(r)'\mathbf{A}_k' d\mathbf{W}(r) + \frac{\theta^2 p(\alpha^k - \alpha^{s-k})}{1-\alpha^2} - \sigma_\eta^2 I_k\end{aligned}$$

where $I_k = 1$ for $k = s$ and is zero otherwise. ■

Proof of Corollary 2.

The proof of a) and b) for the case with no noise is given in Osborn and Rodrigues (1999). Using the same line of arguments as in the proof of Corollary 1, result b) follows immediately. In a) the task is to show how the localization parameters $\mathcal{K}_j$ appear. Naturally these parameters will depend upon which filter $z^s_{jt-1} = \phi_j(B)z_t$ is applied to the original series, c.f. (14) and (15). For instance, when considering the filter $z^s_{1t-1} = \sum_{i=1}^{s} B^i z_t = (B + B^2 + ... + B^s)z_t = z_{t-1} + z_{t-2} + ... + z_{t-s}$ for $s = 1, 2, 4,$ and 12, it follows that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} z^s_{1t-1}\Delta_s z_t &= \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{s}\left(B^i z_t\right)\Delta_s z_t = \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(y_{t-1} + y_{t-2} + ... + y_{t-s}\right)\varepsilon_t + \\
&\quad \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{s}\left(\frac{\theta}{1-\alpha B}\delta_{t-i}\right)\left(\frac{\theta}{1-\alpha B}\Delta_s\delta_t\right) + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{s}\eta_{t-i}\Delta_s\eta_t + o_p(1) \\
&\Rightarrow \frac{\sigma^2_\varepsilon}{s}\int_0^1 \mathbf{W}(r)'\mathbf{B}_1 d\mathbf{W}(r) + \mathcal{K}_1 - \sigma^2_\eta
\end{aligned}$$

from which it can be seen that

$$\begin{aligned}
\mathcal{K}_1 &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{s}\left(\frac{\theta}{1-\alpha B}\delta_{t-i}\right)\left(\frac{\theta}{1-\alpha B}\Delta_s\delta_t\right) \\
&= \theta^2 p\sum_{i=1}^{s}\frac{(\alpha^i - \alpha^{s-i})}{1-\alpha^2} = -\theta^2 p\frac{1-\alpha^s}{(1-\alpha^2)}.
\end{aligned}$$

Also note that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{s}\eta_{t-i}\Delta_s\eta_t = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\eta_{t-s}(-\eta_{t-s}) = -\sigma^2_\eta.$$

In general, the location parameters are calculated according to the following formula:

$$\mathcal{K}_j = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\theta^2 p\sum_{i=1}^{s} b_{ij}\frac{(\alpha^i - \alpha^{s-i})}{1-\alpha^2}$$

where the b_{ij} coefficients are defined from (14) and (15). The second location parameter, which we may denote κ_j, can be deduced from the expression

$$\kappa_j = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{s} b_{ij}\eta_{t-i}\Delta_s \eta_t = -b_{sj}\sigma_\eta^2.$$

Note that b_{sj} is the coefficient associated with B^s in the filter $\phi_j(B)$ where $z_{jt-1}^s = \phi_j(B)z_t$. The remaining $\mathcal{K}_j$ coefficients can thus be found by direct application to the filters described in Appendix A.

It remains to prove c). It will be shown in the proof of Theorem 3 that the estimates $\widehat{\boldsymbol{\pi}}^s$ given in (10) are consistent. Hence $\widehat{\sigma}_u^2 = \frac{1}{T}\sum_{t=1}^{T} \widehat{u}_t^2$ is a consistent estimator of σ_u^2, and

$$\begin{aligned} \lim_{T\to\infty} \widehat{\sigma}_u^2 &= E\left(\left(\varepsilon_t + \frac{\theta}{1-\alpha B}\Delta_s \delta_{t-k} + \Delta_s \eta_{t-k}\right)^2\right) \\ &= \sigma_\varepsilon^2 + \theta^2 p \frac{2-\alpha^s}{1-\alpha^2} + 2\sigma_\eta^2. \end{aligned}$$

This ends the proof of Corollory 2. ■

Proof of Theorem 3.

The proof is straightforward given Lemma 2. It follows from (10) that

$$T\widehat{\boldsymbol{\pi}}^s = \text{diag}\left\{ \frac{1}{T^2}\sum_{t=1}^{T} (z_{1t}^s)^2, \frac{1}{T^2}\sum_{t=1}^{T} (z_{2t}^s)^2, ..., \frac{1}{T^2}\sum_{t=1}^{T} (z_{st}^s)^2 \right\}^{-1} \left(\frac{1}{T}\sum_{t=1}^{T} \mathbf{z}_t^{s\prime}\Delta_s z_t \right) \tag{18}$$

and hence $\widehat{\boldsymbol{\pi}}^s$ is consistent even when outliers and measurement errors are present. The distribution of the $t-$ratios (11) now follow directly from Lemma 2, the continuous mapping theorem, and the formula

$$t_{\pi_i^s} = \frac{T\widehat{\pi}_i^s}{\widehat{\sigma}_u \left(\frac{1}{T^s}\sum_{t=1}^{T} z_{it}^{s2}\right)^{1/2}}.$$

■

Proof of Theorem 4.

Let y_t^* denote the residual series from the projection of y_t on a constant, seasonal dummies, and possibly a time trend (in fact, any combination of these deterministic regressors may be considered). Similarly, we let $D^*(T_0)_t$ denote the projection residuals of the outlier dummy. According to well-known results using the Frisch-Waugh lemma, see Smith and Taylor (1999), y_t^* will tend to appropriately detrended and de-seasonalized Brownian Motion processes $\mathbf{W}^*(r)$ rather than $\mathbf{W}(r)$, the limit of the process y_t after appropriate scaling.

The least squares estimate of θ will be diverge because essentially the regression (17) is a spurious regression. In particular, we have that

$$\widehat{\theta} = \frac{\sum_{t=1}^{T} y_t^* D^*(T_0)_t}{\sum_{t=1}^{T} D^*(T_0)_t^2} = O_p(T^{1/2}) \tag{19}$$

since

$$T^{-1/2} \sum_{t=1}^{T} y_t^* D^*(T_0)_t = T^{-1/2} y_{T_0}^* = \frac{1}{(Ns)^{1/2}} \mathbf{e}_k' \mathbf{S}^*_{\left[\frac{T_0}{T}N\right]} \Rightarrow \frac{\sigma_\varepsilon}{s^{1/2}} \mathbf{e}_k' \mathbf{W}^*(\lambda)$$

and $\sum_{t=1}^{T} D^*(T_0)_t^2 = 1$. In the above expression $\mathbf{e}_k$ is $s \times 1$ and selects row number $k = T_0 - \left[\frac{T_0}{s}\right]$ of the matrix $\mathbf{W}^*(r) = (W_1^*(r), W_2^*(r), ..., W_s^*(r))'$ ([.] takes the integer value of its argument), see the results (6) and (7) presented in section 3. Also, we have defined $\lambda = \frac{T_0}{T}$.

Now the $t-$ratio of ψ for a zero coefficient null will have the distribution

$$\begin{aligned} t_\theta &= \frac{T^{-1/2} \sum_{t=1}^{T} y_t^* D^*(T_0)_t}{\left[T^{-2} \sum_{t=1}^{T} y_t^{*2} + T^{-2} \widehat{\psi}^2 \sum_{t=1}^{T} D^*(T_0)_t^2 - 2T^{-2} \widehat{\psi} \sum_{t=1}^{T} y_t^* D^*(T_0)_t\right]^{1/2} \left[\sum_{t=1}^{T} D^*(T_0)_t^2\right]^{1/2}} \\ &= \frac{T^{-1/2} \sum_{t=1}^{T} y_t^* D^*(T_0)_t}{\left[T^{-2} \sum_{t=1}^{T} y_t^{*2} + o_p(1)\right]^{1/2}} \\ &\Rightarrow \frac{s^{1/2} \mathbf{e}_k' \mathbf{W}^*(\lambda)}{\int_0^1 \mathbf{W}^*(r)' \mathbf{W}^*(r) dr}. \end{aligned}$$

The denominator of the expression follows straightforwardly from Osborn and Rodrigues (1999), Lemma A.1, applied to the 'corrected' data y_t^*. Hence the limiting result of Theorem 4 follows accordingly by use of the continuous mapping theorem.

■

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11. TABLES

Panel A: Measurement errors						
σ^2_η	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
0.4	48	0.097	0.101	0.133	0.015	0.072
	100	0.121	0.133	0.17	0.009	0.103
	200	0.145	0.147	0.157	0.009	0.081
	400	0.155	0.175	0.175	0.010	0.097
0.8	48	0.156	0.145	0.24	0.009	0.135
	100	0.216	0.208	0.299	0.007	0.183
	200	0.224	0.248	0.293	0.006	0.158
	400	0.252	0.260	0.300	0.003	0.181

Panel B: Additive Outliers						
$p\theta^2$	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
0.4	48	0.105	0.095	0.145	0.017	0.087
	100	0.101	0.12	0.168	0.010	0.100
	200	0.143	0.134	0.170	0.009	0.100
	400	0.149	0.162	0.173	0.012	0.093
0.8	48	0.159	0.153	0.242	0.014	0.131
	100	0.189	0.215	0.287	0.01	0.178
	200	0.229	0.222	0.296	0.006	0.186
	400	0.239	0.256	0.287	0.005	0.178

Table 1. Rejection frequencies of the HEGY tests for quarterly data with measurement errors and additive outliers. Data is generated according to the data generating process $z_t = y_t + \frac{\theta}{(1-\alpha B)}\delta_t + \eta_t$ with $\Delta_4 y_t = \varepsilon_t$, $\alpha = 0$, $\eta_t \sim N(0, \sigma^2_\eta)$, and $\varepsilon_t \sim N(0, 1)$. The number of Monte Carlo repetitions was 1000 and no standard errors exceeded 0.015.

Temporary Change Outliers							
$p\theta^2$	α	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
0.4	0.25	48	0.094	0.112	0.149	0.027	0.090
		100	0.102	0.139	0.170	0.019	0.113
		200	0.146	0.148	0.178	0.024	0.109
		400	0.155	0.172	0.180	0.017	0.111
	0.75	48	0.103	0.146	0.182	0.057	0.165
		100	0.114	0.188	0.208	0.079	0.198
		200	0.190	0.193	0.241	0.106	0.213
		400	0.199	0.218	0.239	0.088	0.221
0.8	0.25	48	0.147	0.180	0.243	0.024	0.162
		100	0.193	0.228	0.301	0.035	0.205
		200	0.235	0.232	0.313	0.029	0.217
		400	0.243	0.270	0.308	0.024	0.217
	0.75	48	0.162	0.260	0.293	0.106	0.277
		100	0.228	0.297	0.382	0.194	0.368
		200	0.295	0.314	0.397	0.253	0.402
		400	0.312	0.348	0.417	0.267	0.424

Table 2. Rejection frequencies of the HEGY tests for quarterly data with temporary change outliers. Data is generated according to the data generating process $z_t = y_t + \frac{\theta}{(1-\alpha B)}\delta_t$ with $\Delta_4 y_t = \varepsilon_t$, and $\varepsilon_t \sim N(0,1)$. The number of Monte Carlo repetitions was 1000 and no standard errors exceeded 0.015.

	Probability of a smaller value								
T	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
					Constant				
48	1.29	1.35	1.40	1.49	1.99	2.64	2.86	3.07	3.35
100	1.40	1.47	1.54	1.64	2.12	2.73	2.94	3.14	3.39
200	1.50	1.57	1.65	1.75	2.21	2.80	3.00	3.18	3.44
400	1.57	1.64	1.73	1.83	2.28	2.85	3.05	3.23	3.44
∞	1.63	1.70	1.79	1.90	2.35	2.91	3.11	3.30	3.53
					Constant and seasonal dummies				
48	1.78	1.86	1.93	2.02	2.44	3.10	3.34	3.60	3.91
100	1.84	1.92	2.00	2.09	2.49	3.10	3.33	3.53	3.83
200	1.88	1.96	2.04	2.14	2.54	3.12	3.33	3.55	3.80
400	1.92	1.99	2.07	2.17	2.58	3.16	3.38	3.57	3.85
∞	1.94	2.03	2.10	2.20	2.61	3.20	3.41	3.63	3.89
					Constant, seasonal dummies, and trend				
48	1.78	1.86	1.94	2.03	2.46	3.12	3.38	3.62	3.99
100	1.84	1.93	2.00	2.11	2.52	3.15	3.36	3.60	3.88
200	1.88	1.97	2.05	2.16	2.58	3.17	3.38	3.60	3.86
400	1.92	2.00	2.10	2.20	2.62	3.21	3.42	3.62	3.88
∞	1.95	2.01	2.12	2.23	2.66	3.25	3.46	3.64	3.91

Table 3. Asymptotic and finite sample critical values of the supremum τ statistic for bi-annual data ($s = 2$). The critical values were computed from 20,000 replications of the data generating process $\Delta_2 y_t = \varepsilon_t$ with $\varepsilon_t \sim N(0, 1)$.

	Probability of a smaller value								
T	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
Constant									
48	1.53	1.55	1.69	1.77	2.10	2.61	2.81	3.04	3.27
100	1.68	1.75	1.82	1.91	2.25	2.79	2.99	3.16	3.42
200	1.78	1.85	1.92	2.01	2.37	2.91	3.11	3.30	3.58
400	1.86	1.94	2.01	2.09	2.47	3.02	3.22	3.41	3.66
∞	1.93	2.00	2.09	2.18	2.57	3.12	3.31	3.50	3.76
Constant and seasonal dummies									
48	1.92	2.01	2.08	2.18	2.64	3.37	3.64	3.87	4.24
100	2.03	2.11	2.18	2.28	2.72	3.40	3.65	3.88	4.19
200	2.11	2.19	2.27	2.35	2.78	3.42	3.65	3.90	4.15
400	2.16	2.25	2.32	2.42	2.84	3.47	3.70	3.91	4.16
∞	2.20	2.28	2.36	2.46	2.89	3.51	3.73	3.94	4.19
Constant, seasonal dummies, and trend									
48	1.93	2.02	2.09	2.19	2.64	3.36	3.64	3.90	4.22
100	2.05	2.13	2.20	2.30	2.72	3.39	3.64	3.89	4.16
200	2.12	2.20	2.28	2.37	2.78	3.43	3.67	3.88	4.15
400	2.19	2.26	2.34	2.43	2.85	3.48	3.71	3.92	4.20
∞	2.23	2.31	2.39	2.48	2.89	3.51	3.73	3.92	4.20

Table 4. Asymptotic and finite sample critical values of the supremum τ statistic for quarterly data ($s = 4$). The critical values were computed from 20,000 replications of the data generating process $\Delta_4 y_t = \varepsilon_t$ with $\varepsilon_t \sim N(0, 1)$.

	Probability of a smaller value								
T	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
Constant									
48	1.81	1.88	1.95	2.04	2.45	3.05	3.27	3.50	3.72
100	1.97	2.05	2.12	2.20	2.62	3.23	3.44	3.63	3.84
200	2.11	2.18	2.26	2.35	2.79	3.41	3.63	3.82	4.07
400	2.22	2.30	2.38	2.47	2.91	3.55	3.76	3.97	4.22
∞	2.32	2.40	2.48	2.58	3.03	3.69	3.91	4.11	4.34
Constant and seasonal dummies									
48	2.02	2.10	2.19	2.30	2.80	3.61	3.92	4.18	4.57
100	2.23	2.32	2.40	2.50	3.00	3.74	4.00	4.30	4.59
200	2.39	2.48	2.56	2.67	3.14	3.87	4.12	4.36	4.68
400	2.50	2.59	2.67	2.77	3.24	3.96	4.19	4.44	4.69
∞	2.59	2.69	2.76	2.86	3.34	4.03	4.26	4.50	4.75
Constant, seasonal dummies, and trend									
48	2.02	2.11	2.20	2.31	2.82	3.63	3.91	4.24	4.54
100	2.23	2.33	2.41	2.52	3.01	3.77	4.03	4.26	4.60
200	2.39	2.48	2.56	2.67	3.14	3.87	4.12	4.35	4.68
400	2.51	2.60	2.67	2.77	3.24	3.93	4.17	4.43	4.66
∞	2.59	2.70	2.76	2.86	3.34	4.02	4.26	4.51	4.77

Table 5. Asymptotic and finite sample critical values of the supremum τ statistic for monthly data ($s = 12$). The critical values were computed from 20,000 replications of the data generating process $\Delta_{12} y_t = \varepsilon_t$ with $\varepsilon_t \sim N(0, 1)$.

Single fixed Outlier											
		HEGY-test (non corrected)					HEGY-test (corrected)				
θ	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
5	48	.043	.065	.077	.007	.098	.027	.036	.029	.022	.049
	100	.072	.084	.092	.015	.099	.060	.068	.077	.022	.085
	200	.059	.074	.063	.027	.059	.057	.073	.062	.029	.062
10	48	.138	.146	.337	.000	.357	.022	.020	.015	.036	.038
	100	.202	.230	.365	.000	.329	.037	.044	.036	.041	.055
	200	.159	.174	.220	.005	.178	.071	.093	.088	.028	.078
15	48	.256	.274	.678	.000	.688	.022	.020	.015	.036	.038
	100	.466	.468	.716	.000	.663	.028	.036	.030	.044	.050
	200	.350	.375	.495	.000	.414	.040	.059	.038	.050	.048

Table 6. Size of HEGY tests for $s = 4$ in the presence of single fixed additive outliers. The cases with and without outlier correction are considered.

Notes: The data-generating process is given by $z_t = y_t + v_t$, $t = 1, 2, ...T$, where $\Delta_4 y_t = \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, and $v_t = \theta D(0.5T)_t$. In the model with no correction for outliers the HEGY test was conducted using the regression $\Delta_s z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + u_t$.The tests with outlier correction were based on the regression $\Delta_4 z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + \sum_{i=0}^{4} \gamma_i D(0.5T)_{t-i} + u_t$, where the τ supremum statistic was used to identify the outliers. 1000 replications were used to construct the sizes.

Additive Outliers														
				HEGY-test (non corrected)					HEGY-test (corrected)					
p	θ	α	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
.01	5	0	48	.044	.040	.049	.021	.081	.046	.033	.040	.026	.069
			100	.076	.079	.121	.018	.108	.071	.075	.101	.020	.096
			200	.097	.096	.124	.024	.107	.094	.093	.122	.023	.104
	15	0	48	.142	.158	.259	.022	.278	.055	.055	.055	.035	.080
			100	.373	.390	.475	.016	.462	.068	.078	.070	.041	.079
			200	.506	.529	.657	.016	.604	.089	.091	.103	.041	.101
.05	5	0	48	.119	.125	.209	.021	.261	.093	.107	.166	.023	.206
			100	.278	.297	.411	.014	.402	.250	.279	.376	.012	.366
			200	.385	.363	.535	.008	.464	.383	.364	.538	.006	.468
	15	0	48	.430	.453	.695	.028	.739	.184	.180	.232	.047	.253
			100	.885	.891	.959	.017	.953	.223	.227	.259	.053	.265
			200	.970	.970	.996	.022	.994	.392	.396	.473	.036	.463

Table 7. Size of HEGY tests for $s = 4$ in the presence of additive outliers. The cases with and without outlier correction are considered.

Notes: The data-generating process is given by $z_t = y_t + v_t$, $t = 1, 2, ...T$, where $\Delta_4 y_t = \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, and $v_t = \frac{\theta}{(1-\alpha B)}\delta_t$. δ_t is a Bernoulli variable with parameter p. In the model with no correction for outliers the HEGY test was conducted using the regression $\Delta_4 z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + u_t$.The tests with outlier correction were based on the regression $\Delta_4 z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + \sum_{i=0}^{4} \sum_{j=1}^{q} \gamma_{ij} D(T_0^j)_{t-i} + u_t$ where the τ supremum statistic was used to identify the outliers. 1000 replications were used to construct the sizes.

Temporary Change Outliers													
				HEGY-test (non corrected)					HEGY-test (corrected)				
p	θ	α	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
.01	5	.75	48	.029	.075	.064	.077	.166	.068	.060	.059	.070	.131
			100	.030	.122	.168	.101	.228	.082	.109	.148	.082	.198
			200	.049	.154	.191	.113	.238	.089	.145	.183	.103	.225
	15	.75	48	.046	.210	.217	.285	.368	.268	.064	.075	.077	.120
			100	.083	.449	.533	.549	.602	.465	.190	.230	.188	.258
			200	.233	.607	.737	.737	.802	.703	.326	.391	.354	.415
.05	5	.75	48	.021	.229	.246	.288	.554	.081	.211	.242	.242	.506
			100	.059	.445	.532	.470	.720	.127	.424	.520	.412	.692
			200	.152	.507	.691	.635	.828	.232	.507	.689	.584	.820
	15	.75	48	.062	.536	.571	.694	.879	.339	.358	.380	.418	.568
			100	.335	.909	.974	.955	.989	.612	.772	.840	.800	.876
			200	.810	.996	.998	.996	.999	.908	.922	.977	.959	.981

Table 8. Size of HEGY tests for $s = 4$ in the presence of temporary change outliers. The cases with and without outlier correction are considered.

Notes: See note of table 7.

Additive Outliers													
				Aug-HEGY-test (non corrected)					Aug-HEGY-test (corrected)				
p	θ	α	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
.01	5	0	48	.049	.051	.059	.043	.099	.047	.052	.058	.045	.095
			100	.056	.076	.069	.046	.087	.055	.075	.066	.045	.084
			200	.051	.051	.058	.061	.069	.045	.049	.058	.058	.069
	15	0	48	.142	.158	.245	.039	.284	.091	.097	.137	.047	.181
			100	.228	.231	.287	.037	.292	.100	.130	.141	.045	.153
			200	.073	.090	.118	.038	.114	.049	.060	.088	.044	.087
.05	5	0	48	.118	.139	.188	.026	.242	.112	.131	.181	.024	.227
			100	.139	.138	.185	.023	.179	.131	.136	.180	.020	.173
			200	.070	.061	.080	.033	.077	.068	.061	.075	.032	.074
	15	0	48	.377	.405	.644	.032	.677	.312	.318	.496	.031	.523
			100	.548	.554	.661	.022	.666	.476	.463	.562	.014	.563
			200	.249	.245	.332	.011	.302	.209	.220	.283	.016	.255

Table 9. Size of augmented HEGY tests for $s = 4$ in the presence of additive outliers with and without outlier correction. Lags of $\Delta_4 z_t$ are included in the auxiliary regression.

Notes: The data-generating process is given by $z_t = y_t + v_t$, $t = 1, 2, ...T$, where $\Delta_4 y_t = \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, and $v_t = \frac{\theta}{(1-\alpha B)}\delta_t$. δ_t is a Bernoulli variable with parameter p. In the model with no correction for outliers the augmented HEGY test was conducted using the regression $\Delta_4 z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + \sum_{j=1}^{k} \gamma_j \Delta_4 z_{t-j} + u_t$.The tests with outlier correction were based on the regression $\Delta_4 z_t = \mu_4 + \sum_{j=1}^{s-1} \mu_j D_{jt} + \sum_{j=1}^{4} \pi_j^4 z_{j,t-1}^4 + \sum_{j=1}^{k} \gamma_j \Delta_4 z_{t-j} + \sum_{i=0}^{k+4} \sum_{j=1}^{q} \gamma_{ij} D(T_0^j)_{t-i} + u_t$ where the τ supremum statistic on $S(B)z_t = (1 + B + B^2 + B^3)z_t$ was used to identify the outliers. The lag length for the parametric correction was chosen by testing the significance of the last included lag using a 10% two-tail test based on the asymptotic normality. $k_{\max} = 8 * [T/100]$, except for T=48 where $k_{\max} = 4$. 1000 replications were used to construct the sizes.

Temporary Change Outliers													
				Aug-HEGY-test (non corrected)					Aug-HEGY-test (corrected)				
p	θ	α	T	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$	$t_{\pi_1^4}$	$t_{\pi_2^4}$	$t_{\pi_3^4}$	$t_{\pi_4^4}$	$F_{\pi_3^4,\pi_4^4}$
.01	5	.75	48	.053	.060	.067	.085	.150	.057	.048	.042	.083	.113
			100	.127	.066	.068	.064	.092	.078	.061	.051	.069	.074
			200	.116	.052	.059	.072	.073	.063	.050	.048	.069	.066
	15	.75	48	.074	.183	.209	.263	.338	.063	.063	.061	.095	.111
			100	.282	.152	.239	.161	.303	.114	.052	.043	.066	.062
			200	.339	.061	.106	.108	.138	.099	.024	.028	.049	.043
.05	5	.75	48	.102	.143	.170	.176	.320	.097	.132	.143	.172	.281
			100	.276	.071	.162	.081	.192	.215	.062	.133	.075	.154
			200	.244	.050	.072	.075	.082	.181	.040	.068	.073	.073
	15	.75	48	.110	.487	.486	.584	.760	.096	.377	.345	.444	.555
			100	.437	.404	.497	.442	.624	.320	.312	.362	.343	.462
			200	.544	.123	.186	.212	.284	.324	.098	.135	.166	.197

Table 10. Size of augmented HEGY tests for $s = 4$ in the presence of temporary change outliers with and without outlier correction. Lags of $\Delta_4 z_t$ are included in the auxiliary regression.

Notes: See note of table 9.

Outlier Correction	t_{π_1}	t_{π_2}	$F_{3,4}$	F_{234}	F_{1234}
No	-3.62^{**}	-1.14	4.85	3.67	6.60^{**}
Yes	-2.80	-1.33	5.01	4.07	5.20

Table 11. HEGY tests for the Production Index of the US Transportation Industry 1934:1-1991:2.

Note: This table reports the values of the HEGY tests, quarterly case, with and without the correction of the outliers that have been detected by the τ statistic. All the regressions include an intercept, a deterministic trend and the seasonal dummy variables, whilst the value of lag truncation parameter has been selected by way of the use of the *k(t)* method recommended in Ng and Perron (1995), with *kmax*=24.

Rejection of the unit root null hypothesis using a 1,5 and 10 % significance level is denoted by "***", "**", and "*", respectively.

Outlier Correction	t_{π_1}	t_{π_2}	$F_{3,4}$	$F_{5,6}$	$F_{7,8}$	$F_{9,10}$	$F_{11,12}$
No	-0.99	-1.43	2.25	5.58^{*}	1.84	15.62^{***}	7.97^{**}
Yes	-1.31	-1.80	1.50	1.30	1.65	4.06	2.04

Table 12. HEGY tests for the Brazilian Inflation Rate 1974:1-1993:5.

Note: This Table reports the values of the HEGY tests, monthly case, without and with the correction of the outliers that have been detected by the τ statistic. All the regressions include an intercept, a deterministic trend and the seasonal dummy variables, whilst the value of lag truncation parameter has been selected by way of the use of the *k(t)* method recommended in Ng and Perron (1995), with *kmax*=36 for the non-outlier correction case and kmax=48 when the influence of the outliers is corrected.

Rejection of the unit root null hypothesis using a 1,5 and 10 % significance level is denoted by "***", "**", and "*", respectively.

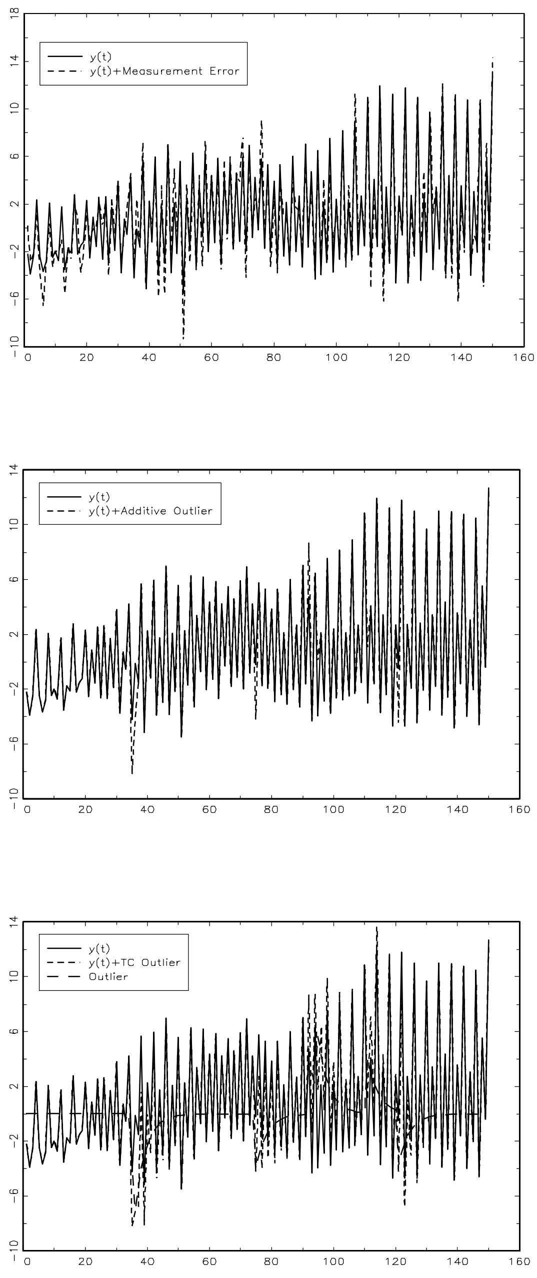

Figure 1. Examples of simulated series: moderate noise. For each panel, the quarterly random walk y_t has been contaminated with measurement errors $\left((\sigma_\eta/\sigma_\varepsilon)^2 = 2\right)$, additive outliers $(p = .05, \theta = 4)$, and temporary change outliers $(p = .05, \theta = 4, \alpha = .75)$, respectively.

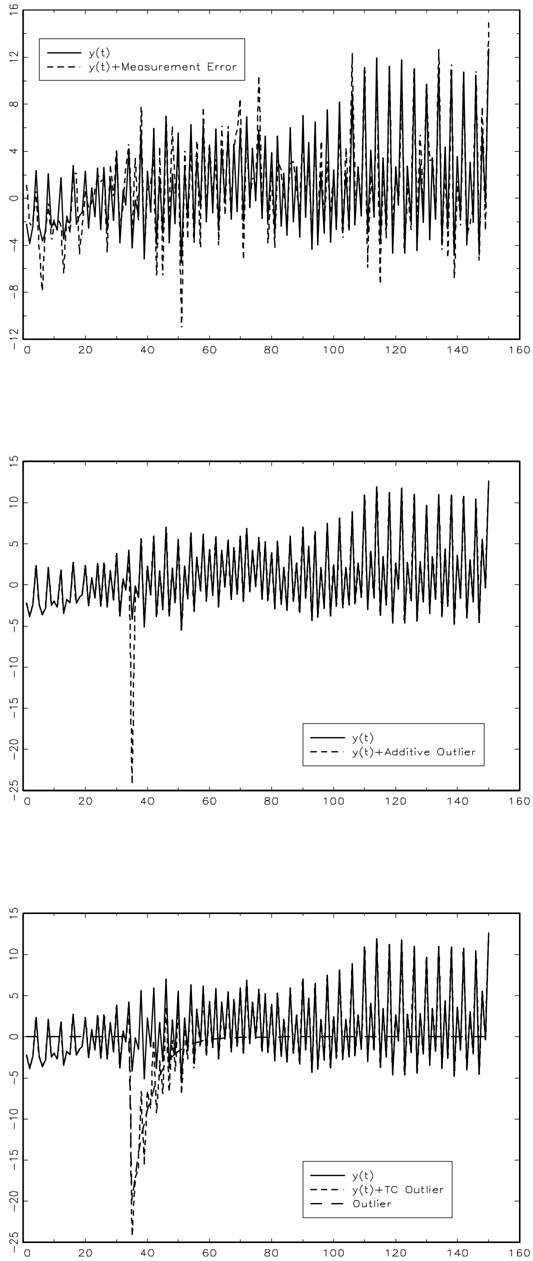

Figure 2. Examples of simulated series: strong noise. For each panel, the quarterly random walk y_t has been contaminated with measurement errors $((\sigma_\eta/\sigma_\varepsilon)^2 = 4)$, additive outliers $(p = .01, \theta = 20)$, and temporary change outliers $(p = .01, \theta = 20, \alpha = .75)$, respectively.

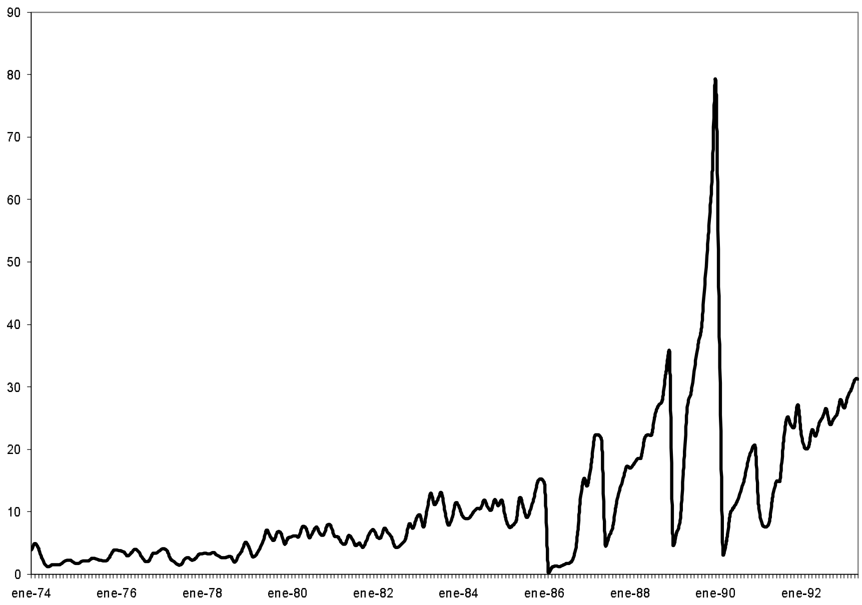

Figure 3. Brazilian inflation rate, Monthly Data 1974:1-1993:5

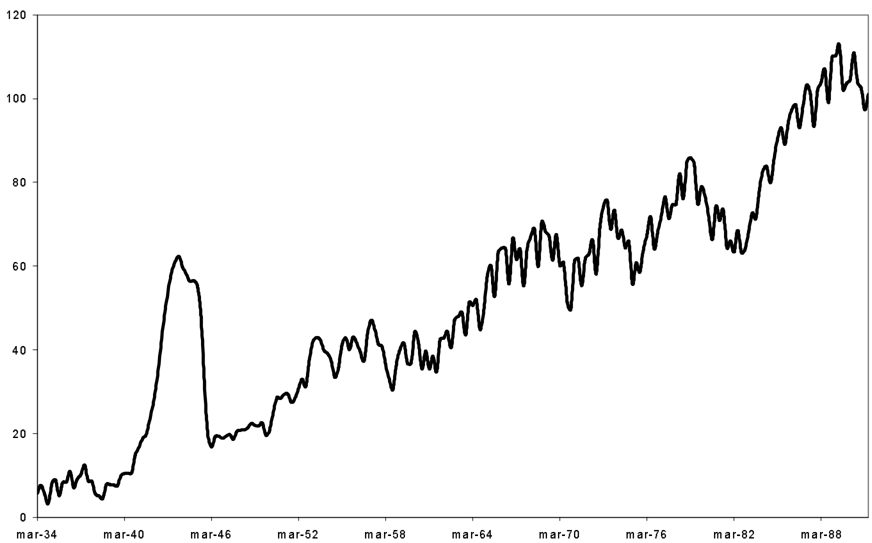

Figure 4. Production Index of the US Transportation Industry, Quarterly Data 1934:1-1991:2