Title

Optimal Spectrum Management in Multiuser Interference Channels

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Abstract

In this paper, we investigate the optimal spectrum management problem in multiuser frequency selective interference channels. First, a simple pairwise condition for FDMA to be optimal is discovered: for any two among all the users, as long as the normalized cross couplings between them two are both larger than or equal to 1/2, orthogonalization between these two users is optimal for every existing user. Therefore, this single condition applies to achieving all Pareto optimal points of the rate region. Furthermore, not only is this condition sufficient, but in symmetric channels, it is also necessary for FDMA to be always optimal. When the normalized cross couplings are less than 1/2, the optimal spectrum management strategy can be a mixture of frequency sharing and FDMA, depending on users' power constraints. We first explicitly solve the sum-rate maximization problem in two user symmetric flat channels by solving a closed form equation, providing the optimal spectrum management with a clear intuition as the optimal combination of flat FDMA and flat frequency sharing. Next, we show that this result leads to a primal domain convex optimization formulation for generalizations to frequency selective channels. Finally, we show that all the general optimization problems with $n \geq 2$ users and an arbitrary weighted sum-rate objective function in non-symmetric frequency selective channels can be solved by primal domain convex optimization with the same methodology.

Contents

I. Introduction

In multiuser communications systems, interference coupling between different users remains a major problem that limits the performance from the perspectives of both a user group and an individual user. A general multiuser interference channel is depicted in Figure 1.1. When the interference signal is strong enough to decode, interference cancellation techniques [1] [8] [12] [16] can be applied. However, to implement interference cancellation, not only is the complexity high, but also the users need to have prior knowledge of each other's transmission schemes such as code books. In this paper, we make the assumption that the receivers do not apply interference cancellation. In this case, interference is treated as noise, and the interference limited nature of a multiuser communications system becomes even harsher.

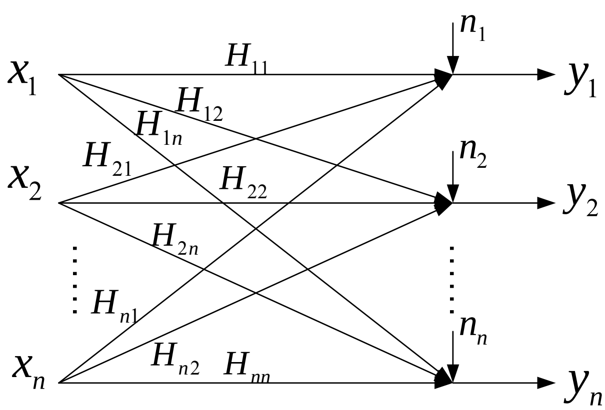


Fig. 1.1 Multiuser Interference Channel

We consider the scenario of multiple multicarrier communications systems contending in a common frequency band. There may sometimes be practical reasons to channelize the resources in some other fashion, e.g. in time. Here, we regard any such alternatives as equivalent to channelizing in the frequency domain [9] [15]. We investigate the optimal spectrum and power allocation that achieves an arbitrary

Pareto optimal point of the rate region. In this paper, the terms *spectrum management*, *spectrum and power allocation*, and *co-existence strategy* are interchangeable for purposes of this discussion.

There are essentially two strategies for multiple users to co-exist:

1. To avoid each other in the frequency domain, i.e. *FDMA.*

2. To occupy a common band at the same time, i.e. *frequency sharing*.

When the cross coupling gains are strong, users can co-exist in an FDMA manner so that there is no mutual interference. When the cross coupling gains are weak, they can share the same bandwidth, while the mutual interference is insignificant. This basic idea of interference management is applied in cellular networks by frequency re-use. It can be generalized to any wireless communication networks: the more densely the frequency-reused users are packed without loss of their rates, the higher is the spatial throughput achieved. As the cross coupling grows from being extremely strong to extremely weak, the preferable co-existence strategies intuitively shift from complete avoidance (FDMA) to pure frequency sharing. The characterization of the *optimal* co-existence strategies under *arbitrary* cross coupling conditions between the two extremes is the key problem this paper focuses on.

We start from one extreme of the interference condition which is the strong coupling scenario, and investigate the weakest interference condition under which FDMA is still guaranteed to be optimal. In the literature, a *pairwise* coupling condition for FDMA to be optimal is proposed, and it applies to *all Pareto optimal points* of the rate region [7]. By pairwise we mean that whether two users should avoid each other only depends on the interference condition between those two users. For one typical Pareto optimal point which is the *sum-rate* maximization point, the required coupling strengths for FDMA to be optimal are further lowered [10]. (It is further claimed that these lowered coupling conditions also apply to the weighted sum-rate maximization problem [11]). However, this condition is a *group-wise* one, meaning that the couplings between all users are required to be strong.

We relax the previous results to a simple pairwise condition for FDMA to be optimal: for any two users, as long as the two normalized cross couplings between them are both larger than or equal to 1/2, FDMA between these two users is optimal for *every existing user*. Thus, this condition applies to achieving all Pareto optimal points of the rate region. We also obtain an interesting related result: no matter what the cross coupling conditions are, from any individual user's point of view, it always prefers its interferers, i.e. the other users, to coexist in an FDMA manner. (Notice that from the other users' points of view, however, an FDMA among themselves is not necessarily always preferable, since the couplings among them might be weak.)

With the proposed condition, the problem space is divided into two parts along the axis of interference coupling. When the interference coupling is less than 1/2 in symmetric channels, we provide a precise characterization of the non-empty power constraint region within which frequency sharing between two users leads to a higher rate than an FDMA between them. That is to say, the proposed condition for FDMA to be always optimal is not only sufficient in all channels, but also necessary at least in symmetric channels.

With the interference coupling less than 1/2, the form of the optimal spectrum management strategy depends on the power constraints of the users. Toward one extreme, with users' power constraints approaching infinity, FDMA outperforms frequency sharing. This is because frequency sharing is interference limited and leads to a finite upper bound on the achievable rate, whereas FDMA avoids the interference completely and allows the rate to go logarithmically to infinity. Toward the other extreme, with users' power constraints approaching zero, frequency sharing becomes (if not exactly optimal) infinitely close to optimal. This is because the interference becomes negligible compared to the noise level, and each user should simply perform waterfilling over the noise across the whole band, resulting in frequency sharing of all users. Generally, the optimal spectrum management can be FDMA, or frequency sharing, or a mixture of the two. Finding the optimal spectrum and power allocation that maximizes a weighted sum-rate is in the form of a non-convex optimization problem [18]. Although a non-convex optimization is hard to solve, the Lagrangian dual problem is always a convex optimization [2]. It is shown in the literature that the duality gap actually goes to *zero* when the number of sub-channels goes to infinity [19]. This fundamentally justifies the asymptotic optimality of solving the problem in the dual domain, and many spectrum balancing algorithms using dual methods have been developed [3] [4] [13] [19].

We approach this non-convex optimization problem from the primal perspective, finding explicit characterizations of the optimal spectrum management. We develop a new method of solving this non-convex optimization by formulating it into an equivalent *primal domain convex optimization*. We start with the sum-rate maximization problem in two-user symmetric flat channels. We provide the solution, namely the optimal spectrum and power allocation scheme by *solving a closed form equation.* Both the method we use and the solution we get have a clear intuition of combining FDMA and frequency sharing in an optimal way, and this solution is obtained analytically instead of by waiting for the resulting spectrum and power allocation that an algorithm converges to. With this solution, the optimal spectrum and power allocation for any frequency selective channel can be naturally obtained by a primal domain convex optimization.

The key idea of the above method is a two-step procedure:

1. Explicitly solve the flat channel case.

2. Obtain the solution of frequency selective channels by forming a primal

domain convex optimization based on the solutions of flat channels.

By generalizing this method, we show that all the general optimization problems with $n \geq 2$ users and an arbitrary weighted sum-rate objective function in non-symmetric frequency selective channels can be solved by formulating an equivalent primal domain convex optimization. Its solution then provides the performance limit for all practical algorithms. In retrospect, the methodology we provide shares some common insight with the time sharing condition discussed in [19].

Table 1 summarizes the various forms of the multiuser interference channel coexistence problems, the prior art, and in which sections we present solutions that improve upon these prior results. In the conclusion, we suggest research directions for the problems in the table that are not addressed in this paper.

TABLE 1
PROBLEMS, PRIOR ARTS, AND RELATED SECITIONS IN THIS PAPER

<table>
<tr><th colspan="4">Problems</th><th>Prior Art</th><th>Our Results</th></tr>
<tr><td colspan="4">Spectrum Management in Cooperative Scenarios</td><td colspan="2"></td></tr>
<tr><td rowspan="5"></td><td rowspan="2">Strong Interference Scenarios</td><td colspan="2">Conditions for FDMA schemes to be optimal</td><td>[7][10][11]</td><td>Section III</td></tr>
<tr><td colspan="2">Finding optimal schemes with FDMA constraints</td><td>[17][10]</td><td></td></tr>
<tr><td rowspan="3">General Interference Scenarios</td><td rowspan="2">Continuous Frequency Scenarios</td><td>Primal domain solution:
Convex formulation and its solution as the optimal combination of FDMA and frequency sharing</td><td></td><td>Section IV
Section V</td></tr>
<tr><td>Dual domain methods</td><td>[19]</td><td></td></tr>
<tr><td colspan="2">Discrete Frequency Scenarios:
Approximation Algorithms</td><td>[3][4][5]
[13][14][19]</td><td></td></tr>
<tr><td colspan="4">Spectrum Sharing in Non-cooperative Scenarios: Nash Equilibriums</td><td>[6][7]18]</td><td></td></tr>
<tr><td colspan="4">Incorporating Interference Cancellation</td><td>[8][12] [1][16]</td><td></td></tr>
</table>

II. Channel Model and Two Basic Co-existence Strategies

A. Interference Channel Model

As depicted in Figure 1.1, an n user interference channel is modeled by

$$y_i = H_{ii} x_i + \sum_{j \neq i} x_j H_{ji} + n_i, \quad i = 1, 2, \ldots, n.$$

where x_i is the transmitted signal of user i, and y_i is the received signal of user i including additive Gaussian noise n_i, (a user corresponds to a pair of transmitter and receiver). H_{ii} is the direct channel gain from the transmitter to the receiver of user i. H_{ji} is the cross coupling gain from the transmitter j to the receiver i. We assume the transmission is over the interference channel without interference cancellation: for every user i, only the signal from its transmitter is decodable, and interference from other users is treated as noise. We assume that the channel is frequency selective over the band (f_1, f_2). The channel gains H_{ii} and H_{ji} are denoted as $H_{ii}(f)$ and $H_{ji}(f),\ f \in (f_1, f_2)$. Denote the transmit power spectrum density of user i by $P_i(f)$, and the noise power spectrum density at receiver i by $\sigma_i(f)$.

We have the achievable rate for user i:

$$R_i = \int_{f_1}^{f_2} \log\left(1 + \frac{P_i(f)\left|H_{ii}(f)\right|^2}{\sigma_i(f) + \sum_{j \neq i} P_j(f)\left|H_{ji}(f)\right|^2}\right) df.$$

Normalizing the channel gains and noise power by the direct channel gains, we have

$$R_i = \int_{f_1}^{f_2} \log\left(1 + \frac{P_i(f)}{N_i(f) + \sum_{j \neq i} P_j(f)\alpha_{ji}(f)}\right) df \,,$$

where $N_i(f) \triangleq \frac{\sigma_i(f)}{|H_{ii}(f)|^2}$ and $\alpha_{ji}(f) \triangleq \frac{|H_{ji}(f)|^2}{|H_{ii}(f)|^2}$.

Thus, finding the optimal coexistence strategy corresponds to optimizing over the power and spectrum allocation functions $P_i(f),\ i = 1, 2, ..., n$.

B. Two Basic Co-existence Strategies and One Basic Transformation

There are essentially two co-existence strategies for users to reside in a common band: frequency sharing and FDMA. We introduce two basic forms of these two strategies: *Flat Frequency Sharing* and *Flat FDMA*, both defined in flat channels. We will see that these two basic strategies are the building blocks of all non-flat co-existence strategies in frequency selective channels.

Consider a flat channel in the frequency band $(f_1,\ f_2)$:

$$N_1(f) = n_1,\ N_2(f) = n_2,\ \ \alpha_{21}(f) = \alpha_{21}, \alpha_{12}(f) = \alpha_{12},\ \ \forall f \in (f_1,\ f_2).$$

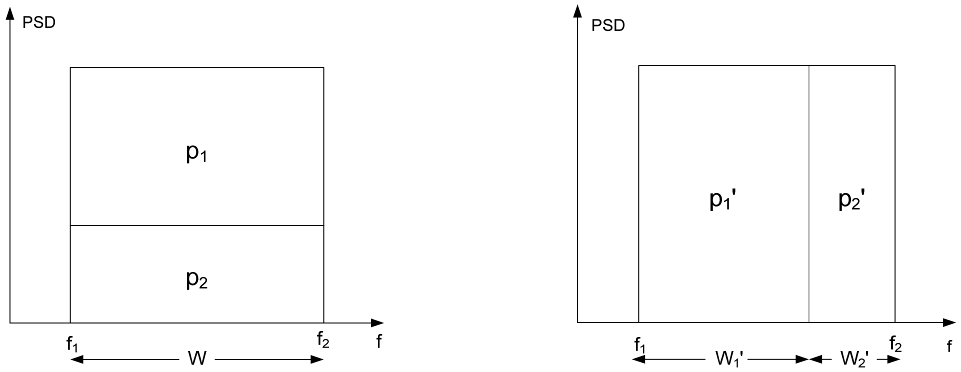
A flat frequency sharing scheme of two users is defined as any power allocation in the form of

$$P_1(f) = p_1,\ P_2(f) = p_2,\ \forall f \in (f_1,\ f_2)\,.$$

A flat FDMA scheme of two users is defined as any power allocation in the form of

$$\begin{cases} P_1(f)P_2(f) = 0 \\ P_1(f) + P_2(f) = p \end{cases},\quad \forall f \in (f_1, f_2)$$

Illustrations of the power allocations of these two basic co-existence strategies are depicted in Figure 2.1.



Flat Frequency Sharing (Before flat FDMA re-allocation) Flat FDMA (After flat FDMA re-allocation)

Fig. 2.1 power allocations of flat frequency sharing and flat FDMA, also an illustration of flat FDMA re-allocation

Next, we introduce a basic transformation from flat frequency sharing to flat FDMA: *flat FDMA re-allocation*. A flat FDMA re-allocation is defined to be the following scheme that transforms a flat frequency sharing to a flat FDMA:

user 1 re-allocates all of its power within a sub-band $W_1' = \frac{p_1}{p_1 + p_2} W$ with a flat power spectral density (PSD) $p_1' = p_1 + p_2$; user 2 re-allocates all of its power within another disjoint sub-band $W_2' = \frac{p_2}{p_1 + p_2} W$ with the same flat PSD $p_2' = p_1 + p_2$.

The power allocations before and after a flat FDMA re-allocation are also illustrated in Figure 2.1. Clearly, the total power of each user does not change after this re-allocation, i.e. $P_1 = p_1 W = p_1' W_1'$, $P_2 = p_2 W = p_2' W_2'$.

Similarly, we define flat frequency sharing schemes, flat FDMA schemes, and flat FDMA re-allocation in n*-user* flat channel cases.

The reasons we introduce and investigate these three basic concepts are as follows:

1. Flat frequency sharing and flat FDMA are the building blocks of all non-flat cases: with an arbitrary multiuser spectrum and power allocation in a frequency selective channel, by looking at the infinitesimal sub-channels around every frequency point, the channel becomes flat, and the power allocation scheme becomes either flat frequency sharing or flat FDMA.

2. Flat FDMA re-allocation has the key property of *power invariance*. It serves as a powerful tool while comparing frequency sharing schemes with FDMA schemes under the same power constraint.

III. The Conditions for the Optimality of FDMA

In this section, we investigate the conditions under which the optimal spectrum and power allocation is FDMA in the general communication environments. Instead of working with a specific optimization goal, we show that our results apply to all Pareto optimal points of the achievable rate region.

We first show a coupling condition under which FDMA is optimal within a group of strongly coupled users. In real communication networks, however, there are usually users not strongly enough (maybe just moderately) coupled with some other users. For these users outside the strongly coupled group, we show that they always benefit from an FDMA within the strongly coupled group. Interestingly, we show that this result actually does not require the strongly coupled condition within this group. With these results, a simple *pairwise* condition is naturally obtained: for any two users, as long as the normalized cross couplings between them are both larger than or equal to 1/2, *every existing user* will benefit from an FDMA between these two users. In this section, we show the sufficiency of this condition. In Section IV, we show that it is also necessary at least in symmetric channels.

A. The Optimality of FDMA within Strongly Coupled Users

In this section, we prove a sufficient condition for the n-user scenarios under which the optimal spectrum and power allocation must be FDMA. This condition requires that between every pair of users, the cross couplings normalized by the direct channel gains must be stronger than a threshold. We begin with two-user flat channels, and extend the results to n-user frequency selective channels.

Theorem 1 Consider a two-user flat interference channel: $N_1(f) = n_1$, $N_2(f) = n_2$, $\alpha_{21}(f) = \alpha_{21}, \alpha_{12}(f) = \alpha_{12}$, $\forall f \in (f_1, f_2)$. Suppose the two users co-exist in a flat frequency sharing manner: $P_1(f) = p_1,\ P_2(f) = p_2,\ \forall f \in (f_1,\ f_2)$. If

$$\alpha_{12} \ge \frac{1}{2} \ \ and \ \ \alpha_{21} \ge \frac{1}{2},$$

then with a flat FDMA power re-allocation, *both* users' rates will be higher or kept the same.

Before proving Theorem 1, we provide the following lemma first:

Lemma 1 If $f(x) = \frac{1}{x}\log(\frac{c+x}{c-x})$, $c > 1$, then $f(1) \ge f(x)$, $\forall x \in (0,\ 1]$

Proof:

We want to show $f'(x) = \dfrac{2cx - (c^2 - x^2)\log(\frac{c+x}{c-x})}{x^2(c^2 - x^2)} \ge 0,\ x \in (0,1]$

Since $c > 1$, it is equivalent to show $\dfrac{2cx}{c^2 - x^2} \ge \log(\dfrac{c+x}{c-x}),\ x \in (0,1]$.

Let $g(x) = \dfrac{2cx}{c^2 - x^2},\ x \in [0,\ 1]$, and $h(x) = \log(\dfrac{c+x}{c-x}),\ x \in [0,\ 1]$. We have

$$g'(x) = \frac{2c(c^2 + x^2)}{(c^2 - x^2)^2},\ \ h'(x) = \frac{2c}{c^2 - x^2}$$

We see that $g'(x) = h'(x)\dfrac{c^2 + x^2}{c^2 - x^2} \ge h'(x)$.

Since $g(0) = h(0) = 0$, we have $g(x) \ge h(x),\ x \in [0,\ 1]$, i.e.

$$\frac{2cx}{c^2 - x^2} \ge \log(\frac{c+x}{c-x}),\ \ x \in (0,1]$$

$$\Rightarrow f'(x) \ge 0,\ \ x \in (0,1]$$

Therefore, $f(1) \ge f(x)$, $\forall x \in (0,\ 1]$. ■

Proof of Theorem 1:

As shown in Figure 3.1, at the receiver of user 1, the received PSD is the sum of p_1, $\alpha_{21}p_2$, and n_1. Similarly at the receiver of user 2, the received PSD is the sum of

p_2, $\alpha_{12} p_1$, and n_2.

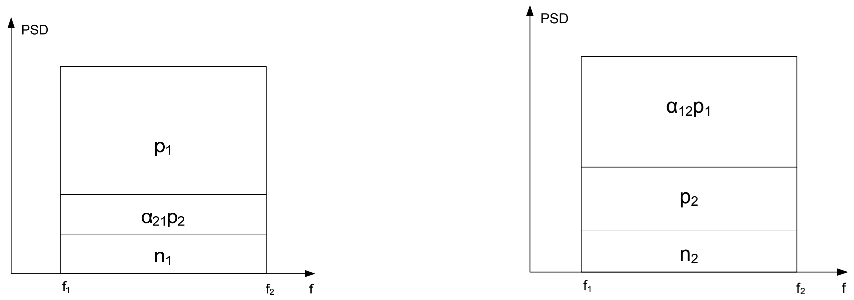


Fig. 3.1 The PSD composition at receiver 1 and receiver 2

The rate of user 1 is

$$R_1 = \int_{f_1}^{f_2} \log\left(1+\frac{p_1}{n_1 + p_2\alpha_{21}}\right)df = (f_2 - f_1)\log\left(1+\frac{p_1}{n_1 + p_2\alpha_{21}}\right) = W\log\left(1+\frac{p_1}{n_1 + p_2\alpha_{21}}\right)$$

where $W = f_1 - f_2$ is the bandwidth. Similarly, $R_2 = W\log\left(1+\frac{p_2}{n_2 + p_1\alpha_{12}}\right)$.

With a *flat FDMA power re-allocation*, we have $W_1' = \frac{p_1}{p_1 + p_2}W$, $p_1' = p_1 + p_2$, $W_2' = \frac{p_2}{p_1 + p_2}W$, $p_2' = p_1 + p_2$. The power allocations before and after this re-allocation are depicted in Figure 2.1.

Now, we prove that after the flat FDMA re-allocation, both user's rates can only be higher or the same. It is sufficient to prove it for user 1, since user 2's case is symmetric to user 1. Denote user 1's rate after re-allocation by

$$R_1' = W_1'\log\left(1+\frac{p_1'}{n_1}\right) = \frac{p_1}{p_1 + p_2}W\log\left(1+\frac{p_1 + p_2}{n_1}\right)$$

Notice that

$$R_1 = W\log\left(1+\frac{p_1}{n_1+p_2\alpha_{21}}\right) = W\log\left(1+\frac{\frac{p_1}{p_1+p_2}}{\frac{n_1}{p_1+p_2}+\frac{p_2}{p_1+p_2}\alpha_{21}}\right) = W\log\left(1+\frac{\hat{p}_1}{\hat{n}_1+\hat{p}_2\alpha_{21}}\right),$$

$$R_1' = \frac{p_1}{p_1+p_2}W\log\left(1+\frac{p_1+p_2}{n_1}\right) = \frac{p_1}{p_1+p_2}W\log\left(1+\frac{\frac{p_1}{p_1+p_2}+\frac{p_2}{p_1+p_2}}{\frac{n_1}{p_1+p_2}}\right) = \hat{p}_1 W\log\left(1+\frac{1}{\hat{n}_1}\right)$$

where the power and noise normalized by the sum-power are

$$\hat{p}_1 \triangleq \frac{p_1}{p_1+p_2},\ \hat{p}_2 \triangleq \frac{p_2}{p_1+p_2}, \hat{n}_1 \triangleq \frac{n_1}{p_1+p_2}.$$

WLOG, we will use the normalized power and noise in the remainder of the proof. Since n_1 can be *arbitrarily* chosen in our problem, using normalized terms is equivalent to adding the assumption that $p_1+p_2=1$. With this assumption, we can re-express the rates as $R_1 = W\log\left(1+\frac{p_1}{n_1+(1-p_1)\alpha_{21}}\right)$, $R_1' = p_1 W\log\left(1+\frac{1}{n_1}\right)$.

We want to show that if $\alpha_{21} \ge \frac{1}{2}$, we have $R_1' \ge R_1$. Since $R_1\big|_{\alpha_{21}=\frac{1}{2}} \ge R_1\big|_{\alpha_{21}>\frac{1}{2}}$, it is sufficient to show that $\alpha_{21}=\frac{1}{2} \Rightarrow R_1' \ge R_1$.

With $p_1+p_2=1,\ \alpha_{21}=\frac{1}{2}$,

$$R_1' \ge R_1 \Leftrightarrow p_1 W\log\left(1+\frac{1}{n_1}\right) \ge W\log\left(1+\frac{p_1}{n_1+\frac{1-p_1}{2}}\right)$$

$$\Leftrightarrow \log\left(\frac{n_1+1}{n_1}\right) \ge \frac{1}{p_1}\log\left(\frac{2n_1+1+p_1}{2n_1+1-p_1}\right)$$

Notice that $\log\left(\frac{n_1+1}{n_1}\right) = \frac{1}{1}\log\left(\frac{2n_1+1+1}{2n_1+1-1}\right)$. Define function $f(x) = \frac{1}{x}\log(\frac{c+x}{c-x})$, where $c = 2n_1+1$. Thus, $R_1' \ge R_1 \Leftrightarrow f(1) \ge f(p_1)$.

Since we assume $p_1 + p_2 = 1$, $p_1 \in (0,1]$. From lemma 1, $f(1) \ge f(p_1)$. Thus, we conclude that

$$\alpha_{21} \ge \frac{1}{2} \Rightarrow R_1' \ge R_1 .$$

By symmetry, we also have $\alpha_{12} \ge \frac{1}{2} \Rightarrow R_2' \ge R_2$.

That is to say, when $\alpha_{12} \ge \frac{1}{2}, \alpha_{21} \ge \frac{1}{2}$, a flat FDMA power re-allocation leads to rates higher than or equal to the flat frequency sharing for both users. ■

Theorem 1 can be generalized to the n-user case as the following corollary.

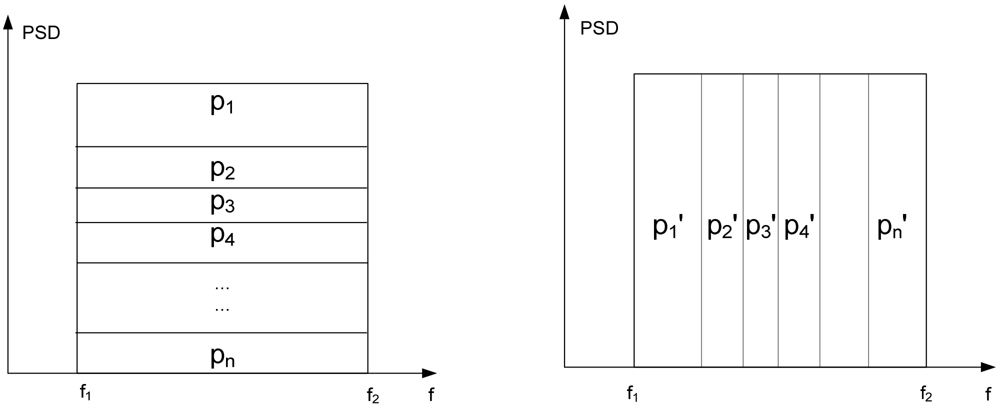
Corollary 1.1 Consider an *n*-user flat interference channel: $N_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the n users co-exist in a flat frequency sharing manner: $P_i(f) = p_i, \forall f \in (f_1, f_2)$, $i = 1,2,3,...,n$. If

$$\alpha_{ji} \ge \frac{1}{2}, \forall j \ne i ,$$

then with a flat FDMA power re-allocation scheme, *all* users' rates will be higher or kept the same.

Proof:

Suppose we have n users. We first generalize the flat FDMA power re-allocation to n users: Each user i re-allocates all its power within a sub-band $W_i' = \frac{p_i}{\sum_{j=1}^{n} p_j} W$ disjoint from all other users' frequency occupation, with a flat PSD $p_i' = \sum_{j=1}^{n} p_j$. The power before and after re-allocation are depicted in Figure 3.2.



Before re-allocation After re-allcoation

Fig. 3.2 PSDs before and after flat FDMA re-allocation of flat frequency sharing

Clearly, the total power of each user does not change after re-allocation.

Next, we show that every user's rate can only increase or be the same after this power re-allocation. It is sufficient to show $R_1' \geq R_1$, with $\alpha_{ji} \geq \frac{1}{2},\ \forall j \neq i$, the normalization assumption $\sum_{i=1}^{n} p_i = 1$, and R_1' and R_1 defined as follows:

$$R_1 = W \log\left(1 + \frac{p_1}{n_1 + \sum_{j=2}^{n} p_j \alpha_{j1}}\right)$$ is user 1's rate before the FDMA re-allocation.

$$R_1' = W_1' \log\left(1 + \frac{p_1'}{n_1}\right) = p_1 W \log\left(1 + \frac{1}{n_1}\right)$$ is user 1's rate after the FDMA re-allocation.

From the results we derived in the two user case (Theorem 1),

$$R_1' \geq W \log\left(1 + \frac{p_1}{n_1 + \frac{1}{2}(1 - p_1)}\right) = W \log\left(1 + \frac{p_1}{n_1 + \frac{1}{2}\sum_{j=2}^{n} p_j}\right) \geq W \log\left(1 + \frac{p_1}{n_1 + \sum_{j=2}^{n} p_j \alpha_{j1}}\right) = R_1$$

The second inequality comes from $\alpha_{j1} \geq \frac{1}{2},\ j = 2,3,\ldots,n$.

From symmetry, we have $R_i' \geq R_i, \ \forall i = 1, 2, ..., n$

That is to say, a flat FDMA power re-allocation (Figure 3.2) leads to higher rates than the flat frequency sharing for all the users. ■

This sufficient condition can also be generalized to frequency selective channels.

Corollary 1.2 Consider an n-user frequency selective interference channel. We make one assumption for pure mathematical convenience which does not affect the generality of the problem: $\forall i, j, \alpha_{ji}(f)$ and $N_i(f)$ are uniformly continuous in the band (f_1, f_2). Suppose we have an arbitrary spectrum and power allocation scheme $P_i(f), \ i = 1, 2, 3, ..., n$ with some frequency sharing (overlapping) in the band (f_1, f_2). Again we assume all $P_i(f)$ are uniformly continuous in the band (f_1, f_2). If

$$\alpha_{ji}(f) \geq \frac{1}{2}, \forall j \neq i, \ \forall f \in (f_1, f_2),$$

then we can always find an FDMA power re-allocation scheme $\tilde{P}_i(f), \ i = 1, 2, ..., n$ satisfying $\int_{f_1}^{f_2} \tilde{P}_i(f)df = \int_{f_1}^{f_2} P_i(f)df, \ \ i = 1, 2, ..., n$ with which *all* user's rates are higher or kept the same.

Proof:

An arbitrary power and spectrum allocation with frequency sharing is depicted in Figure 3.3.

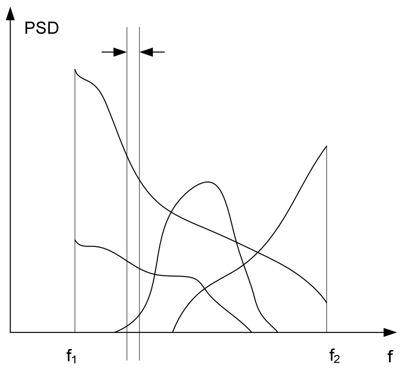


Fig. 3.3 An arbitrary n user power allocation

Now we divide the band (f_1, f_2) into infinitely many frequency bins. From uniform continuity, let all frequency bins have sufficiently infinitesimal widths so that within any bin $(f_1', f_2'), \left|f_1' - f_2'\right| \to 0$ we have the following:

$\forall f \in (f_1', f_2'), \forall i \neq j$

1. channel gains become flat: $\alpha_{ji}(f) \to \alpha_{ji}(f_0) \geq \frac{1}{2}$, for some $f_0 \in (f_1', f_2')$

2. noise power become flat: $N(f) \to N(f_0)$, for some $f_0 \in (f_1', f_2')$

3. users' power become flat: $P_i(f) \to P_i(f_0)$, for some $f_0 \in (f_1', f_2')$

Next we construct an FDMA power re-allocation by constructing an FDMA power re-allocation for every bin in (f_1, f_2): if there's no user or only one user's power in it, we do nothing; Otherwise we apply Corollary 1.1 within this bin and create an FDMA power re-allocation for all users within this bin. The three conditions we just obtained from letting every bin sufficiently infinitesimal enables that theorem 1 can be successfully applied.

With this procedure, we finally obtain an FDMA power re-allocation scheme of all users satisfying $R_i' \geq R_i,\ i = 1, 2, ..., n$. In other words, the rate of every user will be higher or unchanged after this FDMA power re-allocation. ■

We conclude with re-stating Corollary 1.2 as follows: pick any sub-band (f_1', f_2'), as long as all the users having power within this sub-band are strongly coupled - precisely characterized by $\alpha_{ji}(f) \ge \frac{1}{2}, \forall j \ne i, \forall f \in (f_1', f_2')$, then for any power allocation scheme having frequency sharing happening anywhere within this sub-band, there always exists an FDMA power re-allocation scheme (with the total allocated power unchanged for each user) that leads to a rate higher than or equal to the original sharing scheme *for every existing user.*

B. FDMA Within a Subset of Users Benefits All Other Users

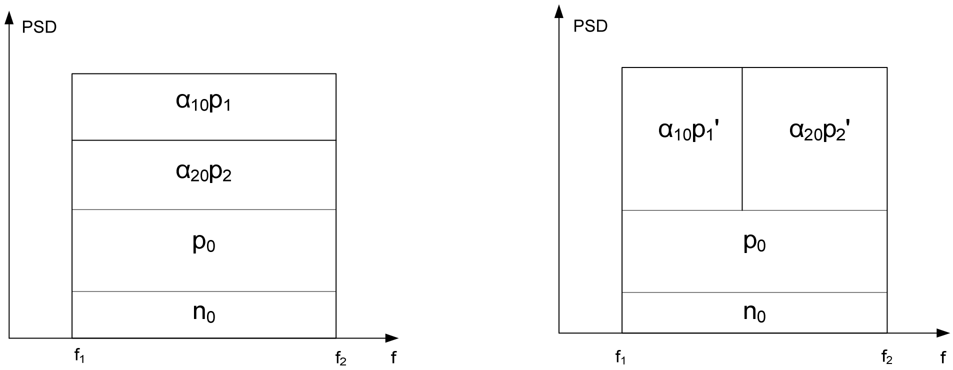
We have seen that by properly separating strongly coupled users (with $\alpha_{ji}(f) \ge \frac{1}{2}$) to orthogonal channels (FDMA), every individual user among them will have a rate higher than or equal to the rate of any frequency sharing (overlapping) scheme. In this section, we show that an FDMA among a group of users not only benefits those users inside the group if they are strongly coupled, but also benefits *every other user outside the group* regardless of all the coupling conditions. This result completes the fundamental truth that the optimal spectrum and power allocation must have all the strongly coupled users (among all the users) orthogonally separated in the frequency band.

We begin with two-interferer flat channels, and extend the results to n-interferer frequency selective channels.

Theorem 2 Consider a three-user (one user + two interferers) flat channel: $N_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the three users co-exist in a flat frequency sharing manner: $P_i(f) = p_i, \forall f \in (f_1, f_2), i = 0,1,2$. From user 0's perspective, a flat FDMA power re-allocation of its two interferers, namely user 1 and user 2, always leads to a rate higher than or equal to the original rate for user 0.

Proof:

At the receiver of user 0, the received PSDs before and after the flat FDMA power re-allocation of its interferers are depicted in Figure 3.4.



Before re-allocation After re-allocation

Fig. 3.4 PSD compositions at receiver 0 before and after the flat FDMA re-allocation of user 1 and user 2

Thus user 0's rate before re-allocation is $R = W\log\left(1+\dfrac{p_0}{\alpha_{10}p_1+\alpha_{20}p_2+n_0}\right)$.

User 0's rate after re-allocation is

$$R' = W_1'\left(1+\frac{p_0}{\alpha_{10}p_1'+n_0}\right)+W_2'\left(1+\frac{p_0}{\alpha_{20}p_2'+n_0}\right)$$

$$=\frac{p_1}{p_1+p_2}W\left(1+\frac{p_0}{\alpha_{10}(p_1+p_2)+n_0}\right)+\frac{p_2}{p_1+p_2}W\left(1+\frac{p_0}{\alpha_{20}(p_1+p_2)+n_0}\right)$$

Notice that

$$R = W\log\left(1+\frac{1}{\left(\frac{p_1+p_2}{p_0}\alpha_{10}\right)\frac{p_1}{p_1+p_2}+\left(\frac{p_1+p_2}{p_0}\alpha_{20}\right)\frac{p_2}{p_1+p_2}+\frac{n_0}{p_0}}\right)$$

$$= W\log\left(1+\frac{1}{\hat{\alpha}_{10}\hat{p}_1+\hat{\alpha}_{20}\hat{p}_2+\hat{n}_0}\right)$$

$$R' = \frac{p_1}{p_1+p_2}W\left(1+\frac{1}{\left(\frac{p_1+p_2}{p_0}\alpha_{10}\right)+\frac{n_0}{p_0}}\right)+\frac{p_2}{p_1+p_2}W\left(1+\frac{1}{\left(\frac{p_1+p_2}{p_0}\alpha_{20}\right)+\frac{n_0}{p_0}}\right),$$

$$= \hat{p}_1 W\left(1+\frac{1}{\hat{\alpha}_{10}+\hat{n}_0}\right)+\hat{p}_2 W\left(1+\frac{1}{\hat{\alpha}_{20}+\hat{n}_0}\right)$$

where the normalized power, noise and coupling gains are

$$\hat{p}_0 \triangleq 1,\ \hat{p}_1 \triangleq \frac{p_1}{p_1+p_2},\ \hat{p}_2 \triangleq \frac{p_2}{p_1+p_2},\ \hat{\alpha}_{10} \triangleq \frac{p_1+p_2}{p_0}\alpha_{10},\ \hat{\alpha}_{20} \triangleq \frac{p_1+p_2}{p_0}\alpha_{20},\ \hat{n}_0 \triangleq \frac{n_0}{p_0}$$

WLOG, we will use the normalized power, channel gains, and noise in the remainder of the proof. Since $\alpha_{10}, \alpha_{20}, n_0$ can be *arbitrarily* chosen in our problem, using normalized terms is equivalent to adding the assumption $\begin{cases} p_0 = 1 \\ p_1 + p_2 = 1 \end{cases}$. With this assumption, we can re-express the rates as

$$R = W\log\left(1+\frac{1}{\alpha_{10}p_1+\alpha_{20}(1-p_1)+n_0}\right)$$

$$R' = p_1 W\left(1+\frac{1}{\alpha_{10}+n_0}\right)+(1-p_1)W\left(1+\frac{1}{\alpha_{20}+n_0}\right)$$

Notice that $R\big|_{p_1=0} = W\left(1+\frac{1}{\alpha_{20}+n_0}\right) = R'\Big|_{p_1=0}$, $R\Big|_{p_1=1} = W\left(1+\frac{1}{\alpha_{10}+n_0}\right) = R'\Big|_{p_1=1}$.

Furthermore, R' is a *linear* function of p_1, while R is a *convex* function of p_1: $\frac{\partial^2}{\partial p_1}R = W\frac{(\alpha_{10}-\alpha_{20})^2(1+2n_0+2\alpha_{20}(1-p_1)+2\alpha_{10}p_1)}{(\alpha_{20}+n_0+\alpha_{10}p_1-\alpha_{20}p_1)^2(1+\alpha_{20}+n_0+\alpha_{10}p_1-\alpha_{20}p_1)^2} \geq 0$.

Therefore, $\forall p_1 \in [0,1],\ R' \geq R$.

A graphical interpretation is depicted in Figure 3.5.

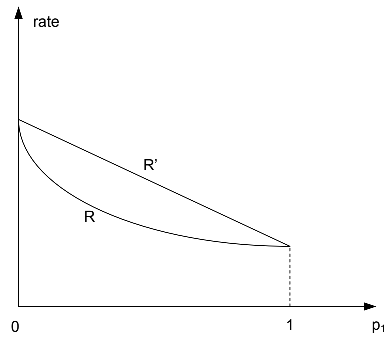


Fig. 3.5 Graphical illustration of $R' \geq R$ for user 0

We conclude that a flat FDMA power re-allocation of the two interferers of one user (Figure 3.4) always leads to a rate higher than or equal to the original rate for this user. ■

Theorem 2 can be generalized to an arbitrary number of users case as stated in the following corollary.

Corollary 2.1 Consider an $n+1$-user (one user + n interferers) flat channel: $N_i(f) = n_i$, $\alpha_{ji}(f) = \alpha_{ji}$. Suppose the $n+1$ users co-exist in a flat frequency sharing manner: $P_i(f) = p_i,\ \forall f \in (f_1, f_2),\ i = 0,1,2,...,n$. From user 0's perspective, a flat FDMA power re-allocation of its n interferers, namely user 1, user 2, … , user n, always leads to a rate higher than or equal to the original rate for user 0.

Proof:

Suppose we have one user 0 and n interferers of it: user 1, user 2, … , user n.

Assume within the band of interest $(f_1, f_2), W = f_2 - f_1$, the channel and noise are flat: $N_0(f) = n_0$, $\alpha_{i0}(f) = \alpha_{i0},\ i = 1,2,...,n$. Suppose we have a flat frequency sharing power allocation scheme, in which user 0, 1, 2, …, n's power allocation are all flat over the band $(f_1, f_2): P_i(f) = p_i,\ i = 0,1,2,...,n$.

Next, we will prove that by a flat FDMA power re-allocation of the n interferers of user 0, user 0's rate will be higher or kept the same. We prove it by induction starting from $n = 2$.

i) When there are $n = 2$ interferers, this is the case we proved in Theorem 2.

ii) Assume when there are $n = k$ interferers, the statement is proved.

When $n = k + 1$:

At the receiver of user 0, the received PSD is depicted in Figure 3.6.

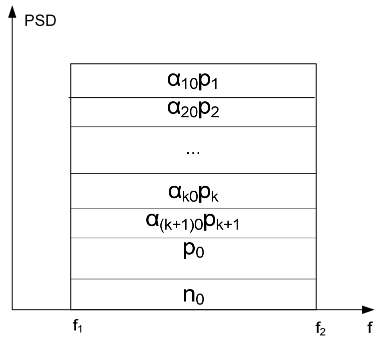


Fig. 3.6 PSD at receiver 0 with flat frequency sharing of its n interferers

Now, we apply the flat FDMA re-allocation in Theorem 2 to the interference from user 1 and user 2 only. In other words, we treat $n_0 + \sum_{i=3}^{k+1} \alpha_{i0} p_i$ all as noise. The PSD at the receiver 0 after re-allocation of user 1 and user 2 is depicted in Figure 3.7.

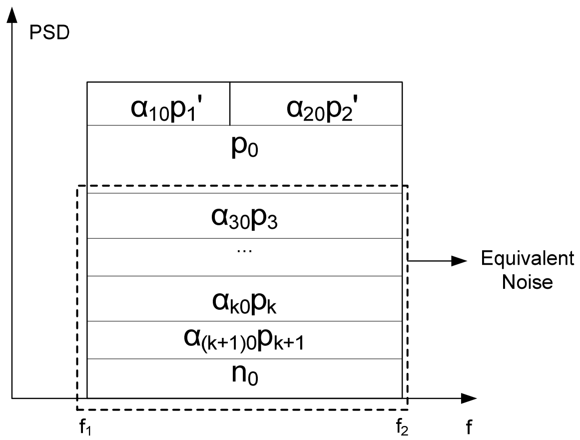


Fig. 3.7 PSD at receiver 0 after flat FDMA re-allocation of user 1 and 2

From Theorem 2, we know that this step of separating user 1 and user 2 (out of total $k+1$ interferers) can only lead to a higher or an unchanged rate of user 0.

With the separation of user 1 and user 2, the band (f_1, f_2) can be divided into 2 disjoint sub-bands, each with bandwidth $W_1' = \frac{p_1}{p_1 + p_2} W$ and $W_2' = \frac{p_2}{p_1 + p_2} W$. We denote them as $(f_1^{(1)}, f_2^{(1)})$ and $(f_2^{(1)}, f_3^{(1)})$. The PSD within the two sub-bands are depicted in Figure 3.8.

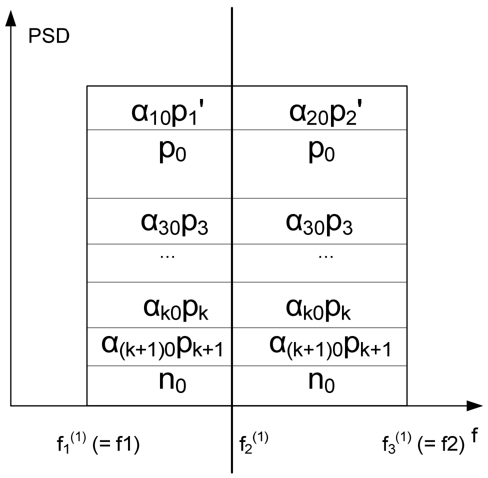


Fig. 3.8 PSDs at receiver 0 viewed as in two disjoint sub-bands after flat FDMA re-allocation of user 1 and 2

As a result, we see that within each sub-band, the problem is exactly in an $n = k$ interferers situation. By the induction assumption, within each sub-band, a flat FDMA re-allocation of the k interferers will have user 0's rate higher or unchanged.

By combining the two-step FDMA re-allocation, we finally obtain a flat FDMA power re-allocation of the $k+1$ interferers with which user 0's rate is higher or unchanged.

■

Finally, the benefits of an FDMA within a subset of users to the other users can be generalized to frequency selective channels.

Corollary 2.2 Consider an $n+1$-user (one user + n interferers) frequency selective channel. As in Corollary 1.2, we assume $\alpha_{ji}(f)$, $N_i(f)$ are uniformly continuous in the band (f_1, f_2). Given an arbitrary (uniformly continuous) spectrum and power allocation scheme $P_i(f),\ i = 0,1,2,...,n$, in which user 1, user 2, … , user n are *not* completely FDMA, then from user 0's perspective, there is always a corresponding FDMA power re-allocation of its n interferers, namely user 1, user 2, … , user n, that leads to a rate higher than or equal to the original rate for user 0.

Proof:

The proof in this general case follows the same idea as we did in Corollary 1.2:

We divide the band (f_1, f_2) into infinitely many frequency bins. From uniform continuity, let all frequency bins have sufficiently infinitesimal widths so that within any bin $(f_1', f_2'),\ |f_1' - f_2'| \to 0$ we have the following:

$\forall f \in (f_1', f_2'), \forall i \neq j$

1. channel gains become flat: $\alpha_{ji}(f) \to \alpha_{ji}(f_0) \geq \frac{1}{2}$, for some $f_0 \in (f_1', f_2')$

2. noise power become flat: $N(f) \to N(f_0)$, for some $f_0 \in (f_1', f_2')$

3. users' power become flat: $P_i(f) \to P_i(f_0)$, for some $f_0 \in (f_1', f_2')$

Next we construct an FDMA power re-allocation of the interferers $1,2,...,n$ by constructing an FDMA power re-allocation of them for every bin in (f_1, f_2): if there's no interferer or only one interferer's power in it, we do nothing; Otherwise we apply the FDMA re-allocation from Corollary 2.1 for all users within this bin. The three conditions we just obtained from letting every bin sufficiently infinitesimal enables that the results from Corollary 2.1 can be successfully applied.

With this procedure, we finally obtain an FDMA power re-allocation scheme of all $1,2,...,n$ interferers, with which user 0's rate will be higher or unchanged. ■

Notice that Theorem 2 and its two corollaries do *not* have any assumption on the strength of the cross couplings $\alpha_{ji}(f)$. Thus, from any one particular user's perspective, an FDMA of its interferers is always preferred regardless of all the cross coupling conditions.

Yet in the case that $\exists i, j$ with $\alpha_{ji}(f) \geq \frac{1}{2}$ and $\alpha_{ij}(f) \geq \frac{1}{2}$, combining Theorem 1 and Theorem 2 does give us another very strong insight into the conditions under which the optimal co-existence strategies must be FDMA:

Suppose there are $n(\geq 2)$ users, for any two users i, j, for any frequency band (f_1', f_2'), if the normalized cross coupling gains $\alpha_{ji}(f) = \frac{|H_{ji}(f)|^2}{|H_{ii}(f)|^2} \geq \frac{1}{2}$ and $\alpha_{ij}(f) = \frac{|H_{ij}(f)|^2}{|H_{jj}(f)|^2} \geq \frac{1}{2}$ $\forall f \in (f_1', f_2')$, then no matter from which user's point of view, an FDMA of user i and user j within this band is always preferred. The reason is as follows: Suppose the spectrum and power allocation for user i and j are not FDMA, then with a proper FDMA re-allocation of user i and j,

1. Theorem 1 guarantees that user i and j's rates will be higher or kept the same).

2. Theorem 2 guarantees that all the other $n-2$ users' rates will be higher or kept the same.)

The pairwise condition $\alpha_{ji}(f) \geq \frac{1}{2}$ and $\alpha_{ij}(f) \geq \frac{1}{2}$ is very convenient to use because it makes determining whether any two users should be orthogonally channelized depend only on the coupling conditions between the two of them. On the other hand, since this condition guarantees that an FDMA between user i and user j benefits every existing user, we conclude that under this condition, all the Pareto optimal points of the achievable rate region must be achieved with these two users being orthogonal (FDMA).

IV. Optimal Spectrum Management in Two-User Symmetric Channels

In this section, we continue to analyze the optimal spectrum management in the cases with $\alpha(f) < \frac{1}{2}$. We give a complete analysis of the two-user symmetric Gaussian interference channels defined as follows:

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) < \frac{1}{2}, \ \forall f \in (f_1, f_2) \\ N_1(f) = N_2(f), \ \forall f \in (f_1, f_2) \end{cases} \tag{4.1}$$

We choose the objective to be the *sum-rate* of the two users $f(R_1, R_2) = R_1 + R_2$. General problems with $n \geq 2$ users and an arbitrary weighted sum-rate objective function in non-symmetric channels are discussed later in Section V.

Here, a *sum*-power constraint $\int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \leq P$, or equivalently, *equal* power constraints $\int_{f_1}^{f_2} P_i(f) df \leq \frac{P}{2}, i = 1, 2$ are assumed. (Equivalency are shown later in this section.) We begin with flat channels, and obtain the optimal spectrum and power allocation by *solving a closed form equation*. With this result, we show that the non-convex optimization problem over the spectrum and power allocation in symmetric frequency selective channels can be equivalently transformed into a convex optimization in the *primal* domain. The key insights are twofold:

1. *Achievability*: In flat channels, for any sum-rate function (as a function of power constraints) that is achievable, the convex hull of this sum-rate function is also achievable.
2. *Optimality*: In flat channels, the convex hull of the sum-rate functions of flat frequency sharing and flat FDMA is in fact an upper bound (and hence optimal from its achievability) of the actual sum-rate, essentially due to the fact that all

spectrum management schemes are built with flat frequency sharing and flat FDMA. Consequently the optimal solution can be obtained by solving a closed form equation. This result then naturally leads to a primal domain convex optimization formulation without loss of optimality in frequency selective channels.

A. Solution of the Flat Channel Cases with Sum-Power Constraint, or Equivalently, Equal Power Constraints

Consider a two-user flat symmetric Gaussian interference channel model:

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) = \alpha < \dfrac{1}{2}, \ \forall f \in (f_1, f_2) \\ N_1(f) = N_2(f) = n, \ \forall f \in (f_1, f_2) \end{cases}$$

First, we have the following theorem on the condition under which a flat FDMA scheme is better than a flat frequency sharing scheme.

Theorem 3 For any flat frequency sharing power allocation, a flat FDMA power re-allocation (Figure 2.1) leads to a higher or unchanged sum-rate if and only if

$$\frac{p_1 + p_2}{n} \geq 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$$

Proof:

The rates of user 1 and user 2 with flat frequency sharing are

$$R_1 = W\log(1 + \frac{p_1}{n + p_2\alpha}),\ R_2 = W\log(1 + \frac{p_2}{n + p_1\alpha})$$

As we did in Section III, if we normalize the power and noise by the sum-power: $\hat{p}_1 = \dfrac{p_1}{p_1 + p_2}$, $\hat{p}_2 = \dfrac{p_2}{p_1 + p_2}$, $\hat{n} = \dfrac{n}{p_1 + p_2}$, we get

$$R_1 = W\log\left(1 + \frac{\hat{p}_1}{\hat{n} + (1 - \hat{p}_1)\alpha}\right),\ R_2 = W\log\left(1 + \frac{1 - \hat{p}_1}{\hat{n} + \hat{p}_1\alpha}\right)$$

WLOG, we add the assumption that $p_1 + p_2 = 1$. The sum-rate with flat frequency sharing is then given by

$$f(p_1) = R_1 + R_2 = W\left(\log(1 + \frac{p_1}{n + (1 - p_1)\alpha}) + \log(1 + \frac{(1 - p_1)}{n + p_1\alpha}) \right)$$

a. Convexity/Concavity of the sum-rate $f(p_1)$.

Compute the second derivative of $f(p_1)$,

$$f''(p_1) = \frac{\alpha^2}{\left(n + \alpha(1 - p_1)\right)} + \frac{\alpha^2}{(n + \alpha p_1)} - \frac{(1-\alpha)^2}{\left(1 + n - (1-\alpha)p_1\right)^2} - \frac{(1-\alpha)^2}{\left(\alpha + n + (1-\alpha)p_1\right)^2},\ p_1 \in [0,1]$$

Now $\forall p_1 \in [0,1]$, $f''(p_1)$ is a function of α, denote it by $g(\alpha)$.

Compute $g'(\alpha)$, we get

$$g'(\alpha) = \frac{2(1-\alpha)(1+n)}{\left(1 + n - (1-\alpha)p_1\right)^3} + \frac{2(1-\alpha)(1+n)}{\left(\alpha + n + (1-\alpha)p_1\right)^3} + \frac{2\alpha n}{\left(n + \alpha(1 - p_1)\right)^3} + \frac{2\alpha n}{(n + \alpha p_1)^3}$$

Since $\alpha < \frac{1}{2}, p_1 < 1$, we have

$$g'(\alpha) > 0,\ \forall p_1 \in [0,1], 0 < \alpha < \frac{1}{2}$$

In conclusion, with *any* p_1 and n fixed, $f''(p_1)$ increases when α increases (Figure 4.1).

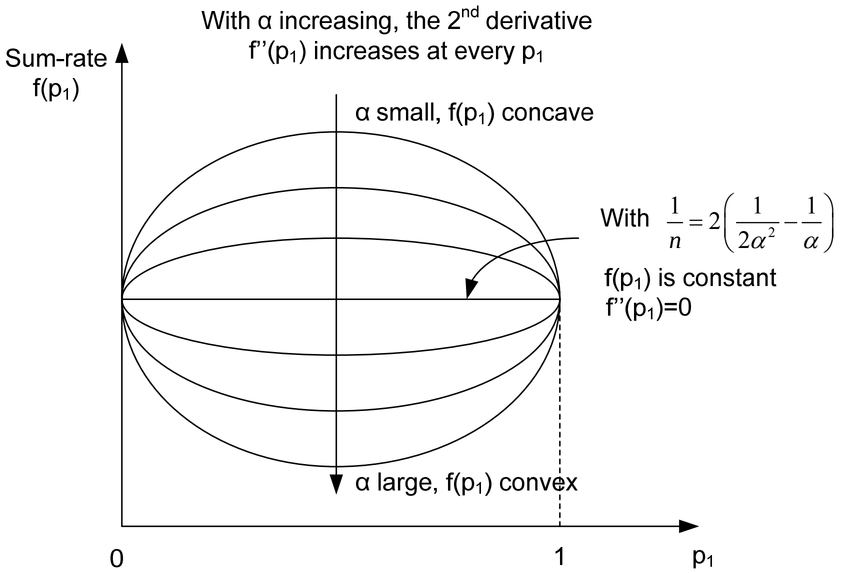


Fig. 4.1 Changes of the shape of $f(p_1)$ as α changes

When α has the critical value that satisfies $2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)=\frac{1}{n}$, or equivalently, $n=\left(2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)\right)^{-1}$, we have

$$f(p_1)=W\left(\log(1+\frac{p_1}{n+(1-p_1)\alpha})+\log(1+\frac{(1-p_1)}{n+p_1\alpha})\right)$$

$$=W\left(\log\left(\frac{(1-\alpha)(p_1+\alpha(1-2p_1))}{\alpha(1-p_1+\alpha(2p_1-1))}\right)+\log\left(\frac{(1-\alpha)(1-p_1+\alpha(2p_1-1))}{\alpha(p_1+\alpha(1-2p_1))}\right)\right)$$

$$=W\log\left(\frac{(1-\alpha)(p_1+\alpha(1-2p_1))}{\alpha(1-p_1+\alpha(2p_1-1))}\frac{(1-\alpha)(1-p_1+\alpha(2p_1-1))}{\alpha(p_1+\alpha(1-2p_1))}\right)$$

$$=W\log\frac{(1-\alpha)^2}{\alpha^2} \tag{4.2}$$

In this case, we see that $f(p_1)$ is a constant function that does *not* depend on p_1. Clearly, $f''(p_1)=0$.

Denote this critical value by α_0, $\frac{1}{n} = 2\left(\frac{1}{2\alpha_0^2} - \frac{1}{\alpha_0}\right)$. Notice that $\frac{1}{2\alpha^2} - \frac{1}{\alpha}$ is a decreasing function of α when $0 < \alpha < \frac{1}{2}$.

Therefore, as depicted in Fig. 4.1,

when $\alpha > \alpha_0$, i.e. $\frac{1}{n} > 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, $f''(p_1) > 0,\ \forall p_1 \in [0,1],\ f(p_1)$ is convex;

when $\alpha < \alpha_0$, i.e. $\frac{1}{n} < 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, $f''(p_1) < 0,\ \forall p_1 \in [0,1],\ f(p_1)$ is concave.

when $\alpha = \alpha_0$, i.e. $\frac{1}{n} = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, $f''(p_1) = f'(p_1) = 0,\ \forall p_1 \in [0,1],\ f(p_1)$ is constant.

b. Comparison between sum-rates of flat frequency sharing and flat FDMA re-allocation.

With a flat FDMA re-allocation and the normalization $p_1 + p_2 = 1 = p_1' = p_2'$, the rates of user 1 and user 2 become

$$\begin{cases} R_1' = W_1' \log(1 + p_1') = p_1 W \log(1 + \frac{1}{n}) \\ R_2' = W_2' \log(1 + p_2') = (1 - p_1) W \log(1 + \frac{1}{n}) \end{cases}$$

The sum-rate with the flat FDMA re-allocation is given by

$$h(p_1) = R_1' + R_2' = W \log(1 + \frac{1}{n}) \tag{4.3}$$

It is consistent with the fact that the sum-rate of FDMA does not depend on the cross interference gain α, since there is no interference when FDMA is used.

When $\alpha = \alpha_0$, i.e. $\frac{1}{n} = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, substitute this into (4.3), we get

$$h(p_1) = W\log(1+\frac{1}{n}) = W\log\left(1+2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)\right) = W\log\left(\frac{(1-\alpha)^2}{\alpha^2}\right)$$

Compared it with (4.2), we see that in this case $f(p_1) = h(p_1) = constant,\ p_1 \in [0,1]$.

In other words, when $\frac{1}{n} = 2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)$, the flat frequency sharing and its flat FDMA re-allocation have the same sum-rate $W\log\left(\frac{(1-\alpha)^2}{\alpha^2}\right)$, regardless of p_1.

Furthermore, since $f(0) = f(1) = h(p_1) = W\log(1+\frac{1}{n})$ always holds, with the convexity/concavity conditions derived above, we have

when $\frac{1}{n} > 2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)$, $f(p_1)$ is convex $\Rightarrow$ $f(p_1) \le h(p_1),\ \forall p_1 \in [0,1]$;

when $\frac{1}{n} < 2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)$, $f(p_1)$ is concave $\Rightarrow$ $f(p_1) \ge h(p_1),\ \forall p_1 \in [0,1]$.

Finally, we remove the normalization $p_1 + p_2 = 1$, and concludes that for a flat frequency sharing power allocation, a flat FDMA re-allocation leads to a higher or unchanged sum-rate if and only if $\frac{p_1+p_2}{n} \ge 2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)$. ■

When $\alpha < \frac{1}{2}$, we have $2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right) > 0$. In this case, as long as $0 < \frac{p_1+p_2}{n} < 2\left(\frac{1}{2\alpha^2}-\frac{1}{\alpha}\right)$, FDMA schemes are clearly *not* optimal because flat frequency sharing has a higher rate than flat FDMA (which is the optimal FDMA scheme in flat channels). Thus, to guarantee the optimal spectrum and power allocation to be FDMA regardless of the power constraints, the coupling conditions $\alpha_{ji}(f) \ge \frac{1}{2}$ are not only sufficient, but also necessary in symmetric channels.

Given noise level n and cross coupling gains α, Theorem 3 provides us a power region P_{FDMA} within which flat FDMA has a higher sum-rate than flat frequency sharing, depicted as the shaded area in Figure 4.2 (with complement region $\overline{P}_{FDMA}$ also depicted).

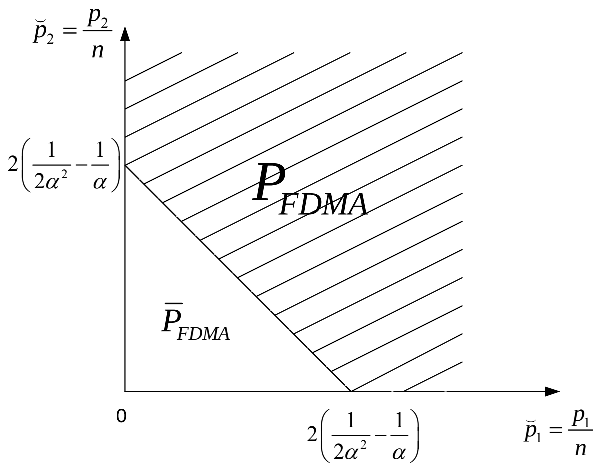


Fig. 4.2 The region in which flat FDMA has higher sum-rate than flat frequency sharing

If we normalize the power by the noise instead of the sum-power: $\breve{p}_1 = \dfrac{p_1}{n}$, $\breve{p}_2 = \dfrac{p_2}{n}$, the region becomes $P_{FDMA} = \begin{cases} \breve{p}_1 + \breve{p}_2 \geq 2\left(\dfrac{1}{2\alpha^2} - \dfrac{1}{\alpha}\right) \\ \breve{p}_1 \geq 0, \breve{p}_2 \geq 0 \end{cases}$. Since p_1, p_2 and n are considered to be *arbitrary* here, using the noise-normalized terms $\breve{p}_1, \breve{p}_2$ and $n = 1$ is equivalent to the original formulation. *From now on, instead of adding the assumption that* $p_1 + p_2 = 1$*, we add the assumption that* $n = 1$ WLOG. The sum-rate of flat frequency sharing becomes

$$f(p_1, p_2) = R_1 + R_2 = W\left(\log(1 + \frac{p_1}{1 + \alpha p_2}) + \log(1 + \frac{p_2}{1 + \alpha p_1})\right)$$

With the sum-power constraint $p_1 + p_2 \leq p$ only, the maximum achievable sum-rate with flat frequency sharing $f^*(p)$ is defined as the optimal value of the following optimization problem:

$$\begin{aligned} f^*(p) = \ & \max \ f(p_1, p_2) \\ & s.t. \ p_1 + p_2 \le p \\ & \quad p_1 \ge 0, p_2 \ge 0 \end{aligned}$$

Next, we show the form and the concavity of $f^*(p)$ in the region of $\overline{P}_{FDMA}$.

Lemma 2 When $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, $\quad f^*(p) = 2W\log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$ (4.4)

is a concave function of the constraint p. The optimal flat frequency sharing scheme is $p_1 = p_2 = \frac{p}{2}$.

Proof:

Clearly, the condition of p implies $(p_1, p_2) \in \overline{P}_{FDMA}$.

First, we find the solution to the optimization problem with equality sum-power constraint rather than inequality, i.e.

$$\max \ f(p_1, p_2), \ \ s.t. \ p_1 + p_2 = p, \ p_1 \ge 0, \ \ p_2 \ge 0$$

where $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$.

With $p_1 + p_2 = p$, $f(p_1, p_2) = W\left(\log(1 + \frac{p_1}{1+\alpha(p-p_1)}) + \log(1 + \frac{p-p_1}{1+\alpha p_1})\right) = f(p_1)$.

Directly from the proof of Theorem 3, we know that since $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, $f(p_1)$ is a concave function of p_1, $\forall p_1 \in [0, p]$. Furthermore, $f(p_1) = f(p - p_1)$, i.e. $f(p_1)$ is symmetric about $p_1 = \frac{p}{2}$. Therefore, $f(p_1)$ takes the maximum value when $p_1 = \frac{p}{2} = p_2$, and the maximum value is $f^* = 2W \log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$.

We see that within the region $\overline{P}_{FDMA}$, i.e. $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$, with sum-power of the two users unchanged, the maximum sum-rate is always achieved when the two users' power are the same. It implies that the constraint $p_1 + p_2 \le p$ in the definition problem of $f^*(p)$ can be equivalently replaced by $p_1 + p_2 = p$. With this change of constraint, we have the expression of $f^*(p)$:

$$f^*(p) = 2W \log\left(1 + \frac{p/2}{1+\alpha p/2}\right)$$

Computing the second derivative we get

$$\frac{d^2}{dp^2} f^*(p) = -\frac{8(1+2\alpha+\alpha p+\alpha^2 p)}{(2+\alpha p)^2(2+p+\alpha p)^2} < 0$$

Therefore, $f^*(p)$ is a concave function of the constraint p, when $0 < p \le 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$. ■

In comparison, we compute the maximum achievable sum-rate with sum-power constraint for FDMA schemes. Clearly, with general sum-power constraint $\int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P$, the sum-rate of both users are equivalent to the rate of a single user with a power constraint P. With our flat channel assumption, according to the water-filling principle, the maximum sum-rate of both users is achieved when the power spectrum density over the whole channel is flat. In other words, both users' powers are allocated mutually non-overlapped and collectively filling the whole band uniformly. Denote the flat power spectral density by $p = \frac{P}{W}$. We summarize the above result in the following lemma.

Lemma 3 The maximum achievable sum-rate with FDMA is

$$h^*(p) = W\log(1+p) \tag{4.5}$$

where $h^*(p)$ is defined as the optimal value of the following optimization problem:

$$\begin{aligned} h^*(p) = \quad & \max\ R_1 + R_2 \\ & s.t. \int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P, \quad p \triangleq \frac{P}{W} \\ & P_1(f)P_2(f) = 0,\ P_1(f) \ge 0, P_2(f) \ge 0,\ \forall f \in (f_1, f_2) \\ & R_1 = \int_{f_1}^{f_2} \log\left(1 + P_1(f)\right) df,\ R_2 = \int_{f_1}^{f_2} \log\left(1 + P_2(f)\right) df \end{aligned}$$

Define the critical point $p_0 = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)$. As a direct implication of Theorem 3, it can be easily verified that $f^*(p_0) = h^*(p_0)$.

We define the upper envelope of $f^*(p)$ and $h^*(p)$ to be

$$r(p) = \max\{f^*(p), h^*(p)\} = \begin{cases} f^*(p), & p \in [0, p_0] \\ h^*(p), & p \in [p_0, \infty) \end{cases}.$$

Furthermore,

$$\left.\frac{d}{dp} f^*(p)\right|_{p=p_0} = \frac{4\alpha^3}{1-\alpha} < \frac{\alpha^2}{(1-\alpha)^2} = \left.\frac{d}{dp} h^*(p)\right|_{p=p_0} \text{ when } 0 < \alpha < \frac{1}{2}. \tag{4.6}$$

Thus, although $f^*(p)$ is concave in $[0, p_0]$ as proved in lemma 2, and $h^*(p)$ is clearly concave in $[p_0, \infty]$, (4.6) implies that the upper envelope $r(p)$ of those two functions is *not* concave in $[0, \infty)$.

Next, we define $r^*(p)$ to be the unique *convex hull* of $r(p)$:

1) Define the set of functions

$$S = \{\tilde{r}(p) \mid \tilde{r}(p) \text{ concave; } \tilde{r}(p) \ge r(p),\ \forall p \in [0, \infty)\}.$$

2) $r^*(p)$ is the unique function satisfying

$$\begin{cases} r^*(p) \in S \\ r^*(p) \le \tilde{r}(p),\ \forall p \in [0,\infty),\ \forall \tilde{r}(p) \in S \end{cases}$$

A typical plot of $f^*(p)$, $h^*(p)$, and the convex hull of their upper envelope $r^*(p)$ is given in Figure 4.3. Since $f^*(p)$ and $h^*(p)$ are themselves concave, finding the convex hull of the upper envelope boils down to finding their *common tangent line*.

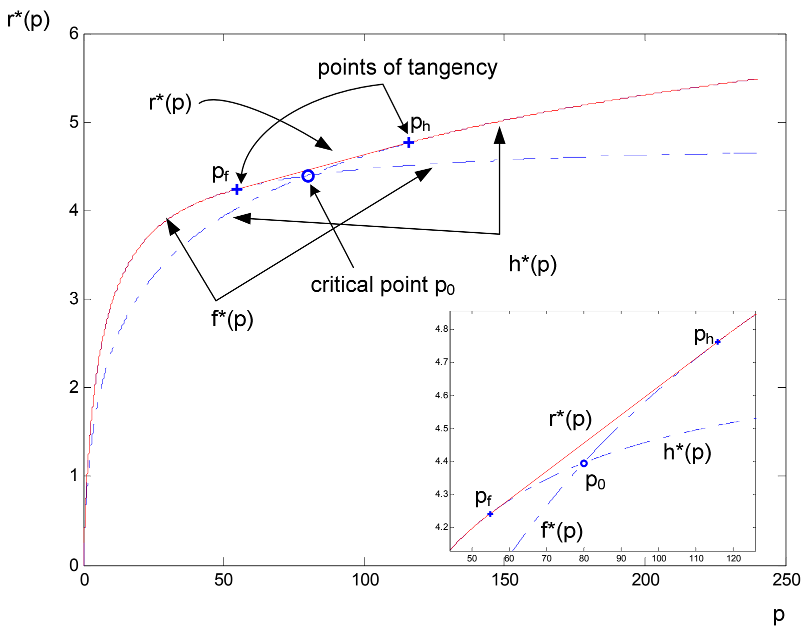


Fig. 4.3 The maximum achievable rate as the convex hull of the rates of flat FDMA and flat frequency sharing

In Figure 4.3, α is chosen to be 0.1. $f^*(p)$ and $h^*(p)$ intersects at $p_0 = 2\left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right) = 80$. The two points of tangency are $p_f = 54.931,\ p_h = 115.938$.

The slope of the common tangent line is 0.00855.

In order to find the common tangent line of $f^*(p)$ and $h^*(p)$, the two points of tangency p_f and p_h are determined by

$$\frac{d}{dp} f^*(p)\bigg|_{p=p_f} = \frac{d}{dp} h^*(p)\bigg|_{p=p_h} = \frac{h^*(p_h) - f^*(p_f)}{p_h - p_f},$$

which simplifies to finding p_f by solving

$$\frac{p_f\left(\alpha(1+\alpha)p_f + 4\alpha - 2\right)}{\left(\alpha p_f + 2\right)\left((1+\alpha)p_f + 2\right)} = \log\left(\frac{(\alpha p_f + 2)^3}{4\left((1+\alpha)p_f + 2\right)}\right) \tag{4.7}$$

and computing p_h by

$$p_h = \frac{1}{4} p_f\left(\alpha(1+\alpha)p_f + 4\alpha + 2\right)$$

p_f and p_h can be obtained by solving the closed form equation (4.7) where various numerical methods can be applied.

Next, we provide the main theorem of this sub-section, showing that $r^*(p)$ is in fact the *maximum* sum-rate that the two users can achieve with a sum-power constraint.

Theorem 4 The Maximum Sum-Rate and the Optimal Spectrum and Power Allocation in Flat Symmetric Gaussian Interference Channel with Sum-Power Constraint.

In a flat symmetric Gaussian interference channel with $\alpha < \frac{1}{2}$ (4.1), the maximum sum-rate defined as the optimal value of the following optimization problem

$$\begin{aligned}
&\max\ R_1 + R_2 \\
&s.t.\ \int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P,\quad p \triangleq \frac{P}{W} \\
&P_1(f) \ge 0, P_2(f) \ge 0,\ \forall f \in (f_1, f_2) \\
&R_1 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_1(f)}{1+\alpha P_2(f)}\right) df,\ R_2 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_2(f)}{1+\alpha P_1(f)}\right) df
\end{aligned} \tag{4.8}$$

is $r^*(p)$, i.e. the convex hull of the upper envelope of $f^*(p)$ and $h^*(p)$.

Proof:

i) $r^*(p)$ *is achievable.*

As in Figure 4.3, by definition of $r^*(p)$, we see that

$$\begin{cases} r^*(p) = f^*(p) & ,0 < p \le p_f \\ r^*(p) = f^*(p_f) + \dfrac{h^*(p_h) - f^*(p_f)}{p_h - p_f}(p - p_f) & ,p_f \le p \le p_h \\ r^*(p) = h^*(p) & ,p \ge p_h \end{cases}$$

Clearly, when $0 < p \le p_f$, $r^*(p) = f^*(p)$ is achievable with flat frequency sharing as in lemma 2; when $p \ge p_h$, $r^*(p) = h^*(p)$ is achievable with flat FDMA as in lemma 3. When $p_f \le p \le p_h$, $r^*(p) > \max\{f^*(p),\ h^*(p)\}$ strictly. In this case, choose λ such that $p = p_f + \lambda(p_h - p_f),\ 0 \le \lambda \le 1$. We separate the band of the original channel into two orthogonal channels: C_1 with bandwidth $(1-\lambda)W$, and C_2 with bandwidth λW.

In C_1, we impose the sum-power constraint

$$\int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P_{C_1} \triangleq (1-\lambda)W \cdot p_f .$$

Define $p_{C_1} \triangleq \dfrac{P_{C_1}}{(1-\lambda)W} = p_f$. Thus, *the rate with the optimal flat frequency sharing* can be achieved as in lemma 2:

$$f_{C_1}^*(p_{C_1}) = 2(1-\lambda)W \log\left(1 + \frac{p_f/2}{1+\alpha p_f/2}\right) = (1-\lambda) f^*(p_f)$$

In C_2, we impose the sum-power constraint

$$\int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P_{C_2} \triangleq \lambda W \cdot p_h$$

Define $p_{C_2} \triangleq \dfrac{P_{C_2}}{\lambda W} = p_h$. Thus, *the rate with the optimal flat FDMA* can be achieved as in lemma 3:

$$h_{C_2}^*(p_{C_2}) = \lambda W \log(1+p_h) = \lambda h^*(p_h)$$

With these two power constraints in the two orthogonal sub-channels, the original power constraint $\int_{f_1}^{f_2} \left(P_1(f)+P_2(f)\right)df \le P = pW$ is automatically satisfied:

$$P_{C_1} + P_{C_2} = (1-\lambda)W \cdot p_f + \lambda W \cdot p_h = \left(p_f + \lambda(p_h - p_f)\right)W = pW = P\ .$$

Therefore, the sum-rate $f_{C_1}^*(p_{C_1}) + h_{C_2}^*(p_{C_2})$ can be achieved in the original problem (4.8). Substitute p by $p_f + \lambda(p_h - p_f)$ in the expression of $r^*(p) = f^*(p_f) + \frac{h^*(p_h) - f^*(p_f)}{p_h - p_f}(p - p_f)$, we have

$$r^*(p) = (1-\lambda) f^*(p_f) + \lambda h^*(p_h) = f_{C_1}^*(p_{C_1}) + h_{C_2}^*(p_{C_2})$$

Therefore, $r^*(p)$ is achievable.

The optimal spectrum and power allocation to achieve the sum-rate of $r^*(p)$ is depicted in Figure 4.4.

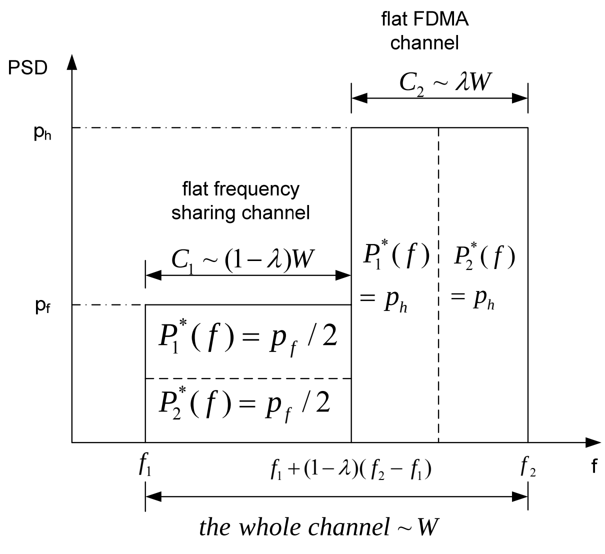


Fig. 4.4 The optimal spectrum and power allocation as a mixture of flat FDMA and flat frequency sharing

ii) $r^*(p)$ *is optimal.*

Suppose the maximum achievable sum-rate is $r^o(p)$, with the optimal spectrum and power allocation $P_1^o(f)$ and $P_2^o(f)$ satisfying $\int_{f_1}^{f_2}\left(P_1^o(f)+P_2^o(f)\right)df \le P = pW$ and achieving

$$r^o(p)=\int_{f_1}^{f_2}\left(\log\left(1+\frac{P_1^o(f)}{1+\alpha P_2^o(f)}\right)+\log\left(1+\frac{P_2^o(f)}{1+\alpha P_1^o(f)}\right)\right)df$$

WLOG, we add the assumption of $P_1^o(f)$ and $P_2^o(f)$ being uniformly continuous as in Theorem 2 and Theorem 3.

Define $r^o(f;p)=\log\left(1+\frac{P_1^o(f)}{1+\alpha P_2^o(f)}\right)+\log\left(1+\frac{P_2^o(f)}{1+\alpha P_1^o(f)}\right)$, then

$$r^o(p)=\int_{f_1}^{f_2} r^o(f;p)df$$

Define $p^o(f)=P_1^o(f)+P_2^o(f)$. Notice that the expression $r^o(f;p)$ has the form of either a flat frequency sharing (if $P_1^o(f)>0,\ P_2^o(f)>0$) or a flat FDMA (if $P_1^o(f)P_2^o(f)=0$), with unit bandwidth. It corresponds to the fact we have mentioned in Section II: flat frequency sharing and flat FDMA are the two building blocks of all non-flat cases. Thus, with the above definition of $p^o(f)$,

$$\begin{aligned} r^o(f;p)&=\frac{1}{f_2-f_1}\cdot(f_2-f_1)\left(\log\left(1+\frac{P_1^o(f)}{1+\alpha P_2^o(f)}\right)+\log\left(1+\frac{P_2^o(f)}{1+\alpha P_1^o(f)}\right)\right)\\ &\le r^*(p^o(f))\cdot\frac{1}{f_2-f_1}. \end{aligned}$$

Thus, $\int_{f_1}^{f_2} r^o(f;p)df \le \int_{f_1}^{f_2} r^*(p^o(f))\cdot\frac{1}{f_2-f_1}df$.

Since $r^*(p)$ is a concave function over $[0,\infty)$, by Jensen's Inequality,

$$\int_{f_1}^{f_2} r^*(p^o(f))\cdot\frac{1}{f_2-f_1}df \le r^*\left(\int_{f_1}^{f_2} p^o(f)\cdot\frac{1}{f_2-f_1}df\right),$$

and $\int_{f_1}^{f_2} p^o(f)\cdot\frac{1}{f_2-f_1}df = \frac{1}{f_2-f_1}\cdot\int_{f_1}^{f_2}\left(P_1^o(f)+P_2^o(f)\right)df \le \frac{1}{f_2-f_1}\cdot pW = p$,

where $W = f_2 - f_1$, and the inequality comes from the sum-power constraint (4.8).

Since $r^*(p)$ is a strictly increasing function over $[0,\infty)$, we have

$$r^*\left(\int_{f_1}^{f_2} p^o(f)\cdot\frac{1}{f_2-f_1}df\right) \le r^*(p)$$

Therefore, we have in conclusion $\int_{f_1}^{f_2} r^o(f;p)df \le r^*(p)$, i.e. $r^o(p) \le r^*(p)$.

Since we assume $r^o(p)$ to be the maximum achievable sum-rate, and $r^*(p)$ is achievable as proved previously in i), we conclude that $r^*(p)$ is the maximum achievable sum-rate. Furthermore, the mixture of a flat frequency sharing and a flat FDMA shown in Figure 4.4 is the optimal spectrum and power allocation achieving $r^*(p)$. ■

From Theorem 4, we know that given the following set of conditions,

1) sum-power constraint P (with noise power normalized to 1),

2) bandwidth W,

3) cross interference gain α (with direct channel gains normalized to 1),

with the definition of $p = \frac{P}{W}$, $f^*(x) = 2W\log\left(1+\frac{x/2}{1+\alpha x/2}\right)$, $h^*(x) = W\log(1+x)$, the maximum achievable sum-rate $r^*(p)$ is computed through the following steps:

Procedure 4.1,

1) Solve the two points of tangency p_f and p_h of the convex hull of $f^*(x)$ and $h^*(x)$:

a. Solve equation (4.7) numerically to find p_f

$$\frac{p_f\left(\alpha(1+\alpha)p_f+4\alpha-2\right)}{\left(\alpha p_f+2\right)\left((1+\alpha)p_f+2\right)}=\log\left(\frac{(\alpha p_f+2)^3}{4\left((1+\alpha)p_f+2\right)}\right).$$

b. Compute p_h by $p_h=\frac{1}{4}p_f\left(\alpha(1+\alpha)p_f+4\alpha+2\right)$.

2) Compute the maximum achievable sum-rate $r^*(p)$:

If $p\le p_f$, $r^*(p)=f^*(p)=2W\log\left(1+\frac{p/2}{1+\alpha p/2}\right)$;

If $p\ge p_h$, $r^*(p)=h^*(p)=W\log(1+p)$;

If $p_f<p<p_h$, $r^*(p)=f^*(p_f)+\frac{h^*(p_h)-f^*(p_f)}{p_h-p_f}(p-p_f)$.

The optimal power allocation scheme $\left\{P_1^*(f),P_2^*(f)\right\}$ that achieves the maximum achievable sum-rate $r^*(p)$ is obtained through the following steps: ***Procedure 4.2,***

1) If $p\le p_f$, allocate $P_1^*(f)=P_2^*(f)=\frac{p}{2},\forall f$, i.e. flat frequency sharing in the whole band with equal power spectral density for the two users.

2) If $p\ge p_h$, allocate $P_1^*(f)$ and $P_2^*(f)$ such that $\begin{cases}P_1^*(f)P_2^*(f)=0,\ \forall f\\ P_1^*(f)+P_2^*(f)=p,\ \forall f\end{cases}$, i.e. flat FDMA of the two users (with no specific requirement for each individual user's power.)

3) if $p_f<p<p_h$,

a. Compute $\lambda=\frac{p-p_f}{p_h-p_f}$

b. Separate the bandwidth W into two disjoint channels:

C_1 with bandwidth $(1-\lambda)W$ and C_2 with bandwidth λW.

c. Allocate power as follows (Figure 4.4):

In C_1, $P_1^*(f)=P_2^*(f)=\dfrac{p_f}{2}$, i.e. flat frequency sharing with equal power spectral density.

In C_2, $\begin{cases} P_1^*(f)P_2^*(f)=0, \forall f \\ P_1^*(f)+P_2^*(f)=p_h, \forall f \end{cases}$, i.e. flat FDMA.

This optimal spectrum and power allocation scheme $P_1^*(f)$ and $P_2^*(f)$ has the following two properties:

1) The power constraint P is always strictly met.

2) There always exists an optimal power allocation with *equal* total power of the two users, i.e. $\int_{f_1}^{f_2} P_1^*(f)df = \int_{f_1}^{f_2} P_2^*(f)df = \dfrac{P}{2}$, because

a. The optimal flat frequency sharing always enforces equal PSD of the two users;

b. The optimal flat FDMA does not have any requirement on how the sum-power is divided among the two users, meaning that we can always choose to divide the sum-power equally.

Theorem 4 and the above procedures tell us the best we can do to maximize the sum-rate with sum-power constraint (4.8). Another form of optimization problem that often appears is with the *individual* power constraints as follows:

$$\begin{aligned} &\max\ R_1+R_2 \\ &s.t.\ \int_{f_1}^{f_2} P_1(f)df \le P_1,\ \ p_1 \triangleq \frac{P_1}{W} \\ &\int_{f_1}^{f_2} P_2(f)df \le P_2,\ \ p_2 \triangleq \frac{P_2}{W} \\ &P_1(f)\ge 0, P_2(f)\ge 0,\ \forall f\in(f_1,f_2) \\ &R_1=\int_{f_1}^{f_2}\log\left(1+\frac{P_1(f)}{1+\alpha P_2(f)}\right)df,\ R_2=\int_{f_1}^{f_2}\log\left(1+\frac{P_2(f)}{1+\alpha P_1(f)}\right)df \end{aligned}$$

Clearly, with individual power constraints, the optimal solution can be different from the problems with the sum-power constraint. However, as we have alluded to in property 2) of the optimal spectrum and power allocation, the equal power constraints problems, namely $P_1 = P_2$, can be equivalently transformed into problems with sum-power constraint as in the next corollary.

Corollary 4.1 The Maximum Sum-Rate and the Optimal Spectrum and Power Allocation in Flat Symmetric Gaussian Interference Channels with Equal Power Constraints.

In a flat symmetric Gaussian interference channel with $\alpha < \frac{1}{2}$ (4.1), the maximum sum-rate defined as the optimal value of the following optimization problem

$$\begin{aligned}
&\max\ R_1 + R_2 \\
&s.t.\ \int_{f_1}^{f_2} P_1(f)df \le \frac{P}{2},\quad p \triangleq \frac{P}{W} \\
&\int_{f_1}^{f_2} P_2(f)df \le \frac{P}{2} = \frac{pW}{2} \\
&P_1(f) \ge 0, P_2(f) \ge 0,\ \forall f \in (f_1, f_2) \\
&R_1 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_1(f)}{1+\alpha P_2(f)}\right)df,\ R_2 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_2(f)}{1+\alpha P_1(f)}\right)df
\end{aligned} \tag{4.9}$$

is $r^*(p)$, i.e. the convex hull of the upper envelope of $f^*(p)$ and $h^*(p)$.

At this maximum sum-rate point, both users have the same rate $\frac{r^*(p)}{2}$.

Proof:

First, the individual power constraints $\begin{cases} \int_{f_1}^{f_2} P_1(f)df \le \frac{P}{2} = \frac{p}{2}W \\ \int_{f_1}^{f_2} P_2(f)df \le \frac{P}{2} = \frac{p}{2}W \end{cases}$ *imply* the sum-power constraint $\int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right)df \le P = pW$. In other words, the domain of this problem with equal power constraints (4.9) is contained in the domain of which in Theorem 4 with sum-power constraint (4.8). Thus, the optimal value of the problem in Theorem 4 (4.8), namely $r^*(p)$, is an upper bound of the optimal value of the problem here (4.9).

Recall the property of the optimal power allocation scheme with sum-power constraint: the optimal flat FDMA does not have any requirement on how the sum-power is divided among the users, meaning that we can always choose to divide the sum-power equally. We know that for the problem in Theorem 4 (4.8), we can always achieve the maximum achievable sum-rate $r^*(p)$ with *equal* power for each user, i.e. $\int_{f_1}^{f_2} P_1(f)df = \int_{f_1}^{f_2} P_2(f)df = \frac{P}{2}$. In other words, $r^*(p)$ is always achievable in the problem with equal individual power constraints here (4.9).

Therefore, $r^*(p)$ is the maximum achievable sum-rate for problem with equal power constraints (4.9).

To obtain the optimal power allocation scheme with equal power constraints, we first obtain an optimal power allocation scheme with sum-power constraint by Procedure 4.1. Then, we simply divide the flat-FDMA portion of the sum-power equally to the two users. ■

In this sub-section, we see that the maximum achievable sum-rate $r^*(p)$ with sum-power constraint (4.8) or equal power constraints (4.9) can be computed with Procedure 4.1 efficiently. Concurrently, the optimal spectrum and power allocation scheme is readily obtained with Procedure 4.2, and is precisely characterized as in Figure 4.4.

B. Generalization to the Cases of Frequency Selective Channels

In this sub-section, we extend the sum-rate maximization problem to the symmetric frequency selective Gaussian interference channel.

$$\begin{cases} \alpha_{12}(f) = \alpha_{21}(f) = \alpha(f) < \frac{1}{2}, \ \forall f \in (f_1, f_2) \\ N_1(f) = N_2(f) = N(f), \ \forall f \in (f_1, f_2) \end{cases} \tag{4.10}$$

where $\alpha(f)$ and $N(f)$ are assumed to be uniformly continuous. The maximum sum-rate with sum-power constraint is then defined as the optimal value of the following optimization problem:

$$\begin{aligned} &\max \ R_1 + R_2 \\ &s.t. \ \int_{f_1}^{f_2} \left(P_1(f) + P_2(f)\right) df \le P \\ &\quad P_1(f) \ge 0, P_2(f) \ge 0, \ \forall f \in (f_1, f_2) \\ &\quad R_1 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_1(f)}{N(f) + P_2(f)\alpha(f)}\right) df \\ &\quad R_2 = \int_{f_1}^{f_2} \log\left(1 + \frac{P_2(f)}{N(f) + P_1(f)\alpha(f)}\right) df \end{aligned} \tag{4.11}$$

Cleary, the expression of the objective function $R_1 + R_2$ are highly *non-concave* in terms of $P_1(f)$ and $P_2(f)$ and hard to deal with. However, since we have already explicitly solved the optimization problem in flat channels, we show that this non-convex optimization in frequency selective channels can be equivalently transformed into a *convex optimization in the primal domain*. The idea is that at every frequency point, the infinitesimal sub-band around this point is flat. Applying the solution we obtained for flat channels, the maximum sum-rate within this sub-band is an increasing concave function of the power constraint within this sub-band.

Define the normalized PSD to be $\tilde{P}_1(f) = \frac{P_1(f)}{N(f)}, \tilde{P}_2(f) = \frac{P_2(f)}{N(f)}$, and the normalized sum-PSD to be $\tilde{p}(f) = \tilde{P}_1(f) + \tilde{P}_2(f)$. At every frequency $f \in (f_1, f_2)$,

1. in the same form of (4.4) and (4.5) with $\alpha(f)$ instead of α, and $W = 1$:

$$f^*(p;f) \triangleq 2\log\left(1+\frac{p/2}{1+\alpha(f)p/2}\right),\ h^*(p;f) \triangleq \log(1+p),$$

and for every $f \in (f_1, f_2)$, $r^*(p;f)$ is defined as the convex hull of $\max\left\{f^*(p;f), h^*(p;f)\right\}$.

2. $p_f(f)$, $p_h(f)$, and finally $r^*(p;f)$ are computed in the same way as in Procedure 4.1 with $\alpha(f)$ instead of α.

Now we have the following theorem transforming (4.11) into a convex optimization.

Theorem 5 The problem of maximizing the sum-rate in symmetric frequency selective Gaussian interference channels (4.11) has the same optimal value as the following optimization problem:

$$\begin{aligned} &\max_{\tilde{p}(f)} \int_{f_1}^{f_2} r^*(\tilde{p}(f);f)df \\ &s.t.\ \int_{f_1}^{f_2} \tilde{p}(f)N(f)df \le P, \quad \tilde{p}(f) \ge 0, \forall f \in (f_1, f_2) \end{aligned} \tag{4.12}$$

Proof:

Denote the optimal value of (4.11) by R^*, and the optimal value of (4.12) by R^o.

1) $R^* \le R^o$

(4.11) is equivalent to

$$\begin{aligned} R^* = &\max_{\tilde{P}_1(f),\tilde{P}_2(f)} \int_{f_1}^{f_2} r(f)df \\ &s.t.\ r(f) \triangleq \log\left(1+\frac{\tilde{P}_1(f)}{1+\tilde{P}_2(f)\alpha(f)}\right) + \log\left(1+\frac{\tilde{P}_2(f)}{1+\tilde{P}_1(f)\alpha(f)}\right), \\ &\tilde{P}_1(f)+\tilde{P}_2(f) = \tilde{p}(f) \\ &\int_{f_1}^{f_2} \tilde{p}(f)N(f)df \le P,\ \tilde{P}_1(f) \ge 0, \tilde{P}_2(f) \ge 0,\ \forall f \in (f_1, f_2) \end{aligned}$$

Notice that the expression of $r(f)$ has the form of either a flat frequency sharing (if $\tilde{P}_1(f) > 0,\ \tilde{P}_2(f) > 0$) or a flat FDMA (if $\tilde{P}_1(f)\tilde{P}_2(f) = 0$). Thus,

$$r(f) \le r^*(\tilde{p}(f); f)$$

It leads directly to,

$$R^* \le \max_{\tilde{p}(f)} \int_{f_1}^{f_2} r^*(\tilde{p}(f); f) df$$
$$s.t. \int_{f_1}^{f_2} \tilde{p}(f) N(f) df \le P, \ \tilde{p}(f) \ge 0, \ \forall f \in (f_1, f_2)$$

i.e. $R^* \le R^o$.

2) R^o is achievable, i.e. $R^* \ge R^o$

Denote the optimal solution of (4.11) by $\tilde{p}^*(f), \ f \in (f_1, f_2)$, $\int_{f_1}^{f_2} r^*(\tilde{p}^*(f); f) df = R^o$.

We divide the band (f_1, f_2) into n equal width frequency bins:

$$(f_1' \triangleq f_1, f_2'), (f_2', f_3'), (f_3', f_4'), \dots, (f_n', f_{n+1}' \triangleq f_2), \quad f_{k+1}' - f_k' = \frac{W}{n}, \ k = 1, 2, \dots, n.$$

Let $n \to \infty$. Within any bin $(f_k', \ f_{k+1}'), \ |f_k' - f_{k+1}'| \to 0$. From uniform continuity we have the following:

$\forall f \in (f_k', \ f_{k+1}')$

1. channel gain becomes flat: $\alpha(f) \to \alpha(\hat{f}_k) < \frac{1}{2}$, for some $\hat{f}_k \in (f_k', \ f_{k+1}')$

2. noise power becomes flat: $N(f) \to N(\hat{f}_k)$, for some $\hat{f}_k \in (f_k', \ f_{k+1}')$

Setting the sum-power constraint within this frequency bin to be $\tilde{p}^*(\hat{f}_k) \cdot (f_{k+1}' - f_k')$, an optimal power allocation as in Figure 4.4 for this frequency bin $(f_k', \ f_{k+1}')$ can be obtained by Procedure 4.2, and a sum-rate of $r^*(\tilde{p}^*(\hat{f}_k); \hat{f}_k) \cdot (f_{k+1}' - f_k')$ can be achieved within this frequency bin.

Thus, a total sum-rate within the entire frequency band (f_1, f_2)

$$\sum_{k=1}^{n} r^*(\tilde{p}^*(\hat{f}_k); \hat{f}_k) \cdot (f_{k+1}' - f_k')$$

can be achieved, with a total weighted sum-power constraint of

$$\sum_{k=1}^{n} \tilde{p}^*(\hat{f}_k) \cdot N(\hat{f}_k) \cdot (f'_{k+1} - f'_k) .$$

With $n \to \infty$, $f'_{k+1} - f'_k = \dfrac{W}{n} \to 0$, we have

$$\sum_{k=1}^{n} r^*(\tilde{p}^*(\hat{f}_k); \hat{f}_k) \cdot (f'_{k+1} - f'_k) \to \int_{f_1}^{f_2} r^*(\tilde{p}^*(f); f) df = R^o ,$$

with

$$\sum_{k=1}^{n} \tilde{p}^*(\hat{f}_k) \cdot N(\hat{f}_k) \cdot (f'_{k+1} - f'_k) \to \int_{f_1}^{f_2} \tilde{p}^*(f) N(f) df \le P$$

Therefore, with the weighted sum-power constraint $\int_{f_1}^{f_2} \tilde{p}^*(f) N(f) df \le P$ satisfied, there exists a power allocation scheme achieving a sum-rate of

$$\int_{f_1}^{f_2} r^*(\tilde{p}^*(f); f) df = R^o ,$$

i.e. $R^* \ge R^o$.

In conclusion, $R^* = R^o$ ■

In the optimization problem (4.12) in Theorem 5, we have

1. For any fixed $f \in (f_1, f_2)$, the integrand of the objective function, namely $r^*(\tilde{p}(f); f)$ is an increasing concave function of $\tilde{p}(f)$.
2. The power constraint $\int_{f_1}^{f_2} \tilde{p}(f) N(f) df \le P$ is linear in $\tilde{p}(f)$

Thus, (4.12) is in the form of a *convex optimization*. The intuition of this convex optimization is clear:

1. The total power needs to be optimally distributed over the whole band.

2. The power allocated to every sub-band needs to be optimally used within this sub-band to achieve the maximum sum-rate.

In Section IV.A., we explicitly solved the 2nd step above, providing an increasing concave sum-rate function of $p \quad r^*(p;f)$ at every frequency point *f*. The original non-convex optimization problem is then naturally reduced to the 1st step above: a primal domain convex optimization. Since we also obtain the explicit characterization of the optimal spectrum and power allocation in flat channels (Procedure 4.2, Figure 4.4), the optimal spectrum and power allocation in frequency selective channels are directly obtained after solving this convex optimization (4.12).

Finally, for the same reason as in Section IV.A., the optimal solution with equal power constraints is the same as that with the corresponding sum-power constraint.

V. Optimal Spectrum Management in the General Cases

In Section IV, we solved the sum-rate maximization problem in two-user symmetric frequency selective channels with equal power (or sum-power) constraints. In this section, we make the following four generalizations:

1. two-user $\rightarrow$ n-user
2. equal power constraints $\rightarrow$ arbitrary individual power constraints
3. symmetric channel $\rightarrow$ arbitrary non-symmetric channel
4. sum-rate $\rightarrow$ arbitrary weighted sum-rate

Furthermore, the generality of frequency selective channels is preserved.

The general optimization problem is thus the following:

$$\begin{aligned} &\max_{P_i(f), i=1,2,\dots,n} \sum_{i=1}^{n} w_i R_i \\ &s.t. \int_{f_1}^{f_2} \mathbf{P}(f) df \le \mathbf{P},\ \mathbf{P}(f) \ge 0, \forall f \in (f_1, f_2) \quad , \\ &R_i = \int_{f_1}^{f_2} \log\left(1 + \frac{P_i(f)}{N_i(f) + \sum_{j \ne i} P_j(f) \alpha_{ji}(f)}\right) df \end{aligned} \tag{5.1}$$

where $\mathbf{P}(f) = \left(P_1(f), \dots, P_n(f)\right)$ and $\mathbf{P} = \left(P_1, \dots, P_n\right)$.

Define the *rate density function* as

$$r(\mathbf{P}(f); f) \triangleq \sum_{i=1}^{n} w_i \log\left(1 + \frac{P_i(f)}{N_i(f) + \sum_{j \ne i} P_j(f) \alpha_{ji}(f)}\right). \tag{5.2}$$

Problem (5.1) can then be rewritten as

$$\max_{P_i(f),i=1,2,...,n} \int_{f_1}^{f_2} r(\mathbf{P}(f);f)df$$
$$s.t. \int_{f_1}^{f_2} \mathbf{P}(f)df \le \mathbf{P},\ \mathbf{P}(f) \ge 0, \forall f \in (f_1, f_2) \tag{5.3}$$

At every frequency point *f*, $r(\mathbf{P}(f);f)$ is a non-concave function of $\mathbf{P}(f)$, and this non-concavity is the key difficulty of all problems in this area of rate maximization. Clearly, the domain of $r(\mathbf{P}(f);f)$ has $n+1$ dimensions: n dimensions of users' power and one dimension of frequency. For every $f \in (f_1, f_2)$ along the frequency dimension, we define $r^*(\mathbf{P}(f);f)$ as the convex hull of $r(\mathbf{P}(f);f)$ along the n dimensions of users' power:

1) Define the set of functions

$$S' = \{\tilde{r}(\mathbf{P}(f);f) \mid \tilde{r}(\mathbf{P}(f);f) \text{ concave in } \mathbf{P}(f);$$
$$\tilde{r}(\mathbf{P}(f);f) \ge r(\mathbf{P}(f);f),\ \forall \mathbf{P}(f) \ge 0\}$$

2) $r^*(\mathbf{P}(f),f)$ is the unique function satisfying

$$\begin{cases} r^*(\mathbf{P}(f);f) \in S' \\ r^*(\mathbf{P}(f);f) \le \tilde{r}(\mathbf{P}(f);f),\ \forall \mathbf{P}(f) \ge 0,\ \forall \tilde{r}(\mathbf{P}(f);f) \in S' \end{cases}$$

Next, we replace the original non-concave rate density function $r(\mathbf{P}(f);f)$ in (5.3) by its convex hull $r^*(\mathbf{P}(f);f)$:

$$\max_{P_i(f),i=1,2,...,n} \int_{f_1}^{f_2} r^*(\mathbf{P}(f);f)df$$
$$s.t. \int_{f_1}^{f_2} \mathbf{P}(f)df \le \mathbf{P},\ \mathbf{P}(f) \ge 0, \forall f \in (f_1, f_2) \tag{5.4}$$

Now we have the following theorem generalizing Theorem 5 to all general cases:

Theorem 6 The convex optimization (5.4) has the same optimal value as the original non-convex optimization (5.3).

Proof:

Denote the optimal value of (5.3) by R^*, and the optimal value of (5.4) by R^o,

1) $R^* \le R^o$, because $r(\mathbf{P}(f); f) \le r^*(\mathbf{P}(f); f)$, $\forall f, \mathbf{P}(f)$.

2) We show that R^o is achievable, i.e. $R^* \ge R^o$.

At any frequency point f, $\forall \mathbf{P}(f) \ge 0$, by definition of the convex hull, there exists a set of points C in the n-dimensional space of users' power, and a weighting function $w(\tilde{\mathbf{P}}(f))$ that satisfy

i) $\int_C r(\tilde{\mathbf{P}}(f); f) w(\tilde{\mathbf{P}}(f)) d\tilde{\mathbf{P}}(f) = r^*(\mathbf{P}(f); f)$ $\forall \mathbf{p} \in C,\ r^*(\mathbf{p}) = r(\mathbf{p})$;

ii) $\int_C \tilde{\mathbf{P}}(f) w(\tilde{\mathbf{P}}(f)) d\tilde{\mathbf{P}}(f) = \mathbf{P}(f)$,

iii) $\int_C w(\tilde{\mathbf{P}}(f)) d\tilde{\mathbf{P}}(f) = 1$.

i.e., $r^*(\mathbf{P}(f); f)$ is generated as the weighted average of $r(\tilde{\mathbf{P}}(f); f)\big|_{\tilde{\mathbf{P}}(f)\in C}$ with the weighting function $w(\tilde{\mathbf{P}}(f))$.

We divide the band (f_1, f_2) into n equal width frequency bins:

$$(f_1' \triangleq f_1, f_2'), (f_2', f_3'), (f_3', f_4'), \dots, (f_n', f_{n+1}' \triangleq f_2), \quad f_{k+1}' - f_k' = \frac{W}{n},\ k = 1, 2, \dots, n.$$

Let $n \to \infty$. Within any bin $(f_k',\ f_{k+1}')$, $|f_k' - f_{k+1}'| \to 0$. From uniform continuity we have the following:

$\forall f \in (f_k',\ f_{k+1}')$

1. channel gain becomes flat: $\alpha(f) \to \alpha(\hat{f}_k)$, for some $\hat{f}_k \in (f_k',\ f_{k+1}')$

2. noise power becomes flat: $N(f) \to N(\hat{f}_k)$, for some $\hat{f}_k \in (f_k',\ f_{k+1}')$

Setting the power constraint within this frequency bin to be $\mathbf{P}(\hat{f}_k) \cdot (f_{k+1}' - f_k')$, then a weighed sum-rate of $r^*(\mathbf{P}(\hat{f}_k); \hat{f}_k)(f_{k+1}' - f_k')$ can be achieved by first dividing (f_k', f_{k+1}') into sub-bins with bandwidths allocated according to $w(\tilde{\mathbf{P}}(\hat{f}_k))$, and then applying flat frequency sharing with the corresponding $\tilde{\mathbf{P}}(f) \in C$ as the power spectral density in each sub-bin. In the mean time, the power constraint $\mathbf{P}(\hat{f}_k)\cdot(f'_{k+1}-f'_k)$ is met because $\int_C w(\tilde{\mathbf{P}}(\hat{f}_k))d\tilde{\mathbf{P}}(\hat{f}_k)=1$ and $\int_C \tilde{\mathbf{P}}(\hat{f}_k)w(\tilde{\mathbf{P}}(\hat{f}_k))d\tilde{\mathbf{P}}(\hat{f}_k)=\mathbf{P}(\hat{f}_k)$.

Thus, a total sum-rate within the entire frequency band (f_1, f_2)

$$\sum_{k=1}^{n} r^*(\mathbf{P}(\hat{f}_k);\hat{f}_k)(f'_{k+1}-f'_k)$$

can be achieved, with a total sum-power constraint of

$$\sum_{k=1}^{n} \mathbf{P}(\hat{f}_k)\cdot(f'_{k+1}-f'_k).$$

With $n\to\infty$, $f'_{k+1}-f'_k=\dfrac{W}{n}\to 0$, we have

$$\sum_{k=1}^{n} r^*(\mathbf{P}(\hat{f}_k);\hat{f}_k)\cdot(f'_{k+1}-f'_k)\to\int_{f_1}^{f_2} r^*(\mathbf{P}(f);f)df$$

with

$$\sum_{k=1}^{n} \mathbf{P}(\hat{f}_k)\cdot(f'_{k+1}-f'_k)\to\int_{f_1}^{f_2}\mathbf{P}(f)df\le\mathbf{P}$$

Thus, any rate of (5.4) can be achieved in (5.3) with the same power constraint satisfied. Therefore, the optimal value of (5.4) is achievable in optimization (5.3), i.e. $R^*\ge R^o$. ■

A fully worked out example as a special case of Theorem 6 is the case we addressed in Section IV. From Theorem 4, we obtained the convex hull $r^*(p(f);f)$ by solving a closed form equation. From Theorem 5, we use $r^*(p(f);f)$ in the objective function and obtain the equivalent convex optimization. As a consequence of Theorem 6, although the primal objective function is non-concave in users' power at every frequency point (5.3), we can simply replace it with the convex hull of it, and solve the resulting convex optimization (5.4) without loss of optimality. The optimal spectrum and power allocation of (5.4) can be transformed to that of (5.3) according to the weighting function $w(\tilde{\mathbf{P}}(f))$ in the above proof.

In this section, we formulated the optimal spectrum management in all general cases into an equivalent primal domain convex optimization. Several remarks on the consequences of our results are below.

Remark 1. In Section IV, $r^*(p(f);f)$ is defined as the convex hull of the upper envelope of flat frequency sharing and flat FDMA, whereas in this section $r^*(\mathbf{P}(f);f)$ is defined as the convex hull of simply flat frequency sharing, both leading to the same optimal value as the original non-convex optimization. The latter definition gives the most concise convex formulation of the originally non-convex problem, whereas the former definition is more explicit in characterizing the optimal spectrum management by capturing how it is optimally composed of FDMA and frequency sharing. In other words, there is a trade-off between simplicity of the model and the explicitness of the solution: If we use a model that captures more characteristics of the solution ($r^*(p(f);f)$), although we need to obtain more knowledge (not only flat frequency sharing rates but also flat FDMA rates) to build this model, we end up with more explicit results (optimal combination of FDMA and frequency sharing) after solving the problem with this model.

Remark 2. In wireless networks, there are often users excluded from the optimization group for practical reasons. These practical reasons can be i) the existence of other heterogeneous systems that are not cooperating, ii) the asynchronism of distributed adaptation algorithms that makes some users not able to cooperate instantaneously, iii) far away weakly coupled users are sometimes neglected, and a local optimization is performed so that a faster decision can be made with loss of some but not much optimality. The excluded users' arbitrary interference become non-symmetric noise to the group of cooperating users, and the normalized cross couplings can also be non-symmetric due to users' different direct channel gains. When dealing with general non-symmetric cases, the generalized primal domain convex optimization we provide in this section serves as a baseline formulation that preserves optimality. However, as listed above, there are often good reasons for practical solutions to use a model that can be solved with lower complexity algorithms (e.g. reducing the scale of optimization by focusing on local cooperation) at the cost of some optimality. The fully worked out two-user case in Section IV (Procedures 4.1, 4.2) serves as a first order optimization with the simplest realization. It is worth pointing out that although the optimal two-user co-existence strategy in Section IV does not guarantee global optimality for all users, the pairwise condition in Section III for any two users to use FDMA, as we proved, does guarantee global optimality.

Remark 3. We have derived all results in the form of spectrum management. All these results also apply to resource management problems in other forms, e.g. finding optimal time division and sharing schemes. Since OFDMA is by nature managing the orthogonalization and sharing of time-frequency resource for all users, with its flexibility and low complexity in implementation, it is a strong contender for optimal multiuser time-frequency management (a combination of orthogonalization and sharing).

Remark 4. Since interference cancellation is not assumed in this paper, inclusion of it could lead to higher capacity at some cost in implementation complexity. As a conjectured consequence if interference cancellation is applied, the greater part of the strong interferences will be cancelled, and the remaining optimization problem is then largely shifted into the sharing regime with low (residual) interference couplings.

VI. Concluding Remarks

In this paper, we have analyzed the evolution of the optimal spectrum and power allocation from FDMA to frequency sharing as the coupling conditions change from extremely strong to extremely weak. We have shown that for any two users, as long as the two normalized cross couplings between them are both larger than or equal to 1/2, an FDMA between these two users benefits every existing user regardless of all users' powers, and hence can be used to achieve any Pareto optimal point of the achievable rate region. Because this interference condition has a pairwise nature, viz. whether any two users should be orthogonalized only depends on the interference coupling between themselves, it leads foreseeably to distributed implementation.

This condition cannot be further lowered as shown in two user symmetric flat channels: when this coupling condition is not satisfied, flat frequency sharing has a higher sum-rate than flat FDMA if and only if the power constraints fall in a precisely characterized region $\overline{P}_{FDMA}$ (which shrinks to zero as the interference coupling rises up to $\geq 1/2$). In the sum-rate maximization problem in two-user symmetric channels with equal power constraints, by solving a closed form equation, we obtained the optimal spectrum and power allocation (which has the clear intuition of combining flat FDMA and flat frequency sharing in an optimal way) for the flat channel cases. Based on this result, we provided an equivalent primal domain convex optimization formulation of the frequency selective channel cases.

For the general *n*-user weighted sum-rate maximization problems in frequency selective channels with arbitrary individual power constraints, we generalized our method in the two-user case and formulated the originally non-convex optimization into an equivalent primal domain convex optimization by replacing the non-concave objective function at every frequency point with its convex hull. This result provides the performance limit and a new perspective into optimal algorithm designs in spectrum management. It also gives a direct insight in understanding why the convex dual problem has a zero duality gap: the primal problem is equivalent to a convex one. Understanding the optimization with the convex hull at every frequency point as the objective function and developing efficient algorithms to solve it are very interesting topics for future research.

This paper has worked on the continuous frequency domain problems, and hence has infinite-dimension variables. The ideas from both the condition for the optimality of FDMA and the primal domain convex optimization formulation can be applied to discrete frequency spectrum management via approximation. With the new insights we obtained for this optimization problem, the rich literature in spectrum balancing algorithms [3] [4] [13] [19] can be reinterpreted, and the design of novel optimal algorithms is an interesting future research direction.

Finally, while we have focused on optimal management schemes, in reality the performance loss resulting from use of FDMA when sharing is optimal can be quite low until well past the interference coupling threshold of 1/2. This may be exploited in the future design of practical low complexity distributed allocation algorithms.

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