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Author
Yellott, John I

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Probability Learning with Noncontingent Success[1]

JOHN I. YELLOTT, JR.

University of Minnesota
Minneapolis, Minnesota 55455

A noncontingent success (NCS) reinforcement schedule for binary prediction is one in which the subject has the same probability (δ) of being correct regardless of which response he makes. These schedules may be contrasted with the more commonly studied noncontingent event (NCE) schedules in which the event probabilities are not contingent on the subject's choice of response, but the probability of his being correct is. The NCS schedules are examined here in connection with the problem of deciding experimentally between the linear and N element models for probability learning. It is shown that for mathematical reasons there is essentially no possibility of making such a decision on the basis of experiments with NCE schedules. Predictions for NCS schedules are then derived from the two models, and an experiment with two such schedules ($\delta = .8$ and $\delta = 1.0$) is reported. The results unequivocally support the N element model over the linear model, but under $\delta = 1$ contingencies subjects generate patterned response sequences—"superstitious solutions"—that cannot be explained by any of the current models.

The present paper is concerned with a class of "noncontingent success" reinforcement schedules for binary prediction experiments. The original motivation for studying this class of schedules was provided by a problem that arises in attempting to evaluate the two most widely studied models for probability learning: the N element pattern model (Atkinson and Estes, 1963) and the linear model with experimenter controlled reinforcing events (Bush and Mosteller, 1955). Briefly stated, that problem is as follows. Although the N element and linear models are similar to the extent that both are simple path-independent conditioning models, they make very different assumptions about the nature of the learning process. According to the N element model, changes in a subject's response probabilities can occur only on trials when the subject makes an incorrect prediction. That is, learning occurs only on errors. According to the linear model, on the other hand, response probability can change on any trial,

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regardless of whether the subject's prediction is correct or incorrect. In view of these different assumptions one might expect that the two models would make quite different predictions about behavior, and that consequently an experimental decision could be made between the two without much difficulty. Surprisingly, this is not the case if one employs the most commonly studied class of reinforcement schedules—schedules with noncontingent events. Under schedules of this sort the two models turn out to be very similar, and in fact it can be shown (see Sect. 2) that in these cases the N element is for all practical purposes indistinguishable from a simple generalization of the linear model. Consequently, to achieve an experimental decision between the two models one must look to some other class of reinforcement schedules. Noncontingent success schedules turn out to be nicely suited to this purpose. Under these schedules the linear and N element models make dramatically different predictions about observable behavior, and consequently a straightforward experimental comparison can easily be achieved.

The organization of the paper is as follows. Section 1 describes the various models and reinforcement schedules and discusses the background of the comparison problem. Section 2 contains a proof that the N element and linear models, and also a third model called the N element-linear model, are all "practically" indistinguishable from one another under reinforcement schedules with noncontingent events. Section 3 gives the predictions of the various models for experiments with noncontingent success reinforcement, and Sec. 4 describes an experiment in which noncontingent success schedules were employed. To anticipate the results of that section, it was found that the N element model is clearly superior to the linear model in a simple comparison between the two, but that both models fail to account for certain rather charming results that one obtains under a noncontingent success schedule in which any response the subject makes is reinforced with probability one.

1. LINEAR AND PATTERN MODELS FOR BINARY PREDICTION

Reinforcement Schedules

Throughout the paper we will be concerned with a standard binary prediction experiment in which the subject is required on each trial to predict which one of two events, e_1 or e_2, will occur. Following each prediction the experimenter causes one or the other event to occur. (A reference experiment is Friedman, Burke, Cole, Keller, Millward, and Estes, 1964.) Let a_1 and a_2 denote the responses associated with the prediction of e_1 and e_2, respectively, $A_{i,n}$ and $E_{j,n}$ the occurrence of a_i and e_j on trial n, and let A_n and E_n denote indicator random variables corresponding to the events $A_{1,n}$ and $E_{1,n}$: $A_n = 1$ if $A_{1,n}$ occurs, and zero otherwise; $E_n = 1$ if $E_{1,n}$ occurs, and zero otherwise. A reinforcement schedule then is simply a sequence of functions $\{\pi_n\}$, where π_n determines the conditional probability of $E_{1,n}$ as a function of the observable history of the experiment up through the response on trial n. This paper deals with two kinds of schedules. The first is the well-known case of "noncontingent reinforcement" in which e_1 occurs with a fixed probability π on each trial independent of the previous history of responses and events:

$$P(E_{1,n} \mid A_n E_{n-1} A_{n-1} \cdots E_1 A_1) = \pi. \tag{1.1}$$

We will refer to schedules satisfying (1.1) as *noncontingent event* (NCE) schedules. This terminology underlines the fact that in this case it is the events $E_{i,n}$ which are independent of the subject's predictions, rather than "reinforcements" in the sense of rewards for correct responses. The distinction is emphasized because in the second class of schedules that concern us here it is the case that correct predictions (and hence rewards) are independent of the subject's choice of response. By a *noncontingent success* (NCS) schedule with parameter δ we mean a schedule satisfying the following condition:

$$P(E_{1,n} \mid A_{1,n} H_{n-1}) = P(E_{2,n} \mid A_{2,n} H_{n-1}) = \delta, \tag{1.2}$$

where H_{n-1} is any history of the form $E_{n-1}A_{n-1} \cdots E_1 A_1$. In other words, under an NCS schedule the probability that the event on any trial n will agree with the prediction on that trial is δ, regardless of which prediction is made, and regardless of the history of the experiment up through trial $n - 1$. Obviously in this case there is no way for the subject to increase or decrease his percentage of correct predictions, so the situation is quite different than that which obtains under NCE schedules, where the probability of a correct response is directly proportional to the probability of predicting the event having the higher probability of occurrence. The NCE and NCS schedules are equivalent only in the special case $\pi = \delta = .5$.

Models for Binary Prediction

Within the class of stochastic learning models introduced by Bush and Mosteller the model which has had the widest application to binary prediction experiments is the *linear model with experimenter controlled reinforcing events* (Bush and Mosteller, 1955, Ch. 10; Estes and Suppes, 1959). Letting p_n denote A_1 response probability (for a single subject) on trial n, this model can be written

$$p_{n+1} = (1 - \theta) p_n + \theta E_n . \tag{1.3}$$

Throughout the paper we use the term "linear model" to refer specifically to (1.3). Historically the linear model has been by far the most widely studied model for probability learning, and in a good number of experiments (e.g., Estes and Straughn, 1954; Friedman *et al.*, 1964; Suppes and Atkinson, 1960) its predictions for a broad range of statistics were surprisingly accurate. In one respect, however, the linear model is consistently deficient: it seriously under-predicts the intersubject variance in asymptotic response probabilities. A model which does a great deal better in this respect, and which also has certain computational advantages, is the *N element pattern model* (Atkinson and Estes, 1963; Estes, 1959). As applied to binary prediction situations, (Atkinson and Estes, 1963) this model assumes that the effective stimulus on each prediction trial can be represented by a fixed set of N stimulus elements. Each element is conditioned either to a_1 or a_2. On each trial the subject samples exactly one of the elements and makes the response conditioned to that element. The probability of sampling any element is assumed to be $1/N$. If the response on a trial is correct the conditioning state of the element sampled on that trial remains unchanged. If the response is incorrect (i.e., disagrees with the event) the conditioning state of the sampled element changes, with probability c, to agree with the event that occurred, while with probability $1 - c$ the conditioning state of the element remains unchanged. In either case the sampled element is returned to the pool of stimulus elements and the same process is repeated on the next trial. Letting k_n denote the number of elements conditioned to a_1 on (at the beginning of) trial n, we have

$$P(A_{1,n} \mid k_n, H_{n-1}) = \frac{k_n}{N},$$

(where H_{n-1} is any history up through trial $n - 1$), and

$$\frac{k_{n+1}}{N} = \begin{cases} \dfrac{k_n}{N} + \dfrac{1}{N} & \text{if } A_{2,n} E_{1,n} C_n \\ \dfrac{k_n}{N} - \dfrac{1}{N} & \text{if } A_{1,n} E_{2,n} C_n, \\ \dfrac{k_n}{N} & \text{otherwise} \end{cases} \tag{1.4}$$

where C_n is an "effectiveness of conditioning" event which has probability c of occurring on trial n independent of the rest of the process.

Although comparisons of the N element and linear models for NCE schedules have found the N element model to be somewhat superior (Friedman *et al.*, 1964, p. 271; Suppes and Atkinson, 1960, Ch. 10), the issue between the two models has not been resolved by these experiments. To begin with, the N element model has two free parameters: c and N, while the linear model has only the learning rate parameter θ. A natural way to equate the number of parameters in the models is to weaken the linear model's assumption that the occurrence of an event always leads to a change in the p value. In place of this assumption we can suppose that the events $E_{1,n}$ and $E_{2,n}$ lead to the transformations $(1 - \theta) p_n + \theta$ and $(1 - \theta) p_n$ only with some probability

c, while with probability $1 - c$, $p_{n+1} = p_n$ regardless of the event on trial n. In this case we have

$$p_{n+1} = \begin{cases} (1-\theta)\,p_n + \theta & \text{if} \quad E_{1,n}\,C_n \\ (1-\theta)\,p_n & \text{if} \quad E_{2,n}\,C_n\,, \\ p_n & \text{otherwise} \end{cases} \tag{1.5}$$

where C_n is defined as in (1.4). In terms of Estes and Suppes' formulation of the linear model (1959) this is equivalent to assuming that with probability $1 - c$ the outcome on any trial leads to an E_0 reinforcing event. The effects of such an assumption in the context of paired associates learning have been studied by Norman (1964), who called this extended linear model the *random trial increment* (RTI) model. In paired associates learning the reason for introducing the parameter c is to account for the relatively small changes observed in precriterion response probabilities. From the standpoint of probability learning, however, it is clear that this extension does not change anything important in the structure of the linear model, or reduce the fundamental conceptual differences between that model and the N element model. In particular the RTI model retains the assumption that changes in response probability can occur on any trial regardless of whether the subject's prediction is correct or incorrect. Intuitively, the only effect of introducing the parameter c is to increase the number of possible sample paths in the p_n process, and thereby increase the variance in the response process. Nevertheless it can be shown that the RTI model and the N element model are virtually identical for experiments using NCE reinforcement schedules. Specifically, it will be shown in Sec. 2 that in the case of NCE reinforcement the two models imply exactly the same functional relationships among the expectations of a large class of statistics, including all those normally considered in the analysis of binary prediction experiments. A consequence of this result is that if any pair of statistics in this class is used (via the method of moments) to estimate the parameters of both models, the two models will predict exactly the same values for every other statistic in the class. From a practical point of view this result all but eliminates the possibility of differentiating between the two models on the basis of results from experiments using NCE schedules.

The fact that it is so difficult in principle to compare the N element and RTI models under NCE reinforcement is not altogether surprising, since a schedule of this sort does not exploit the fundamental difference between the two models. The situation under NCS reinforcement is quite different. Consider the case $\delta = 1$. Since the subject is always "correct" in this condition the N element model predicts that no change will occur over trials in individual subject response probabilities. If a subject begins a sequence of $\delta = 1$ trials with k of the N elements conditioned to a_1 his subsequent responses will form a sequence of Bernoulli trials with parameter k/N. Under the assumptions of the RTI model, on the other hand, a sequence of such trials will cause an individual subject's sequence of p_n values to converge to zero or one as $n \to \infty$. This convergence follows from the fact that when $\delta = 1$, (1.5) implies that $\{p_n\}$ is a martingale, i.e.,

$$E(p_{n+1} \mid p_n, p_{n-1}, ..., p_1) = p_n . \tag{1.6}$$

It is well known that every bounded martingale converges with probability one to a limiting random variable—which we call p_∞ here. (An accessible proof is provided in Khinchin, 1957.) In the case of the RTI model the distribution of p_∞ is concentrated at zero and one, with

$$P(p_\infty = 1 \mid p_1 = p) = p. \tag{1.7}$$

To prove (1.7) it is sufficient to note first that (1.6) implies $E(p_\infty) = E(p_1)$, and second that if $c\theta > 0$, the probability of a p_n sequence converging to a point in the open interval (0,1) is zero. Since every sequence converges to some point in [0,1], every sequence must converge to zero or one.

The convergence of individual p_n sequences in the case $\delta = 1$ is reflected at the level of responses by a decrease over trials in the probability of an alternation—i.e., of a response which differs from the preceding response. It will be shown in Sec. 3 that this probability decreases geometrically as $(1 - c\theta^2)^{n-1}$. The N element model, of course, predicts that the probability of an alternation remains constant over any number of $\delta = 1$ trials. Thus the NCS schedule $\delta = 1$ provides a situation in which the N element and RTI models make sharply different predictions.

The case $\delta = 1$ is not the only NCS schedule which permits a clear comparison of the N element and RTI models. For any δ the RTI model predicts that the probability of an alternation following m consecutive "success" trials (that is, trials on which the schedule generates events agreeing with whatever response the subject happens to make) decreases geometrically with m, whereas the N element model implies that this probability does not depend on m. If δ is relatively large, so that the event "m consecutive success trials" occurs a substantial number of times for several values of m, the observed conditional proportions for $m = 1, 2, ...$ can be used to compare the two models.

Previous experimental studies of NCS schedules for binary prediction appear to be confined to the cases $\delta = .5$ and $\delta = 1$. A good number of experiments have employed sequences of $\delta = .5$ trials, but this has always been regarded as the $\pi = .5$ case of NCE reinforcement, and no one seems to have reported sequential data that would permit a comparison of the sort mentioned in the previous paragraph. The only experiment in which the reinforcement schedule was NCS : $\delta = 1$ appears to be one reported by Arima (1965). In this experiment 40 subjects were run through 30 consecutive $\delta = 1$ trials as part of a larger study dealing with the effects of free versus forced choice trials in probability learning. Before beginning the $\delta = 1$ phase all subjects had fifty $\pi = .5$ trials as a warm-up. The nature of the results in the $\delta = 1$

phase is suggested by Arima's comment: "Closer inspection of the performance of the 40 free-choice control [$\delta = 1$] subjects revealed that 10 subjects perseverated positi on responses [i.e., made the same response on all 30 trials] and 10 subjects perseverated patterns of single, double, and triple alternations over the 30 trials." Arima's results appear to be quite consistent with those obtained in the present study, as will be seen in Sec. 4.

2. NCE SCHEDULES: PROBLEMS IN COMPARING MODELS[2]

Roughly speaking, the object of this section is to show that in practice it is virtually impossible to distinguish between the N element and RTI models on the basis of experiments with NCE schedules of reinforcement. "Virtually" is a necessary qualifier here because strictly speaking it is *not* the case that the two models are identical (i.e., "equivalent" in Greeno and Steiner's, 1964, terminology) under NCE reinforcement. What can be shown instead is that the two are strictly identical so long as one confines his attention to a certain class of statistics Σ, and in addition that the differences between their predictions for statistics outside the set Σ are so small as to be essentially undetectable in any feasible experiment. From a practical standpoint these results effectively rule out the possibility of making any unambiguous decision between the two models based on NCE reinforcement, even though the two are not actually identical under such schedules.

The results of binary prediction experiments are usually analyzed in terms of a number of summary statistics. These generally include the mean learning curve and various "asymptotic sequential statistics." The latter are the averages (across subjects) of the averages (over trials) of such random variables as $(A_n E_n A_{n+1})$. Normally these are evaluted for $n \geqslant n^*$, where n^* is relatively large, say 100, in order that they may be assumed to be estimates of corresponding asymptotic probabilities, e.g., $\lim_{n\to\infty} P(A_n E_n A_{n+1})$ (see, for example, Friedman *et al.*, 1964). Typically a model specifies the expectations of these statistics as functions of its parameters. The parameters are estimated by equating a sufficient number of observed statistics to their theoretical expectations and solving these equations for the parameters. The estimates obtained in this way are then used to predict the remaining statistics by substituting into the expressions for their expectations. In this sense evaluation involves using the model to predict relationships between observed statistics; the model fits the data to the extent that the observed statistics satisfy the predicted relationships between their expectations. Other estimation methods can be used, e.g., least squares minimization and pseudo maximum likelihood techniques (Suppes and Atkinson, 1960), but these achieve essentially the same result. A comparison between two models involves exactly the same procedure; predictions are computed for both models and compared

[2] I thank J. L. Myers for suggesting a very helpful simplification of the results of this section.

to the data; one model is judged superior to another if its predictions are substantially closer to the data.

Clearly the results of the sort of comparison procedure just described depend on the class of statistics considered. To take an extreme example, if only the mean learning curve is examined in an NCE experiment, the one element case of the N element model cannot be distinguished from the RTI model, since in both cases the appropriate theoretical expression is

$$P(A_{1,n}) = \pi + (1 - \alpha)^{n-1} (P(A_{1,1}) - \pi),$$

where α equals c for the one element and $c\theta$ for the RTI model. However, the two models are easily distinguished on the basis of the sequential statistic $\hat{P}(A_{1,n+1} \mid A_{1,n}, E_{1,n})$. This suggests that the more statistics considered the better. In practice, however, the number of statistics employed in evaluation is limited by the amount of computation involved in obtaining theoretical expressions, and, more significantly, by the requirement that statistics used for this purpose be based on a substantial number of cases. Consequently, in practice, the evaluation of models for the NCE case of binary prediction has almost always been confined to a fairly restricted class of statistics—in particular, statistics corresponding to the quantities:

$$P(A_{1,n});\ n = 1, 2,..., \tag{2.1.1}$$

$$\lim_{n\to\infty} P(A_{1,n} \mid E_{i,n-1}A_{j,n-1}), \qquad i = 1, 2; j = 1, 2, \tag{2.1.2}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid A_{j,n}), \qquad j = 1, 2; m = 1, 2,..., \tag{2.1.3}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{j,n}), \qquad j = 1, 2; m = 1, 2,..., \tag{2.1.4}$$

$$\lim_{n\to\infty} \mathrm{Var}\left(\sum_{k=1}^{m} A_{n+k}\right), \qquad m = 1, 2,..., \tag{2.1.5}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{i,n} \cdots E_{i,n+m-1}), \qquad i = 1, 2; m = 1, 2,.... \tag{2.1.6}$$

We are aware of only one study which has considered statistics other than those corresponding to (2.1) (Suppes and Atkinson, 1960, Ch. 10). Now let Σ denote the class of statistics having as their expectations the quantities (2.1.1)–(2.1.6), and let s denote an arbitrary member of Σ. The following result shows that the N element and RTI models are indistinguishable over Σ:

Let $n_s(c, N)$ denote the value of s predicted by the N element model as a function of its parameters c and N, and *$r_s(c, \theta)$ the value predicted by the RTI model as a function of its parameters c and θ. Then for every s in Σ there exists a function $p_s(U, V)$ such that*

$$n_s(c, N) = p_s(U, V) = r_s(c, \theta), \tag{2.2.1}$$

where

$$U = \frac{c}{N}, \qquad V = \frac{1-c}{N}, \tag{2.2.2}$$

in the N element model, and

$$U = c\theta, \qquad V = \frac{\theta(1-c\theta)}{2-\theta}, \tag{2.2.3}$$

in the RTI model.

In other words, the predictions of the N element and RTI models within Σ depend on their respective parameters c, N and c, θ only through U and V, and they are the same functions of U and V in all cases. This implies, first of all, that the two models will make identical predictions throughout Σ whenever

$$\frac{c}{N} = c'\theta, \qquad \frac{1-c}{N} = \frac{\theta(1-c'\theta)}{2-\theta}. \tag{2.3}$$

(The parameter c of the RTI model is denoted here by c'.) In addition, (2.2) implies that if the same pair of Σ statistics is used in estimating the parameters of both models (via the method of moments), then their predictions based on these estimates will agree for every other Σ statistic. This follows from the fact that a pair of statistics which generates separate estimates of c and N, and c' and θ, must generate separate estimates of U and V, and since these estimates (by 2.2.1) will be the same for both models, (2.3) will hold and their remaining predictions will be identical.

The proof of (2.2) involves simply showing that the theoretical expressions for (2.1.1)–(2.1.6) satisfy (2.2.1)–(2.2.3) under both models. In order to do this it is convenient to consider a very general and rather cumbersome model for binary prediction which we call the *c-θ-N model.* This model contains as special cases the linear model, the N element model, the RTI model, and a third two-parameter model: the N element-linear model, which is equivalent to the other two over the class Σ. The c-θ-N model was originally suggested by Estes as a device for comparing these special cases. By estimating all the parameters of the general model from the results of an experiment one could, hopefully, determine which of the special cases came closest to the data without having to perform a separate analysis for each case. The difficulties which arose in attempting to carry out this program led to the result which is the subject of this section.

In the N element model we assume that each of the stimulus elements is conditioned in an all-or-none fashion to a_1 or a_2. The c-θ-N model generalizes this notion by assuming that corresponding to every element $W_i (i = 1, 2, \ldots, N)$ there is a number $0 \leqslant w_i \leqslant 1$ which represents the probability of making response a_1 when W_i is sampled. Let $w_{i,n}(i = 1, 2, \ldots, N)$ denote the a_1 response probability attached to element W_i at the beginning of trial n. We assume that:

A1. On every trial the subject samples exactly one element; the probability of sampling any element is $1/N$.

A2. If W_i is sampled on trial n, the probability of $A_{1,n}$ is $w_{i,n}$.

A3. If W_i is sampled on trial n, then regardless of the response on trial n,

$$w_{i,n+1} = (1 - \theta) w_{i,n} + \theta E_n \qquad \text{with probability } c,$$

and

$$w_{i,n+1} = w_{i,n} \qquad \text{with probability } 1-c,$$

where c is a constant for all n and i.

A4. If W_i is not sampled on trial n, then with probability one

$$w_{i,n+1} = w_{i,n}.$$

Relevant special cases of the c-θ-N model arise when $c = N = 1$ (the linear model); $N = 1$ (the RTI model); $\theta = 1$ (the N element model); and $c = 1$ (the N element-linear model). Occasionally, below, we refer to the three two-parameter cases in terms of their free parameters:

The c-θ model $\sim$ RTI model,
c-N model $\sim$ N element model,
θ-N model $\sim$ N element-linear model.

The following expressions can be derived from the c-θ-N model for a binary prediction experiment with NCE reinforcement and e_1 probability π (Yellott, 1965, App. *B*):

$$P(A_{1,n}) = (1 - U)^{n-1} P(A_{1,1}) + \pi[1 - (1 - U)^{n-1}], \tag{2.4.1}$$

$$\lim_{n\to\infty} P(A_{1,n+1} \mid E_{1,n} A_{1,n}) = (1 - \pi)(U + V) + \pi, \tag{2.4.2}$$

$$\lim_{n\to\infty} P(A_{1,n+1} \mid E_{2,n} A_{1,n}) = (1 - \pi)V + \pi(1 - U), \tag{2.4.3}$$

$$\lim_{n\to\infty} P(A_{1,n+1} \mid E_{1,n} A_{2,n}) = \pi(1 - U - V) + U, \tag{2.4.4}$$

$$\lim_{n\to\infty} P(A_{1,n+1} \mid E_{2,n} A_{2,n}) = \pi(1 - U - V), \tag{2.4.5}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid A_{1,n}) = (1 - \pi) V(1 - U)^{m-1} + \pi, \tag{2.4.6}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid A_{2,n}) = \pi - \pi V(1 - U)^{m-1}, \tag{2.4.7}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{1,n}) = \pi + (1 - \pi)\, U(1 - U)^{m-1}, \tag{2.4.8}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{2,n}) = \pi - \pi U(1 - U)^{m-1}, \tag{2.4.9}$$

$$\lim_{n\to\infty} \operatorname{Var} \sum_{k=1}^{m} A_{n+k} = m\pi(1 - \pi) + 2\pi(1 - \pi)\frac{V}{U}\left[m - \frac{1}{U}(1 - (1 - U)^m)\right], \tag{2.4.10}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{1,n}, \ldots, E_{1,n+m-1}) = (1 - U)^m\pi + [1 - (1 - U)^m], \tag{2.4.11}$$

$$\lim_{n\to\infty} P(A_{1,n+m} \mid E_{2,n}, \ldots, E_{2,n+m-1}) = \pi(1 - U)^m, \tag{2.4.12}$$

where

$$U = \frac{c\theta}{N}, \qquad V = \frac{\theta(1 - c\theta)}{N(2 - \theta)}.$$

A comparison of (2.1.1)–(2.1.6)and Eqs. 2.4 will show that this list contains theoretical expressions for all of the statistics in Σ. Since all of the expressions in (2.4) are functions of U and V alone, specialization to the appropriate two-parameter cases of the c-θ-N model completes the proof of (2.2). In addition it follows immediately from Eqs. 2.4 that within Σ the θ-N model is identical to the N element and RTI models in the sense of (2.2).

It remains now to show that the N element and RTI models are not strictly identical under NCE reinforcement. Earlier we mentioned that there is one type of asymptotic sequential statistic that has been used to evaluate models for NCE experiments (Suppes and Atkinson, 1960, Ch. 10), but is not in Σ. This is the set of second order sequential statistics corresponding to

$$\lim_{n\to\infty} P(A_n \mid E_{n-1}\, A_{n-1}\, E_{n-2}\, A_{n-2}).$$

These quantities were excluded from Σ because (2.2) does not, in fact, hold for them. A moment's thought will indicate the reason for this. It will be recalled from Sec. 1 that the conditional probability of an alternation following m success trials under NCS reinforcement is a constant for all m in the N element model, but declines geometrically with m in the RTI model whenever $c\theta > 0$. In particular consider the quantities

$$\lim_{n\to\infty} P(A_n \neq A_{n-1} \mid A_{n-1} = E_{n-1}), \tag{2.5}$$

$$\lim_{n\to\infty} P(A_n \neq A_{n-1} \mid A_{n-1} = E_{n-1}, A_{n-2} = E_{n-2}). \tag{2.6}$$

Under NCS reinforcement with $0 < \delta < 1$ the RTI model implies that (2.6) is strictly less than (2.5) if $c\theta > 0$: the ratio (2.6) over (2.5) is $(1 - c\theta^2)$. The N element model, on the other hand, implies (2.6) = (2.5) for all values of c and N, if $c > 0$ (see Sec. 3). Since the NCS schedule $\delta = .5$ is equivalent to the NCE schedule $\pi = .5$ it is clear that the N element and RTI models are not strictly identical for all NCE schedules. In particular, (2.2) cannot hold if Σ includes statistics whose expectations are equal to (2.5) and (2.6). Since (2.6) is a compound of the second order sequential probabilities $P(A_n \mid E_{n-1}A_{n-1}E_{n-2}A_{n-2})$, it is clear that these quantities must be excluded from Σ if (2.2) is to hold. However it should be noted that, for the range of parameters normally encountered in practice, the ratio of (2.6) to (2.5) implied by the RTI model will be very close to 1, i.e., $(1 - c\theta^2)$ will be on the order of .95 (using parameter estimates based on asymptotic sequential statistics, cf. Sec. 4). Consequently, we would not expect the second order sequential statistics of a $\pi = .5$ experiment to support a clear choice of one model over the other. One could, of course, consider longer runs of successes, calculate the corresponding probabilities analogous to (2.6), and compare these to (2.5). But the number of observations contributing to these estimates would decrease by one-half for each additional success, while the relevant comparison ratio would decrease each time by a factor of only .95. This seems at best an inefficient procedure; certainly it would require a fairly massive experiment. The next section deals with NCS reinforcement and shows that for these schedules the N element and RTI models make nonparametrically different predictions which can easily be compared in an experiment of reasonable dimensions.

3. PREDICTIONS FOR NONCONTINGENT SUCCESS SCHEDULES

This section describes the predictions of the N element, RTI, and N element-linear models for noncontingent success schedules. For the most part the results presented here have been obtained by methods commonly employed with these models (Atkinson and Estes, 1963; Estes and Suppes, 1959; Sternberg, 1963), and consequently most of the derivations have been omitted. They can be found in Yellott (1965, pp. 32–59). The predictions of primary relevance to the purposes of this paper are those describing the effects of consecutive $\delta = 1$ trials on response alternation probability, since as indicated in Sec. 1, these differentiate between the models in a nonparametric fashion. To deal with these predictions some additional notation is required. The NCS schedules can be thought of as generating Bernoulli sequences of "successes" and "failures" in the same way that NCE schedules generate Bernoulli sequences of e_1's and e_2's. We will use Δ_n to denote the event corresponding to a noncontingent success on trial n, and $\tilde{\Delta}_n$ to denote the occurrence of a noncontingent failure on trial n. We define indicator random variables $\{S_n\}_{n=2}^{\infty}$ by

$$S_n = A_n(1 - A_{n-1}) + A_{n-1}(1 - A_n).$$

Clearly $S_n = 1$ if a response alternation occurs on trial n (i.e., if $A_n \neq A_{n-1}$), and

$S_n = 0$ otherwise. We will be concerned with the conditional probability of an alternation given m consecutive success trials: $P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m})$. Occasionally the intersection $\Delta_{n-1}\Delta_{n-2} \cdots \Delta_{n-m}$ will be denoted by Δ_{n-1}^m; in this case we write $P(S_n = 1 \mid \Delta_{n-1}^m)$.

Since NCS schedules are a special case of simple contingent schedules, and the latter have been widely studied, we can make use of a number of previous results. In particular, whenever $0 < \delta < 1$ and the learning rate parameter is greater than zero (i.e., $c\theta > 0$, $c/N > 0$, $\theta/N > 0$ in the c-θ, c-N, θ-N models, respectively) the response probability random variables p_n, k_n/N, and $W(n)$ have asymptotic distributions that depend only on δ and the learning parameters. In the case of the N element model this is shown in Atkinson and Estes (1963). For the RTI model it follows from a theorem of Lamperti and Suppes (1959, Theorem 4.1) if we let the events $C_nE_{1,n}$ and $C_nE_{2,n}$ correspond to the events "$E_n = 1$" and "$E_n = 2$" in their formulation, and interpret the event $\bar{C}_n$ (C_n complement) to be their "$E_n = 0$" event. A proof for the N element-linear model follows from the ergodicity of the RTI model. Consider the process $\{w_{i,n}\}_{n=1}^{\infty}$ corresponding to stimulus element W_i. If W_i is sampled (with probability $1/N$), then $w_{i,n+1} = (1 - \theta)\, w_{i,n} + \theta E_n$, otherwise $w_{i,n+1} = w_{i,n}$. Since E_n depends only on A_n, the change in $\{w_{i,n}\}$ when W_i is sampled depends only on the current value of that subprocess. It is clear then that $\{w_{i,n}\}$ is an RTI process, with $c = 1/N$. And it is clear that as $n \to \infty$, the random variables $w_{i,n}$, $i = 1, 2,..., N$, become independent and identically distributed, so that

$$\lim_{n\to\infty} P(w_{1,n} \leqslant \omega_1 ,..., w_{N,n} \leqslant \omega_N) = \prod_{i=1}^{N} \lim_{n\to\infty} P(w_{1,n} \leqslant \omega_i).$$

The case $\delta = 1$ leads to a non-ergodic process in all of the models considered. Consequently we deal with it as a special case, beginning with the N element model.

Before considering the various models separately, however, we note that the mean learning curve under $\delta = 1$ contingencies is a nonparametric prediction of the c-θ-N model, and hence is common to all three of its two-parameter specializations. Let $\bar{w}_n$ denote the mean of the N A_1 response probabilities $w_{i,n}$. Then $P(A_{1,n}) = E(\bar{w}_n)$. When $\delta = 1$, an easy calculation shows that

$$E(w_{i,n}) = E(E(w_{i,n} \mid w_{i,n-1})) = E(w_{i,n-1})$$

(where the inner expectation is over the possible events on trial $n - 1$ and the outer one is over possible values of $w_{i,n-1}$). Consequently, under $\delta = 1$ contingencies the c-θ-N model predicts

$$P(A_{1,n}) = P(A_{1,1}), \tag{3.1}$$

for all n. This common prediction of the c-θ-N models has been referred to as *marginal constancy*. Some implications of requiring that a model predict marginal constancy under $\delta = 1$ reinforcement have been explored by Rose (1965), and Norman and Yellott (1966).

$\delta = 1$: *The N Element Model*

As before, k_n denotes the number of elements conditioned to a_1 at the beginning of trial n. Clearly in the case $\delta = 1, k_n = k_1$, since no element can change its conditioning state on a success trial. Consequently, for a fixed value of k_1 the sequence $\{A_n ; n = 1, 2, \ldots\}$ is a Bernoulli sequence, with $P(A_n = 1) = k_1/N$. The probability of an alternation in such a sequence is

$$P(S_n = 1 \mid k_1) = 2\frac{1}{N}k_1\left(1 - \frac{1}{N}k_1\right) \qquad n = 2, 3, \ldots \tag{3.2}$$

In general we regard k_1 as a random variable, with $V_1 = (1/N)\,E(k_1)$, $V_2 = (1/N^2)\,E(k_1{}^2)$. In this case $P(A_{1,n}) = V_1$, and

$$P(S_n = 1) = 2(V_1 - V_2). \tag{3.3}$$

Suppose, in particular, that a group of subjects is run for a large number of trials on an NCE schedule with e_1 probability π_0, and then shifted to an NCS : $\delta = 1$ schedule. The asymptotic distribution of the number of conditioned elements in the NCE phase will be binomial with parameters π_0 and N (Atkinson and Estes, 1963), so this will also be the distribution of k_1 in the $\delta = 1$ phase. If $\bar{A}_n$ and $\bar{S}_n$ denote the proportions of A_1 responses and alternations over the group on trial n, then $E(\bar{A}_n) = \pi_0$, and

$$E(\bar{S}_n) = 2\pi_0(1 - \pi_0)\frac{N-1}{N}. \tag{3.4}$$

$\delta = 1$: *The RTI Model*

Rose (1965) provides a very thorough treatment of the case $c = 1$ (the linear model). We will confine our discussion to a few predictions on which the RTI and N element models disagree. Let $V_1 = E(p_1) = P(A_{1,1})$, $V_{2,n} = E(p_n{}^2)$, and $V_2 = V_{2,1}$. Then it is easily shown that

$$V_{2,n} = V_1 + (V_2 - V_1)(1 - c\theta^2)^{n-1}. \tag{3.5}$$

The convergence of the p_n random variables can be measured by $E(p_n - p_{n-1})^2$. Using (3.5),

$$E(p_n - p_{n-1})^2 = (V_1 - V_2)\,c\theta(1 - c\theta^2)^{n-2}, \tag{3.6}$$

i.e., the convergence is geometric in $(1 - c\theta^2)$. At the level of responses this geometric convergence is reflected in a geometric decrease in alternation probability:

$$P(S_n = 1) = 2(1 - c\theta)(V_1 - V_2)(1 - c\theta^2)^{n-2}. \tag{3.7}$$

Note that in (3.7) the quantities V_1 and V_2 can be regarded either as moments of a nondegenerate p_1 distribution, or as p and p^2 in the case of a particular sequence with initial a_1 probability $p_1 = p$.

A comparison of (3.7) and (3.3) indicates a clear difference between the RTI and N element models. According to (3.7) the expected proportion of alternations per trial should decrease geometrically to zero whenever the learning rate $c\theta$ is positive, whereas according to (3.3) there should be no trend at all over a sequence of success trials of any length.

$\delta = 1$: *The N Element-Linear Model*

The asymptotic behavior of the θ-N model under double reward is determined by the fact that each of the response probability processes $\{w_{i,n}\}$, $i = 1, 2,..., N$, is in effect an RTI process, consequently a martingale, and hence converges to one or zero, with

$$P(\lim_{n\to\infty} w_{i,n} = 1 \mid w_{i,1}) = w_{i,1} \,.$$

In the limit then, each of the w_i will be zero or one, and the behavior of the subject will be determined by random sampling from a collection of what amount to conditioned elements, just as it is in the N element model. The difference between the two models is that in the θ-N model the probability of an alternation *decreases* to the asymptotic value, whereas the N element model implies that this probability is a constant over the entire sequence of success trials.

In computing $P(S_n = 1)$ it is convenient to assume that $\{w_{i,1}\}_{i=1}^{N}$ are identically distributed and pairwise uncorrelated. This will be true if the distribution of $(w_{i,1}, ..., w_{N,1})$ is the asymptotic distribution corresponding to any NCE schedule (e.g., if the subjects are run to asymptote on an NCE schedule before starting the $\delta = 1$ series), so the assumption is not unduly restrictive. Its motivation is the fact that when the $\{w_{i,n}\}$ are initially uncorrelated they remain uncorrelated over a $\delta = 1$ series, which simplifies things considerably. The assumption also allows us to write $V_2 = E(w_{i,1}^2)$ for $i = 1, 2,..., N$. To compute $P(S_n)$ one can show first that regardless of the initial distribution

$$E(A_n A_{n-1}) = E[(\bar{w}_{n-1})^2] + \frac{\theta}{N} E(\bar{w}_{n-1}) - \frac{\theta}{N^2} E\left(\sum_i w_{i,n-1}^2\right), \tag{3.8}$$

and from (3.8) and the assumption just mentioned it follows that

$$E(A_n A_{n-1}) = \frac{V_1}{N} + \frac{N-1}{N} V_1^2 + (1-\theta)\frac{(V_2 - V_1)}{N}\left(1 - \frac{\theta^2}{N}\right)^{n-2}, \tag{3.9}$$

where $V_1 = P(A_{1,1})$. Using (3.9), (3.1), and the definition of S_n, it follows that

$$P(S_n = 1) = 2V_1(1 - V_1)\frac{N-1}{N} + 2\frac{(1-\theta)}{N}(V_1 - V_2)\left(1 - \frac{\theta^2}{N}\right)^{n-2}. \quad (3.10)$$

The second term in (3.10) vanishes geometrically as $n \to \infty$, so that in the limit the probability of an alternation is simply $2V_1(1 - V_1)[(N - 1)/N]$. Assuming prior training to asymptote on an NCE schedule with $\pi = \pi_0$, this limit becomes $2\pi_0[(1 - \pi_0)(N - 1)/N]$, which is exactly $P(S_n = 1)$ for the N element model under the same assumption.

This completes the discussion of the case $\delta = 1$. We next consider NCS schedules with $\delta < 1$, starting with the RTI model.

$\delta < 1$: *The RTI Model*

For any c-θ-N model with $0 < (c\theta/N) < 1$, the mean learning curve in the case NCS: $\delta < 1$ is

$$P(A_{1,n}) = \tfrac{1}{2} + (1 - \lambda)^{n-1}(V_{1,1} - \tfrac{1}{2}), \quad (3.11)$$

where $\lambda = [2c\theta(1 - \delta)/N]$, and $V_{1,1} = \bar{w}_1 = P(A_{1,1})$.

Equation 3.11 follows from the fact that, for any c, θ, N:

$$E(w_{i,n+1}) = 1 - \frac{2c(1-\delta)\theta}{N}E(w_{i,n}) + \frac{c\theta(1-\delta)}{N}.$$

When $0 < c\theta/N < 1$ this has the solution

$$E(w_{i,n}) = \tfrac{1}{2} + (1 - \lambda)^{n-1}[E(w_{i,1}) - \tfrac{1}{2}],$$

and, since $P(A_{1,n}) = E(\bar{w}_n)$, (3.11) follows. If $c = \theta = N = 1$, (3.11) holds provided $0 < \delta < 1$. If $\delta = 0$ and $c\theta/N$ is 1, the predicted response–outcome process is an alternating sequence of the form (a_1, e_2), (a_2, e_1), (a_1, e_2),..., and A_n is determined by A_1. In experimental applications the learning rate is never close to one, and it is convenient to have (3.11) hold for all $\delta < 1$, so in the remainder of the section we will assume $0 < c\theta/N < 1$. In this case for every $\delta < 1$ (3.11) implies

$$\lim_{n\to\infty} P(A_{1,n}) = \tfrac{1}{2}. \quad (3.12)$$

The learning curve for the RTI model is given by (3.11) with $\lambda = c\theta$, and $V_{1,1} = E(p_1) = P(A_{1,1})$. Generalizing the V notation, let $V_{j,n} = E(p_n^{\,j})$ and $V_{j,\infty} = \lim_{n\to\infty} V_{j,n}$.

To compute the conditional probability of an alternation given m consecutive success trials, note that $P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m} p_{n-m} = p)$ is simply the probability of an alternation on trial $m + 1$ of a $\delta = 1$ sequence, with $V_1 = p$. Using (3.7), it follows that

$$P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m}) = 2(1 - c\theta)(1 - c\theta^2)^{m-1} (V_{1,n-m} - V_{2,n-m}), \tag{3.13}$$

so that asymptotically

$$\lim_{n\to\infty} P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m}) = 2(1 - c\theta)(1 - c\theta^2)^{m-1}(\tfrac{1}{2} - V_{2,\infty}). \tag{3.14}$$

where $V_{2,n}$ can be obtained from the following difference equation:

$$\begin{aligned} V_{2,n+1} = {} & V_{2,n}\{1 - c\theta[4 - 3\theta - 4\delta(1 - \theta)]\} \\ & + V_{1,n}\{\theta(2\delta - 1) + 2(1 - \delta)(1 - \theta)\} + c(1 - \delta)\,\theta^2. \end{aligned} \tag{3.15}$$

This has the asymptotic solution:

$$V_{2,\infty} = \frac{2(1 - \theta)(1 - \delta) + \theta}{2[4 - 3\theta - 4\delta(1 - \theta)]}\,. \tag{3.16}$$

The conditional probability of an alternation given m successive *failure* trials can be shown to be

$$\begin{aligned} & \lim_{n\to\infty} P(S_n = 1 \mid \bar{\Delta}_{n-1} \cdots \bar{\Delta}_{n-m}) \\ & \quad = 1 - 2(1 - c\theta)\{(1 - x)^{m-1}(V_{2,\infty} - y) + y\}, \end{aligned} \tag{3.17}$$

where

$$x = c\theta(4 - 3\theta), \qquad y = \frac{(2 - \theta)}{2(4 - 3\theta)}\,.$$

Let $s(\bar{m})$ denote the right side of (3.17) as a function of m. Then for any $\delta > 0$, $(V_{2,\infty} - y) > 0$, and $s(\bar{m})$ is increasing in m. In other words, the conditional probability of an alternation, given m consecutive failure trials, increases with m. It will be shown below that this is not the case in the N element model; there the probability is independent of m.

The last results we will present for the RTI model are the asymptotic first order sequential probabilities. These are useful in parameter estimation.

$$\lim_{n\to\infty} P(A_{1,n+1} \mid E_{1,n}A_{1,n}) = 2(1 - c\theta)V_{2,\infty} + c\theta, \tag{3.18}$$

$$\lim P(A_{1,n+1} \mid E_{2,n}A_{1,n}) = 2(1 - c\theta)V_{2,\infty}\,, \tag{3.19}$$

$$\lim P(A_{1,n+1} \mid E_{2,n}A_{2,n}) = (1 - c\theta)(1 - 2V_{2,\infty}), \tag{3.20}$$

$$\lim P(A_{1,n+1} \mid E_{1,n}A_{2,n}) = 1 - 2(1 - c\theta)V_{2,\infty}\,. \tag{3.21}$$

$\delta < 1$: *The N Element Model*

The learning curve is given by (3.11), with $\lambda = c/N$ and $V_{1,1} = (1/N) E(k_1)$. The asymptotic distribution of k_n can be shown to be binomial and independent of δ :

$$\lim_{n\to\infty} P(k_n = k) = \binom{N}{k}\left(\frac{1}{2}\right)^N. \tag{3.22}$$

Let $V_{j,n} = (1/N^j) E(k_n{}^j)$, and $V_{j,\infty} = \lim_{n\to\infty} V_{j,n}$. Equation 3.22 implies

$$V_{1,\infty} = \frac{1}{2}; \qquad V_{2,\infty} = \frac{N+1}{4N}; \qquad V_{3,\infty} = \frac{3+N}{8N}. \tag{3.23}$$

Then using (3.2) it is easily shown that

$$P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m}) = 2(V_{1,n-m} - V_{2,n-m}),$$

so that asymptotically

$$\lim_{n\to\infty} P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m}) = 1 - 2V_{2,\infty} = \frac{N-1}{2N}. \tag{3.24}$$

Note that for the N element model, in contrast to the RTI model, the asymptotic conditional probability of an alternation given m consecutive success trials is independent of m. (cf. Eq. 3.14.) This also holds for the conditional probability of an alternation given m consecutive failure trials; for any m

$$\lim_{n\to\infty} P(S_n = 1 \mid \bar{\Delta}_{n-1} \cdots \bar{\Delta}_{n-m}) = \frac{N-1}{2N} + \frac{c}{N}. \tag{3.25}$$

For our purposes (3.24) and (3.25), and the corresponding Eq. (3.14) and (3.17), provide the major points of comparison between the N element and RTI models for the case $\delta < 1$. The remaining results to be presented in this section are the asymptotic first and second order sequential probabilities of the N element model for $\delta < 1$, and the asymptotic variance of the total number of a_1 responses in blocks of M trials. These are primarily useful in evaluating the goodness of fit of the N element model, apart from its comparative merits vis-a-vis the RTI model. Following standard practice the trial indices are omitted, and the sequence of events in the conditionalization is given in decreasing time order from left to right. Thus:

$$P(A_i \mid E_j A_k E_s A_t) = \lim_{n\to\infty} P(A_{i,n} \mid E_{j,n-1} A_{k,n-1} E_{s,n-2} A_{t,n-2}).$$

V_2 and V_3 are used to denote $V_{2,\infty}$ and $V_{3,\infty}$ as given by (3.23).

$$P(A_1 \mid E_1 A_1) = 2V_2, \tag{3.26}$$

$$P(A_1 \mid E_2A_1) = 2V_2 - \frac{c}{N}, \tag{3.27}$$

$$P(A_1 \mid E_1A_1E_1A_1) = \frac{V_3}{V_2}, \tag{3.28}$$

$$P(A_1 \mid E_1A_1E_2A_2) = \frac{2(V_2 - V_3)}{1 - 2V_2}, \tag{3.29}$$

$$P(A_1 \mid E_2A_1E_1A_1) = \frac{(V_3 - (c/N)V_2)}{V_2} \tag{3.30}$$

$$P(A_1 \mid E_1A_1E_2A_1) = \frac{V_3 - (2c/N)V_2 + (c/2N^2)}{V_2 - (c/2N)}, \tag{3.31}$$

$$P(A_1 \mid E_1A_2E_1A_1) = 2\,\frac{(V_2 - V_3) + (c/N)(1/2 - V_2)}{1 - 2V_2}, \tag{3.32}$$

$$P(A_1 \mid E_1A_1E_1A_2) = \frac{2(V_2 - V_3) + (4c/N)(1/2 - V_2) + (c/N^2)}{1 - [2V_2 - (c/N)]}, \tag{3.33}$$

$$P(A_1 \mid E_2A_1E_2A_1) = \frac{V_3 + (c/N^2)(1 + 2c)(1/2 - 2V_2)}{V_2 - (c/2N)}, \tag{3.34}$$

$$P(A_1 \mid E_1A_2E_2A_1) = \frac{V_2 - V_3 + (1/N)\,cV_2 - (1/2N^2)\,c(1 - c)}{1/2 + (1/2N) - V_2}, \tag{3.35}$$

$$\lim_{n\to\infty} \operatorname{var} \sum_{j=n+1}^{n+M} A_j = \frac{M}{4} + 2Q\left[V_2 - \frac{1}{4}(1 + \lambda)\right], \tag{3.36}$$

where

$$Q = M\left[\frac{1 - \beta^{M-1}}{1 - \beta}\right] - \sum_{i=1}^{M-1} i\beta^{i-1},$$

$$\beta = 1 - \lambda = 1 - \frac{2c(1 - \delta)}{N}.$$

Note that this list contains only 10 of the 20 possible first- and second-order sequential probabilities. The remaining 10 can be obtained from those given by substituting A_2 for A_1, A_1 for A_2, E_2 for E_1, and E_1 for E_2, in each of the Eqs. 3.26–3.35. Thus, for example,

$P(A_2 \mid E_1A_2) = P(A_1 \mid E_2A_1)$ (Eq. 3.27), $P(A_2 \mid E_2A_2E_1A_2) = P(A_1 \mid E_1A_1E_2A_1)$ (Eq. 3.31), etc.

$\delta < 1$: *The N Element-Linear Model*

The learning curve is given by (3.11) with $c = 1$ and $V_{1,1} = E(w_1) = P(A_{1,1})$. The asymptotic probability of an alternation given m consecutive success trials is

$$\lim_{n\to\infty} P(S_n = 1 \mid \Delta_{n-1} \cdots \Delta_{n-m}) = \frac{N-1}{2N} + 2\,\frac{(1-\theta)}{N}\left(\frac{1}{2} - V_{2,\infty}\right)\left(1 - \frac{\theta^2}{N}\right)^{m-1}, \tag{3.37}$$

where

$$V_{2,\infty} = \lim_{n\to\infty} E(w_{i,n}^2).$$

Equation (3.37) indicates that the asymptotic conditional probability of an alternation decreases geometrically to the N element model value of $[(N-1)/2N]$. It is clear that $V_{2,\infty}$ for the θ-N model will be the same function of θ and δ as $V_{2,\infty}$ in the RTI model (Eq. 3.16), since the latter is independent of c and we have already seen that $\{w_{i,n}\}_{n=1}^{\infty}$ can be regarded as an RTI process with $c = 1/N$.

The following results are useful in parameter estimation:

$$\lim_{n\to\infty} P(A_{1,n} \mid A_{1,n-1}E_{1,n-1}) = 2\left(1 - \frac{\theta}{N}\right) z_\infty + \frac{\theta}{N}, \tag{3.38}$$

$$\lim_{n\to\infty} P(A_{1,n} \mid A_{1,n-1}E_{2,n-1}) = 2\left(1 - \frac{\theta}{N}\right) z_\infty, \tag{3.39}$$

where

$$z_\infty = \lim_{n\to\infty} E[(\bar{w}_n)^2].$$

4. AN EXPERIMENT WITH NONCONTINGENT SUCCESS SCHEDULES

The last section contained a number of predictions for NCS experiments based on the N element, RTI, and N element-linear models—the three two-parameter specializations of the c-θ-N model. It was shown that each of these models predicts marginal constancy in the case $\delta = 1$, and, in the case $\delta < 1$, each of them predicts that $P(A_{1,n}) \to \frac{1}{2}$ as $n \to \infty$, regardless of the initial distribution of response probability. In addition, it was shown that the models disagree in certain of their first-order predictions; most importantly, in their predictions concerning the effects of a sequence of success trials on the probability of a response alternation. We now report an experiment designed, first of all, to test the common implications of the c-θ-N family, and secondly, assuming that the common predictions do well, to determine whether any one of the two-parameter members of that family is differentially supported.

Experimental Design

Two NCS schedules were studied: $\delta = .8$ and $\delta = 1$. Each of these schedules was employed in a single block of trials, with the two NCS blocks forming two phases of a standard binary prediction experiment. The subject's task was to predict which of two letters, "X" or "Y", would appear in a display window. Each trial began with the occurrence of a "ready" signal; as soon as this signal appeared the subject made his prediction by pressing one of two keys, and following his response either "X" or "Y" appeared briefly in the display window. The experiment consisted of 450 such trials and took about 45 min. Subjects were paid a fixed amount for participation, but there were no payoffs for correct responses, or penalties for errors.

From the experimenter's point of view the 450 prediction trials were divided into six phases, each phase corresponding to a particular reinforcement schedule. The schedule and number of trials in each phase are indicated in Table 1.

TABLE 1

Reinforcement Schedules in Phases I through VI

Phase	Schedules	Trials
I	NCE: $\pi = .5$	1–20
II	NCE: $\pi = .8$	21–70
III	NCE: $\pi = .2$	71–150
IV	NCS: $\delta = .8$	151–350
V	NCE: $\pi = .8$	351–400
VI	NCS: $\delta = 1.0$	401–450

The Y event was arbitrarily designated e_1, so that in the NCE phases, π refers to the probability of a Y event. The randomizations in each schedule were constrained by phase length; in each NCE phase the proportion of Y events was exactly π, and in each NCS phase the proportion of successes was exactly δ. For each phase, every subject's outcome sequence was an independent random permutation of a fixed sequence of events; either X's and Y's, or successes and failures.

The subjects were given no indication of the existence of distinct phases in the experiment, except what could be learned from the response–event sequence itself. The trial format was exactly the same on NCE and NCS trials, and the transitions between phases were not marked in any way. In running himself as a subject, the experimenter could not determine with any certainty where one phase had ended and another begun. It was expected that the fact that all trials were ostensibly the same would lead subjects to employ the same strategies in both the NCE and NCS phases of the experiment.

Apparatus. The experiment made use of an automated verbal associative learning apparatus at the Institute for Mathematical Studies in the Social Sciences, Stanford University. The subject sat alone in a soundproofed, air-conditioned room which contained stimulus display and response recording equipment. An adjoining room contained an IBM 526 Summary Punch and additional storage and timing equipment. On a table directly in front of the subject was a response panel containing two $\frac{7}{8} \times 1$-in. response keys, 3 in. apart. These keys could be illuminated to remind the subject which response he had made. The left hand key was labeled X, the right hand key, Y. On a second table 5.5 ft away were two stimulus display boxes, one on top of the other. The front of each box was a 2×12 in. display window in which alphanumeric characters (Sylvania Electroluminescent displays) could be made to appear to serve as the ready signal and event stimulus. The ready signal on each trial was a minus sign "–" in the center of the upper display window; the event stimulus was an "X" or "Y" appearing in the left-most position of the lower display window.

Procedure. The sequence of events on every trial was as follows: At the beginning of each trial both response keys and both display windows were dark. The trial began with the ready signal appearing in the upper display window. The apparatus then waited for the subject's response. As soon as the subject pressed a response key this key was illuminated, and the apparatus processed the response to determine which event (X or Y) to present. The event stimulus was then displayed. The delay between the response and the appearance of the X or Y event was 1.5 sec. The event stimulus remained on for 1 sec. At the end of this time both it and the illuminated response key went off, and after a 1.25-sec interval the ready signal appeared to begin the next trial. Assuming a 2-sec latency, the time between one ready signal onset and the next was 5.75 sec.

On arrival at the experimental room a subject was seated in front of the response panel and asked to read the following instructions:

"This experiment consists of a series of trials. On each trial exactly one of two possible events, "X" or "Y", will occur. Every trial will begin with the appearance of two small illuminated lines in the upper display window in front of you, as shown in Fig. 1 below. [Note: Figs. 1 and 2 were schematic drawings of the stimulus displays.] This is the signal for you to indicate your prediction by pressing either the X or the Y button on your response panel. After you make your prediction it will be automatically recorded by the IBM equipment, and the actual trial event—either an X or a Y—will appear in the lower display window as shown in Fig. 2. If the trial event agrees with your prediction you have made a correct response. The upper and lower display windows will extinguish after a brief period, and the next trial will begin shortly thereafter. It is important that you indicate your prediction as rapidly as possible after the signal light appears at the start of the trial, otherwise the experiment may run overtime.

We will not give you any information on how best to make correct predictions, except to urge you to pay close attention to the event on each trial. If you have any questions about the procedure, please ask the experimenter. There will be two short breaks during the session."

After the subject finished reading, he was asked if he had any questions. If he did, these were answered by paraphrasing the appropriate section of the instructions. The experimenter then ran through several practice trials (NCE: $\pi = .5$ trials; not part of Phase I) to make sure that the task was completely clear to the subject. As soon as this was certain the experimenter reminded the subject to respond "as quickly as possible," and then returned to the control room. The sequence of experimental trials began immediately and continued without interrruption for at least 250 trials. At a variable point between trials 250 and 280 the sequence was stopped to allow a 2-min rest period. The trial sequence then began again and continued without further interruptions through trial 450. As soon as the last trial had ended the subject was questioned to determine whether he had become suspicious of the task as a result of the final $\delta = 1$ phase. This interview was quite informal, but it always began with, "I'd like you to tell me on what basis you were making your predictions." This question appeared to be sufficient to partition the subjects into two distinctly different groups: those who had recognized that their responses (in Phase VI) were actually determining the outcomes (5 out of 55), and those who either had not noticed anything especially different at the end of the experiment, or who thought that they had finally solved the problem (50 out of 55). Subjects in the first group were questioned further to try to determine what had caused them to become suspicious; subjects in the second group were asked if they had managed to find any "pattern" in the outcome sequence "especially towards the end of the experiment." (Some volunteered this information before being asked.) If a subject answered that he had, he was asked to describe the pattern and the information was recorded. Finally, the experimenter explained the actual nature of the task, paid the subject, and dismissed him.

Subjects. Fifty-five subjects were obtained from the student employment offices of Stanford University and a nearby junior college. Each subject was paid $ 1.75 for his participation. The subjects were students; they ranged from first quarter freshmen to graduate students (none in psychology). Five subjects' data were not included in the analysis because they indicated in the postexperimental interview that they had recognized the $\delta = 1$ contingencies in Phase VI.

RESULTS

Data bearing on the predictions common to all c-θ-N models are examined first. Figure 1 shows the mean learning curve for the entire experiment. Each point represents the proportion of a_1 responses made by all 50 subjects in a given 10-trial block. To a first approximation the data are consistent with the predictions of the *c-θ-N* model: The learning curve begins in the neighborhood of .5 in Phase I, rises toward the probability matching value of .8 in Phase II, decreases towards the matching value of .2 in Phase III, and then moves back to .5 as predicted in Phase IV. In Phase V the curve rises again towards .8, this time somewhat more smoothly, and then remains quite constant during Phase VI. The data for Phases IV and VI are of particular

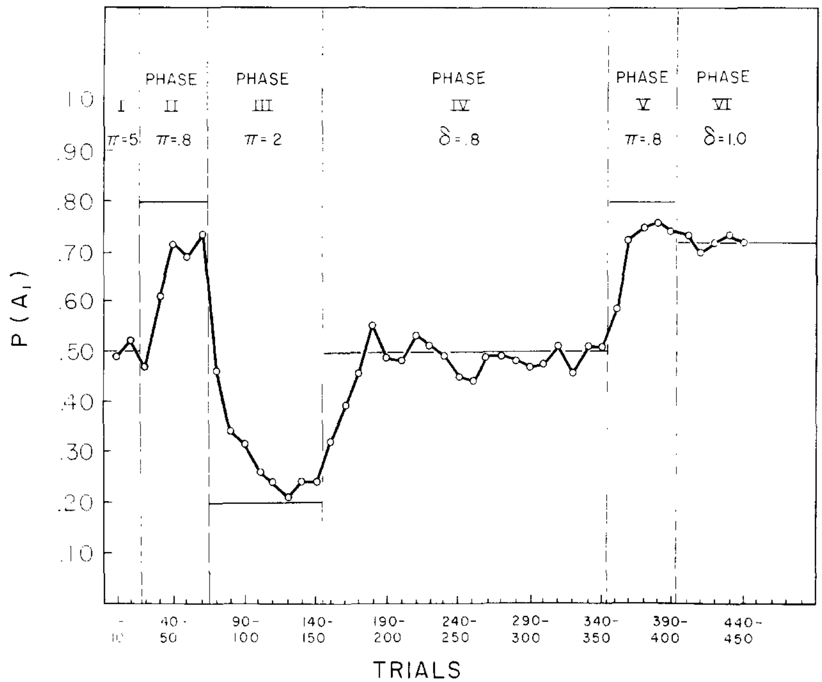

FIG. 1. Proportion of a_1 responses in 10-trial blocks: broken vertical lines separate phases; horizontal lines indicate the predicted asymptote in each phase.

interest. Over the five 10-trial blocks of Phase VI the a_1 proportions were (starting with the first block) .73, .70, .72, .73, and .72. These values clearly support the prediction of marginal constancy under $\delta = 1$ contingencies (Eq. 3.1); there is no trend in response probability over trials and all the values are satisfactorily close to the mean of .72 (the predicted line in Fig. 1 is at .72). The prediction $\lim_{n\to\infty} P(A_{1,n}) = .5$ for $\delta = .8$ was also confirmed to a good approximation; the overall proportion of a_1 responses in the last 100 trials of Phase IV was .486, and in the last 50 trials, .494. Individual subject response probabilities were distributed fairly symmetrically around .5: Over the last 100 trials 21 subjects had a_1 probabilities in the interval .41–.50, 15 in the interval .51–.60, and the remaining 14 subjects were scattered fairly evenly over the other eight intervals. To anticipate a bit, the variance of this distribution was quite accurately predicted by the N element model.

Turning now to results which ought to differentiate between the c-θ, c-N, and θ-N models, it is first necessary to estimate the learning rate parameter [i.e., the appropriate specialization of $(c\theta/N)$] for each model. On the basis of Eqs. 3.18 and 3.19, 3.26 and 3.27, and 3.38 and 3.39, this parameter can be estimated for all the models by taking the difference between the asymptotic sequential statistics $\hat{P}(A_1 \mid E_1 A_1)$ and $\hat{P}(A_1 \mid E_2 A_1)$ in Phase IV. The last 100 trials of Phase IV were assumed (both for this purpose and in subsequent analyses) to be asymptotic, and the estimate obtained on this basis was $\widehat{(c\theta/N)} = .175$. This value is comfortably close to estimates obtained in other experiment: Suppes and Atkinson (1960) estimated a learning rate of .174 from data generated by an NCE: $\pi = .6$ schedule, and Friedman *et al.* (1964) reported an estimate of .172 based on an NCE: $\pi = .80$ sequence. (These estimates were also based on asymptotic sequential statistics.)

Now, since $c\theta^2 \geqslant c^2\theta^2$, according to the RTI model (Eq. 3.7) the probability of an alternation on the nth trial of Phase VI should decrease at least as fast as $(.97)^{n-2}$, i.e., at least as fast as $[1 - \widehat{(c\theta)}^2]^{n-2}$. Consequently, according to the RTI model the proportion of alternations on Trial 50 of Phase VI should be no more than 23% of the corresponding proportion on the second trial. According to the N element model, of course, there should be no tendency for alternations to decrease over these trials. Table 2 shows the proportion of alternations in Phase VI averaged in seven-trial blocks. The data are clearly inconsistent with the RTI model: The maximum deviation of any proportion from the mean of .250 is .03, and the last entry is .242, whereas

TABLE 2

PROPORTION OF ALTERNATION RESPONSES IN 7 TRIAL BLOCKS: PHASE VI

Trials	Proportion of alternations
402–408	.257
409–415	.271
416–422	.226
423–429	.283
430–436	.242
437–443	.223
444–450	.242

according to the RTI model it should be around .06. Moreover there is no evidence of any consistent decrease over trials; the rank order correlation between block number and ordinal position in a ranking based on proportion of alternations is .43, which is not significant ($p > .10$). Consequently there is no evidence to support the θ-N model's prediction of a monotonic decrease in alternation probability (Eq. 3.10).

Examination of the effects of consecutive success trials in the $\delta = .8$ Phase IV leads to the same conclusion. The RTI model prediction here is given by Eq. 3.14. On the basis of this equation and the parameter estimate $c\theta = .175$ the RTI model implies that $P(S_n = 1 \mid \Delta_{n-1}^m) \leqslant (.97)^{m-1} P(S_n \mid \Delta_{n-1}^1)$. For $m = 10$ this implies that the estimated asymptotic conditional probability of an alternation given 10 success trials ought to be no more than about 75% of the corresponding estimate for 1 success trial. Table 3 gives the asymptotic proportions of alternations following m consecutive success trials for $m = 1, 2, \ldots, 10$. On the basis of the entry .313 for $m = 1$, the RTI

TABLE 3

OBSERVED CONDITIONAL ALTERNATION PROBABILITIES: PHASE IV, LAST 100 TRIALS

m	$\hat{P}(S_n \mid \Delta_{n-1}^m)$	Observations
1	.313	4021
2	.322	3200
3	.326	2555
4	.335	2039
5	.330	1637
6	.326	1317
7	.329	1062
8	.317	871
9	.312	711
10	.311	578
Av	.322	

model predicts that the entry for $m = 10$ should be on the order of .23; the observed value is .311. The rank order correlation between m and $\hat{P}(S_n \mid \Delta_{n-1}^m)$ is $\approx .3$, which is not significant, and none of the entries differs from the mean of .322 by more than .013. These results are entirely consistent with the N element model prediction of a constant alternation probability (Eq. 3.24), and quite inconsistent with the RTI model prediction of a geometric decrease with m.

It appears then that the RTI model can be rejected as incompatible with the results of Phases IV and VI. The case against the N element-linear model is naturally not as strong, since the issue between that model and the N element model is less sharp. However, there is no evidence in Tables 2 and 3 for the decreases in alternation probability predicted by the θ-N model, and these would be the only justification for accepting it in preference to the N element model. Consequently we reject it also, and proceed to a more detailed evaluation of the N element model. For this purpose the parameters c and N have been re-estimated in a manner that uses somewhat more of the data than our earlier estimation. To estimate N we make use of the fact that each of the entries in Table 3 is an estimate of $(N - 1)/2N$, (cf. Eq. 3.24); taking the average of these values leads to the estimate $\hat{N} = 2.78$. This is to be interpreted as an estimate of the average value of (the integer) N across subjects. (We note by way of comparison that the estimates of N obtained in the previously cited experiments by Suppes and Atkinson, and Friedman *et al.*, were 3.48 and 1.84, respectively.) To estimate c, we use the fact that for $\delta < 1$, the difference between $P(S_n = 1 \mid \tilde{\Delta}_{n-1})$ and $P(S_n = 1 \mid \Delta_{n-1})$ is equal to c/N (Eqs. 3.24 and 3.25). The observed asymptotic value of $P(S_n = 1 \mid \tilde{\Delta}_{n-1})$ was .494 (based on 979 observations). From this we obtain a c/N estimate of .173, which is virtually identical to the previous estimate of .175. Combining these c/N and N estimates we have $\hat{c} = .48$. These estimates will be used in subsequent computations of predicted values.

Table 4 shows the predicted and observed values of a number of asymptotic sequential statistics from Phase IV. All first and second order sequential probabilities are included, except those for which the observed proportions in the last 100 trials of Phase IV were based on fewer than 125 observations.

TABLE 4

N ELEMENT MODEL PREDICTIONS AND OBSERVED SEQUENTIAL DATA: PHASE IV, LAST 100 TRIALS ($\widehat{c/N} = .173, \hat{N} = 2.78$)

	Pred.	Obs.	No. of observations
$P(A_1 \mid E_1A_1)$	.680	.674	1918
$P(A_1 \mid E_2A_1)$	.507	.499	493
$P(A_1 \mid E_1A_2)$	.493	.488	486
$P(A_1 \mid E_2A_2)$	.320	.301	2103
$P(A_1 \mid E_1A_1E_1A_1)$	.760	.730	1012
$P(A_1 \mid E_2A_2E_2A_2)$	.240	.254	1165
$P(A_1 \mid E_1A_1E_2A_2)$	.500	.538	507
$P(A_1 \mid E_2A_2E_1A_1)$	.500	.438	516
$P(A_1 \mid E_2A_1E_1A_1)$	.580	.500	264
$P(A_1 \mid E_1A_2E_2A_2)$	.420	.473	294
$P(A_1 \mid E_1A_1E_2A_1)$	.681	.677	198
$P(A_1 \mid E_2A_2E_1A_2)$	.319	.282	195
$P(A_1 \mid E_1A_1E_1A_2)$	.675	.740	181
$P(A_1 \mid E_2A_2E_2A_1)$	.325	.257	206
$\text{Var} \sum_{n=251}^{350} A_n$	206.	233.	50

By and large the fit between predicted and observed values in Table 4 is quite good; the largest deviation is .08, and as the number of observations increases the deviations tend to become smaller. The fact that the model predicts a variance of the proper order of magnitude is particularly impressive; this prediction is very sensitive to small changes in the parameter values. (If the variance prediction were computed in terms of the proportion of a_1 responses over the 100 trial block it would be .021, as compared to an observed value of .023.) In evaluating the goodness of fit of the model to the Phase IV data, it should be recalled that Table 3 also contains relevant asymptotic sequential statistics: $P(S_n = 1 \mid \Delta_{n-1}^{m})$ for 10 values of m. The model predicts all 10 of these values with a maximum deviation of .02.

A persistent problem in fitting c-θ-N models to binary prediction data has been that the learning rate parameter estimated from asymptotic statistics normally does not permit a good prediction of the acquisition portion of the learning curve (see, e.g., Friedman *et al.*, 1964). Typically (although not always) it has been found that the learning rate parameter needed to fit the learning curve is about one-third the value of the same parameter as estimated from asymptotic sequential statistics. In this respect the present experiment is something of an exception; the asymptotic estimate $c/N = .173$ fit the learning curve for Phase IV relatively well: Except for two points which could not be fit by any choice of parameters the discrepancies between predicted and observed values in the first 11 blocks of Phase IV were all $\leqslant .02$. Interestingly, however, and more in line with previous research, $c/N = .173$ would not fit the learning curves in the NCE phases (II, III, and V). There the required value was, in fact, about one-third of .173: an adequate fit of the first four points of the Phase V curve was obtained with $c/N = .058$. This is the value that was estimated from the acquisition phase of the $\pi = .8$ series of the Friedman *et al.* experiment—the same series which yielded an asymptotic c/N estimate of .174.

We turn next to an evaluation of the N element model predictions for Phase VI—the $\delta = 1$ phase. We saw earlier that the trial by trial proportions of a_1 response and alternations computed over all subjects (Fig. 1 and Table 2) were consistent with the N element model. At the individual subject level the model makes the very strong prediction that each subject's response sequence is a Bernoulli process. To evaluate this prediction chi-square tests were performed on the Phase VI data to test for stationarity over trials in the individual subject response probabilities, and to determine whether the individual response sequences exhibited the predicted trial by trial independence. In the test for stationarity ($df = 1$) six subjects had χ^2 values significant at the .05 level. The total χ^2 for all subjects was 58.52, which is not quite significant at the .05 level for $df = 40$. (Ten subjects made all X's or all Y's in Phase VI and hence did not figure into this test.) The test for response independence led to somewhat sharper results. The total χ^2 for all subjects ($df = 80$) was 208, and six subjects had χ^2's significant at the .01 level. The total χ^2 for these six alone ($df = 12$) was 122. These results suggest a more serious deviation from the model on the part of some subjects.

As it turns out, these deviations can be observed quite directly in the response protocols themselves, and in the relationship between these protocols and the subjects' descriptions of their own behavior.

In the interview at the end of Phase VI, several subjects volunteered the information that they had solved the prediction problem "near the end of the experiment" by discovering a "pattern" in the event sequence. By this they meant either that the sequence appeared to be formed by repeating a fixed cycle of events—e.g.,

$$\cdots \overbrace{e_1 e_1 e_2}\ \overbrace{e_1 e_1 e_2} \cdots,$$

or that the events came in runs which were of increasing, but predictable length, e.g.,

$$\overbrace{e_2}\, e_1\, \overbrace{e_2 e_2}\, e_1\, \overbrace{e_2 e_2 e_2}\, e_1 \cdots.$$

We will refer to reported patterns of the first type as periodic patterns, and to the others as aperiodic patterns. (The cycle of a periodic pattern may contain more than two runs, e.g.,

$$\overbrace{e_1 e_2 e_2 e_1 e_2}\ \overbrace{e_1 e_2 e_2 e_1 e_2} \cdots.)$$

Of course the patterns discovered by these subjects were entirely their own creations, since the event sequence in Phase VI was simply a reflection of their responses. For just this reason they are of particular interest; we would expect the subject to create a pattern of some sort if his behavior in the prediction situation were determined by an effort to find such a pattern, i.e., if he were trying out plausible patterns in an attempt to find one that fit the event sequence.

Figure 2 shows each subject's Phase VI response protocol as a sequence of 1's and 0's. The 1's denote Y predictions, the 0's X predictions. Subject 10 reported that towards the end of the experiment he discovered the periodic pattern $\langle 5Y, 2X \rangle$ ... (i.e., the periodic pattern consisting of a repeating cycle of five consecutive Y's followed by two X's.) Figure 2 shows that his responses followed this pattern throughout trials 7-50 of Phase VI. Using a bit more ingenuity, subject 12 discovered the periodic pattern $\langle 10Y, X, 9Y, X, \ldots, 2Y, X, Y, X \rangle$, and his responses correspond to this pattern beginning with trial 17 of Phase VI. (He starts at $5Y$). Altogether 26 subjects claimed to have discovered some pattern in the event sequence "near the end of the experiment." (This does not include subjects who reported "all X" or "all Y.") These reports were made with varying degrees of conviction and precision; some were volunteered and others came only after the experimenter specifically mentioned the possibility of patterns. The protocols of these 26 subjects were examined to determine whether they had actually responded according to the patterns they claimed to have discovered. A subject's pattern was classified as confirmed if the last two complete runs of responses in his protocol, together with the final run (i.e., the run terminated by the

Sub	TRIALS (1 … 25 … 50)
1	10111011010100100010000010000000100000000010000000
2	10010001000010000010000001000000010000000010000000
3	11111111110100111111111101001111111111010011111111
4	11011111111101111111110111111111011111111101111111
5	11111111111011001111111111011111111011111110111101
6	10010101111111111111111111111111111111111111111111
7	11111111111111111101111111111111111111101111111111
8	11111101100100000000001010011011111111101001101000
9	10111111111111111110111111111011111111101111111111
10	11111011111001111100111110011111001111100111110011
11	11101111111011111011111110111111111110111111111111
12	11010111111111101111101111011101101011111111110111
13	11111110101111111111011111111101111111011110110100
14	01111111111111111111111111111111111111111111111111
15	01101011011011011010110110110110110110110110110110
16	11111111111111111111111111111111111111111111111111
17	00110000111100000000001111111111000000000111111111
18	00000000001000010101010101001110000111111100000011
19	00110011001100110011001100110011001100110011001100
20	11111111111111111111111111111111111111111111111111
21	10011110101011111111101010101010101010101010101010
22	11111111111111111111111111111111111111111111111111
23	00001111111111000111111111000000011111111111100000
24	11010100000000001010111111111101010000000000010101
25	11111111111111111111111111111111111111111111111111
26	11010100100001000000000010000000000010000010000000
27	00010001110000000001011110000101011111100011000111
28	00111100111011100111011100111011100111011100111011
29	00000000001111000111111001111110011111100111111001
30	10010111111001011111100101111110010111111001011111
31	10111011111111111000000000011111111000001111011111
32	11111111111111111111111111111111111111111111111111
33	11111111111011001001110111111111111111110100101111
34	10011111111111111110111001111111111111110111001111
35	11111111111111111111111111111111111111111111111111
36	11101010010110110100001001010110100000011011010111
37	11111111111111110101101111111111111111010111111111
38	11111111111111111111111111111111111111111111111111
39	11111011111110011111100111111001111110011111100111
40	01111011111011111101111111011111111011111111110111
41	11111111101011011101111011111011111101111111011111
42	11111111010101111111111101010111111111111111010111
43	10101111100010111111001011111101010001111101000001
44	11111111111111111111111111111111111111111111111111
45	10111111111111111111111101011111010111111111010111
46	11111111111111111111111111111111111111111111111111
47	00001111010000101111010110101111100101111110111101
48	10000010110010000001100100110001001000110011000111
49	10001111111111111111111000011111111111111111111111
50	00000000000000000000000000000000000000000000000000

FIG. 2. Phase VI response protocols: 1 = Y, 0 = X.

end of the experiment) formed a sequence consistent with any portion of the reported pattern. Thus, for example, subject 39 claimed to have discovered the periodic solution $\langle 6Y, 2X \rangle$, and this was confirmed; subject 34 reported the periodic pattern $\langle 2Y, 3X \rangle$, and this was not confirmed. On this basis 18 of the 26 reported patterns were confirmed. The patterns themselves covered a wide range of complexity and sophistication. Subject 21, for example, reported solving the problem with a simple alternation *YXYX*..., while subject 1 reported a periodic pattern based on the first six prime numbers: $\langle X, Y, 2X, Y, 3X, Y, 5X, Y, 7X, Y, 11X, Y \rangle$. (In the protocol it appears he either miscounted at $11X$ or mistook 9 for a prime.) An example of an aperiodic pattern is provided by subject 2: he reported X, Y, $2X$, Y, $3X$, Y... and this describes his protocol.

It would be difficult to give any summary characterization of the various patterns that subjects reported. Some idea of the range of complexity is provided by examples already cited. In the case of the periodic patterns the proportion of Y's in the repeating cycle could be calculated; the mean of these proportions across all reported patterns (including confirmed and nonconfirmed) was .68. With "all X" and "all Y" included as patterns; the proportion of Y's across all reported patterns was .76. It will be recalled that the schedule in Phase V was NCE: $\pi = .8$; consequently there is a suggestion that the patterns discovered by the subjects, or at least the patterns that seemed plausible to the subjects, tended to probability match in the mean. This would be expected if the effect of an NCE schedule were to restrict subjects "working hypotheses" to those that are consistent with the observed sequence of (noncontingent) events; either in a general sense (e.g., matching the observed proportions of e_1 and e_2 events), or specifically in the sense of being compatible with the actual sequence of previous events over the last few trials. However, it should be noted that there was a good deal of variability in the proportions; they ranged from zero to 1.00.

DISCUSSION

The results of the experiment can be summarized as follows. First, the mean learning curve predictions common to all c-θ-N models were confirmed for the NCS schedules $\delta = .8$ and $\delta = 1.0$. Second, under NCS reinforcement consecutive success trials did not produce any decrease in the probability of a response alternation. In this respect the predictions of the N element model proved to be correct and those of the RTI model incorrect. Third, the N element model predicted the asymptotic sequential statistics of the NCS: $\delta = .8$ phase with considerable accuracy using an estimate of the learning rate parameter c/N (.173) which was virtually identical to estimates obtained in previous experiments by Friedman *et al.* (1964; $c/N = .172$) and Suppes and Atkinson (1960; $c/N = .174$). Fourth, under NCS : $\delta = 1$ contingencies a number of subjects fixated on deterministic response patterns consisting of runs of a_1's and a_2's. In some cases these patterns formed periodic sequences (e.g., $\langle 5a_1, 2a_2, 5a_1, 2a_2 ...\rangle$), in other cases aperiodic sequences (e.g., $\langle a_1, a_2, 2a_1, a_2, ..., na_1, a_2, ...\rangle$). For convenience in the balance of the discussion we refer to these structured response patterns which emerged under $\delta = 1$ contingencies as "superstitious solutions." The analogy to superstitious behavior in operant conditioning does not seem too far-fetched: In both cases reinforcement is not contingent on any particular configuration of responses, but subjects nevertheless fixate on more or less complex, idiosyncratic, response patterns (Skinner, 1948).

As far as the original problem of deciding between the N element and linear (i.e., RTI) models is concerned, these results are unequivocal: the N element model is clearly superior. The data of Phase IV ($\delta = .8$) are particularly relevant to the conclusion, since the experimental conditions in that phase were similar to those in an ordinary prediction task with noncontingent events and the N element model was able to provide a good account of all aspects of the data. The fact that the parameter estimates from Phase IV were equal to those obtained under NCE reinforcement in other experiments suggests that the behavior observed in this experiment was generated by learning mechanisms common to all prediction experiments with simple contingent schedules. On these grounds it seems reasonable to expect that the N element model will in general be superior to the RTI model in any direct comparison. One basis for comparison which was not exploited in this study is provided by Eqs. 3.17 and 3.25. These give the respective predictions of the RTI and N element models for the probability of a response alternation following m consecutive noncontingent failures. A comparison similar to that provided by Phase IV of this experiment could easily be achieved using, e.g., an NCS : $\delta = .2$ schedule.

The results of Phase VI ($\delta = 1$), however, raise some doubts as to the usefulness of further comparisons between models of the c-θ-N family, except perhaps in the context of psychophysical experiments. The highly structured response sequences produced by a number of subjects in that phase clearly cannot be explained by any of the c-θ-N models (cf. in particular subjects 2, 3, 4, 10, 15, 19, 21, 28, 34, and 41). They suggest instead that the underlying learning processes in even this very simple prediction task are fairly complex, and that the good fit of the N element model under certain schedules is not so much the solution to a problem as a problem in itself. It has, of course, always been recognized that path independent models of the c-θ-N variety cannot provide a general theory of probability learning because they do not permit the subject to learn deterministic event sequences involving alternations. But one might have hoped that these models, or something similar, would be at least asymptotically adequate for experiments with simple contingent schedules in which there is no reinforcement pressure to support more complex strategies. The results of Phase VI suggest instead that the mechanisms which enable subjects to learn deterministic sequences remain active throughout any prediction experiment, even when the reinforcement schedule does not allow them to be of any use to the subject. Of course, it is not clear whether this is true for all subjects, since not every subject fixated on a superstitious solution during the 50 $\delta = 1$ trials of Phase VI. Further investigation will be required to determine whether this is due to individual differences, or simply to a random selection caused by the relatively small number of $\delta = 1$ trials.

The occurrence of superstitious solutions under $\delta = 1$ contingencies is not entirely surprising. Arima's experiment (1965) apparently led to the same result, although his description (cf. Sec. 1) suggests that the solutions generated after 50 $\pi = .5$ trials were somewhat simpler than those which appeared here after 400 trials on a variety of schedules. It has often been reported that subjects in binary prediction tasks claim to be searching for patterns in the event sequence (e.g., Jarvik, 1951; Nies, 1962), and Feldman (1963) has attempted to account for binary prediction behavior with a simulation model based solely on hypothesis testing mechanisms. Feldman's experimental situation is somewhat different than the standard prediction task, since his subjects were required to provide a rationale for each response. The results of Phase VI make it unlikely that hypothesis testing alone can account for ordinary probability learning, since not all subjects fixated on superstitious solutions, and even those who eventually fixated did not always do so as soon as $\delta = 1$ contingencies were introduced (e.g., subject 21). It may be that sporadic hypothesis testing can account for this variability, but in that case some supplementary mechanism would be required to account for responses that occur between hypotheses.

As an alternative to hypothesis testing, one might suppose that superstitious solutions arise simply as the result of straightforward conditioning processes such as those suggested by Burke and Estes (1957) and, more recently, Gambino and Myers (1967). In these models probability learning involves simply the conditioning of a_1 and a_2 responses to event sequences of a fixed length (the "K-span" model in Rose and Vitz's, 1966, terminology), or to runs of homogeneous events (in Gambino and Myers' extension of a model proposed originally by Restle, 1961, Ch. 6). Neither of these models can account for the entire range of periodic solutions observed in Phase VI, and they are also known to be inadequate for experiments with deterministic (Restle, 1967) and quasi-deterministic (Rose and Vitz, 1966) event sequences. But it is not difficult to imagine that a model similar to that of Gambino and Myers could be devised which would account for periodic solutions and also predict the standard statistical results of experiments with simple contingent schedules (i.e., probability matching, positive recency, and so on). In fact, it can be shown that if the model proposed by Gambino and Myers is altered to allow generalization only after incorrect responses, simulated subjects fixate on periodic superstitious solutions of the form $\langle ma_1, na_2, ma_1, na_2, ... \rangle$ in a way that closely resembles the behavior of human subjects. (Generalization can be permitted to occur only on errors because generalization on successes precludes fixation on patterns other than "all a_1," "all a_2," and "$a_1a_2a_1a_2$....") However it is difficult to see how any such model could account for fixation on aperiodic solutions, even if we allow sequences of runs to operate as the effective stimuli, as Restle (1967) has suggested. For if we suppose that, for example, subject 41 would have continued with the aperiodic sequence $\langle a_1, a_2, 2a_1, a_2, ..., na_1, a_2 ... \rangle$ for an indefinite number of trials, then it is necessary to assume that the appropriate responses must somehow have become conditioned to run sequences which the subject had never previously experienced. Considerations of this sort suggest that a complete theory may need to include mechanisms by which the subject can make long range predictions, perhaps in the form of hypothesized algorithms capable of generating infinite sequences of events.

One other point seems worth mentioning. Restle (1966) reported experiments in which the event sequences contained both random and deterministic features. In one version the sequence consisted entirely of runs of length 2 and length 5. After a run of $2e_i$ events the run either continued (with probability .5) or broke off. In this experiment then subjects could learn to always predict an e_2 following $5e_1$'s, to always predict an e_1 following a run of $4e_1$'s, and so on. They could not learn anything useful about what to do after a run of $2e_1$'s. Rather surprisingly, subjects did not learn to perform perfectly on runs of length 5; even after 600 trials the probability of incorrectly predicting another e_1 after $5e_1$'s remained greater than .3 and the learning curve appeared asymptotic. One way to explain this failure is to suppose that subjects were not able to discriminate perfectly between runs of length 5 and shorter runs. But in this case subjects could not maintain stable periodic superstitious solutions of the form $\langle 5a_1, 2a_2 \rangle$, and the results of Phase VI suggest that in fact they can (e.g., subject 10). This discrepancy may simply be due to individual differences in memory span. Another possibility is that prediction errors, which were unavoidable in Restle's experiment because of the random choice point at runs of length 2, have a particularly disruptive effect on memory. According to this interpretation a subject could have essentially perfect recall for the current run if he had predicted all of its events correctly, but incorrect predictions could cause him to lose track of the actual current run length.

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